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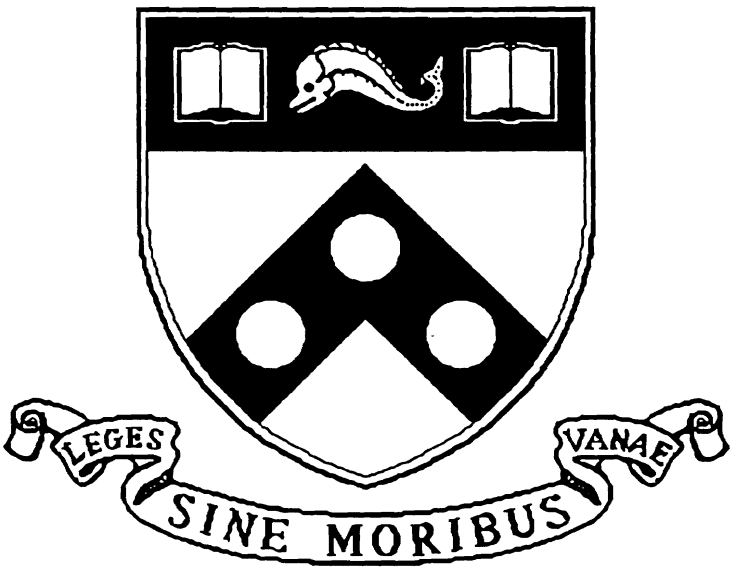

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Semantic Representations and Query Languages for Or-sets*

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Abstract

Or-sets were introduced by Imielinski, Naqvi and Vadaparty for dealing with limited forms of disjunctive information in database queries. Independently, Rounds used a similar notion for representing disjunctive and conjunctive information in the context of situation theory. In this paper we formulate a query language with adequate expressive power for or-sets. Using the notion of normalization of or-sets, queries at the "structural" and "conceptual" levels are distinguished. Losslessness of normalization is established for a large class of queries. We have obtained upper bounds for the cost of normalization. An approach related to that of Rounds is used to provide semantics for or-sets.

1 Introduction

Applications within design, planning, and scheduling areas have motivated Imielinski, Naqvi, and Vadaparty to introduce the notion of or-set [15, 16]. Although or-sets are in essence disjunctive information, they are distinguished from the latter by having two distinct interpretations. An or-set can either be treated at a *structural* level or at a *conceptual* level. The structural level concerns the precise way in which an or-set is constructed. The conceptual level sees an or-set as representing an object which is equal to a member of the or-set. For example, the or-set $\langle 1, 2, 3\rangle$ is structurally a collection of numbers; however it is conceptually a number that is either 1, 2, or 3. (In this report angle brackets $\langle\rangle$ are used for or-sets and $\{\}$ for the usual sets.)

The two views of or-sets are complementary. Consider a design template used by an engineer. The template may indicate that component A can be built by either module B or module C. Such a template, as explained in [15], is structurally a complex object whose component A is the or-set

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containing B and C. A designer employing such a template should be allowed to query the structure of the template, for example, by asking what are the choices for component A. On the other hand, the designer should also be allowed to query about possible completed designs, for example, by asking is there a cheap complete design. In the latter case, as the designer is still in the process of creating a design, the "completed design" is purely conceptual. Both views of or-sets are important and should be supported.

The structural interpretation of or-sets is quite clear. However, the conceptual interpretation requires further exposition. A few operators at the structural level prescribing the interaction of or-sets, products and ordinary sets are needed for this purpose. These operators are used to express transformations among objects that are conceptually equivalent. As will be seen in Section 3, these operators are the only crucial ones for the passage from the structural to the conceptual level.

The operator $or\_\mu^s : \langle\langle s\rangle\rangle \to \langle s\rangle$ flattens an or-set of or-sets of type s. For example, applying or_μ to $\langle\langle 1,2,3\rangle,\langle 2,4\rangle\rangle$ produces the or-set $\langle 1,2,3,4\rangle$. The most important thing to note here is that or_μ *preserves the conceptual value of the input.* First $\langle 1,2,3\rangle$ is conceptually either 1, 2, or 3. Similarly, $\langle 2,4\rangle$ is conceptually either 2 or 4. The input is conceptually either $\langle 2,4\rangle$ or $\langle 1,2,3\rangle$; that is, it conceptually represents 1, 2, 3, or 4. This is of course what the output is at the conceptual level.

The operator $or\_\rho_2{}^{s,t} : s \times \langle t\rangle \to \langle s \times t\rangle$ takes in a pair of type $s \times \langle t\rangle$ and pairs the first component with every item in the second component, which is an or-set. For example, $or\_\rho_2\ (1,\langle 2,3\rangle)$ yields the or-set $\langle(1,2),(1,3)\rangle$. Here the input stands conceptually for a pair whose first component is 1 and whose second component is either 2 or 3. That is, the input is conceptually either $(1,2)$ or $(1,3)$. Hence or_ρ_2 also has the important property of preserving meaning at the conceptual level. We also use $or\_\rho_1{}^{s,t} : \langle s\rangle \times t \to \langle s \times t\rangle$ for the operator that does pairing the other way round.

The operator $\alpha^s : \{\langle s\rangle\} \to \langle\{s\}\rangle$ takes in an ordinary set containing or-sets of type s and produces an or-set containing sets of type s obtained by combining the or-sets componentwise in all possible ways. For example, $\alpha\ \{\langle 2,3\rangle,\langle 4,5,3\rangle\}$ produces the or-set $\langle\{2,4\},\{2,5\},\{2,3\},\{3,4\},\{3,5\},\{3\}\rangle$. This is also an operator that preserves conceptual meaning. In the above example, the input is conceptually a set of two elements such that one of them is either 2 or 3 and the other is either 4, 5, or 3. This is precisely what the output is conceptually. Note that sets such as $\{2\}$, $\{4\}$, etc. are not part of the output, even though $\{3\}$ is because it arises by letting both the first and second elements be 3.

As a further example, consider the result of applying α to $\{\langle 1,2\rangle,\langle\rangle,\langle 3\rangle\}$. It is *not* $\langle\{1,3\},\{2,3\}\rangle$. The *correct* output is the empty or-set $\langle\rangle$. To see this, let us find out what the input is at the conceptual level. It represents a set of three elements, they are conceptually the values represented respectively by $\langle 1,2\rangle$, $\langle\rangle$, and $\langle 3\rangle$. Hence the first element is either 1 or 2 and the third is 3. But what is the second element? Recall that an or-set represents at the conceptual level an object that is equal to one of its elements. Since $\langle\rangle$ has no element, it does not represent any object at the conceptual level. Consequently, our input represents at the conceptual level "a set having an element which is not anything." As there is no such set, the input does not represent any object either. This coincides precisely with the meaning of the output. An item which does not represent any object at the conceptual level indicates a conceptual *inconsistency.* (But note that it is still structurally meaningful.)

The above operators provide an idea of what to include in a *structural* query language. But what kind of operators should be provided in a *conceptual* query language? Should there be an operator for testing whether two objects are conceptually equivalent? Should there be an operator for testing whether one object is amongst the objects that a second object can conceptually be?

Fortunately, it is not necessary to make such chaotic "enhancements." It is found that any two objects which are conceptually equivalent can be reduced to the same object by repeated applications of the above operators. The normal form induced happens to be *independent* of the precise sequence of applications of these operators. Moreover, given the type of any object, the type of its normal form can be read off. Therefore, one can robustly take the conceptual meaning of any object to be its normal form under the rewriting induced by the above operators. Consequently, a conceptual query language can be built by extending a structural language with a single operator *normalize* which takes the input object to its normal form. A query at the conceptual level is then simply a query performed on normal forms.

Related work. Imielinksi, Naqvi, and Vadaparty stressed the applications of or-sets in design and planning areas and informally explained the distinction between structural and conceptual queries [15, 16]. The semantics and query language proposed by [15] are rather involved. They defined a concept of order-independence which is related to the notion of normalization but is based on assigning object identifiers, and demonstrated a sufficient condition for order-independence. In addition, they were able to demonstrate a $co\mathcal{NP}$-complexity result for that particular proposal. In [16] they studied some intrinsic lower bounds on complexity of $\mathcal{LDL}$-style [22] queries on or-sets. The language can express queries of hyper-exponential complexity. Nevertheless, they were successful in identifying certain restricted tractable fragments that are useful in real-life applications.

Rounds [24] studied complex object databases from the situation-theoretic point of view. Connections with natural language problems motivated him to introduce the notions of conjunctive and disjunctive information which correspond exactly to our notions of sets and or-sets. He studied order relations on complex objects and their logical representations. However, it is unclear whether his use of non-well-founded set theory is helpful in designing a database programming language.

Organization. The main contributions and organization of this report are summarized below. A query language *or-*$\mathcal{NRA}$ that cleanly integrates or-sets and more traditional types of data at the structural level is proposed in section 2.

In Section 3 we give two semantic representations which are in the spirit of Rounds' work [24] but using simpler machinery. For example, using our representations we were able to provide a simple proof that α^s is the isomorphism of semantic domains of types $\{\langle s\rangle\}$ and $\langle\{s\}\rangle$.

A query at the conceptual level is exactly a query on an object that is in a certain normal form. In section 4, the normal form is properly characterized. Moreover, we show that the process of normalization is *coherent*. That is, the normal form of any object is independent of how the object is normalized. This allows us to define a query language *or-*$\mathcal{NRA}^+$ at the conceptual level by adding a new operator *normalize* to *or-*$\mathcal{NRA}$.

Since objects of different structures may have the same normal form, it is clear that certain structural

information is lost by normalization. In section 5, a *losslessness* theorem is proved. Consequently, loss of structural information has no effect with respect to a general class of queries.

Queries at the conceptual level are much simpler than those at the structural level. Unfortunately, conceptual queries must be performed on normalized data. In section 6, we study a few important costs of normalization. In particular, an upper bound on the number of elements in normal forms of complex objects and an upper bound on the actual size of normal forms of complex objects are given. Also significant is that we have been able to demonstrate that every definable query in *or*-$\mathcal{NRA}^+$ is at most exponential in the size of input, in contrast to the proposal of Imielinski, Naqvi, and Vadaparty [16] which contains some hyperexponential queries.

2 Structural Query Language

A nested relational language based on the idea of structural recursion [4, 3] and on monads [20, 25] was proposed in [5]. This language is of polynomial time complexity and smoothly generalizes many approaches to nested relational algebras, cf. [2]. It is extensible and has an appealing syntax. For example, $\langle x \mid x \in \mathit{normalize}(DB), \mathit{is\_cheap}(x)\rangle$ selects cheap completed designs assuming that *is_cheap* and *normalize* are defined. (In section 4, *normalize* is added as a primitive to obtain the conceptual query language.)

The algebraic version of the language is used in this report. We denote this language by $\mathcal{NRA}(\Sigma)$ where Σ are some additional primitives like operations on integers. As observed by Wadler in [25], the same syntax can be used for many "collection" types besides sets. In particular, by replacing the set operators of $\mathcal{NRA}$ by the corresponding operators for or-sets, a language for programming with or-sets can be obtained. This language is denoted by $\mathcal{NRA}_{or}$.

For example, the above query becomes $\mathit{or\_}\mu \circ \mathit{or\_map}(\mathit{cond}(\mathit{is\_cheap}, \mathit{or\_}\eta, K\langle\rangle \circ !)) \circ \mathit{normalize}$. Here *cond* is a primitive: $\mathit{cond}(p,t,f)(x) = t(x)$ if $p(x)$ is true and $f(x)$ otherwise. Then $\mathit{cond}($ *is_cheap*, $\mathit{or\_}\eta$, $K\langle\rangle \circ !)$ (x) is $\langle x\rangle$ if x is cheap and $\langle\rangle$ otherwise. *or_map* applies it to every element in the normalized database, returning $\langle x\rangle$ for each cheap x and $\langle\rangle$ for each expensive one. $\mathit{or\_}\mu$ flattens this or-set of or-sets, producing an or-set containing precisely the cheap completed designs.

In this section, the language for sets $\mathcal{NRA}$ and the language for or-sets $\mathcal{NRA}_{or}$ are integrated into a single language we called *the structural query language*, denoted by *or*-$\mathcal{NRA}$. *or*-$\mathcal{NRA}$ supports *structural* manipulations of complex objects containing a mixture of freely combined tuples, sets, and or-sets. This language is obtained by the union of $\mathcal{NRA}$ and $\mathcal{NRA}_{or}$ and an operator α prescribing the interaction between sets and or-sets.

Types. A type in *or*-$\mathcal{NRA}$ is either an object type or a function type $s \to t$, where s and t are both object types. The object types are given by the grammar: $t ::= b \mid t \times t \mid \{t\} \mid \langle t\rangle$, where b denotes a collection of base types such as booleans and integers. Included in b is a special base type *unit* containing precisely one element. In this report $\langle t\rangle$ stands for the or-set of type t, while $\{t\}$ is the ordinary set of type t.

