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Amarnath Mukherjee
University of Pennsylvania

Jaffar Rehman
University of Pennsylvania

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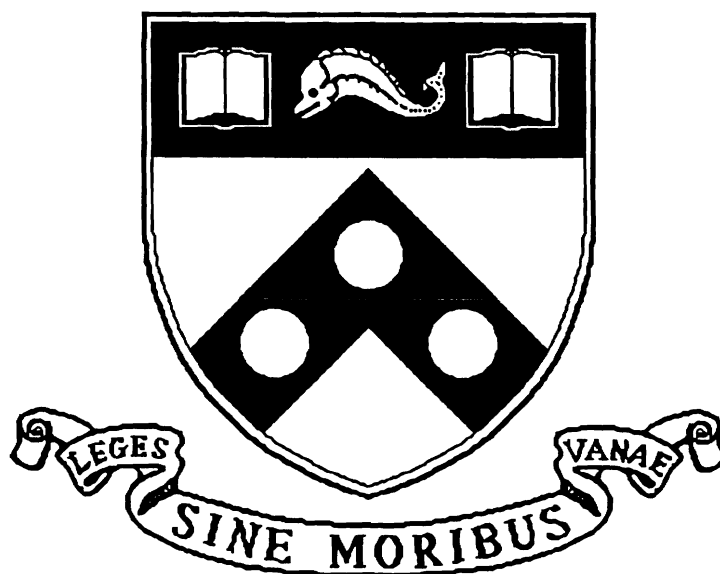
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DISTRIBUTED SYSTEMS LAB 10

Amarnath Mukherjee
Jaffar Rehman

University of Pennsylvania
School of Engineering and Applied Science
Computer and Information Science Department

Philadelphia, PA 19104-6389

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Amarnath Mukherjee
Jaffar Rehman

Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA 19104

January 27, 1992

Abstract

This study presents simple and accurate heuristics for determining the equivalent bandwidth for multiplexed variable bit rate (VBR) video sources. The results are based on empirical studies of measurement data of various classes of VBR video sources. They are also validated through extensive simulations. The principal result is that the equivalent bandwidth per source for n independent and identically distributed VBR video sources may be approximated by a hyperbolic function of the form: $a \coth^{-1} n + b$ where a and b are independent of n. Further, assuming ϵ is the acceptable loss tolerance, statistical regression shows that b is a linear function of mean and $\log(\epsilon)$, while a is a polynomial in $\log(\epsilon)$.

The capacity assignment problem is further augmented with a scheduling algorithm that is an extension of the Virtual Clock Algorithm. The new algorithm belongs to a class of algorithms which we refer to as Generalized Virtual Clock (GVC) algorithms. The particular GVC algorithm investigated in this paper estimates the instantaneous rate of transmission of each source, and uses this estimate instead of the static average rates, for prioritizing packets. In so doing, it attempts to synchronize the switch scheduling rates and the packet arrival rates of each source, and improves upon the spatial loss distribution characteristics of Virtual Clock. The combined allocation and scheduling algorithms are proposed as means for guaranteeing Quality of Service in high speed networks.

1 Introduction

Optical fibers have provided large technological advances in the field of telecommunications. Network prototypes with bandwidths of few hundreds of Mbps have been available for sometime and prototyping of networks with Gbps data rates is in progress. The availability of enormous bandwidth in high speed networks allows support for applications with very high bit rates such as motion video. Video traffic, in fact, is likely to be the most dominant load component of the future multimedia communications environment. Whereas the support for video applications is not entirely new, the multitude of end-users who would be able to use these services simultaneously will be a new phenomenon. This potential scale of availability of services however, poses many new challenges in networking. Bandwidth allocation and proper scheduling are two such challenges. Efficient bandwidth allocation is a difficult task because it involves simultaneous pursuit of two conflicting goals, namely provision of desired Quality of Service (QOS) to the users and efficient utilization of network resources. This is the subject of this paper.

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The capacity assignment problem is further augmented with a scheduling algorithm that is an extension of the Virtual Clock Algorithm. This algorithm belongs to a class of algorithms that is referred to as Generalized Virtual Clock (GVC) Algorithms. The particular GVC algorithm investigated in this paper is shown to improve upon the spatial loss distribution characteristics of Virtual Clock. The combined allocation and scheduling algorithms are proposed as means for guaranteeing Quality of Service in high speed networks.

In the rest of this section, we first give the problem statement and then overview the issue involved in capacity allocation and scheduling.

1.1 The problem statement

The capacity allocation problem addressed in this paper may be stated as follows. Given a known probability distribution of the number of bits per frame transmitted by a variable bit rate video source and the need to ensure that a switching node drops less than some $\epsilon > 0$ number of bits, what is the necessary switching capacity to support N such sources? The sources may be homogeneous (i.e., i.i.d.) or heterogeneous.

The scheduling problem may be stated as follows. Once a certain capacity is allocated for N sources, what are the tradeoffs involved in using one scheduling strategy over another?

1.2 Capacity Allocation

The easiest algorithm is to allocate bandwidth to a connection based on its peak bit rate. Such an allocation results in deterministic guarantees for the quality of service but results in underutilization of network bandwidth. This is because it ignores dynamics of the workload in an environment where several sources are being multiplexed over the same communication channel. The loss of the potential statistical multiplexing advantages actually may be quite significant if peak rate is allocated. For instance, as reported in [36], the peak bandwidth allocation for a video conferencing source was 14 Mbits/sec. For 16 sources, this allocation would sum up to 224 Mbits/sec. The probability of achieving a combined peak bit rate of 224 Mbits/sec, however, was found to be only 10^{-48}. In fact, for a cell loss rate of 10^{-8}, a total allocation of 108.8 Mbits/sec or approximately 7 Mbits/sec per source was found to be sufficient. The statistical multiplexing gain in this case would result in a bandwidth saving of a factor of 2.

The other extreme to peak rate allocation is to allocate bandwidth based on the sum of the average bandwidths of the sources. Average bandwidth allocation addresses the concern of under-utilization of bandwidth, but may result in intolerable penalties on Quality of Service, unless the number of sources being multiplexed (degree of multiplexing) over a link is large. An appropriate capacity allocation algorithm must take into account both the desired Quality of Service requirements and the efficiency of the communication channel, and should allocate an amount somewhere between the sum of the peak and the sum of the average rates of the sources. This amount is usually referred to as equivalent bandwidth. (A precise definition of equivalent bandwidth will be given in Section 3.)

Several schemes have been proposed recently [4, 6, 10, 11, 12, 35, 36, 37] for determining equivalent bandwidth. [4, 6, 10, 11] propose using precomputed bandwidth requirement curves defined through simulation. [12] proposes a combination of fluid approximation method and a stationary approximation method for on-off sources with exponential holding times in each state. [36, 37] suggest the use of average rate and variance of sources to compute the equivalent bandwidth. [35] suggests the use of probability density functions (pdf's) of sources to calculate the equivalent bandwidth. The pdf's are to be calculated using sources' average and peak data rates. The

convolved pdf of these individual pdf's gives the bandwidth requirements. Such on-line calculation of the convolved pdf may, however, not be computationally feasible in a real time environment.

In this paper, we investigate simple and accurate heuristics for determining the equivalent bandwidth for variable bit rate video sources. We use steady state bit-rate distributions of sources to determine these formulae. The bit-rate distributions are extracted from measurement data of traffic from fifteen different VBR video sources described in the next section. Our study shows that equivalent bandwidth may be closely approximated by a class of Hyperbolic functions. Specifically, if Z_N is the bandwidth needed to guarantee Quality of Service (precise definition of Quality of Service is given in the next section) for N i.i.d. sources, then

$$\frac{Z_N}{N} \propto \coth^{-1}(N).$$

The rationale for this is also discussed, although a rigorous theoretical proof is not available at this time.

1.3 Scheduling

A related issue in resource management of variable bit rate video sources is that of scheduling and policing. Several new algorithms have been proposed recently, e.g., Virtual Clock [40], Hierarchical Round Robin [24], Pulse [27], Earliest Due Date [8] and Stop-and-Go Queueing [13, 14]. While the exact details differ (e.g., Stop-and-Go is designed to be non-work-conserving, others are not), these algorithms assume a contract between the network and end-users and attempt to prioritize packets according to that contract. The contract is static, in that it does not adapt to the time correlation structure of the packet arrival process. However, they are effective policing devices against misbehaving transmitters.

The algorithm proposed and studied here belongs to a class of Generalized Virtual Clock (GVC) algorithms. Earlier, we have studied a GVC algorithm that was suitable for data traffic [27]. The particular algorithm proposed here is applicable to variable bit rate sources with real-time deadlines, and is centered around the idea of exploiting the time-correlation structure of the source arrival processes.

The outline of the paper is as follows. Section 2 describes the data used in the study, the model used to characterize the sources, and the precise definitions of various measures used later. Section 3 investigates the capacity assignment problem, while Section 4 investigates the scheduling problem. Section 5 briefly compares related work and Section 6 presents the conclusions of this paper and outlines future work. The appendices document some of the details omitted from the main sections for clarity of presentation.

2 Preliminaries

2.1 Classification of Measurement Data

The study reported here is based on measurement data on video traffic [38]. The data comprised of sequences of numbers indicating the number of bits transmitted in successive frames. We have data for a total of fifteen different video sources. The data may be categorized into four different classes according to their statistical properties:

- Video Telephony: head and shoulder view of one person.

- Video Conferencing: head and shoulder view of two people.

- Normal Quality Broadcast Video: Different TV programs captured from the Belgian Cable Television network.

- High Quality Broadcast Video: Different TV programs taken directly from the studios of the Belgian Radio and Television.

Important statistical characteristics of each class of traffic is summarized in the following table.

Table 1. Statistical Characteristics of VBR Video Data			
Service Class	Average Rate(s) (Mbits/s)	Peak Rate(s) (Mbits/s)	Length Data Used
Video telephony	4.5	28.5	25 min's
Video conference	4.4	28.5	26 min's, 26 sec's
Normal quality broadcast video	15.4 - 19.1	24.4 - 39.4	30 min's, 52 sec's to 35 min's, 38 sec's
High quality broadcast video	25 - 30.6	36.6 - 43.2	18 min's, 30 sec's to 31 min's, 27 sec's

As the table indicates, the data used was collected over long periods of time. The length varied from a minimum of 18 minutes and 30 seconds (27750 frames) to a maximum of 35 minutes and 38 seconds (53450 frames) of video activity. Each source generated 25 frames per second, i.e, the interframe gap was 40 ms.

2.2 Source Characterization

A bandwidth management algorithm needs to know the characteristics of a source based on which it will try to allocate appropriate amount of bandwidth to satisfy the requested QOS by the source during the connection. The most widely suggested parameters to characterize a source include average bit rate, peak bit rate, burst length and inter burst gap. These parameters may be considered sufficient to accurately describe the behavior of (i) continuous bit stream sources that transmit at a fixed or a fairly consistent rate, (ii) sources which oscillate between idle and busy periods with transmission at a fixed or fairly consistent rate (peak rate) during the busy periods. The type (ii) sources are often characterized as two-state Markov processes [1, 12, 7].

Capturing all the statistical characteristics of a source which does not transmit at a fixed rate is, however, non-trivial. Sources such as variable bit rate video, transmit across a wide range of bit-rates, and their characterization by only peak and average rates cannot sufficiently capture their statistical behavior. Consider, for instance, Figure 1 which shows the number of bits transmitted in each frame as a function of time.

Some researchers have quantized the bit rates per frame into M levels and then modeled the sources as M-state Markov processes [32]. Such modeling also captures the temporal behavior of a source. The knowledge of the temporal characteristics of very high bit rate bursty sources may, however, not be useful, as suggested in [36, 37]. Their argument is that network queues are too small for any significant absorption of very high bit rate bursts of sources such as VBR video, and should be used only to absorb the asynchronicities generated by the network itself [37]. It is argued in [25, 36] that the contribution of network queues for statistical multiplexing of VBR video sources is marginal, and that they may be omitted while describing the process. This implies that no knowledge of the temporal behavior of the sources is required. The knowledge of stationary bit-rate probability distributions of the sources is however, needed, to determine the joint probability distribution of bit-rates when the sources are multiplexed.

