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Motion generation for groups of robots: a centralized, geometric approach

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Motion generation for groups of robots: a centralized, geometric approach

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We develop a method for generating smooth trajectories for a set of mobile robots. We show that, given two end configurations of the set of robots, by tuning one parameter, the user can choose an interpolating trajectory from a continuum of curves varying from that corresponding to maintaining a rigid formation to motion of the robots toward each other. The idea behind this method is to change the original constant kinetic energy metric in the configuration space and can be summarized into three steps. First, the energy of the motion as a rigid structure is decoupled from the energy of motion along directions that violate the rigid constraints. Second, the metric is "shaped" by assigning different weights to each term, and, third, geodesic flow is constructed for the modified metric. The optimal motions generated on the manifolds of rigid body displacements in 3-D space ($SE(3)$) or in plane ($SE(2)$) and the uniform rectilinear motion of each robot corresponding to a totally uncorrelated approach are particular cases of our general treatment.

1 Introduction

Multi-robotic systems are versatile and efficient in exploration missions, military surveillance, and cooperative manipulation tasks. Recent research on such systems include work on cooperative manipulation [9], multi-robot motion planning [14], mapping and exploration [8], behavior-based formation control [1], and software architectures for multi-robotic systems [10]. In all these paradigms, the motion planning and control of the *team* of robots in formation can be modelled as a triple $(\mathbf{g}, \mathbf{r}, \mathbf{H})$, where $\mathbf{g} \in SE(3)$ represents the gross position and orientation of the team (for example, the pose of the leader), $\mathbf{r}$ is a set of shape variables that describes the relative positions of the robots in the team, and $\mathbf{H}$ is a control graph which describes the control strategy used by each robot [7]. In this paper, we are primarily interested in trajectory and shape, *i.e.*, in $\mathbf{g}$ and $\mathbf{r}$.

First, we consider the formation of robots as a rigid body, and investigate its motion. Virtual structures have

been proposed in [12] and used for motion planning and coordination and control of space-crafts in [2]. Our definition of a rigid formation requires the distances between robots, or reference points on robots, to remain fixed. Such a rigid formation is geometrically defined as a polyhedron formed by the reference points of each robot. The relative orientations of each robot are not restricted in such a rigid formation. To this end, we build on the results from [15] to generate trajectories that satisfy the rigid formation constraint and the overall energy of motion is minimized. The optimal problem is reduced to generating geodesics on $SO(3)$ and $SE(3)$. Due to the geometrical framework that we use, the generated trajectories are left invariant *i.e.*, independent of the choice of the inertial frame $\{F\}$.

The rigid formation constraint is too restrictive in many applications. We would like robots to be able to break formation, cluster together or string themselves out to avoid obstacles, and to regroup to achieve a desired goal formation at the destination. This paper develops a family of trajectories ranging from the trajectories that are optimal for a rigid formation on one extreme to independent trajectories that are optimal for each robot on the other.

We build the geodesic flow of a new metric in the whole configuration space given by collecting the configuration spaces of all robots. This new metric is obtained from the naturally induced (constant) kinetic energy metric dependent on the inertial properties of the robots by first decomposing each tangent space into two metric-orthogonal subspaces and then assigning different weights to the terms corresponding to rigid and non-rigid instantaneous motions. This idea of a "decomposition" and a subsequent "modification" is closely related to the methodology of controlled Lagrangians described in [5, 13]. The optimal motions generated on the manifolds of rigid body displacements in 3-D space ($SE(3)$) or in plane ($SE(2)$) and the uniform rectilinear motion of each robot corresponding to a completely independent approach are also particular cases of this general treatment.

2   Problem statement and notation

Consider N robots moving (rotating and translating) in 3-D space with respect to an inertial frame $\{F\}$. We choose a reference point on each robot at its center of mass O_i. A moving frame $\{M_i\}$ is attached to each robot at O_i (see Figure 1).

Robot i has mass m_i and matrix of inertia H_i with respect to frame $\{M_i\}$. Let $R_i \in SO(3)$ denote the rotation of $\{M_i\}$ in $\{F\}$ and $q_i \in \mathbb{R}^3$ the position vector of O_i in $\{F\}$. Let ω_i denote the expression in $\{M_i\}$ of the angular velocity of $\{M_i\}$ with respect to $\{F\}$. Sometimes it is also useful to define a *formation frame* $\{M\}$, attached at some virtual point O' and with pose $(R, d) \in SE(3)$ in $\{F\}$. Let r_i denote the position vectors of O_i in $\{M\}$.

The configuration space is the 6N-dimensional $SE(3)^N$, given by the poses of each robot. Given two configurations at times $t = 0$ and $t = 1$ respectively, the goal is to generate smooth interpolating motion for each robot so that the total kinetic energy is minimized.

The kinetic energy T of the system of robots is the sum of the individual energies. Since the frames $\{M_i\}$ were
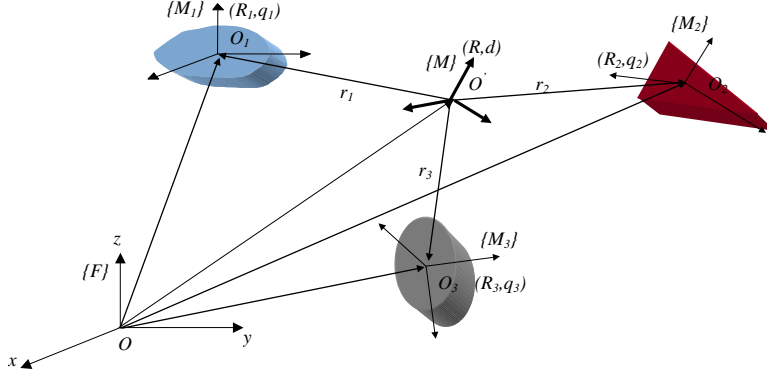
Figure 1: A set of $N = 3$ robots.

