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Mohamed Babaali
University of Pennsylvania

George J. Pappas
University of Pennsylvania, pappasg@seas.upenn.edu

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Observability of Switched Linear Systems in Continuous Time

Mohamed Babaali and George J. Pappas

Electrical and Systems Engineering
University of Pennsylvania,
Philadelphia, PA USA
{babaali,pappasg}@grasp.cis.upenn.edu

Abstract. We study continuous-time switched linear systems with unobserved and exogenous mode signals. We analyze the observability of the initial state and initial mode under arbitrary switching, and characterize both properties in both the autonomous and non-autonomous cases.

1 Introduction

The general model being considered here is[1]

$$\begin{aligned} \dot{x}_t &= A(r_t)x_t + B(r_t)u_t \\ y_t &= C(r_t)x_t + D(r_t)u_t \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^p$, and where $A(\cdot)$, $B(\cdot)$ and $C(\cdot)$ are real matrices of compatible dimensions. The input signals $u : [0, \infty) \rightarrow \mathbb{R}^m$ are assumed to be analytic. The exogenous, yet unobserved, mode (or switching) signal

$$r : [0, \infty) \rightarrow Q \triangleq \{1, \dots, s\} \tag{2}$$

is furthermore assumed to be right-continuous, so that all trajectories of vector-valued variables are well defined and infinitely right-differentiable over $[0, \infty)$. Note that, even though we impose no minimum separation between consecutive switches (jumps of r), Zeno behaviors cannot occur since r is exogenous, thus always well defined over $[0, \infty)$.

While observability is well understood in classical linear system theory [14], it becomes more complex in the switched case. One reason is that the switching gives rise to a richer set of problems. First, the discrete modes may or may not be observed, giving rise to two sets of problems. Second, in the latter case, since one may also want to recover the modes, a distinction must be made

[1] For notational convenience, we have chosen to subscript time: We will denote the value of some signal x at time t by x_t instead of the standard $x(t)$, while x and $x_{[t,t']}$ will denote the whole signal and its restriction to $[t, t']$, respectively.

between recovering the modes and recovering the states. Moreover, one can no longer decouple observation from control, which makes for the need to distinguish between the autonomous and non-autonomous cases, creating the problem of existence of controls allowing observation. Finally, two sets of problems arise from the fact that one may want the observability properties to hold for either all possible mode signals (i.e. *universal problems*) or for some mode signal (i.e. *existential problems*), in which case a characterization of the class of signals may be desired. In this paper, we assume that the mode signals are unobserved (i.e. unknown), and study the mode and state observability properties under arbitrary switching.

Observability of hybrid systems has recently been the center of a great deal of attention. However, most of the resulting literature is not related to the problems under consideration here. For instance, while the work in [6, 11, 12, 16, 23] was carried out in a stochastic setting, the papers [3, 5, 9, 18, 13] studied observability of hybrid linear systems, where the modes depend on the state trajectory, and deterministic discrete-time switched linear systems were considered in [1, 21]. However, in contrast to classical linear systems, there are differences between the discrete and continuous time cases in switched linear systems, which require them to be studied independently. For example, in continuous-time, taking successive time derivatives of the output allows the current mode to fully express itself in infinitesimal time, i.e. provide all the information it can provide about the current state. It is thus possible to decouple the modes in the known modes case, as we will see later in this paper. However, arbitrary switching removes such a luxury in discrete-time (see, e.g., [1]).

Returning to continuous-time switched linear systems, we first report the results for observed switching. First, observability under arbitrary switching has long been known to be equivalent to standard observability of every pair $(A(q), C(q))$ (see, e.g., [8]). However, the existence of a mode signal making the initial state observable, which has proven to be a challenging problem, has only recently been characterized, and shown to be decidable, in [10, 19]. It was shown to be equivalent to $\mathcal{V} = \mathbb{R}^n$, $\mathcal{V}$ being the minimal subspace of $\mathbb{R}^n$ invariant with respect to each $A(q)^T$, $q \in Q$, and containing $\sum_{q \in Q} \operatorname{Im} C(q)^T$. Furthermore, a constructive procedure for designing the mode signal r was given in [19], along with an upper bound on the minimum number of switches necessary to achieve observability.

It appears that the unobserved switching case has only been analyzed in [2, 7, 22]. In [22], the problem of recovering, simultaneously, the initial mode and state was considered along with the switch detection problem, but for autonomous systems. In [2], sufficient conditions were given for *generic final state determinability*, which we do not consider here. Finally, in [7], notions of observability and detectability were proposed in the framework of *linear switching systems*, of which our model is a special case. The authors considered the problem of recovering both the initial state and initial mode for some input, again simultaneously, and the problem of detecting the switches, generalizing the results of [22] to the non-autonomous case.

In this paper, we give linear-algebraic characterizations of mode observability and state observability under arbitrary and unobserved switching. The fact that we analyze them separately not only provides criteria for simultaneous state/mode observability (since such a property is characterized by the intersection of both sets of criteria), but provides some additional insight into the specific problems. In particular, by showing that mode and state observability are not necessary for each other, we relax the conditions previously given in the literature.