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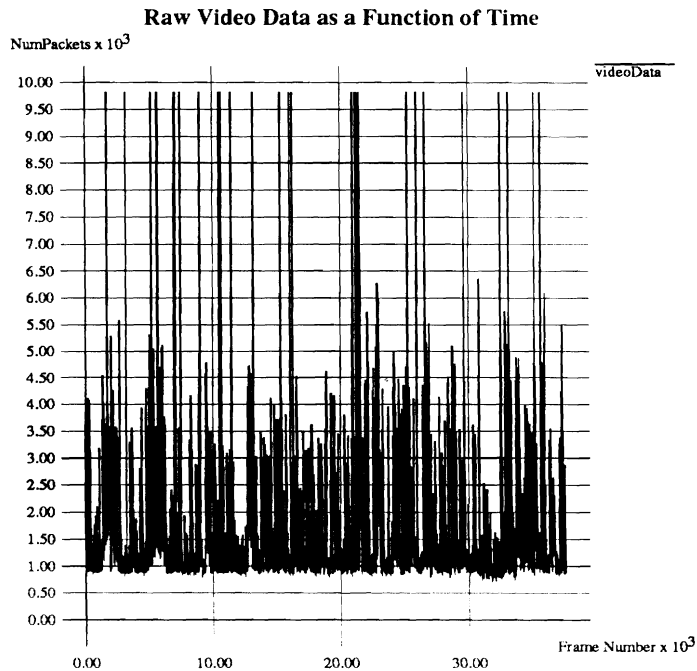**Raw Video Data as a Function of Time**

NumPackets x 10³

Figure 1: The number of bits transmitted per frame as a function of time for a typical VBR video source.

Our approach has been to characterize the variable bit rate video sources by their stationary bit-rate probability density functions for determining equivalent bandwidth and to incorporate the time correlation structure into the scheduling algorithm.

2.3 Assumptions and Definitions

The capacity allocation problem assumes that the variable bit rate sources have known stationary distributions, $F_X(x)$, of the the number of bits transmitted in a frame. For the case when the sources are independent and identically distributed, the sum of the bits transmitted by N such sources in the duration of a frame time is assumed to be S_N (irrespective of whether the frames are synchronized or not). That is, if X_i is the number of bits transmitted by the ith source, then

$$S_N = X_1 + X_2 + \cdots + X_N.$$

It is assumed that the maximum tolerable probability of loss for each source is ϵ. The capacity allocation problem may now be viewed in two equivalent ways.

One view is to assume that (a) the switch has infinite buffers, and that (b) all sources are synchronized (technically, with infinite buffers, it is easy to show that this is not necessary, but it simplifies exposition). The problem then is to allocate an equivalent bandwidth Z_N such that

$$Pr\{S_N > Z_N\} < \epsilon \tag{1}$$

Z_N is the number of bits that are cleared out by the end of a frame. Notice that since ϵ is assumed to be the same for all sources, the individual loss probabilities per source given by (1), may be made equal to ϵ with a proper scheduler (because $\epsilon S_N = \epsilon X_1 + \epsilon X_2 + \cdots + \epsilon X_N$).

An alternate but equivalent view is to assume that there are very few buffers compared to the amount received over a frame [25, 36]. Let the instantaneous transmission rates of the sources be

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$\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_N$, where $\tilde{\lambda}_i$ are independent of each other. If S_N is set to $\tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_N$, then the equivalent bandwidth Z_N may again be defined by (1). Now ϵ would indicate the proportion of dropped packets given lack of availability of buffers. (In this case, however, the worst case scenario of synchronized frames must be assumed.) The two views therefore, lead to the same analysis methodology. An analysis with finite buffers is not addressed in this paper.

3 Capacity Allocation

We assume that the transmission of bit streams of all the sources being multiplexed over a single link are statistically independent of each other.

As mentioned in Section 2, the problem of capacity allocation may be stated as follows. Given a known probability distribution of the number of bits per frame transmitted by a variable bit rate video source and the need to ensure that a switching node drops less than some $\epsilon > 0$ number of bits, what is the necessary switching capacity to support N such sources? The sources may be homogeneous (i.e., i.i.d.) or heterogeneous.

3.1 Homogeneous Multiplexing

Homogeneous multiplexing refers to multiplexing of i.i.d. video sources. For instance, multiplexing of traffic from several video telephony sources which are defined by the same density function, would be categorized as homogeneous multiplexing.

Let $X_1, X_2, \cdots, X_N$ be i.i.d. random variables with a known distribution $F_X(x)$. Figure 2 shows the stationary probability density function, $f_X(x)$ $(= dF(x)/dx)$, of the number of bits transmitted in a frame of a particular source. Let $S_N = X_1 + X_2 + \cdots + X_N$. The problem is to determine the minimum capacity (Z_N) required by the switch such that most of the bits transmitted in a frame may be cleared out by the end of frame. i.e., if the video sources require a probability of loss less than ϵ, then

$$Z_N = \inf_x : Pr\{S_N > x\} < \epsilon \tag{2}$$

One approach to solving this is to assume that the central limit theorem holds. i.e.,

$$Pr\left\{ \frac{\frac{S_N}{N} - \mu}{\sigma/\sqrt{N}} > x \right\} - \eta(x) \tag{3}$$

where $\eta(x)$ is the Standard Normal distribution, and μ and σ^2 are the mean and variance of X_1.

Assuming (3) holds, let $\epsilon = \eta(x) \Rightarrow x = \eta^{-1}(\epsilon)$ which together with (2) yields

$$\frac{Z_N}{N} = \mu + \eta^{-1}(\epsilon)\frac{\sigma}{\sqrt{N}} \tag{4}$$

To check the validity of (4). $f_X(x)$ may be convolved N times to give the density of S_N. As an example, for $N = 2$, the joint density of $X_1 + X_2$ is

$$f^{*2}(x) = f_{X_1 + X_2}(x) = \int_{-\infty}^{\infty} f_X(y) f_X(x - y) dy \tag{5}$$

The convolutions as a function of N for Figure 2 are shown in Figure 3. Notice that these indeed look similar to the Normal density function. However, when the tail is integrated to determine Z_N, and Z_N/N is plotted as a function of N. we see an apparent discrepancy between (4) and the actual Z_N/N computed from the convolved distribution (see Figure 4).
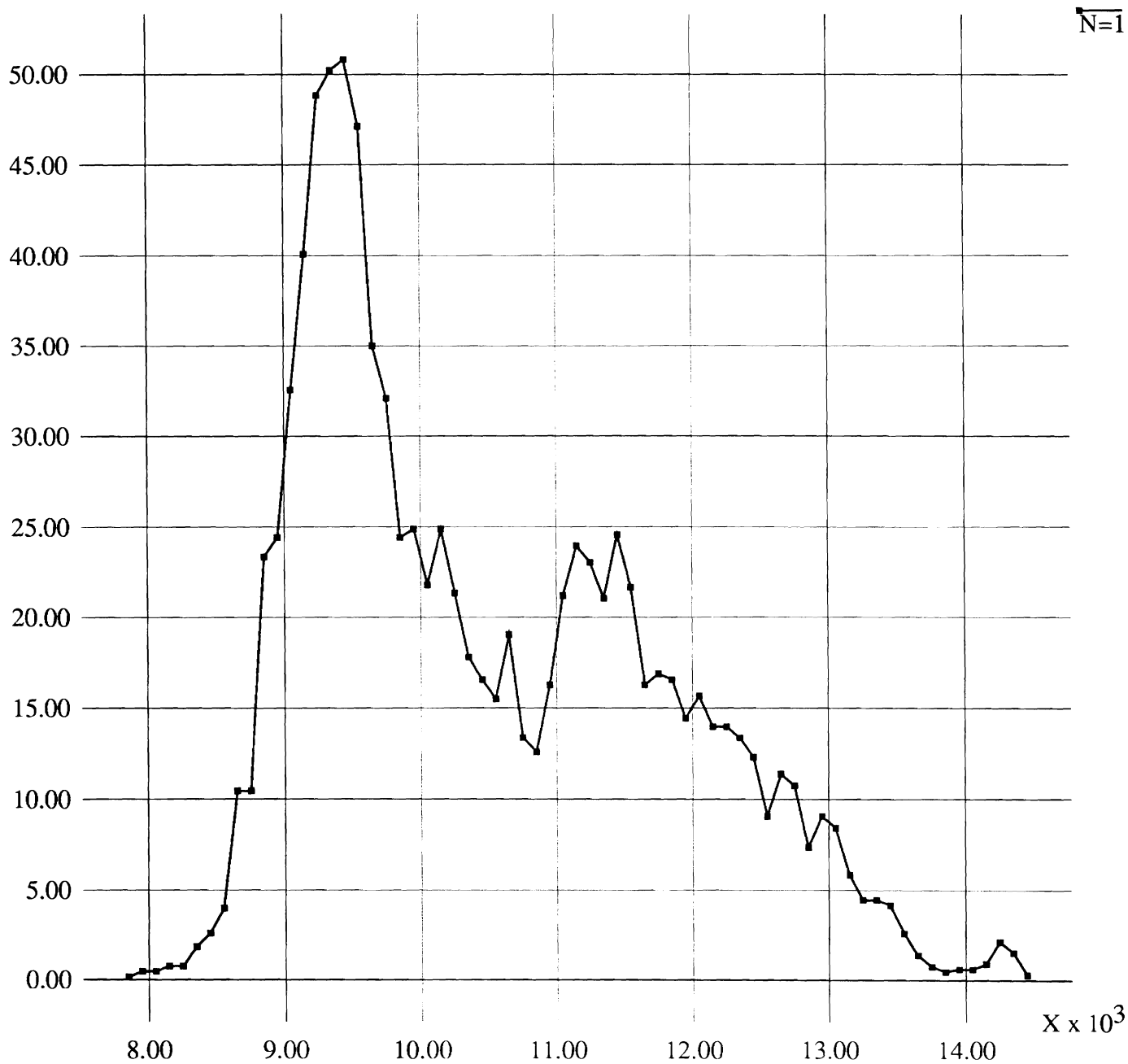
Probability Density Function

f(x) x 10⁻³

Figure 2: Probability density of a VBR video source.

Probability Density Function
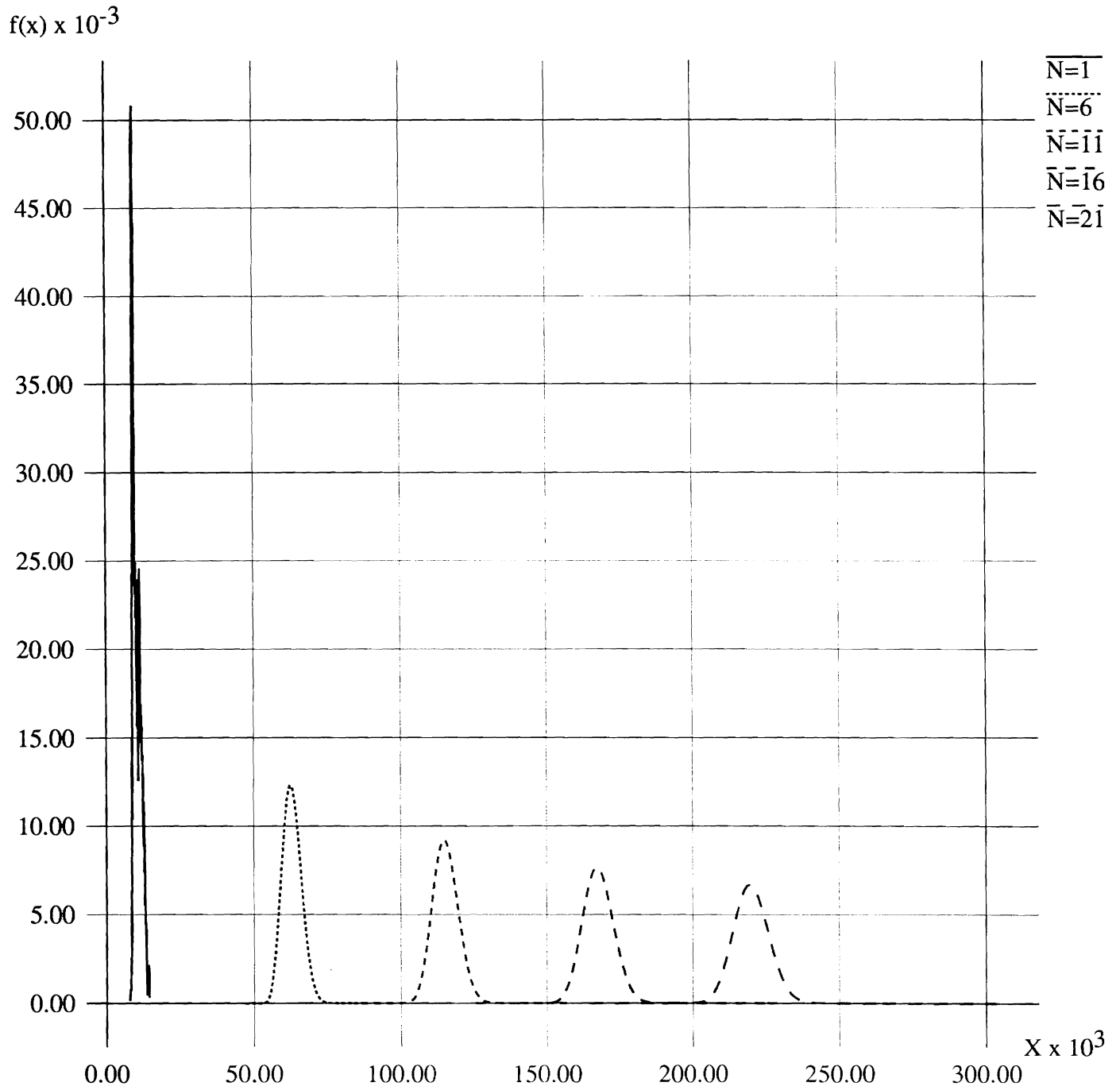
f(x) x 10^-3

Figure 3: Convolved densities of the same video source.