placed at the centroids O_i of the robots, T can be written as the sum of the total rotational energy T_r and the total translational energy T_t in the form:

$$T = T_r + T_t, \ \ T_r = \frac{1}{2}\sum_{i=1}^{N}(\omega_i^T H_i \omega_i), \ T_t = \frac{1}{2}\sum_{i=1}^{N}(m_i \dot{q}_i^T \dot{q}_i) \tag{1}$$

Due to the decomposition in equation (1), minimizing the total energy is equivalent to solving $N + 1$ independent optimization subproblems:

$$\min_{\sigma_i} \int_0^1 \omega_i^T H_i \omega_i dt, \ i = 1, \ldots, N, \tag{2}$$

$$\min_{q_i} \int_0^1 T_t dt \tag{3}$$

where σ_i is some parameterization of the rotation of $\{M_i\}$ in $\{F\}$, *i.e.,* some local coordinates on $SO(3)$. The solutions to equations (2) are given by N geodesics on $SO(3)$ with left invariant metrics with matrices H_i. Two different methods to obtain the solution are given in [15, 3] and a short example is given in Section 6.

Our main focus in this paper is solving problem (3) while certain constraints on the positions of the reference points O_i are satisfied. This is why throughout the paper the configuration space we are interested in is just the $3N$ dimensional $Q = \{q|q = (q_1, \ldots, q_N)\}$, which collects all the position vectors. Even though we envision that more general problems can be approached, in this paper we restrict to maintaining a rigid formation (virtual structure) and relaxing the constraint as necessary.

3 Approach

Our approach is geometric. To impose some degree of rigidity to the motion of the reference points O_i, we need to be able to separate the infinitesimal rigid motion from the non-rigid motion at each point in the position configuration space $q \in Q$. It is well known that the rigid body displacements in $\mathbb{R}^3$ are elements of a Lie group, the special Euclidean $SE(3)$, which can be seen as acting on Q. Then, if the set of points O_i described by $q \in Q$ are assumed as rigidly coupled at some instant, then the subset of Q that they can reach at further times is given by

3

the orbit of q under the actions of $SE(3)$. With this formulation, the separation of instantaneous rigid and non-rigid motion is simple: at each point $q \in Q$, in the tangent space $T_q Q$, decompose the velocity $\dot{q}$ into a component which is tangent to the orbit at q (this will induce rigid motion) and one which is orthogonal in some metric to the first (this will produce motion violating the rigid restriction).

The orthogonality of the two components of the velocity will asssure the separability of the terms corresponding to rigid and non-rigid motion in the kinetic energy, if the metric used for orthogonal decomposition is the kinetic energy metric. These two terms can be differently weighted to produce a new metric on the configuration space, for which we can control the amount of energy spent for rigid and non-rigid motions. The geodesic flow for the modified metric gives the optimal trajectories of the system of robots in the form of a continuum of curves varying from optimal motion in rigid formation to independent minimum energy motion of each robot.

The mathematical tools that we use are outlined in the next section.

4 Background

4.1 Velocity decomposition

Let Q be the configuration space of a system and $\mathcal{G}$ a Lie group that acts on Q so that the Lagrangean defined on TQ is invariant under this action. The state of the system can be described by a pair (g, s), where $g \in \mathcal{G}$ and s is an element in the complementary space $Q/\mathcal{G}$, which we will call the shape space. At any point $q \in Q$, a tangent vector $V_q \in T_q Q$ can be decomposed into a component which is tangent to Orb_q (the orbit of q under actions of $\mathcal{G}$), and a component which is orthogonal (in some metric $<, >$) to this first component (see Figure 2). Following
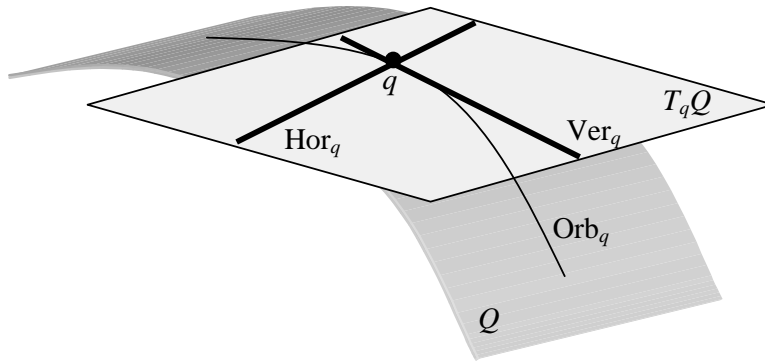

Figure 2: Pointwise decomposition of the tangent space in the vertical and horizontal subspaces

the notation in [5, 13], the space $T_q \mathrm{Orb}_q$ is called the vertical space at q, Ver_q, and its orthogonal complement is the horizontal space at $q \in Q$, Hor_q. The decomposition of the tangent vector V_q into $\mathrm{Ver}V_q$ (projection onto Ver_q) and $\mathrm{Hor}V_q$ (projection onto Hor_q) is uniquely defined by requiring that metric $<, >$ satisfies

$$< V_q^1, V_q^2 > = < \mathrm{Hor}V_q^1, \mathrm{Hor}V_q^2 > +$$
$$< \mathrm{Ver}V_q^1, \mathrm{Ver}V_q^2 >, \ V_q^1, V_q^2 \in T_q Q \qquad (4)$$

4.2 The geometry of rigid body motion

The Lie group $\mathcal{G}$ that we are interested in is the special Euclidean group $SE(3)$, the set of all rigid displacements in $\mathbb{R}^3$:

$$SE(3) = \left\{ g \mid g = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, \ R \in \mathbb{R}^{3\times 3}, \ RR^T = I, \ \det R = 1, \ d \in \mathbb{R}^3 \right\}.$$

The Lie algebra of $SE(3)$, denoted by $se(3)$, is given by:

$$se(3) = \left\{ \zeta \mid \zeta = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}, \ \hat{\omega} \in \mathbb{R}^{3\times 3}, \ \hat{\omega}^T = -\hat{\omega}, \ v \in \mathbb{R}^3 \right\}$$

where $\hat{\omega}$ is the skew-symmetric matrix form of the vector $\omega \in \mathbb{R}^3$. Given a curve

$$g(t) = \begin{bmatrix} R(t) & d(t) \\ 0 & 1 \end{bmatrix} \in SE(3)$$

an element $\zeta(t)$ of the Lie algebra $se(3)$ can be associated to the tangent vector $\dot{g}(t)$ at an arbitrary point t by:

$$\zeta(t) = g^{-1}(t)\dot{g}(t) = \begin{bmatrix} \hat{\omega}(t) & R^T \dot{d} \\ 0 & 0 \end{bmatrix} \tag{5}$$

where $\hat{\omega}(t) = R^T \dot{R}$.

