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Economics of Spectrum Allocation in Cognitive Radio Networks

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Abstract
Cognitive radio networks (CRNs) are emerging as a promising technology for the efficient use of radio spectrum. In these networks, there are two levels of networks on each channel, primary and secondary, and secondary users can use the channel whenever the primary is not using it. Spectrum allocation in CRNs poses several challenges not present in traditional wireless networks; the goal of this dissertation is to address some of the economic aspects thereof. Broadly, spectrum allocation in CRNs can be done in two ways- (i) one-step allocation in which the spectrum regulator simultaneously allocates spectrum to primary and secondary users in a single allocation and (ii) two-step allocation in which the spectrum regulator first allocates spectrum to primary users, who in turn, allocate unused portions on their channels to secondary users. For the two-step allocation scheme, we consider a spectrum market in which trading of bandwidth among primaries and secondaries is done. When the number of primaries and secondaries is small, we analyze price competition among the primaries using the framework of game theory and seek to find Nash equilibria. We analyze the cases both when all the players are located in a single small location and when they are spread over a large region and spatial reuse of spectrum is done. When the number of primaries and secondaries is large, we consider different types of spectrum contracts derived from raw spectrum and analyze the problem of optimal dynamic selection of a portfolio of long-term and short-term contracts to sell or buy from the points of view of primary and secondary users. For the one-step allocation scheme, we design an auction framework using which the spectrum regulator can simultaneously allocate spectrum to primary and secondary users with the objective of either maximizing its own revenue or maximizing the social welfare. We design different bidding languages, which the users can use to compactly express their bids in the auction, and polynomial-time algorithms for choosing the allocation of channels to the bidders.

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ECONOMICS OF SPECTRUM ALLOCATION

IN COGNITIVE RADIO NETWORKS

To my dear family

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ABSTRACT

ECONOMICS OF SPECTRUM ALLOCATION

IN COGNITIVE RADIO NETWORKS

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Supervisor: Saswati Sarkar

Cognitive radio networks (CRNs) are emerging as a promising technology for the efficient use of radio spectrum. In these networks, there are two levels of networks on each channel, primary and secondary, and secondary users can use the channel whenever the primary is not using it. Spectrum allocation in CRNs poses several challenges not present in traditional wireless networks; the goal of this dissertation is to address some of the economic aspects thereof. Broadly, spectrum allocation in CRNs can be done in two ways– (i) one-step allocation in which the spectrum regulator simultaneously allocates spectrum to primary and secondary users in a single allocation and (ii) two-step allocation in which the spectrum regulator first allocates spectrum to primary users, who in turn, allocate unused portions on their channels to secondary users.

For the two-step allocation scheme, we consider a spectrum market in which trading of bandwidth among primaries and secondaries is done. When the number of primaries and secondaries is small, we analyze price competition among the primaries using the framework of game theory and seek to find Nash equilibria. We analyze the cases both when all the players are located in a single small location and when they are spread over a large region and spatial reuse of spectrum is done. When the number of primaries and

secondaries is large, we consider different types of spectrum contracts derived from raw spectrum and analyze the problem of optimal dynamic selection of a portfolio of long-term and short-term contracts to sell or buy from the points of view of primary and secondary users.

For the one-step allocation scheme, we design an auction framework using which the spectrum regulator can simultaneously allocate spectrum to primary and secondary users with the objective of either maximizing its own revenue or maximizing the social welfare. We design different bidding languages, which the users can use to compactly express their bids in the auction, and polynomial-time algorithms for choosing the allocation of channels to the bidders.

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Chapter 1

Introduction

1.1 Motivation

The last decade has seen a tremendous proliferation in the use of different wireless network technologies such as cellular networks, Wireless Local Area Networks, Wireless Meteropolitan Area networks etc, resulting in a proportionate increase in demand for radio spectrum. As a result, there is a widespread belief that radio spectrum is becoming increasingly crowded. However, spectrum measurements indicate that the allocated spectrum is under-utilized, *i.e.*, at any given time and location, much of the spectrum is unused [21]. This is because, in the traditional spectrum licensing model, a spectrum regulator (*e.g.* the Federal Communications Commission (FCC) in the United States), allocates spectrum by assigning *exclusive* licenses to service providers to operate their networks on different bands. So a band lies idle when not in use by the license holder on the band.

Cognitive radio networks (CRNs) [2] are emerging as a promising solution to this dilemma. In these networks, there are two levels of networks on a channel– *primary networks* and *secondary networks*. A primary network has priortized access to the band, whereas a secondary network can access the band when the primary is not using it. CRNs allow a more efficient use of spectrum than traditional networks in which each band is used by a single network– spectrum that would have been idle in the latter case can be used by secondary networks in the former. CRNs have been enabled by the cognitive radio technology [28], [29] that allows secondary nodes to detect which channel is not being used by primary nodes, share this channel with other nodes and vacate the channel when a primary node is detected. Surveys on CRNs can be found in [2] and [75].

Spectrum allocation in CRNs poses several challenges not present in traditional wireless networks; the goal of this dissertation is to address some of the economic aspects thereof. Broadly, there are two possibilities for spectrum allocation in CRNs [51]. In the first possibility, which we refer to as *one-step allocation*, the regulator simultaneously allocates the rights to be the primary and secondary networks on the channels in a single allocation, *e.g.* by an auction. This is a natural extension of spectrum allocation in traditional wireless networks– *e.g.* the FCC has been conducting spectrum auctions [1] since 1994 to allocate (exclusive) licenses to service providers. In the other possibility, which we refer to as *two-step allocation*, the regulator allocates channels to primary networks, which then independently allocate unused portions on their channels to secondary networks. The transition from the spectrum allocation process in traditional wireless networks to the two step allocation process in CRNs is perhaps more imminent owing to the decomposition of the allocation process into two steps, as opposed to the one-step allocation process which will require the involvement of a larger set of players in auctions involving the regulator. Hence, we first study the two-step allocation scenario in Chapters 2 to 6 and then the one-step allocation scenario in Chapter 7.

We now describe some challenges that arise in each of the above two possibilities. In the two-step allocation scenario, trading of bandwidth between primaries and secondaries can be done through a market mechanism, in which primaries quote prices at which they are willing to sell bandwidth, and then sell it to interested secondaries. There are two possible cases, depending on whether the number of players (primaries and secondaries) is large or small. When the number of players is small in the two-step allocation scenario, each player exerts a considerable amount of influence on the market. In this case, the price is not necessarily set at a competitive level by the market. Instead, there may be price competition in which each primary must decide how to price its bandwidth, the tradeoff being that a low price will attract more buyers for its bandwidth and a high price will fetch a high profit if the bandwidth is sold. A CRN has several distinguishing features, which makes the price competition in CRNs different from that in traditional commodity markets, *e.g.* (i) there is uncertainty about whether a primary has unused bandwidth in a given time slot and (ii) spectrum is a commodity that allows spatial reuse, *i.e.* the same band can be used at far-off locations without interference. A problem of interest is to study the behavior of players in this price competition setup. This problem constitutes the bulk of this dissertation, we address it using the framework of game theory [43], and it is the subject of Chapters 2 to 5.

When there are a large number of primaries and secondaries, the amount of influence that an individual player exerts on the market is typically small, and the price of bandwidth is set at a competitive level, which is determined by the market. In this case, a problem facing a primary that owns multiple channels [1] is to dynamically select the durations for which to lease each of these to secondaries and whether to provide service guarantees on these "bandwidth contracts". The corresponding problem facing the secondaries is to buy an appropriate mix or portfolio of different types of contracts. Since a primary's demand for bandwidth evolves stochastically over time, if it sells a long-term contract on a band and guarantees availability of the bandwidth over this duration, then it may need to pay a hefty penalty as compensation to the buyer if the primary is later forced to use the band to satisfy its own demand for bandwidth. On the other hand, it may wish to sell a lot of long-term contracts if their market price is much higher than that of short-term contracts. Similar tradeoffs are faced by the secondaries. We address this problem of selection of a portfolio of spectrum contracts for both the primary and the secondary using the stochastic dynamic programming framework, as discussed in

[1] When there are a small number of players, we assume for tractability that each primary owns only one channel. When there are a large number of players, we allow a primary to own multiple channels.

Chapter 6.

Now, in the one-step allocation scenario mentioned above, the regulator needs to select the networks that will be the primary and secondary networks on each band, with the goal of either maximizing the social welfare or its own revenue. Different networks may have different traffic demands and utilities, and hence may assign different valuations to a given allocation of primary and secondary rights on the bands. A problem is to design a mechanism that allows networks to compactly express their valuations for different channel allocations, and efficient algorithms that allocate the channels based on the submitted valuations. We have designed an auction mechanism for this problem, which we discuss in Chapter 7.

1.2 Our Contributions and Related Work

As explained above, this dissertation consists of three parts– (i) spectrum pricing games, (ii) dynamic contract trading in spectrum markets and (iii) spectrum auction framework for access allocation. Now we outline our contributions and overview related work in each of these parts.

