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Combinatorial Structures and Generating Functions of Fishburn Numbers, Parking Functions, and Tesler Matrices.

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Abstract

This dissertation reflects the author's work on two problems involving combinatorial structures.

The first section, which was also published in the Journal of Combinatorial Theory, Series A, discusses the author's work on several conjectures relating to the Fishburn numbers. The Fishburn numbers can be defined as the coefficients of the generating function

begin{align*}

1+sum_{m=1}^{infty} prod_{i=1}^{m}(1-(1-t)^{i}).

end{align*} Combinatorially, the Fishburn numbers enumerate certain supersets of sets enumerated by the Catalan numbers.

We add to this work by giving an involution-based proof of the conjecture of Claesson and Linusson that the Fishburn numbers enumerate non-2-neighbor-nesting matchings. We begin by proving that a map originally defined by Claesson and Linusson gives a bijection between non-2-neighbor-nesting matchings and $textbf{(2-1)}$-avoiding inversion tables. We then define a set of diagrams, which we call Fishburn diagrams, that give a natural interpretation to the generating function of the Fishburn numbers. Using an involution on Fishburn diagrams, we then prove that the Fishburn numbers enumerate $textbf{(2-1)}$-avoiding inversion tables. By using two variations of this involution on two subsets of Fishburn diagrams, we then give a visual proof of the conjecture of Remmel and Kitaev that two bivariate refinements of the generating function of the Fishburn numbers are equivalent. In an appendix, we give an inductive proof of the conjecture of Claesson and Linusson that the distribution of left-nestings over the set of all matchings is given by the second-order Eulerian triangle.

The conjecture of Remmel and Kitaev was independently proved by Jelin'ek and by Yan with a matrix interpretation defined by Dukes and Parviainen. Bijections surveyed by Callan can lead to a similar proof of the conjecture of Claesson and Linusson giving the distribution of left-nestings over matchings, using a result on the Stirling permutations due to Gessel and Stanley. This work was done independently.

The second section, some of which was presented at the Formal Power Series and Algebriac Combinatorics conference (FPSAC), discusses the author's work on several conjectures relating to parking functions and to Tesler matrices. Parking functions are combinatorial objects which generalize both permutations and Catalan paths. Haglund and Loehr conjectured that the generating functions of two statistics, $area$ and $dinv$, over the set of parking functions (the q, t-parking functions) gives the Hilbert series of the diagonal coinvariants. Haglund recently proved that this Hilbert series is given by another generating function over the set of matrices with every hook sum equal to one (``Tesler matrices''). We prove several structural results on parking functions inspired by Tesler matrices, including a near-recursive generation of the q, t-parking functions. We also give consistent bijective proofs of several special cases of Haglund's Tesler function identity, giving a combinatorial connection between parking functions and Tesler matrices, and discuss related conjectures. A connection between the $q=1$ special case and a result of Kreweras was later pointed out by Garsia et al, and some of the original ideas on the $q=1$ special case arose from a discussion between the author, Haglund, Bandlow, and Visontai.

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COMBINATORIAL STRUCTURES AND GENERATING FUNCTIONS OF FISHBURN NUMBERS, PARKING FUNCTIONS, AND TESLER MATRICES.

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I am also thinking of Herb Wilf. I read "Generatingfunctionology", or some of it, in high school. Looking back, I'm sure I never thought I'd be fortunate enough to be able to present my own work to him, and he gave me a lot of confidence when I needed it.

ABSTRACT

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This dissertation reflects the author's work on two problems involving combinatorial structures.

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$$1+\sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-t)^{i}).$$

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function of the Fishburn numbers. Using an involution on Fishburn diagrams, we then prove that the Fishburn numbers enumerate **(2-1)**-avoiding inversion tables. By using two variations of this involution on two subsets of Fishburn diagrams, we then give a visual proof of the conjecture of Remmel and Kitaev that two bivariate refinements of the generating function of the Fishburn numbers are equivalent. In an appendix, we give an inductive proof of the conjecture of Claesson and Linusson that the distribution of left-nestings over the set of all matchings is given by the second-order Eulerian triangle.

The conjecture of Remmel and Kitaev was independently proved by Jelinék and by Yan with a matrix interpretation defined by Dukes and Parviainen. Bijections surveyed by Callan can lead to a similar proof of the conjecture of Claesson and Linusson giving the distribution of left-nestings over matchings, using a result on the Stirling permutations due to Gessel and Stanley. This work was done independently.

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with every hook sum equal to one ("Tesler matrices"). We prove several structural results on parking functions inspired by Tesler matrices, including a near-recursive generation of the q,t-parking functions. We also give consistent bijective proofs of several special cases of Haglund's Tesler function identity, giving a combinatorial connection between parking functions and Tesler matrices, and discuss related conjectures. A connection between the $q = 1$ special case and a result of Kreweras was later pointed out by Garsia et al, and some of the original ideas on the $q = 1$ special case arose from a discussion between the author, Haglund, Bandlow, and Visontai.

Contents

Chapter 1

Introduction: Fishburn Numbers

Let $[m]$ be the set of integers from 1 to m. Let a *matching* on $[2n]$ be an involution on $[2n]$ with no fixed points, or equivalently a partition of $[2n]$ into disjoint pairs. Let a *nesting* in a matching X on $[2n]$ be a pair of pairs (a, b) and (c, d) in X such that $a < c < d < b$.

It is well-known that the Catalan numbers C_n enumerate the set of non-nesting matchings on $[2n]$. Let a *neighbor nesting* in a matching X be a nesting $(a, b)(c, d) \in X$ such that $c = a + 1$ or $d = b + 1$. Non-neighbor-nesting matchings can be seen as a superset of non-nesting matchings. Let f_n be the number of non-neighbor-nesting matchings on $[2n]$. The numbers $f_1, f_2, \ldots$ are known as the *Fishburn numbers* (after Claesson and Linusson [13]; the citations and references to Zagier, Stoimenow, Bosquet-Mélou et al., and Dukes and Parviainen below are taken from, and used as cited by, [13], which we also follow for the convention of diagramming matchings).

The Fishburn numbers can also be defined as the coefficients of the following generating function (see Zagier [5] and Stoimenow [4]):

$$\sum_{n=0}^{\infty} f_n t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1-(1-t)^i). \tag{1.0.1}$$

Bosquet-Mélou et al. [9] proved that the Fishburn numbers enumerate the following sets. Each set can be seen as a superset of a *Catalan set*, or a set enumerated by the Catalan numbers:

- The set of **(2+2)**-avoiding posets with n elements. This can be seen as a superset of the Catalan set of **(2+2)**- and **(3+1)**-avoiding posets with n elements.

- The set of permutations $\pi = \pi_1\pi_2 \cdots \pi_n$ of $[n]$ such that there exist no $i < j$ with $\pi_j = \pi_i - 1$ and $\pi_{i+1} > \pi_i$. This can be seen as a superset of the Catalan set of 231-avoiding permutations of $[n]$.

- The set of *ascent sequences* of length n, where an ascent sequence is a sequence of non-negative integers $x_1x_2 \cdots x_n$ such that $x_1 = 0$ and x_{i+1} is less than or equal to the number of ascents in the first i terms. This can be seen as a superset of the Catalan set of sequences of non-negative integers $x_1x_2 \cdots x_n$ such that $x_1 = 0$ and $x_{i+1} < x_i + 1$.

Dukes and Parviainen [10] proved that the Fishburn numbers also enumerate upper-triangular matrices with non-negative integer entries and no empty rows or columns.

Let a *k-neighbor-nesting* on a matching X be a nesting $(a, b)(c, d) \in X$ such that $c - a \leq k$. Claesson and Linusson [13] recently conjectured that the Fishburn numbers also enumerate the set of non-2-neighbor-nesting matchings on $[2n]$.

Let $f_{n,d}$ be the number of ascent sequences with precisely d zeroes. The bivariate generating function of $f_{n,d}$ was recently proved by Remmel and Kitaev [12] to be:

$$\sum_{n=0}^{\infty}\sum_{d=1}^{n} f_{n,d}t^n z^d = 1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^{k}(1-(1-t)^i). \tag{1.0.2}$$

Remmel and Kitaev conjectured that the bivariate generating function of $f_{n,d}$ has the following simpler form:

$$\sum_{n=0}^{\infty} f_{n,d}t^n z^d = 1 + \sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-t)^{i-1}(1-zt)). \tag{1.0.3}$$

(Note that Equation (1.0.3), unlike Equation (1.0.2), is trivially seen to reduce to Equation (1.0.1) at $z = 1$.)

In the following article, we will prove both conjectures and show how they are related:

- First, we will give a simple bijection (a restriction of a bijection from [13]) between the set of non-2-neighbor-nesting matchings on $[2n]$ and the set of **(2-1)**-avoiding inversion tables $a_1 a_2 \cdots a_n$ of length n, where the pattern **(2-1)** is said to occur if there exists a $p < q$ such that $a_p = a_q + 1$.

- We will then give an involution-based proof that the generating function of **(2-1)**-avoiding inversion tables with respect to length is given by Equation (1.0.1). This will prove the conjecture of Claesson and Linusson.

- We will then restrict this involution to prove that the bivariate generating function of **(2-1)**-avoiding inversion tables with respect to length and the number of $a_i = i - 1$ is given by Equation (1.0.3).

- Finally, we will use a variation of this involution to prove that the bivariate generating function of **(2-1)**-avoiding inversion tables is also given by Equation (1.0.2). This will prove the conjecture of Remmel and Kitaev.

Let a *left-nesting* of a matching X be a nesting $(a, b)(c, d) \in X$ such that $c = a + 1$. Claesson and Linusson [13] also conjectured that the distribution of left-nestings over the set of all matchings is given by the second-order Eulerian triangle. In an appendix, we will give a short inductive proof of this conjecture.

Note: Distinct proofs of the conjecture of Remmel and Kitaev using the upper-triangular matrices defined by Dukes and Parviainen were recently given independently by Jelínek [17] and by Yan [14]. Callan [8] surveyed bijections among objects equinumerous with matchings, as well as interpretations of the second-order Eulerian triangle among these objects. One of these bijections, when combined with a result of Gessel and Stanley [2], also cited by Callan, that the distribution of descents over Stirling permutations is given by the second-order Eulerian triangle, effectively proves the conjecture of Claesson and Linusson of the distribution of left-nestings over matchings and is similar to the proof included here. This work was done independently and we thank Mirkó Visontai for bringing the latter to our attention. See also [18]. A number of other articles or preprints relating to the

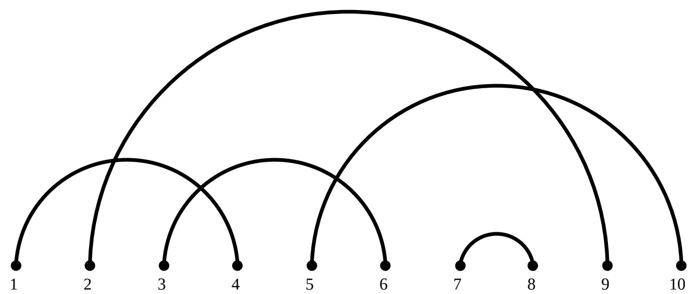

Figure 1.1: The matching $(1,4)(2,9)(3,6)(5,10)(7,8)$ on $[10]$.

combinatorics of Fishburn numbers have appeared during the review and editing processes for the final published version [19] of this article.

Chapter 2

A bijection between factorial matchings and inversion tables

Recall that a *matching* on $[2n]$ is a partition of $[2n]$ into a disjoint pairs. Throughout this article, we will specify a matching on $[2n]$ with a list of n disjoint ordered pairs. We will illustrate the matching $(a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)$ on $[2n]$ with a diagram of the n semicircular arcs joining a_i to b_i for all i. For example, the diagram of 5 arcs in Figure 1.1 illustrates the matching $(1, 4)(2, 9)(3, 6)(5, 10)(7, 8)$ on $[10]$. Note that the arcs do not have to be labeled for the diagram to illustrate a unique matching.

For a matching X on $[2n]$ and an integer a in $[2n]$, let $X(a)$ be the partner of a, i.e., the other integer in the same pair in X as a. For example, if X is the matching in Figure 1.1, then $X(2) = 9$. Note that $X(X(a)) = a$ for all a. Let a be an *opener* of X if $X(a) > a$ and let a be a *closer* of X if $a < X(a)$. For example, the openers

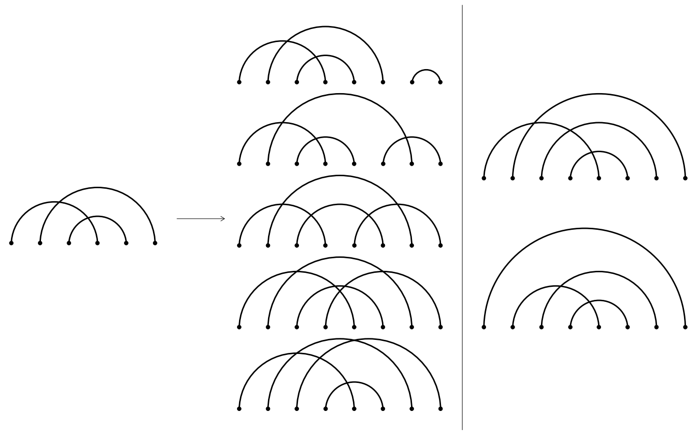

Figure 2.1: Matchings on [6] and [8]

of the matching in Figure 1.1 are $1, 2, 3, 5, 7$ and the closers are $4, 6, 8, 9, 10$. When referring to a pair (a, b) in a matching X, we will always mean that the first integer is the opener and the second integer is the closer, so we assume without loss of generality that $a < b$.

Given a matching X on $[2n]$ and an integer k in $[2n + 1]$, let a matching X' on $[2n + 2]$ be *constructed* from X and k by adding the arc $(k, 2n + 2)$ to X and re-labeling if necessary. For example, if X is the matching on the left-hand side of Figure 2.1, then the matchings on the right-hand side of Figure 2.1 are constructed from X and $7, 6, 5, 4, 3, 2, 1$, respectively. Alternately, let a matching X' be constructed from X and k if and only if it consists of the pair $(k, 2n+2)$ and the pairs

- (a, b) if $(a, b) \in X$ and $b < k$. Then let (a, b) in X' be the corresponding arc of (a, b) in X.

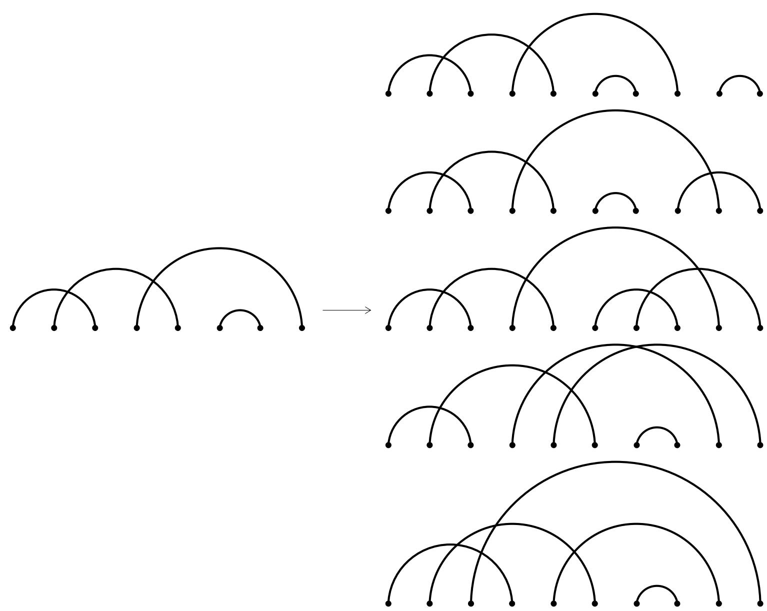

Figure 2.2: The non-left-nesting matching $(1,3)(2,5)(4,8)(6,7)$ on $[8]$ and the non-left-nesting matchings on $[10]$ constructed from it.

- $(a, b+1)$ if $(a, b) \in X$ and $a < k \leq b$. Then let $(a, b+1)$ in X' be the corresponding arc of (a, b) in X.

- $(a+1, b+1)$ if $(a, b) \in X$ and $a \leq k$. Then let $(a+1, b+1)$ in X' be the corresponding arc of (a, b) in X.

Clearly every matching on $[2n+2]$ is constructed from a matching on $[2n]$ and an integer in $[2n+1]$. Note that if X' is a matching on $[2n+2]$ constructed from X and k then every arc in X' except for $(k, 2n+2)$ corresponds to an arc in X.

Recall that a *left-nesting* of a matching X is a pair of pairs $(a, b)(a+1, d)$ in X with $d < b$. For example, $(2, 9)(3, 6)$ is the only left-nesting in Figure 1.1. Claesson and Linusson [13] proved that there are $n!$ non-left-nesting-matchings on $[2n]$. In fact, they proved a deeper structural connection between non-left-nesting matchings and permutations, which requires a few more definitions to state. Let an *inversion table* of length n be a sequence of non-negative integers $a_1 a_2 \cdots a_n$ such that $a_i \leq i - 1$.

Let the function ϕ from the set of non-left-nesting matchings on $[2n]$ to the set of inversion tables be defined as follows:

Given a non-left-nesting matching X on $[2n]$, let $c_1 < c_2 < \ldots < c_n$ be the increasing arrangement of the closers of X. For each i in $[n]$, let a_i be the number of closers of X to the left of the partner of the i-th closer of X. Let $\phi(X) = a_1 a_2 \cdots a_n$. For example, ϕ gives 0021 when applied to the matching on the left half of Figure 2.2, and ϕ gives $00214, 00213, 00212, 00211, 00210$ (respectively) when applied to the matchings on the right half of Figure 2.2 from top to bottom. Alternately, letting $c_0 = 0$ and $c_{n+1} = n$, let $a_i = j$ if and only if $c_j < X(c_i) < c_{j+1}$.

Note that the closers of the matching X on the left-hand side of Figure 2.2 are $3, 5, 7, 8$, and that the matchings on the right-hand side are constructed from X and the integers $9, 8, 7, 5, 3$. This suggests the structural relationship between non-left-nesting matchings and inversion tables proved by Claesson and Linusson.

Claim 2.0.1. *Let X be a non-left-nesting matching on $[2n]$. Let $c_1 < c_2 < \ldots < c_n$ be the increasing arrangement of the closers of X. Let $\phi(X) = a_1 a_2 \cdots a_n$. Let the matching X' on $[2n+2]$ be constructed from X and the integer k in $[2n+1]$.*

1. *X' is a non-left-nesting matching if and only if k is a closer of X or $k = 2n+1$.*

2. *If $k = c_i$, then $\phi(X') = a_1 a_2 \cdots a_n(i-1)$. If $k = 2n+1$, then $\phi(X') = a_1 a_2 \cdots a_n(n)$.*

Therefore, by induction, ϕ is a bijection between non-left-nesting matchings on $[2n]$ and inversion tables of length n. In particular, there are $n!$ non-left-nesting matchings on $[2n]$.

Proof. 1. If k is not an opener of X, then the corresponding arcs of a left-nesting of X' form a left-nesting of X, and vice-versa. Since X is a non-left-nesting matching, so is X'. If k is an opener of X, then $(k, 2n+2)(k+1, X(k)+1)$ is a left-nesting of X'. Therefore X' is a non-left-nesting matching if and only if k is not an opener of X, or if and only if k is a closer of X or $k = 2n+1$.

2. Adding the arc $(k, 2n+2)$ to X adds a new opener, k, to the integers from 1 to $2n$. This does not change how many closers are before the partners of the first n closers of X. Therefore the first n entries of $\phi(X')$ are $a_1 a_2 \cdots a_n$. If $k = c_i$ for some i in $[n]$, then there are $i-1$ closers to the left of k, the partner of the $(n+1)$-st closer $2n+2$, and therefore $\phi(X') = a_1 a_2 \cdots a_n(i-1)$. If $k = 2n+1$, then there are n closers to the left of k, and therefore $\phi(X') = a_1 a_2 \cdots a_n(n)$. This exhausts all $n+1$ possible choices for k. By induction, this proves that ϕ is a bijection between non-left-nesting matchings and inversion tables.