Consider a rigid body moving in free space. Assume any inertial reference frame $\{F\}$ fixed in space and a frame $\{M\}$ fixed to the body at point O' as shown in Figure 3. A curve on $SE(3)$ physically represents a motion of
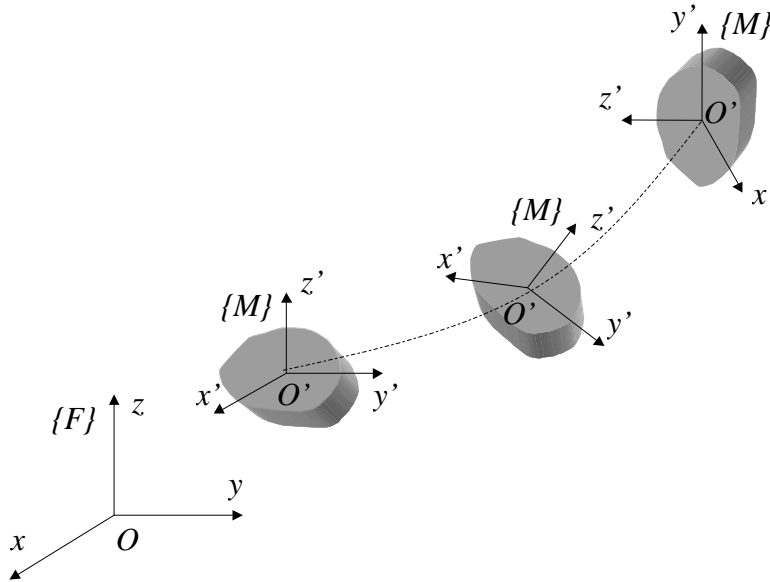

Figure 3: The inertial frame and the moving frame

the rigid body. If $\{\omega(t), v(t)\}$ is the vector pair corresponding to $\zeta(t)$, then ω corresponds to the angular velocity of the rigid body while v is the linear velocity of O', both expressed in the frame $\{M\}$. In kinematics, elements

of this form are called twists and $se(3)$ thus corresponds to the space of twists. The twist $\zeta(t)$ computed from Equation (5) does not depend on the choice of the inertial frame $\{F\}$.

If P is an arbitrary point on the rigid body with position vector r in frame $\{M\}$, then the velocity of P in frame $\{M\}$ is given by

$$v_P = \left[\begin{array}{cc} -\hat{r} & I_3 \end{array} \right] \zeta \tag{6}$$

where ζ is the twist of the rigid body written in vector form $\zeta = [\omega^T \; v^T]^T$.

4.3   The kinetic energy of a moving body - a left invariant metric on $SE(3)$

For any $\zeta_1, \zeta_2 \in se(3)$ and G a positive definite matrix, we can define a metric $\zeta_1^T G \zeta_2$ on the Lie algebra $se(3)$ and extend it through left translation throughout the manifold $SE(3)$. Let m be the mass of the rigid body from (Figure 3) and H its matrix of inertia with respect to the body frame $\{M\}$, assumed at the center of mass. Then, with

$$G = \frac{1}{2} \left[\begin{array}{cc} H & 0 \\ 0 & mI_3 \end{array} \right] \tag{7}$$

the norm induced by the above metric $\zeta^T G \zeta$ is exactly the kinetic energy of the moving (rotating and translating) rigid body. Moreover, if $\{M\}$ is aligned with the principal axes, then H is diagonal. In the most general case, when the frame $\{M\}$ is displaced by some (R_0, d_0) from the centroid and the orientation parallel with the principal axes, we have [15]:

$$G = \frac{1}{2} \left[\begin{array}{cc} R_0^T H R_0 - m R_0^T \hat{d}_0 R_0 & \hat{a} \\ -\hat{a} & mI_3 \end{array} \right], \; a = -m R_0 d_0$$

For a rigid system of N particles with masses $m_1, \ldots, m_N$ and position vectors $r_1, \ldots, r_N$ in the body fixed frame $\{M\}$, the matrix of the (left invariant) kinetic energy metric on $SE(3)$ is [4]:

$$\mathcal{M} = \left[\begin{array}{cc} -\sum_{i=1}^{N} m_i \hat{r}_i^2 & \sum_{i=1}^{N} m_i \hat{r}_i \\ -\sum_{i=1}^{N} m_i \hat{r}_i & \sum_{i=1}^{N} m_i I_3 \end{array} \right] \tag{8}$$

The upper left 3×3 submatrix of $\mathcal{M}$ is the inertia matrix of the system of particles with respect to $\{M\}$. If frame $\{M\}$ is placed at the center of mass and aligned with the principal axes of the structure, then $\mathcal{M}$ becomes diagonal.

4.4   Geodesics - minimum energy curves

The geodesics on a differentiable manifold can be defined as minimum length, or equivalently, minimum energy curves [6]. A useful (local) characterization of a geodesic curve on a n-dimensional Riemannian manifold (locally) parameterized by $x_1, \ldots, x_n$ is the following set of differential equations:

$$\ddot{x}_i + \sum_{j,k} \Gamma_{jk}^i \dot{x}_j \dot{x}_k = 0, \; i = 1, \ldots, n \tag{9}$$

where Γ_{jk}^i are the Christoffel symbols of the unique symmetric connection associated to the metric on the manifold.