1.2.1 Spectrum Pricing Games

We study price competition in a CRN when there are a small number of primaries and secondaries. Each primary tries to attract secondaries by setting a lower price for its bandwidth than other primaries. A CRN has several distinctive features, which makes the price competition very different from that in traditional commodity markets. First, in every time slot, each primary may or may not have unused bandwidth available. Second, the number of secondaries will be random and not known apriori as each secondary may be a local spectrum provider or even a user shopping for spectrum in a futuristic scenario, *e.g.*, users at airports, hotspots, etc. Thus, each primary who *has* unused bandwidth is uncertain about the number of primaries from whom it will face competition as well as the demand for bandwidth; it may only have access to imperfect information such as statistical distributions about either. A low price will result in unnecessarily low revenues in the event that very few other primaries have unused bandwidth or several secondaries are shopping for bandwidth, because even with a higher price the primary's bandwidth would have been bought, and vice versa. Third, spectrum is a commodity that allows spatial reuse: the same band can be simultaneously used at far-off locations without interference; on the other hand, simultaneous transmissions at neighboring locations on the same band interfere with each other. As a result, a primary cannot offer bandwidth at all locations, but must select an independent set of locations at which to offer it. Also, the choice of the independent set and the prices at those locations must be made jointly. We formulate price competition in a CRN as a game, taking into account bandwidth uncertainty, a random number of secondaries and spatial reuse. We analyze the game in a single slot, as well as its repeated version.

In the one-shot game at a single location, we explicitly compute the Nash Equilibrium [43] (NE) and show its uniqueness (Chapter 3). The proof is complicated by the fact that the strategies of the primaries being prices, the strategy sets are continuous; also the utility functions are not continuous. Also, we allow the probabilities with which different primaries have unused bandwidth to be arbitrary and unequal; this asymmetry further complicates the analysis. The NE turns out to be of a mixed-strategy type, *i.e.* each primary randomly chooses his price from a range. The structure of the NE provides several insights into the price competition among primaries (discussed in Section 3.3.5).

Next, we analyze the repeated game version of the one-shot game (Section 3.4), and show that there exists an efficient NE in which each primary sets the highest possible price and as a result, the sum of expected revenues of the primaries is maximized. This is achieved through a *threat mechanism*: if any primary lowers its price in a slot, all others retaliate in future slots by playing the one-shot game NE strategy and hence the primary suffers in the long run.

We then analyze a generalization of the basic model in which the valuations of secondary users for unit bandwidth are not constant, but random variables whose distributions are known (Chapter 4). We explicitly compute the symmetric NE in this game and show its uniqueness in the class of symmetric NE.

Finally, we consider the game with spatial reuse (Chapter 5), in which each primary owns bandwidth over a large region containing several smaller locations, which we model as an undirected graph. Each primary must simultaneously choose a set of mutually non-interfering (an independent set of) locations at which to offer bandwidth and the price of bandwidth at each of those locations. We focus on a special class of graphs, which we refer to as mean valid graphs, such that the conflict graphs corresponding to a large number of topologies that arise in practice are mean valid graphs. We explicitly compute a NE in mean valid graphs and show its uniqueness in the class of NE with symmetric independent set selection probability mass functions of the primaries.

Fig. 1.1 summarizes the main results.

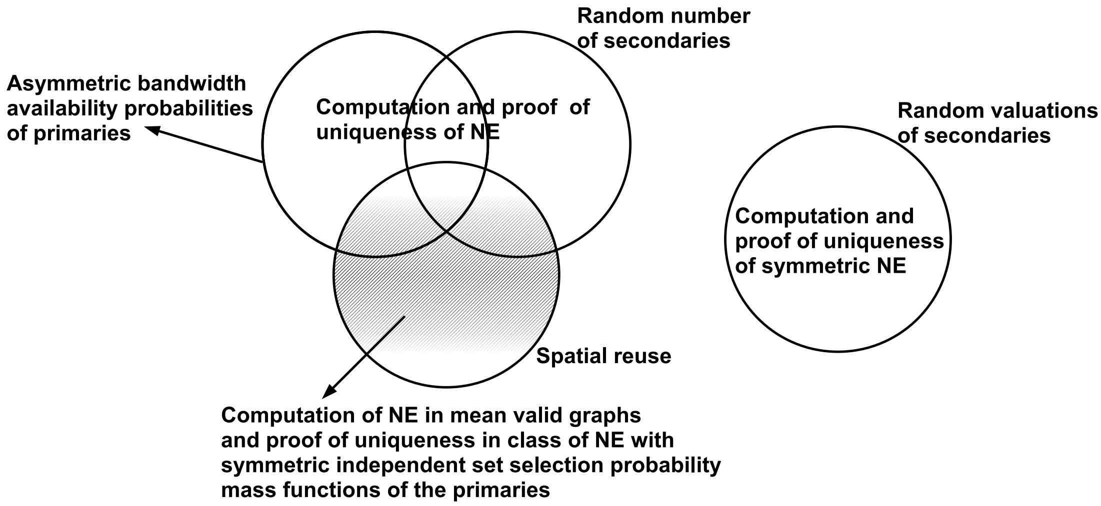

Figure 1.1: The figure summarizes the main results obtained in the “spectrum pricing games” part of the dissertation (Chapters 2 to 5).

Related Work: Pricing related issues have been extensively studied in the context of wired networks and the Internet; see [12] for an overview. Price competition among spectrum providers in wireless networks has been studied in [30], [40], [41], [74], [46], [47]. Specifically, Niyato *et. al.* analyze price competition among multiple primaries in CRNs [46], [47]. However, neither uncertain bandwidth availability, nor spatial reuse is modeled in any of the above papers. Also, most of these papers do not explicitly find a NE (exceptions are [40], [46]). Our model incorporates both uncertain bandwidth availability and spatial reuse, which makes the problem challenging; despite this, we are able to explicitly compute a NE. Zhou *et. al.* [77] have designed double auction based spectrum trades in which an auctioneer chooses an allocation taking into account spatial reuse and bids. However, in the price competition model we consider, each primary independently sells bandwidth, and hence a central entity such as an auctioneer is not required.

In the economics literature, the *Cournot game* and the *Bertand game* are two basic models that have been widely used to study competition among sellers in oligopolies [42]. In a Cournot game, sellers choose the quantity of a good to produce as opposed to prices in a Bertrand game, and hence the latter is more relevant to our model. In a Bertrand game, each seller quotes a price for a good, and the buyers buy from the seller that quotes the lowest price [2] [42]. Several variants of the Bertrand game have been studied, *e.g.*, [48], [36], [31], [34], [9]. Osborne et al [48] consider price competition in a duopoly, when the capacity of each firm is constrained. Chawla *et al.* [9] consider price competition in networks where each seller owns a capacity-constrained link, and decides the price for using it; the consumers choose paths they would use in the networks based on the prices declared and pay the sellers accordingly. The capacities in both cases are deterministic, whereas the availability of bandwidth is random in our model. The closest to our work are [31], [34], which analyze price competition where each seller may be inactive with some probability and find a Nash equilibrium [42]

[2] If two or more sellers quote the lowest price, the demand is equally shared between them.

(NE), which they show to be unique. However, the results in [31], [34] are restricted to the case of one buyer; but, a CRN is likely to have multiple secondaries, which we consider. Also, [31], [34] analyze only the *symmetric* model where the probability of owning the good is the same for each seller. Also, in [31], it is only shown that the NE is unique in the class of *symmetric* NE. In [34], uniqueness in the class of *all* NE is shown only for the case of a single buyer (and symmetric good availability probabilities). Moreover, unlike [31], [34], we consider repeated interactions among primaries, unequal probabilities of availability of unused bandwidth and random valuations for secondaries (Chapters 3 and 4).

Finally, none of the above papers [48], [36], [31], [34], [9] consider the spectrum-specific issue of spatial reuse, which introduces a new dimension, that each player not only needs to determine the price of the commodity he owns (as in [48], [36], [31], [34], [9]), but also select an independent set to compete in. The joint decision problem significantly complicates the analysis.

1.2.2 Dynamic Contract Trading in Spectrum Markets

We consider a spectrum market with a large number of primary and secondary providers [3]. Providers in both categories have their subscriber (*e.g.*, TV or mobile communication subscriber) bases whom they need to serve using the spectrum they respectively license from the FCC or buy in the spectrum market.

[3] We use the terms primary network (respectively, secondary network), primary provider (respectively, secondary provider) and primary (respectively, secondary) interchangeably.

A question that is key to the efficient operation of the spectrum market is how the primary and the secondary providers should trade bandwidth contracts dynamically, based on time-varying demand patterns arising from their subscribers, to maximize their returns while satisfying their subscriber base. We consider two basic forms of contracts that are used for selling/buying spectral resources: i) *Guaranteed-bandwidth (Type-G)* contracts, and (ii) *Opportunistic-access (Type-O)* contracts. Under the Type-*G* contracts, a secondary provider purchases a guaranteed amount of bandwidth (in units of frequency bands or sub-bands) for a specified duration of time (typically a "long term") from a primary provider, and pays a fixed fee (either as a lump-sum or as a periodic payment through the duration of the contract) irrespective of how much it uses this bandwidth. If after selling the contract, the primary is unable to provide the promised bandwidth (this may for example happen when the primary is forced to use a band it has sold due to an unexpected rise in its subscriber demand), the primary financially compensates the secondary for contractual violation. On the other hand, Type-*O* contracts are short-term (one time unit in our model), and a secondary which buys a Type-*O* contract pays only for the amount of bandwidth it actually uses on the corresponding band. The primary does not provide any guarantee on a Type-*O* contract and may use the channel sold as a Type-*O* contract without incurring any penalty. Thus, a Type-*O* contract provides the secondary the right to use the channel if the primary is not using it.

The *spectrum contract trading* problem that we formulate and solve allows the primary (respectively, secondary) provider to dynamically adjust its *spectrum contract portfolio*, i.e, choose how much of each type of contract to sell (respectively, buy) at any time, so as to maximize (respectively, minimize) its profit (respectively, cost) subject to satisfying its own subscriber demand that varies with time, and given the current market prices of Type-G and Type-O contracts which also vary with time.