Morphisms (expressions). The "morphisms" (or expressions) of *or-NRA* are formed according to the rules in Figure 1. The language is parameterized by a collection of primitives p of function type $Type(p)$, amongst them are the equality tests $=_s$: $s \times s \to bool$ for each object type s, and a collection of constants c of base type $Type(c)$. Type superscripts are usually omitted because the most general type of any given morphisms can be inferred (see [12] for example).

Semantics. π_1 and π_2 are first and second projections. ! maps anything to the unique element of type *unit*. (f, g) is pair formation, $f \circ g$ is the composition of f and g. *id* is the identity function. or_ρ_2, or_μ and α have already been described. or_η is the singleton formation: $or\_\eta(x) = \langle x \rangle$. $or\_\cup$ makes union of two or-sets. $or\_map(f)$ applies f to all elements of an or-set. $K\langle\rangle$ produces an empty or-set. or_ρ_1 has been omitted because it is definable $or\_map(\pi_2, \pi_1) \circ or\_\rho_2 \circ (\pi_2, \pi_1)$. The operators from *NRA* have similar meaning for the usual sets.

We have included $K\langle\rangle$, the morphism which produces the empty or-set, in *or-NRA*. We note that if f is a morphism of *or-NRA* such that $K\langle\rangle$ does not occur in it and such that each p in it does not involve or-sets, then f applied to any complex object x not containing any empty or-set yields a complex object $f(x)$ containing no empty or-set.

One of *or-NRA*'s primitives, α, is essentially a translation of conjunctive normal form into disjunctive normal form. This operation may be very expensive. Indeed, if its argument is a collection of n two-element or-sets, all $2n$ elements being distinct, then α produces an or-set containing 2^n n-element sets. Several query languages use expensive (exponential) operations. For example, in the Abiteboul-Beeri algebra [1, 5], one of the primitives is *powerset*: $\{t\} \to \{\{t\}\}$ which takes a set and returns the set of all its subsets. The result that we are going to formulate can be intuitively understood as follows: the expressive power of α is that of *powerset*. However, *powerset* does not use the $\langle\rangle$ type constructor. To be able to speak of the equivalence of expressive power of languages one of which uses or-sets and the other does not, for technical purposes only, we introduce the functions $or\_to\_set : \langle t \rangle \to \{t\}$ and $set\_to\_or : \{t\} \to \langle t \rangle$ with the obvious semantics: $or\_to\_set(\langle x_1, \ldots, x_n \rangle) = \{x_1, \ldots, x_n\}$ and $set\_to\_or(\{x_1, \ldots, x_n\}) = \langle x_1, \ldots, x_n \rangle$. We remark here that, if *or_to_set* and *set_to_or* are given, then *NRA* and NRA_{or} are interdefinable. That is, $NRA(or\_to\_set, set\_to\_or) \cong NRA_{or}(or\_to\_set, set\_to\_or)$.

Proposition 1 $NRA(or\_to\_set, set\_to\_or, \alpha) \cong NRA(or\_to\_set, set\_to\_or, powerset)$.

Proof. It is not hard to see that *powerset* can be expressed as follows:

$$powerset = or\_to\_set \circ \alpha \circ map(or\_\cup \circ (or\_\eta \circ K\{\} \circ !, or\_\eta \circ \eta))$$

Conversely, we must show that α is definable in $NRA(or\_to\_set, set\_to\_or, powerset)$. For the sake of clarity we use *cond* to show that α is definable. A clumsier proof that does not use *cond* is also possible. It is known that the test for equal cardinality can be implemented (see [5]). To check whether $|X| \leq |Y|$, notice that

$$\mu \circ map(\lambda Z.cond(equal\_card?(X, Z), X, \{\}))(powerset(Y))$$

returns X if $|X| \leq |Y|$ and $\{\}$ otherwise, thus giving us the test for lesser cardinality.

Operators shared by $\mathcal{NRA}$ and $\mathcal{NRA}_{or}$

$$\frac{g : u \to s \quad f : s \to t}{f \circ g : u \to t} \qquad \frac{}{\pi_1^{s,t} : s \times t \to s} \qquad \frac{}{\pi_2^{s,t} : s \times t \to t} \qquad \frac{f : u \to s \quad g : u \to t}{(f,g) : u \to s \times t}$$

$$\frac{}{!^t : t \to unit} \qquad \frac{}{Kc : unit \to Type(c)} \qquad \frac{}{p : Type(p)} \qquad \frac{}{id^t : t \to t}$$

Operators from $\mathcal{NRA}$

$$\frac{}{\rho_2^{s,t} : s \times \{t\} \to \{s \times t\}} \qquad \frac{}{\eta^t : t \to \{t\}} \qquad \frac{}{\cup^t : \{t\} \times \{t\} \to \{t\}}$$

$$\frac{}{\mu^t : \{\{t\}\} \to \{t\}} \qquad \frac{}{K\{\}^t : unit \to \{t\}} \qquad \frac{f : s \to t}{map\ f : \{s\} \to \{t\}}$$

Operators from $\mathcal{NRA}_{or}$

$$\frac{f : s \to t}{or\_map\ f : \langle s \rangle \to \langle t \rangle} \qquad \frac{}{or\_\rho_2{}^{s,t} : s \times \langle t \rangle \to \langle s \times t \rangle} \qquad \frac{}{or\_\eta^t : t \to \langle t \rangle}$$

$$\frac{}{or\_\cup^t : \langle t \rangle \times \langle t \rangle \to \langle t \rangle} \qquad \frac{}{or\_\mu^t : \langle \langle t \rangle \rangle \to \langle t \rangle} \qquad \frac{}{K\langle\rangle^t : unit \to \langle t \rangle}$$

Interaction of sets and or-sets

$$\frac{}{\alpha^t : \{\langle t \rangle\} \to \langle \{t\} \rangle}$$

Figure 1: Syntax of *or*-$\mathcal{NRA}$

Now, given an input of type $\{\langle t \rangle\}$, first apply *map(or_to_set)* to it and then flatten the result, thus obtaining the set of elements that occur in the input. Applying *powerset* now gives the set of all sets of those elements. A set of elements of the input makes it to the output if and only if two conditions hold: first, its cardinality does not exceed the cardinality of the input (i.e. the number of or-sets) and it has a nonempty intersection with any element of the input, unless the input is $\{\}$. Since selection, lesser cardinality test, intersection and test for nonemptiness are definable in $\mathcal{NRA}$ (see [5] and above), selection over the powerset followed by an application of *set_to_or* yields the desired result. □

3 Partial Information and Or-sets

In this section we address some semantic issues. Presence of or-sets in a database means presence of partial information. We assume that partiality can be expressed by means of a partial order on database objects, i.e. $x \leq y$ expresses the fact that x is more partial than y or y is more informative than x. The idea of using partially ordered sets to model partial information has been around since early 80s: Codd's tables, for example, can be captured by so-called flat domains which are obtained from unordered sets by adding a unique bottom element (null). Zaniolo's approach of having three kinds of nulls – unknown, nonexistent, existent unknown – is another example of ordering on objects. In fact, a general approach to the treatment of partial information as ordering on the set of objects was proposed in [7] and further developed in [6, 18]. We remark here that this approach is also suitable for databases *without* partial information. In such a case, values of base types are totally unordered.

Assume that orders on values of the base types are given. It is clear how to order pairs: $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$. However, there is no immediate answer to the question how to extend the ordering to the set and or-set types. In [7, 6, 18, 24] two ways to extend an ordering to subsets of a partially ordered set were studied. Let $\langle X, \leq \rangle$ be a poset and $A, B \subseteq X$. *The Hoare* and *the Smyth* orderings are defined as follows:

$$A \leq^{\flat} B \Leftrightarrow \forall a \in A\ \exists b \in B : a \leq b$$

$$A \leq^{\sharp} B \Leftrightarrow (\forall b \in B\ \exists a \in A : a \leq b)\,\&\,(B = \emptyset \Rightarrow A = \emptyset)$$

Traditionally the condition $B = \emptyset \Rightarrow A = \emptyset$ is omitted because the Smyth powerdomain does not contain the empty set. Observe that if X is totally unordered, $\leq^{\flat}$ is the subset and $\leq^{\sharp}$ is the superset ordering on non-empty sets. The Hoare ordering was also used in [14] to order relations with partial information. We will try to justify using $\leq^{\flat}$ to order values of the set types and $\leq^{\sharp}$ to order the values of the or-set types.

Assume that a set $A \subseteq X$ is given. How can we improve our knowledge about the real world situation represented by this set? There are two ways to do so: first, by replacing an element $a \in A$ by a set A' of elements greater than a. For example, if a record [Name $\Rightarrow \perp$, Office $\Rightarrow$ '515'] is contained in the database, we can improve our knowledge about the office assignment by replacing this record by [Name $\Rightarrow$ 'Joe', Office $\Rightarrow$ '515'] and [Name $\Rightarrow$ 'Mary', Office $\Rightarrow$ '515']. Secondly, we can add an element to the set. For example, adding a record [Name $\Rightarrow$ 'Bill', Office $\Rightarrow$ '212'] gives us more information about office allocation.

Define a binary relation $\rightsquigarrow$ on subsets of X as follows: $A \rightsquigarrow (A - \{a\}) \cup A'$, where $a \leq a'$ for all

$a' \in A'$, and $A \rightsquigarrow A \cup \{a\}$. A set B is said to be *more informative* than A, denoted $A \overset{*}{\rightsquigarrow} B$, if B can be obtained from A by a sequence of transformations $\rightsquigarrow$. In other words, $\overset{*}{\rightsquigarrow}$ is the transitive closure of $\rightsquigarrow$.

Similarly for or-sets we define $\mapsto$ by $A \mapsto (A - \{a\}) \cup A'$, where $a \leq a'$ for all $a' \in A'$, and $A \mapsto A - \{a\}$ provided that $A - \{a\}$ is not empty (removing an element from an or-set makes it more informative). Again, $\overset{*}{\mapsto}$ is defined as the transitive closure of $\mapsto$.

Proposition 2 *$\overset{*}{\rightsquigarrow}$ coincides with $\leq^\flat$ and $\overset{*}{\mapsto}$ coincides with $\leq^\sharp$.*

Proof. First notice that $\rightsquigarrow \subseteq \leq^\flat$ and $\mapsto \subseteq \leq^\sharp$. Therefore, transitivity of $\rightsquigarrow$ and $\mapsto$ implies $\overset{*}{\rightsquigarrow} \subseteq \leq^\flat$ and $\overset{*}{\mapsto} \subseteq \leq^\sharp$.

To prove the reverse inclusion, let $A \leq^\flat B$. The case of empty sets is obvious, so assume $A, B \neq \emptyset$. Let $B_a = \{b \in B \mid a \leq b\}$ and $B_A = \bigcup_{a \in A} B_a$. Notice that $B_A \neq \emptyset$. For each $a \in A$, apply the following transformations to A: $A \rightsquigarrow (A - \{a\}) \cup (B_a \cup \{a\})$ for each $a \in A$ in any order. This shows $A \overset{*}{\rightsquigarrow} (A \cup B_A)$. For any $a \in A$, pick $b_a \in B_a$ and apply transformations $A \cup B_A \rightsquigarrow (A - \{a\}) \cup \{b_a\}$ in any order, thus obtaining $A \overset{*}{\rightsquigarrow} B_A$. Finally, if $B - B_A \neq \emptyset$ and $B - B_A = \{b_1, \ldots, b_k\}$, $B_A \rightsquigarrow B_A \cup \{b_1\} \rightsquigarrow \ldots \rightsquigarrow B_A \cup \{b_1, \ldots, b_k\} = B$, i.e. $A \overset{*}{\rightsquigarrow} B$. This shows $\leq^\flat \subseteq \overset{*}{\rightsquigarrow}$. The proof of $\leq^\sharp \subseteq \overset{*}{\mapsto}$ is similar. □

This proposition justifies the semantics defined inductively below on types. Notice that the semantics for or-sets is given in such a way that the empty or-set is incomparable with any other or-sets. This matches very well the intention that the empty or-set stands for inconsistency.

- For each base type b a poset $\langle [\![b]\!], \leq_b \rangle$ is given;
- $[\![s \times t]\!] = \langle [\![s]\!] \times [\![t]\!], \leq_s \times \leq_t \rangle$;
- $[\![\{t\}]\!] = \langle \mathbf{P}_{\text{fin}}([\![t]\!]), \leq_t^\flat \rangle$;
- $[\![\langle t \rangle]\!] = \langle \mathbf{P}_{\text{fin}}([\![t]\!]), \leq_t^\sharp \rangle$.