In [15] it has been proved that a geodesic $g(t)$ on $SE(3)$ equipped with a left invariant product metric of the type (7) is composed of the geodesics in the component spaces $SO(3)$ and $\mathbb{R}^3$ with the corresponding component metrics, and are described by the following set of differential equations:

$$\frac{d\omega}{dt} = -H^{-1}(\omega \times (H\omega)) \tag{10}$$

$$\ddot{d} = 0. \tag{11}$$

If $H = \alpha I$, an analytical expression for the geodesic passing through $g(0) = (R(0), d(0))$ and $g(1) = (R(1), d(1))$ at $t = 0$ and $t = 1$ respectively, is given by [15] $g(t) = (R(t), d(t))$, where $R(t) = R(0)\exp(\hat{\omega}_0 t)$, $d(t) = (d(1) - d(0))t + d(0)$ and $\hat{\omega}_0 = \log(R(0)^T R(1))$ In the case when $H \neq \alpha I$, there is no closed form expression for the corresponding geodesic and numerical methods or the projection method [3] should be employed.

5 Motion decomposition: rigid vs. non-rigid

We first define a metric $<,>$ in the position configuration space, which is the same at all points $q \in Q$:

$$< V_q^1, V_q^2 > = V_q^{1^T} M V_q^2, \tag{12}$$

$$V_q = \dot{q} \in T_q Q, \quad M = \frac{1}{2}\text{diag}\{m_1 I_3, \ldots, m_N I_3\}$$

Metric (12) is called the *kinetic energy metric* because its induced norm ($V_q^1 = V_q^2 = \dot{q}$) assumes the familiar expression of the total kinetic energy of the system $1/2 \sum_{i=1}^N m_i \dot{q}_i^T \dot{q}_i$.

If no restrictions are imposed on Q, the geodesic for metric (12) is obviously a straight line uniformly parameterized in time interpolating between q^0 and q^1 in Q.

In this paper, the Lie group $\mathcal{G}$ as defined in Section 4.1 is $SE(3)$. The left action of $\mathcal{G}$ on $Q = (q_1, \ldots, q_N)$ is the rigid body displacement applied to each q_i written in homogeneous form. The $\mathcal{G}$-orbit at q is the set of all poses that the structure $(q_1, q_2, \ldots, q_N)$ can reach if it was assumed rigidly attached to $\{M\} \equiv \{F\}$ at that instant.

At each point q in the configuration space Q, in the corresponding tangent space $T_q Q$, the velocity corresponding to infinitesimal rigid motion is given by $\text{Ver}V_q$. Therefore, Ver_q locally describes the set of all rigid body motion directions. The orthogonal complement to Ver_q, Hor_q will be the set of all directions violating the rigid body constraints.

Using (6) for each q_i, $i = 1, \ldots, N$ and $\{M\} \equiv \{F\}$, it is easy to see that Ver_q is the range of the following $3N \times 6$ matrix:

$$\text{Ver}_q = \text{Range}(A(q)), \quad A(q) = \begin{bmatrix} -\hat{q}_1 & I_3 \\ \ldots & \ldots \\ -\hat{q}_N & I_3 \end{bmatrix} \tag{13}$$

The coordinates of the expansion of $\text{Ver}V_q \in \text{Ver}_q$ along the columns of $A(q)$ are exactly the components of the left invariant twist $\zeta \in se(3)$ of the virtual structure formed by $(q_1, \ldots, q_N)$ and $\{M\} \equiv \{F\}$ at that instant:

$$\text{Ver}V_q = A\zeta$$

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Using metric (12), the orthogonal complement of Ver_q is

$$\text{Hor}_q = \text{Null}(A(q)^T M) \tag{14}$$

Let $B(q)$ denote a matrix whose columns are a basis of Hor_q. Let ψ denote the components of $\text{Hor}V_q$ in this basis: $\text{Hor}V_q = B(q)\psi$. Therefore, the velocity at point q can be written as:

$$V_q = \text{Ver}V_q + \text{Hor}V_q = A(q)\zeta + B(q)\psi \tag{15}$$

Then, requirement (4) is satisfied. Indeed, for any $V_q^1,\ V_q^2 \in T_q Q$,

$$< V_q^1, V_q^2 >= V_q^{1^T} M V_q^2 = \zeta^{1^T} A^T M A \zeta^2 +$$
$$+\zeta^{1^T} A^T M B \psi^2 + \psi^{1^T} B^T M A \zeta^2 + \psi^{1^T} B^T M B \psi^2 =$$
$$= \zeta^{1^T} A^T M A \zeta^2 + \psi^{1^T} B^T M B \psi^2 =$$
$$=< B\psi^1, B\psi^2 > + < A\zeta^1, A\zeta^2 >=$$
$$=< \text{Hor}V_q^1, \text{Hor}V_q^2 > + < \text{Ver}V_q^1, \text{Ver}V_q^2 >$$

because both $A^T M B$ and $B^T M A$ are zero from (14). Also, note that

$$\zeta = (A^T M A)^{-1} A^T M V,$$

$$\tag{16}$$

$$\psi = (B^T M B)^{-1} B^T M V$$

where the explicit dependence of A and B on q was omitted for simplicity. Therefore, the translational kinetic energy (which is the square of the norm induced by metric (12)) becomes:

$$T_t(q, \dot{q}) = \dot{q}^T M \dot{q} = \zeta^T A^T M A \zeta + \psi^T B^T M B \psi \tag{17}$$

Straightforward calculation shows that $A^T M A$ is the same as (8), (when $r_i = q_i$, $\{M\} = \{F\}$) *i.e.*, the matrix of the left invariant kinetic energy metric if the system of particles is assumed rigid. Therefore, in (17), $\zeta^T A^T M A \zeta$ captures the energy of the motion of the system of particles as a rigid body, while the remaining part $\psi^T B^T M B \psi$ is the energy of the motion that violates the rigid body restrictions. For example, in the obvious case of a system of $N = 2$ particles, the first part corresponds to the motion of the two particles connected by a rigid massless rod, while the second part would correspond to motion along the line connecting the two bodies.