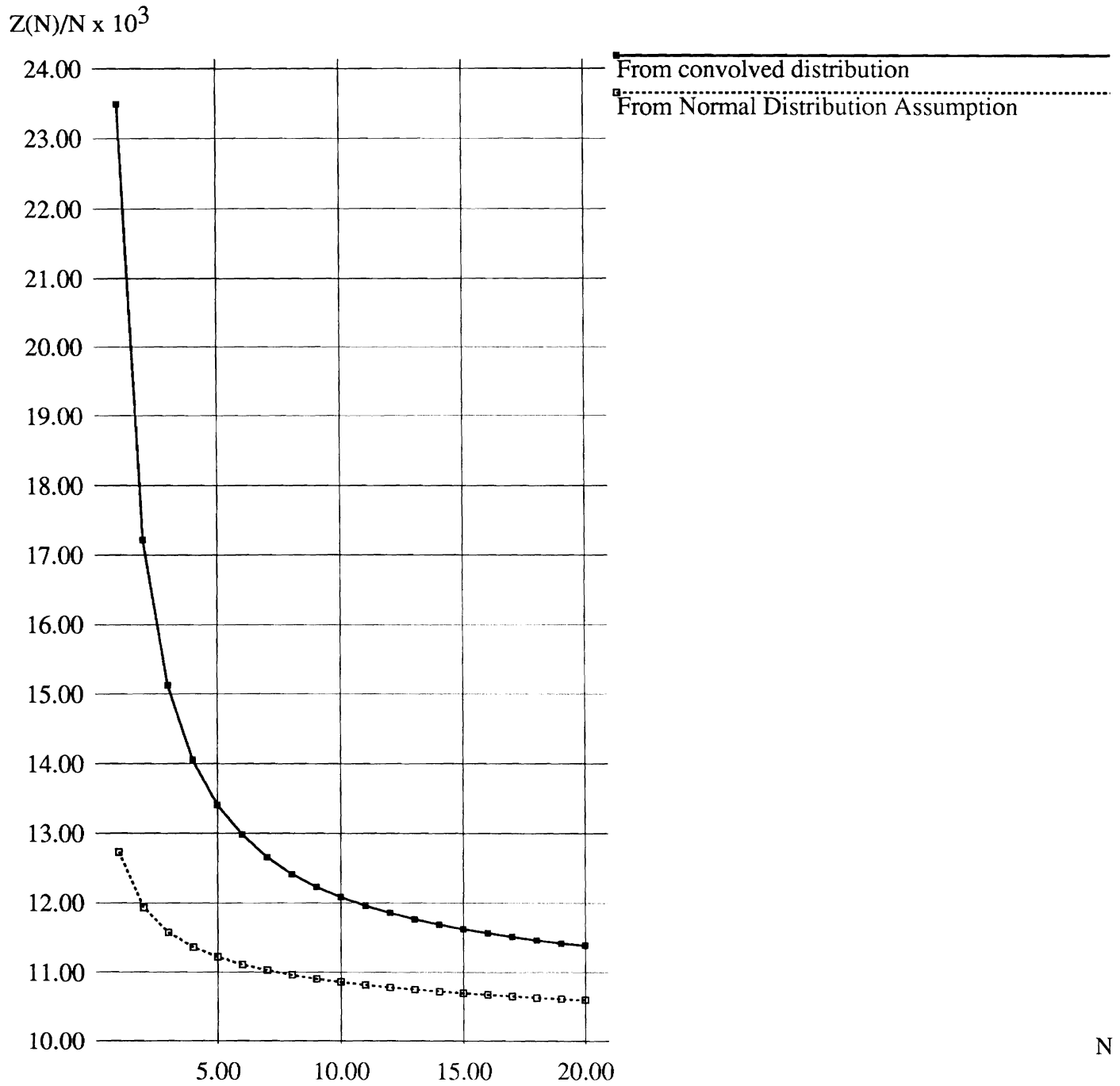# Z(N)/N vs. N

Z(N)/N x 10^3

Figure 4: Z_N/N vs. N from convolution and from Normal assumption.

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This may be explained by checking for the validity of the observed distribution to see if it indeed is Normal. Figure 5 shows a quantile-quantile plot of the convolved distribution for $N = 11$. To obtain this plot, one computes $F_X^{*N}(x)^{-1}$ vs. $\eta(x)^{-1}$, i.e., consider the points (x_i, y_i) such that for some q_i, $0 \leq q_i \leq 1$, $F^{*N}(y_i) = q_i$ and $\eta(x_i) = q_i$. Then the plot of y_i vs. x_i should be a straight line if the two distributions being compared match. (For the purposes of comparison, this may be carried out with a Standard Normal distribution with mean 0 and variance 1, see [23]. Figure 5 shows that the convolved distributions are indeed Normal for the most part (in the middle), but diverge from it sharply at the tails. Further, it shows that the tail of the experimental distribution is longer than that of the Normal distribution. Since our interest is exactly on this part of the distribution, the central limit theorem does not hold, and in fact predicts lower capacity than is actually required. The important question is what is the correct equation for Z_N/N. Experiments with curve fitting show that

$$\frac{Z_N}{N} \propto \coth^{-1} N \tag{6}$$

Figure 6 demonstrates this: Here we have plotted Z_N/N (as obtained from the convolution) vs. $\coth^{-1} N$ and the result is a straight line. Surprisingly, this relationship holds for all the fifteen different VBR video sources whose data was available to us.

From (6) we have

$$\frac{Z_N}{N} = a \coth^{-1} N + b \tag{7}$$

The coefficients a and b were obtained using linear regression. These were then investigate further to determine possible relationships between them, ϵ, and the properties of the distributions. We investigate these in detail next.

3.2 The coefficient b

To determine the relationship between b and the parameters of the distribution and the Quality of Service parameter ϵ, first, we need to identify some dominant patterns similar to the one for Z_N/N vs. N.

Let m denote the mean and σ the standard deviation of a distribution. Let ϵ denote the loss tolerance of a source, and let $\xi = -\log_{10} \epsilon$.

Figure 7 shows b vs. m for several sources with ϵ as a parameter. Notice that b vs. m is a straight line.

Figure 8 shows b vs. ξ for the different sources. Again, notice that they have a linear relationship.

Experiments were conducted to check if b depended on any of the higher moments of the distributions. No significant patterns were observed for the next 16 higher moments and central moments, at which point the experiment was stopped.

The above observations suggest the following form for b :

$$b = b_0 + b_1 m + b_2 \xi \tag{8}$$

Statistical regression from the data gives[1]

$$b_0 = 35974.286(-155145.434, 227094.007)$$

---

[1]Appendix B overviews Multiple Linear Regression, Analysis of Variation, Analysis of Variance and generation of confidence intervals. The key idea in regression is to minimize the sum of squares of errors between the actual data points and the model that is developed from it. If SSE is the sum of squares of errors and SST is the sum of squares of the total variation, then an amount equal to $SSR = SST - SSE$, is successfully accounted for by the regression. The coefficient of determination R^2 is defined as SSR/SST, and it gives a measure of the goodness of the regression. Acceptance on rejection of a hypothesis is based on the analysis of variance which yield $F_{computed}$. This value must be greater than $F_{table}(\beta)$ for the regression to be accepted at a confidence level β.
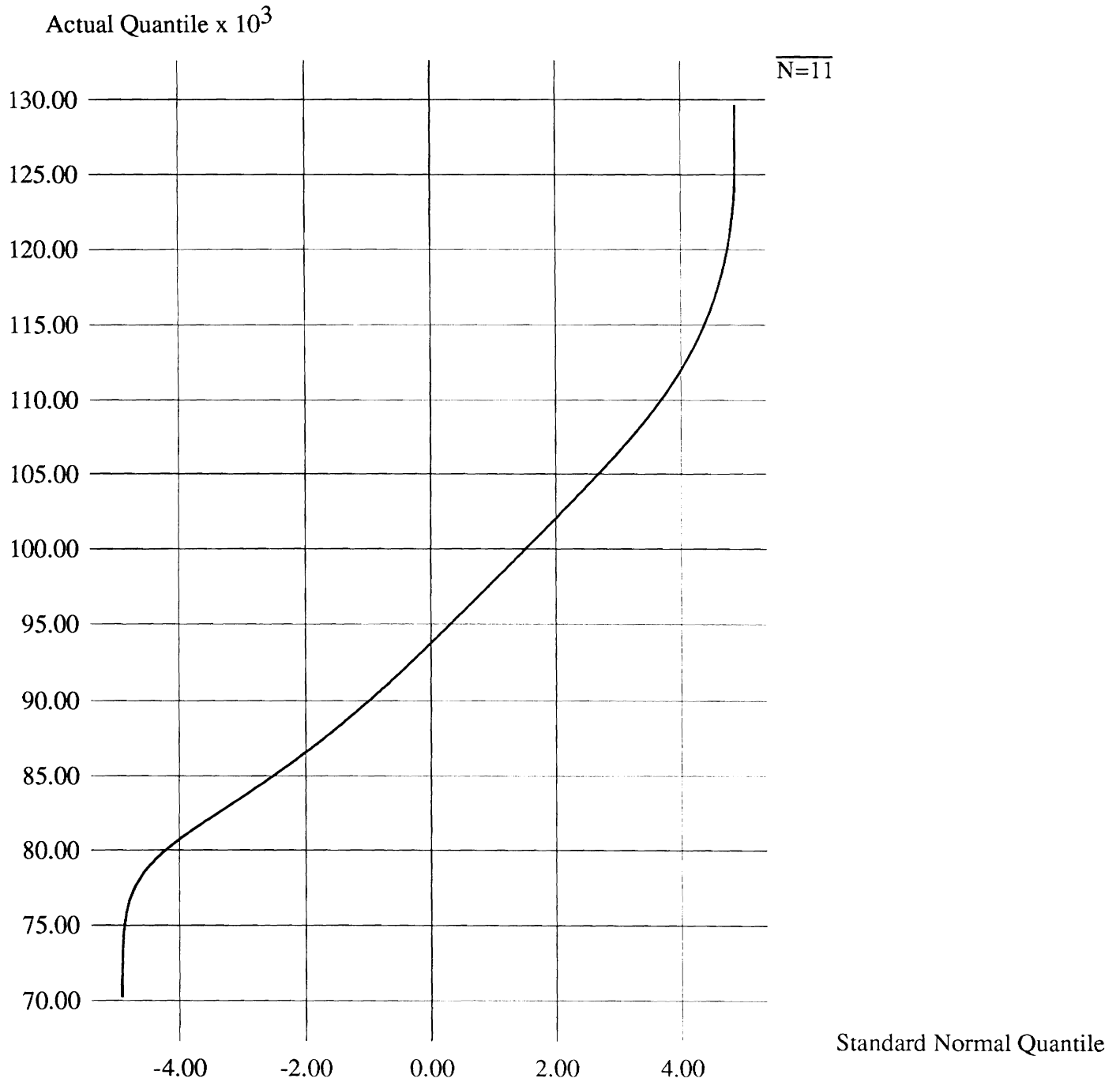
Quantile-Quantile Plot