We separately address the *Primary's Spectrum Contract Trading (Primary-SCT)* problem and the *Secondary's Spectrum Contract Trading (Secondary-SCT)* problem. We formulate each problem as a finite horizon stochastic dynamic program whose computation time is polynomial in the input size. We *prove* several structural properties of the optimum solutions. For example, we show that the optimal number of Type-G contracts, for both primary and secondary providers, are monotone (increasing or decreasing) functions of the subscribers' demands and the contract prices. These structural results provide more insight into the problems, and allow us to develop faster algorithms for solving the dynamic programs. Also, using numerical evaluations, we investigate properties of the optimal solutions and demonstrate that the revenues they earn substantially outperform static spectrum portfolio optimization strategies that determine the portfolio based on the steady-state statistics of the contract price and subscriber demand processes.

Note that the spectrum contract trading problem differs in several key aspects from the problem of trading traditional goods such as stocks, bonds, foodgrains etc. First, both the primary and the secondary must decide their trading strategies considering their subscriber demand which changes with time. For example, a primary (respectively, secondary) cannot simply decide to sell (respectively, buy) a large number of Type-G contracts at any given time at which their market prices are high (respectively, low). This is because a primary will need to pay a hefty penalty if it can not deliver the promised bandwidth owing to an increase in its subscriber demand, and the secondary will need to pay for the contract even if it does not use the corresponding bands owing to a decrease in its subscriber demand. Next, spectrum usage must satisfy certain spatial constraints that are perhaps unique. Specifically, a frequency band cannot be simultaneously successfully used at neighboring locations (without causing significant interference), but can be simultaneously successfully used at geographically disparate locations. Thus, the spectrum trading solution for the primary provider must also take into account spatial constraints for spectrum reuse, and therefore the computation of the optimal trading strategy requires a joint optimization across all locations. We prove a surprising *separation theorem* in this context: when the same signal is broadcast at all locations, the Primary-SCT problem can be solved separately for each location and the individual optimal solutions can subsequently be combined so as to optimally satisfy the global reuse constraints, and obtain the same revenue as the solution of a computationally prohibitive joint optimization across locations.

Related Work: The need for bringing market-based reform in spectrum trading, with the goal of ensuring efficient use of spectrum and fairness in allocation and pricing of bandwidth, is being increasingly recognized by both economists and engineers [8, 20, 49, 50, 52, 68]. The literature on the economics of spectrum allocation has so far mostly focused on the debate of spectrum commons [38, 49, 50] and spectrum auction mechanism design [33, 53, 65, 64]. Spectrum sharing games and/or pricing issues have been considered in [14, 18, 44, 24, 59]. A clear design of the spectrum market structure, precise definition of spectrum contracts, or how the different contracts can be optimally traded in a dynamic market environment is yet to emerge. This is the space in which we contribute in this dissertation (Chapter 6).

The question we address in Chapter 6 also differs significantly from existing related work in the Economics and Operations Research literature. In the *inventory problem* [60], [63], a firm maintains an inventory of some good to meet customer demand, which is uncertain. The firm needs to decide the amount to purchase in every slot of a finite or infinite horizon. There is a tradeoff between purchasing and storing costs of the inventory and the cost of not satisfying customers. This is somewhat related to our model, in which a secondary provider needs to decide the number of Type-G and Type-O contracts to buy in every time slot to meet its subscriber demand. However, contracts in our model have a different nature from goods in the inventory model: *e.g.*, Type-G contracts, once bought, can be used in every subsequent time slot to satisfy subscriber demand, whereas goods in an inventory can be used only once to satisfy customer demand. This aspect of Type-G contracts is loosely related to production capacity: once a firm installs capacity, it can be used to manufacture goods in all subsequent time periods. In *capacity expansion* problems [16], [39], a firm needs to optimally decide the volumes, times, and locations of production plants; the tradeoff is that if capacity falls short of demand, the demand cannot be met; on the other hand, if capacity exceeds demand, the excess capacity is wasted. However, our model differs in several aspects from the capacity expansion problem: *e.g.*, (i) there is no counterpart of Type-O contracts in the capacity expansion model, (ii) Type-G contracts can be bought on the spot, whereas capacity installation typically needs to be planned in advance. Finally, spatial reuse constraints being spectrum-specific, are not considered in either inventory or capacity expansion models.

1.2.3 Spectrum Auction Design

We consider a scenario in which the regulator conducts an auction to sell the rights to be the primary and secondary networks on a set of channels. Networks can bid for these rights based on their utilities and traffic demands. The regulator uses these bids to solve the *access allocation* problem, *i.e.*, the problem of deciding which networks will be the primary and secondary networks on each channel. The goal of the regulator may be either to maximize its revenue or to maximize the social welfare of the bidding networks. Now, networks can have utilities or valuations that are functions of the number of channels on which they get primary and secondary rights, how many and which other networks they share these channels with etc. The number of valuations of a network may be large and an exponential amount of space may be required to express a bid for each valuation. So we design bidding languages, that is, compact formats for networks to express bids for their valuations. For different bidding languages, we design algorithms for the access allocation problem.

We first consider the case when the bids of a network depend on which other networks it will share channels with. When there is only one secondary network on each channel, we design an optimal polynomial-time algorithm for the access allocation problem based on reduction to a maximum matching problem in weighted graphs. When there can be two or more secondary networks on a channel, we show that the optimal access allocation problem is NP-Complete. Next, we consider the case when the bids of a network are independent of which other networks it will share channels with. We design a polynomial-time dynamic programming algorithm to optimally solve the access allocation problem when the number of possible cardinalities of the set of secondary networks on a channel is upper-bounded. Finally, we design a polynomial-time algorithm that approximates the access allocation problem within a factor of 2 when the above upper bound does not exist.

Related Work: Spectrum auctions have been studied in [22], [76], [61], [27], [32], [62]. In [22], [76] a framework is developed to distribute spectrum in real-time to a set of wireless users. Channel allocation is done under interference constraints, in which the same channel cannot be allocated to two or more users whose transmissions interfere with each other. The mechanism in [76] is strategy-proof, that is, under the mechanism buyers find it in their best interest to bid according to their true valuations. In [61], there is a set of bidders and multiple chunks of spectrum. The paper investigates sequential and concurrent auction mechanisms to allocate the chunks of spectrum to the bidders such that each bidder is allocated at most one chunk. In [27], a set of spread spectrum users is considered, who share the spectrum with the owner of the spectrum. The goal is to design auctions to allocate the transmit power to each user subject to a limit on the interference at a measurement location. In [32], there are multiple primary users who own the licenses to channels in a region and multiple secondary users who are interested in leasing the unused portions of the channels of the primaries. The paper proposes a double auction mechanism with multiple sellers (the primaries) and multiple buyers (the secondaries). In [62], a knapsack based auction model is proposed to allocate spectrum to providers while maximizing revenue and spectrum usage.

We now explain how our work differs from previous work. In some of the existing work on spectrum auctions [22], [76], [61], [62] each channel is assigned to a single network, *i.e.*, there is no notion of primary and secondary networks on a single channel. We consider the case when there is a primary network and one or more secondary networks on each channel. As explained above, there are two possibilities, one-step and two-step allocation, for allocating secondary rights on channels [51]. Auctions have been designed for the two-step allocation scenario in [27] and [32]. To the best of our knowledge, our work is the first to design an auction for the one-step allocation scenario.

1.3 Organization

The rest of the dissertation is organized as follows. In Chapter 2, to gain insight, we analyze a simplified symmetric model for the two-step allocation scenario in which there are a small number of primaries and secondaries in a region. We also analyze a simplified symmetric model, which provides insight. Chapter 3 analyzes the general asymmetric model– both the game in a single slot and its infinitely repeated version. Chapter 4 considers a generalization in which the valuations of the secondaries are random and Chapter 5 considers the model with spatial reuse of spectrum. Chapter 6 is on dynamic contract trading in spectrum markets and Chapter 7 describes our auction framework for the one-step allocation scenario. Finally, we conclude in Chapter 8.

Chapter 2

Spectrum Pricing Games: A Symmetric Analysis

In this chapter and the next three chapters, we consider Cognitive Radio Networks (CRNs) with a small number of primaries and secondaries; in this scenario, each player exerts a significant influence on the market price of bandwidth.

2.1 Introduction

Consider a CRN with multiple primaries and multiple secondaries. Price competition in a CRN has the distinguishing feature that in every slot, each primary may or may not have bandwidth available. We model this price competition as a game [43] and seek a Nash Equilibrium (NE) in it. Since prices can take real values, the *strategy sets of players are continuous*. In addition, the utilities of the primaries are not continuous functions of their actions. Thus, classical results, including those for concave and potential games, do not establish the existence and uniqueness of NE in the resulting game, and there is no standard algorithm for finding a NE unlike when each player's strategy set is finite [43]. Nevertheless, as described in Chapter 3, we are able to explicitly compute a NE and show that it is unique in the class of *all* NE, allowing for player strategies that are arbitrary mixtures of continuous and discrete probability distributions.

Our results also apply to any price competition setting where the sellers' supply is uncertain. In particular, *microgrids* [37] are a newly emerging technology for distributed electricity generation, which consist of a connected network of generators (*e.g.*, solar panels, wind turbines) and loads (*e.g.*, households, factories). There is uncertainty in the power generated by a generator at a given time, *e.g.*, the power produced by a solar panel on a given day depends on the availability of sunlight. Our results characterize NE in pricing games in such electricity markets.