□

Let a *k-neighbor-nesting* be a nesting $(a, b)(c, d)$ with $c - a \leq 2$. If $(a, b)(c, d)$ is a 2-neighbor-nesting and $c - a = 1$, then $(a, b)(c, d)$ is a left-nesting. Therefore the set of matchings on $[2n]$ with no 2-neighbor-nestings is a subset of the matchings on $[2n]$ with no left-nestings. We can therefore restrict the bijection ϕ to the set of non-2-neighbor-nesting matchings.

Let the inversion table $a_1 a_2 \cdots a_n$ have a *j-occurrence* of the pattern **(2-1)** for a fixed integer j if there exist integers $p < q$ such that $a_p = j + 1$ and $a_q = j$. Let an inversion table be ***(2-1)**-avoiding* if it does not have a j-occurrence of **(2-1)** for any j. Let T_n be the set of **(2-1)**-avoiding inversion tables of length n.

Theorem 2.0.2. *Let X be a non-left-nesting matching on $[2n]$.*

1. If X has at least one 2-neighbor-nesting, then $\phi(X) \notin T_n$.

2. If $\phi(X) \notin T_n$, then X has at least one 2-neighbor-nesting.

Therefore ϕ is a bijection between non-2-neighbor-nesting matchings on $[2n]$ and T_n.

Proof. Let $c_1 < c_2 < \ldots < c_n$ be the closers of X. Let $\phi(X) = a_1 a_2 \cdots a_n$.

1. Assume X has at least one 2-neighbor-nesting. This 2-neighbor-nesting must be of the form $(v, c_b)(v+2, c_d)$ with $v+2 < c_d$ and $d < b$. If $v+1$ is an opener of X, then there are two possibilities, as $X(v+1)$ cannot equal c_b:

   - $c_b > X(v+1)$, and so the pair $(v, c_b)(v+1, X(v+1))$ is a left-nesting of X, or

   - $X(v+1) > c_b > c_d$, and so the pair $(v+1, X(v+1))(v+2, c_d)$ is a left-nesting of X.

   Therefore $v+1$ must be a closer of X. Let $c_i = v+1$. Therefore there are $i-1$ closers less than $v = X(c_b)$ and i closers less than $v+2 = X(c_d)$. Therefore $a_d = i$, $a_b = i-1$, and there is an $(i-1)$-occurrence of **(2-1)** in $a_1 a_2 \cdots a_n$, so $\phi(X) \notin T_n$.

2. Assume $\phi(X) \notin T_n$. Then for some $p < q$ and some j, $a_p = j+1$ and $a_q = j$. Therefore $c_j < X(c_q) < c_{j+1} < X(c_p)$. Therefore every integer in the intervals $[X(c_q)+1, c_{j+1}-1]$ and $[c_{j+1}+1, X(c_p)-1]$ is an opener.

   We will examine the partners of the integers in each interval in turn. We begin with the interval $[X(c_q)+1, c_{j+1}-1]$. The smallest integer in the interval is

$X(c_q)+1$. If $X(X(c_q)+1) < c_q$, then the pair $(X(c_q), c_q)(X(c_q)+1, X(X(c_q)+1))$ is a left-nesting of X. Therefore $X(X(c_q)+1) > c_q$. Similarly, the second-smallest integer in the interval is $X(c_q)+2$. If $X(X(c_q)+2) < X(X(c_q)+1)$, then the pair $(X(c_q)+1, X(X(c_q)+1))(X(c_q)+2, X(X(c_q)+2))$ is a left-nesting of X. Therefore $X(X(c_q)+2) > X(X(c_q)+1) > c_q$. By repeating this argument up to the largest integer in the interval, $c_{j+1}-1$, we have that $X(c_{j+1}-1) > X(c_{j+1}-2) > \ldots > X(X(c_q)+2) > X(X(c_q)+1) > c_q$. Therefore $X(c_{j+1}-1) > c_q$.

We now examine the partners of the integers in the interval $[c_{j+1}+1, X(c_p)-1]$. The largest integer in the interval is $X(c_p)-1$. If $X(X(c_p)-1) > c_p$, then the pair $(X(c_p)-1, X(X(c_p)-1))(X(c_p), c_p)$ is a left-nesting of X. Therefore $X(X(c_p)-1) < c_p$. Similarly, the second-largest integer in the interval is $X(c_q)-2$. If $X(X(c_p)-2) > X(X(c_p)-1)$, then the pair $(X(c_p)-2, X(X(c_p)-2))(X(c_p)-1, X(X(c_p)-1))$ is a left-nesting of X. Therefore $X(X(c_p)-2) < X(X(c_p)-1) < c_p$. By repeating this argument down to the smallest integer in the interval, $c_{j+1}+1$, we have that $X(c_{j+1}+1) < X(c_{j+1}+2) < \ldots < X(X(c_p)-2) < X(X(c_p)-1) < c_p$. Therefore $X(c_{j+1}+1) < c_p$.

By combining these two inequalities, we have that $X(c_{j+1}+1) < c_p < c_q < X(c_{j+1}-1)$. Therefore the pair $(c_{j+1}-1, X(c_{j+1}-1))(c_{j+1}+1, X(c_{j+1}+1))$ is a 2-neighbor-nesting of X. Therefore, if $\phi(X) \notin T_n$, then X has at least one 2-neighbor-nesting.

This proves that ϕ is a bijection between the set of non-2-neighbor-nesting matchings on $[2n]$ and T_n.

$\square$

Chapter 3

The main generating function

In this section, we will prove that the Fishburn numbers enumerate the set of **(2-1)**-avoiding inversion tables. In particular, we will prove that

$$\sum_{n=0}^{n} |T_n| t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1-t)^i).$$

First, we will define a class of diagrams with statistics that will give a visual interpretation to the right-hand-side of this equation. Next, we will define a statistic-preserving signed involution on this class of diagrams such that there is a trivial bijection between the set of fixed points and T_n. This will complete our proof.

3.1 Fishburn diagrams

Let a *Fishburn diagram* be a Young diagram of a staircase partition with dots placed in the squares such that each square has at most one dot and each column has at

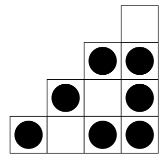

Figure 3.1: A Fishburn diagram $\{0\}\,\{1\}\,\{0,2\}\,\{0,1,2\} \in F_7$ with signed t-weight $(-1)^3t^7$.

least one dot. See Figure 3.1 for an example with the staircase partiton $(1, 2, 3)$. Let F be the set of all Fishburn diagrams, with F_n the subset of F consisting of Fishburn diagrams with precisely n dots. Alternately, let F be the set of sequences of sets $A = A_1A_2 \cdots A_m$, with m unfixed, such that, for all i, $A_i \subset \{0, 1, \ldots, i-1\}$ and $A_i \neq \emptyset$. Let F_n be the subset of F consisting of Fishburn diagrams $A_1A_2 \cdots A_m$ such that $\sum_{i=1}^{m} |A_i| = n$. We will use these definitions interchangeably using the obvious correspondence that the Fishburn diagram in Figure 3.1 corresponds to the sequence $\{0\}\,\{1\}\,\{0,2\}\,\{0,1,2\}$.

Note that Fishburn diagrams in F_n with precisely one dot per column must have length n, and that $A_i = \{a_i\}$ gives a trivial bijection between the set of such Fishburn diagrams and the set of inversion tables of length n.

3.2 A weighted sum over Fishburn diagrams

Let the two statistics dots and columns on F be defined as follows: If $A = A_1A_2 \cdots A_m$ is a Fishburn diagram in F, then let $\text{dots}(A) = \sum_{i=1}^{m} |A_i|$, and let columns$(A) = m$. Let the signed t-weight of A be given by $t^{\text{dots}(A)}(-1)^{\text{dots}(A)-\text{columns}(A)}$, or, by $(-t)^{\text{dots}(A)}(-1)^{\text{columns}(A)}$. We can think of the unsigned t-weight as the number of dots in the Fishburn diagram. We can think of the sign as the parity of the number of "extra" dots beyond the minimum of one per column. For example, the signed t-weight of the Fishburn diagram in Figure 3.1 is $(-1)^3 t^7$.

We can now interpret the right-hand-side of Equation (1.0.1) as a weighted sum over F, using the following lemma.

Lemma 3.2.1.

$$\begin{aligned}\sum_{A \in F} t^{\text{dots}(A)}(-1)^{\text{dots}(A)-\text{columns}(A)} &= \sum_{n=0}^{\infty} \sum_{A_1 A_2 \cdots A_m \in F_n} t^n (-1)^{n-m} \\ &= 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1-(1-t)^i).\end{aligned}$$

Proof. The first equality is trivial.

For any fixed integer m, consider the weighted sum of all Fishburn diagrams with m columns. Begin by assigning each column a weight of -1. A given column A_i has i squares, each of must contain either nothing, contributing a weight of 1, or a dot, contributing a weight of $-t$. These choices are made independently, with the only condition being that A_i cannot be empty. Therefore the weighted sum over all possible choices for A_i is $1-(1-t)^i$, with the 1 serving to cancel out the possibility that A_i is empty.

Taking the product of these weighted sums, we have that

$$(-1)^m \sum_{A_1 A_2 \cdots A_m \in F} (-t)^{|A_1|+|A_2|+\cdots+|A_m|} = \prod_{i=1}^{m} (1-(1-t)^i).$$

Taking the summation over m on each side, we have that

$$\sum_{m=0}^{\infty}(-1)^m \sum_{A_1A_2\cdots A_m\in F}(-t)^{|A_1|+|A_2|+\cdots+|A_m|} = 1+\sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-t)^i).$$

By changing the order of summation, and using the fact that $(-t)^n(-1)^m = t^n(-1)^{n-m}$, we have that

$$\sum_{m=0}^{\infty}(-1)^m \sum_{A_1A_2\cdots A_m\in F}(-t)^{|A_1|+|A_2|+\cdots+|A_m|} = \sum_{n=0}^{\infty}\sum_{A_1A_2\cdots A_m\in F_n} t^n(-1)^{n-m}.$$

This proves the lemma.

□

3.3 An Involution on Fishburn Diagrams

We can now prove that the Fishburn numbers enumerate **(2-1)**-avoiding inversion tables. Extending our earlier definitions, let a Fishburn diagram $A = A_1A_2\cdots A_m$ have a j-occurrence of the pattern **(2-1)** if there exist integers $p < q$ such that $j+1 \in A_p$ and $j \in A_q$. Let a Fishburn diagram be **(2-1)**-avoiding if it does not have a j-occurrence of **(2-1)** for any j.

Given $A = A_1A_2\cdots A_m \in F_n$, let the function ψ_n on F_n be defined as follows: If A is **(2-1)**-avoiding and has precisely one dot per column, then let $\psi_n(A) = A$. Otherwise, let j be the smallest integer such that at least one of the following two conditions hold:

- There is a column containing j and at least one other integer. In this case,

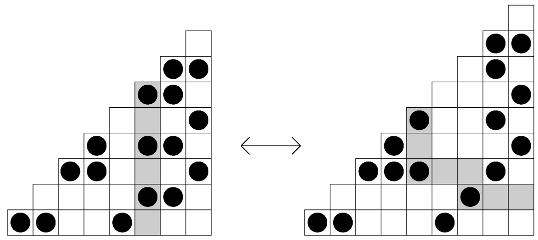

Figure 3.2: $\psi_{16}(\{0\}\,\{0\}\,\{2\}\,\{2,3\}\,\{0\}\,\{1,3,5\}\,\{1,3,5,6\}\,\{2,4,6\}) = \{0\}\,\{0\}\,\{2\}\,\{2,3\}\,\{2,4\}\,\{0\}\,\{1\}\,\{2,4,6,7\}\,\{3,5,7\}$.

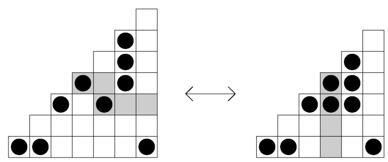

Figure 3.3: $\psi_9(\{0\}\,\{0\}\,\{2\}\,\{3\}\,\{2\}\,\{3,4,5\}\,\{0\}) = \{0\}\,\{0\}\,\{2\}\,\{2,3\}\,\{2,3,4\}\,\{0\}$.

let A_i be the leftmost such column. Alternately, i is the smallest integer such that $j \in A_i$ and $|A_i| > 1$.

- There is at least one j-occurrence of **(2-1)**. In this case, let the q-th column be the rightmost column containing j. Let the p-th column be the rightmost column to its left containing $j+1$. Alternately, q is the largest integer such that $j \in A_q$ and p is the largest integer less than q such that $j+1 \in A_p$.

We distinguish the two possible cases:

Case 1: There is *at least one* column in A containing j and at least one other integer, with A_i the leftmost such column. We will construct a new Fishburn diagram with a corresponding occurrence of **(2-1)**.

By minimality, j is the smallest integer in A_i. Let $j + R$ be the second-smallest integer in A_i. Let $B = B_1 B_2 \cdots B_{m+1}$ be the Fishburn diagram defined as follows: Add a new column B_{i-R+1} in between the $(i-R)$-st and $(i-R+1)$-st columns of the original Fishburn diagram A. Move the dots in A_i above height j to this new column, lowering them if necessary so that the dot originally at height $j+R$ in A_i is now at height $j+1$ in B_{i-R+1}. Add a blank row at height $j+1$ in between B_{i-R+1} and A_i (including A_i), moving any dots at height $j+1$ or above up one row. Add a blank row at height j after A_i, moving any dots at height j or above up one row. Let $B = B_1 B_2 \cdots B_{m+1}$ be the resulting Fishburn diagram. Let $\psi_n(A) = B$.

For example, see Figure 3.2. The Fishburn diagram on the left is

$$A = \{0\}\,\{0\}\,\{2\}\,\{2,3\}\,\{0\}\,\{1,3,5\}\,\{1,3,5,6\}\,\{2,4,6\} \in F_{16}.$$

Then the smallest integer j such that there is either a column containing j and at least one other integer or a j-occurrence of **(2-1)** is 1. There is a column containing 1 and at least one other integer, with A_6 the leftmost such column. Because 3 is the second-smallest integer in A_6, $R = 2$. We add a new column B_5 in between A_4 and A_5 and move the dots 3 and 5 from A_6 into this column, lowering them so the dot originally at height 3 in A_6 is now at height 2 in B_5 (and therefore the dot originally at height 5 in A_6 is now at height 4 in B_5). We add a blank row at height 2 in between B_5 and A_6 (including A_6), moving any dots at height 2 or height up one row, although as it happens there are no such dots here. We also add a blank row at height 1 after A_6, moving any dots at height 1 or above up one row, so A_7 becomes $\{2,4,6,7\}$ and A_8 becomes $\{3,5,7\}$. Let $B = B_1B_2\cdots B_9$ be the resulting Fishburn diagram, which is on the right-hand side of Figure 3.2. Then

$$\psi_n(A) = B = \{0\}\,\{0\}\,\{2\}\,\{2,3\}\,\{2,4\}\,\{0\}\,\{1\}\,\{2,4,6,7\}\,\{3,5,7\} \in F_{16}.$$

Alternately, the sets B_L are defined as follows for each $L \in [m+1]$:

- $B_L = A_L$ for $L \in [1, i-R]$.
- $B_L = \{s - R + 1 : s \in A_i, s \neq j\}$ for $L = i - R + 1$.
- $B_L = \{s : s \in A_{L-1}, s < j+1\} \cup \{s+1 : s \in A_{L-1}, s \geq j+1\}$ for $L \in [i-R+2, i]$.
- $B_L = \{j\}$ for $L = i+1$.
- $B_L = \{s : s \in A_{L-1}, s < j\} \cup \{s+1 : s \in A_{L-1}, s \geq j\}$ for $L \in [i+2, m+1]$.

Let $B = B_1B_2\cdots B_{m+1}$. Let $\psi_n(A) = B$.

Case 2: There are *no* columns in A containing j and at least one other integer. Therefore there is at least one j-occurrence of **(2-1)**, with A_q the rightmost column containing j. We will construct a new Fishburn diagram with a corresponding column that contains j and at least one other integer.

Because there is at least one j-occurrence of **(2-1)**, with A_q the rightmost column containing j, A has a blank at height j after A_q. Remove this blank row, moving any dots at heights $j+1$ or higher down one row. The column A_p is the rightmost column to the left of A_q containing $j+1$. Therefore there is a blank row in A at height $j+1$ in between A_p and A_q (including A_q). Remove this blank row, moving any dots at heights $j+2$ or higher down one row. By minimality, $j+1$ is the smallest integer in A_p. Move the dots in A_q to A_p, raising them if necessary so that the dot originally at height $j+1$ in A_p is now at height $j+q-p$ in A_q. Remove the column A_p. Let $B = B_1 B_2 \cdots B_{m-1}$ be the resulting Fishburn diagram.

For example, see Figure 3.3. The Fishburn diagram on the left is

$$A = \{0\}\,\{1\}\,\{2\}\,\{3\}\,\{2\}\,\{3,4,5\}\,\{0\} \in F_9.$$

Then the smallest integer j such that there is either a column containing j and at least one other integer or a j-occurrence of **(2-1)** is 2. There are no columns containing 2 and at least one other integer. Therefore we need to find the rightmost column containing 2 and the rightmost column to its left containing 3. The rightmost column containing 2 is A_5. The rightmost column containing to its left containing 3 is A_4. Therefore we remove the blank row at height 2 after A_5 and the blank row at height 3 in between A_3 and A_4 (including A_4). We then move the dot 3 from A_4 into A_5, and in this case it is not necessary to lower it so that it is at height 3 in A_5, so A_5 is now

$\{2,3\}$. Finally, we remove the (now-empty) column A_4. Let B the resulting Fishburn diagram. Let $B = B_1B_2\cdots B_6$ be the resulting Fishburn diagram, which is on the right-hand side of Figure 3.3. Then

$$\psi_9(A) = B = \{0\}\,\{1\}\,\{2\}\,\{2,3\}\,\{2,3,4\}\,\{0\}\,.$$

Alternately, the sets B_L are defined as follows for each $L \in [m-1]$:

- $B_L = A_L$ for $L \in [1, p-1]$.
- $B_L = \{s : s \in A_{L+1}, s < j+1\} \cup \{s-1 : s \in A_{L+1}, s > j+1\}$ for $L \in [p, q-2]$.
- $B_L = \{s+p-q-1 : s \in A_p\} \cup \{j\}$ for $L = q-1$.
- $B_L = \{s : s \in A_{L+1}, s < j\} \cup \{s-1 : s \in A_{L+1}, s > j\}$ for $L \in [q, m-1]$.

Let $B = B_1B_2\cdots B_{m-1}$. Let $\psi_n(A) = B$.

(See Figure 4.2 and Figure 4.6 for two other examples of ψ.)

The Fishburn diagram $\psi_n(A)$ has the same number of dots as A in every case. Therefore ψ_n is a map from F_n to F_n.

Let $\mathrm{Fix}(\psi_n)$ be the set of fixed points of ψ_n. Let the function ψ on F be defined by $\psi(A) = \psi_n(A)$ for $A \in F_n$. Let $\mathrm{Fix}(\psi) = \cup\,\mathrm{Fix}(\psi_n)$ be the set of fixed points of ψ.

Theorem 3.3.1. *1. The function ψ_n is an involution on F_n such that*

- *A Fishburn diagram* $A \in \mathrm{Fix}(\psi_n)$ *if and only if it is* ***(2-1)****-avoiding and has precisely one dot per column.*

- *If* $A = A_1 A_2 \cdots A_m \notin \mathrm{Fix}(\psi_n)$ *and* $\psi_n(A) = B = B_1 B_2 \cdots B_r$*, then* $r = m \pm 1$.