The outline of this paper is as follows. In Section 2, we establish some notation in order to simplify the subsequent exposition. In Section 3, we study the initial mode and initial state observability problems for autonomous systems. The same treatment is then repeated in the non-autonomous case in Section 4.

2 Notation

Letting w denote a trajectory (or execution) of some system comprising all signals of interest, including inputs, outputs and states, we decompose w into three collections of signals or portions of signals over time segments as $w = (w_d, w_o, w_r)$, and we say a system $\Sigma = \{w_i\}_{i \in I}$ is

$$(w_d/w_o) - \text{observable} \tag{3}$$

if w_d, the "desired" set of quantities, can be uniquely recovered when w_o is "observed", while w_r, i.e. the 'rest", is neither observed nor desired. In other words, it means that

$$\forall w, w' \in \Sigma,\ (w_o = w'_o \Rightarrow w_d = w'_d). \tag{4}$$

By default, the domains of all variables are the full spaces of definition, which is often too restrictive since one may find systems that are not (w_d/w_o)-observable, and yet exhibit trajectories for which w_d can be observed from w_o. Of course the "golden" solution to the observation problem is to actually determine all such trajectories, i.e., find $\Sigma_0 \triangleq \{w \in \Sigma \mid \forall w' \in \Sigma,\ w_o = w'_o \Rightarrow w_d = w'_d\}$, the "observable" subset of trajectories. However, we will take a different approach in this paper, and will instead isolate some components of interest (typically inputs, known or unknown) and either restrict them *a priori* or ask whether the system is observable for some value or for generic values of those components.

We thus define $(w_d \in W_d/w_o \in W_o/w_r \in W_r)$-observability as

$$\forall w, w' \in \Sigma,\ (w_d \in W_d,\ w_o \in W_o,\ w_r \in W_r,\ w_o = w'_o \Rightarrow w_d = w'_d). \tag{5}$$

Note that w' in (5) ranges over Σ: Indeed, for any execution w to determine w_d, one needs to rule out $w'_d \neq w_d \wedge w'_0 = w_o$ for all $w' \in \Sigma$. In particular, restricting, say w_r, to $\{0\}$ will be denoted $\underline{w_r}$ instead of $w_r \in \{0\}$. Moreover, since, any two restricting sets being fixed (say $\overline{W_o}$ and W_d), one can compute the largest possible third one (i.e., W_r) such that the system remains $(w_d \in W_d/w_o \in W_o/w_r \in W_r)$-observable, we will set to compute it, and we will then say the system is

$$(w_d \in W_d/w_o \in W_o/w_r^*) - \text{observable or} \tag{6}$$

$$(w_d \in W_d/w_o \in W_o/\overline{w_r}) - \text{observable} \tag{7}$$

according as W_r is nonempty or generic (when w_r lies in a vector space). Informally, (6) reads "is w_d observable from w_o for *some* w_r?", while (7) reads "is w_d observable from w_o for generic w_r?" Finally, extending the previous conventions to the case where the three components of w themselves have components, we can summarize what has been studied in the following table.

Property	**Paper**
$(r_0, x_0 \neq 0/y, \underline{u})$-observability	[22]
$(r_0, x_0/y, u^*)$-observability	[7]

Table 1. Observability Concepts

Finally, we establish the following notational conventions to ease the discussion. First, let $y(r, x_0, u)$ be the output signal y of (1) when the initial state is x_0, the input signal is u and the mode signal is r. For any vector-valued signal z, let $z^{(N)}$ denote its N^{th} right-derivative with respect to time, and let

$$z^{[N]} \triangleq \begin{pmatrix} z \\ z' \\ \vdots \\ z^{(N-1)} \end{pmatrix}. \tag{8}$$

Now, let the N^{th}*-order observability matrix* of a mode $q \in Q$ be

$$\mathcal{O}_N(q) \triangleq \begin{pmatrix} C(q) \\ \vdots \\ C(q)A(q)^{N-1} \end{pmatrix}, \tag{9}$$

the N^{th}*-order Hankel (or behavior) matrix* of a mode q be

$$\mathcal{H}_N(q) \triangleq \begin{pmatrix} D(q) & \cdots & 0 & 0 \\ C(q)B(q) & \cdots & 0 & 0 \\ C(q)A(q)B(q) & \cdots & \vdots & 0 \\ \vdots & \cdots & D(q) & \vdots \\ C(q)A(q)^{N-1}B(q) & \cdots & C(q)B(q) & D(q) \end{pmatrix},$$

and define the following mapping as

$$Y_N(q, x, U) \triangleq \mathcal{O}_N(q)x + \mathcal{H}_N(q)U, \tag{10}$$

where $U \in \mathbb{R}^{mN}$, so that

$$y_t^{[N]}(r, x_0, u) = Y_N(r_t, x_t, u_t^{[N]}). \qquad (11)$$

In words, $Y_N(q, x, U)$ is the stack of the first N derivatives of the output y_t when $r_t = q$, $x_t = x$, and $u_t^{[N]} = U$.