In this chapter, we intuitively analyze a simplified symmetric model, which provides insight. In the next chapter, we provide a formal analysis of the general model.

2.2 Model

Suppose there are $n \geq 2$ primaries and $k \geq 1$ secondaries in a region. Each primary owns 1 channel, which corresponds to 1 unit of bandwidth. Each secondary may constitute a customer who requires 1 unit of bandwidth, or may simply be a demand for 1 unit of bandwidth. For simplicity, in this chapter, we assume that the number of secondaries k

is a constant that is known to the primaries, and in the next chapter generalize our results to the case where the number of secondaries is random and unknown. Time is divided into slots of equal duration. In every slot, the channel of each primary is independently free (unused) with probability (w.p.) $q \in (0,1)$; *i.e.*, each primary independently has 1 unit of unused bandwidth w.p. q. For simplicity, in this chapter, we assume that this probability is the same for all the primaries. In the next chapter, we generalize our results to the case in which the primaries have unused bandwidth with arbitrary and possibly different probabilities. A primary i who has unused bandwidth in a slot can lease it out to a secondary for the duration of the slot, in return for an access fee of p_i. Leasing in a slot incurs a cost of $c \geq 0$. This cost may arise, for example, if the secondary uses the primary's infrastructure to access the Internet. We assume that $p_i \leq v$ for each primary, for some constant $v > c$. This upper bound v may arise as follows:

1. The spectrum regulator may impose this upper bound to ensure that primaries do not excessively overprice bandwidth even when competition is limited owing to bandwidth scarcity or high demands from secondaries, or when the primaries collude.

2. Alternatively, the *valuation* of each secondary for 1 unit of bandwidth may be v, and no secondary will buy bandwidth at a price that exceeds his valuation.

We initially assume that the primaries know this upper limit v, which is likely to be the case for the first interpretation. For the second interpretation, the primaries need not know the secondaries' valuations– we consider this generalization in Chapter 4.

Secondaries buy bandwidth from the primaries that offer the lowest price. More precisely, in a given slot, let Z be the number of primaries who offer unused bandwidth. Then the bandwidth of the $\min(Z, k)$ primaries that offer the lowest prices is bought (ties are resolved at random).

2.3 Game Formulation

We formulate the above price competition among primaries as a *game*, which is any situation in which multiple individuals called *players* interact with each other, such that each player's welfare depends on the *actions* of others [42]. In our model, the primaries are the players, and the action of primary i is the price p_i that he chooses [4]. For the most part of this dissertation, we study the interaction of the primaries in a single slot, which is referred to as the *one-shot game*. In Section 3.4 of Chapter 3, we consider a setting where the one-shot game is repeated an infinite number of times, referred to as the *repeated game*.

The *utility* or *payoff* of a player in a game is a numerical measure of his satisfaction level [42], which in our context is the corresponding primary's net revenue. In (the one-shot version of) our game, the utility of primary i is 0 if he has no unused bandwidth.

[4] If primary i has no unused bandwidth, it does not matter what price p_i he sets. Yet, for convenience, we speak of p_i as being his action.

Let $u_i(p_1,\ldots,p_n)$ denote his utility [5] if he has unused bandwidth [6] and primary j sets a price of p_j, $j=1,\ldots,n$. Thus,

$$u_i(p_1,\ldots,p_n) = \begin{cases} p_i - c & \text{if primary } i \text{ sells his bandwidth} \\ 0 & \text{otherwise} \end{cases}$$

Recall that the distribution function (d.f.) [19] of a random variable (r.v.) X is the function:

$$G(x) = P(X \le x),\ x \in R$$

where R is the set of real numbers. Now, a *strategy* [42] for primary i is a plan for choosing his price p_i. We allow each primary i to choose his price randomly from a set of prices using an arbitrary d.f. $\psi_i(.)$, which is referred to as the strategy of primary i. A d.f. that concentrates its entire mass on a single value allows a primary to deterministically choose this value as his price– such a $\psi(.)$ is referred to as a *pure strategy*. The

[5]The utility of any primary i who has unused bandwidth depends on whether he sells his bandwidth, which depends on other primaries' bandwidth availabilities that are random and not included in the action space. The expected utility (if the expectation is taken over the availabilities) however depends only on the primaries' actions. For example, if $n=2, k=1$, if primary 1 has unused bandwidth, his expected utility is

$$E[u_1(p_1,p_2)] = \begin{cases} p_1 - c & \text{if } p_1 < p_2 \\ (p_1-c)/2 & \text{if } p_1 = p_2 \\ (1-q_2)(p_1-c) & \text{if } p_1 > p_2. \end{cases}$$

So, in game theoretic terminology, the above expected utility ought to be considered as the utility of a primary. We will consider the expectation in defining the Nash equilibrium and also in our proofs. Finally, note that the expected utility is not a continuous function of the actions (prices).

[6]If instead, $u_i(p_1,\ldots,p_n)$ were defined to be primary i's net revenue, unconditional on whether he has unused bandwidth or not, then his expected utility in the one-shot game analysis would be scaled everywhere by q.

vector $(\psi_1(.),\ldots,\psi_n(.))$ of strategies of the primaries is called a *strategy profile* [42]. Let $\psi_{-i} = (\psi_1(.),\ldots,\psi_{i-1}(.),\psi_{i+1}(.),\ldots,\psi_n(.))$ denote the vector of strategies of primaries other than i. Let $E\{u_i(\psi_i(.),\psi_{-i})\}$ denote the expected utility of player i when he adopts strategy $\psi_i(.)$ and the other players adopt ψ_{-i}.

We use the *Nash Equilibrium* (NE) solution concept, which has been extensively used in game theory in general and wireless network applications in particular to predict the outcome of a game. Several arguments have been proposed in the literature for using NE as a solution concept, *e.g.* it is a necessary condition if there is a unique predicted outcome to a game, a strategy profile can be a "focal point" only if it is a NE etc. (see Section 8.D in [42] for a detailed discussion). A NE is a strategy profile such that no player can improve his expected utility by unilaterally deviating from his strategy [42]. Thus, $(\psi_1^*(.),\ldots,\psi_n^*(.))$ is a NE if for each primary i:

$$E\{u_i(\psi_i^*(.),\psi_{-i}^*)\} \geq E\{u_i(\tilde{\psi}_i(.),\psi_{-i}^*)\},\ \forall\ \tilde{\psi}_i(.) \tag{1}$$

When players other than i play ψ_{-i}, $\psi_i^*(.)$ maximizes i's expected utility and is thus his *best-response* [42] to ψ_{-i}.

2.4 Symmetric NE

Since the bandwidth availability probability of each primary is the same (equal to q), the game in Section 2.2 is a *symmetric game*, which is one in which all players have the same action sets and utility functions.

We focus on a specific class of Nash equilibria, known as *symmetric Nash equilibria*. A NE $(\psi_1^*(.),\ldots,\psi_n^*(.))$ is a symmetric NE if all players play identical strategies under it, i.e., $\psi_1^*(.) = \psi_2^*(.) = \ldots = \psi_n^*(.)$. In practice it is challenging to implement any other NE– the simple example of two primaries and a NE of $(\psi_1^*(.),\psi_2^*(.))$ elucidates the inherent complications in the current context. If $\psi_1^*(.) \neq \psi_2^*(.)$, then since players have the same action sets, utility functions and probability of having unused bandwidth (*i.e.*, the game is a *symmetric game*), $(\psi_2^*(.),\psi_1^*(.))$ also constitutes a NE. If player 1 knows that player 2 is playing $\psi_2^*(.)$ ($\psi_1^*(.)$ respectively), he would choose the best response $\psi_1^*(.)$ ($\psi_2^*(.)$ respectively), but he cannot know player 2's choice between the two options without explicitly coordinating with him, which is again ruled out due to the competition between the two. Under symmetric NE, all players play the same strategy, and thus this quandary is somewhat limited– symmetric NE has indeed been advocated for symmetric games by several game theorists [10]. The natural question now is whether there exists at least one symmetric NE, and also whether there is a unique symmetric NE (only uniqueness will eliminate the above quandary). Note that some symmetric games are known to have multiple symmetric NE. For example, consider the simple "Meeting in New York game" [42] with two players, where each player can either be at Grand Central or at Empire State Building, and both receive unit utility if they meet and zero utility otherwise. The strategies where each player is at Grand Central, and where each player is at Empire State Building, both constitute symmetric NE. We prove existence of a symmetric NE, by explicitly computing one, and subsequently prove that it is the unique symmetric NE in our context.

2.5 Nash Equilibria

If $k \geq n$, then the number of buyers is always greater than or equal to the number of sellers. So a primary i will sell his unused bandwidth even when he chooses the maximum possible price v. So the strategy profile under which all primaries deterministically choose the price v is the unique NE. So henceforth, we assume that $k \leq n-1$.

Theorem 1. *There is no pure strategy NE (i.e., one where every primary selects his price deterministically) in the above game.*

Before proving Theorem 1, we state a definition. A strategy p_i of player i is said to *strictly dominate* [42] another strategy p_i' if:

$$E\{u_i(p_i, p_{-i})\} > E\{u_i(p_i', p_{-i})\}, \ \forall p_{-i}$$

Proof of Theorem 1. For every primary i, and any p_{-i}, $u_i(c, p_{-i}) = 0$. Also, $E\{u_i(p_i, p_{-i})\} > 0$ for all $p_i \in (c, v]$ because primary i gets a positive payoff in the event that no other primary has unused bandwidth, which happens with positive probability. Thus, the strategy $p_i = c$ is strictly dominated by each $p_i \in (c, v]$, and hence no primary sets $p_i = c$ in any pure-strategy Nash equilibrium.