Therefore $|T_n| = |\mathrm{Fix}(\psi_n)|$.

2. *A Fishburn diagram is therefore either fixed by* ψ *or paired with another Fishburn diagram with the same unsigned t-weight but with the opposite sign. Therefore we have that*

$$\sum_{n=0}^{\infty} |T_n| t^n = \sum_{n=0}^{\infty} |\mathrm{Fix}(\psi_n)| t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1-t)^i).$$

Therefore the Fishburn numbers enumerate ***(2-1)****-avoiding inversion tables, and therefore also enumerate non-*2*-neighbor-nesting matchings.*

Proof. 1. We need only prove that ψ_n is an involution on F_n, or that, if $A = A_1 A_2 \cdots A_m \notin \mathrm{Fix}(\psi_n)$ and $\psi_n(A) = B = B_1 B_2 \cdots B_r$, then $\psi_n(B) = A$. Let j again be the smallest integer such that A has a column containing j and at least one other integer or a j-occurrence of **(2-1)**. For each of the two possible cases, we will prove that j is also the smallest integer such that B has a column containing j and at least one other integer or a j-occurrence of **(2-1)**, and that therefore $\psi_n(B) = A$.

Case 1: If there is at least one column in A containing j and at least one other integer, then $r = m + 1$, so $\psi_n(A) = B = B_1 B_2 \cdots B_{m+1}$. Let A_i

again be the leftmost column containing j and at least one other integer, with $j + R$ again the second-smallest integer in A_i. By construction, $B_{i+1} = \{j\}$ and B_{i-R+1} contains $j + 1$. Therefore there is at least one j-occurrence of **(2-1)** in B. Also, the distribution of dots at heights lower than j is unchanged by ψ_n from A to B, except that some dots at height j in A may now be at height $j + 1$ in B. This cannot add any new occurrences of **(2-1)** with an integer less than j or columns with more than one dot and an integer less than j. Therefore j remains the smallest integer such that B has a column containing j and at least one other integer or a j-occurrence of **(2-1)**.

As a result, we can calculate $\psi_n(B)$: Recall that, by construction, A_i is the leftmost column containing j and at least one other integer. To construct B, a blank row is added at height j after A_i, so there is no column after B_{i+1} containing j. The columns to the left of A_i in A are unchanged in B except for the addition of B_{i-R+1} and the blank row after it at height $j + 1$. Therefore no column in B contains j and at least one other integer. Therefore, to calculate $\psi_n(B)$, we need to find the rightmost column in B containing j and the rightmost column to its left containing $j + 1$.

By construction, the rightmost column in B containing j is B_{i+1} and the rightmost column to its left containing $j + 1$ is B_{i-R+1}. We therefore construct $\psi_n(B)$ by removing the blank row at height $j+1$ in between B_{i-R+1} and B_{i+1} (moving any dots at height $j+2$ or higher down one row) and the blank row at height j after B_{i+1} (moving any dots at height $j+1$ or higher down one row). We then move the dots in B_{i-R+1} to B_{i+1}, raising them if necessary so that the dot at height $j+1$ in B_{i-R+1} is now at height $j+(i+1)-(i-R+1)=j+R$ in B_{i+1}. Finally, we remove the now-empty column B_{i-R+1}. The resulting Fishburn diagram is $\psi_n(B)$. These operations simply reverses the construction of B from A, and therefore $\psi_n(B)=A$.

Case 2: If there is no column in A containing j and at least one other integer, then $r=m-1$, so $\psi_n(A)=B=B_1B_2\cdots B_{m-1}$. Let A_q again be the rightmost column containing j and A_p again be the rightmost column to its left containing $j+1$. By construction the smallest integer in B_{q-1} is j and the second-smallest integer is $j+q-p$. Therefore B has a column containing j and at least one other integer. Also, the distribution of dots at heights lower than j is unchanged by ψ_n from A to B, except that some dots at height $j+1$ in A may now be at height j in B. This change cannot add any new columns with more than one dot and an integer less than j.

We need only verify that this change cannot add new occurrences of **(2-1)** with an integer less than j. Because the distribution of dots at heights lower than j is unchanged, the only possibility is that there might be a new $(j-1)$-occurrence of **(2-1)** in B formed by a dot at height $j+1$ in A which is now at height j in B and a dot at height $j-1$ in B to its right. Assume there is such a $(j-1)$-occurrence of **(2-1)** in B. Then there exist integers $X < Y$ such that $j \in B_X$ and $j-1 \in B_Y$ with $j+1 \in A_{X+1}$ and $j-1 \in A_{Y+1}$. Since the dot at height $j \in B_X$ was at height $j+1$ in A, it must have been pushed down one row by the operation of ψ_n. Since there was a blank row at height $j+1$ between A_p and A_q, this dot must have been to the right of A_q. However, the dot at height $j-1$ in A_{Y+1}, to its right, would then form a $(j-1)$-occurrence of **(2-1)** in A along with the dot $j \in A_q$. By the minimality of j, this is impossible. Therefore j remains the smallest integer such that B has a column containing j and at least one other integer or a j-occurrence of **(2-1)**.

As a result, we can calculate $\psi_n(B)$: Recall that, by construction, A has no columns containing j and at least one other integer. The column B_{q-1} contains j and at least one other integer. The columns to the left of A_q in A are unchanged in B, except that A_p has been removed, as has the blank row in between A_p and A_q at height $j+1$. Therefore the leftmost column in B containing j and at least one other integer is B_{q-1}. The second-smallest integer in B_{q-1} is $j+q-p$. We therefore construct $\psi_n(B)$

by adding a new $(q-1-(q-p)+1)$-st column, or p-th column, and moving the dots above j in B_{q-1} to it, lowering them if necessary so the dot originally at height $j+q-p$ in B_{q-1} is now at height $j+1$. We then add a blank row at height $j+1$ in between the new p-th column and B_{q-1} (the q-th column of the new diagram), moving any dots at height $j+1$ or higher up one row, and a blank row at height j after B_{q-1}, moving any dots at height j or higher up one row. The resulting Fishburn diagram is $\psi_n(B)$. These operations simply reverse the construction of B from A, and therefore $\psi_n(B) = A$.

This proves that ψ_n is an involution on F_n.

2. Clearly, if ψ_n is an involution on F_n, then ψ is an involution on F. For all $A \in F$, we have proved that $\text{dots}(\psi(A)) = \text{dots}(A)$ and, if $A \notin \text{Fix}(\psi)$, then $\text{columns}(\psi(A)) = \text{columns}(A) \pm 1$. This proves that a Fishburn diagram is therefore either fixed by ψ or paired with another Fishburn diagram with the same unsigned t-weight but with the opposite sign.

Therefore we have that

$$\sum_{A \in F} t^{\text{dots}(A)}(-1)^{\text{dots}(A)-\text{columns}(A)} = \sum_{A \in \text{Fix}(\psi)} t^{\text{dots}(A)}(-1)^{\text{dots}(A)-\text{columns}(A)}.$$

If $A \in \text{Fix}(\psi)$, then $\text{columns}(A) = \text{dots}(A)$, and therefore

$$\sum_{A \in \text{Fix}(\psi)} t^{\text{dots}(A)}(-1)^{\text{dots}(A)-\text{columns}(A)} = \sum_{A \in \text{Fix}(\psi)} t^{\text{dots}(A)} = \sum_{n} |\text{Fix}(\psi_n)| t^n.$$

By Lemma 3.2.1, we have that

$$\sum_{A \in F} t^{\text{dots}(A)} (-1)^{\text{dots}(A) - \text{columns}(A)} = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1-t)^i).$$

Therefore

$$\sum_n |T_n| t^n = \sum_n |\operatorname{Fix}(\psi_n)| t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1-t)^i).$$

Therefore the Fishburn numbers enumerate **(2-1)**-avoiding inversion tables and non-2-neighbor-nesting matchings.

□

Chapter 4

The refined generating functions of the Fishburn numbers

Let $T_{n,d}$ be the subset of T_n consisting of **(2-1)**-avoiding inversion tables $a_1 a_2 \cdots a_n$ such that $a_i = i - 1$ for precisely d integers i. Using variations of the proof of Theorem 3.3.1 we will prove that

$$\sum_{n=0}^{\infty} \sum_{d=1}^{n} |T_{n,d}| t^n z^d = \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1-t)^{i-1}(1-zt)), \tag{4.0.1}$$

and also that

$$\sum_{n=0}^{\infty} \sum_{d=1}^{n} |T_{n,d}| t^n z^d = 1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^{k} (1 - (1-t)^i). \tag{4.0.2}$$

This will prove the conjecture of Remmel and Kitaev that Equation (1.0.2) and Equation (1.0.3) are equivalent.

4.1 The simpler refined generating function

We begin with a proof of the first refinement, Equation (4.0.1). Let $F_{n,d}$ be the set of Fishburn diagrams A with n dots, precisely d of which are on the top diagonal. For example, both Fishburn diagrams in Figure 3.2 are in $F_{16,5}$ and both Fishburn diagrams in Figure 3.3 are in $F_{9,4}$. Alternately, let $F_{n,d}$ be the subset of F_n consisting of Fishburn diagrams $A = A_1 A_2 \cdots A_m$ such that $i - 1 \in A_i$ for precisely d integers i. The natural bijection $A_i = \{a_i\}$ between T_n and the subset of F_n consisting of **(2-1)**-avoiding Fishburn diagrams with precisely one dot per column is also a bijection between $T_{n,d}$ and the subset of $F_{n,d}$ consisting of **(2-1)**-avoiding Fishburn diagrams with precisely one dot per column.

Let $A \in F$ be a Fishburn diagram. Let the statistic diagonal be defined by $\text{diagonal}(A) = |\{i : i - 1 \in A_i\}|$. We can think of diagonal(A) as the number of dots on the top diagonal of A. Let the signed t, z-weight of A be

$$t^{\text{dots}(A)} z^{\text{diagonal}(A)} (-1)^{\text{dots}(A) - \text{columns}(A)}.$$

We can now interpret the right-hand-side of Equation (4.0.1) as a weighted sum over F, using the following lemma.

Lemma 4.1.1.

$$\begin{aligned} \sum_{A \in F} t^{\text{dots}(A)} z^{\text{diagonal}(A)} (-1)^{\text{dots}(A) - \text{columns}(A)} &= \sum_{n=0}^{\infty} \sum_{A_1 A_2 \cdots A_m \in F_{n,d}} t^n z^d (-1)^{n-m} \\ &= 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1-t)^{i-1}(1 - zt)). \end{aligned}$$

Proof. This proof is identical to the proof of Lemma 3.2.1, except that a dot placed at the top of its column now contributes a weight of $-tz$ instead of a weight of $-t$.

□

We can now prove Equation (4.0.1). Let $\psi_{n,d}$ be the restriction of ψ_n to $F_{n,d}$. Let $\mathrm{Fix}(\psi_{n,d})$ be the set of fixed points of $\psi_{n,d}$.

Theorem 4.1.2. *1. $\psi_{n,d}$ is an involution on $F_{n,d}$, and therefore $|T_{n,d}| = |\mathrm{Fix}(\psi_{n,d})|$.*

2. A Fishburn diagram A is therefore either fixed by ψ or paired with another Fishburn diagram with the same unsigned t,z-weight but with the opposite sign. Therefore we have that

$$\sum_{n=0}^{\infty}\sum_{d=1}^{n}|T_{n,d}|t^n z^d = \sum_{n=0}^{\infty}\sum_{d=1}^{n}|\mathrm{Fix}(\psi_{n,d})|t^n z^d = \sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-t)^{i-1}(1-zt)).$$

Proof. We need only prove that $\psi_{n,d}$ is an involution on $F_{n,d}$, as the proof of the rest is identical to the proof of Theorem 3.3.1. Because ψ_n is an involution, we can assume that $A = A_1A_2\cdots A_m \in F_{n,d}$ and $\psi_n(A) = B = B_1B_2\cdots B_{m+1}$. Then we need only prove that $B \in F_{n,d}$. In other words, we need only consider one of the two cases in the proof of Theorem 3.3.1.

Once again, let j be the smallest integer such that A has a column containing j and at least one other integer or a j-occurrence of **(2-1)**. By assumption, $\psi_n(A) = B = B_1B_2\cdots B_{m+1}$. Therefore, by the proof of Theorem 3.3.1, A has a column containing j and at least one other integer. Let A_i again be the leftmost column containing j with $j+R$ again the second-smallest integer in A_i. There is a dot at the top of its column in B if and only if there is a dot at the top of the corresponding column in A, letting B_{i-R+1} correspond to A_i, with no column in A corresponding to the column B_{i+1} (since $B_{i+1} = \{j\}$, there cannot be a dot at the top, so we do not need a corresponding column in A).

Alternately, $L-1 \in B_L$ if and only if

- $L-1 \in A_L$ and $1 \leq L \leq i-R$, or
- $i-1 \in A_i$ and $L = i-R+1$, or
- $L-2 \in A_{L-1}$, $L \neq i+1$, and $i-R+2 \leq L \leq m+1$.

Since A has d dots on the top diagonal, B does as well, and therefore $B \in F_{n,d}$. Therefore $\psi_{n,d}$ is an involution on $F_{n,d}$, and the rest follows.

□

This proves Equation (4.0.1).

4.2 Three ways of defining $F^*_{n,d}$

We will now turn our attention to proving Equation (4.0.2). We will begin by defining a set $F^*_{n,d}$ that will give a visual interpretation to the right-hand side of the equation. Let $F^*_{n,d}$ be the subset of $F_{n,d}$ consisting of Fishburn diagrams A such that no dot on the top diagonal of A contributes to a column with more than one dot or to an occurrence of **(2-1)**. For example, Figure 4.1 shows the Fishburn diagram

$$\{0\}\,\{0\}\,\{2\}\,\{3\}\,\{3\}\,\{3,4\}\,\{6\}\,\{7\}\,\{8\}\,\{0,3,8\} \in F^*_{13,6}.$$

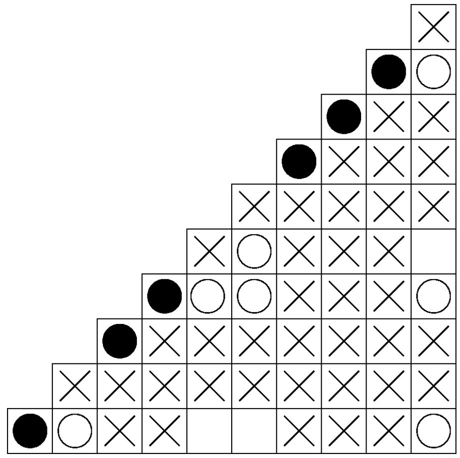

Figure 4.1: $\{0\}\,\{0\}\,\{2\}\,\{3\}\,\{3\}\,\{3,4\}\,\{6\}\,\{7\}\,\{8\}\,\{0,3,8\} \in F^*_{13,6}$.

The 6 dots on the top diagonal have been colored black and the 7 other dots have been colored white. We have placed an X in any square in every square below a black dot and in every square one row below, and to the right of, a black dot. Placing any dots in these squares would result in a Fishburn diagram that is not in $F^*_{n,d}$.

Alternately, let $F^*_{n,d}$ be the subset of $F_{n,d}$ consisting of Fishburn diagrams $A = A_1A_2\cdots A_m \in F_{n,d}$ such that, for all L, if $L-1$ is in A_L, then $A_L = \{L-1\}$ and, if $i > L$, then $L-2 \notin A_i$.

Note that the squares in Figure 4.1 without an X or a black dot form a staircase partition $(1, 2, 3, 4)$. In particular, the 4 indexes of columns without a dot at the top are $2, 5, 6$, and 10, and there is an X in every square of of these columns except at the heights $0, 3, 4$ and 8. This suggests the following claim, which will allow an alternative definition of the set $F^*_{n,d}$.

Claim 4.2.1. *Let $A = A_1A_2\cdots A_m$ be a Fishburn diagram in $F^*_{n,d}$. Let $d_1 < d_2 < \ldots < d_{m-d}$ be the increasing rearrangement of the $m-d$ indexes of columns of A without a dot at the top. Then $s \in A_{d_i}$ only if $s+2 \in \{d_1, d_2, \cdots, d_i\}$.*

In particular, a given column A_{d_i} can only have dots at the i possible heights $d_1-2, d_2-2, \ldots, d_i-2$, and these squares form a Fishburn diagram within A.

Proof. Let s be an integer in A_{d_i} for some fixed s and i. Because A is a Fishburn diagram, $s \leq d_i - 1$. Because d_i is, by definition, the index of a column without a dot at the top, $s \leq d_i - 2$. Therefore $s+2 \notin \{d_{i+1}, d_{i+2}, \ldots, d_{m-d}\}$.

Assume that $s+2 \neq d_i$. Then $s \leq d_i - 3$. Therefore A_{d_i} is to the right of A_{s+2}. Since $s \in A_{d_i}$, s appears to the right of A_{s+2}. Further assume that $s+2 \notin \{d_1, d_2, \cdots, d_{i-1}\}$. Then $s+2$ cannot be the index of any column without a dot at the top. Therefore it must be the index of a column with a dot at the top. Therefore $s+1 \in A_{s+2}$ and s appears to the right of A_{s+2}, so there is an s-occurrence of **(2-1)** in A. This contradicts the assumption that $A \in F^*_{n,d}$. Therefore either $s+2 = d_i$ or $s+2 \in \{d_1, d_2, \ldots, d_{i-1}\}$. More concisely, $s+2 \in \{d_1, d_2, \ldots, d_i\}$.

□

We can use Lemma 4.2.1 to give another alternative definition of the set: Let $G^*_{n,d}$ be the set of ordered pairs (α, A^*) such that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ is a composition of d into $k+1$ parts with α_1 nonzero and $A^* = A^*_1A^*_2\cdots A^*_k$ is a Fishburn diagram in F_{n-d}.

Let the map $f_{n,d}$ on $F^*_{n,d}$ be defined as follows: Given a Fishburn diagram

$A = A_1A_2\cdots A_m \in F^*_{n,d}$, let $d_1 < d_2 < \ldots < d_{m-d}$ again be the increasing rearrangement of the $m-d$ indexes of columns of A without a dot at the top. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m-d+1})$ be the distribution of dots on the top diagonal of A, with α_i the number of dots between $A_{d_{i+1}}$ and $A_{d_{i+2}}$, all of which must be on the top diagonal. There is a $(m-d)$-column Fishburn diagram within A consisting of the intersection of the columns $A_{d_1}, A_{d_2}, \ldots, A_{d_{m-d}}$ and the rows $d_1-2, d_2-2, \ldots, d_{m-d}-2$. Let A^* be this Fishburn diagram. Let $f_{n,d}(A) = (\alpha, A^*)$. For example, if A is the Fishburn diagram in Figure 4.1, then

$$f_{13,6}(A) = ((1,2,0,3,0), \{0\}\,\{1\}\,\{1,2\}\,\{0,1,3\}).$$

Alternately, let $\alpha_1 = d_1 - 1$, let $\alpha_i = d_i - d_{i-1} - 1$ for all i from 2 to $m-d$, and let $\alpha_{m-d+1} = m - d_{m-d}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m-d+1})$. Let $A^*_i = \{r - 1 : d_r - 2 \in A_{d_i}\}$ for all i in $[m-d]$. Let $A^* = A^*_1 A^*_2 \cdots A^*_{m-d}$. Let $f_{n,d}(A) = (\alpha, A^*)$.