For further reference, we define the following coupled system parameters

$$A(q,q') \triangleq \begin{pmatrix} A(q) & 0 \\ 0 & A(q') \end{pmatrix} \qquad B(q,q') \triangleq \begin{pmatrix} B(q) \\ -B(q') \end{pmatrix}$$
$$C(q,q') \triangleq (C(q)\ C(q')) \qquad D(q,q') \triangleq D(q) - D(q'),$$

and we note that the N^{th}-order Kalman observability matrix of the pair $(A(q,q'), C(q,q'))$ is $(\mathcal{O}_N(q)\ \mathcal{O}_N(q'))$ and that the Hankel matrix of the tuple $(A(q,q'), B(q,q'), C(q,q'), D(q,q'))$ is simply $\mathcal{H}_N(q) - \mathcal{H}_N(q')$.

Finally, we let $\rho(M)$, $\Re(M)$ and $M^{\{1\}}$ denote the rank, the column range space, and a (generalized) $\{1\}$ -inverse of any real matrix M (see [4]). A matrix N is a $\{1\}$-inverse of M if $MNM = M$. The pseudo-inverse is thus always a $\{1\}$-inverse, and whenever M is of full column rank, any $\{1\}$-inverse N of M is also a left inverse of M in the sense that $M^{\{1\}}M$ equals the identity matrix. Moreover, x is a solution to the equation $Y = Mx$ if and only if $x = M^{\{1\}}Y$ for some $\{1\}$-inverse $M^{\{1\}}$ of M. Given a subspace V of $\mathbb{R}^n$, we let P_V denote the matrix of the orthogonal projection on V.

3 Autonomous Systems

In this section we assume that $u = 0$, hence the autonomous case. We start with the important observation that the SLS (1) cannot be $(r_0/y, \underline{u})$-observable. Indeed, if $x_0 = 0$, then $y = 0$ identically for all r, and so the measurements give no information about r_0. We therefore need to lower our expectation on the observability of the initial mode, and relax the previous requirements. We thus consider observability of the initial mode for *generic* initial states, and define *discernibility* as follows.

Definition 1. *The mode q is* discernible *from another mode q' over $T > 0$ if whenever $r_{[0,T]} \equiv q$ and $r'_{[0,T]} \equiv q'$, the set*

$$\{x_0 \in \mathbb{R}^n \mid \forall x_0' \in \mathbb{R}^n,\ y_{[0,T]}(r, x_0, 0) \neq y_{[0,T]}(r', x_0', 0)\}. \qquad (12)$$

is generic in $\mathbb{R}^n$.

In words, q is discernible from q' if, for generic initial states x_0, one can rule out q' when observing $y(r, x_0, 0)$ over $[0, T]$. Before giving a characterization of discernibility, let us establish the following straightforward lemma:

Lemma 1 *Let M and M' be two real $N \times n$ matrices, and define $V \triangleq \Re(M) \cap \Re(M')$. Then*

$$\dim M^{-1}(V) = n - \rho((M\ M')) + \rho(M'), \qquad (13)$$

where M^{-1} denotes the set-valued inverse of M.

Proof. Minkowski's equality gives $\dim(V) = \rho(M) + \rho(M') - \rho((M\ M'))$, the Rank Plus Nullity Theorem $\dim(M^{-1}(V)) = \dim(V) + \dim\ker(M)$ and $n = \rho(M) + \dim\ker(M)$, and the lemma follows. □

We have:

Lemma 2 *A mode q is discernible from q' over T if and only if*

$$\rho((\mathcal{O}_{2n}(q)\ \mathcal{O}_{2n}(q'))) > \rho(\mathcal{O}_{2n}(q')). \tag{14}$$

Proof. Fix $T > 0$. We need to show that

$$\left\{x_0 \in \mathbb{R}^n \mid \forall x_0' \in \mathbb{R}^n,\ y_{[0,T]}(r, x_0, 0) = y_{[0,T]}(r', x_0', 0)\right\} \tag{15}$$

is a generic set if and only if q is discernible from q'. Recalling that $(\mathcal{O}_{2n}(q)\ \mathcal{O}_{2n}(q'))$ is the Kalman observability matrix of the pair $(A(q,q'), C(q,q'))$ and that $y_{[0,T]}(r, x_0, 0) - y_{[0,T]}(r', x_0', 0)$ is its output in free evolution with initial state $\begin{pmatrix} x_0 \\ -x_0' \end{pmatrix}$, we have

$$y_{[0,T]}(r, x_0, 0) = y_{[0,T]}(r', x_0', 0) \Leftrightarrow (\mathcal{O}_{2n}(q)\ \mathcal{O}_{2n}(q')) \begin{pmatrix} x_0 \\ -x_0' \end{pmatrix} = 0 \tag{16}$$