6 Motion generation for rigid formations

In this section, we will assume that the robots are required to move in *rigid formation*, *i.e.*, the distances between any two reference points O_i are preserved, or, equivalently, the reference points form a rigid polyhedron.

In our geometric framework, the rigid body requierement means restricting the trajectory $q(t) \in Q$ to be a $SE(3)$ - orbit, or equivalently, $\text{Hor}V_q = 0$ for all q.

In this case, to reduce the dimension of the problem and make the invariance properties obvious, we will make use of the formation frame $\{M\}$ as defined in Section 2, which is attached at the centroid O' and aligned with the principal axes of the virtual structure determined by the points O_i. Moving in rigid formation is therefore equivalent to $\dot{r}_i = 0$ for all $i = 1, \ldots, N$. Let v_i denote the velocity of O_i in $\{F\}$ expressed in $\{M\}$, *i.e.*, $v_i = R^T \dot{q}_i$.

Then, using the invariance of the 2-norm to orthogonal transformations and equation (6), the kinetic energy T_t becomes:

$$T_t = \frac{1}{2} \sum_{i=1}^{N} (m_i \dot{q}_i^T \dot{q}_i) = \frac{1}{2} \sum_{i=1}^{N} (m_i v_i^T v_i) = \zeta^T A(r)^T M A(r) \zeta$$

where $\zeta \in se(3)$ is the instantaneous twist of the virtual structure and $A^T M A$ is the constant metric of the left invariant kinetic energy metric as in (8).

Therefore, if the set of robots is required to move while maintaining a constant shape r, the optimization problem is reduced from dimension $6N$ to dimension $3N + 6$, and consists of solving for N geodesics on $SO(3)$ with metrics H_i (individual rotations) and one geodesic on the $SE(3)$ of the virtual structure with metric $A^T M A$.

We are now in the position to outline a procedure for generating smooth motion for each robot to interpolate between two given positions while preserving the formation and minimizing the kinetic energy.

Assume the initial ($t = 0$) and final ($t = 1$) poses of each robot are given in frame $\{F\}$. Obviously, the initial and final poses should have the same shape, *i.e.* have the same r_i's.

First, we locate the centroids O_i, attach the frames $\{M_i\}$ and, from the geometric properties of each robot, calculate the inertia matrices H_i. If $\{M_i\}$ is aligned with the principal axes of robot i, then H_i is diagonal, but this is not necessary. If the initial and final rotations R_i^0, R_i^1 of robot i are given, then the rotation of each robot is the interpolating geodesic on $SO(3)$ equipped with metric H_i.

Given $q_1^0, q_2^0, \ldots, q_N^0$ at $t = 0$ and $q_1^1, q_2^1, \ldots, q_N^1$ at $t = 1$, the initial and final positions of the centroid of the fictitious rigid body are given by $d^j = (\sum_{i=1}^{N} m_i q_i^j)/(\sum_{i=1}^{N} m_i)$, $j = 0, 1$. A frame $\{M\}$ is fixed to the virtual rigid body at its center of mass, which will give the initial and final orientations of the formation, R^0 and R^1. The r_i's are then determined by $r_i = R^{0T}(q_i^0 - d^0)$, which will induce a metric G on the $SE(3)$ of the formation, according to (8). The geodesic $(d(t), R(t))$ on the $SE(3)$ of the formation with boundary conditions (d^0, R^0), (d^1, R^1) can be found as described in Section 4.4. The trajectory of the centroid of each robot is finally determined in the form $q_i(t) = d(t) + R(t)r_i$.

Example: Five identical robots in 3D space

For illustration, we consider five identical parallelepipedic robots $m_i = m$, $i = 1, \ldots 5$ required to move in formation while minimizing energy. The initial and final poses together with the geometrical properties of the robots are given in Figure 4.

As outlined in the previous section, generating optimal motion for this group of robots reduces to generating
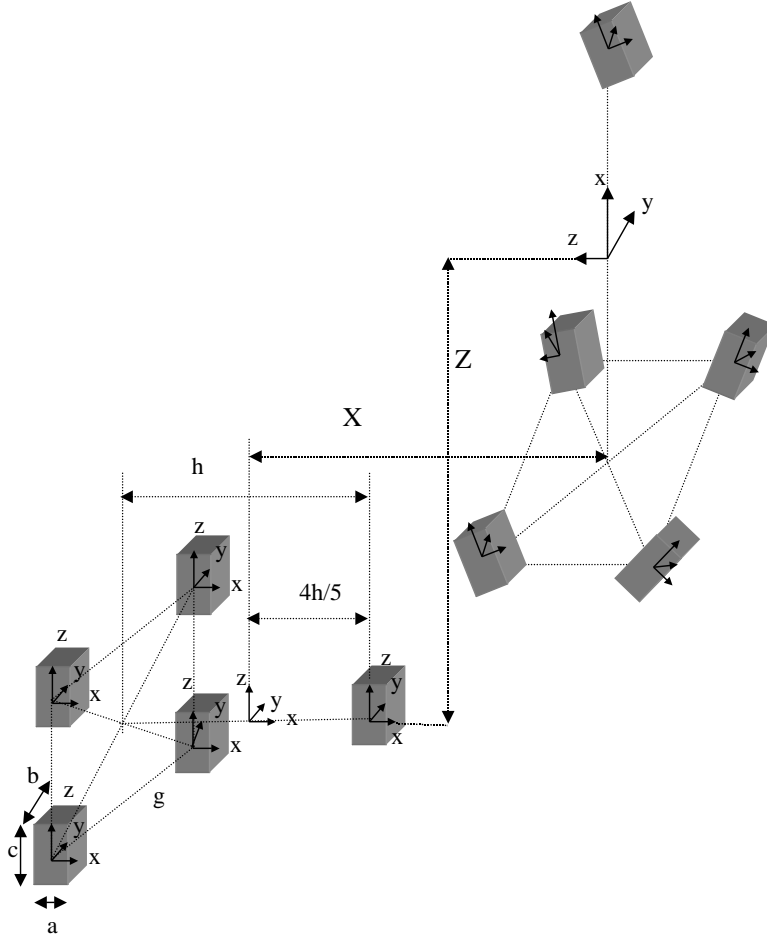
Figure 4: Geometry of the robots and of the virtual structure: a=c=2, b=10, h=20, l=10, X=20, Z=20, m=12