Claim 4.2.2. *$f_{n,d}$ is a bijection between $F^*_{n,d}$ and $G^*_{n,d}$.*

Proof. Let $A = A_1A_2\cdots A_m$ be a Fishburn diagram in $F^*_{n,d}$. Let $d_1 < d_2 < \ldots < d_{m-d}$ again be the increasing rearrangement of the $m-d$ indexes of columns of A without a dot at the top. Let $f_{n,d}(A) = (\alpha, A^*)$. The diagram $A^* = A^*_1A^*_2\cdots A^*_{m-d}$ is a Fishburn diagram with one dot for every dot in A other than the d dots on the top diagonal. Therefore $A^* \in F_{n-d}$. The composition α is a composition of d into $m-d+1$ parts. By definition, $0 \notin A_{d_1}$, and since A_1 must equal $\{0\}$ for any Fishburn diagram, $d_1 > 1$. Therefore $\alpha_1 = d_1 - 1 > 0$, and $(\alpha, A^*) \in G^*_{n,d}$.

We need only define an inverse map $g_{n,d} : G^*_{n,d} \to F^*_{n,d}$ to prove that f is a bijection. Let (β, B^*) be an ordered pair in $G^*_{n,d}$ with $\beta = (\beta_1, \beta_2, \cdots, \beta_{k+1})$. Begin with a staircase partition with $k + d$ columns. Place β_1 dots into the top diagonal, beginning with the left-hand-corner. Skip one square of the top diagonal, then place β_2 dots on it. Continue in this fashion until all d dots have been placed according to α. Place an X in every square below a dot on the top diagonal and in the same column, as well as in every square one row below, and to the right of, a dot on the top diagonal. The remaining squares form a staircase partition with k columns. Put B^* onto this staircase partition. Let $g_{n,d}(\beta, B^*)$ be the resulting Fishburn diagram.

Alternately, let $e_i = \beta_1 + \beta_2 + \cdots + \beta_i + i$ for all i in $[k]$. Let $B^* = B^*_1 B^*_2 \cdots B^*_k$. Let sets B_i be defined as follows for all i in $[k + d]$: If $i \notin \{e_1, e_2, \ldots, e_k\}$, then let $B_i = \{i - 1\}$. If $i \in \{e_1, e_2, \ldots, e_k\}$, then let $i = e_j$. Let $B_{e_j} = \left\{e_r - 2 : r - 1 \in B^*_j\right\}$. Let $B = B_1 B_2 \cdots B_{k+d} = g_{n,d}(\beta, B^*)$.

There are d indexes not in $\{e_1, e_2, \ldots, e_k\}$, so diagonal$(B) = d$. There are d dots in the columns with these d indexes and $n - d$ dots in the columns with the indexes $e_1, e_2, \ldots, e_k$, so dots$(B) = n$. Therefore $B \in F_{n,d}$.

By construction, if $i - 1 \in B_i$, then $B_i = \{i - 1\}$. If $B_i = \{i - 1\}$ and $i - 2$ were in a column to the right of B_i, then it could not be at the top. Therefore $i - 2$ could only be in B_{e_j} for some j. The integer $i - 2 \in B_{e_j}$ if and only if, for some r, $i = e_r$ and $r \in B^*_j$. However, $i - 1 \in B_i$, so i cannot equal e_r for any r. Therefore $i - 2$ is not in any column to the right of B_i, and therefore $B \in F^*_{n,d}$. It is trivial that $g_{n,d}$ and $f_{n,d}$ simply reverse the operations of each other, and therefore $f_{n,d}$ is a bijection between $F^*_{n,d}$ and $G^*_{n,d}$.

□

For conciseness, if $f_{n,d}(A) = (\alpha, A^*)$, then we will sometimes refer to the latter as the "ordered pair interpretation" of A.

4.3 The more complicated refinement

We can now prove Equation (4.0.2). Let $F^* = \cup_{n,d} F^*_{n,d}$. Let $f(A) = f_{n,d}(A)$ for $A \in F^*_{n,d}$. Let Comp_1^M be the set of compositions with M parts, the first of which is nonzero. Along the lines of the previous proofs of Equation (1.0.1) and Equation (4.0.1), we can now interpret the right-hand side of Equation (4.0.2) as a weighted sum over F^*:

Lemma 4.3.1. *1. Let A be a Fishburn diagram in F^*. Let $f(A) = (\alpha, A^*)$. Then* $\text{diagonal}(A) = |\alpha|$, $\text{dots}(A) = \text{dots}(A^*) + |\alpha|$, *and* $\text{columns}(A) = \text{columns}(A^*) + |\alpha|$.

2. Therefore,

$$\sum_{A \in F^*} t^{\text{dots}(A)} z^{\text{diagonal}(A)} (-1)^{\text{dots}(A) - \text{columns}(A)} = 1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^{k} (1-(1-t)^i).$$

Proof. 1. If $A = A_1 A_2 \cdots A_m \in F^*_{n,d}$ and $f(A) = (\alpha, A^*) \in G^*_{n,d}$, then $\text{diagonal}(A) = d = |\alpha|$, $\text{dots}(A) = n = (n-d) + d = \text{dots}(A^*) + |\alpha|$, and $\text{columns}(A) = m = (m-d) + d = \text{columns}(A^*) + |\alpha|$.

2. By the first part of this lemma, and by Claim 4.2.2, we have that

$$\sum_{A\in F^*} t^{\text{dots}(A)} z^{\text{diagonal}(A)} (-1)^{\text{dots}(A)-\text{columns}(A)}$$

$$= \sum_{(\alpha,A^*)\in G^*} t^{\text{dots}(A^*)+|\alpha|} z^{|\alpha|} (-1)^{\text{dots}(A^*)-\text{columns}(A^*)}.$$

Since G^* can be seen as $\cup_{k=0}^{\infty} \left(\text{Comp}_1^{k+1} \times \{A^* \in F : \text{columns}(A^*) = k\}\right)$, we have that

$$\sum_{(\alpha,A^*)\in G^*} t^{\text{dots}(A^*)+|\alpha|} z^{|\alpha|} (-1)^{\text{dots}(A^*)-\text{columns}(A^*)} =$$
$$\sum_{k=0}^{\infty} \left(\sum_{\alpha\in\text{Comp}_1^{k+1}} (zt)^{|\alpha|} \right) \left(\sum_{\text{columns}(A^*)=k} t^{\text{dots}(A^*)} z^{\text{columns}(A^*)} (-1)^{\text{dots}(A^*)-\text{columns}(A^*)} \right).$$

By standard generating functions, we have that

$$\sum_{\alpha\in\text{Comp}_1^{k+1}} (zt)^{|\alpha|} = \frac{zt}{(1-zt)^{k+1}}.$$

Finally, from the proof of Lemma 3.2.1, we have that

$$\sum_{\text{columns}(A^*)=k} t^{\text{dots}(A^*)} z^{\text{columns}(A^*)} (-1)^{\text{dots}(A^*)-\text{columns}(A^*)} = \prod_{i=1}^{k} (1-(1-t)^i).$$

Therefore,

$$\sum_{A\in F^*} t^{\text{dots}(A)} z^{\text{diagonal}(A)} (-1)^{\text{dots}(A)-\text{columns}(A)} = 1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^{k} (1-(1-t)^i).$$

□

We can now prove Equation (4.0.2) using an involution. Unfortunately, we cannot use $\psi_{n,d}$, since $\psi_{n,d}$ is not a map from $F^*_{n,d}$ to itself. To illustrate this, see Figure 4.2. Let the Fishburn diagram on the left-hand side of Figure 4.2 be

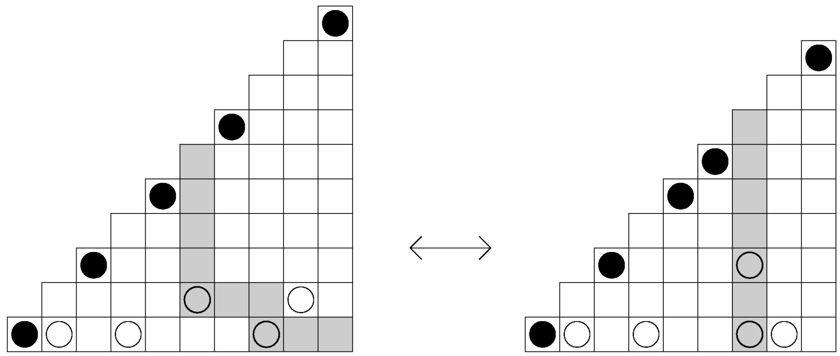

Figure 4.2: $\psi_{10,5}(\{0\}\{0\}\{2\}\{0\}\{4\}\{1\}\{6\}\{0\}\{1\}\{9\}) = \{0\}\{0\}\{2\}\{0\}\{4\}\{5\}\{0,2\}\{0\}\{8\}$.

$$A = \{0\}\{0\}\{2\}\{0\}\{4\}\{1\}\{6\}\{0\}\{1\}\{9\} \in F_{10,5}.$$

Note that $A = A_1A_2\cdots A_{10} \notin F^*_{10,5}$, because $2 \in A_3$ and $1 \in A_6$. However,

$$\psi_{10,5}(A) = \{0\}\{0\}\{2\}\{0\}\{4\}\{5\}\{0,2\}\{0\}\{8\},$$

which is the Fishburn diagram on the right-hand side of Figure 4.2. The Fishburn diagram $\psi_{10,5}(A) \in F^*_{10,5}$. Since $\psi_{n,d}$ is an involution, it follows that $\psi_{n,d}$ is not, in general, a map from $F^*_{n,d}$ to itself.

Therefore, we must define a new involution on $F^*_{n,d}$. To ensure that the new involution is a map from $F^*_{n,d}$ to itself, we will use the ordered-pair interpretations of Fishburn diagrams in $F^*_{n,d}$. First, we must understand the ordered pair interpretations of **(2-1)**-avoiding Fishburn diagrams with precisely one dot per column.

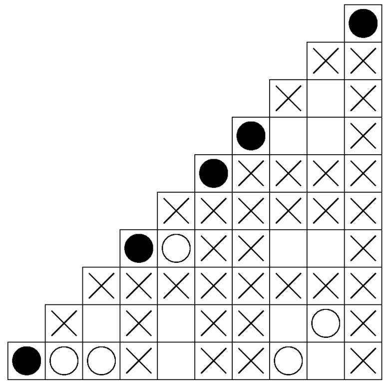

Figure 4.3: The **(2-1)**-avoiding $((1,0,1,2,0,1),\{0\}\{0\}\{2\}\{0\}\{1\}) \in F^*_{10,5}$.

For example, Figure 4.3 shows the Fishburn diagram

$$A = \{0\}\{0\}\{0\}\{3\}\{3\}\{5\}\{6\}\{0\}\{1\}\{9\}.$$

Let the ordered pair interpretation of A be

$$(\alpha, A^*) = ((1,0,1,2,0,1),\{0\}\{0\}\{2\}\{0\}\{1\}).$$

Note that $A^* = A^*_1A^*_2A^*_3A^*_4A^*_5$ has precisely one dot per column but also has a 1-occurrence of **(2-1)**, since $2 \in A^*_3$ and $1 \in A^*_5$. However, because $\alpha_3 > 0$, there is a row in A (with an X in every square) in between the white dot corresponding to the $2 \in A^*_3$ and the white dot corresponding to the $1 \in A^*_5$. This breaks up the occurrence of **(2-1)**, so A is both **(2-1)**-avoiding and has precisely one dot per column.

This suggests the following lemma, which allows us to interpret "problems" in a Fishburn diagram A as problems in its ordered pair interpretation (α, A^*).

Lemma 4.3.2. *Let $A = A_1 A_2 \cdots A_m$ be a Fishburn diagram in $F^*_{n,d}$ with ordered pair interpretation $(\alpha, A^*) \in G^*_{n,d}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{m-d+1})$ and let $A^* = A^*_1 A^*_2 \cdots A^*_{m-d}$.*

1. *A has precisely one dot per column if and only if A^* has precisely one dot per column.*
2. *A is **(2-1)**-avoiding if and only if $\alpha_{i+2} > 0$ for all i such that there is an i-occurrence of **(2-1)** in A^*.*

Proof.

1. Each column in A with a dot at the top has no other dots, and $|A_{d_i}| = |A^*_i|$ for all i in $[m-d]$.
2. Assume that there is a j-occurrence of **(2-1)** in A. Because $A \in F^*_{n,d}$, the dot at height $j+1$ must not be at the top of its column. Therefore there exist integers $p < q$ with $j+1 \in A_{d_p}$ and $j \in A_{d_q}$. By Claim 4.2.1, there exist Y and Z such that $d_Y - 2 = j$ and that $d_Z - 2 = j+1$, with $Y - 1 \in A^*_q$ and $Z - 1 \in A^*_p$. Therefore $d_Z = d_Y + 1$, so $Z = Y + 1$ and $Y \in A^*_p$. Since $Y \in A^*_p$ and $Y - 1 \in A^*_q$, A^* has a $(Y-1)$-occurrence of **(2-1)**. We also have that $d_{Y+1} = d_Z = d_Y + 1$. By definition, $\alpha_{Y+1} = d_{Y+1} - d_Y - 1$, and so $\alpha_{Y+1} = 0$. Therefore, if A has an occurrence of **(2-1)**, then there exists an integer Y such that A^* has a $(Y-1)$-occurrence of **(2-1)** and $\alpha_{Y+1} = 0$.

   Similarly, let i be such that there is an i-occurrence of **(2-1)** and $\alpha_{i+2} = 0$. Then there exist integers $R < S$ such that $i+1 \in A^*_R$ and $i \in A^*_S$. Then

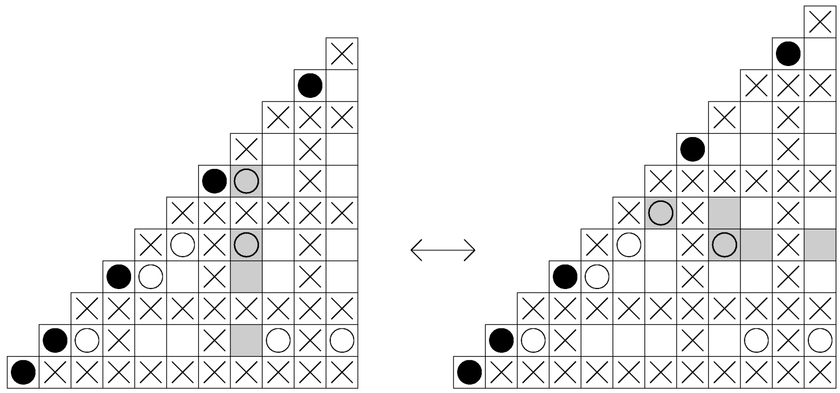

Figure 4.4: $\psi^*_{12,5}((2,1,0,1,0,1,0),\{0\}\{1\}\{2\}\{2,3\}\{0\}\{0\}) = ((2,1,0,0,1,0,1,0),\{0\}\{1\}\{2\}\{3\}\{2\}\{0\}\{0\})$.

$d_{i+2}-2 \in A_{d_R}$ and $d_{i+1}-2 \in A_{d_S}$. Because $\alpha_{i+2} = d_{i+2} - d_{i+1} - 1 = 0$, we have that $d_{i+2} = d_{i+1} + 1$. Therefore $d_{i+2} - 2 = d_{i+1} - 1 \in A_{d_R}$ and $d_{i+1} - 2 \in A_{d_S}$. Therefore A has a $(d_{i+1}-2)$-occurrence of **(2-1)**. Therefore A is **(2-1)**-avoiding if and only if $\alpha_{i+2} > 0$ for all i such that there is an i-occurrence of **(2-1)** in A^*.

□

We can now proceed to prove Equation (4.0.2) using an involution. Let the function $\psi^*_{n,d}$ be defined on $F^*_{n,d}$ as follows:

Let A be a Fishburn diagram in $F^*_{n,d}$ with ordered pair interpretation (α, A^*), with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ and $A^* = A^*_1 A^*_2 \cdots A^*_k$. If A^* has precisely one dot per column and $\alpha_{i+2} > 0$ for all i such that there is an i-occurrence of **(2-1)** in A^* then let $\psi^*_{n,d}(\alpha, A^*) = (\alpha, A^*)$.

Otherwise, let j be the smallest integer such that at least one of the following two conditions hold:

- There is a column in A^* containing j and at least one other integer. In this case, let A_i^* be the leftmost such column.
- There is at least one j-occurrence of **(2-1)** in A^* with $\alpha_{j+2} = 0$. In this case, let A_q^* be the rightmost column containing j. Let the A_p^* be the rightmost column to its left containing $j+1$.

We distinguish the two possible cases:

Case 1: There is at least one column in A^* containing j and at least one other integer, with A_i^* the leftmost such column. By minimality, j is the smallest integer in A_i^*. Let $j+R$ be the second-smallest integer in A_i^*. Let $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_{j+1}, 0, \alpha_{j+2}, \ldots, \alpha_{k+1})$. To construct a Fishburn diagram B^* from A^*, perform the same operation on A^* that ψ_{n-d} would, except using this j (which might lead to a different result, see below): Add a new column B_{i-R+1}^* in between the $(i-R)$-st and $(i-R+1)$-st columns of A^*. Move the dots in A_i^* above height j to this new column, lowering them if necessary so that the dot originally at height $j+R$ in A_i^* is now at height $j+1$ in B_{i-R+1}^*. Add a blank row at height $j+1$ in between B_{i-R+1}^* and A_i^* (including A_i^*), moving any dots at height $j+1$ or above up one row. Add a blank row at height j after A_i^*, moving any dots at height j or above up one row. Let B^* be the resulting Fishburn diagram. Let $\psi_{n,d}^*(\alpha, A^*) = (\beta, B^*)$.

For example, see Figure 4.4. Let the ordered pair interpretation of the Fishburn diagram on the left-hand side be

$$(\alpha, A^*) = ((2,1,0,1,0,1,0), \{0\}\,\{1\}\,\{2\}\,\{2,3\}\,\{0\}\,\{0\}) \in F^*_{12,5}.$$

The minimal j such that there exists a column in A^* with j and at least one other integer or a j-occurrence of **(2-1)** in A^* with $\alpha_{j+2} = 0$ is $j = 2$. The Fishburn diagram A^* has a column containing 2 and at least one other integer, with A^*_4 the leftmost such column. Therefore, let $\beta = (2,1,0,0,1,0,1,0)$. To construct a Fishburn diagram B^* from A^*, we perform the same operation on A^* as ψ_7, except that we use $j = 2$. More explicitly, the second-smallest integer in A^*_4 is 3. Therefore we insert a new column B^*_4 into A^* and move the dot at height 3 from A^*_4 into this new column, and it is not necessary to lower it. We add an blank row at height 3 in between the new column B^*_4 and A^*_4 (including A^*_4) and a blank row at height 2 after A^*_4. Let $B^* = \{0\}\,\{1\}\,\{2\}\,\{3\}\,\{2\}\,\{0\}\,\{0\}$ be the resulting Fishburn diagram. Then $\psi^*_{12,5}(\alpha, A) = (\beta, B^*)$. The Fishburn diagram with ordered pair interpretation (β, B^*) is on the right-hand side of Figure 4.4.

Alternately, let the sets B^*_L be defined as follows for each $L \in [k+1]$:

- $B^*_L = A^*_L$ for $L \in [1, i-R]$.
- $B^*_L = \{s - R + 1 : s \in A^*_i, s \neq j\}$ for $L = i - R + 1$.
- $B^*_L = \left\{s : s \in A^*_{L-1}, s < j+1\right\} \cup \left\{s+1 : s \in A^*_{L-1}, s \geq j+1\right\}$ for $L \in$

$[i-R+2,i]$.

- $B^*_L = \{j\}$ for $L = i+1$.
- $B^*_L = \{s : s \in A^*_{L-1}, s < j\} \cup \{s+1 : s \in^* A_{L-1}, s \geq j\}$ for $L \in [i+2, k+1]$.

Let $B^* = B^*_1 B^*_2 \cdots B^*_{k+1}$. Let $\psi^*_{n,d}(\alpha, A^*) = (\beta, B^*)$.