Actual Quantile x 10^3

$\overline{N=11}$

Standard Normal Quantile

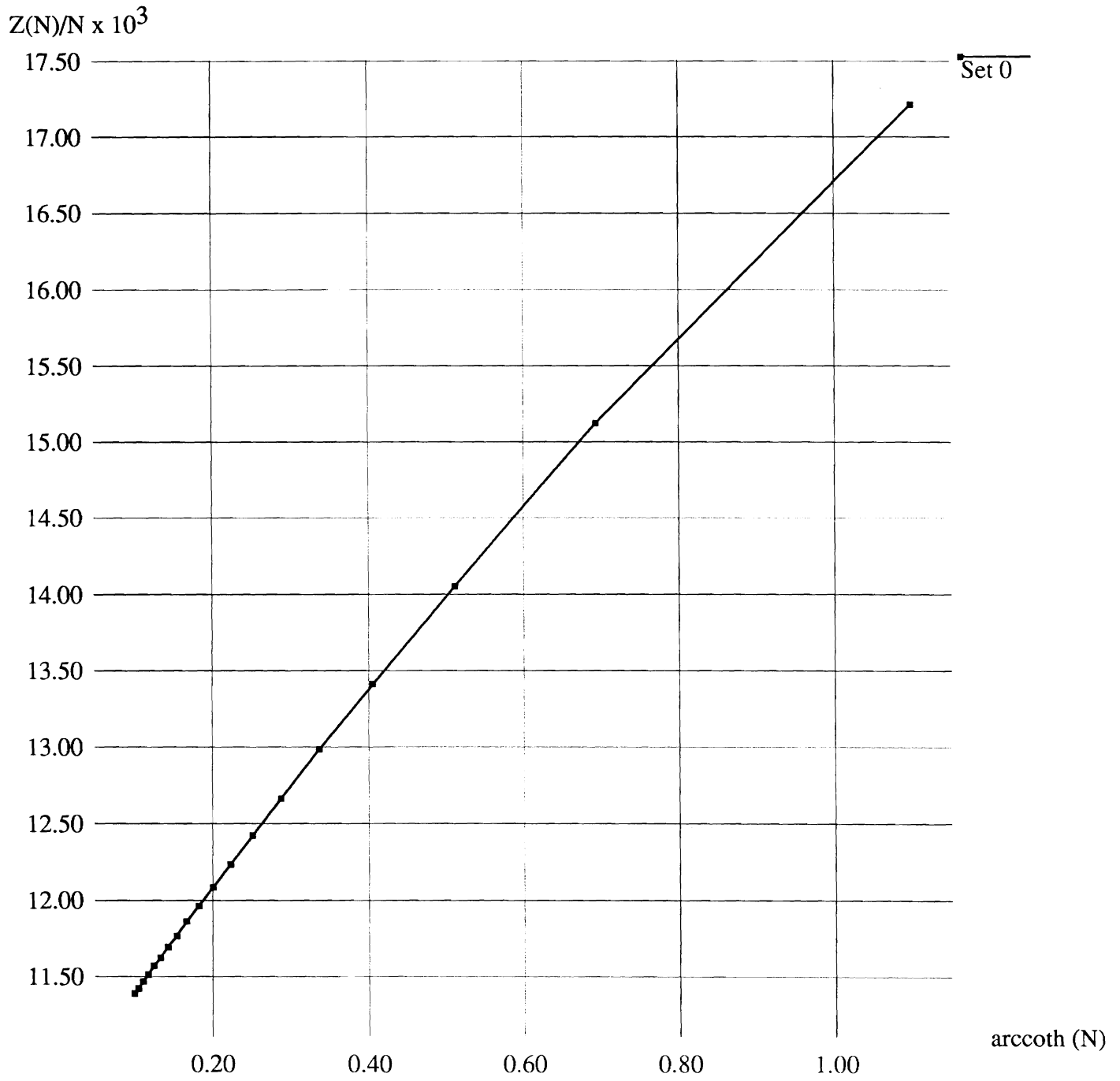Figure 5: Quantile-Quantile plot of observed and Standard Normal distributions.

Figure 6: Empirical validation of the Arccoth assumption for Equivalent bandwidth.
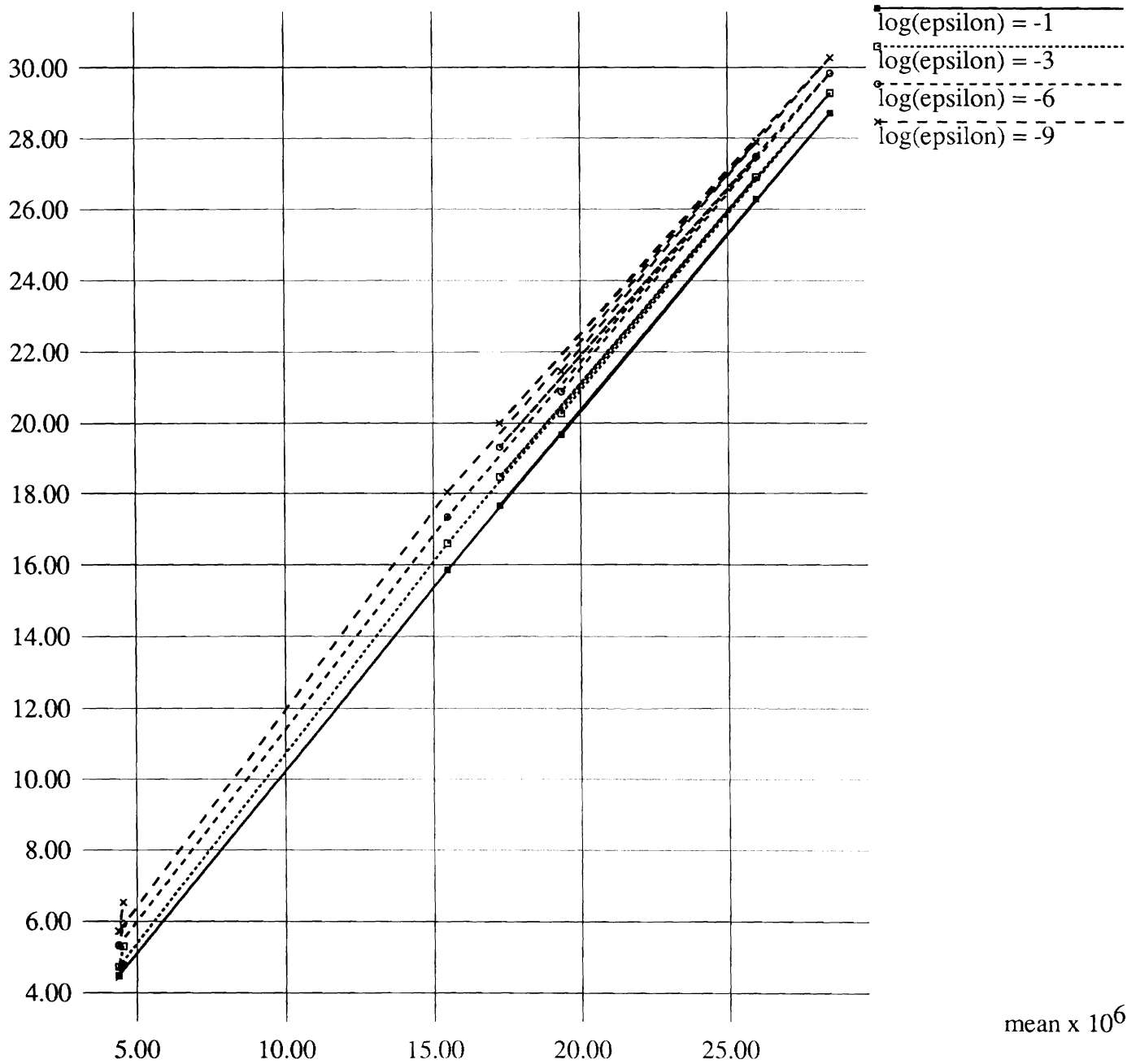
b vs. mean

b x 10^6

Figure 7: b vs. mean for the data set.

mean x 10^6

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b vs. - log(epsilon)
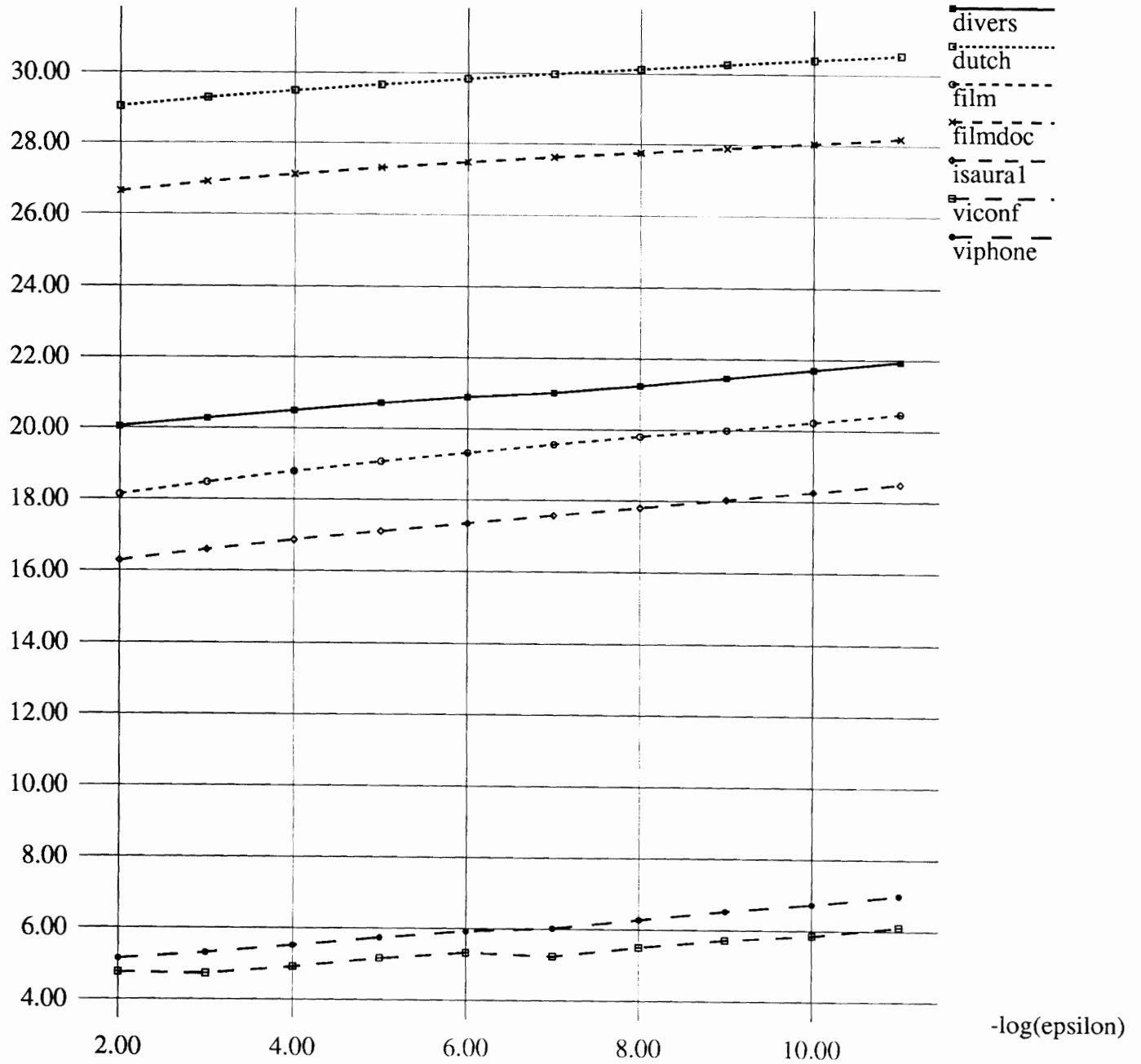

Figure 8: b vs. $-\log_{10} \epsilon$ for the same data set.

$$b_1 = 1.011(1.004, 1.019)$$
$$b_2 = 207023.555(185885.406, 228161.704)$$

where the numbers in parentheses are the 90% confidence intervals for the coefficients. The coefficient of determination, $R^2 = 0.998$ and $F_{computed} = 23682.89$. From tables, $F_{table} = 9.48$ for a 90% confidence level, and $F_{table} = 99.48$ for a 99% confidence level. Since $F_{computed} >> F_{table}$, for both these confidence levels, this regression can be accepted with a high level of confidence. Further, the value of R^2 also indicates that the regression explains over 99% ($= \sqrt{R}$) of the variance.

Notice that the confidence interval for b_0 includes 0, so 0 cannot be rejected at the 90% confidence level, (i.e., b_0 might as well be set to 0 for all practical purposes). Hence we have

$$b = 1.011m + 207023.6[-log_{10}(\epsilon)] \tag{9}$$

This then is the equation for the coefficient b in (7).

3.3 The coefficient a

Figure 9 shows a vs. ξ for several of the sources. This figure suggests a non-linear relationship between the two. In fact it looks similar to a hypoexponential. As we shall see shortly, it turns out that a polynomial approximation for a vs. ξ yields pretty accurate results.

Figure 10 shows a vs. σ for $\xi = -1$ which suggests a linear relationship. However, this does not hold for smaller values of ξ as seen in Figure 11.

Regression of a involving both σ and ξ in fact, did not yield satisfactory results. For instance, experiments with the form

$$a = a_0 + a_1\sigma + \sum_{i=1}^{m} a_{i+1}\xi^i$$

yielded a coefficient of determination. R^2. of 0.64 or less (i.e, the confidence level one could place on this regression was only $\sqrt{R} = 0.8$ or 80%, at best).