In several papers dealing with partial information in databases it was proposed that instances of type $\{t\}$ be restricted to those containing no comparable elements, commonly called *antichains*, see [7, 18]. For example, if one field of a record plays the role of object identifier, then instead of having two comparable elements with the same oid their join should be taken, provided the records with the same oid are consistent. One way to obtain an antichain from an arbitrary finite set is to take all the maximal elements. Dually, we can take the minimal elements. Thus obtained antichains will be denoted by $\max_\leq A$ and $\min_\leq A$ or just max and min if the ordering is understood. We suggest using max for the usual sets and min for or-sets (cf. [24]). Then the relations $\rightsquigarrow$ and $\mapsto$ must be redefined as follows: $A \rightsquigarrow_a \max((A - a) \cup A')$, $A \rightsquigarrow_a \max(A \cup a)$ and $A \mapsto_a \min((A - a) \cup A')$, $A \mapsto_a \min(A - a)$.

Proposition 3 *On the family of finite antichains of* $\langle X, \leq \rangle$, $\overset{*}{\rightsquigarrow}_a$ *coincides with* $\leq^\flat$ *and* $\overset{*}{\mapsto}_a$ *coincides with* $\leq^\sharp$.

Proof. Again, as in the proof of proposition 2, only the case of nonempty should be considered and only one direction, namely $\leq^\flat \subseteq \overset{*}{\rightsquigarrow}$ and $\leq^\sharp \subseteq \overset{*}{\mapsto}$ must be proved as the other direction is immediate. The empty set case is immediate, so throughout this proof all the sets are nonempty. We also need the following ordering on sets, called *the Plotkin ordering* (cf. [11]): $A \leq^\natural B \Leftrightarrow A \leq^\flat B$ and $A \leq^\sharp B$.

Let $A, B \neq \emptyset$, $A \cap B = \emptyset$. Define B_A is in the proof of proposition 2. Similarly, $A_b = \{a \in A \mid a \leq b\}$ and $A_B = \bigcup_{b \in B} A_b$.

Claim 1: Let $A \leq^\natural B$, $A \cap B = \emptyset$. *Then* $A \overset{*}{\rightsquigarrow} B$ *and, moreover, only elements of* $A \cup B$ *are used in the transformations.*
Proof of claim 1 is by induction on the size of $A \cup B$. The base case $|A \cup B| = 2$ is obvious. In the general case, let B' be a minimal subset of B such that $A \leq^\natural B'$. Our goal is to show that there exist $a \in A$ and $b \in B'$ such that $A - A_b \leq^\natural B' - b$. Then, by induction hypothesis, $A - A_b \overset{*}{\rightsquigarrow} B' - \{b\}$ and, therefore, $A \overset{*}{\rightsquigarrow} A_b \cup (B' - \{b\}$ (the same transformations can be used). Since $A_b \cup (B' - \{b\}) \rightsquigarrow B'$, $A \overset{*}{\rightsquigarrow} B'$ follows, and adding elements of $B - B'$ gives us $A \overset{*}{\rightsquigarrow} B$.

Assume that no pair (a, b) such that $A - A_b \leq^\natural B' - b$ exists. $A - A_b \leq^\flat B' - b$ always hold. Therefore, whenever $a \leq b$, there exists $b_1 \in B'$ such that $b_1 \geq a_1$ implies $a_1 \in A_b$. Since $a_1 \leq b_1$, there exists $b_2 \in B'$ such that $a_2 \in A_{b_1}$ whenever $a_2 \leq b_2$. Notice that $a_2 \leq b$ since $a_2 \leq b_1$ and therefore must be in A_b. Since A and B are finite, continuing this process, we obtain a finite number of elements $b_1, \ldots, b_k \neq b$ (k may not be zero) such that $A_{b_i} \subseteq A_b$ for all $i = 1, \ldots, k$. We now claim that $A \leq^\natural B' - \{b_1, \ldots, b_k\}$. Clearly, $A \leq^\sharp B' - \{b_1, \ldots, b_k\}$. To prove $A \leq^\flat B' - \{b_1, \ldots, b_k\}$, let $a_0 \in A$. There is $b_0 \in B'$ such that $a_0 \leq b_0$. If b_0 is one of b_i's, $i = 1, \ldots, k$, then $a_0 \leq b$. Hence, b_0 can be chosen from $B' - \{b_1, \ldots, b_k\}$. Therefore, $A \leq^\natural B' - \{b_1, \ldots, b_k\}$ which contradicts minimality. This contradiction finishes the proof of claim 1.

Claim 2: Let $A \leq^\natural B$, $A \cap B = \emptyset$. *Then* $A \overset{*}{\mapsto} B$ *and, moreover, only elements of* $A \cup B$ *are used in the transformations.*
Proof of claim 2 is similar to the proof of claim 1. Again, use induction on $|A \cup B|$. Since removal is now allowed, assume w.l.o.g. that no proper subset of A is less than B w.r.t. $\leq^\natural$. We claim that there exists $a \in A$ such that $A - \{a\} \leq^\natural B - B_a$. Suppose not; then for every $a \in A$ there exists $a_1 \in A$ such that $B_{a_1} \subseteq B_a$. Continuing, we obtain $B_a \supseteq B_{a_1} \supseteq B_{a_2} \supseteq \ldots$. Since all the sets are finite, $B_{a_i} = B_{a_j}$ for some distinct a_i and a_j which contradicts minimality of A for $A - \{a_i\} \leq^\natural B$. Now, given $a \in A$ such that $A - \{a\} \leq^\natural B - B_a$, apply the hypothesis to $A - \{a\}$ and $B - B_a$ and observe that a is not under any element of $B - B_a$. Hence, $A \overset{*}{\mapsto} (B - B_a) \cup \{a\}$ since only elements of $(A - \{a\}) \cup (B - B_a)$ were used in transformation. $(B - B_a) \cup \{a\} \mapsto B$ finishes the proof of claim 2.

Claim 3: Let $A \overset{*}{\rightsquigarrow} B$ *(or* $A \overset{*}{\mapsto} B$*) and all* $\rightsquigarrow$ *and* $\mapsto$ *transformations use only elements of* A *and* B. *If* C *is a finite set such that both* $A \cup C$ *and* $B \cup C$ *are antichains, then* $A \cup C \overset{*}{\rightsquigarrow} B \cup C$ *(or* $A \cup C \overset{*}{\mapsto} B \cup C$*).*
Proof of claim 3. Clearly, C does not interact with any $\rightsquigarrow$ or $\mapsto$ transformation, provided they use only elements of $A \cup B$.

Now, let $A \leq^\flat B$. Since A, B are antichains, for $A' = A - B$ and $B' = B - A$ one has $A' \leq^\flat B'$. Therefore, $A' \leq^\natural B'_A$ and by claim 1 $A' \stackrel{*}{\rightsquigarrow} B'_A$. Moreover, all transformations use only elements from $A' \cup B'_A$. Then, by claim 3, $A \stackrel{*}{\rightsquigarrow} B'_A \cup (A \cap B)$. Adding elements to the right hand side one obtains $A \stackrel{*}{\rightsquigarrow} B$. The proof that $A \leq^\sharp B$ implies $A \stackrel{*}{\mapsto} B$ is similar and it relies on claims 2 and 3. The proposition is proved. □

This proposition shows that if we deal with antichains, we can change the last two clauses in the inductive definition of the semantics of types into

- $[\![\{t\}]\!]_a = \langle \mathbf{A}_{\mathrm{fin}}([\![t]\!]), \leq^\flat_t \rangle$;
- $[\![\langle t \rangle]\!]_a = \langle \mathbf{A}_{\mathrm{fin}}([\![t]\!]), \leq^\sharp_t \rangle$;

where $\mathbf{A}_{\mathrm{fin}}(X)$ is the set of finite antichains of X. It is clear how to define semantics of *or-NRA* expressions if either semantics for types is used. In the case of the antichain semantics, if an application produces a set (or or-set), max (or min) operation is used to make the resulting object into an antichain.

α in the case of the antichain semantics requires some care: $\alpha_a \equiv [\![\alpha]\!]_a$ is a function from $[\![\{\langle t \rangle\}]\!]_a$ to $[\![\langle \{t\} \rangle]\!]_a$. Given an element of $[\![\{\langle t \rangle\}]\!]_a$, i.e. an antichain $\mathcal{A} = \{A_1, \ldots, A_n\}$ w.r.t. $\leq^\flat$, of antichains from $[\![t]\!]_a$, let $A_i = \{a^i_1, \ldots, a^i_{n_i}\}$. Let $\mathcal{F}_\mathcal{A}$ be the set of all choice functions $f : \{1, \ldots, n\} \to \mathbb{N}$ such that $1 \leq f(i) \leq n_i$. For $f \in \mathcal{F}_\mathcal{A}$, $f(\mathcal{A})$ is defined to be $\{a^1_{f(1)}, \ldots, a^n_{f(n)}\}$. Then

$$\alpha_a(\mathcal{A}) = \min_{f \in \mathcal{F}_\mathcal{A}} {}_{\leq^\flat}(\max f(\mathcal{A}))$$

Furthermore, the famous result that iterated powerdomains are isomorphic [13] can now be given a very simple description (cf. [19]):

Theorem 1 *α_a establishes an isomorphism between $[\![\{\langle t \rangle\}]\!]_a$ and $[\![\langle \{t\} \rangle]\!]_a$. The converse β_a is*

$$\beta_a(\mathcal{A}) = \max_{f \in \mathcal{F}_\mathcal{A}} {}_{\leq^\natural}(\min f(\mathcal{A})), \quad \mathcal{A} \in [\![\langle \{t\} \rangle]\!]_a.$$

Proof. We have to show that α_a maps $[\![\{\langle t \rangle\}]\!]$ to $[\![\langle \{t\} \rangle]\!]$, β_a maps $[\![\langle \{t\} \rangle]\!]$ to $[\![\{\langle t \rangle\}]\!]$ and α_a and β_a are mutually inverse and monotone. The first two claims follow immediately from the definitions of α_a and β_a. To complete the proof, show that α_a is monotone and $\beta_a \circ \alpha_a = \mathsf{id}$. By duality the proof of monotonicity of β_a and $\alpha_a \circ \beta_a = \mathsf{id}$ can be obtained.

We start with two easy observations. If Y_1, Y_2 are finite subsets of an arbitrary poset, then
1) $Y_1 \leq^\flat Y_2$ iff $\max Y_1 \leq^\flat \max Y_2$;
2) $Y_1 \leq^\sharp Y_2$ iff $\min Y_1 \leq^\sharp \min Y_2$.

Throughout this proof, $\mathcal{A}$ is defined as above, i.e. $\mathcal{A} = \{A_1, \ldots, A_n\}$ and each A_i consists of elements a^i_j, $j = 1, \ldots, k_i$.
Claim 1: α_a is monotone.

Proof of claim 1: Let $\mathcal{A}, \mathcal{B} = \{B_1, ..., B_m\} \in [\![\{\langle t \rangle\}]\!]$ and $\mathcal{A} \leq^\flat \mathcal{B}$. We must prove $\alpha_a(\mathcal{A}) \leq^\sharp \alpha_a(\mathcal{B})$. In view of the above observations, it is enough to show that for any $f \in \mathcal{F}_\mathcal{B}$ there exists $g \in \mathcal{F}_\mathcal{A}$ such that $g(\mathcal{A}) \leq^\flat f(\mathcal{B})$. Since for each $i = 1, \ldots, n$ there exists j_i such that $A_i \leq^\sharp B_{j_i}$, there is an element $a^i_{p_i} \in A_i$ such that $a^i_{p_i} \leq b^{j_i}_{f(j_i)}$. Let $g(i) = p_i$. Then for this function g one has $\{a^i_{g(i)} \mid i = 1, \ldots, n\} \leq^\flat \{b^i_{f(i)} \mid i = 1, \ldots, m\}$, i.e. $g(\mathcal{A}) \leq^\flat f(\mathcal{B})$. Claim 1 is proved.

Let $\mathcal{A} \in [\![\{\langle t \rangle\}]\!]$ and $\mathcal{B} = \{B_1, \ldots, B_m\} = \alpha_a(\mathcal{A}) \in [\![\langle\{t\}\rangle]\!]$. By 1) and 2) above, to show that $\beta_a \circ \alpha_a = \mathsf{id}$, i.e. that $\beta_a(\mathcal{B}) = \mathcal{A}$, it suffices to prove
Claim 2: For any $f \in \mathcal{F}_\mathcal{B}$ there exists $A_i \in \mathcal{A}$ such that $f(\mathcal{B}) \leq^\sharp A_i$.
Claim 3: Every A_i is in $\beta_a(\mathcal{B})$.

Proof of claim 2: Let $\mathcal{C}$ be the collection of all sets $f(\mathcal{A})$ where $f \in \mathcal{F}_\mathcal{A}$; $\mathcal{C} = \{C_1, \ldots, C_k\}$. Then for any $g \in F_\mathcal{C}$, there exists $A_i \in \mathcal{A}$ such that A_i is contained in $g(\mathcal{C})$ because, if this is not the case, for any $A_i \in \mathcal{A}$ there exists $j_i \leq k_i$ such that $a^i_{j_i} \in A_i$ and, for any $f \in \mathcal{F}_\mathcal{A}$, g on $f(\mathcal{A})$ picks an element different from $a^i_{j_i}$. If we define f_0 such that $f_0(i) = j_i$, g may pick only elements of form $a^i_{j_i}$ on $f_0(\mathcal{A})$, a contradiction. Therefore, $g(\mathcal{C}) \leq^\sharp A_i$ for some i.