five geodesics on the $SO(3)$ of each robot with left invariant metric

$$H_i = \frac{m}{24} \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}, \ i = 1, \ldots, 5$$

and one geodesic on the $SE(3)$ of the virtual structure endowed with a left invariant metric with matrix

$$G = \frac{m}{2} \begin{bmatrix} 2l^2 & 0 & 0 & 0 \\ 0 & l^2 + \frac{4h^2}{3} & 0 & 0 \\ 0 & 0 & l^2 + \frac{4h^2}{3} & 0 \\ 0 & 0 & 0 & 3I_3 \end{bmatrix}$$

The resulting motion is presented in Figure 5.

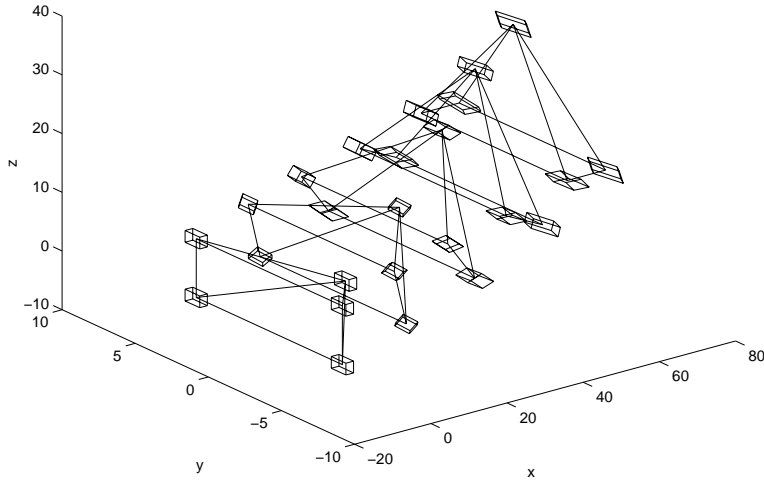We used exponential coordinates $sigma_i$ as local parameterization of $SO(3)$.

Figure 5: Optimal motion for five identical bodies required to maintain a rigid formation

7 Motion generation by kinetic energy shaping

By shaping the kinetic energy, we mean smoothly changing the corresponding metric (12) at T_qQ so that motion along some specific directions is allowed while motion along some other directions is penalized. The new metric will no longer be constant - the Christoffel symbols of the corresponding symmetric connection will be non-zero. The associated geodesic flow gives optimal motion.

In this paper, we "shape" the original metric (12)) by putting different weights on the terms corresponding to the rigid and non-rigid motions:

$$< V_q^1, V_q^2 >_\alpha = \alpha < \text{Hor} V_q^1, \text{Hor} V_q^2 > + (1 - \alpha) < \text{Ver} V_q^1, \text{Ver} V_q^2 > \tag{18}$$

Using (16) to go back to the original coordinates, we get the modified metric in the form:

$$< V_q^1, V_q^2 >_\alpha = V_q^{1T} M_\alpha(q) V_q^2, \tag{19}$$

where the new matrix of the metric is now dependent on the artificially introduced parameter α and the point on the manifold $q \in Q$:

$$M_\alpha(q) = \alpha M A (A^T M A)^{-T} A^T M + (1 - \alpha) M B (B^T M B)^{-T} B^T M \tag{20}$$

The influence of the parameter α can be best seen by examining the significance of α taking on the values of 0, 0.5 and 1. The two extreme values of α, 0 and 1, cause the metric (19) to become singular. $\alpha = 1$ reduces to the rigid formation metric (8) on $\mathcal{G}$, while $\alpha = 0$ yields a metric for motions along the fiber $Q/\mathcal{G}$. The intermediate case, $\alpha = 0.5$, yields the kinetic energy of a system of independent robots.

As α tends to 0, the preferred motions will be ones where robots cluster together through much of the duration of the trajectory, thus minimizing the *rigid body energy* consumption. As α approaches 0.5, the motions degenerate

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toward uncoordinated, independent motions. As α tends to 1, the preferred motions are ones where the robots stay in rigid formation through most of the trajectory, thus minimizing the energy associated with *deforming* the formation.

We use the geodesic flow of metric (19) to produce smooth interpolating motion between two given configurations:

$$q^0 = q(0), \quad q^1 = q(1) \ \in \mathbb{R}^{3N} \tag{21}$$

To simplify the notation, let x_i, $i = 1, \ldots, 3N$ denote the coordinates $q_i \in \mathbb{R}^3$, $i = 1, \ldots, N$ on the configuration manifold Q. In this coordinates, the geodesic flow is described by the following differential equations [6]:

$$\ddot{x}_i + \sum_{j,k} \Gamma^i_{jk} \dot{x}_j \dot{x}_k = 0, \ i = 1, \ldots, 3N \tag{22}$$

where Γ^k_{ij} are the Christoffel symbols of the unique symmetric connection associated to metric (19):

$$\Gamma^k_{ij} = \frac{1}{2} \sum_h \left(\frac{\partial m_{hj}}{\partial x^i} + \frac{\partial m_{ih}}{\partial x^j} - \frac{\partial m_{ij}}{\partial x^h} \right) m^{hk} \tag{23}$$

m_{ij} and m^{ij} are elements of M_α and M_α^{-1}, respectively.

Because $alpha = 0$ and $alpha = 1$ make the metric singular, (23) can only be used for $0 < \alpha < 1$.