since $\ker((\mathcal{O}_{2n}(q)\ \mathcal{O}_{2n}(q')))$ is $A(q,q')$-invariant. We can therefore shift our attention to showing that the complement of

$$v(q,q') \triangleq \left\{x_0 \in \mathbb{R}^n \mid \exists x_0' \in \mathbb{R}^n, (\mathcal{O}_{2n}(q)\ \mathcal{O}_{2n}(q')) \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = 0\right\} \tag{17}$$

in $\mathbb{R}^n$ is generic if and only if q is discernible from q'. Defining $V(q,q') \triangleq \Re(\mathcal{O}_{2n}(q)) \cap \Re(\mathcal{O}_{2n}(q'))$, noting that $v(q,q') = \mathcal{O}_{2n}(q)^{-1}(V(q,q'))$, and then using Lemma 1, we get

$$\dim v(q,q') = n - \rho((\mathcal{O}_{2n}(q)\ \mathcal{O}_{2n}(q'))) + \rho(\mathcal{O}_{2n}(q')). \tag{18}$$

Therefore, by definition of discernibility, we see that $\dim(v(q,q')) < n$, thus that its complement is generic, if and only if q is discernible from q', which completes the proof. □

A consequence of this result is that discernibility is independent of the duration of observation. Therefore, we will from now on simply say "q is discernible from q'", thus omitting the dependence on T.

Theorem 1 *The SLS (1) is $(r_0/y, \underline{u}/\overline{x_0})$-observable if and only if every pair of different modes is mutually discernible.*

Proof. $(r_0/y, \underline{u}/\overline{x_0})$-observability means that the set

$$P \triangleq \{x_0 \in \mathbb{R}^n \mid \forall r, r',\ \forall x_0',\ r_0' \neq r_0 \Rightarrow y(r, x_0, 0) \neq y(r', x_0', 0)\}. \tag{19}$$

is generic in $\mathbb{R}^n$. Now, letting

$$Q(q,q') \triangleq \{x_0 \in \mathbb{R}^n \mid \exists r, r',\ r_0 = q,\ r'_0 = q',\ \exists x'_0,\ y(r, x_0, 0) \neq y(r', x'_0, 0)\},$$

we get

$$P = \mathbb{R}^n \setminus \cup_{q \neq q'} Q(q,q'). \tag{20}$$

Now, by right-continuity of the mode signals, for every pair r, r', there exists $0 < T \leq \infty$ such that $r_{[0,T]} \equiv q$, $r'_{[0,T]} \equiv q'$, and so $v(q,q') \subset Q(q,q')$ (see Lemma 2). On the other hand, $Q(q,q') \subset v(q,q')$ follows by considering $r \equiv q$ and $r' \equiv q'$. Consequently,

$$P = \mathbb{R}^n \setminus \cup_{q \neq q'} v(q,q'), \tag{21}$$

and is generic if and only if each $v(q,q')$ is a proper subspace of $\mathbb{R}^n$, and thus if and only if every pair of modes is mutually discernible. □

Example 1. Consider (1), where $s = 2$, $B(1) = B(2) = 0$, $D(1) = D(2) = 0$, and where

$$\begin{array}{ll} A(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & A(2) = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \\ C(1) = \begin{pmatrix} 1 & 0 \end{pmatrix} & C(2) = \begin{pmatrix} 1 & 0 \end{pmatrix}. \end{array} \tag{22}$$

Then

$$(\mathcal{O}_4(1)\ \mathcal{O}_4(2)) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 8 \\ 1 & 3 & 1 & 26 \end{pmatrix}, \tag{23}$$

and has rank 3, while $\rho(\mathcal{O}_4(1)) = \rho(\mathcal{O}_4(2)) = 2$. Therefore, it is possible to recover the initial mode for generic initial states. For instance,

$$y_0^{[4]}(r_0, x, o) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \tag{24}$$

could only have been produced by $r_0 = 1$ (with $x_0 = (1,1)$). It is actually possible to recover r_0 uniquely whenever the second entry of x_0 is not zero, which constitutes a generic subset of $\mathbb{R}^2$.

We now turn to the study of the ability to recover the initial state x_0 of the system, based only on the output signal y. A first route for that is, first, to recover the initial mode r_0, and, then, to invert the Gramian to get x_0. Noting that this can only be done for generic x_0, we state the following corollary to Proposition 1.