When regression was tried on individual sources however, the coefficient of determination shot up to 0.99. For instance, consider a quadratic model for a as an approximation for the data:

$$a = a_0 + a_1\xi + a_2\xi^2 \tag{10}$$

(10) was regressed individually against each source distribution. The coefficient values computed, yielded a high level of accuracy ($R^2 > 0.99$). The statistical test results for one particular source is included in Appendix C. The results for the coefficients and their confidence intervals for several sources are summarized in Table 2. The analysis of variation and variance for the corresponding coefficients is shown in Table 3. Notice that in all cases, the coefficient of determination R^2 is large, and $F_{computed} >> F_{table}$ indicating that the regression may be accepted with a high level of confidence. Investigation is in progress to determine how to combine all the equations into one.

Table 2. Estimation of a : Coefficients and 90 percent Confidence Intervals.		
Source	a_0 and 90% Conf. Int.	a_1 and 90% Conf. Int.	a_2 and 90% Conf. Int.
divers	-2309786 (3045880, 1573692)	4709942 (4428012, 4991872)	-161427 (-184310, -138545)
dutch	1257487 (855991, 1658983)	1499605 (1345829, 1653381)	-77611 (-90092 , -65130)
film	673455 (58790,1288121.018)	3404385 (3168963, 639806)	-132489.516 (-151597, -113381)
filmdoc	970699 (674823.952, 1266575)	1866265 (1752942.777, 1979588)	-71539 (-80737, -62341)
isaural	-145835 (-407089, 115418)	3655070 (3555007, 3755132)	-159839 (-167960, -151717)
viconf	-3678783 (-6268656, -1088910)	4492990 (3501048, 5484932)	-143843 (-224353, -63333)
viphone	-2590536 (-3620172, -1560899)	4750290 (5931.971, 5144649)	-178501 (-210509, -146494)

Table 3. Analysis of Variation and Analysis of Variance for (10)			
Source	R^2	F_{computed}	F_{table}
divers	0.998	2614.38	99.37
dutch	0.990	411.91	99.37
film	0.998	1629.29	99.37
filmdoc	0.998	2164.36	99.37
isaura1	0.999	8458.92	99.37
viconf	0.981	209.19	99.37
viphone	0.997	1193.94	99.37

3.4 Heterogeneous Source Multiplexing

Heterogeneous source multiplexing refers to multiplexing sources with different distributions. This subject is currently under investigation and only preliminary results are available at this time.

If the Quality of Service parameter ϵ is the same (and this need not be in a real world situation), the convolution approach may be used to generate the data for the capacity needed. Let $Z(n_1, n_2, \cdots, n_k)$ be the minimum capacity needed for supporting n_1 sources of type 1, n_2 sources of type 2, etc., such that the Quality of Service guarantees are met. Then, preliminary experiments with $k = 2$ sources show that $Z(n_1, n_2)/n_1$ is proportional to $\coth^{-1}(n_1)$ and $Z(n_1, n_2)/n_1$ is proportional to $\coth^{-1}(n_2)$, see Figures 12 and 13.

The exact relationship is currently under investigation and will be reported when available.

4 Scheduling

Once sufficient capacity has been allocated, it is of interest to know what role a scheduling discipline plays at the switch. Specifically, is it necessary to provide a scheduler that is any more complicated than First-Come-First-Serve (FCFS) to achieve Quality of Service guarantees? The answer is positive, as discussed below.

4.1 Role of A Scheduling Discipline

The purpose of a scheduling discipline is to regulate competing traffic with some form of fairness. In the context of VBR video transmission, this role translates to the following:

- Provide firewall against misbehaving transmitters. In this role, the scheduler acts as a policing agent and ensures that those who violate an initially agreed upon contract do not adversely affect those who are well behaved.

- Provide priority disciplines in order to facilitate different rates to different transmitters. There are two cases that need to be addressed, one when all transmitters are well behaved, and one when they are not.

  If transmitters could misbehave, a priority discipline could provide the necessary firewall to each individual class of traffic. For instance, Virtual Clock, Hierarchical Round Robin and Pulse try to match usage rate of resource with the contract agreed upon and prevent misbehaving users from monopolozing resources.

  If all VBR video transmitters are well behaved however, and enough capacity has been allocated, then is a scheduling discipline still potentially useful? The answer appears to be positive, given a variable bit rate transmission. The next item addresses this subject.
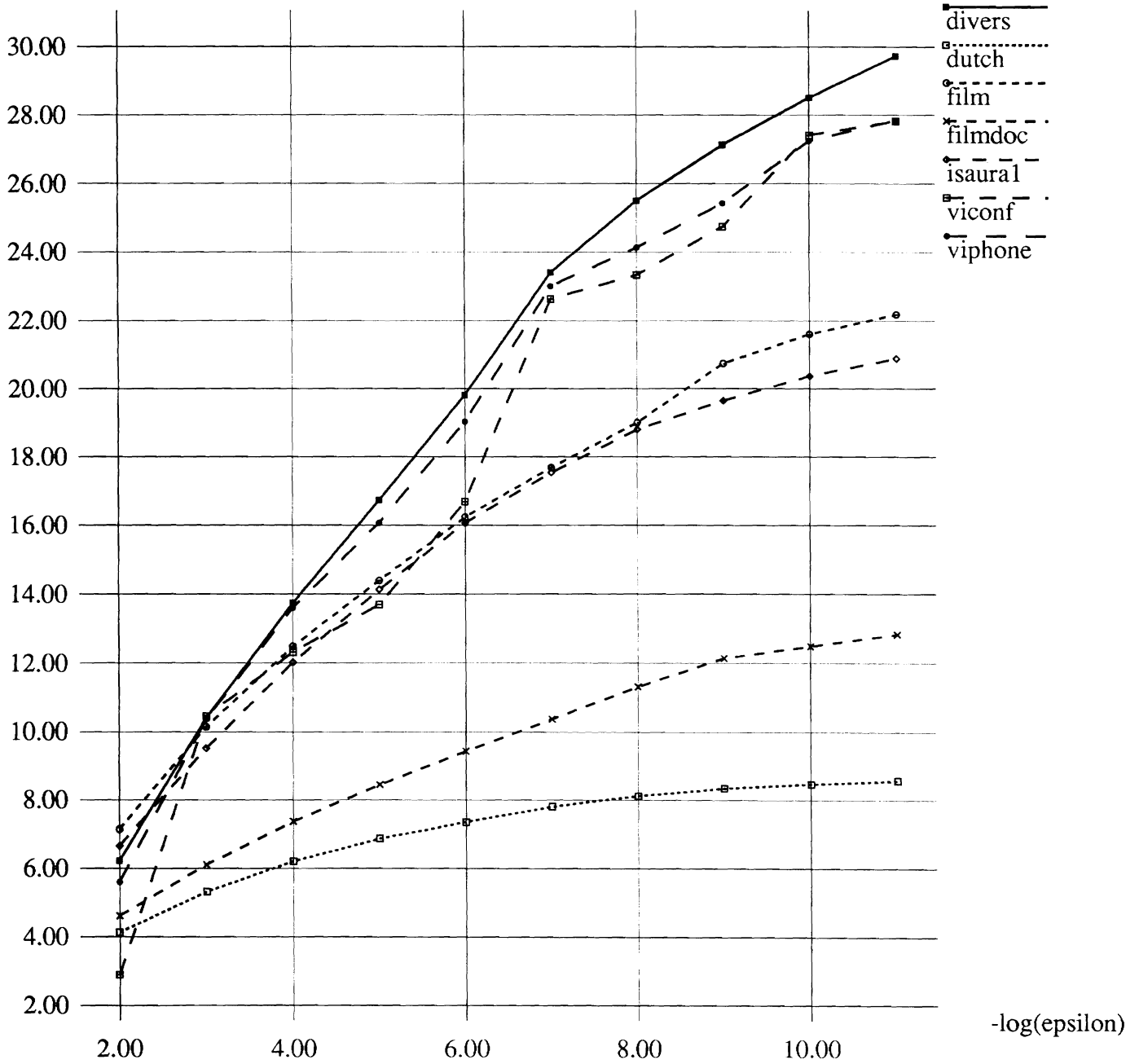
16

a vs. - log(epsilon)

a x 10^6

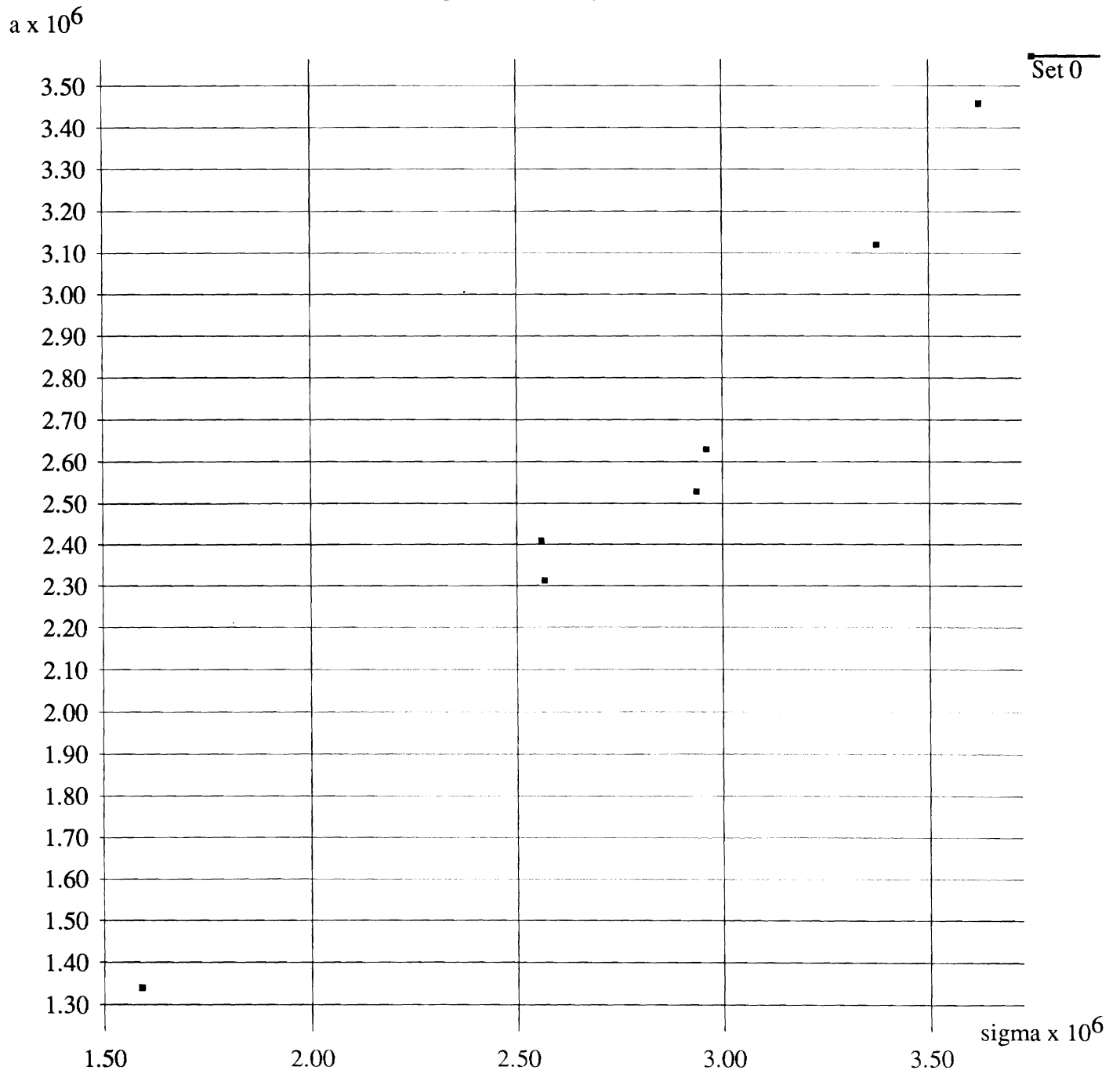Figure 9: a vs. $-\log_{10}\epsilon$ for the data set.