Let $f \in \mathcal{F}_\mathcal{B}$. Let H be the set of functions in $\mathcal{F}_\mathcal{A}$ that correspond to elements of $\mathcal{B} = \alpha_a(\mathcal{A})$ or, in other words, $\max h(\mathcal{A}) \in \mathcal{B}$ for $h \in H$. Then, for any $h' \in \mathcal{F}_\mathcal{A} - H$, there exists a function $h \in H$ such that $\max h(\mathcal{A}) \leq^\flat \max h'(\mathcal{A})$, i.e. $h(\mathcal{A}) \leq^\flat h'(\mathcal{A})$. Since $h \in H$, $\max h(\mathcal{A}) \in \mathcal{B}$, i.e. $\max h(\mathcal{A}) = B_i$. If $f(i) = j$, then there is an element in $h'(\mathcal{A})$ that is greater than b^i_j. Define a function $g \in F_\mathcal{C}$ to coincide with f on those C_i's that are given by functions in H. On C_i that corresponds to $f \in \mathcal{F}_\mathcal{A} - H$, let g pick an element which is greater than some b^i_j where $f(i) = j$ (we have just shown it can be done). Then $f(\mathcal{B}) \leq^\sharp \{c^i_{g(i)} \mid i = 1, \ldots, k\} = g(\mathcal{C})$. We know that there exists $A_i \in \mathcal{A}$ such that $g(\mathcal{C}) \leq^\sharp A_i$. Thus, $f(\mathcal{B}) \leq^\sharp A_i$. Claim 2 is proved.

Proof of claim 3: Prove that for any $a^i_j \in A_i$ there exists $B_l \in \mathcal{B}$ such that $a^i_j \in B_l$. Consider the set $F^{ij}_\mathcal{A}$ of functions $f \in \mathcal{F}_\mathcal{A}$ such that $f(i) = j$. If for no $f \in F^{ij}_\mathcal{A}$: $a^i_j \in \max f(\mathcal{A})$, then there exists $A_p \in \mathcal{A}$ such that all elements of A_p are greater than a^i_j, i.e. $A_i \leq^\sharp A_p$ which contradicts our assumption that $\mathcal{A}$ is an antichain w.r.t. $\leq^\sharp$. Hence, $a^i_j \in \max f(\mathcal{A})$ for at least one function in $F^{ij}_\mathcal{A}$. Since $\mathcal{A}$ is an antichain, for any $p \neq i$ there exists $a^p_q \in A_p$ which is not greater than any element of A_i. Change f to pick such an element for any $p \neq i$. Then a^i_j is still in $\max f(\mathcal{A})$. There exists a function $f' \in \mathcal{F}_\mathcal{A}$ such that $\max f'(\mathcal{A}) \leq^\flat \max f(\mathcal{A})$ and $\max f'(\mathcal{A}) \in \alpha_a(\mathcal{A})$. If $f'(i) = j' \neq j$, then, since $f'(\mathcal{A}) \leq^\flat f(\mathcal{A})$ and A_i is an antichain, $a^i_{j'} \leq a^p_q$ for some p and q, where $p \neq i$. But this contradicts the definition of f. Hence, $f'(i) = j$ and $a^i_j \in \max f'(\mathcal{A})$ because $a^i_j \in \max f(\mathcal{A})$. Since $\max f'(\mathcal{A}) = B_l$ for some index l, $a^i_j \in B_l \in \mathcal{B}$.

Let $\mathcal{B}'$ be the collection of elements of $\mathcal{B}$ that contain elements of A_i. Then we can define a function $f \in \mathcal{F}_\mathcal{B}$ on elements of $\mathcal{B}'$ to pick all elements of A_i. Each $B_j \in \mathcal{B} - \mathcal{B}'$ either contains an element of A_i or contains an element which is greater than some $a^i_p \in A_i$. Let f pick any such element. Then $\min f(\mathcal{B}) = A_i$. Suppose $A_i \notin \beta_a(\mathcal{B})$. Then $A_i \leq^\sharp \min g(\mathcal{B})$ for some function $g \in \mathcal{F}_\mathcal{B}$ such that $\min g(\mathcal{B}) \in \beta_a(\mathcal{B})$. By claim 2, $g(\mathcal{B}) \leq^\sharp A_j$ for some A_j. Hence, $\min g(\mathcal{B}) \leq^\sharp A_j$ and since $\mathcal{A}$ is an antichain w.r.t. $\leq^\sharp$, $A_i = A_j = \min g(\mathcal{B}) \in \beta_a(\mathcal{B})$. This finishes the proof of claim 3 and the theorem. □

It was shown in [26] that the orderings $\leq^\flat$ and $\leq^\sharp$ can be given a logical interpretation. Motivated by applications in the semantics of concurrent programming, Winskel used the modal connectives $\Box$ and $\Diamond$ to describe $\leq^\flat$ and $\leq^\sharp$. Rounds [24] used a similar logic to show the interaction between derivable properties of complex objects and their ordering. Here we present what we believe is the simplest interpretation of logics of [24, 26] for complex objects with or-sets.

Start with an unspecified language $\mathcal{L}$ that contains the symbol $\vee$ for disjunction but does not contain $\&, \Box$ and $\Diamond$. With each element $x \in [\![b]\!]$, where b is a base type, associate a collection of formulas in $\mathcal{L}$ closed under $\vee$, called the *theory of* x and denoted $\mathrm{Th}(x)$ in such a way that $x < y$ implies $\mathrm{Th}(x) \supset \mathrm{Th}(y)$ and $x \neq y$ implies $\mathrm{Th}(x) \neq \mathrm{Th}(y)$. For example, if $[\![b]\!]$ is a *flat domain*, i.e. an unordered collection of values with added bottom element $\perp$ which is less than anything else, the above requirement says that theories of distinct nonbottom elements do not coincide and the theory of $\perp$ contains all other theories (i.e. bottom implies everything).

The theory of a pair is a collection of pairs of statements from the theories of the components. The theory of a set is informally defined as those facts which are true of all elements of the set. A theory of an or-set contains facts which are true of at least one element of the or-set. These descriptions are known as unary connectives in modal logic usually denoted by $\Box$ and $\Diamond$.

Now we can give a formal definition of theories of objects in an extended language $\mathcal{L} \cup \{\&, \Box, \Diamond\}$. A theory of an object x, $\mathrm{Th}(x)$, is the minimal collection of formulas closed under $\vee$ which contains

- $\{\phi_1 \& \phi_2 \mid \phi_i \in \mathrm{Th}(x_i), i = 1, 2\}$ if $x = (x_1, x_2)$;
- $\{\Box\phi \mid \forall i : \phi \in \mathrm{Th}(x_i)\}$ if $x = \{x_1, \ldots, x_n\}$;
- $\{\Diamond\phi \mid \exists i : \phi \in \mathrm{Th}(x_i)\}$ if $x = \langle x_1, \ldots, x_n \rangle$.

Proposition 4 *Given two objects x, y of the same type,*

$$x \leq y \Leftrightarrow \mathrm{Th}(x) \supseteq \mathrm{Th}(y)$$

Proof. Induction on the type of x and y. The base case follows immediately from the definition. The case of pair is easy. Let $x = \{x_1, \ldots, x_n\}$, $y = \{y_1, \ldots, y_m\}$, $x \leq y$ means $x \leq^\flat y$. If $\Box\phi \in \mathrm{Th}(y)$, then for all $i = 1, \ldots, m$: $\phi \in \mathrm{Th}(y_i)$. Given any x_j, there exists y_i such that $x_j \leq y_i$; hence $\phi \in \mathrm{Th}(x_j)$ and therefore $\Box\phi \in \mathrm{Th}(x)$. Conversely, let $\mathrm{Th}(x) \supseteq \mathrm{Th}(y)$. Suppose that $x \not\leq^\flat y$, i.e. there exists x_i such that $x_i \leq y_j$ for no y_j. Then, by hypothesis, there exists a formula $\phi_j \in \mathrm{Th}(y_j)$ such that $\phi_j \notin \mathrm{Th}(x_i)$. Let $\phi = \phi_1 \vee \ldots \vee \phi_m$. Then $\phi \in \mathrm{Th}(y_j)$ for all $j = 1, \ldots, m$. Therefore, $\Box\phi \in \mathrm{Th}(y) \subseteq \mathrm{Th}(x)$, i.e. $\phi_1 \vee \ldots \vee \phi_m \in \mathrm{Th}(x_i)$ which means that for at least one j: $\phi_j \in \mathrm{Th}(x_i)$. This contradiction proves $x \leq^\flat y$. A similar proof for the case of or-sets which is based on the properties of $\leq^\sharp$ is omitted. $\Box$

Since $X \leq^\flat Y$ iff $\max X \leq^\flat \max Y$ and $X \leq^\sharp Y$ iff $\min X \leq^\sharp \min Y$, proposition 4 is true if either $[\![\cdot]\!]$ or $[\![\cdot]\!]_a$ semantics is used.

4 Conceptual Query Language and Normalization

As it has been pointed out already, there are two levels of manipulation of objects – structural and conceptual. This section is dedicated to the query language for the conceptual level.

We start with a few examples. If a pair (x, y) of or-sets is given, say, $(\langle 1,2\rangle, \langle 3,4\rangle)$, on conceptual level we must deal with all possible objects it can conceptually stand for, that is, with or-set of pairs $\langle (1,3),(1,4),(2,3),(2,4)\rangle$. In this case the function that carries out transformation of structural representation to conceptual one can be given as $or\_\mu \circ or\_map(or\_\rho_1) \circ or\_\rho_2$. Another example of the passage from structural to conceptual level is given by the primitive $\alpha^s : \{\langle s\rangle\} \to \langle\{s\}\rangle$, provided that s is in the or-set free fragment.

Let us consider a more sophisticated example. Given an object $x = (\{\langle 1,2\rangle, \langle 3\rangle\}, \langle 1,2\rangle)$ of type $\{\langle int\rangle\} \times \langle int\rangle$. Denote the first component by y. Applying or_ρ_2 to x first yields $\langle (y,1),(y,2)\rangle$ which is an object of type $\langle \{\langle int\rangle\} \times int\rangle$. Applying $or\_map(\alpha \circ \pi_1, \pi_2)$ yields an object

$$\langle (\langle \{1,3\},\{2,3\}\rangle, 1), (\langle \{1,3\},\{2,3\}\rangle, 2)\rangle$$

of type $\langle \{\langle int\rangle\} \times int\rangle$. Finally, applying $or\_\mu \circ or\_map(or\_\rho_1)$ yields

$$\langle (\{1,3\},1), (\{1,3\},2), (\{2,3\},1), (\{2,3\},2)\rangle$$

of type $\langle \{int\} \times int\rangle$. This can be considered as a conceptual level object for all the possibilities are listed.

However, one could have used another strategy to list all the possibilities. For example, to apply $(\alpha \circ \pi_1, \pi_2)$ first to obtain an object of type $\langle \{int\}\rangle \times \langle int\rangle$ and then $or\_\mu \circ or\_map(or\_\rho_1) \circ or\_\rho_2$ to obtain an object of type $\langle \{int\} \times int\rangle$. It is easy to check that such a strategy results in precisely the same object as the previous one.

In fact, there is a general result saying that each type has a unique representation at the conceptual level - such that no or-set type occurs in the type expression except as the outermost type constructor. For reasons that should emerge shortly we call such a type *a normal form.* Furthermore, for each object of type t there exists its unique representation at the conceptual level whose type is the normal form of t.

To state these results precisely, we need some definitions about rewrite systems, see [8]. If a signature is fixed, a *rewrite system* is a set of rules of form $\tau_1 \longrightarrow \tau_2$ where τ_1, τ_2 are terms. If σ is obtained from τ by rewriting a subterm of τ, we also write $\tau \longrightarrow \sigma$. If σ is obtained from τ by a (possibly empty) sequence of applications of rewrite rules, we write $\tau \longrightarrow\!\!\!\!\!\rightarrow \sigma$.

A term τ is called a *normal form* if there is no other term σ such that $\tau \longrightarrow \sigma$. A rewrite system is called *terminating* if there is no infinite sequence of terms $\tau_1 \longrightarrow \tau_2 \longrightarrow \ldots$. It is called *Church-Rosser* if, whenever $\tau \longrightarrow\!\!\!\!\!\rightarrow \tau_1$ and $\tau \longrightarrow\!\!\!\!\!\rightarrow \tau_2$, there exists a term τ' such that $\tau_1 \longrightarrow\!\!\!\!\!\rightarrow \tau'$ and $\tau_2 \longrightarrow\!\!\!\!\!\rightarrow \tau'$. In a Church-Rosser terminating system for every term τ there exists a unique normal form $nf(\tau)$ such that $\tau \longrightarrow\!\!\!\!\!\rightarrow nf(\tau)$.