Corollary 1 *The SLS (1) is $(\overline{x_0}/y, \underline{u})$-observable if every mode is observable and every pair of modes is mutually discernible.*

Even though this route may seem to be the natural way to proceed, we will now show that it is neither necessary nor sufficient for $(x_0/y, \underline{u})$-observability, which is in fact possible. In other words, it is possible to determine the initial state from the output *globally*, for *all* mode signals, and *without* necessarily recovering the modes. To this end, we define *joint observability* as follows:

Definition 2. *Two different modes q and q' are jointly observable over $T > 0$ if whenever $r_{[0,T]} \equiv q$ and $r'_{[0,T]} \equiv q'$,*

$$\forall x_0,\ \forall x_0',\ x_0 \neq x_0' \Rightarrow y_{[0,T]}(r, x_0, 0) \neq y_{[0,T]}(r', x_0', 0). \tag{25}$$

Note that, in contrast to discernibility, joint observability is symmetric. That two modes are jointly observable means that one can recover the initial state from the output without knowledge of the initial mode. We have:

Lemma 3 *q and q' are jointly observable over T if and only if they are both observable (i.e., $\rho(\mathcal{O}_n(q)) = \rho(\mathcal{O}_n(q')) = n$) and the left inverses of their $2n^{th}$-order observability matrices agree on $V(q, q')$, i.e.*

$$(\mathcal{O}_{2n}(q)^{\{1\}} - \mathcal{O}_{2n}(q')^{\{1\}})P_{V(q,q')} = 0. \tag{26}$$

Proof. Assume that q and q' are both observable and satisfy (26), and suppose that $y_{[0,T]}(r, x_0, 0) = y_{[0,T]}(r', x_0', 0)$ (with $T > 0$ and $r_{[0,T]} \equiv q$ and $r'_{[0,T]} \equiv q'$). Then, recalling (16), we get

$$\mathcal{O}_{2n}(q)x_0 = \mathcal{O}_{2n}(q')x_0'. \tag{27}$$

Furthermore, q and q' being observable, (26) implies that $v(q, q') = v(q', q)$ and that $(\mathcal{O}_{2n}(q) - \mathcal{O}_{2n}(q'))P_{v(q,q')} = 0$, which, in turn, implies that

$$\mathcal{O}_{2n}(q)x_0 = \mathcal{O}_{2n}(q')x_0, \tag{28}$$

since $x_0 \in v(q, q')$. Combining (27) and (28), we get

$$\mathcal{O}_{2n}(q')(x_0 - x_0') = 0, \tag{29}$$

hence that $x_0 = x_0'$ since q' is observable.

Conversely, assume that, say q, is not observable. Then taking $x_0 \in \ker(\mathcal{O}_n(q)) \setminus \{0\}$, we get $y_{[0,T]}(r, x_0, 0) = y_{[0,T]}(r', 0, 0) = 0$ while $x_0 \neq 0$, hence that q and q' are not jointly observable. Finally, assuming q and q' are both observable but that (26) does not hold, we have the existence of $Y \in V(q, q')$ such that $(\mathcal{O}_{2n}(q)^{\{1\}} - \mathcal{O}_{2n}(q')^{\{1\}})Y \neq 0$. Letting $x_0 = \mathcal{O}_{2n}(q)^{\{1\}}Y$ and $x_0' = \mathcal{O}_{2n}(q')^{\{1\}}Y$, we have $x_0 \neq x_0'$ but $\mathcal{O}_{2n}(q)x_0 = \mathcal{O}_{2n}(q')x_0' = Y$, and thus $y_{[0,T]}(r, x_0, 0) = y_{[0,T]}(r', x_0', 0)$ and q and q' are not jointly observable. □

What Lemma 3 has established is that joint observability is independent of the observation horizon T. Once again, we will from now on omit T when saying two modes are jointly observable. A characterization of $(x_0/y, \underline{u})$-observability follows.

Theorem 2 *The SLS (1) is $(x_0/y, \underline{u})$-observable if and only if every mode is observable and any two different modes are jointly observable.*

Proof. $(x_0/y, \underline{u})$-observability means that

$$\forall r,\ \forall r',\ \forall x_0,\ \forall x_0',\ x_0' \neq x_0 \Rightarrow y(r, x_0, 0) \neq y(r', x_0', 0). \tag{30}$$

Assume that every mode is observable, that any pair is jointly observable, and that $y(r, x_0, 0) = y(r', x_0', 0)$. First, by right-continuity of both mode signals, there exist $0 < T \leq \infty$ and two modes q, q' such that $r_{[0,T]} \equiv q$, $r'_{[0,T]} \equiv q'$. Then $x_0 = x_0'$ is implied by observability of each mode or joint observability of each pair of modes according as $q = q'$ or $q \neq q'$, by definition.

Conversely, assume that, say q, is not observable. Then letting $r = r' \equiv q$, and choosing $x_0 \in \ker(\mathcal{O}_n(q)) \setminus \{0\}$, we have $y(r, x_0, 0) \neq y(r', 0, 0)$ even though $x_0 \neq 0$. On the other hand, assuming the existence of a jointly unobservable pair q, q', letting $r \equiv q$ and $r' \equiv q'$, there must exist $x_0 \neq x_0'$ such that $y(r, x_0, 0) \neq y(r', x_0', 0)$, by definition of joint observability. □

Remark 1. In [22], it was established that $(r_0, x_0 \neq 0/y, \underline{u})$-observability was equivalent to the so-called "rank-$2n$" condition

$$\forall q, q',\ q \neq q',\ \rho((\mathcal{O}_{2n}(q)\mathcal{O}_{2n}(q'))) = 2n. \tag{31}$$

Since $\rho([\mathcal{O}_{2n}(j)]) \leq n$ for both $j = q$ and $j = q'$, (31) is sufficient for mutual discernibility of q and q', and therefore for $(r_0/y, \underline{u}/\overline{x_0})$-observability. In fact, by (18), it is equivalent to

$$\forall q, q',\ q \neq q',\ v(q, q') = \{0\}, \tag{32}$$

which is the least-dimensional possible subspace of conflict, and hence to $(r_0/y, \underline{u}/x_0 \neq 0)$-observability. What we have thus shown is that it is possible to recover r_0 even if $v(q, q') \neq \{0\}$, and we have relaxed (31) to account for such cases and fully characterize the observability of the initial mode in the autonomous case.