a vs. sigma for epsilon=0.1

a x 10^6

Figure 10: a vs. Standard Deviation (σ) for the data set for $\epsilon = 0.1$.
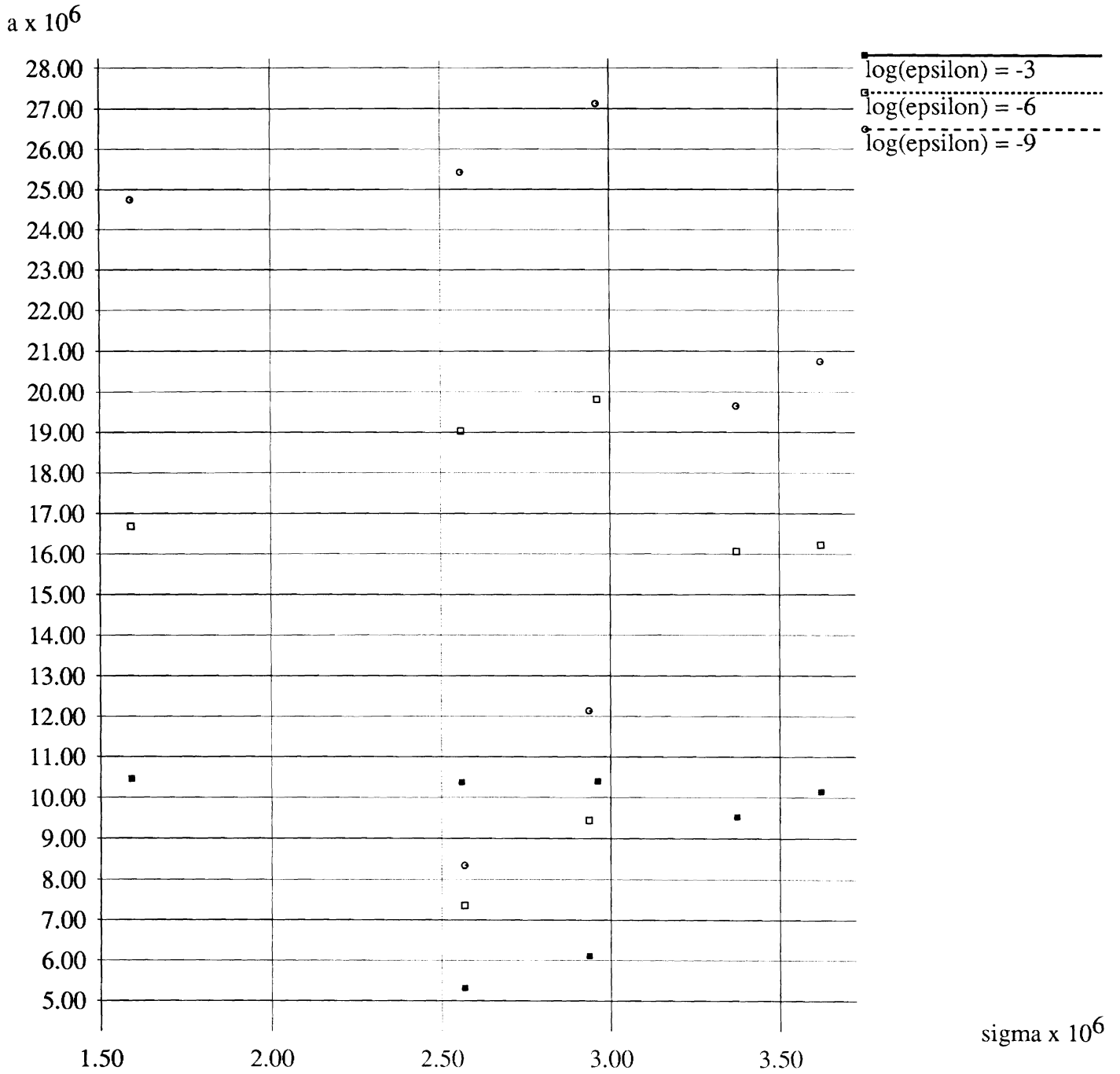
a vs. sigma

a x 10^6

28.00

27.00

26.00

25.00

24.00

23.00

22.00

21.00

20.00

19.00

18.00

17.00

16.00

15.00

14.00

13.00

12.00

11.00

10.00

9.00

8.00

7.00

6.00

5.00

log(epsilon) = -3

log(epsilon) = -6

log(epsilon) = -9

1.50   2.00   2.50   3.00   3.50

sigma x 10^6

Figure 11: a vs. Standard Deviation (σ) for the data set for other values of ϵ.
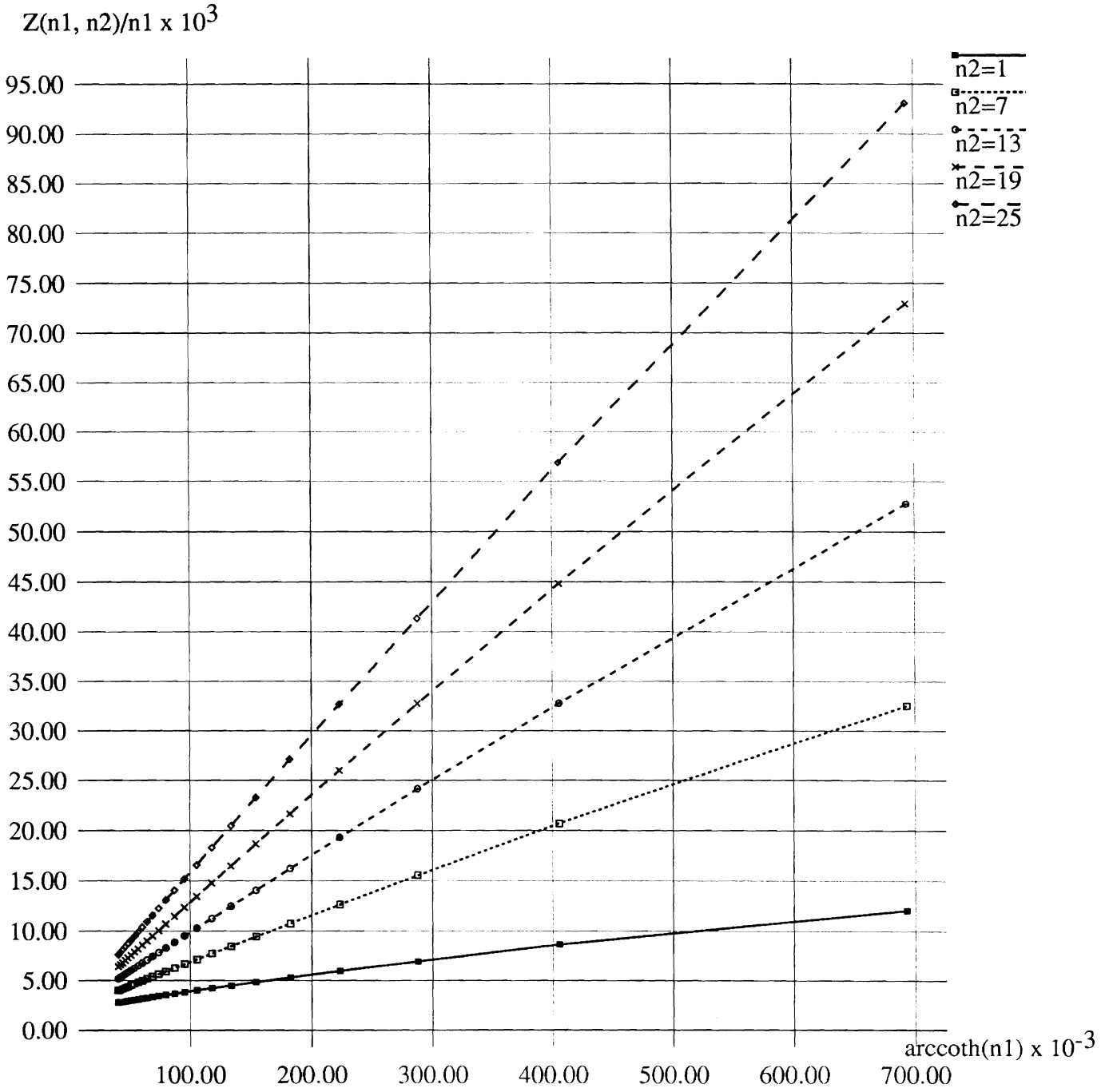
Z(n1, n2)/n1 vs. arccoth(n1)

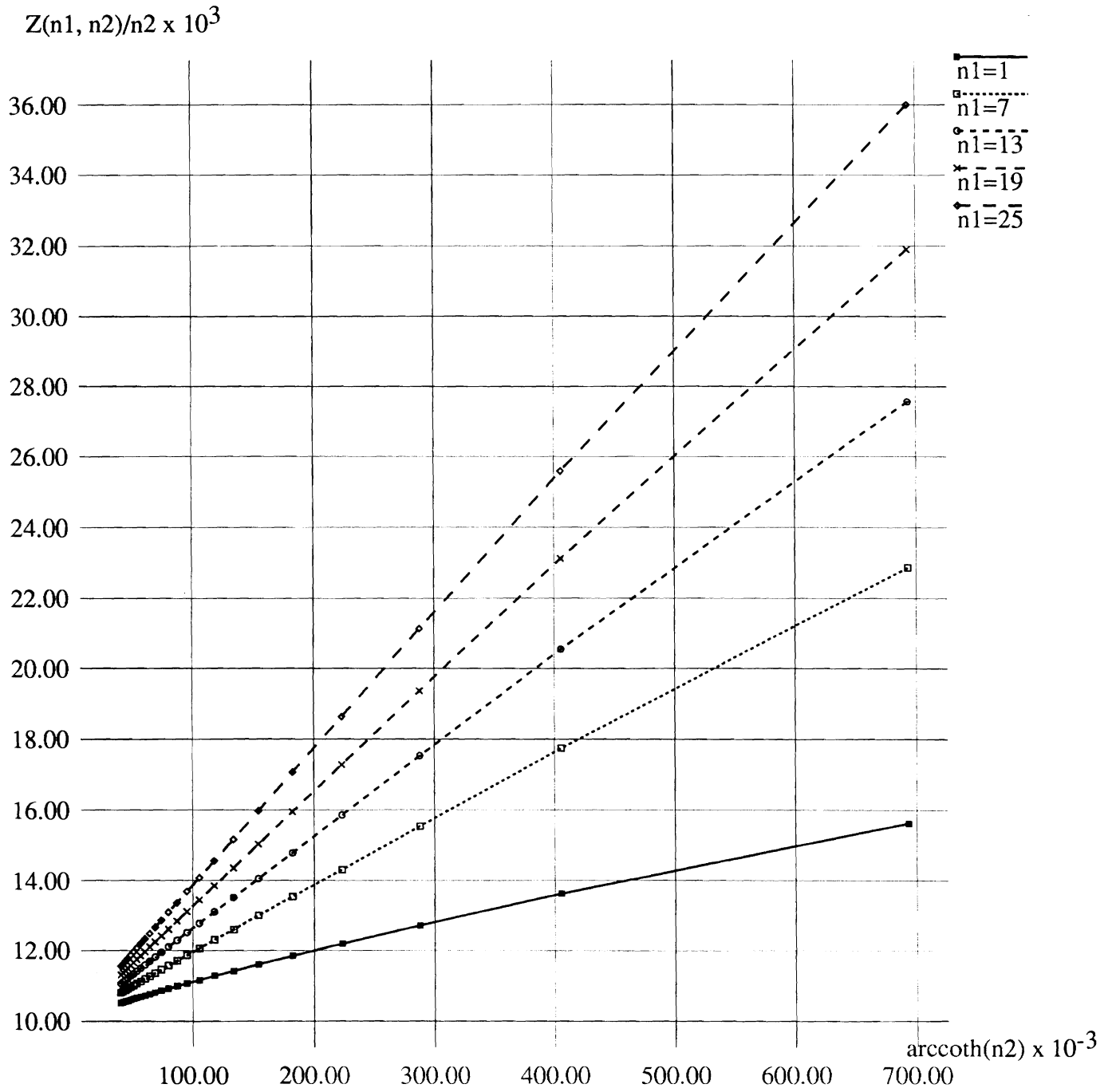Figure 12: $Z(n_1, n_2)/n_1$ vs. n_1

Z(n1, n2)/n2 vs. arccoth(n2)

Figure 13: $Z(n_1, n_2)/n_2$ vs. n_2

- Provide good spatial loss distribution characteristics. In the context of VBR video sources, this means that the scheduler should not relegate all the losses to large frames. The motivation for this comes from these observations:

  - compression of bits in VBR video sources usually implies that, the larger the number of bits transmitted, the more important they are in that there are more differences across frames. Screen refreshes are the extreme example of this scenario. Therefore, it may in fact be a good idea give large frames a break, but this could lead to difficulties in distinguishing large frames from misbehaving sources. Fortunately, the next bullet suggests a property of stochastic processes, that may be used to distinguish well behaved sources from misbehaving ones.