Now introduce the rewrite rules for type expressions:

$$t \times \langle s \rangle \longrightarrow \langle t \times s \rangle \qquad \langle t \rangle \times s \longrightarrow \langle t \times s \rangle$$

$$\langle\langle t \rangle\rangle \longrightarrow \langle t \rangle \qquad \{\langle t \rangle\} \longrightarrow \langle \{t\} \rangle$$

Proposition 5 *The above rewrite system is terminating and Church-Rosser. The normal form $nf(t)$ for type t can be found as follows: If t does not use $\langle\rangle$, then $nf(t) = t$. Otherwise, remove all angle brackets from t. If the resulting type is t', then $nf(t) = \langle t' \rangle$.*

Proof. To show that the rewrite system is terminating, define the following function on types. Considering types as their derivation trees, let k_i be the number of occurrences of $\langle\rangle$ on the ith level of the derivation tree of type t. If the height of the derivation tree is n, define $\varphi(t)$ as $\Sigma_{i=1}^{n} k_i \cdot i$. It is easy to see that if $t \longrightarrow t_0$, then $\varphi(t) > \varphi(t_0)$. Hence, any rewriting terminates.

To prove Church-Rosserness, one has to find so-called *critical pairs*, see [8], which in essence are pairs of terms that can give rise to ambiguity in rewriting, and show that for any critical pair (τ_1, τ_2) there exists a term τ such that $\tau_1 \longrightarrow\!\!\!\!\!\rightarrow\ \tau$ and $\tau_2 \longrightarrow\!\!\!\!\!\rightarrow\ \tau$. We refer the interested reader to [8] for the definitions and proof of the critical pair lemma. A straightforward analysis of our rewrite system reveals the following critical pairs: 1) $(\langle\{\langle t \rangle\}\rangle, \{\langle t \rangle\})$; 2) $(\langle t \times \langle s \rangle\rangle, t \times \langle s \rangle)$; 3) $(\langle\langle s \rangle \times t\rangle, \langle s \times \langle t \rangle\rangle)$ and 4) $(\langle\langle s \rangle \times t\rangle, \langle\langle s \rangle\rangle \times t)$ and their symmetric analogs. The terms to which both components of critical pairs rewrite are $\langle\{t\}\rangle$ for 1), $\langle t \times s \rangle$ for 2) and $\langle s \times t \rangle$ for 3) and 4). Thus, the rewrite system is Church-Rosser and, therefore, has unique normal forms.

The proof of the last statement is by induction on the structure of a given type. We limit ourselves only to types containing $\langle\rangle$. The base case is immediate. In general case, consider three subcases: 1) $t = t_1 \times t_2$, 2) $t = \{t_1\}$, 3) $t = \langle t_1 \rangle$. In subcase 1, $t' = t'_1 \times t'_2$, hence, if both t_1 and t_2 contain or-sets, $nf(t_1) = \langle t'_1 \rangle, nf(t_2) = \langle t'_2 \rangle$ and $t \longrightarrow\!\!\!\!\!\rightarrow\ \langle t'_1 \rangle \times \langle t'_2 \rangle \longrightarrow \langle t'_1 \times t'_2 \rangle = \langle t' \rangle$ which is a normal form. Thus, $nf(t) = \langle t' \rangle$. The simple proofs of other cases are omitted. □

Having defined rewrite rules for types, we must show how to apply these rules to instances. First, associate a morphism with each rule as follows:

$$or\_\rho_2 : t \times \langle s \rangle \longrightarrow \langle t \times s \rangle \qquad or\_\rho_1 : \langle t \rangle \times s \longrightarrow \langle t \times s \rangle$$

$$or\_\mu : \langle\langle t \rangle\rangle \longrightarrow \langle t \rangle \qquad \alpha : \{\langle t \rangle\} \longrightarrow \langle \{t\} \rangle$$

Let t be a type and p a position in the derivation tree for t such that applying a rewrite rule with associated function f to t at p yields type s. Our aim is to define a function $\mathsf{app}(t, p, f) : t \to s$ showing the action of rewrite rules on objects. Define it by induction on the structure of t:

- if p is the root of the derivation of t, then $\mathsf{app}(t, p, f) = f$;
- if $t = t_1 \times t_2$ and p is in t_1 , then $\mathsf{app}(t, p, f) = (\mathsf{app}(t_1, p, f) \circ \pi_1, \pi_2)$;
- if $t = t_1 \times t_2$ and p is in t_2, then $\mathsf{app}(t, p, f) = (\pi_1, \mathsf{app}(t_2, p, f) \circ \pi_2)$;

- if $t = \{t'\}$ then $\mathsf{app}(t, p, f) = map(\mathsf{app}(t', p, f))$;
- if $t = \langle t' \rangle$ then $\mathsf{app}(t, p, f) = or\_map(\mathsf{app}(t', p, f))$.

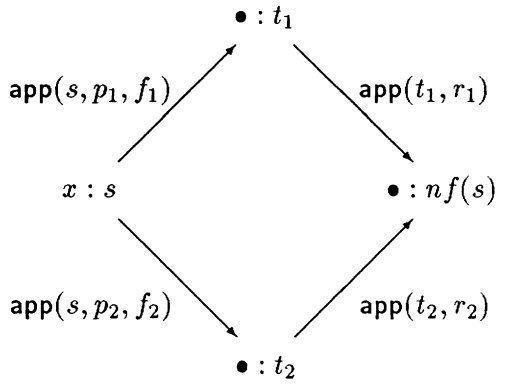
Notice that the definition of app relies on the fact that the functions associated with the rewrite rules are polymorphic.

Given a type t and a rewriting strategy $r := t \xrightarrow{f_1} t_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} t_n = nf(t)$ such that the rewrite rule with associated function f_i is applied at a position p_i, we can extend the function app to $\mathsf{app}(t, r) : t \to nf(t)$ by $\mathsf{app}(t, r) = \mathsf{app}(t_{n-1}, p_n, f_n) \circ \ldots \circ \mathsf{app}(t_1, p_2, f_2) \circ \mathsf{app}(t, p_1, f_1)$

Notice that $\mathsf{app}(t, p, id) = id$. Therefore, $\mathsf{app}(t, p, f)$ destroys our informal notion of conceptual meaning if and only if f does. Since all of the f_i's we use in the rewriting rules preserve conceptual meaning (see section 1), so does $\mathsf{app}(t, r)$. Now we can formalize this informal notion of conceptual meaning by proving the following theorem.

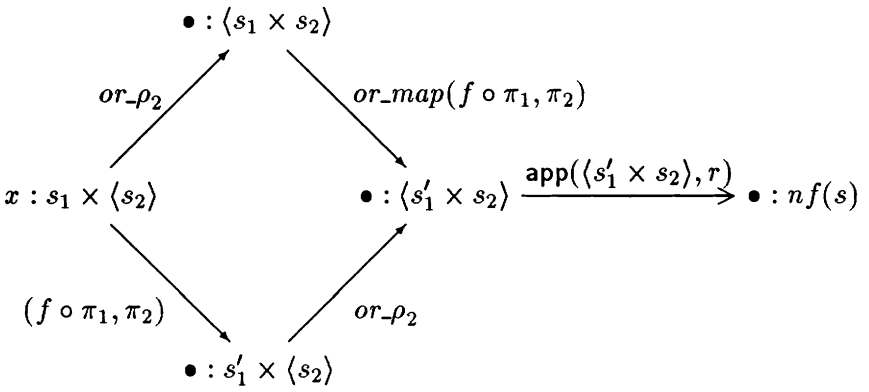
Theorem 2 (Coherence) *Given a type t, any two rewrite strategies $r_1, r_2 : t \longrightarrow\!\!\!\!\!\rightarrow nf(t)$ yield the same result on objects. That is, for any object x of type t, $\mathsf{app}(t, r_1)(x) = \mathsf{app}(t, r_2)(x)$.*

Proof. By proposition 5, it suffices to prove the weaker property that for any complex object $x : s$ and any two rewrite steps $\mathsf{app}(s, p_1, f_1) : s \to t_1$ and $\mathsf{app}(s, p_2, f_2) : s \to t_2$, there are $\mathsf{app}(t_1, r_1) : t_1 \to nf(s)$ and $\mathsf{app}(t_2, r_2) : t_2 \to nf(s)$ such that the diagram below commutes.



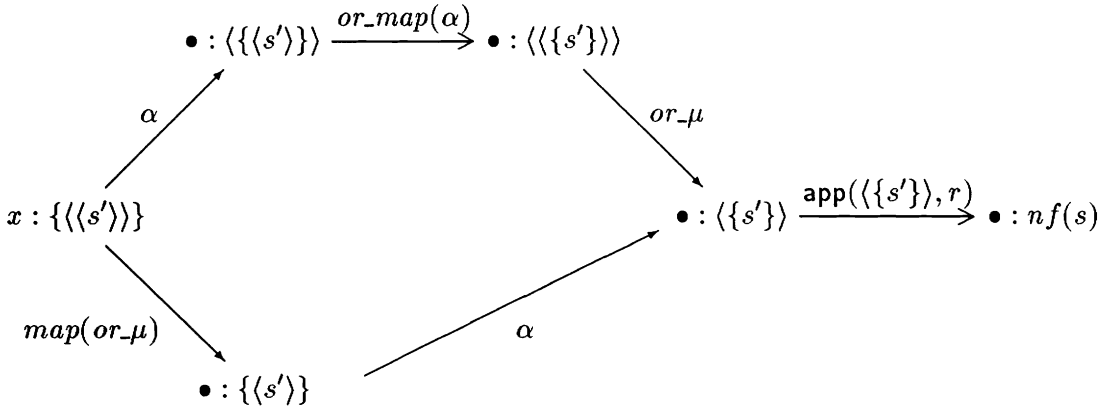
The proof is a straight forward case analysis. We present two cases for illustration. Suppose s is $s_1 \times \langle s_2 \rangle$, $\mathsf{app}(s, p_1, f_1)$ is or_ρ_2 and $\mathsf{app}(s, p_2, f_2)$ is $(f \circ \pi_1, \pi_2)$, where $f : s_1 \to s_1'$. Then $t_1 = \langle s_1 \times s_2 \rangle$

and $t_2 = \langle s_1' \times \langle s_2 \rangle \rangle$. Moreover, the diagram



commutes for any strategy r. Then r_1 and r_2 can be obtained readily. Hence the case.

Suppose s is $\{\langle\langle s' \rangle\rangle\}$, $\mathsf{app}(s, p_1, f_1)$ is α, and $\mathsf{app}(s, p_2, f_2)$ is $map(or\_\mu)$. Then $t_1 = \langle\{\langle s' \rangle\}\rangle$ and $t_2 = \{\langle s' \rangle\}$. The diagram



commutes for any strategy r because $or\_\mu \circ or\_map(\alpha) \circ \alpha = \alpha \circ map(or\_\mu)$. From which r_1 and r_2 can be derived. So the case holds. □

Therefore, all objects with the same meaning at the conceptual level rewrite to the same normal form. The intuitive notion of the conceptual meaning can now be rigorously defined as the normal form. So now we can define the *conceptual query language* or-$\mathcal{NRA}^+$ by adding the new construct

$$\frac{}{normalize^t : t \to nf(t)}$$

to or-$\mathcal{NRA}$. By the coherence theorem, $normalize^t$ can be implemented as $\mathsf{app}(t, r)$ where $r : t \longrightarrow\!\!\!\!\!\!\rightarrow nf(t)$. Notice that, for any given t, $normalize^t$ can be expressed in or-$\mathcal{NRA}$ (maybe in more than one way) but it is impossible to express it *polymorphically.*

There are two questions to be asked about this new query language. First, how much information is lost by normalization? There are different objects that normalize to the same one, i.e. information from the structural level could be lost. Secondly, how costly is normalization? We address these problems in the subsequent sections. In the next section it is shown that normalization is *lossless*, i.e. practically all queries are unaffected by the loss of structural information. In Section 6 the upper bounds for the size of normalized objects are found.

5 Losslessness of Normalization

This section investigates whether the process of normalization loses anything "that can be regarded as critical." If loss of information is inevitable in the general case, then one would like to obtain a set of general sufficient (and, if possible, necessary) conditions that guarantee losslessness of normalization. In order to proceed, a criterion on what normalization can be regarded as "losing nothing essential" has to be formulated. The following is a reasonable choice.