As for state observability, it turns out that (31) is not necessary for $(x_0/y, \underline{u})$-observability, simply because it is not necessary to recover the initial mode in order to infer the initial state when the initial state is not trivial. For instance, the system in Example 1 is $(x_0/y, \underline{u})$-observable, but does not satisfy (31). Recall that

$$v(1, 2) = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}. \tag{33}$$

If $x_0 \notin v(1,2)$, then one can uniquely infer r_0 and recover x_0, since every mode is observable. However, if $x_0 \in v(1,2)$, then

$$y_0^{[4]}(r, x_0, 0) = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix} \Rightarrow x_0 = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \tag{34}$$

for all r, hence the claim.

4 Non-Autonomous Systems

We now turn to the non-autonomous case, and study both existence and generic problems in u. We will show that existence and generic properties will be equivalent for the initial mode observability properties, and that the genericity requirement on x_0 can actually be waived. We will need the following definition and lemma.

Definition 3. *Two different modes q and q' are controlled-discernible over $T > 0$ if whenever $r_{[0,T]} \equiv q$ and $r'_{[0,T]} \equiv q'$, there exists an input u such that*

$$\forall x_0,\ \forall x_0',\ y_{[0,T]}(r, x_0, u) \neq y_{[0,T]}(r', x_0', u). \tag{35}$$

In other words, q and q' are controlled-discernible if there exists a control making it possible to distinguish them by their outputs.

Lemma 4 *The two modes q and q' are controlled-discernible if and only if there exists a positive integer N such that*

$$(I - P_N(q, q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big) \neq 0, \tag{36}$$

where $P_N(q, q')$ is the matrix of the orthogonal projection on $\Re(\mathcal{O}_N(q)) \cap \Re(\mathcal{O}_N(q'))$. Moreover, (35) is then satisfied for generic u.

Proof. First, note that since the inputs u are analytic, we have

$$y_{[0,T]}(r, x_0, u) = y_{[0,T]}(r', x_0', u) \tag{37}$$

$$\Leftrightarrow \forall N,\ y_0^{[N]}(r, x_0, u) = y_0^{[N]}(r', x_0', u) \tag{38}$$

$$\Leftrightarrow \forall N,\ Y_N(q, x_0, u_0^{[N]}) = Y_N(q', x_0', u_0^{[N]}). \tag{39}$$

Therefore, q and q' are controlled-discernible if and only if there exists u such that

$$\forall x_0,\ x_0',\ \exists N,\ \mathcal{O}_N(q)x_0 + \mathcal{H}_N(q)u_0^{[N]} \neq \mathcal{O}_N(q')x_0' + \mathcal{H}_N(q')u_0^{[N]}, \tag{40}$$

which is equivalent to

$$\exists N,\ \forall x_0,\ x_0',\ \mathcal{O}_N(q)x_0 + \mathcal{H}_N(q)u_0^{[N]} \neq \mathcal{O}_N(q')x_0' + \mathcal{H}_N(q')u_0^{[N]}, \tag{41}$$

since

$$\Big\{\big(\, x_0 - x_0' \,\big) \;\Big|\; (\mathcal{O}_N(q)\ \mathcal{O}_N(q'))x_0 + (\mathcal{H}_N(q) - \mathcal{H}_N(q'))u_0^{[N]} = 0\Big\} \in \\ \Big\{\big(\, x_0 - x_0' \,\big) \;\Big|\; (\mathcal{O}_{N'}(q)\ \mathcal{O}_{N'}(q'))x_0 + (\mathcal{H}_{N'}(q) - \mathcal{H}_{N'}(q'))u_0^{[N]} = 0\Big\} \quad (42)$$

if $N > N'$. Equation (41) is equivalent to the existence of an integer N and of a vector $U \in \mathbb{R}^{mN}$ such that

$$\Big(\Re(\mathcal{O}_N(q)) + \mathcal{H}_N(q)U\Big) \cap \Big(\Re(\mathcal{O}_N(q')) + \mathcal{H}_N(q')U\Big) = \emptyset, \quad (43)$$

which, by elementary linear algebra, is equivalent to

$$(I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big)U \neq 0 \quad (44)$$

which proves that (35) holds if and only if there exist u and N such that

$$(I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big)u_0^{[N]} \neq 0, \quad (45)$$

which is equivalent to (36). Moreover, any input u such that (45) holds works in (35). The set of such inputs is characterized by

$$u_0^{[N]} \in \mathbb{R}^{mN} \setminus \ker\Big((I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big)u_0^{[N]}\Big), \quad (46)$$

which is generic if and only if (36) holds. □

We can now prove the following theorem.