  - the sum of the bits transmitted over several frames by well behaved video sources would come close to each other every so often as stated in the following theorem:

Theorem 1 (Griffeath 1978 [16]) *Let $i_1, i_2, \cdots$ and $j_1, j_2, \cdots$ be independent and identically distributed random variables. Let*

$$S_i(k) = i_1 + i_2 + \cdots + i_k$$

and

$$S_j(k) = j_1 + j_2 + \cdots + j_k.$$

Then $\exists N$ such that

$$S_i(N) = S_j(N) \qquad a.s. \qquad (11)$$

i.e., after some finite number of intervals. the total number of packets transmitted by i and j must be equal with probability one.

As a consequence of this theorem, the two $i.i.d.$ sources would have transmitted the same number of packets for some N with probability one. (Experiments show that this convergence occurs for reasonably small values of N.) In between, however, they may follow different sample paths.

The algorithm proposed next, will be developed under the assumption that sources are well behaved. A method to build a firewall against misbehaving sources will be discussed subsequently.

4.2 A Generalized Virtual Clock Algorithm for Variable Rate Video

The key idea in the scheduling algorithm is to estimate the instantaneous rate of transmission of a source and use it to drive the Virtual Clock algorithm (or Hierarchical Round Robin algorithm). We shall call this, an instance of the class of Generalized Virtual Clock (GVC) Algorithms.

The classical Virtual Clock algorithm uses the average transmission rate of a source to prioritize the latter's packets, i.e.. if $\bar{\lambda}$ denotes the average rate of a source. then its kth packet is stamped with a priority number $k/\bar{\lambda}$. The scheduler then gives the highest priority to the packet that has the smallest priority number. The GVC algorithm proposed here for VBR video sources would prioritize packets of a source according to its instantaneous transmission rate $\hat{\lambda}$ instead of the average rate $\bar{\lambda}$. This is to ensure that the scheduler does not excessively penalize larger frames (e.g., refresh screens) of well behaved sources. In fact, if the capacity is exceeded, (and this is designed to happen very infrequently). the losses are distributed across the transmitters, with the hope that due to Griffeath's coupling theorem. the scenario will be reversed shortly. Simulation results on the performance of this algorithm will verify this.

The first step in the GVC algorithm is to devise a good estimator for the instantaneous rate. One possibility is to assume that the instantaneous rate for frame $n + 1$ is the same as the observed rate during frame n but this does not work well in practice because

a) there is noise in the measured rate, and

b) the algorithm does not model the correlation structure of the data accurately.

It turns out that the Filter given in Appendix A estimates the rates quite accurately. The performance of this estimator is shown in Figure 14. The y-axis represents the actual and the predicted rates, while the x-axis represents time, in units of frames.

With the rate-estimator incorporated into the Virtual Clock algorithm, we next address the performance of the GVC algorithm. Specifically, the spatial loss distributions of GVC and Virtual Clock are investigated with the help of simulation.

A simple and efficient simulation algorithm was written for this purpose. It's description is not in the scope of this paper. The algorithm is simple, and takes only a few minutes of real time to run on a Sparc II, to simulate a 30 minute video trace.

The simulator kept track of the number of packets lost in a frame on a per source basis whenever the allocated capacity was exceeded. Let X be the number of packets dropped in a frame for a designated source, given that capacity was exceeded. Figures 15 and 16 show $Pr\{X = k\}$ vs. k for GVC and Virtual Clock for a particular simulation. The x-axis is k, and the y-axis is $Pr\{X = k\}$. Notice that GVC reduces the probability of $\{X = k\}$ for large k, as compared to Virtual Clock (as it should). This, however, comes at the cost of increasing the loss probabilities for smaller values of k.

4.3   Protection of GVC from Abuse

It is possible that some sources might attempt to abuse the rate estimation feature of the Generalized Virtual Clock algorithm. Therefore some form of protection needs to be in place. We are currently investigating this problem, and only some initial thoughts are presented.

We believe that the law of large numbers may be used to detect misbehaving sources. For suppose, $X_1, X_2, \cdots, X_k$ are the number of bits transmitted over successive frames by a particular source and let

$$S_k = X_1 + X_2 + \cdots + X_k. \tag{12}$$

Then, by the (weak) law of large numbers,

$$\frac{S_k}{k} - EX_1 \tag{13}$$

Now, for large k, the curve S_k vs. k should essentially have a constant slope EX_1. (We verified this for the data and it was true even for the data in Figure 14.) To protect well behaved sources from misbehaving ones, one possibility is to prescribe an envelope around the average slope line, and ascertain that S_k remain within this envelope. (In other words, prescribe two functions $l(k)$ (lowMark) and $h(k)$ (highMark) and use GVC only if $l(k) < S_k < h(k)$.) The envelope should be wider for small k and approach the average slope line as k increases, ultimately merging with it as $k \to \infty$. (An example envelope is the 99% confidence interval around S_k.) Simulation studies are are currently in progress, and the results will be reported in the journal version of the paper.

5   Related Work

Resource allocation and equivalent bandwidth computations have been under considerable investigation lately [1, 2, 3, 9, 39, 17, 18, 19, 32, 26, 30, 31, 33, 34, 20, 21, 22, 4, 6, 10, 11, 12, 35, 36, 37]. [4, 6, 10, 11] propose using precomputed bandwidth requirement curves defined through simulation.

Predicted and Actual Rates
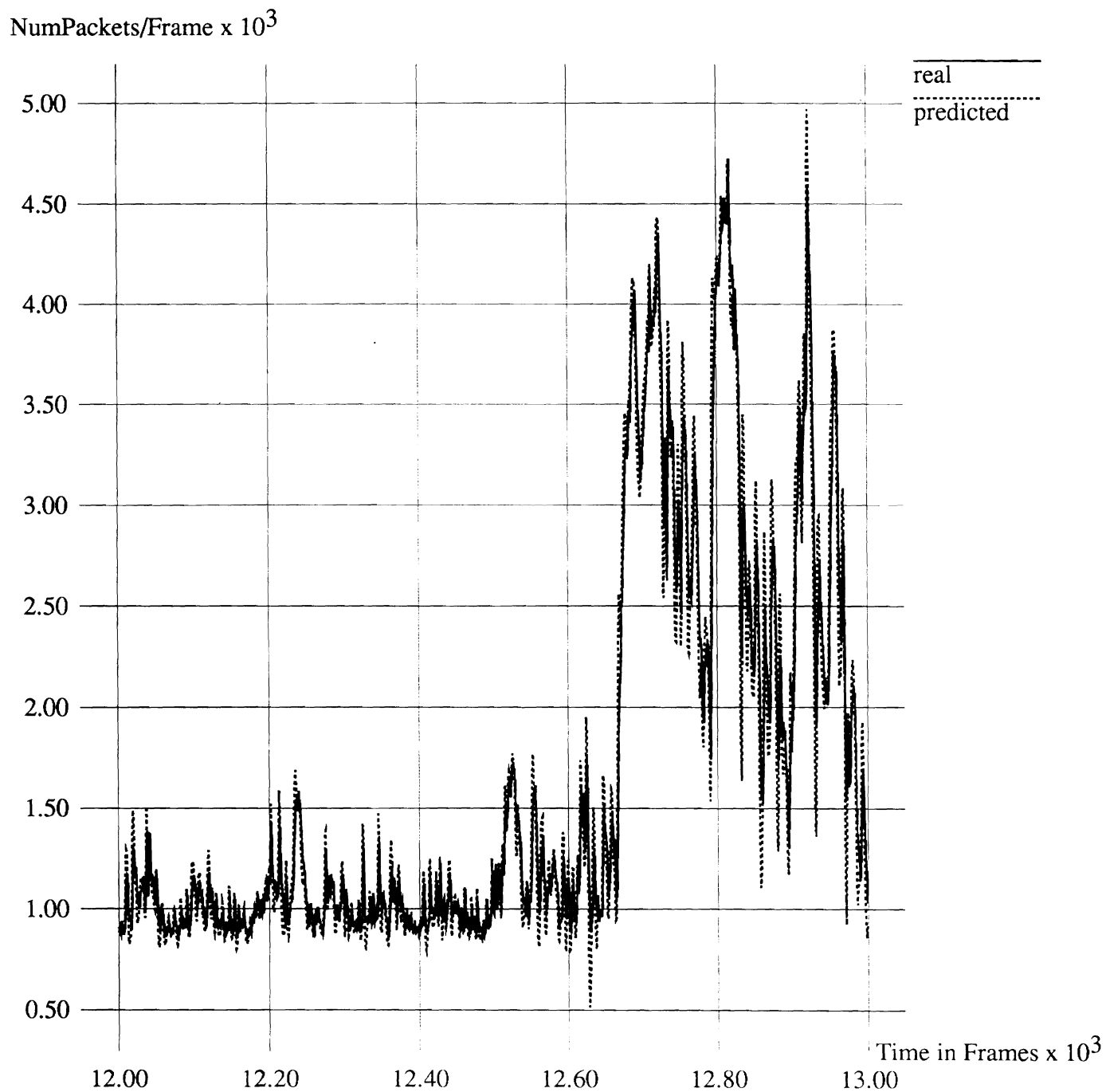
NumPackets/Frame x 10^3

Figure 14: Performance of the rate estimation algorithm for a video-phone source. Only a small section of the data is shown to increase readability. Other sources performed similarly.
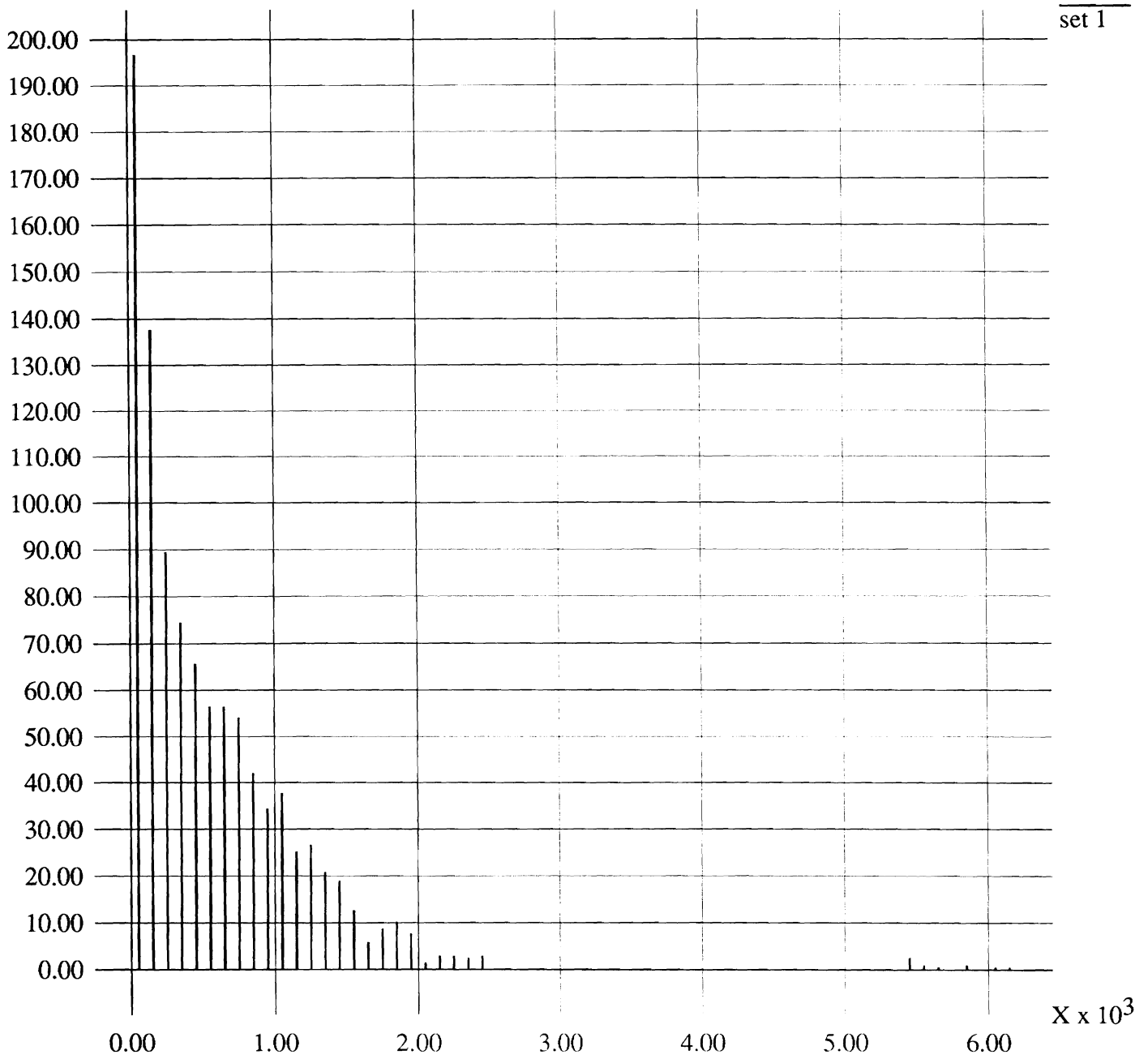
Figure 15: Spatial loss distribution of GVC. The figure shows the probability of number of packets lost per frame, given that there was loss.