Case 2: There are no columns in A^* containing j and at least one other integer. Therefore there is at least one j-occurrence of **(2-1)** in A^* with $\alpha_{j+2} = 0$, with A^*_q the rightmost column containing j and A^*_p the rightmost column to its left containing $j+1$. Let $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_{j+1}, \alpha_{j+3}, \ldots, \alpha_{k+1})$. To construct a Fishburn diagram B^* from A^*, perform the same operation on A^* that ψ_{n-d} would, except using this j: There is a blank row in A^* at height j after A^*_q. Remove this blank row, moving any dots at heights $j+1$ or higher down one row. There is a blank row in A^* at height $j+1$ in between A^*_p and A^*_q (including A^*_q). Remove this blank row, moving any dots at heights $j+2$ or higher down one row. By minimality, $j+1$ is the smallest integer in A^*_p. Move the dots in A^*_q to A^*_p, raising them if necessary so that the dot originally at height $j+1$ in A^*_p is now at height $j+q-p$ in A^*_q. Remove the column A^*_p. Let B^* be the resulting Fishburn diagram. Let $\psi^*_{n,d}(\alpha, A^*) = (\beta, B^*)$.

For example, see Figure 4.5. Let the ordered pair interpretation of the Fish-

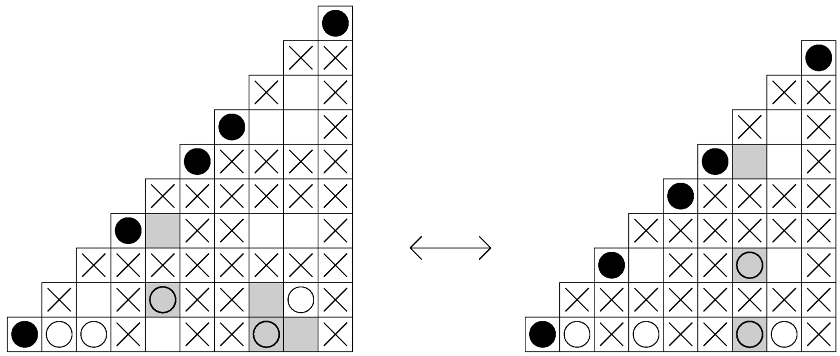

Figure 4.5: $\psi^*_{10,5}((1,0,1,2,0,1), \{0\}\{0\}\{1\}\{0\}\{1\}) = ((1,1,2,0,1), \{0\}\{0\}\{0,1\}\{0\})$.

burn diagram on the left-hand side be

$$(\alpha, A^*) = ((1,0,1,2,0,1), \{0\}\{0\}\{1\}\{0\}\{1\}) \in F^*_{10,5}.$$

The minimal j such that there exists a column in A^* with j and at least one other integer or a j-occurrence of **(2-1)** in A^* with $\alpha_{j+2} = 0$ is $j = 0$. The Fishburn diagram A^* does not have a column containing 0 and at least one other integer. Therefore, let $\beta = (1,1,2,0,1)$. The rightmost column in A^* containing 0 is A^*_4. The rightmost column to its left containing 1 is A^*_4. To construct a Fishburn diagram B^* from A^*, we perform the same operation on A^* as ψ_5, except that we use $j = 0$. More explicitly, we remove the blank row at height 0 after A^*_3 and the blank row at height 1 in between A^*_3 and A^*_4 (including A^*_4). We then move the dot 1 from A^*_3 into A^*_3, and in this case it is not necessary to lower it so that it is at height 1 in A^*_4. Finally, we remove the (now-empty) column A^*_3. Let $B^* = \{0\}\{0\}\{0,1\}\{0\}$ be the resulting

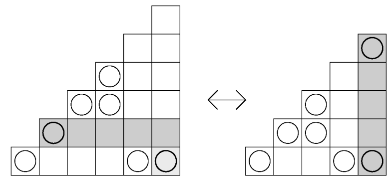

Figure 4.6: $\psi_7(\{0\}\,\{1\}\,\{2\}\,\{2,3\}\,\{0\}\,\{0\}) = \{0\}\,\{1\}\,\{1,2\}\,\{0\}\,\{0,4\}$.

Fishburn diagram. Then $\psi^*_{10,5}(\alpha, A) = (\beta, B^*)$. The Fishburn diagram with ordered pair interpretation (β, B^*) is on the right-hand side of Figure 4.5.

(Note that the Fishburn diagram on the right-hand side of Figure 4.5 is also the Fishburn diagram on the right-hand side of Figure 4.2.)

Alternately, let the sets B^*_L be defined as follows for each $L \in [k-1]$:

- $B^*_L = A^*_L$ for $L \in [1, p-1]$.
- $B^*_L = \{s : s \in A^*_{L+1}, s < j+1\} \cup \{s-1 : s \in A^*_{L+1}, s > j+1\}$ for $L \in [p, q-2]$.
- $B^*_L = \{s + p - q - 1 : s \in A^*_p\} \cup \{j\}$ for $L = q-1$.
- $B^*_L = \{s : s \in A^*_{L+1}, s < j\} \cup \{s-1 : s \in A^*_{L+1}, s > j\}$ for $L \in [q, k-1]$.

Let $B^* = B^*_1 B^*_2 \cdots B^*_{k-1}$. Let $\psi^*_{n,d}(\alpha, A^*) = (\beta, B^*)$.

In every case, the composition part of $\psi_{n,d}(\alpha, A^*)$ has the same sum as α, and the Fishburn diagram part of $\psi_{n,d}(\alpha, A^*)$ has the same number of dots as A^*. Therefore $\psi_{n,d}$ is a map from $F^*_{n,d}$ to $F^*_{n,d}$.

Note that the smallest j such that A^* has a column with j and at least one other integer or a j-occurrence of **(2-1)** with $\alpha_{j+2} = 0$ is not necessarily the smallest j such that A^* has a column with j and at least one other integer or a j-occurrence of **(2-1)**. In other words, $\psi^*_{n,d}$ is not simply an alternate description of ψ_{n-d}. To illustrate the difference, compare Figure 4.4 to Figure 4.6. Recall that the ordered pair interpretation of the Fishburn diagram on the left-hand side of Figure 4.4 is (α, A^*), with $\alpha = (2,1,0,1,0,1,0)$ and $A^* = \{0\}\{1\}\{2\}\{2,3\}\{0\}\{0\}$. Recall also that $\psi^*_{12,5}(\alpha, A) = (\beta, B^*)$, with $\beta = (2,1,0,0,1,0,1,0)$ and $B^* = \{0\}\{1\}\{2\}\{3\}\{2\}\{0\}\{0\}$.

However, $\psi_7(A^*) \neq B^*$. The left-hand side of Figure 4.6 shows A^* alone. The minimal j such that A^* has a column with j and at least one other integer or a j-occurrence of **(2-1)** is $j = 0$. Following the construction of ψ, we have that $\psi_7(A^*) = \{0\}\{1\}\{1,2\}\{0\}\{0,4\}$, the Fishburn diagram on the right-hand side of Figure 4.6.

Let $\mathrm{Fix}(\psi^*_{n,d})$ be the set of fixed points of $\psi^*_{n,d}$. Let the function ψ^* on F^* be defined by $\psi^*(A) = \psi^*_{n,d}(A)$ for $A \in F^*_{n,d}$. Let $\mathrm{Fix}(\psi^*) = \cup_{n,d}\,\mathrm{Fix}(\psi^*_{n,d})$ be the set of fixed points of ψ.

Theorem 4.3.3. *1. $\psi^*_{n,d}$ is an involution on $F^*_{n,d}$ such that, if (α, A^*) is a Fishburn diagram in $F^*_{n,d}$ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$, then*

- *(α, A^*) is in $\mathrm{Fix}(\psi^*_{n,d})$ if and only if A^* has precisely one dot per column and $\alpha_{i+2} > 0$ for all i such that there is an i-occurrence of **(2-1)** in A^*.*

- *If* $(\alpha, A^*) \notin \mathrm{Fix}(\psi^*_{n,d})$*, and* $\psi^*_n(\alpha, A^*) = (\beta, B^*)$ *with* $\beta = (\beta_1, \beta_2, \ldots, \beta_{r+1})$*,*

  then $r = k \pm 1$.

Therefore $|T_{n,d}| = |\mathrm{Fix}(\psi^*_{n,d})|$.

2. *A Fishburn diagram* $(\alpha, A^*) \in F^*$ *is therefore either fixed by* ψ^* *or paired with another Fishburn diagram with the same unsigned* t, z*-weight but with the opposite sign. Therefore we have that*

$$\sum_{n=0}^{\infty}\sum_{d=1}^{n} |T_{n,d}|t^n z^d = \sum_{n=0}^{\infty}\sum_{d=1}^{n} |\mathrm{Fix}(\psi^*_{n,d})|t^n z^d = 1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^{k}(1-(1-t)^i).$$

Proof. 1. As in the proof of Theorem 3.3.1, we need only prove that $\psi^*_{n,d}$ is an involution, or that, if $(\alpha, A^*) \notin \mathrm{Fix}(\psi^*_{n,d})$ and $\psi^*_n(\alpha, A^*) = (\beta, B^*)$, then $\psi^*_{n,d}(\beta, B^*) = (\alpha, A^*)$. Let j again be the smallest integer such that A^* has a column containing j and at least one other integer or a j-occurrence of **(2-1)** with $\alpha_{j+2} = 0$. For each of the two possible cases, we will prove that j is also the smallest integer such that B^* has a column containing j and at least one other integer or a j-occurrence of **(2-1)** with $\beta_{j+2} = 0$. Using an identical argument to the proof of Theorem 3.3.1, we will then conclude that $\psi^*_{n,d}(\beta, B^*) = (\alpha, A^*)$.

We again distinguish the two possible cases:

Case 1: If there is at least one column in A^* containing j and at least one other integer, then $r = k+1$, so $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_{j+1}, 0, \alpha_{j+2}, \ldots, \alpha_{k+1})$ and $B^* = B_1 B_2 \cdots B_{k+1}$. Let A^*_i again be the leftmost column containing

j and at least one other integer, with $j+R$ again the second-smallest integer in A_i^*. By construction, $B_{i+1}^* = \{j\}$ and B_{i-R+1}^* contains $j+1$. Therefore there is at least one j-occurrence of **(2-1)** in $B_1B_2\cdots B_{m+1}$ and the $(j+2)$-nd entry of β is equal to zero. The distribution of dots at heights lower than j is unchanged by ψ_n, except that some dots at height j in A^* may now be at height $j+1$ in B^*. This cannot add any new occurrences of **(2-1)** with an integer less than j or columns with more than one dot and an integer less than j, and the first $j+1$ entries of α and β are identical. Therefore j remains the smallest integer such that B^* has a column containing j and at least one other integer or a j-occurrence of **(2-1)** with $\beta_{j+2}=0$.

We can now calculate $\psi_{n,d}^*(\beta, B^*)$. An identical argument to the argument used in the proof of Theorem 3.3.1 shows that, by construction, there is no column in B^* containing j and at least one other integer. Therefore the composition part of $\psi_{n,d}^*(\beta, B^*)$ is β with the $(j+2)$-nd entry removed, or α. An identical argument to the argument used in the proof of Theorem 3.3.1 shows that the Fishburn diagram part of $\psi_{n,d}^*(\beta, B^*)$ is A^*. Therefore $\psi_{n,d}^*(\beta, B^*) = (\alpha, A^*)$.

Case 2: If there is no column in A^* containing j and at least one other integer, then $r=k-1$ and there is a j-occurrence of **(2-1)** in A^* with $\alpha_{j+2}=0$. Therefore $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_{j+1}, \alpha_{j+3}, \ldots, \alpha_{k+2})$ and $B^* = B_1B_2\cdots B_{k-1}$.

Let A^*_q again be the rightmost column containing j in A^* and A^*_p again be the rightmost column to its left containing $j+1$. By construction the smallest integer in B^*_{q-1} is j and the second-smallest integer is $j+q-p$. Therefore B^* has a column containing j and at least one other integer. The distribution of dots at heights lower than j is unchanged by $\psi^*_{n,d}$, except that some dots at height $j+1$ in A^* may now be at height j in B^*. This cannot add any new columns with more than one dot.

We need only verify that this change cannot add new i-occurrences of **(2-1)** with $\beta_{i+2} = 0$ and $i < j$. Because the distribution of dots at heights lower than j is unchanged, the only possibility is that there might be a new $(j-1)$-occurrence of **(2-1)** in B^*, with $\beta_{j+1} = 0$, formed by a dot at height $j+1$ in A^* which is now at height j in B^* and a dot at height $j-1$ in B^* to its right. Assume there is such a $(j-1)$-occurrence of **(2-1)** in B^* with $\beta_{j+1} = 0$. Then there exist integers $X < Y$ such that $j \in B^*_X$ and $j-1 \in B^*_Y$, with $j+1 \in A^*_{X+1}$ and $j-1 \in A^*_{Y+1}$. Since the dot at height $j \in B^*_X$ was at height $j+1$ in A^*, it must have been pushed down one row by the operation of $\psi^*_{n,d}$. Since there was a blank row at height $j+1$ between A^*_p and A^*_q, this dot must have been to the right of A^*_q. However, the dot at height $j-1$ in A^*_{Y+1}, to its right, would then form a $(j-1)$-occurrence of **(2-1)** in A^* along with the dot $j \in A^*_q$. By the minimality of j, this implies that $\alpha_{j+1} > 0$. Since the first $j+1$

entries of α and β are identical, this is impossible. Therefore j remains the smallest integer such that B has a column containing j and at least one other integer or a j-occurrence of **(2-1)** with $\beta_{j+2} > 0$.

We can now calculate $\psi^*_{n,d}(\beta, B^*)$. By construction, there is at least one column in B^* containing j and at least one other integer. Therefore the composition part of $\psi^*_{n,d}(\beta, B^*)$ is β with a zero inserted as the $(j+2)$-nd entry, or α. An identical argument to the argument used in the proof of Theorem 3.3.1 shows that the Fishburn diagram part of $\psi^*_{n,d}(\beta, B^*)$ is A^*. Therefore $\psi^*_{n,d}(\beta, B^*) = (\alpha, A^*)$.

This proves that $\psi^*_{n,d}$ is an involution.

2. This follows from an identical argument to that used in the proofs of the second parts of Theorem 3.3.1 and Theorem 4.1.2.

□

This proves Equation (4.0.2), and therefore the conjecture of Remmel and Kitaev that Equation (1.0.2) and Equation (1.0.3) are equivalent.

Chapter 5

Further Research Directions

Our initial approach to prove that the Fishburn numbers enumerate non-2-neighbor-nesting matchings was to try to find a bijection to non-neighbor-nesting matchings. Our second approach was to try to find a bijection to ascent sequences. Specifically, we conjectured that there exists a bijection from non-2-neighbor-nesting matchings X on $[2n]$ such that the inversion table $\phi(X)$ has precisely k distinct integers to ascent sequences of length n with $k-1$ non-ascents, which would make inductive sense. We were unable to find elegant bijections in either case. It would be interesting if any existed. (A bijection could be constructed from the proofs in this paper by using an involution on Fishburn diagrams to prove that the Fishburn numbers enumerate ascent sequences and then going back and forth between the two involutions. See Section 2.6 of [3]. However, this would not necessarily be the most elegant bijection.)

A second interesting question is whether it is possible to define analogues of the bounce, area, or dinv statistics on the various Fishburn sets to generalize the statistics defined on the Catalan sets (see [7]). It would be particularly interesting, as the combinatorics of the symmetry of the q,t-Catalan polynomials are famously poorly understood, if the $(\text{bounce}, \text{area})$ or $(\text{dinv}, \text{area})$ ordered pair of statistics resulted in symmetric q,t-polynomials.

Using our proof that Equation (1.0.2) and Equation (1.0.3) are equivalent, we defined statistics similar to area and dinv on **(2-1)**-avoiding inversion tables, as well as slightly different statistics on ascent sequences. More specifically, we associated each **(2-1)**-avoiding Fishburn diagram with one dot per column to a composition of n, where each entry of the composition gave the number of dots in a diagonal of the Fishburn diagram. We then tried to extend the following identity:

$$\sum_{\pi} q^{\text{dinv}(\pi)} t^{\text{area}(\pi)} = \sum_{\alpha} q^{\sum(\alpha_i - 1)} \prod_i \begin{bmatrix} \alpha_i + \alpha_{i+1} - 1 \\ \alpha_i \end{bmatrix}_t .$$

where the first sum is over Dyck paths and the second sum is over compositions of n (again, see Haglund [7]).

In both cases we obtained symmetric q,t-polynomials for all $n \leq 5$, but not for $n = 6$. In both cases the symmetric difference was only a few terms long, with the symmetric difference of the q,t-polynomial resulting from ascent sequences slightly shorter. This question is therefore likely to be outside of the bounds of this article, but it is certainly possible that working with another interpretation of the

Fishburn numbers could inspire statistics that did, in fact, result in symmetric q,t-polynomials.

Chapter 6

Introduction: Parking Functions

Parking functions are combinatorial objects generalizing permutations, with two statistics, *area* and *dinv*, which are conjectured (see [7]) to give the Hilbert series of the diagonal co-invariants $Hilb(DR_n)$, much as Mahonian statistics over permutations gives the Hilbert series of the quotient of polynomials and symmetric polynomials. The conjecture dates back to Haglund and Loehr [6].

Haglund [15] recently proved that this Hilbert series is given by a generating function over particular matrices ("Tesler matrices"), which reduces the parking functions conjecture to the purely combinatorial form:

$$\sum_{X=[a_{ij}]\in T_n} (q+t-1-qt)^{extra(X)} \prod_{a_{ij}>0} [a_{ij}]_{q,t} = \sum_{a\in P^n} q^{dinv(a)} t^{area(a)}.$$

This combinatorial generating function motivated the below research. For an overview of parking functions, see [7].

Note: Some of the below research first appeared in the conference proceedings for FPSAC 2011 [16]. A superior exposition, which points out a connection to an overlapping classical result of Kreweras, is given in a summary of Tesler matrix research [1]. The original insight that there are $n!$ Tesler matrices with one entry per row, and that they may be thought of as an "incidence matrix", is from a meeting between the author, Haglund, Bandlow, and Visontai, who used the phrase.

6.1 Parking Functions

Let a sequence $a_1 a_2 \cdots a_n$ be a "parking function" if and only if, for all $i \in [n]$,

$$i \leq \# \{j : a_j \leq i\} .$$

For example, 31433 is not a parking function, since there are is only one entry less than or equal to 2, but 31413 is a parking function.

Let:

- P^n be the set of parking functions of length n.
- $$P_i^n = \{a_1 a_2 \cdots a_n \in P^n | a_i = 1 < a_1, a_2, \cdots, a_{i-1}\}$$

  for each $i \in [n]$.

For example,

$$P_2^3 = \{213, 312, 212, 211, 311\} .$$

6.2 Parking Functions as labeled Dyck paths

Given a parking function $a = a_1 a_2 \cdots a_n \in P_n$, let

$$DP_a = \begin{bmatrix} b_1 & b_2 & \dots & b_{2n} \end{bmatrix}$$

be the unique word in $[n] \cup \{\cdot^n\}$ such that:

- if $b_i, b_{i+1} \in [n]$, then $b_i < b_{i+1}$, and
- j appears after k dots if and only if $a_j = k + 1$.

For example, if $a = 31413$, then

$$DP_a = \begin{bmatrix} 2 & 4 & \cdot & \cdot & 1 & 5 & \cdot & 3 & \cdot & \cdot \end{bmatrix}.$$

Let a multi-set permutation of $\begin{bmatrix} c_1 c_2 \cdots c_{2n} \end{bmatrix}$ on the alphabet $[n] \cup \{\cdot^n\}$ be a "labeled Dyck path" if it the following conditions hold:

- If $c_i, c_{i+1} \in [n]$, then $c_i < c_{i+1}$.
- For all $R \in [2n]$,

$$\#\{i < R : c_i \in [n]\} \geq \#\{i < R : c_i = \cdot\}.$$

Claim 6.2.1. *1. DP_a begins with i if and only if $a \in P_i^n$.*

2. The map

$$a \to DP_a$$

is a bijection between parking functions and labeled Dyck paths.