Example: two bodies in the plane

Consider two bodies of masses m_1 and m_2 moving in the $x - y$ plane. The configuration space is $Q = R^4$ with coordinates $q = [x_1, \ y_1, \ x_2, \ y_2]^T$. The symmetry group $\mathcal{G}$ is the three-dimensional $SE(2)$. The A and B matrices describing Ver_q and Hor_q as in (13) and (14) are:

$$A = \begin{bmatrix} -y_1 & 1 & 0 \\ x_1 & 0 & 1 \\ -y_2 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{m_2(x_2 - x_1)}{m_1(y_1 - y_2)} \\ -\frac{m_2}{m_1} \\ \frac{x_1 - x_2}{y_1 - y_2} \\ 1 \end{bmatrix}$$

The 64 Christoffel symbols $\Gamma^k = (\Gamma^k_{ij})_{ij}$ of the connection associated with the modified metric at $q \in Q$ become:

$$\Gamma^1 = \frac{2(1 - 2\alpha)}{\alpha} \frac{m_2}{m_1 + m_2} \frac{d_x}{(d_x^2 + d_y^2)^2} \Gamma$$

$$\Gamma^2 = \frac{2(1 - 2\alpha)}{\alpha} \frac{m_2}{m_1 + m_2} \frac{d_y}{(d_x^2 + d_y^2)^2} \Gamma$$

$$\Gamma^3 = -\frac{2(1 - 2\alpha)}{\alpha} \frac{m_1}{m_1 + m_2} \frac{d_x}{(d_x^2 + d_y^2)^2} \Gamma$$

$$\Gamma^4 = -\frac{2(1 - 2\alpha)}{\alpha} \frac{m_1}{m_1 + m_2} \frac{d_y}{(d_x^2 + d_y^2)^2} \Gamma$$

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where

$$
\Gamma = \begin{bmatrix}
-d_y^2 & d_x d_y & d_y^2 & -d_x d_y \\
d_x d_y & -d_x^2 & -d_x d_y & d_x^2 \\
d_y^2 & -d_x d_y & -d_y^2 & d_x d_y \\
-d_x d_y & d_x^2 & d_x d_y & -d_x^2
\end{bmatrix}
$$

and $d_x = x_1 - x_2$, $dy = y_1 - y_2$. It can be easily seen that, as expected, all Christoffel symbols are zero if $\alpha = 0.5$. Also, the actual masses of the robots are not relevant, it's only the ratio m_1/m_2 which is important.

In this example, we assume $m_2 = 2m_1$ and the boundary conditions:

$$
q^0 = \begin{bmatrix} 1 \\ 0 \\ -0.5 \\ 0 \end{bmatrix}, \quad
q^1 = \begin{bmatrix} 3 - \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 3 + \frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} \end{bmatrix}
$$

which correspond to a rigid body displacement so that we can compare our results to the optimal motion corresponding to a rigid body.

If the structure was assumed rigid, then the optimal motion is described by uniform rectilinear translation of the center of mass between $(0,0)$ and $(3,0)$ and uniform rotation between 0 and $3\pi/4$ around $-z$ placed at the center of mass. The corresponding trajectories of the robots are drawn in solid line in all the pictures in Figure 6. It can be easily seen that there is no difference between the optimal motion of the virtual structure solved on $SE(2)$ and the geodesic flow of the modified metric with $\alpha = 0.99$ (Figure 6, bottom). If $\alpha = 0.5$, all bodies move in straight line as expected (Figure 6, middle). For $\alpha = 0.2$, the bodies go toward each other first, and then split apart to attain the final positions (Figure 6, top).

Example: three bodies in the plane

The calculation of the trajectories for three bodies moving in the plane is simplified by assuming that the robots are identical, and, without loss of generality, we assume $m_1 = m_2 = m_3 = 1$. The vertical and the horizontal spaces at a generic configuration
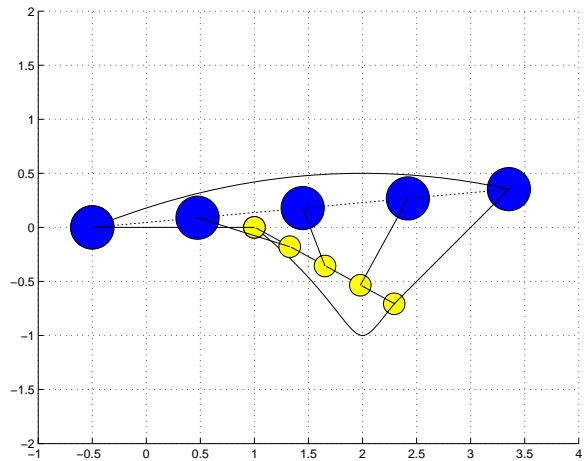
$$
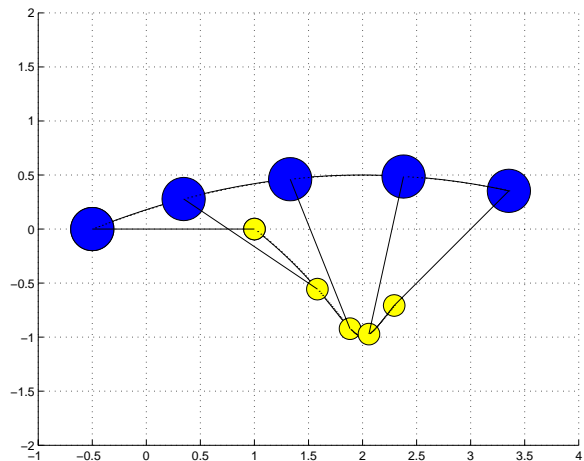q = [x_1,\, y_1,\, x_2,\, y_2,\, x_3,\, y_3]^T \in Q = \mathbb{R}^6
$$

are given by

$$
\mathrm{Ver}_q = \mathrm{Range}(A), \quad A = \begin{bmatrix}
-y_1 & 1 & 0 \\
x_1 & 0 & 1 \\
-y_2 & 1 & 0 \\
x_2 & 0 & 1 \\
-y_3 & 1 & 0 \\
x_3 & 0 & 1
\end{bmatrix},
$$