Suppose $(p_1, \ldots, p_n)$ is a pure-strategy Nash equilibrium, where $c < p_i \leq v$ for $i = 1, \ldots, n$. Let $p_{min} = \min(p_1, \ldots, p_n)$, $S_{min} = \{i : p_i = p_{min}\}$, and $n_{min} = |S_{min}|$. Note that S_{min} is the set of primaries who set the lowest price p_{min}, and n_{min} is its cardinality. One of the following two cases must hold:

Case (i): $n_{min} \leq k$

Since $k \leq n-1$, $n_{min} \leq n-1$ and hence at least one primary sets a price above p_{min}. Since $p_i \leq v$, $i = 1, \ldots, n$, it follows that $p_{min} < v$.

Let $p_j = \min\{p_i : i \notin S_{min}\}$ be the second lowest price. Now, note that $\forall i \in S_{min}$, $u_i(p_{min}, p_{-i}) = p_{min} - c$ and $u_i(p_i', p_{-i}) = p_i' - c \ \forall p_i' \in (p_{min}, p_j)$. This is because the bandwidth of primary i always gets sold for any $p_i' < p_j$, since it is among the primaries with the $n_{min} \leq k$ lowest prices. So $\forall i \in S_{min}$:

$$u_i(p_{min}, p_{-i}) < u_i(p_i', p_{-i}) \ \forall p_i' \in (p_{min}, p_j)$$

Hence $p_i = p_{min}$ is not a best response to p_{-i}, which contradicts the assumption that $(p_1, \ldots, p_n)$ is a Nash equilibrium.

Case (ii): $n_{min} > k$

In this case, for $i \in S_{min}$:

$$E\{u_i(p_{min}, p_{-i})\} = (p_{min} - c)P(E_1)$$

where E_1 is the event that primary i's bandwidth is bought by a secondary. Note that $P(E_1) < 1$ because with a positive probability, k or more primaries, other than i, in S_{min} have unused bandwidth. In this case, k randomly selected primaries, out of the primaries in S_{min} who have unused bandwidth, sell their bandwidth, and with a positive probability, primary i is not among them. Also, note that primary i's bandwidth is always sold if it sets a price less than p_{min} and the vector of prices of primaries other than i is p_{-i}. Hence, for small enough $\varepsilon > 0$:

$$\begin{aligned} E\{u_i(p_{min} - \varepsilon, p_{-i})\} &= (p_{min} - \varepsilon - c) \\ &> (p_{min} - c)P(E_1) \\ &= E\{u_i(p_{min}, p_{-i})\} \end{aligned}$$

Thus, $p_i = p_{min}$ is not a best response, which contradicts the assumption that $(p_1, \dots, p_n)$ is a Nash equilibrium. $\square$

In contrast, in the Bertrand game, which corresponds to $q = 1$ in our model, the pure strategy profile under which each primary deterministically selects c as his price is the unique NE [42]. This strategy profile is not a NE in our context as this provides 0 utility for each primary, whereas by quoting any price above c (and less than or equal to v) each primary can attain a positive utility since he will sell his unused bandwidth at least when he is the only primary that has unused bandwidth which happens with positive probability (since $q < 1$).

In the rest of this chapter, we intuitively derive the symmetric NE in the above game, which turns out to be the unique symmetric NE. We defer the formal proofs until Section 3.3.4 in the next chapter.

For convenience, we introduce the notion of "pseudo-price" for each primary. The pseudo-price of primary j, p'_j, is the price he selects if he has unused bandwidth; $p'_j = v + 1$ otherwise [7]. Consider primary 1 and let $p'_{(k),1}$ denote the k'th smallest pseudo-price among the pseudo-prices of the rest of the primaries, i.e., $p'_j, j \in \{2, \dots, n\}$ (which

[7]The choice $v + 1$ is arbitrary. Any other value greater than v would also work.

primary 1 will know only after choosing his price or equivalently pseudo-price). Since the primaries choose their prices randomly and since their bandwidth availabilities are random, $p'_{(k),1}$ is a random variable; let $F(.)$ be its d.f. If primary 1 offers a price of x, he sells his bandwidth only if $p'_{(k),1} > x$ (since there are k secondaries who opt for the lowest available prices), which happens with probability $(1-F(x))$; the sale fetches a utility of $x-c$. Hence, primary 1's expected utility is $(x-c)(1-F(x))$. Now, under NE, primary 1's price distribution being his best response to those of others, he must attain the same expected utility for the entire range of prices he is randomly choosing his price from, more technically, in the entire support set [8] of his price distribution; this is because his best response price distribution will never select from the less profitable ones which will not therefore be in its support set. Thus, $(x-c)(1-F(x))$ is the same (i.e., a constant) for all x in the support set for his NE price distribution. Hence, $F(x)$ is fully specified once this constant is known, which we determine by considering $F(v)$. Note that $F(v)$ is the probability that $p'_{(k),1} \leq v$, which happens when k or more primaries have unused bandwidth (among those in $\{2,\ldots,n\}$); this probability therefore is $w(q,n)$, where:

$$w(q,n) = \sum_{i=k}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} q^i (1-q)^{n-1-i}. \quad (2)$$

Thus, $F(v) = w(q,n)$. Hence, the constant in question is $(v-c)(1-F(v)) = (v-c)(1-w(q,n))$. Thus, in the support set of $F(.)$, $F(x) = 1 - \frac{(v-c)(1-w(q,n))}{x-c}$. The x at

[8]The support set of a d.f. is the smallest closed set such that its complement has probability zero under the d.f. [19].

which $F(x) = 0$ provides the lower limit of this support set, which, from the above expression, is:

$$\tilde{p} = v - w(q,n)(v-c). \tag{3}$$

Thus,

$$F(x) = \begin{cases} 0, & x \leq \tilde{p} \\ \frac{x-\tilde{p}}{x-c}, & \tilde{p} < x \leq v. \end{cases} \tag{4}$$

We now only need to determine a price d.f. $\psi(.)$ for each primary that leads to the above d.f. $F(.)$ for the kth smallest pseudo-price of $n-1$ primaries. Note that the pseudo-price for any given primary is less than or equal to x (where $x \leq v$) whenever he has unused bandwidth and he quotes a price of x or less: the probability that both these events occur is $q\psi(x)$. Thus, since $F(x)$ is the probability that k or more pseudo-prices (among those $n-1$) are less than or equal to x, $F(x)$ equals

$$\sum_{i=k}^{n-1} \binom{n-1}{i} [q\psi(x)]^i [1-q\psi(x)]^{n-1-i},$$

for all $x \leq v$. Thus, since we know $F(.)$ from (4), we can compute $\psi(x) = (1/q)\phi(x)$, where $\phi(x)$ is the solution of the following equation:

$$\sum_{i=k}^{n-1} \binom{n-1}{i} [\phi(x)]^i [1-\phi(x)]^{n-1-i} = F(x). \tag{5}$$

We can in fact formally prove that:

Lemma 1. *Equation (5) has a unique solution $\phi(x) \in [0,1]$. The function $\phi(x)$ is strictly increasing and continuous on $[\tilde{p}, v]$. Also, $\phi(\tilde{p}) = 0$ and $\phi(v) = q$.*

And, the symmetric NE price d.f. $\psi(.)$ is:

$$\psi(x) = \begin{cases} 0, & x \leq \tilde{p} \\ \frac{1}{q}\phi(x), & \tilde{p} < x \leq v \\ 1, & x \geq v \end{cases} \tag{6}$$

From the properties of the $\phi(.)$ function obtained in Lemma 1, $\psi(x)$ is a continuous d.f[9].

The above intuitive justification however glosses over some technical, nonetheless important, details: we implicitly assume that $F(.)$ is continuous and that the set of best responses of a primary is a convex set. In the formal proof, we prove both the above for any symmetric NE and subsequently establish that:

Theorem 2. *The strategy profile in which each primary i chooses his price p_i according to $\psi(.)$, where $\psi(.)$ is defined by* (6), (5), (4) *is the unique symmetric NE.*

This random selection of prices as per $\psi(.)$ can be interpreted as follows: each primary i sets a base price v and randomly holds "sales" to attract secondaries by lowering the price to some value $p_i \in [\tilde{p}, v]$[10].

Example: For $n = 2$ and $k = 1$, we have $w(q,n) = q$, $\tilde{p} = v - q(v-c)$, and

$$\psi(x) = \begin{cases} 0 & x \leq \tilde{p} \\ \frac{1}{q}\left(\frac{x-\tilde{p}}{x-c}\right) & \tilde{p} < x \leq v \\ 1 & x \geq v \end{cases} \tag{7}$$

[9] A function $f(x)$ is a d.f. iff it is increasing, right continuous, and has limits 0 and 1 as x tends to $-\infty$ and ∞ respectively [19].

[10] This interpretation has been suggested in [69] for random selection of prices in a different context.

Chapter 3

Spectrum Pricing Games with Asymmetric Bandwidth Availability Probabilities

3.1 Introduction

In Chapter 2, we described a simplified spectrum pricing games model and provided an intuitive analysis. In this chapter, we analyze the general asymmetric model in which the bandwidth availability probabilities of the primaries need not be all equal. In Section 3.3, we analyze the game in a single time slot and in Section 3.4, we study its repeated version in which the game is repeated an infinite number of times.