Proof. 1. This much should be clear.

2. First, we will show that, for a given $a \in P^n$, $DP_a = \begin{bmatrix} b_1 & b_2 & \cdots & b_{2n} \end{bmatrix}$ is, in fact, a labeled Dyck path.

   The first condition is satisfied by definition.

   Given $R \in [2n]$, assume that $b_R \in [n]$ and b_R is after k dots. Therefore $a_{b_R} = k+1$. The number of j such that $a_j < k+1$ must be at least k, and each will be an integer appearing before b_R. This suffices. A similar argument shows the inverse.

□

P^n can therefore be thought of as the set of labeled Dyck paths, and P_i^n can be thought of as the set of labeled Dyck paths with first upstep labelled by i.

6.2.1 Statistics on parking functions: area

Given a parking function $a = a_1 a_2 \cdots a_n$, define

$$area(a) = \binom{n+1}{2} - \sum_{i=1}^{n} a_i.$$

For example, if $a = 31413$, then $area(a) = 15 - 12 = 3$. Alternately, given an integer $i \in [n]$, if $DP_a = \begin{bmatrix} b_1 & b_2 & \cdots & b_{2n} \end{bmatrix}$ and $b_k = i$, then define $f_a(i)$ by:

$$f_a(i) = \#\{j < k : b_j \in [n]\} - \#\{j \leq k : b_j = \cdot\}.$$

For example, if $a = 31413$ and $DP_a = \begin{bmatrix} 2 & 4 & \cdot & \cdot & 1 & 5 & \cdot & 3 & \cdot & \cdot \end{bmatrix}$, then $f_a(1) = 0, f_a(2) = 0, f_a(3) = 1, f_a(4) = 1$, and $f_a(5) = 1$.

Note that these sum to the area of a.

Claim 6.2.2. *If $a \in P^n$, then*

$$area(a) = \sum_{i=1}^{n} f_a(i).$$

Proof. Assume that $DP_a = \begin{bmatrix} b_1 & b_2 & \cdots & b_{2n} \end{bmatrix}$ and that $b_1^*, b_2^*, \cdots, b_n^*$ is the permutation of $[n]$ that is a subsequence of $b_1 b_2 \cdots b_{2n}$. Note that there are $i-1$ integers before b_i^*. Note also that the number of dots before b_i^* is equal to $a_{b_i^*} - 1$.

Therefore

$$\sum_{i=1}^{n} f_a(b_i^*) = \sum_{i=1}^{n} ((i-1) - (a_{b_i^*} - 1))$$
$$= \sum_{i=1}^{n} (i - a_{b_i^*}) = \binom{n+1}{2} - \sum_{i=1}^{n} a_i = area(a).$$

□

6.2.2 Statistics on parking functions: dinv

Given a parking function $a = a_1 a_2 \cdots a_n$ with $DP_a = \begin{bmatrix} b_1 & b_2 & \cdots & b_{2n} \end{bmatrix}$, let the functions $f_a(i)$ be defined as above.

Let an ordered pair (b_i, b_j) with $i < j$ be a "dinv pair" if and only if either:

- $b_i < b_j$ and $f_a(b_i) = f_a(b_j)$ or

- $b_i > b_j$ and $f_a(b_i) = f_a(b_j) + 1$.

Define $dinv(a)$ to be the number of dinv pairs.

For example, if $a = 31413$ and $DP_a = \begin{bmatrix} 2 & 4 & \cdot & \cdot & 1 & 5 & \cdot & 3 & \cdot & \cdot \end{bmatrix}$, recall that $f_a(2) = 0, f_a(4) = 1, f_a(1) = 0, f_a(5) = 1$, and $f_a(3) = 1$.

Therefore the dinv pairs are $(4, 1)$ and $(4, 5)$, and so $dinv(a) = 2$.

6.2.3 Symmetry?

Define the generating function $PF^n(t, q)$ by

$$PF^n(t, q) = \sum_{a \in P^n} t^{area(a)} q^{dinv(a)}.$$

For example, it can be shown that $PF^1(q, t) = 1$, $PF^2(q, t) = 1 + q + t$, and:

$$PF^3(q, t) = \begin{array}{c} q^3 + q^2t + qt^2 + t^3 + \\ 2q^2 + 3qt + 2t^2 + \\ 2q + 2t + \\ 1 \end{array}.$$

Claim 6.2.3. *For all n, $PF^n(t, q) = PF^n(q, t)$.*

6.3 Tesler Matrices

Given an upper triangular matrix $X = [a_{ij}]_{i \geq j}$ with n rows and columns of non-negative integers, let the k-th "hook sum" be the sum

$$\sum_{j=k+1}^{n} a_{kj} - \sum_{i=1}^{k} a_{ik}.$$

Let $X = [a_{ij}]_{i \geq j}$ be in T_n, the set of *Tesler matrices* if all hook sums are equal to one. In other words,

$$T_n = \left\{ [a_{ij}]_{i \geq j} | \forall k \in [n], \sum_{j=k+1}^{n} a_{kj} - \sum_{i=1}^{k} a_{ik} = 1 \right\}.$$

Note that the first row of a matrix in T_n must consist of one 1 and $n-1$ 0's. The second row consists of one 1 and $n-2$ 0's, unless the 1 in the first row is in the second column. If the 1 in the first row is in the second column, then the second row must consist of either one 2 or two 1's, and so on.

For example,

$$T_3 = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ & 1 & 0 \\ & & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 3 \end{bmatrix}.$$

Note that each matrix must have at least one non-zero entry per row, and that each matrix in T_n must therefore have at least n non-zero entries.

Given $X = [a_{ij}] \in T_n$, let

$$extra(X) = \# \{a_{ij} | a_{ij} > 0\} - n.$$

We define the following as the "Haglund generating function" for a given Tesler matrix $X = [a_{ij}] \in T_n$:

$$(q + t - 1 - qt)^{extra(X)} \prod_{a_{ij} > 0} [a_{ij}]_{q,t}.$$

The full Haglund generating function is the sum of this function over the set of Tesler matrices.

$$H_n(q,t) = \sum_{X=[a_{ij}] \in T_n} (q + t - 1 - qt)^{extra(X)} \prod_{a_{ij} > 0} [a_{ij}]_{q,t}.$$

For example, the Haglund generating function applied to the matrices in T_3 gives, respectively,

$$\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ & 0 & 1 \\ & & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ & 1 & 1 \\ & & 2 \end{bmatrix}$$

$$1 \qquad q + t \qquad q + t \qquad (q + t)(q + t - 1 - qt)$$

$$\begin{bmatrix} 0 & 1 & 0 \\ & 0 & 2 \\ & & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ & 1 & 0 \\ & & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 1 \\ & & 3 \end{bmatrix}.$$

$$(q+t)(q^2+qt+t^2) \quad (q+t) \quad q^2+qt+t^2$$

Therefore

$$H_3(q,t) = \begin{matrix} q^3+q^2t+qt^2+t^3+ \\ 2q^2+3qt+2t^2+ \\ 2q+2t+ \\ 1 \end{matrix} .$$

Notice that $H_3(q,t) = PF^3(q,t)$.

6.4 Our Results

As a result of Haglund's work, the parking function conjecture can be written as this purely combinatorial identity on generating functions:

Conjecture 6.4.1. *For all n,*

$$\sum_{X=[a_{ij}]\in T_n} (q+t-1-qt)^{extra(X)} \prod_{a_{ij}>0} [a_{ij}]_{q,t} = \sum_{a\in P^n} q^{dinv(a)} t^{area(a)}.$$

We will begin by giving several results on the combinatorics of parking functions that have been inspired by our efforts to prove this identity, including a quasirecursive generation of parking functions and the *area* and *dinv* statistics and, as a result, explicit formulas for the distribution of *area* and *dinv* in special cases.

We will then give a combinatorial proof of the $t = 0$ special case of this identity:

$$\sum_{X=[a_{ij}]\in T_n} (q-1)^{extra(X)} \prod_{a_{ij}>0} q^{a_{ij}-1} = \sum_{\pi\in S_n} q^{dinv(\pi)}.$$

Our proof will use a bijection between Tesler matrices and a new combinatorial object we call a “Tesler array”, with the generating function given by a particular kind of “filled Tesler array”. This interpretation also allows for a much more efficient generation of Tesler matrices.

We will then give a related combinatorial proof of the $q = 1$ special case of this identity, expanding on classical results on parking functions:

$$\sum_{X=[a_{ij}]\in T_n^*} \prod_{a_{ij}>0} [a_{ij}]_t = \sum_{\pi\in P_n} t^{area(\pi)}.$$

We will conclude by giving additional patterns and conjectures we have noticed over the course of this research.

Chapter 7

Structural Results

7.1 A near-recursive generation

We begin with some definitions. Recall that P^n is the set of parking functions of length n, and P_i^n is the subset of P^n consisting of parking functions whose first upstep is labelled by i.

1. Let $\overbrace{P^n}$ be the subset of P_n consisting of parking functions π such that $f_\pi(n) > 0$. (In Dyck path terms, n is not on the lowest diagonal.)

2. Let $\overbrace{P_i^n}$ be the intersection of P_i^n and $\overbrace{P^n}$.

Examining specific cases suggests the existence of a bijection between P_i^n and $\overbrace{P_{i-1}^n}$ that preserves *dinv* and increases *area* by one.

We will define such a bijection below.

Given a parking function $\pi \in P_i^n$, with $i > 1$, define the map ϕ as follows:

Let

$$DP_\pi = \begin{bmatrix} b_1 & b_2 & \cdots & b_{2n} \end{bmatrix}$$

be the unique way of writing π as a labelled Dyck path, with $b_1 b_2 \cdots b_{2n}$ a multiset permutation of $[n] \cup \{\cdot^n\}$.

Let k be such that $b_k = 1$. Because π does not start with 1, $k > 1$. Since b_{k-1} would be less than b_k if it were an integer, $b_{k-1} = \cdot$.

Let π^* be the unique parking function such that DP_{π^*} is formed by transposing 1 and the $\cdot$ before, replacing 1 with n, and subtracting 1 from every other integer.

More formally,

$$DP_{\pi^*} = \begin{bmatrix} b_1^* & b_2^* & \cdots & b_{k-1}^* & b_k^* & \cdots & b_{2n}^* \end{bmatrix},$$

where:

- If $1 < b_i \in [n]$, then $b_i^* = b_i - 1$.
- If $b_i = \cdot$ and $i \neq k-1$, then $b_i^* = \cdot$.
- $b_{k-1}^* = n$, and
- $b_k^* = \cdot$.

Let $\phi(\pi) = \pi^*$.

For example, if $\pi = 35112 \in P_3^5$, then

$$DP_\pi = \begin{bmatrix} 3 & 4 & \cdot & 5 & \cdot & 1 & \cdot & \cdot & 2 & \cdot \end{bmatrix}.$$

Therefore

$$DP_{\phi(\pi)} = \begin{bmatrix} 2 & 3 & \cdot & 4 & 5 & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix},$$

and so $\phi(\pi) = 51122 \in \widehat{P_2^5}$.

For a parking function π, recall that $f_\pi(j)$ is the "partial sum" at j in π.

Theorem 7.1.1. *1. For $i > 1$, ϕ is a bijection from $P_i^n \to \widehat{P_{i-1}^n}$.*

2. For $j > 1$, $f_{\phi(\pi)}(j-1) = f_\pi(j)$, and $f_{\phi(\pi)}(n) = f_\pi(1) + 1$.

3. $area(\phi(\pi)) = area(\pi) + 1$.

4. $dinv(\phi(\pi)) = dinv(\pi)$.

Proof. 1. ϕ is clearly a map from P_i^n to P_{i-1}^n.

Following the notation in the definiton of ϕ, let $b_k = 1$. There will be one fewer $\cdot$ before n in $DP_{\phi(\pi)}$ then there is before 1 in DP_π. Therefore $f_{\phi(\pi)}(n) = f_\pi(1) + 1 > 0$. Therefore $\phi(\pi) \in \widehat{P_{i-1}^n}$.

It suffices to define an inverse map ϕ^{-1}.

Given $\pi \in \widehat{P_{i-1}^n}$, let

$$DP_\pi = \begin{bmatrix} b_1 & b_2 & \dots & b_{2n} \end{bmatrix}$$

Let R be the integer such that $b_R = n$. Then $b_{R+1} = \cdot$.

Let π^* be the unique parking function such that DP_{π^*} is formed by transposing n and the $\cdot$ after, replacing n with 1, and adding 1 to every other integer.

More formally

$$DP_{\pi^*} = \begin{bmatrix} b_1^* & b_2^* & \cdots & b_R^* & b_{R+1}^* & \cdots & b_{2n}^* \end{bmatrix},$$

where,

- If $n > b_i \in [n]$, then $b_i^* = b_i + 1$.
- If $b_i = \cdot$ and $i \neq R+1$, then $b_i^* = \cdot$.
- $b_R^* = \cdot$, and
- $b_{R+1}^* = 1$.

Let $\phi^{-1}(\pi) = \pi^*$. This is plainly an inverse map.

2. This follows immediately from the definition of ϕ.

3. This follows immediately from the previous claim and the fact that

$$area(\pi) = \sum_{j=1}^{n} f_\pi(j).$$

4. Recall that, given a parking function π with $DP_\pi = \begin{bmatrix} b_1 & b_2 & \cdots & b_{2n} \end{bmatrix}$, an ordered pair of integers (b_i, b_j) with $i < j$ is a "dinv pair" if and only if either:

- $b_i < b_j$ and $f_\pi(b_i) = f_\pi(b_j)$ or

- $b_i > b_j$ and $f_\pi(b_i) = f_\pi(b_j) + 1$.

For $j > 1$, $f_{\phi(\pi)}(j-1) = f_\pi(j)$, and j is to the left of k in π if and only if $j-1$ is to the left of $k-1$ in $\phi(\pi)$.

Therefore, if $1 < j < k$, (j,k) is a *dinv* pair in π if and only if $(j-1, k-1)$ is a *dinv* pair in π.

$(1, R)$ is a *dinv* pair in π if and only if 1 is to the left of R and $f_\pi(1) = f_\pi(R)$ or 1 is to the right of R and $f_\pi(1) + 1 = f_\pi(R)$.

Also, $f_{\phi(\pi)}(n) = f_\pi(1) + 1$. Therefore, in the first case, n is to the left of $R-1$ and $f_{\phi(\pi)}(n) = f_\pi(R-1) + 1$.

In the second case, n is to the right of $R-1$ and $f_{\phi(\pi)}(n) = f_\pi(R-1)$. Therefore $(1, R)$ is a *dinv* pair in π if and only if $(R-1, n)$ is a *dinv* pair in $\phi(\pi)$.

Therefore $dinv(\phi(\pi)) = dinv(\pi)$.

□

Let $T(\pi)$ be the number of touch points of π.

Theorem 7.1.2. *For $i < n$, there exists a bijection*

$$\psi : \left\{(\pi, k) : \pi \in P_i^{n-1}, k \in [1, T(\pi)]\right\} \to P_i^n - \overbrace{P_i^n},$$

such that $area(\psi(\pi, k)) = area(\pi)$ and $dinv(\psi(\pi, k)) = dinv(\pi) + k$.

Because of these two theorems, we can recursively generate the distribution of *dinv* and *area* over the parking functions in P_i^n, assuming we know the distribution of $T(\pi)$ over P_i^{n-1}:

For example, suppose we want to generate P_1^4, and we know that the distribution of *dinv* and *area* over P_2^4 is:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 2 & \\ 2 & 4 & 4 & 2 & & \\ 1 & 2 & 1 & & & \end{bmatrix}.$$

Technically, so that the result is a rectangular matrix, this matrix gives the distribution of $area + dinv$ (by row) and $area$ (by column).

This will also give the distribution of $area + dinv$ and $area$ over $\overbrace{P_1^4}$.

The distribution of *dinv* and *area* over P_1^3 is given by:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & \end{bmatrix}.$$

The parking functions with $area = 0$ all have 3 touchpoints, so the first column should be multiplied by $q * [3]_q$. The parking functions with $area = 1$ all have 2 touchpoints so the second column should be multiplied by $q * [2]_q$. The other three parking functions all have 1 touchpoint, so they are simply multipled by q.

Therefore, the distribution of *area* and *dinv* over $P_1^4 - \overbrace{P_1^4}$ is given by:

$$\begin{bmatrix} 1 & & & & & & & & & \\ 1 & + & 1 & & & 1 & & & & \\ & & 1 & + & 1 & 2 & + & 1 & 1 & 1 \\ & & & & 1 & & & 2 & 1 & \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 1 & \end{bmatrix}.$$

We then add these two matrices together to give the distribution of *area* and *dinv* over all of P_1^4:

$$P_1^4 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 1 & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 2 & \\ 2 & 4 & 4 & 2 & & \\ 1 & 2 & 1 & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 3 & 2 & \\ 2 & 5 & 5 & 5 & 2 & & \\ 1 & 3 & 3 & 1 & & & \end{bmatrix}$$

Here is a more comprehensive list of examples up to $n = 5$.