Definition. *Given a definable morphism $f : s \to t$. Suppose there is a morphism $preserve(f) : nf(\langle s \rangle) \to nf(\langle t \rangle)$ such that $preserve(f) \circ normalize^{\langle s \rangle} \circ or\_\eta^s = normalize^{\langle t \rangle} \circ or\_\eta^t \circ f$, provided the input is restricted to objects not containing any empty or-set. Then normalization is* lossless *with respect to f.*

Let us first justify the definition given above. The proviso on the input is necessary because all objects containing empty or-set have the same normal form, namely $\langle\rangle$. Recalling that $\langle\rangle$ stands for inconsistency, such objects are conceptually inconsistent and should be omitted. The use of or_η^s and or_η^t is a technical device to ensure that the normal forms produced always look like $\langle d_1, ..., d_n \rangle$ where $d_1, .., d_n$ have no or-sets. This is justified since $or\_\eta\ d$ is conceptually d for any d. conceptual meaning of the input to f and returns the conceptual meaning of the output of f.

It turns out that it is not easy to achieve losslessness of normalization with respect to an arbitrarily given morphism f. There is no simple method to discover the required $preserve(f)$. However, we have been able to isolate the morphisms that can give rise to possible difficulty. Any morphism not containing $K\langle\rangle, p$ where $Type(p)$ contains some or-set, and $\rho_2^{s,t}$ where s contains some or-set does not lead to losslessness.

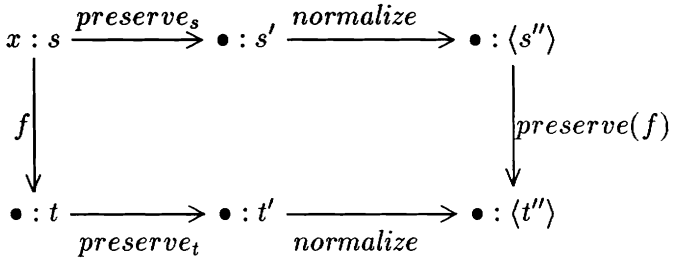
Theorem 3 (Losslessness) *Let $f : s \to t$ be a morphism of or-$\mathcal{NRA}$ not containing any $K\langle\rangle, p$ where some or-set appears in $Type(p)$, and $\rho_2^{u,v}$ where u has some or-sets. Then normalization is lossless with respect to f. Moreover, the $preserve(f)$ that makes normalization lossless has a map-like property; that is, $preserve(f) = or\_\mu \circ or\_map(preserve(f) \circ or\_\eta)$.*

Proof. For each type t, define the type $preserve\ t$ and the morphism $preserve_t : t \to preserve\ t$ as follows.

- $preserve\ b = \langle b \rangle$
- $preserve\ (s \times t) = preserve\ s \times preserve\ t$
- $preserve\ \{t\} = \{preserve\ t\}$
- $preserve\ \langle t \rangle = \langle preserve\ t \rangle$

- $preserve_b = or\_\eta^b$
- $preserve_{(s \times t)} = (preserve_s \circ \pi_1, preserve_t \circ \pi_2)$
- $preserve_{\{t\}} = map(preserve_t)$
- $preserve_{\langle t \rangle} = or\_map(preserve_t)$

Using the fact that normalization is coherent, it is easy to show by induction on t that $normalize \circ or\_\eta^t = normalize \circ preserve_t$. Consequently, we can instead prove the commutativity of



for complex object $x : s$ having no empty or-set and any morphism $f : s \to t$ satisfying the preconditions of the theorem, where $preserve(f)$ is defined by structural induction on f below.

Case f is id. Then $preserve(f) = id$.

Case f is η, π_1, π_2, μ, $K\langle\rangle$, Kc, $!$, $\cup$, ρ_2, or p. Then $preserve(f) = or\_map(f)$.

Case f is (g, h). Then $preserve(g, h) = or\_\mu \circ or\_map(or\_\rho_1) \circ or\_\rho_2 \circ (preserve\ g, preserve\ h)$.

Case f is $g \circ h$. Then $preserve(g \circ h) = preserve(g) \circ preserve(h)$.

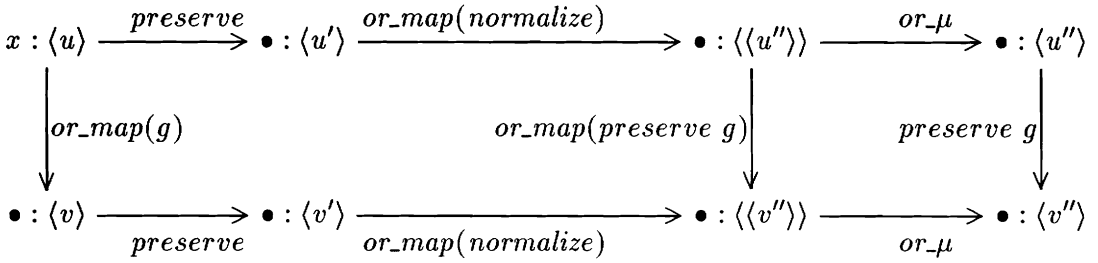
Case f is $map(g)$. Then $preserve(map\ g) = or\_\mu \circ or\_map(\alpha) \circ or\_map(map(preserve(g) \circ or\_\eta))$.

Case f is α, or_η, or_ρ_2, or or_μ. Then $preserve(f) = id$.

Case f is $or\_map(g)$. Then $preserve(or\_map(g)) = preserve(g)$.

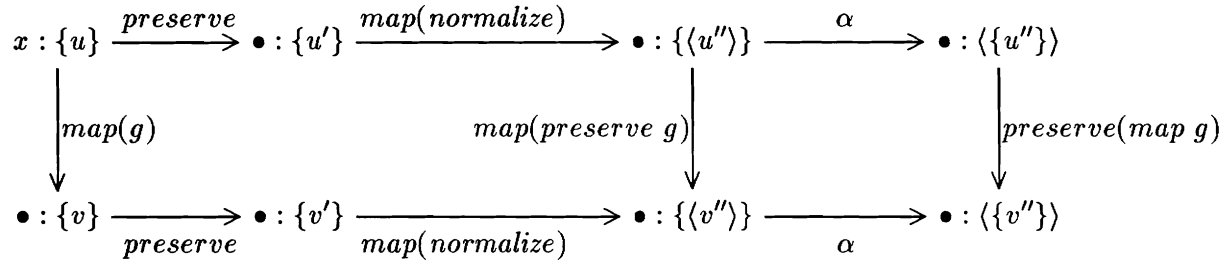
It is readily verified that $preserve(f)$ is map-like. The proof that the diagram commutes is by induction on f and uses the coherence theorem in several places. We present a few illustrative cases below.

Suppose f is $or\_map(g)$, where $g : u \to v$. Then $s = \langle u \rangle$ and $t = \langle v \rangle$. By hypothesis, $preserve(g)$ exists and is map-like. Now consider the diagram below.



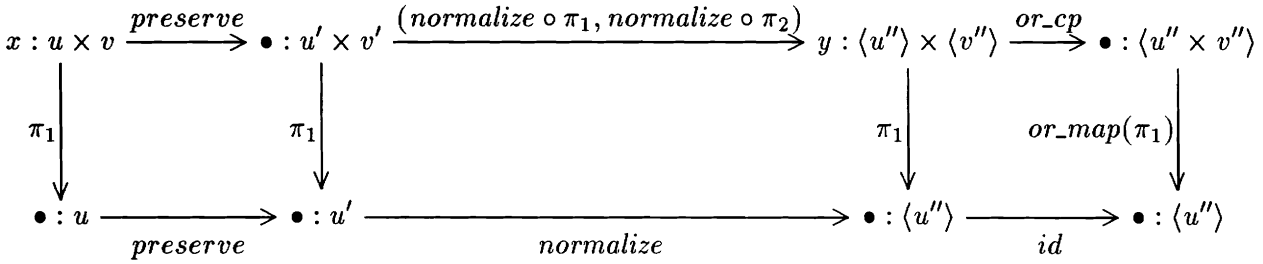
The left rectangle commutes by hypothesis. The right rectangle commutes because $preserve(g)$ is map-like. Hence the entire diagram commutes. By the coherence theorem, $normalize^{\langle u' \rangle} = or\_\mu^{u''} \circ or\_map(normalize^{u'})$ and $normalize^{\langle v' \rangle} = or\_\mu^{v''} \circ or\_map(normalize^{v'})$. So the original diagram commutes and the case follows.

Suppose f is $map(g)$ where $g : u \to v$. Then $s = \{u\}$ and $t = \{v\}$. By hypothesis, $preserve(g)$ exists and is map-like. Consider the diagram below.



The left rectangle commutes by hypothesis. To see that the right rectangle commutes, we calculate as follows: $preserve(map\ g) \circ \alpha = or\_\mu \circ or\_map(\alpha) \circ or\_map(map(preserve(g) \circ or\_\eta)) \circ \alpha = or\_\mu \circ or\_map(\alpha) \circ \alpha \circ map(or\_map(preserve(g) \circ or\_\eta)) = \alpha \circ map(or\_\mu \circ or\_map(preserve(g) \circ or\_\eta)) = \alpha \circ map(preserve\ g)$. The last equality follows from the map-likehood of $preserve(g)$. The second last equality follows from the fact that $or\_\mu \circ or\_map(\alpha) \circ \alpha = \alpha \circ map(or\_\mu)$. Hence the entire diagram commutes. The case then follows by an application of the coherence theorem.

Suppose f is $\pi_1^{u,v}$. Then $s = u \times v$ and $t = u$. Let $or\_cp = or\_\mu \circ or\_map(or\_\rho_1) \circ or\_\rho_2$. Consider the diagram below.



The two left rectangles obviously commutes. By assumption, x has no empty or-set. Thus y has no empty or-sets. Therefore, the right rectangle commutes. Hence the whole diagram commutes. Finally, the coherence theorem is applied to conclude the case. □

The requirement on $\rho_2^{u,v}$ can be relaxed. In particular, a $\rho_2^{u,v}$ such that u has or-set but v has no or-set can appear in f so long as it does not appear in any subexpression of the form $(\cdot, \cdot)$, $or\_map(\cdot)$, or $map(\cdot)$. Losslessness can be maintained in such a situation, although the required $preserve(f)$ is no longer map-like.

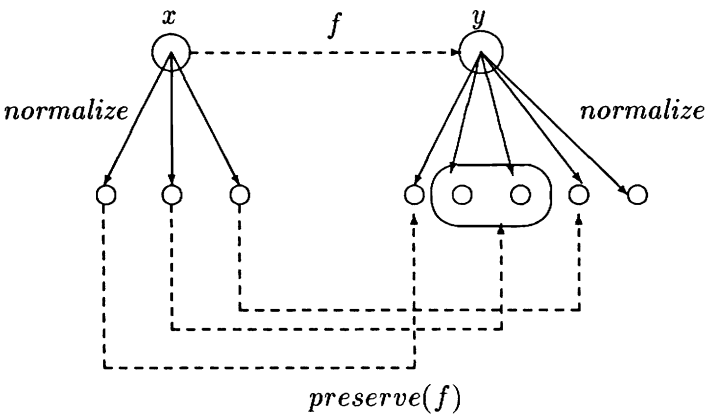


Figure 2: Conceptual analog of morphism f

Since p is generally an uninterpreted primitive, the qualification that $Type(p)$ has no or-set is necessary. This means that equality tests $=^t$ where t has or-set have been excluded. However, $=^t$ is an equality test that is structural. Normalization is a process that removes structural differences from objects that are conceptually identical. Hence one cannot expect normalization to be lossless with respect to such $=^t$.

Given an *or-$\mathcal{NRA}$* morphism $f : s \to t$ and an object $x : s$ containing some or-sets. Then x conceptually represents several values $x_1, ..., x_n$. Suppose $f\ x$ is an object containing or-sets; then it conceptually represents several values $y_1, ..., y_m$. It is desirable to discover which one of $x_1, ..., x_n$ leads to which one of $y_1, ..., y_m$. This is a question of searching for a *conceptual analog* of f that associates each x_i in *normalize* x to a subset of *normalize*$(f\ x)$.

The idea of conceptual analog of morphism is illustrated in Figure 2. One would like to know which combination of the conceptual values of the input give rise to which subset of the conceptual values of the output. However, the ideal situation can only be approximated. As a first attempt, for each possible conceptual value x_i of the input x, we aim only to account for some of the conceptual values in the output that are due to it. This approximation to conceptual analog is illustrated in Figure 2. Some conceptual values y_j in the output may be left unaccounted for. For example, the last element of *normalize* y in the figure. Similarly, the picture given for each input x_i is only partial. For example, the second element of *normalize* x in the figure might in reality contributes to 3 values in the output but the conceptual analog discovers only 2.

This approximation to conceptual analog is formalized below.