For the game in a single slot, we explicitly compute the Nash Equilibrium (NE) and show its uniqueness. Our explicit NE computations provide valuable insights, which we describe in Section 3.3.5.

3.2 Model

In this chapter, we analyze the model described in Section 2.2, with some differences that we now describe. In the model described in Section 2.2, we assumed that every primary has unused bandwidth with the same probability $q \in (0,1)$. Now, instead, suppose primary $i \in \{1,\ldots,n\}$ has unused bandwidth with probability $q_i \in (0,1)$, where we assume without loss of generality that:

$$q_1 \geq q_2 \geq \ldots \geq q_n. \tag{8}$$

Also, in the model described in Section 2.2, we assumed that there are a fixed number of secondaries k. However, in practice, each secondary may be a local spectrum provider or even a user seeking to lease spectrum bands to transmit data on an on-demand basis. So the number of secondaries seeking to buy bandwidth may be random and and also apriori unknown to the primaries, due to user mobility, varying bandwidth requirements of the secondaries, etc. Thus, the number of secondaries seeking to buy bandwidth (henceforth referred to as the number of secondaries for simplicity) is K, where K is a random variable with probability mass function (p.m.f.) $Pr(K=k)=\gamma_k$. The primaries apriori know only the γ_ks, but not the values of K. We will make some technical assumptions on the p.m.f. $\{\gamma_k\}$: (i) $\sum_{k=0}^{n-1}\gamma_k > 0$ (i.e., the total number of primaries exceeds the number of secondaries with positive probability, but not necessarily probability 1) (ii) if $\gamma_0 > 0$, then $\gamma_1 > 0$ (if the event that no secondary requires bandwidth has positive probability, then the event that only 1 secondary requires bandwidth also has positive probability). A large class of probability mass functions, including those generated from the most common scenario, where each local provider or user from a given pool requires bandwidth with a positive probability independent of others, satisfy both the above assumptions.

3.3 One-Shot Game Nash Equilibrium Analysis

Recall from Chapter 2 that the pseudo-price of primary $i \in \{1,\ldots,n\}$, denoted as p'_i, is the price he selects if he has unused bandwidth and $p'_i = v+1$ otherwise. As before, let $\psi_i(.)$ be the distribution function (d.f.) of the price p_i of primary i. Also, let $\phi_i(.)$ be the d.f. of p'_i. For $c \leq x \leq v$, $p'_i \leq x$ for a primary i iff he has unused bandwidth and sets a price $p_i \leq x$. So $\phi_i(x) = q_i P(p_i \leq x) = q_i \psi_i(x)$. Thus, $\psi_i(.)$ and $\phi_i(.)$ differ only by a constant factor on $[c, v]$ and we use them interchangeably wherever applicable.

For a function $f(.)$, we denote the left and right hand side limits at a point a, $\lim_{x\uparrow a} f(x)$ and $\lim_{x\downarrow a} f(x)$ by $f(a-)$ and $f(a+)$ respectively [58].

The proofs of the results in this section are technical and we relegate them to Section 3.5.

3.3.1 Necessary Conditions for a NE

Consider a NE under which the d.f. of the price (respectively, pseudo-price) of primary $i \in \{1,\dots,n\}$ is $\psi_i(.)$ (respectively, $\phi_i(.)$). In Theorem 3 below, we show that the NE strategies must have a particular structure. This is the most challenging part of the analysis; given this structure, the computation of the NE strategies is relatively straightforward. Before stating Theorem 3, we describe some basic properties of the NE strategies.

Property 1. *$\phi_2(.),\dots,\phi_n(.)$ are continuous on $[c,v]$. $\phi_1(.)$ is continuous at every $x \in [c,v)$, has a jump* [11] *of size $q_1 - q_2$ at v if $q_1 > q_2$ and is continuous at v if $q_1 = q_2$.*

Thus, there does not exist a pure strategy NE (one in which every primary selects a single price with probability (w.p.) 1).

Now, let $u_{i,max}$ be the expected payoff that primary i gets in the NE and L_i be the lower endpoint of the support set [12] of $\psi_i(.)$, *i.e.*:

$$L_i = \inf\{x : \psi_i(x) > 0\}. \tag{9}$$

Also, let w_i be the probability of the event that at least K primaries among $\{1,\dots,n\}\backslash i$ have unused bandwidth. Let r be the probability that $K \geq 1$. Note that $r = 1-\gamma_0$, and w_i can be easily computed using the p.m.f $\{\gamma_k\}$ and the fact that each primary j

[11] A d.f. $f(x)$ is said to have a *jump* (discontinuity) of size $b > 0$ at $x = a$ if $f(a) - f(a-) = b$, where $f(a-) = \lim_{x\uparrow a} f(x)$.

[12] The support set of a d.f. is the smallest closed set such that its complement has probability zero under the d.f.

independently has unused bandwidth w.p. q_j.

Property 2. *$L_1 = \ldots L_n = \tilde{p}$, where $\tilde{p} = c + \frac{(v-c)(1-w_1)}{r}$. Also, $u_{i,max} = (\tilde{p} - c)r$, $i = 1, \ldots, n$.*

Thus, the lower endpoints of the support sets of the d.f.s $\psi_1(.), \ldots, \psi_n(.)$ of all the primaries are the same, and every primary gets the same expected payoff in the NE. Note that in the symmetric case $q_1 = \ldots = q_n$, if $\gamma_k = 1$ for some k and $\gamma_l = 0$ $\forall l \neq k$, then $w_1 = \ldots = w_n = w(q,n)$, where $w(q,n)$ is given by (2) in Chapter 2 and $\tilde{p}$ reduces to the expression in (3) in Chapter 2.

Theorem 3. *The following are necessary conditions for strategies $\phi_1(.), \ldots, \phi_n(.)$ to constitute a NE:*

1) $\phi_1(.), \ldots, \phi_n(.)$ satisfy Property 1 and Property 2.

2) There exist numbers R_j, $j = 1, \ldots, n+1$, and a function $\{\phi(x) : x \in [\tilde{p}, v)\}$ such that

$$\tilde{p} = R_{n+1} < R_n \leq R_{n-1} \leq \ldots \leq R_1 \leq v, \tag{10}$$

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x), \ \tilde{p} \leq x < R_j, \tag{11}$$

for each $j \in \{1, \ldots, n\}$,

$$\text{and } \ \phi_j(R_j) = q_j, \ j = 1, \ldots, n. \tag{12}$$

Also, every point in $[\tilde{p}, R_j)$ is a best response for primary j and he plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability. Finally, $R_1 = R_2 = v$.

Theorem 3 says that all n primaries play prices in the range $[\tilde{p}, R_n)$, the d.f. $\phi_n(.)$ of primary n stops increasing at R_n, the remaining primaries $1, \ldots, n-1$ also play prices in the range $[R_n, R_{n-1})$, the d.f. $\phi_{n-1}(.)$ of primary $n-1$ stops increasing at R_{n-1}, and so on. Also, primary 1's d.f. $\phi_1(.)$ has a jump of height $q_1 - q_2$ at v if $q_1 > q_2$. Fig. 3.1 illustrates the structure.

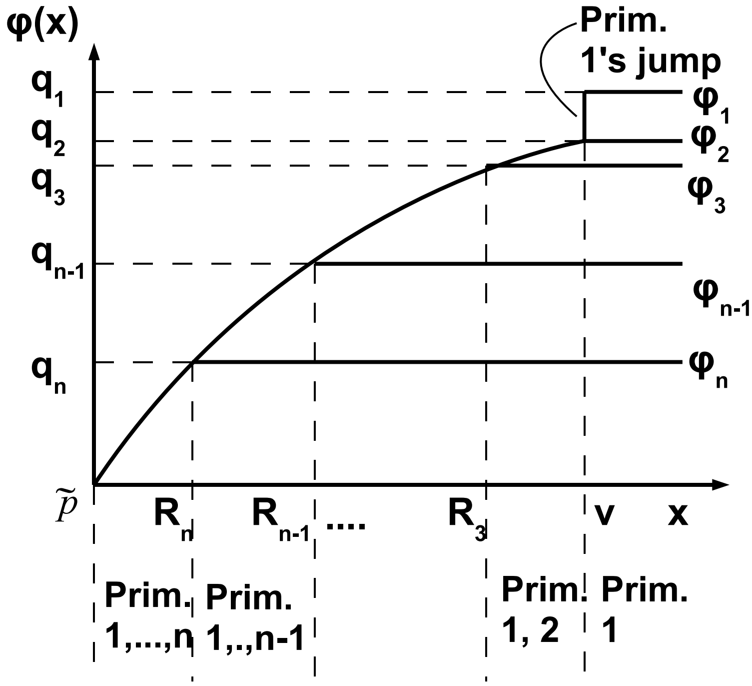

Figure 3.1: The figure shows the structure of a NE described in Theorem 3. The horizontal axis shows prices in the range $x \in [\tilde{p}, v]$ and the vertical axis shows the functions $\phi(.)$ and $\phi_1(.), \ldots, \phi_n(.)$.