Each matrix gives the distribution of *area*+*dinv* (by row) and *area* (by column) over the given subset, and each matrix is built up from the above operations:

- $$P_1^1 = \begin{bmatrix} 1 \end{bmatrix}$$

- $$P_2^2 = P^1 = \begin{bmatrix} 1 \end{bmatrix}$$

- $$P_1^2 = \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- $$P_3^3 = P^2 = \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix}$$

- $$P_2^3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \end{bmatrix}$$

- $$P_1^3 = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & \end{bmatrix}$$

- $$P_4^4 = P^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & \\ 2 & 2 & & \\ 1 & & & \end{bmatrix}$$

- $$P_3^4 = \begin{bmatrix} 1 & \\ 2 & 1 \\ 2 & 1 \\ 1 & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & \\ 2 & 2 & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & \\ 2 & 3 & 2 & & \\ 1 & 1 & & & \end{bmatrix}$$

- $$P_2^4 = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 2 & 2 & 1 \\ 1 & 1 & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 & \\ 2 & 3 & 2 & & \\ 1 & 1 & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 2 & \\ 2 & 4 & 4 & 2 & & \\ 1 & 2 & 1 & & & \end{bmatrix}$$

- $$P_1^4 = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 1 & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 2 & \\ 2 & 4 & 4 & 2 & & \\ 1 & 2 & 1 & & & \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 3 & 3 & 2 & \\ 2 & 5 & 5 & 5 & 2 & & \\ 1 & 3 & 3 & 1 & & & \end{bmatrix}$$

- $$P_5^5 = P^4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 9 & 9 & 5 & & \\ 6 & 11 & 11 & 6 & & & \\ 5 & 8 & 5 & & & & \\ 3 & 3 & & & & & \\ 1 & & & & & & \end{bmatrix}$$

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$$P_4^5 = \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ 5 & 4 & 1 & 1 \\ 6 & 6 & 3 & 1 \\ 5 & 5 & 2 & \\ 3 & 2 & & \\ 1 & & & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 9 & 9 & 5 & & \\ 6 & 11 & 11 & 6 & & & \\ 5 & 8 & 5 & & & & \\ 3 & 3 & & & & & \\ 1 & & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 10 & 10 & 9 & 5 & & \\ 6 & 12 & 14 & 12 & 6 & & & \\ 5 & 10 & 10 & 5 & & & & \\ 3 & 5 & 3 & & & & & \\ 1 & 1 & & & & & & \end{bmatrix}$$

- $$P_3^5 = \begin{bmatrix} 1 \\ 3 & 1 \\ 5 & 4 & 1 \\ 6 & 7 & 4 & 2 & 1 \\ 5 & 7 & 5 & 2 \\ 3 & 4 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 3 \\ 5 & 9 & 10 & 10 & 9 & 5 \\ 6 & 12 & 14 & 12 & 6 \\ 5 & 10 & 10 & 5 \\ 3 & 5 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 4 & 3 \\ 5 & 9 & 10 & 10 & 10 & 9 & 5 \\ 6 & 13 & 16 & 16 & 13 & 6 \\ 5 & 12 & 15 & 12 & 5 \\ 3 & 7 & 7 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

- $$P_2^5 = \begin{bmatrix} 1 & & & & & \\ 3 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 6 & 8 & 4 & 2 & 1 & 1 \\ 5 & 9 & 7 & 4 & 2 & \\ 3 & 6 & 5 & 2 & & \\ 1 & 2 & 1 & & & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 10 & 10 & 10 & 9 & 5 & & \\ 6 & 13 & 16 & 16 & 13 & 6 & & & \\ 5 & 12 & 15 & 12 & 5 & & & & \\ 3 & 7 & 7 & 3 & & & & & \\ 1 & 2 & 1 & & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 10 & 10 & 10 & 10 & 9 & 5 & & \\ 6 & 14 & 17 & 18 & 17 & 14 & 6 & & & \\ 5 & 14 & 19 & 19 & 14 & 5 & & & & \\ 3 & 9 & 12 & 9 & 3 & & & & & \\ 1 & 3 & 3 & 1 & & & & & & \end{bmatrix}$$

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$$P_1^5 = \begin{bmatrix} 1 & & & & & & \\ 3 & 1 & & & & & \\ 5 & 4 & 1 & & & & \\ 6 & 9 & 4 & 2 & 1 & 1 & 1 \\ 5 & 11 & 8 & 6 & 3 & 2 & \\ 3 & 8 & 8 & 5 & 2 & & \\ 1 & 3 & 3 & 1 & & & \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 10 & 10 & 10 & 10 & 9 & 5 & & \\ 6 & 14 & 17 & 18 & 17 & 14 & 6 & & & \\ 5 & 14 & 19 & 19 & 14 & 5 & & & & \\ 3 & 9 & 12 & 9 & 3 & & & & & \\ 1 & 3 & 3 & 1 & & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & \\ 5 & 9 & 10 & 10 & 10 & 10 & 10 & 9 & 5 & & \\ 6 & 15 & 18 & 19 & 19 & 18 & 15 & 6 & & & \\ 5 & 16 & 22 & 25 & 22 & 16 & 5 & & & & \\ 3 & 11 & 17 & 17 & 11 & 3 & & & & & \\ 1 & 4 & 6 & 4 & 1 & & & & & & \end{bmatrix}$$

Let the subsets $P_{i,k}^n$ of P^n be defined by

$$P_{i,k}^n = \{\pi | \pi \in P_i^n, area(\pi) + dinv(\pi) = k\}$$

Theorem 7.1.3. *1. The smallest k such that $P_{i,k}^n$ is non-empty is $n - i$.*

2.

$$\sum_{\pi \in P^n_{i,n-i}} q^{dinv(\pi)} t^{area(\pi)} = (q+t)^{n-i}.$$

3. For $i > 1$,

$$\sum_{\pi \in P^n_{i,n-i+1}} q^{dinv(\pi)} t^{area(\pi)} = (n-2)(q+t)^{n-i+1} - (n-i)qt(q+t)^{n-i-1}.$$

While

$$\sum_{\pi \in P^n_{1,n}} q^{dinv(\pi)} t^{area(\pi)} = (n-2)(q+t)^{n} - (n-1)qt(q+t)^{n-2} - q^2t^2[n-3]_{q,t}.$$

7.1.1 *dinv* Distributions

We can also use the above theorems to give some interesting results on the distribution of *dinv* over parking functions with fixed area.

We begin with a straightforward result:

Theorem 7.1.4. *1.*

$$\sum_{\pi \in P^n_i, area(\pi)=0} q^{dinv(\pi)} = q^{n-i}[n-1]_q!.$$

2. Consequently, for $j < n$,

$$\sum_{\pi \in \overbrace{P^n_j}, area(\pi)=1} q^{dinv(\pi)} = q^{n-j-1}[n-1]_q!.$$

(Note that $\overbrace{P^n_n} = \emptyset$.)

3. *Therefore, for $j < n$,*

$$\sum_{\pi \in P_j^n - \overbrace{P_j^n}, area(\pi)=1} q^{dinv(\pi)} = q[n-2]_q \sum_{\pi \in P_j^{n-1}, area(\pi)=1} q^{dinv(\pi)}.$$

4.

$$\sum_{\pi \in P_j^n, area(\pi)=1} q^{dinv(\pi)} = q^{n-j-1}[n-1]_q! + q[n-2]_q \left(\sum_{\pi \in P_j^{n-1}, area(\pi)=1} q^{dinv(\pi)} \right).$$

Note that this is a recursive formula.

5. *This recursion is solved by*

$$\sum_{\pi \in P_j^n, area(\pi)=1} q^{dinv(\pi)}$$
$$= q^{n-j-1}[n-2]_q! \left(q \sum_{L=1}^{j-2} [L]_q + \sum_{R=j}^{n-1} [R]_q \right).$$

Proof. 1. This is simply a statement about permutations.

2. This follows immediately from the bijection and substituting j for $i-1$.

□

7.1.2 Symmetry

In this section, we will explore the following conjecture:

$$\sum_{a \in P_i^n} q^{dinv(a)} t^{area(a)}.$$

are symmetric in q and t for each i.

Let $a_1 a_2 \cdots a_{k-1}$ be the coefficients the homogeneous part of some

$$\sum_{a \in P_i^n - \overbrace{P_i^n}} q^{dinv(a)} t^{area(a)}$$

for some i. Take $a_k = 0$.

Claim 7.1.5. *Assume that the sequence $a_1 a_2 \cdots a_{k-1} a_k$, when added to some sequence $0 b_1 b_2 \cdots b_{k-1}$, gives some sequence $c_1 c_2 \cdots c_k$, and that both the b_i and c_i sequences are symmetric.*

1. *This uniquely determines both sequences.*
2. $b_j = \sum_{i=1}^{j} a_i - a_{k+1-i}$.
3. $c_j = \sum_{i=1}^{j} a_i - a_{k+2-i}$, *taking* $a_{k+1} = 0$.
4. $a_1 a_2 \cdots a_{k-1} a_k a_{k+1}$, *when added to* $0 c_1 c_2 \cdots c_k$, *gives another symmetric sequence* $d_1 d_2 \cdots d_{k+1}$.

Proof. 1. We assume that

$$b_i = b_{k-i}$$

for all i. We also have

$$c_i = a_i + b_{i-1}.$$

And we also assume that

$$c_i = c_{k+1-i}.$$

Therefore

$$a_i + b_{i-1} = a_{k+1-i} + b_{k-i} \tag{7.1.1}$$

$$= a_{k+1-i} + b_i. \tag{7.1.2}$$

Therefore

$$b_i - b_{i-1} = a_i - a_{k+1-i}.$$

(We take $b_{-1} = 0$.)

Since the a_i are fixed, this uniquely determines the b_i.

2. This follows from

$$b_i - b_{i-1} = a_i - a_{k+1-i}$$

and the fact that we take $b_{-1} = 0$.

3. This follows from:

$$c_j = a_j + b_{j-1} = a_j + \sum_{i=1}^{j-1} a_i - a_{k+1-i} = \sum_{i=1}^{j} a_j - (a_k + \cdots + a_{k+2-j}) = \sum_{i=1}^{j} a_i - a_{k+2-i}.$$

4. Since we have a formula for c_j, this much is trivial.

$\square$

Therefore, assuming the conjecture holds, the parking functions in $P_i^n - \overbrace{P_i^n}$, in some sense, encode the parking functions in P_i^n.

Chapter 8

The $t = 0$ Special Case

8.0.3 Introduction

At $t = 0$, Haglund's generating function gives:

$$H_n(q, 0) = \sum_{X=[a_{ij}]\in T_n} (q-1)^{extra(X)} \prod_{a_{ij}>0} q^{a_{ij}-1}.$$

The $t = 0$ special case of the parking function conjecture is therefore

$$\sum_{X=[a_{ij}]\in T_n} (q-1)^{extra(X)} \prod_{a_{ij}>0} q^{a_{ij}-1} = \sum_{a\in P_n} (0)^{area(a)} q^{dinv(a)} = \sum_{\pi\in S_n} q^{dinv(\pi)},$$

since permutations are precisely parking functions with *area* equal to zero.

Note that this generating function is still over the set of all parking functions, but the result only involves permutations.

While the $t = 0$ special case of the parking function conjecture is easily shown to be true in its own right, we will give a combinatorial proof of this generating function identity that includes a full combinatorial description of Tesler matrices.

In fact, this combinatorial description will allow us to prove a refinement of this generating function.

More precisely, we will define a partition

$$T_n = \coprod_{\pi \in S_n} T_\pi$$

of the set of Tesler matrices into disjoint subsets indexed by the permutations.

For a fixed $\pi \in S_n$, we will then prove that

$$\sum_{X=[a_{ij}] \in T_\pi} (q-1)^{extra(X)} \prod_{a_{ij}>0} q^{a_{ij}-1} = q^{dinv(\pi)}.$$

and this will also serve to prove the original equation.

8.0.4 Decoding Tesler matrices

For the purposes of this section, we will define the coordinates of the diagonal entries of a Tesler matrix in T_n to be $(a, n+1)$ rather than (a, a).

Consider the following Tesler matrix:

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 2 & 0 & 0 & 0 \\ & & 0 & 0 & 2 & 1 \\ & & & 0 & 1 & 0 \\ & & & & 2 & 2 \\ & & & & & 4 \end{bmatrix}$$

Define a multiset C_X, where $(a, b) \in C_X$ with multiplicity m if and only if the (a, b)-coordinate of X is equal to m.

In this case,

$$C_X = (1,2),(2,3),(2,3),(3,5),(3,5),(3,6),(4,5),$$

$$(5,7),(5,7),(5,6),(5,6),(6,7),(6,7),(6,7),(6,7).$$

Note that there are precisely 6 ordered pairs ending in 7. We can write these in weakly-decreasing order from top to bottom:

$$\begin{bmatrix} & & 6 & 7 \\ & & 6 & 7 \\ & & 6 & 7 \\ & & 6 & 7 \\ & 5 & & 7 \\ & 5 & & 7 \end{bmatrix}.$$

Now note that, because X is a Tesler matrix, there are three ordered pairs ending in 6: $(5,6),(5,6),(3,6)$, and there are four ordered pairs beginning with 6: $(6,7),(6,7),(6,7),(6,7)$.

We can summarize this information by replacing the ordered pairs with $567, 567, 367$:

$$\begin{bmatrix} & & 5 & 6 & 7 \\ & & 5 & 6 & 7 \\ 3 & & & 6 & 7 \\ & & & 6 & 7 \\ & & 5 & & 7 \\ & & 5 & & 7 \end{bmatrix}.$$

There are three ordered pairs ending in 5: $(3,5),(3,5)$, $(4,5)$, and there are four things places these might go. Continuing to place things in weakly decreasing order from top to bottom gives:

$$\begin{bmatrix} & 4 & 5 & 6 & 7 \\ 3 & & 5 & 6 & 7 \\ 3 & & & 6 & 7 \\ & & & 6 & 7 \\ 3 & & 5 & & 7 \\ & & 5 & & 7 \end{bmatrix}.$$

There are no ordered pairs ending in 4, but there are two ordered pairs ending in 3: $(2,3),(2,3)$:

$$\begin{bmatrix} & & & 4 & 5 & 6 & 7 \\ & 2 & 3 & & 5 & 6 & 7 \\ & 2 & 3 & & & 6 & 7 \\ & & & & & 6 & 7 \\ & & 3 & & 5 & & 7 \\ & & & & 5 & & 7 \end{bmatrix}.$$

Finally, there is one ordered pair ending in 2, and we have the following array:

$$\begin{bmatrix} & & & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & & 5 & 6 & 7 \\ & 2 & 3 & & & 6 & 7 \\ & & & & & 6 & 7 \\ & & 3 & & 5 & & 7 \\ & & & & 5 & & 7 \end{bmatrix}.$$

If we read the first integer in every row from top to bottom, the result is a permutation, 412635.

We then say that the original Tesler matrix is "associated" with 412635.

For a given permutation π, let T_π be the set of Tesler matrices "associated" with π.

More precisely:

Let A_n be the set of arrays A such that:

1. A has n rows consisting of non-negative integers in increasing order and ending with $(n+1)$.
2. For any integer i, if ai and bi appear in A and $a < b$, then bi appears above ai.
3. Each integer $i < n+1$ appears at the start of precisely one row in A, and does not appear below that row.

We will refer to A_n as the set of "Tesler arrays". The following claim is a trivial consequence of the definition.

Claim 8.0.6. *Given a Tesler array $A \in A_n$, the first entries of the rows of A form a permutation in S_n.*

Theorem 8.0.7.
1. *Given a Tesler matrix $X \in T_n$, there is a unique Tesler array $A \in A_X$ such that ai appears m times in A_X if and only if the (a, i) coordinate of X is equal to m.*
2. *The map $X \to A_X$ is a bijection between T_n and A_n.*

Proof. 1. Let $X = [a_{ij}]$.

Recall that the diagonal of X forms a composition of n Therefore there will be n ordered pairs ending with $n+1$ in A_X, which must be arranged in weakly decreasing order.

There will be $a_{n,n}$ ordered pairs beginning with n and ending with $n+1$. Because X is a Tesler matrix, there will be $a_{n,n}-1$ ordered pairs ending with n, which must be arranged in weakly decreasing order in the top $a_{n,n}-1$ rows beginning with n, leaving the last row blank.

There will be $a_{n-1,n-1} + a_{n-1,n}$ ordered pairs beginning with $n-1$. All of these have been accounted for. Because X is a Tesler matrix, there will be $a_{n-1,n-1} + a_{n-1,n} - 1$ ordered pairs ending with $n-1$, which must be arranged in weakly decreasing order in the top $a_{n,n}-1$ rows beginning with n, leaving the last row blank.

Continuing in this fashion gives a uniquely-defined Tesler array A_X by construction.

2. It suffices to show that the above map is surjective.

Given a Tesler array A, let $[b_{ij}] = Y$ be the matrix uniquely defined by the condition that ij appears precisely b_{ij} times in A. We need only show that Y is a Tesler matrix.

Each integer $i < n+1$ appears at the start of precisely one row in A. Therefore, each integer $i < n+1$ appears at the start of precisely one more consecutive pair than it appears at the end of. This suffices.

□

For each $\pi \in S_n$, let T_π be the subset of T_n consisting of Tesler matrices X such that the first entries of the Tesler array A_X give π.

The following theorem is now trivial.

Theorem 8.0.8.

$$T_n = \coprod_{\pi \in S_n} T_\pi.$$

For conciseness, when writing Tesler arrays, we will henceforth omit the final column, since it is always identically $n+1$.

Here is an example. Consider $\pi = 31524$. What Tesler matrices are in T_{31524}? For one,

$$\begin{bmatrix} & & 3 & & 5 \\ 1 & & & & 5 \\ & & & & 5 \\ & 2 & & 4 & \\ & & & 4 & \end{bmatrix}$$

If 4 isn't in the first row, then it can't be in the second row, but 2 might be:

$$\begin{bmatrix} & & 3 & & 5 \\ 1 & 2 & & & 5 \\ & & & & 5 \\ & 2 & & 4 & \\ & & & 4 & \end{bmatrix}$$

If 4 is in the first row, then we have:

$$\begin{bmatrix} & & 3 & 4 & 5 \\ 1 & & & & 5 \\ & & & & 5 \\ & 2 & & 4 & \\ & & & 4 & \end{bmatrix}$$

$$\begin{bmatrix} & & 3 & 4 & 5 \\ 1 & 2 & & & 5 \\ & & & & 5 \\ & 2 & & 4 & \\ & & & 4 & \end{bmatrix}$$

$$\begin{bmatrix} & & 3 & 4 & 5 \\ 1 & 2 & & 4 & 5 \\ & & & & 5 \\ & 2 & & 4 & \\ & & & 4 & \end{bmatrix}$$

The corresponding Tesler matrices are precisely the elements of T_{31524}, and they are, respectively:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & & 2 & 0 \\ & & & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 \\ & & 0 & 0 & 1 \\ & & & 2 & 0 \\ & & & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 0 \\ & & & 2 & 1 \\ & & & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 \\ & & 0 & 1 & 0 \\ & & & 2 & 1 \\ & & & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 2 & 0 \\ & & 0 & 1 & 0 \\ & & & 2 & 2 \\ & & & & 3 \end{bmatrix}$$

Note that the $t = 0$ Haglund functions of these Tesler matrices sum to a single term:

$$q^3 + (q^4 - q^3) + (q^4 - q^3) + (q^5 - 2q^4 + q^3) + (q^6 - q^5) = q^6.$$

Note also that $q^6 = q^{dinv(31524)}$.

This suggests that

$$\sum_{X \in T_\pi} (q-1)^{extra(X)} \prod_{[a_{ij}>0] \in X} q^{a_{ij}-1} = q^{dinv(\pi)},$$

which would suffice to prove the $t = 0$ special case.

In order to prove this equation, we will define a set of objects called "filled Tesler arrays" along with a weighted sum that gives the left-hand side.

We will then define a sign-reversing involution on the set of filled Tesler arrays such that the weighted sum over the set of fixed points gives the right-hand side.

8.0.5 Filled Tesler Arrays

Let A^*_π be the set of Tesler arrays $A_X \in T_\pi$ that have been "filled" with either q or -1 according to the following conditions:

1. There can only be a q or a -1 immediately before an integer i that is not the first integer in its row.

2. If i is not the first integer in its row, and i immediately precedes j, and this is not the lowest row where i immediately precedes j, then there is a q before i.

3. If i is not the first integer in its row, and i immediately precedes j, and this is the lowest row where i immediately preceds j, then there is either a q or a -1 before i.

For example, consider the following Tesler array,

$$A_X = \begin{bmatrix} 1 & 2 & 3 & 4 & & 6 \\ & 2 & 3 & & & 6 \\ & & & & & 6 \\ & & 3 & 4 & 5 & \\ & & & & 5 & \\ & & & 4 & & \end{bmatrix}$$

corresponding to the Tesler matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 2 & 0 & 0 & 0 \\ & & 0 & 2 & 0 & 1 \\ & & & 1 & 1 & 1 \\ & & & & 2 & 0 \\ & & & & & 3 \end{bmatrix} \in T_{126354}.$$

There are eight possible fillings of this Tesler array.

To summarize these, below is the Tesler array with $q-1$ in the spaces that might be filled by either q *or* -1, and q in the spaces that must be filled by q.

$$\begin{bmatrix} 1 & q & 2 & q & 3 & q-1 & 4 & & & q & 6 \\ & & 2 & q-1 & 3 & & & & & q & 6 \\ & & & & & & & & & & 6 \\ & & & & 3 & q-1 & 4 & q & 5 & & \\ & & & & & & & & 5 & & \\ & & & & & & 4 & & & & \end{bmatrix}.$$

Note that the product of the possibilities is

$$q^5(q-1)^3 = (q-1)^{extra(X)} \prod_{[a_{ij}>0]\in X} q^{a_{ij}-1}.$$

The filled Tesler arrays, in other words, have been defined so that the sum over all fillings gives the $t=0$ Haglund generating function.

For $X \in T_\pi$, let A_X^* be the set of filled Tesler arrays corresponding to fillings of the Tesler array A_X.