Definition. *Let $f : s \to t$ be a definable morphism of or-$\mathcal{NRA}$. Then its* conceptual analog *is a map-like morphism preserve(f) such that for all $x : s$, $(preserve(f) \circ normalize^{\langle s \rangle} \circ or\_\eta^s)(x)$ is included in $(normalize^{\langle t \rangle} \circ or\_\eta^t \circ f)(x)$.*

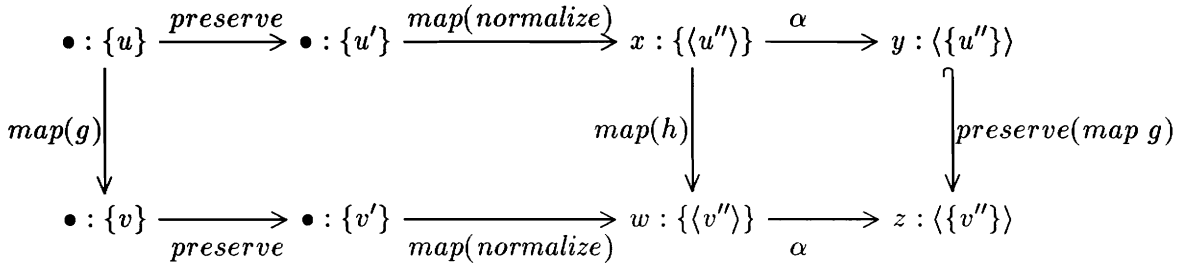
There is some relationship between losslessness and conceptual analog. A conceptual analog of f that

accounts for every element in the output is a morphism that makes normalization lossless with respect to f. Hence the search for a lossless $preserve(f)$ can be generalized as a search for a conceptual analog of f that accounts for each possible conceptual value of the output. A result similar to theorem 3 can be proved.

Proposition 6 *Let $f : s \to t$ be a morphism of or-$\mathcal{NRA}$ not containing any p where $Type(p)$ involves some or-set. Then $preserve(f)$, as defined in the proof of the losslessness theorem, is a conceptual analog of f.*

Proof. The precondition of this proposition is weaker than that of theorem 3 because we merely required that $(preserve(f) \circ normalize \circ or\_\eta)(x)$ be included in $(normalize \circ or\_\eta \circ f)(x)$, as oppose to being equal. The proof is a simple adaptation of the proof of theorem 3. The most involved case is presented below.

Suppose f is $map(g)$ where $g : u \to v$. Then $s = \{u\}$ and $t = \{v\}$. By hypothesis, $preserve(g)$ is a conceptual analog of g. Consider



If x contains some empty or-set, then y is $\langle\rangle$. In this case, the inclusion is trivially satisfied. So assume x has no empty or-set. Let $h : \langle u''\rangle \to \langle v''\rangle$ be a map-like function, not necessarily definable in or-$\mathcal{NRA}$, such that $map\ h\ x = w$ and $preserve\ g\ d$ is included in $h\ d$ for all singleton $d : \langle u''\rangle$. It is easy to see that $(preserve(map\ g) \circ \alpha)(x) = (\alpha \circ map(or\_\mu \circ or\_map(preserve(g) \circ or\_\eta)))(x)$ is included in $(\alpha \circ map(or\_\mu \circ or\_map(h \circ or\_\eta)))(x) = \alpha \circ map(h)$. Since such a h can always be found given x, w, and $preserve(g)$, the case holds. □

6 Costs of Normalization

We have seen before that the complexity of or-$\mathcal{NRA}^+$ queries can be exponential. In particular, the cardinality of $normalize(x)$ can be exponential in the size of x provided that α was used in the course of normalization. In fact, the example given in section 2 shows that even one application of α may result in an or-set of exponential cardinality. If one tries to estimate the cost of normalization by "brute force," a hyperexponential upper bound can be immediately obtained: indeed, if n is the size of x, applying the costly α $O(n)$ times seems to yield a hyperexponential bound.

In this section we show that the fear of hyperexponentiality is not justified. In fact, both cardinality of $normalize(x)$ and its size are in the worst case exponential in the size of x. The first result in this

section explains why consecutive applications of α still yield objects of exponential size. Then we proceed to find upper bounds on the cardinality and the size of normalized objects. The last result in this section shows that there exist existential queries involving normalization which can not be evaluated in polynomial time.

Let x be an object and $y = \mathit{normalize}(x)$. Define $m(y)$ as the number of elements in y if it is an or-set and 1 otherwise. Uniformly, $m(x) = |\mathit{normalize}(\mathit{or\_}\eta(x))|$. The *size* of an object is defined inductively: the size of an atomic object is 1, $\mathsf{size}\,(x,y) = \mathsf{size}\,x + \mathsf{size}\,y$, $\mathsf{size}\,\{x_1,\ldots,x_n\} = \mathsf{size}\,\langle x_1,\ldots,x_n\rangle = \mathsf{size}\,x_1 + \ldots + \mathsf{size}\,x_n$.

To work with objects, it is convenient to associate rooted labeled trees with them. A tree $\mathcal{T}x$ associated with an atomic object x is defined as a one-node tree labeled by x. $\mathcal{T}(x,y)$ is a tree with the root labeled by $\times$ and two subtrees rooted at its children are $\mathcal{T}x$ and $\mathcal{T}y$. $\mathcal{T}\{x_1,\ldots,x_n\}$ (or $\mathcal{T}\langle x_1,\ldots,x_n\rangle$) is a tree whose root is labeled by $\{\}$ (or $\langle\rangle$) and n subtrees rooted at its children are $\mathcal{T}x_1,\ldots,\mathcal{T}x_n$. In view of this definition, $m(x)$ can be redefined as the number of children of the root of $\mathcal{T}\mathit{normalize}(x)$ if the root is labeled by $\langle\rangle$ and 1 otherwise. $\mathsf{size}\,x$ is the number of leaves in $\mathcal{T}x$.

Intuitively, the following proposition says that the "internal" structure of $\mathcal{T}x$ does not contribute to the creation of new possibilities in $\mathit{normalize}(x)$, and the number of such possibilities $m(x)$ is determined by the or-sets which are closest to the leaves.

Proposition 7 *Let x be an object, and $v_1,\ldots,v_k$ the nodes in $\mathcal{T}x$ labeled by $\langle\rangle$, such that the subtrees rooted at v_i's do not have other nodes labeled by $\langle\rangle$ (i.e. they are or-sets closest to the leaves). Let m_i be the number of children of v_i, $i = 1,...,k$. Then, if $k \neq 0$,*

$$m(x) \leq \prod_{i=1}^{k}(m_i + 1)$$

Proof is by induction on the structure of the object. We consider only objects containing or-sets. The base case (i.e. or-sets of objects of base types) is obvious. Let $x = (x_1, x_2)$. Assume that both x_1 and x_2 contain or-sets and $v_1,\ldots,v_p$ are nodes of $\mathcal{T}x_1$ and $v_{p+1},\ldots,v_k$ are nodes of $\mathcal{T}x_2$. Then, by induction hypothesis, $m(x_1) \leq \prod_{i=1}^{p}(m_i+1)$ and $m(x_2) \leq \prod_{i=p+1}^{k}(m_i+1)$. By coherence, $\mathit{normalize}(x) = \mathit{or\_}\rho((\mathit{normalize}(x_1), \mathit{normalize}(x_2)))$ where $\mathit{or\_}\rho$ pairs each item in its first argument with each item in its second argument (it can be easily expressed in or-$\mathcal{NRA}$). Therefore, $m(x) \leq m(x_1)m(x_2) \leq \prod_{i=1}^{k}(m_i+1)$. Two other cases when either x_1 or x_2 contains or-sets are similar.

Let $x = \{x_1,\ldots,x_n\}$. Then all x_i's contain or-sets. Again, by coherence,

$$\mathit{normalize}(x) = \alpha(\{\mathit{normalize}(x_1),\ldots,\mathit{normalize}(x_n)\})$$

Therefore, $m(x) \leq \prod_{i=1}^{n} m(x_i)$ and the result follows from the induction hypothesis.

Finally, if $x = \langle x_1,\ldots,x_n\rangle$, there are two cases. If x_i's do not contain or-sets, then $m(x) = n \leq n+1$. If they do contain or-sets, then by coherence

$$\mathit{normalize}(x) = \mathit{or\_}\mu(\langle\mathit{normalize}(x_1),\ldots,\mathit{normalize}(x_n)\rangle)$$

i.e. $m(x) \leq \Sigma_{i=1}^{n} m(x_i) \leq \prod_{i=1}^{n} m(x_i)$ because $m(\cdot) \geq 2$. The case now follows from the hypothesis. □

This proposition explains why there is an exponential upper bound for $m(x)$ despite the fact that α can be applied many times. The following result finds a sharp upper bound in terms of the size rather than the tree structure.

Theorem 4 *Let x be an object with* $\mathsf{size}\, x = n$. *Then*

$$m(x) \leq \sqrt[3]{3}^{n}$$

Moreover, for any n divisible by 3 there exists an object x such that $\mathsf{size}\, x = n$ *and* $m(x) = \sqrt[3]{3}^{n}$.

Proof. As in the proof of proposition 7, consider only objects containing or-sets. Procced by induction on the number of steps of normalization. If the object is already normalized, we are done. Assume $normalize(x)$ is obtained by one step of normalization. Then this step is one of the maps associated with the rewrite rules, so we have for cases. Notice that in the base cases we may assume w.l.o.g that everything that any element of a set or an or-set is of base type since this will give us the maximal possible $m(x)$ for a given $\mathsf{size}\, x$.

Case 1. $x = (x_1, x_2)$ where $x_1 = \langle x_1^1, \ldots, x_{n-1}^1 \rangle$. Then $normalize(x) = or\_\rho_1(x)$ and it is an easy arithmetic exercise to show that $m(x) = n - 1 \leq \sqrt[3]{3}^{n}$.

Case 2 when or_ρ_2 is applied to obtain the normal form is similar.

Case 3. Let $x = \{X_1, \ldots, X_k\}$ where each X_i is an or-set $\langle x_1^i, \ldots, x_{k_i}^i \rangle$ where all x_j^i are elements of base types. Since we are interested in upper bound, assume w.l.o.g. that all x_j^i's are distinct (if they are not, some of sets in $normalize(x)$ could collapse). Let $X = \bigcup_{i,j} x_j^i$. Define a graph $G = (X, E)$ where $(x_{j_1}^{i_1}, x_{j_2}^{i_2})$ is in E iff $i_1 \neq i_2$. Let $normalize(x) = \alpha(x) = \langle Y_1, \ldots, Y_p \rangle$ (Y_k's are sets). Then it follows from the definition of α that $Y_1, \ldots, Y_p$ are precisely the cliques of G. Since $n = \mathsf{size}\, x = |X|$, applying the upper bound on the number of cliques for a graph with n vertices [21], we obtain $p = m(x) \leq \sqrt[3]{3}^{n}$.

Case 4. $x = \langle X_1, \ldots, X_k \rangle$ where X_i's are or-sets of a base type. Then $normalize(x) = or\_\mu(x)$ and $m(x) \leq n$. Again, simple arithmetic shows that $n \leq \sqrt[3]{3}^{n}$. Hence, $m(x) \leq \sqrt[3]{3}^{n}$.

The proof of the general case is very similar to the proof of proposition 7 and we will show only step. Let $x = \{x_1, \ldots, x_k\}$ where x_i's are not normalized. Then $normalize(x)$ is obtained by applying α to $\{normalize(x_1), \ldots, normalize(x_n)\}$. Let $\mathsf{size}\, x_i = n_i$. By induction hypothesis, $m(x_i) \leq \sqrt[3]{3}^{n_i}$. We now have

$$m(x) \leq \prod_{i=1}^{k} m(x_i) \leq \prod_{i=1}^{k} \sqrt[3]{3}^{n_i} \leq \sqrt[3]{3}^{n}$$

The other cases are similar. To show the sharpness of the upper bound, let $n = 3k, k > 0$. Assume that we have a base type whose domain is infinite (typical example is *int*). Let $b_1, \ldots, b_n$ be n distinct elements of such a type. Let

$$x = \{\langle b_1, b_2, b_3 \rangle, \langle b_4, b_5, b_6 \rangle, \ldots, \langle b_{n-2}, b_{n-1}, b_n \rangle\}$$

Then $\mathsf{size}\, x = n$ and $normalize(x) = \alpha(x)$ contains $3^k = \sqrt[3]{3}^{n}$ elements. The theorem is completely proved. □

Using theorem 4, one can prove the following upper bound on the size of normal forms by induction on the steps of the normalization process:

Theorem 5 *Let x be an object with* $\mathsf{size}(x) = n$ *where* $n > 1$. *Then*

$$\mathsf{size}\, normalize(x) \leq \frac{n}{2}\sqrt[3]{3}^{\,n}$$

Proof. Similarly to the proof of theorem 4, proceed by induction on the steps of normalization. We start with base cases, i.e. consider application of or_ρ_2 or or_ρ_1 or α or or_μ.

Case 1. $x = (x_1, x_2)$ where $x_1 = \langle x_1^1, \ldots, x_k^1 \rangle$. Let $\mathsf{size}\, x_1 = s_1$, $\mathsf{size}\, x_i^1 = \sigma_i$. Then $s_1 + \sigma_1 + \ldots + \sigma_k = n$. Since $normalize(x) = or\_\rho_1(x)$, $\mathsf{size}\, normalize(x) = ks_1 + \sigma_1 + \ldots + \sigma_k = ks_1 + (n - s_1) \leq (n - s_1)s_1 + n - s_1 \leq 2n - 2$. Since empty sets and or-sets are excluded, $n \geq 2$ in this case and therefore $2n - 2 \leq \frac{n}{2}\sqrt[3]{3}^{\,n}$.