Theorem 3 *The following are equivalent.*

1. *The SLS (1) is $(r_0/y, u^*)$-observable.*
2. *The SLS (1) is $(r_0/y, \overline{u})$-observable.*
3. *Every pair of different modes is controlled-discernible.*

Proof. $(r_0/y, u^*)$-observability means that

$$\exists u,\ \forall r,\ \forall r',\ \overline{\forall} x_0,\ \forall x_0',\ r_0' \neq r_0 \Rightarrow y(r, x_0, u) \neq y(r', x_0', u). \quad (47)$$

Fix r and r' such that $r_0 \neq r_0'$, and let $r_0 = q$ and $r_0' = q'$. By right-continuity of both signals, there exists $0 < T \leq \infty$ such that $r_{[0,T]} \equiv q$, $r'_{[0,T]} \equiv q'$. Consider

$$\exists u,\ \overline{\forall} x_0,\ \forall x_0',\ y(r, x_0, 0) \neq y(r', x_0', 0). \quad (48)$$

Necessity of controlled-discernibility of q from q' for (48) to hold for all r, r' such that $r_0 \neq r_0'$ follows from Lemma 4, by taking $T = \infty$. If $T < \infty$, then, again by Lemma 4, controlled-discernibility is sufficient. Furthermore, the set of such controls satisfies

$$u_0^{[N]} \in \mathbb{R}^{mN} \setminus \ker\Big((I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big)u_0^{[N]}\Big), \quad (49)$$

and the set of controls satisfying (47) is therefore characterized by

$$u_0^{[N]} \in \mathbb{R}^{mN} \setminus \cup_{q \neq q'} \ker \Big((I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big) u_0^{[N]} \Big), \tag{50}$$

which is nonempty and generic if and only if every pair of different modes is controlled-discernible by Lemma 4, hence the result. □

Remark 2. In [7, Proposition 6], a necessary and sufficient condition for controlled-discernibility was given as the existence of an integer N such that

$$\mathcal{H}_N(q) - \mathcal{H}_N(q') \neq 0, \tag{51}$$

which is equivalent to the existence of an input u, polynomial in time of degree $N-1$, such that

$$y_{[0,T]}(r, 0, u) \neq y_{[0,T]}(r', 0, u) \tag{52}$$

whenever $r_{[0,T]} \equiv q$ and $r'_{[0,T]} \equiv q'$. Even though (52) does not imply (35) (see the example at the end of this remark), it turns out that

$$\exists N: \ (I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big) \neq 0 \Leftrightarrow \\ \exists N': \ \mathcal{H}_{N'}(q) - \mathcal{H}_{N'}(q') \neq 0, \tag{53}$$

and that the smallest such N' may be strictly smaller than the smallest N. Equivalently, the existence of an input u' satisfying (52) is equivalent to the existence of an input u satisfying (35), though the degree of the polynomial u' of smallest degree may be strictly smaller than the degree of any such polynomial u. To see this, note that

$$(I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big) \neq 0 \Leftrightarrow \\ \rho\left((\mathcal{H}_N(q) - \mathcal{H}_N(q') \ \mathcal{O}_N(q) \ \mathcal{O}_N(q'))\right) > \rho\left((\mathcal{O}_N(q) \ \mathcal{O}_N(q'))\right), \tag{54}$$

which clearly proves the implication in (53). On the otherhand, the sufficiency of (51) stems from the fact that if $\mathcal{H}_N(q) - \mathcal{H}_N(q') \neq 0$, then the rank of $\mathcal{H}_N(q) - \mathcal{H}_N(q')$, thus that of $(\mathcal{H}_N(q) - \mathcal{H}_N(q') \ \mathcal{O}_N(q) \ \mathcal{O}_N(q'))$, grows unbounded in N. Therefore, since the rank of $(\mathcal{O}_N(q) \ \mathcal{O}_N(q'))$ is bounded by $2n$,

$$\rho\left((\mathcal{H}_N(q) - \mathcal{H}_N(q') \ \mathcal{O}_N(q) \ \mathcal{O}_N(q'))\right) - \rho\left((\mathcal{O}_N(q) \ \mathcal{O}_N(q'))\right) \tag{55}$$

is unbounded, hence the sufficiency of (51), by (54).