vc

$Pr\{X\} \times 10^{-3}$

set 1

75.00

70.00

65.00

60.00

55.00

50.00

45.00

40.00

35.00

30.00

25.00

20.00

15.00

10.00

5.00

0.00

0.00   1.00   2.00   3.00   4.00   5.00   6.00   7.00
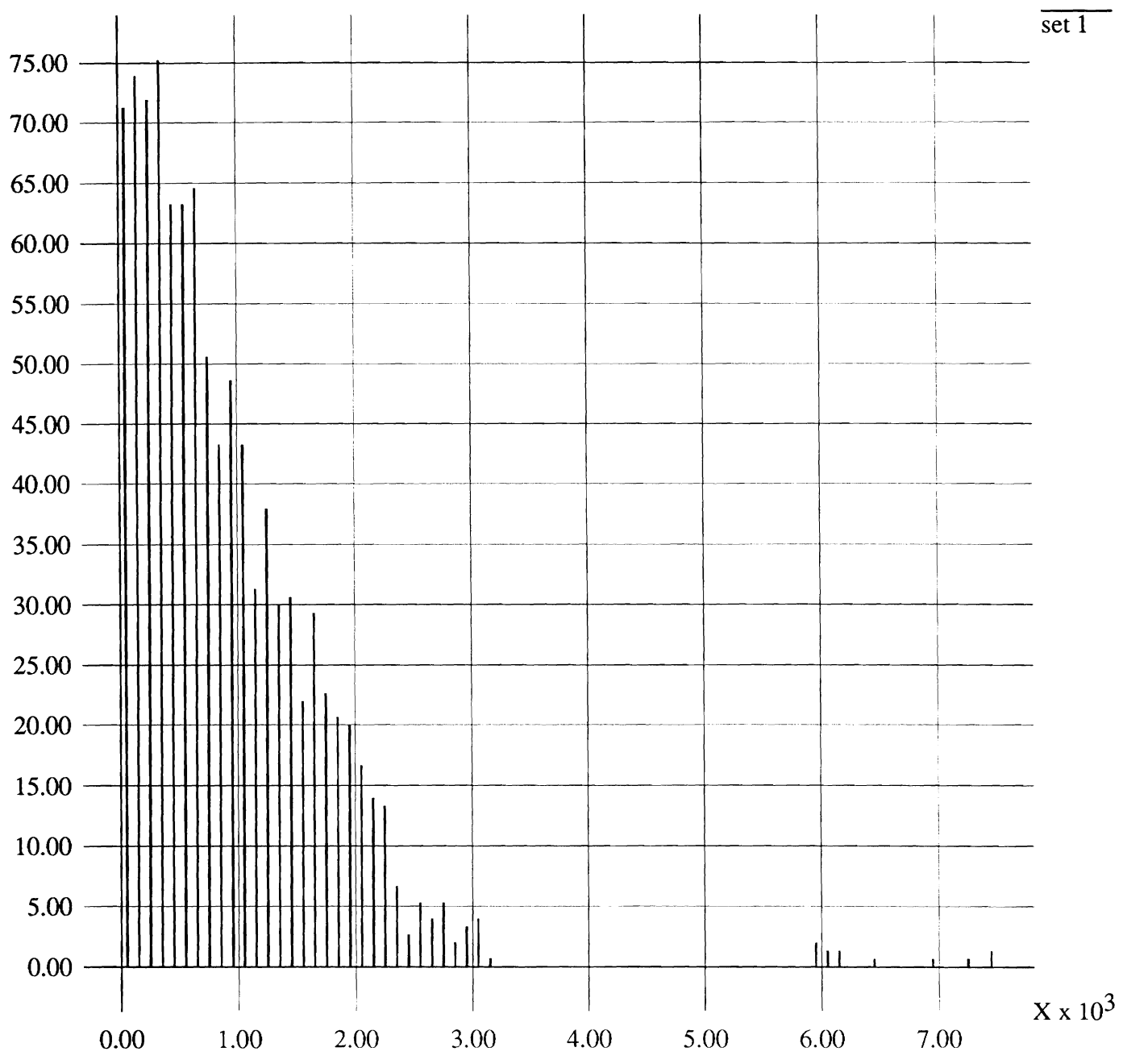
$X \times 10^3$

Figure 16: Spatial loss distribution of Virtual Clock.

Anick et al [1] and Gueren et al [12] study equivalent bandwidth for on/off sources with exponential holding times in each state. The solution proposed in [12] uses a combination of fluid approximation method and what is called stationary approximation method by the authors, to determine the equivalent bandwidth. [36, 37] suggest the use of average rate and variance of sources to compute the equivalent bandwidth. [35] suggests the use of probability density functions of sources to calculate the equivalent bandwidth. The main difference between [35] and this paper is that the latter has derived simple heuristics for capacity allocation from convolved density functions of real data, instead of using the density functions themselves.

Hyman et al [20, 21, 22] view the Quality of Service constraints in terms of an admissible load region. Given a Quality of Service constraint and traffic characteristics, they formulate a constraint optimization problem to maximize the system utility, and this forms an admissible load region. The admissible load region is determined through extensive simulations in [20] and through a linear programming model in [22]. The similarity between [20, 21, 22] and the problem addressed here is that the value of N for which Z_N equals the capacity of the switch is the maximum admissible load. For heterogeneous traffic, a similar mapping holds between the convolved data and their numerical solutions, if ϵ is held constant.

There has also been a considerable amount of work reported lately on scheduling disciplines, see for instance, Earliest-Deadline [8], Fair Queueing [29, 28, 5] Hierarchical-Round-Robin [24] Stop-and-Go [13, 14] and Virtual Clock [40] and Pulse [27].

6   Conclusion

Video applications are expected to dominate the network workload in future broadband networks. Therefore, efficient capacity allocation and scheduling strategies, specific to video traffic, can result in large savings in terms of network resources while simultaneously delivering quality service to users.

This paper presented simple and accurate heuristics for capacity allocation for variable bit rate video sources using detailed traffic analysis of fifteen long traces of video data. The analysis found that if Z_N was the minimum capacity needed to support N i.i.d. sources, each of which had a loss tolerance ϵ, then for small ϵ, Z_N/N could be approximated with $a \coth^{-1} N + b$, where a and b were independent of N. Further a was found to be a linear function in m (mean) and $log(\epsilon)$ whereas b was found to be approximately a quadratic in $log(\epsilon)$. The exact equations for a and b, given by (10) and (9), had a confidence level greater than 99%.

A new algorithm was also proposed for effective scheduling of packets belonging to variable bit rate video sources. This algorithm was a generalization of the Virtual Clock algorithm. It estimated the instantaneous rate of transmission of each source and used this instead of a static average rate to prioritize packets. In so doing, it tried to synchronize the switch scheduling rates and the packet arrival rates of each source. For well behaved sources, this was found to improve the spatial loss distributions.

Much work remains to be done, however. We are currently investigating analysis of heterogeneous traffic mixes, and also a means to combine the different equations for a into a single equation. Parallel work is also in progress to determine a theoretical proof for the main observation of this paper, the $coth^{-1} N$ conjecture. More simulations studies are also underway on algorithms to thwart misbehaving sources.

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Acknowledgement

A Instantaneous Rate Estimator for Generalized Virtual Clock

The following filtering algorithm was used for estimating the instantaneous rate of transmission of a source. The reader is referred to [15] for details.

Let $x(\cdot)$ be the quantity of interest, in this case the rate of transmission. Let $\dot{x} = dx/dt$ be the rate of change of x with respect to time.

Let $\hat{x}(t_1|t_0)$ be the estimated value of x for time t_1 given that the current time is t_0. Similarly, let $\hat{\dot{x}}(t_1|t_0)$ be the estimated value of $\dot{x}$ for time t_1 at time t_0.

Suppose that the current time is t. Then the values $\hat{x}(t|t-1)$ and $\hat{\dot{x}}(t|t-1)$ and a new measurement $m(t)$ is currently available. The filtering algorithm proceeds in two steps:

Step 1: Correct the estimate of $\hat{x}(t|t)$ and $\hat{\dot{x}}(t|t)$:

$$\hat{x}(t|t) = \hat{x}(t|t-1) + \alpha\left(m(t) - \hat{x}(t|t-1)\right) \tag{14}$$

$$\hat{\dot{x}}(t|t) = \hat{\dot{x}}(t|t) + \frac{\beta}{\Delta t}\left(m(t) - \hat{x}(t|t-1)\right) \tag{15}$$

where $0 < \alpha, \beta < 1$. We assume $\Delta t = 1$.

Step 2: Predict for time $t + 1$.

$$\hat{x}(t+1|t) = \hat{x}(t|t) + \Delta t \hat{\dot{x}}(t|t) \tag{16}$$

$$\hat{\dot{x}}(t+1|t) = \hat{\dot{x}}(t|t) \tag{17}$$

B Multiple Linear Regression

This appendix overviews multiple linear regression. For details, see [23] or any statistics book on experimental design.

1. The model for multiple regression is

$$y_i = b_0 + b_1 x_{1i} + b_2 x_{2i} + \cdots + b_k x_{ki} + e_i$$

   or in matrix form: $\mathbf{y} = \mathbf{Xb} + \mathbf{e}$ where

   $$\mathbf{b} = \text{A column vector with } k+1 \text{ elements,}$$
   $$\mathbf{y} = \text{A column vector of } n \text{ observed values,}$$
   $$\mathbf{X} = \text{An } n \ x \ (k+1) \text{ matrix whose } (i, j+1)th \text{ element } X_{i,j+1} = 1 \text{ if } j = 0 \text{ else } x_{ij}$$

2. Parameter estimation: $\mathbf{b} = \left(\mathbf{X^T X}\right)^{-1}\left(\mathbf{X^T y}\right)$

3. Allocation of variance:

   Let SSY = sum of squares of y_i's = $\sum_{i=1}^{n} y_i^2$

   $SS0$ = $n\bar{y}^2$

   SST = $SSY - SS0$

   SSE = $\mathbf{y^T y} - \mathbf{b^T X^T y}$

   SSR = $SST - SSE$

   SSE is the sum of squares of the errors, where error in the ith measurement is defined as the difference between the observed value y_i and the model's prediction. Linear regression minimizes SSE. SST is the sum of squares of the total variation, and the difference between SST and SSE is the variation that is successfully accounted for by regression.

4. Coefficient of determination: $R^2 = SSR/SST$. This fraction gives a goodness measure.

5. Standard deviation of errors: $s_\epsilon = \sqrt{SSR/k}$

6. Standard deviation of parameters: $s_{b_j} = s_\epsilon \sqrt{C_{jj}}$ where C_{jj} is the jth diagonal term of $\mathbf{C} = (\mathbf{X^T X})^{-1}$.

7. Confidence intervals of the parameters: $b_j \pm t_{\{1-\alpha/2, n-k-1\}} \, s_{b_j}$ where $1 - \alpha$ is the desired confidence level, and $t_{\{1-\alpha/2, n-k-1\}}$ is obtained from the t-distribution. A rule of thumb is that for $n - k - 1$ greater than 30, one may use the Standard Normal distribution. The results reported here were therefore computed from the Standard Normal distribution.

8. Analysis of variance: Let $MSR = SSR/k$ and $MSE = SSE/(n - k - 1)$ and let $F_{\text{computed}} = MSR/MSE$. Then regression is significant if F_{computed} is greater than $F_{\{\beta;k,n-k-1\}}$. The latter is the F distribution and its values are given in most statistics books. β is the desired confidence level. For the examples in this paper, F_{computed} was much greater than $F_{\{\beta;k,n-k-1\}}$ even at the 99% confidence level.

C Sample Output of a Regression Test for Coefficient a.

Note: The matrix $\mathbf{b} = (b_0, b_1, b_2)^T$ are the coefficients of interest. The name $\mathbf{b}$ should not be confused with the coefficient b in Section 3. It was chosen in keeping with the notation of Appendix B.

```
b Matrix: estimated coefficients
b[0] = 673455.5597
b[1] = 3404385.2469
b[2] = -132489.5156



 RSq = 0.997551, R= 0.998775
 s_e = 340242.327178


 Std dev in b0, b1, b2 estimates: 373656.813858 143113.493093 11615.681527


90-confidence interval for b0:   673455.560 (   58790.101 ,  1288121.018)
90-confidence interval for b1:  3404385.247 ( 3168963.551 ,  3639806.943)
90-confidence interval for b2:  -132489.516 ( -151597.312 ,  -113381.720)
F_computed = 1629.296154, k = 2, DegreesOfFreedom = 8
```