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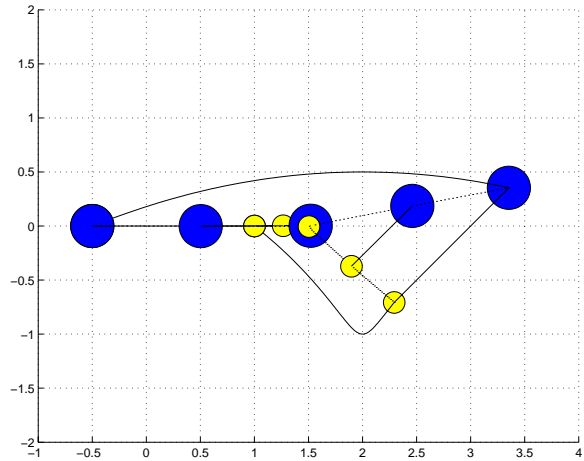
$\alpha = 0.2$

$\alpha = 0.5$

$\alpha = 0.99$

Figure 6: Three interpolating motions for a set of two planar robots as geodesics of a modified metric defined in the configuration space.

$\alpha = 0.2$

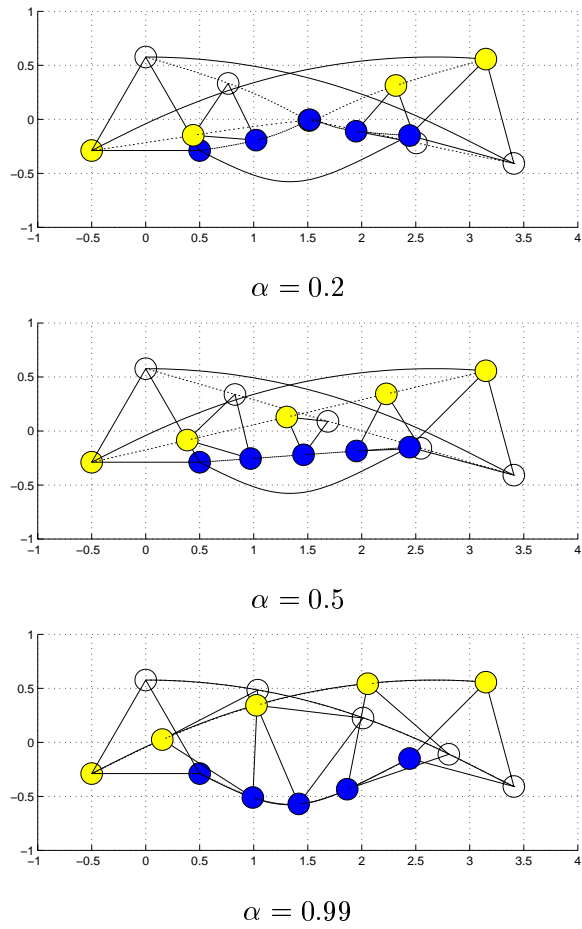$\alpha = 0.5$

$\alpha = 0.99$

Figure 7: Three interpolating motions for a set of three planar robots as geodesics of a modified metric defined in the configuration space.

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$$\text{Hor}_q = \text{Range}(B), \ B = \begin{bmatrix} \frac{x_3-x_1}{y_1-y_2} & \frac{y_2-y_3}{y_1-y_2} & \frac{x_2-x_1}{y_1-y_2} \\ -1 & 0 & -1 \\ \frac{x_1-x_3}{y_1-y_2} & \frac{y_3-y_1}{y_1-y_2} & \frac{x_1-x_2}{y_1-y_2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

For simplicity, we omit the expressions of the modified metric and of the Christoffel symbols. The simulation scenario resembles the one in Section 7: the end poses correspond to a rigid structure consisting of a equilateral triangle with side equal to 1. The optimal trajectory solved on $SE(2)$ corresponds to rectilinear uniform motion of the center of mass (line between (0,0) and (3,0) in Figure 7) and uniform rotation from angle 0 to $3\pi/4$ around axis $-z$. The resulting motion of each robot is shown solid, while the actual trajectory for the corresponding value of α is shown dashed. First note for $\alpha = 0.99$ the trajectories are basically identical with the optimal traces produced by the virtual structure, as expected. In the case $\alpha = 0.5$ the bodies move in straight line (corresponding to the unmodified metric). The tendency to cluster as α decreases is seen for $\alpha = 0.2$. Note also that due to our choice $m_1 = m_2 = m_3$, the geometry of the equilateral triangle is preserved for all values of α, it only scales down when α decreases from 1.

8 Conclusion and future work

We presented a strategy for generating a family of smooth interpolating trajectories for a team of mobile robots. The family is parameterized by a scalar α. As α becomes closer to zero, the robots will tend to cluster together while moving between initial and final positions. The case $\alpha = 0.5$ corresponds to a totally uncoordinated strategy: each robot will move from its initial to its final position while minimizing its own energy. Finally, as α tends to 1, the robots try to preserve the distances between them and minimize the overall energy of the motion. This constitutes an alternative to generating motion for virtual structures by solving an optimization problem on the manifold of rigid body displacements $SE(3)$ [4].

While the paper provides a useful conceptual framework for motion planning and generation of trajectories, there is a practical limitation to this work. As the number of robots, n, increases, the generation of the Christoffel symbols and the solution of the two-point boundary value problem because more complicated. The number of terms in the differential equations governing the motion increases exponentially with the number of robots, although the order of the equations and the quadratic nonlinearity are independent of n.

There are two approaches to overcome the difficulty with scaling. The first is to exploit the known structure in the trajectories for the limiting cases of $\alpha = 0, 0.5$, and 1.0. In these cases, solutions can be obtained efficiently (for $\alpha = 1.0$, approximately) and in closed form. One can interpolate between the trajectories to construct approximate solutions for intermediate cases, a procedure that scales linearly with n.

The second approach is to develop an alternative description of the shape of the formation, a description that

is independent of the exact coordinates of the robots. This would allow the designer to focus on the gross motion $g \in \mathcal{G}$ and the shape $r \in \mathcal{R}$, while the control of the robots to maintain the prescribed shape r can be done at a lower level of control. Both these approaches are directions for future investigation.

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