For example, if $C(1) = C(2) = 1$, $A(1) = B(1) = 1$, and $A(2) = B(2) = 2$, then letting $u_t = -1$ for all t, we get

$$y_0^{[4]}(r, 0, u) = \begin{pmatrix} 0 \\ -1 \\ -1 \\ -1 \end{pmatrix} \neq y_0^{[4]}(r', 0, u) = \begin{pmatrix} 0 \\ -2 \\ -4 \\ -8 \end{pmatrix}, \tag{56}$$

hence (52), but if $x_0 = x'_0 = 1$, then

$$y_t(r, x_0, u) = y_t(r', x'_0, u) = 1 \ \forall t \in [0, T]. \tag{57}$$

In fact, it can be verified that the minimum degree of a polynomial u for (35) to hold is 1, as opposed, obviously, to 0 for (52).

Remark 3. A straightforward consequence of the Cayley-Hamilton Theorem is that $\mathcal{H}_N(q) \neq 0$ for some N if and only if $\mathcal{H}_n(q) \neq 0$. Therefore, recalling that $\mathcal{H}_N(q) - \mathcal{H}_N(q')$ is exactly the N^{th}-order Hankel matrix of the tuple $(A(q,q'), B(q,q'), C(q,q'), D(q,q'))$, we get that (53) holds if and only if it also holds for $N = 2n$. Therefore, controlled discernibility is decidable, and is equivalent to

$$\mathcal{H}_{2n}(q) - \mathcal{H}_{2n}(q') \neq 0. \tag{58}$$

Finally, a straightforward consequence of the reversibility of continuous-time switched linear systems is that if one can recover r_0, then one can recover the whole mode signal r, theoretically, by repeating the analysis for r_0 at every time t. One thus gets, as a corollary to the last two theorems:

Corollary 2 *The following are equivalent.*

1. *The SLS (1) is $(r/y, \underline{u}/\overline{x_0})$-observable.*
2. *Every pair of different modes is discernible.*

The following are also equivalent.

1. *The SLS (1) is $(r/y, u^*)$-observable.*
2. *Every pair of different modes is controlled-discernible.*

As for state observability, we have:

Theorem 4 *The following are equivalent.*

1. *The SLS (1) is $(x_0/y, u^*)$-observable.*
2. *The SLS (1) is $(x_0/y, \overline{u})$-observable.*
3. *Every mode is observable and every pair of modes is either controlled-discernible or jointly observable.*

Proof. $(x_0/y, u^*)$-observability means that

$$\exists u,\ \forall r,\ \forall r',\ \forall x_0,\ \forall x'_0,\ x'_0 \neq x_0 \Rightarrow y(r, x_0, u) \neq y(r', x'_0, u). \tag{59}$$

Necessity of observability of each mode is obvious, and necessity of either controlled-discernibility or joint observability can easily be seen, by Propositions 3 and 2.

We sketch the proof of sufficiency. First, if a pair is controlled-discernible, then we can recover the mode for almost any input, and then invert the observability matrix of each path, which is nonsingular by assumption. The key observation we need to make here is that if a pair of modes is not controlled-discernible, then

$$\mathcal{H}_N(q) - \mathcal{H}_N(q') = 0 \tag{60}$$

for all N, by Remark 2, and can therefore be treated as in the autonomous case, since, if $r_{[1,T]} \equiv q$ and $r'[0,T] \equiv q'$, then

$$y_{[1,T]}(r, x_0, u) - y_{[1,T]}(r', x_0', u) = y_{[1,T]}(r, x_0, 0) - y_{[1,T]}(r', x_0', 0) \tag{61}$$

for all u. Therefore, joint observability becomes sufficient for the inference of x_0. We thus have established that any input u satisfying

$$u_0^{[N]} \in \mathbb{R}^{mN} \setminus \bigcup_{(q,q') \in CD} \ker \Big((I - P_N(q,q'))\big(\mathcal{H}_N(q) - \mathcal{H}_N(q')\big) u_0^{[N]} \Big), \tag{62}$$

where $CD = \{(q,q') | \mathcal{H}_N(q) - \mathcal{H}_N(q') \neq 0\}$, works in (59). This set is of course generic. □

Remark 4. In [7], recall that a necessary and sufficient condition for $(r_0, x_0/y, u^*)$-observability was given as the combination of controlled-discernibility of each pair of modes and observability of each mode. However, by the previous Proposition, this condition, even though sufficient, is not necessary for $(x_0/y, u^*)$-observability, as noted in Remark 1. Even though $\mathcal{H}_N(q) = 0$ for any mode, making controlled-discernibility an impossibility, the constant input $u = 0$ achieves $(x_0/y, u^*)$-observability.

5 Conclusion

We have characterized several observability notions for continuous-time switched linear systems. The analysis is of course still incomplete, and several problems still need to be solved. For instance, mode and state observability properties under fully or partially unknown inputs still have not been investigated in the switched setting. Furthermore, we will be investigating the *existential* counterparts of our current results, i.e. conditions for existence of mode signals allowing the initial or current mode or state to be inferred. It turns out that, in contrast with the universal problems that reduce to instantaneous inversions, such problems will involve observing the outputs *over a period of time*, and will involve the design of switching signals (as pointed out, e.g., in [22], and as is the case in the known modes case [19]). In future work, we will furthermore explore the connection between observability and bisimulation theory for discrete event and hybrid systems [17, 20].

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