3.3.2 Explicit Computation, Uniqueness and Sufficiency

By Theorem 3, for each $i \in \{1, \ldots, n\}$:

$$\phi_i(x) = \begin{cases} \phi(x), & \tilde{p} \leq x < R_i \\ q_i, & x \geq R_i \end{cases} \tag{13}$$

So the candidate NE strategies $\phi_1(.), \ldots, \phi_n(.)$ are completely determined once $\tilde{p}, R_1, \ldots, R_n$ and the function $\phi(.)$ are specified. Also, Property 2 provides the value of $\tilde{p}$, and $R_1 = R_2 = v$ by Theorem 3. First, we will show that there also exist unique $R_3, \ldots, R_n$ and $\phi(.)$ satisfying (10), (11), and (12) and will compute them. Then, we will show that the resulting strategies given by (13) indeed constitute a NE (sufficiency).

Let p'_{-i} be the K'th smallest pseudo-price out of the pseudo-prices, $\{p'_l : l \in \{1, \ldots, n\}, l \neq i\}$, of the primaries other than i (with $p'_{-i} = 0$ if $K = 0$ and $p'_{-i} = v + 2$ if $K > n - 1$). Also, let $F_{-i}(x)$ denote the d.f. of p'_{-i}. Since there are K secondaries, if primary 1 has unused bandwidth and sets $p_1 = x \in [\tilde{p}, v)$, its bandwidth is bought iff [13] $p'_{-1} > x$, which happens w.p. $1 - F_{-1}(x)$. Note that primary 1's payoff is $(x - c)$ if its bandwidth is bought and 0 otherwise. So, letting $E\{u_i(x, \psi_{-i})\}$ denote the expected payoff of primary i if it sets a price x and the other primaries use the strategy profile ψ_{-i}, we have:

$$E\{u_1(x, \psi_{-1})\} = (x - c)(1 - F_{-1}(x)) = (\tilde{p} - c)r, \; x \in [\tilde{p}, v) \tag{14}$$

where the second equality follows from the facts that each $x \in [\tilde{p}, v)$ is a best response for primary 1 by Theorem 3, and $u_{1,max} = (\tilde{p} - c)r$ by Property 2. By (14), we get:

$$F_{-1}(x) = g(x), \; x \in [\tilde{p}, v) \tag{15}$$

$$\text{where, } g(x) = \frac{x - \tilde{p}}{x - c}, \; x \in [\tilde{p}, v). \tag{16}$$

Next, we calculate $R_i, i = 3, \ldots, n$ and $\phi(.)$ using (15).

[13] By Property 1, no primary has a jump at any $x \in [\tilde{p}, v)$. So $P(p'_{-1} = x) = 0$.

3.3.2.1 Computation of $R_i, i = 3, \ldots, n$

For $0 \leq y \leq 1$, let $f_i(y)$ be the probability of K or more successes out of $n-1$ independent Bernoulli events, $(i-1)$ of which have the same success probability y and the remaining $(n-i)$ have success probabilities $q_{i+1}, \ldots, q_n$. An expression for $f_i(y)$ can be easily computed.

Now, to compute $R_i, i \in \{3, \ldots, n\}$, we note that by (13) and (10), $\phi_j(R_i) = q_i,\ j = 2, \ldots, i$, and $\phi_j(R_i) = q_j,\ j = i+1, \ldots, n$. So from the preceding paragraph, with the events $\{p'_j \leq R_i\}, j = 2, \ldots, n$ as the $n-1$ Bernoulli events, and by the definition of $F_{-1}(.)$, we get:

$$F_{-1}(R_i) = f_i(q_i). \tag{17}$$

By (15) and (17):

$$g(R_i) = f_i(q_i). \tag{18}$$

By (16) *and* (18)*, R_i is unique and is given by:*

$$R_i = c + \frac{(\tilde{p} - c)r}{1 - f_i(q_i)}. \tag{19}$$

3.3.2.2 Computation of $\phi(.)$

Now we compute the function $\{\phi(.) : x \in [\tilde{p}, v)\}$ by separately computing it for each interval $[R_{i+1}, R_i)$, $i \in \{2, \ldots, n\}$. If $R_{i+1} = R_i$, then note that the interval $[R_{i+1}, R_i)$ is empty. Now suppose $R_{i+1} < R_i$. For $x \in [R_{i+1}, R_i)$, by (13) and (10):

$$\phi_j(x) = q_j,\ j = i+1, \ldots, n \tag{20}$$

$$\text{and } \phi_1(x) = \ldots = \phi_i(x) = \phi(x). \tag{21}$$

By definition of the function $f_i(.)$, with the events $\{p'_j \leq x\}, j = 2, \ldots, n$ as the $n-1$ Bernoulli events, by definition of $F_{-1}(x)$ and using $P\{p'_j \leq x\} = \phi_j(x)$, (20) and (21):

$$F_{-1}(x) = f_i(\phi(x)),\ R_{i+1} \leq x < R_i. \tag{22}$$

By (15) and (22):

$$f_i(\phi(x)) = g(x),\ R_{i+1} \leq x < R_i. \tag{23}$$

Lemma 2. *For each x,* (23) *has a unique solution* $\phi(x)$*. The function* $\phi(.)$ *is strictly increasing and continuous on* $[\tilde{p}, v)$*. For* $i \in \{2, \ldots, n\}$*,* $\phi(R_i) = q_i$*. Also,* $\phi(\tilde{p}) = 0$*.*

Thus, there is a unique function $\phi(.)$*, and by* (13)*, unique* $\phi_i(.), i = 1, \ldots, n$ *that satisfy the conditions in Theorem 3.*

3.3.2.3 Sufficiency

Theorem 4. *The pseudo-price d.f.s* $\phi_i(.), i = 1, \ldots, n$ *in* (13)*, with* $R_1 = R_2 = v$*,* R_i*,* $i = 3, \ldots, n$ *given by* (19)*, and* $\phi(.)$ *being the solution of* (23)*, constitute the unique NE. The corresponding price d.f.s are* $\psi_i(x) = \frac{1}{q_i}\phi_i(x)$*,* $x \in [c, v]$*,* $i = 1, \ldots, n$*.*

3.3.3 Efficiency of the unique NE

The *efficiency*, η, of a NE quantifies the loss in total revenue incurred owing to lack of cooperation among primaries. η may be defined as $\eta = \frac{R_{\mathrm{NE}}}{R_{\mathrm{OPT}}}$, where R_{NE} is the expected sum of utilities of the primaries at the NE and R_{OPT} is the maximum possible (optimal) expected sum of utilities. Note that R_{OPT} is attained only when all primaries cooperate and each selects the maximum possible price v so as to ensure that bandwidth is always sold at this price. Now, $R_{\text{OPT}} = E[\min(Z,K)](v-c)$, where Z is the number of primaries who have unused bandwidth. Also, from Property 2, at the unique NE, whenever a primary has unused bandwidth, he attains an expected utility of $(v-c)(1-w_1)$. Thus, since primary i has unused bandwidth with probability q_i, $R_{\text{NE}} = (1-w_1)(v-c)\sum_{i=1}^{n} q_i$. Hence,

$$\eta = \frac{(1-w_1)\sum_{i=1}^{n} q_i}{E[\min(Z,K)]}. \tag{24}$$

Now, assume for simplicity that each secondary out of a pool of αn secondaries independently requires bandwidth with some probability, where α is a constant. The following lemma characterizes the asymptotic behavior of the efficiency for a large number of primaries and secondaries.

Lemma 3. *Suppose $K = K_n$, the number of secondaries (who require bandwidth) grows with the number of primaries n. Let $E(K_n) = k_n$. Assume that $k_n \geq \beta$ for all large n for some constant $\beta > 0$.*

1. *If $k_n \leq \sum_{i=2}^{n}(q_i - \varepsilon)$ for all n and some $\varepsilon > 0$, then $\eta \to 0$ as $n \to \infty$.*
2. *If $k_n \geq \sum_{i=2}^{n}(q_i + \varepsilon)$ for all n and some $\varepsilon > 0$, then $\eta \to 1$ as $n \to \infty$.*

Note that $\sum_{i=2}^{n} q_i$ is the expected number of primaries out of primaries $2,\dots,n$ who have unused bandwidth. So the above lemma roughly states that when the expected demand for bandwidth (k_n) is lower than the expected supply of bandwidth, the efficiency of the NE is close to 0 and vice versa. The intuition is that when the demand is low compared to the supply, there is intense price competition among the primaries to sell to the few secondaries who are present, driving down the prices and thereby the efficiency of the NE.

3.3.4 Symmetric q

We now prove the results that were stated but not proved in Section 2.5. We first show that Theorem 2 in Chapter 2 follows as a corollary of Theorem 4 above. When $q_1 = q_2 \ldots = q_n = q$, by definition of the function $f_i(.)$ defined in Section 3.3.2.1, for each $i = 1, \ldots, n$, $f_i(q)$ is the probability of k or more successes out of $n-1$ independent Bernoulli events, each with success probability q. So:

$$f_i(q) = \sum_{j=k}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} q^i (1-q)^{n-i-1} = w(q,n) = w_1 \tag{25}$$

where the second equality follows from (2) and the third equality follows from the definition of w_1, which was defined just before Property 2. So by Property 2,

$$\tilde{p} = v - w(q,n)(v-c). \tag{26}$$

Also, by (19), (25) and (26), for each $i = 1, \ldots, n$:

$$\begin{aligned} R_i &= c + \frac{(v-c)(1-w(q,n))}{(1-w(q,n)} \\ &= v \end{aligned}$$

Thus, all primaries play prices in the range $[\tilde{p}, v]$.