Case 2 when or_ρ_2 is applied is similar.

Case 3. Let $x = \{X_1, \ldots, X_l\}$ where each X_i is an or-set $\langle x_1^i, \ldots, x_{k_i}^i \rangle$ where all x_j^i have types containing no or-set. Let $\mathsf{size}\, x_j^i = s_j^i$ and

$$\sum_{j=1}^{k_i} s_j^i = \sigma_i \qquad\qquad \sum_{i=1}^{l} \sigma_i = n$$

Then an easy calculation shows that $\mathsf{size}\, normalize(x) = \mathsf{size}\, \alpha(x)$ is given by

$$\sigma_1 \cdot k_2 \cdot \ldots \cdot k_l + \sigma_2 \cdot k_1 \cdot k_3 \cdot \ldots \cdot k_l + \ldots + \sigma_l \cdot k_1 \cdot \ldots \cdot k_{l-1} \leq l \cdot \sigma_1 \cdot \ldots \cdot \sigma_l$$

Therefore, we need to maximize $l \cdot \sigma_1 \cdot \ldots \cdot \sigma_l$ under constraint $\sigma_1 + \ldots + \sigma_l = n$. A standard argument shows that such a maximum is bounded above by

$$\begin{cases} 1 & \text{if } n = 1 \\ \frac{n}{2}\sqrt{2}^{\,n} & \text{if } 1 < n < 21 \\ \frac{n}{3}\sqrt[3]{3}^{\,n} & \text{if } n \geq 21 \end{cases}$$

If it easy to see that for $n > 1$, the upper bounds given above are less than $\frac{n}{2}\sqrt[3]{3}^{\,n}$. If $n = 1$, then the size of the normal form is also 1.

Case 4. $x = \langle X_1, \ldots, X_l \rangle$ where X_i's are or-sets of a type that does not contain or-sets. Then $normalize(x) = or\_\mu(x)$. Since the or_μ does not change size, $\mathsf{size}\, normalize(x) < \frac{n}{2}\sqrt[3]{3}^{\,n}$ for all $n \geq 2$. If $n = 1$, then $\mathsf{size}\, normalize(x) = 1$.

To complete the inductive proof, we show that after each step of normalization that produces a normalized subobject x'', that is, $x'' = normalize(x')$ for a subobject x' of x, either $\mathsf{size}\, x'' \leq \frac{n}{2}\sqrt[3]{3}^{\,n}$ is satisfied if $n = \mathsf{size}\, x' > 1$, or $\mathsf{size}\, x'' = 1$ if $n = 1$. This will complete the proof. Two cases corresponding to application of or_ρ_1 or or_ρ_2 are similar to the case of α, so we show here only the case of application of α.

Let $x = \{x_1, \ldots, x_k\}$ where each x_i is an unnormalized object. Let $x'_i = \mathit{normalize}(x_i)$ and k_i be the cardinality of x'_i, i.e. $k_i = m(x_i)$. Let $n_i = \mathbf{size}\, x_i$. By theorem 4, $k_i \leq \sqrt[3]{3}^{n_i}$. First consider the case when all $n_i > 1$.

Let $x'_i = \langle y^i_1, \ldots, y^i_{k_i} \rangle$, $i = 1, \ldots, k$. By s^i_j we denote $\mathbf{size}\, y^i_j$. By induction hypothesis,

$$\forall i = 1, \ldots, k: \quad \sum_{j=1}^{k_i} s^i_j \leq \frac{n_i}{2} \sqrt[3]{3}^{n_i}$$

normalize(x) is obtained by applying α to $\{x'_1, \ldots, x'_k\}$, i.e. its elements are sets of representatives of $x'_1, \ldots, x'_k$. Since we are interested in an upper bound, we may assume that all the elements of $x'_1, \ldots, x'_k$ are distinct. Then each element of x'_i will be present in $k^{(i)} = (\prod_{j=1}^{k} k_j)/k_i$ sets. Therefore, the upper bound for $\mathbf{size}\, \mathit{normalize}(x)$ can be calculated as the sum of the sizes of all elements of $x'_1, \ldots, x'_k$ multiplied by the number of their occurrences in the normalized object, i.e.

$$\begin{aligned} \mathbf{size}\, \mathit{normalize}(x) &\leq \sum_{i=1}^{k} \sum_{j=1}^{k_i} k^{(i)} s^i_j = \sum_{i=1}^{k} k^{(i)} \sum_{j=1}^{k_i} s^i_j \leq \\ \sum_{i=1}^{k} \frac{n_i}{2} k^{(i)} \sqrt[3]{3}^{n_i} &\leq \sqrt[3]{3}^{n_1 + \ldots + n_i} \sum_{i=1}^{k} \frac{n_i}{2} = \frac{n}{2} \sqrt[3]{3}^{n} \end{aligned}$$

If all $n_i = 1$, then $\mathbf{size}\, \mathit{normalize}(x) = k = n$. If $n > 1$, then $n \leq \frac{n}{2} \sqrt[3]{3}^{n}$ and if $n = 1$, that is, $\mathbf{size}\, x = 1$, then $\mathbf{size}\, \mathit{normalize}(x) = 1$.

Now consider the general case, i.e. $n_1, \ldots, n_p > 1$ and $n_{p+1}, \ldots, n_k = 1$. Normalization of x_i for $i > p$ results in a size one object. Let $\sigma_0 = n_1 + \ldots + n_p$ and $\sigma_1 = k - p$. Clearly $\sigma_0 + \sigma_1 = n$. Had we applied α only to $\{x'_1, \ldots, x'_p\}$, it would have resulted in an object whose size is bounded above by $\frac{\sigma_0}{2} \sqrt[3]{3}^{\sigma_0}$ according to the calculations for the case where all $n_i > 1$. But taking into account σ_1 size one objects adds size σ_1 to every element of the or-set *normalize*(x). Since there are at most $\sqrt[3]{3}^{\sigma_0}$ such sets, we obtain

$$\mathbf{size}\, \mathit{normalize}(x) \leq \frac{\sigma_0}{2} \sqrt[3]{3}^{\sigma_0} + \sigma_1 \sqrt[3]{3}^{\sigma_0}$$

Since $\sigma_0 > 1$, $\sigma_0 + 2\sigma_1 \leq (\sigma_0 + \sigma_1) \sqrt[3]{3}^{\sigma_1}$ which shows

$$\mathbf{size}\, \mathit{normalize}(x) \leq \frac{\sigma_0}{2} \sqrt[3]{3}^{\sigma_0} + \sigma_1 \sqrt[3]{3}^{\sigma_0} \leq \frac{n}{2} \sqrt[3]{3}^{n}$$

Finally, if or_μ is applied in the process of normalization, it does not change size. Assume $x = \langle x_1, \ldots, x_k \rangle$ where each x_i is an unnormalized object. Let $x'_i = \mathit{normalize}(x_i)$ and $n_i = \mathbf{size}\, x_i$. Assume $n_1, \ldots, n_p > 1$ and $n_{p+1} = \ldots = n_k = 1$. Define σ_0 and σ_1 as in the case of applying α. Then, by induction hypothesis,

$$\mathbf{size}\, \mathit{normalize}(x) \leq \sum_{i=1}^{p} \frac{n_i}{2} \sqrt[3]{3}^{n_i} + \sigma_1 \leq \frac{\sigma_0}{2} \sqrt[3]{3}^{\sigma_0} + \sigma_1 \leq \frac{n}{2} \sqrt[3]{3}^{n}$$

If all $n_i = 1$, then two cases arise. If $n > 1$, then $\mathsf{size}\, normalize(x) = n \leq \frac{n}{2}\sqrt[3]{3}^n$, and if $n = 1$, then $\mathsf{size}\, normalize(x) = n = 1$.

Theorem is proved. □

Corollary 1 *Let $x = normalize(y)$ and $\mathsf{size}\, x = n$. Then*

$$O(\log n) \leq \mathsf{size}\, y \leq n$$

The upper bound of theorem 5 is not sharp. The following result exhibits a sharp upper bound for a large class of objects. This shows that the previous theorem can not be significantly improved.

Theorem 6 *Let x be an object with $\mathsf{size}\, x = n$ containing or-sets. Assume that every subobject of type $\{\langle t' \rangle\}$ has size at least 21, every subobject of type $t' \times \langle t'' \rangle$ or $\langle t'' \rangle \times t'$ has size at least 6 and every subobject of type $\langle\langle t' \rangle\rangle$ has size at least 3, where t' and t'' do not use the or-set type constructor. Then*

$$\mathsf{size}\, normalize(x) \leq \frac{n}{3}\sqrt[3]{3}^n$$

Moreover, for any n divisible by 3 there exists an object x such that $\mathsf{size}\, x = n$ and $\mathsf{size}\, normalize(x) = \frac{n}{3}\sqrt[3]{3}^n$.

Proof. We have to rework the base cases only. Since no subobject involving or-sets can have size one, the induction step easily goes through, cf. the proof of theorem 5.

The case of applying α was already proved, see proof of theorem 5. For the case of applying or_ρ_1 or or_ρ_2, we established an upper bound $2n - 2$. It is easily seen that $2n - 2 \leq \frac{n}{3}\sqrt[3]{3}^n$ for $n \geq 6$. Finally, applying or_μ does not affect size, and $n \leq \frac{n}{3}\sqrt[3]{3}^n$ for $n \geq 3$.

To show sharpness, consider example from the proof of theorem 5. Let

$$x = \{\langle b_1, b_2, b_3 \rangle, \langle b_4, b_5, b_6 \rangle, \ldots, \langle b_{n-2}, b_{n-1}, b_n \rangle\}$$

where all b_i's are distinct elements of a base type. Then $\alpha(x)$ contains $\sqrt[3]{3}^n$ elements, each having cardinality $\frac{n}{3}$. Thus, $\mathsf{size}\, normalize(x) = \frac{n}{3}\sqrt[3]{3}^n$. □

The importance of existential queries was emphasized in [15, 16]. Essentially, an existential query asks whether there exists a possibility – in the normal form – satisfying a given property. In terms of or-$\mathcal{NRA}^+$, if $nf(s) = \langle t \rangle$ and $p : t \to bool$ is a predicate, $\exists(p) : \langle t \rangle \to bool$ is a predicate which is true of $y : \langle t \rangle$ if $or\_map(p)(y) : \langle bool \rangle$ is an or-set containing the true value. Given an object y of type s, one may ask a query $\exists(p)(normalize(y))$. Clearly, this query can be answered in time polynomial in the size of $normalize(y)$, but can it be answered in time polynomial in the size of y?

The following example gives a negative answer to this question, provided $\mathcal{P} \neq \mathcal{NP}$. Assume $p_k : \{t\} \to bool$ evaluates to *true* if and only if cardinality of the set is at most k. Let b a base type. For an object

x of type $\{\langle b\rangle\}$, one may ask a query $Q(k,x) = \exists(p_k)(normalize(x))$. It is immediately seen that this query evaluates to *true* iff there exists a system of distinct representatives of elements of x (which are or-sets) whose size is at most k. The problem of finding a system of distinct representatives of size $\leq k$ is known to be $\mathcal{NP}$-complete, see [9]. Therefore, the problem whether $Q(k,x)$ evaluates to *true* is $\mathcal{NP}$-complete.

7 Future Work

There are many further problems which we would like to investigate. The languages we have proposed give rise to interesting equational theories which can lead to useful optimizations. For instance, $or\_map(map\ f) \circ \alpha = \alpha \circ map(or\_map\ f)$ and $or\_\mu \circ or\_map(\alpha) \circ \alpha = \alpha \circ map(or\_\mu)$.

There is also an appealing possibility of using or-sets in merging databases. For example, we can use a merge operator for combining databases as follows: $merge(\{(id_1,a),(id_2,b)\},\{(id_1,a'),(id_2,b')\}) = \{(id_1,\langle a,a'\rangle),(id_2,\langle b,b'\rangle)\}$.

There are two further questions concerning losslessness. First, given a query at the conceptual level and an unnormalized object, can we discover an equivalent structural query which does not force normalization on the object. Second, the condition that $K\langle\rangle$ does not appear in f in our losslessness theorem can be removed in some situations. But we do not yet have a full characterization of those situations.

There are various sophisticated order theoretic models of partial information in databases – sandwiches [6], mixes [10], snacks [17, 23]. They enjoy universality properties and therefore can be incorporated into the programming language syntax. We plan to investigate the applicability of such models to the study of or-sets.

Our languages have been extended to include variant types. It is known that the coherence result still holds in the extended languages. The validity of the remaining results of this report remains to be checked for this extension.

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