Theorem 8.0.9. *For F a filled Tesler array in A_X^* for fixed Tesler matrix X, let $b \in F$ refer to the set of t's and -1's in the filling.*

Then

$$\sum_{F \in A_X^*} \prod_{b \in F} b = (q-1)^{extra(X)} \prod_{[a_{ij}>0] \in X} q^{a_{ij}-1}.$$

8.0.6 A sign-reversing involution

Consider the subset A_{1243}^* of filled Tesler arrays.

We will arrange the 9 filled arrays in this subset by sign and by weight.

$$\begin{bmatrix} 1 & -1 & 2 & -1 & 3 & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & & & -1 & 3 & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & q & 2 & & & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}, \begin{bmatrix} 1 & & & q & 3 & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix} 1 & q & 2 & -1 & 3 & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 & q & 3 & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & q & 2 & q & 3 & q & 4 \\ & & 2 & & & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & q & 2 & q & 3 & q & 4 \\ & & 2 & -1 & 3 & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}$$

$$\begin{bmatrix} 1 & q & 2 & q & 3 & q & 4 \\ & & 2 & q & 3 & q & 4 \\ & & & & & & 4 \\ & & & & 3 & & \end{bmatrix}.$$

Note how, as desired, all terms cancel out except for $q^{dinv(1243)} = q^5$.

This suggests how an involution might be defined.

For a given permutation π, we now define a map $\Phi_\pi : \cup_{X \in T_\pi} A_X^* \to \cup_{X \in T_\pi}$ as follows.

Given a filled Tesler array $Q \in A_X^*$ for some $X \in T_\pi$, let k be the integer at the start of the highest row such that, for some integer $m > k$, at least one of the following conditions hold:

- m appears below the row beginning with k but does not appear in the row beginning with k, or

- m appears in the row beginning with k, but has a -1 before it.

If there is no such k, then let $\Phi_\pi(Q) = Q$.

Otherwise, fixing k, without loss of generality, let m be the smallest integer satisfying those conditions.

If the first condition holds, then let $\Phi_\pi(Q)$ be the filled Tesler array that is identical to Q except m is now in the row beginning with k and has a -1 before it.

If the second condition holds, then let $\Phi_\pi(Q)$ be the filled Tesler array that is identical to Q except that the m and -1 before it have been removed.

For example, let

$$Q = \begin{bmatrix} 1 & q & 2 & q & 3 & -1 & 4 & & & q & 6 \\ & & 2 & -1 & 3 & & & & & q & 6 \\ & & & & & & & & & & 6 \\ & & & & 3 & q & 4 & q & 5 & & \\ & & & & & & & & 5 & & \\ & & & & & & 4 & & & & \end{bmatrix}.$$

Then $k = 1$, $m = 4$, and

$$\Phi_\pi(Q) = \begin{bmatrix} 1 & q & 2 & q & 3 & & & & & q & 6 \\ & & 2 & -1 & 3 & & & & & q & 6 \\ & & & & & & & & & & 6 \\ & & & & 3 & q & 4 & q & 5 & & \\ & & & & & & & & 5 & & \\ & & & & & & 4 & & & & \end{bmatrix}.$$

Claim 8.0.10. *If* $Q \in \cup_{X \in T_\pi} A_X^*$*, then* $\Phi_\pi(Q) \in \cup_{X \in T_\pi} A_X^*$*.*

Proof. Fix $Q \cup_{X \in T_\pi} A_X^*$ for fixed π.

We use k and m as in the above definition.

Let a be the largest integer less than m in the row beginning with k, and let b be the smallest integer greater than m in that row.

Assume that m is such that m appears below the row beginning with k but does not appear in the row beginning with k.

If m is added to this row, then it will be a row containing mb. If any row below contains mb, then there must be a row below containing mb in Q. Since the row beginning with k contains ab, and since $a < b$, this is impossible by the definition of Tesler arrays.

Therefore, if m is added to this row to produce $\Phi_\pi(Q)$, a -1 can indeed be placed before it.

Assume that m is such that m appears in the row beginning with k, but has a -1 before it.

There is no integer greater than a and less than m below the row beginning with k, by the minimality of m.

Therefore, if m is removed, then we are replacing mb with ab. This would only be a problem if there were some occurance of cb below the row beginning with k for some $a < c \leq m$, but this is impossible.

Therefore $\Phi_\pi(Q)$ is a filled Tesler array. Since nothing at the start of a row is changed, if $Q \in A^*_X$ for $X \in T_\pi$, then $Q \in A^*_{X'}$ for some $X' \in T_\pi$.

This suffices.

$\square$

Theorem 8.0.11. *1. For a fixed π, Φ_π is an involution on $\cup_{X \in T_\pi} A^*_X$, and therefore*

$$\sum_{X \in T_\pi} \sum_{F \in A^*_X} \prod_{b \in F} b = \sum_{F \in Fix(\Phi_\pi)} \prod_{b \in F} b.$$

2. The set of fixed points of Φ_π is such that

$$\sum_{F \in Fix(\Phi_\pi)} \prod_{b \in F} b = q^{dinv(\pi)}.$$

We have now proved the following theorem.

Theorem 8.0.12. *1. For a given permutation π,*

$$\sum_{X \in T_\pi} (q-1)^{extra(X)} \prod_{[a_{ij}>0] \in X} q^{a_{ij}-1} = q^{dinv(\pi)}.$$

2. As a result,

$$\sum_{X \in T_n} (q-1)^{extra(X)} \prod_{[a_{ij}] \in X} q^{a_{ij}-1} = \sum_{\pi \in S_n} q^{dinv(\pi)} = [n]_q!.$$

This proves the $t = 0$ case of Haglund's theorem.

Unfortunately, the sum of the full Haglund generating function is not q, t-positive over T_π.

However, we have the following conjecture, refining Hanglund's conjecture:

Conjecture 8.0.13.

$$\sum_{\pi_1=i} \sum_{X \in T_\pi} (q + t - 1 - qt)^{extra(X)} \prod_{[a_{ij}] \in X} [a_{ij}]_{q,t} = \sum_{a \in P_i^n} q^{dinv(a)} t^{area(a)}.$$

Chapter 9

The $q = 1$ Special Case

Let T_n^* be the following subset of the set of Tesler matrices T_n:

$$T_n^* = \{X \in T_n : extra(X) = 0\}.$$

In other words, T_n^* is the set of Tesler matrices with precisely one non-zero entry per row.

The Haglund generating function identity, at $q = 1$, reduces to:

$$H_n(1,t) = \sum_{X=[a_{ij}]\in T_n^*} \prod_{a_{ij}>0} [a_{ij}]_t.$$

The parking function conjecture then becomes:

$$\sum_{X=[a_{ij}]\in T_n^*} \prod_{a_{ij}>0} [a_{ij}]_t = \sum_{\pi\in P_n} t^{area(\pi)}.$$

In this section, we will give a combinatorial proof of this special case.

Claim 9.0.14. *1.* $|T_n^*| = n!$.

2. In particular, for a given $\pi \in S_n$*,* $|T_n^* \bigcap T_\pi| = 1$*. In other words, there is precisely one Tesler matrix* X_π *with only one non-zero entry per row in each disjoint subset* T_π*.*

For a fixed $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, let $a_1a_2\cdots a_n$ be the sequence where a_i is the sole non-zero entry in the i-th row of X_π. Let $P_{\pi^{-1}}$ be the set of parking functions that park to π^{-1}.

For each $i \in [n]$, let $g_\pi(i)$ be the number of integers j such that:

- $j < i$,
- j is to the left of i in $\pi_1\pi_2\cdots\pi_n$, and
- there is no integer $k > i$ in between j and i in $\pi_1\pi_2\cdots\pi_n$.

Let $P_{\pi^{-1}}$ be the set of parking functions that "park" to π^{-1}.

Claim 9.0.15. *1.*

$$\prod_{i=1}^{n}[g_\pi(i)]_t = \sum_{r\in P_{\pi^{-1}}} t^{area(r)}.$$

2. For all $i \in [n]$*,* $g_\pi(i) = a_i$*.*

3. Therefore,

$$\prod_{X_\pi=[a_{ij}];a_{ij}>0} [a_{ij}]_t = \sum_{r\in\in P_{\pi^{-1}}} t^{area(r)}.$$

Proof. See the author's [16], or (in a proof which points out the connection, previously unknown to the author, of a result of Kreweras), [1]. □

This proves the $q = 1$ special case.

Chapter 10

The t_n-positivity conjecture

Once again, the full Haglund generating function is:

$$H^n(q,t) = \sum_{X \in T_n} (q+t-1-qt)^{extra(X)} \prod_{[a_{ij}] \in X} [a_{ij}]_{q,t}.$$

We define the following refinement using a new variable t_n:

$$H^n(q,t,t_n) = \sum_{X \in T_n} (q+t-1-qt)^{extra(X)} [a_{n,n}]_{q,t_n} \prod_{[a_{ij}] \in X, i*j<n^2} [a_{ij}]_{q,t}$$

Conjecture 10.0.16. *1. $H^n(q,t,t_n)$ is q,t,t_n-positive.*

2.

$$H^n(q,t,0) = \sum_{a \in P^n - \overbrace{P^n}} q^{dinv(a)} t^{area(a)}.$$

Similarly, we define a further refinement following the above conjectures:

$$H_i^n(q,t,t_n) = \sum_{\pi_1 = i} \sum_{X \in T_\pi} (q+t-1-qt)^{extra(X)} [a_{n,n}]_{q,t_n} \prod_{[a_{ij}] \in X, i*j<n^2} [a_{ij}]_{q,t}$$

And we have parallel conjectures as well:

Conjecture 10.0.17. *1. $H_i^n(q,t,t_n)$ is q,t,t_n-positive.*

2.

$$H_i^n(q,t,0) = \sum_{a \in P_i^n - \overbrace{P_i^n}} q^{dinv(a)} t^{area(a)}.$$

These conjectures have been verified by computer for reasonably large n and give us some confidence that our particular combinatorial interpretation reflects underlying structure.

For example, here is $H_1^4(q,t,t_n)$:

$$\begin{aligned} H_1^4(q,t,t_n) = &(q^6+2q^5+2q^4+q^3+tq^4+3tq^3+2tq^2+t^2q^2+t^2q+t^3q)+ \\ &t_n(q^5+2q^4+2q^3+q^2+tq^3+3tq^2+2tq+t^2q+t^2+t^3)+ \\ &t_n^2(q^4+2q^3+q^2+2q^2t+3q^2+qt^2+qt+t^3)+ \\ &t_n^3(q^3+q^2+q^2t+2qt+qt^2+t^2+t^3). \end{aligned}$$

Note that the "constant term" with respect to t_n gives the distribution of *area* and *dinv* over $P_1^4 - \overbrace{P_1^4}$ that is also given above.

Question 10.0.18. *Do the other coefficients of powers of t_n in $H_i^n(q,t,t_n)$ have similarly straightforward combinatorial interpretations?*

Claim 10.0.19. *1. For a fixed $\pi' \in S_{n-1}$ with $\pi'_j = n-1$, if $\pi \in S_n$ is the unique permutation such that $\pi_1 = n-1$, $\pi_{j+1} = n$, and otherwise $\pi_i = \pi'_{i-1}$, then* *there is a $1-1$ correspondence between $T_{\pi'}$ and T_π.*

2. *In particular, the Tesler matrix $X' = [a'_{ij}] \in T_{\pi'}$ can correspond to the Tesler matrix $X = [a_{ij}] \in T_\pi$ such that:*

$$a_{ij} = a'_{ij}; i < n-1. a_{n-1,n-1} = 0. a_{n-1,n} = 1. a_{n,n} = a'_{n-1,n-1} + 1.$$

3. *Therefore,*

$$H^n_{n-1}(q,t,t_n) = \sum_{i=1}^{n-1} H^{n-1}_i(q,t,0) + t_n * H^n_n(q,t,t_n).$$

Proof. 1. Equivalently, we can simply replace $n-1$ by n in the Tesler array $A_{X'}$, and add the row $n-1n$ at the top, to produce the Tesler array A_X.

2. By definition, and following the same notation,

$$H^n(q,t,t_n)_{n-1} = \sum_{\pi_1=n-1}\sum_{X\in T_\pi}(q+t-1-qt)^{extra(X)}[a_{n,n}]_{q,t_n}\prod_{[a_{ij}]\in X, i*j<n^2}[a_{ij}]_{q,t}$$
$$= \left(\sum_{\pi_1=n-1}\sum_{X\in T_\pi}(q+t-1-qt)^{extra(X)}t_n^{a_{n,n}-1}\prod_{[a_{ij}]\in X, i*j<n^2}[a_{ij}]_{q,t}\right)$$
$$+\sum_{\pi_1=n-1}\sum_{X\in T_\pi}(q+t-1-qt)^{extra(X)}t_n * [a_{n,n}-1]_{q,t_n}\prod_{[a_{ij}]\in X, i*j<n^2}[a_{ij}]_{q,t}$$
$$= \sum_{i=1}^{n-1} H^{n-1}_i(q,t,0) + t_n * H^n_n(q,t,t_n).$$

□

Chapter 11

Appendix

In this Appendix, we will prove the following conjecture of Claesson and Linusson [13]. Recall that a left-nesting of a matching X on $[2n]$ is a nesting $(a,b)(c,d) \in X$ such that $c = a+1$. For example, the matching $(1,4)(2,6)(3,5)$ on the left-hand side of Figure 11.1 has the single left-nesting $(2,6)(3,5)$. Consider Figure 2.1, which we reprint as Figure 11.1, and which shows the matchings constructed from $(1,4)(2,6)(3,5)$ and the integers in $[7]$.

- The integers $7, 6, 5$, and 4 are not openers of $(1,4)(2,6)(3,5)$. Each of the matchings constructed from $(1,4)(2,6)(3,5)$ and one of these integers has 1 left-nesting corresponding to $(2,6)(3,5)$. These matchings are on the left side of the line on the right-hand side of Figure 11.1.

- The integers 2 and 1 are openers of $(1,4)(2,6)(3,5)$, but neither opens the inner arc of the left-nesting $(2,6)(3,5)$. Each of the matchings constructed from $(1,4)(2,6)(3,5)$ and one of these integers has 2 left-nestings, one corresponding to $(2,6)(3,5)$, and one from the new arc. These matchings are on the right side of the line on the right-hand side of Figure 11.1.

- The integer 3 is an opener of $(1,4)(2,6)(3,5)$ and opens the inner arc of the left-nesting $(2,6)(3,5)$. The matching constructed from $(1,4)(2,6)(3,5)$ and this integer has only 1 left-nesting, since a left-nesting is added from the new arc, but this arc also breaks apart the left-nesting $(2,6)(3,5)$. This matching is on the left side of the line on the right-hand side of Figure 11.1.

Therefore, of the 7 matchings on [8] constructed from $(1,4)(2,6)(3,5)$, 5 have 1 left-nesting and 2 have 2 left-nestings. This suggests the distribution of left-nestings over the set of all matchings: Let the *second-order Eulerian triangle* $T(n,i)$ be defined by the recurrence relations $T(n+1,i) = i \cdot T(n,i) + (2n+2-i) \cdot T(n,i-1)$, with $T(n,i) = 0$ if $n < i$, $T(1,1) = 1$, and $T(n,-1) = 0$ (see Sloane [11]).

Claim 11.0.20. *Let $L(n,i)$ be the number of matchings on $[2n]$ with precisely i left-nestings.*

1. *Let X be a matching on $[2n]$ with j left-nestings. Let k be an integer in $[2n+1]$. Let X' be the matching on $[2n+2]$ constructed from X and k. Then X' has j left-nestings if and only if:*

   - *k is not an opener of X, so k is a closer of X or $k = 2n+1$, or*

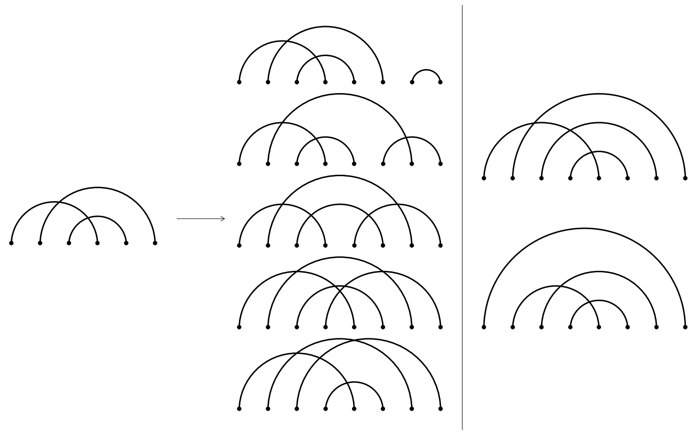

Figure 11.1: Matchings on [6] and [8]

- *k is an opener of X and the arc k opens is the inner arc of a left-nesting of X. Alternately, $X(k-1) > X(k) > k$.*

Otherwise X' has $j+1$ left-nestings.

2. *Therefore $L(n, n-i) = T(n, i)$, where $T(n, i)$ is the second-order Eulerian triangle.*

Proof. 1. Let X be a matching on $[2n]$ with precisely j left-nestings. Let k be an integer in $[2n+1]$. Recall that every arc in X' except for $(k, 2n+2)$ corresponds to an arc of X. There are $2n+1$ choices for k. We distinguish the three possible cases:

Case 1: If k is a closer of X, or if $k = 2n+1$, then the corresponding arcs of every left-nesting of X' give a left-nesting in X, and vice versa (as in the proof of Claim 2.0.1). Therefore X' has j left-nestings.

Case 2: If k is an opener of X, and $X(k-1) > X(k)$, then the corresponding arcs of every left-nesting of X' give a left-nesting of X, except that $(k, 2n+2)(k+1, X(k)+1)$ is a left-nesting of X' and there is no arc corresponding to $(k, 2n+2)$ in X. The corresponding arcs of every left-nesting of X give a left-nesting of X', except that $(k-1, X(k-1))(k, X(k))$ is a left-nesting of X, and the arc $(k, 2n+2)$ is placed in between them. The corresponding arcs in X' are $(k-1, X(k-1))$ and $(k+1, X(k)+1)$, which do not form a left-nesting of X'. Therefore X' still has $j-1+1 = j$ left-nestings.

Case 3: If k is an opener of X and the arc it opens is not the inner arc of a left-nesting, then the corresponding arcs of every left-nesting of X' give a left-nesting of X, except that $(k, 2n+2)(k+1, X(k)+1)$ is a left-nesting of X' and there is no arc corresponding to $(k, 2n+2)$ in X. However, the corresponding arcs of every left-nesting of X give a left-nesting of X'. Therefore X' has $j+1$ left-nestings.

2. Given a matching X on $[2n]$, there are $2n+1$ possible choices for an integer k to construct a matching on $[2n+2]$ with. These $2n+1$ choices are partitioned between the three cases outlined above. Precisely $n+1$ of the possible choices are in the first case, j are in the second case, and $n-j$ are in the third case. Therefore, of the $2n+1$ matchings on $[2n+2]$ formed by adding an arc to X, $n+1+j$ have j left-nestings and $n-j$ have $j+1$ left-nestings.

Alternately, there are $n+1+j$ matchings on $[2n+2]$ with j left-nestings constructible from each of the matchings on $[2n]$ with j left-nestings and $n-(j-1)$ matchings on $[2n+2]$ with j left-nestings constructible from each of the matchings on $[2n]$ with $j-1$ left-nestings. We now have that

$$L(n+1,j)=(n+1+j)\cdot L(n,j)+(n-(j-1))\cdot L(n,j-1).$$

Replacing j with $n+1-i$, we have that

$$L(n+1,n+1-i)=(2n+2-i)\cdot L(n,n-(i-1))+i\cdot L(n,n-i).$$

Since $L(1,1-1)=1$, $L(n,n-(-1))=0$, and $L(n,n-i)=0$ for $n<i$, we see inductively that $L(n,n-i)=T(n,i)$.

$\square$

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