Now, we put $i = n$ in (23) to get:

$$f_n(\phi(x)) = g(x),\ \tilde{p} \leq x < v \tag{27}$$

By definition of the function $f_i(.)$:

$$f_n(\phi(x)) = \sum_{i=k}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} [\phi(x)]^i [1-\phi(x)]^{n-1-i} \tag{28}$$

By (16) and (28), (27) becomes:

$$\sum_{i=k}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} [\phi(x)]^i [1-\phi(x)]^{n-1-i} = \frac{x-\tilde{p}}{x-c}$$

Note that the above equation is the same as (5), where $F(.)$ in (5) is given by (4). Also, $\phi(.)$ is the pseudo-price d.f. of each primary and the corresponding price d.f. is given by (6). So we have proven a strengthening of Theorem 2 in Chapter 2, namely that the strategy profile identified in that Theorem is the unique NE (not only the unique symmetric NE). Also, Lemma 1 in Chapter 2 follows from Lemma 2.

3.3.5 Discussion

The structure of the unique NE identified in Theorems 3 and 4 provides several interesting insights:

1) First, from (8), (10) and the fact that the support set of $\psi_i(.)$ is $[\tilde{p}, R_i]$, it follows that only the primaries with a high bandwidth availability probability (q) play high prices (see Fig. 3.1). Intuitively this is because all the primaries play low prices (near $\tilde{p}$), so if a primary sets a high price, he is undercut by all the other primaries. But a primary

with a high q runs a lower risk of being undercut than one with a low q because of the lower bandwidth availability probabilities of the set of primaries *other than itself.*

2) Second, by Property 1, $\psi_1(.)$ has a jump at v iff $q_1 > q_2$ and is continuous everywhere else, whereas $\psi_2(.), \ldots, \psi_n(.)$ are always continuous on $[c, v]$.

The above insights highlight the differences of the asymmetric case from the symmetric case $q_1 = \ldots = q_n$ discussed in Section 3.3.4, in which the support set of every d.f. $\psi_i(.), i = 1, \ldots, n$ is the same ($[\tilde{p}, v]$) and they are all continuous everywhere.

3.4 Repeated Game

3.4.1 Model

We now consider the repeated game version of the one-shot game analyzed in Section 3.3. Suppose the one-shot game is repeated an infinite number of times, at times $\tau = 1, 2, 3, \ldots$. We refer to the game in each individual time slot as a *stage game*. Each player perfectly recalls the actions of every player in all preceding time slots. The payoff of player i for the overall repeated game is defined to be $u_i = \sum_{\tau=1}^{\infty} \delta^{\tau-1} u_{i,\tau}$, where $u_{i,\tau}$ is his payoff at time τ and $\delta \in (0, 1)$ is the *discount factor*, which is used to discount future payoffs (see [42], [43] for interpretations of the discount factor). The discount factor is usually close to 1 [42].

A strategy of a player in a repeated game is a complete plan for choosing the action in each slot as a function of the actions of all players in all preceding slots. As in a one-shot game, a Nash equilibrium (NE) in a repeated game is a strategy profile in which no player can improve his payoff by unilateral deviation from his strategy [42]. However, NE constitutes a rather weak notion of equilibrium in repeated games [42] and hence we focus on NE with a special property, known as the *Subgame Perfect Nash Equilibria (SPNE)* [42]. A *subgame* [42] of the repeated game is the part of the game starting from some slot $\tau_0 \geq 1$, *i.e.* the stage games in slots $\tau = \tau_0, \tau_0 + 1, \ldots$. An SPNE is an NE of the repeated game that is also an NE of every subgame [42].

3.4.2 Results

It is well-known that for any repeated game, the strategy profile under which every player uses the one-shot game NE strategy in every time slot is a SPNE [42]. Thus, the NE we found in Section 3.3 for the one-shot game provides a SPNE in the repeated game version. *Our main contribution, described in the rest of this section, is to present an SPNE that is also efficient in the sense that the sum of expected utilities of the n primaries at equilibrium equals the maximum possible sum of utilities, provided the discount factor δ is sufficiently high.*

We consider *Nash reversion* [42] type of strategy profiles in which each player plays a specified strategy (called the pre-deviation strategy [42]) at each time until one of the players deviates from it, and all players play the one-shot game NE strategy thereafter.

Strategy for primary i: *Select a price of v at $\tau = 1$, and also for all other τ so long as all other primaries had chosen v in all previous times. Otherwise, play the one-shot* *game Nash equilibrium strategy* $\psi_i(.)$.

Let u_i^{OS} be the expected payoff, conditional on him having unused bandwidth, that primary i receives in the one-shot game Nash equilibrium, which we have shown to be unique in Section 3.3. Let u_i^{PD} be his expected payoff, (conditional on having unused bandwidth), in each stage game of the repeated game when all primaries play the pre-deviation strategy in the above Nash reversion strategy. Also, let u_i^{sup} be the supremum over the possible expected payoffs that primary i can get, (conditional on having unused bandwidth), in a single stage game by using any strategy, when all primaries played the pre-deviation strategy in all slots until the previous stage game, and primaries other than i play the pre-deviation strategy in the current stage game.

It can be shown that a necessary and sufficient condition for the above Nash reversion strategy to be a SPNE (the proof is similar to that of (12.AA.1) in [42]) is that for each primary $i = 1, \ldots, n$:

$$u_i^{sup} + \frac{q_i\delta}{1-\delta}u_i^{OS} \leq u_i^{PD} + \frac{q_i\delta}{1-\delta}u_i^{PD} \tag{29}$$

Note that the left-hand side is primary i's maximum (discounted) payoff starting from a given slot if he deviates from the pre-deviation strategy, and the right-hand side is the payoff if he does not deviate. (The factor q_i appears in the second term on either side to account for the fact that primary i would have free bandwidth in each future slot with probability q_i.) So if condition (29) is met, primary i would not deviate from its pre-deviation strategy. Also, under the pre-deviation strategy, every primary always sets the maximum price of v. So the sum of utilities of the primaries is the maximum possible.

Thus, if the condition in (29) is satisfied, the strategy profile in which all primaries play the above Nash reversion strategy constitutes an efficient SPNE.

In the rest of this section, we simplify the condition in (29). The condition is equivalent to:

$$\frac{1}{\delta} \leq 1 + q_i \left(\frac{u_i^{PD} - u_i^{OS}}{u_i^{sup} - u_i^{PD}} \right), \; i = 1, \ldots, n. \tag{30}$$

Next, we compute u_i^{sup}, u_i^{OS} and u_i^{PD}. To compute u_i^{sup}, note that when primaries other than i set a price of v, primary i's expected payoff is maximized when he sets a price just below v. So:

$$u_i^{sup} = v - c. \tag{31}$$

By Property 2, the payoff that each primary gets in the one-shot game NE is the same, and equals:

$$u_i^{OS} = (v-c)(1-w_1) \tag{32}$$

where w_j is the probability that K or more primaries out of $\{1,\ldots,n\}\backslash j$ have unused bandwidth.

Now we compute u_i^{PD}. Let Z_{-i} be the number of primaries out of $\{1,\ldots,n\}\backslash i$ who have unused bandwidth in a given slot. Let $P_i(win)$ be the probability that primary i's bandwidth is sold if he and each of the other primaries set a price of v [14]. Note that:

$$u_i^{PD} = (v-c)P_i(win). \tag{33}$$

[14] Recall that this computation of u_i^{PD} is conditional on primary i having unused bandwidth, but not conditional on the other primaries having unused bandwidth.

Also recall that $P(K=k)=\gamma_k$; also, if $K=k$ and if more than k primaries have unused bandwidth and set the same price of v, then the bandwidth of k of them, randomly selected, is bought. So:

$$P_i(win)=\begin{cases} 1, & \text{if } K=k \text{ and } Z_{-i}\leq k-1 \\ \frac{k}{Z_{-i}+1}, & \text{if } K=k \text{ and } Z_{-i}\geq k \end{cases} \tag{34}$$

So:

$$P_i(win)=\sum_k \left(P(Z_{-i}\leq k-1)+\sum_{j=k}^{n-1} P(Z_{-i}=j)\frac{k}{j+1} \right)\gamma_k. \tag{35}$$

$P(Z_{-i}=j)$ and $P(Z_{-i}\leq k-1)$, and using them $P_i(win)$, can be easily computed using the fact that primary $l\in\{1,\ldots,n\}\backslash i$ has unused bandwidth w.p. q_l.

By (32) and (33):

$$u_i^{PD}-u_i^{OS}=(v-c)(P_i(win)-(1-w_1)). \tag{36}$$

Also, by (31) and (33):

$$u_i^{sup}-u_i^{PD} = (v-c)(1-P_i(win)) \tag{37}$$

$$> 0 \tag{38}$$

since clearly $P_i(win)<1$.

We claim that by (8) and the definition of $P_i(win)$:

$$P_1(win)\geq\ldots\geq P_n(win). \tag{39}$$

The reason (39) holds is as follows. Consider primaries i and j, where $i<j$. When every primary sets a price of v, a primary's bandwidth is likelier to be sold the fewer

the other primaries who have unused bandwidth. Also, the set of primaries other than i (respectively, j) consists of primaries $\{1,\ldots,n\}\backslash\{i,j\}$ and primary j (respectively, i). Since $q_i \geq q_j$ by (8), more primaries out of the set of primaries other than primary i are likely to have unused bandwidth than out of the set of primaries other than primary j and hence $P_i(win) \geq P_j(win)$. Equation (39) follows.

By (36) and (39):

$$(u_1^{PD} - u_1^{OS}) \geq \ldots \geq (u_n^{PD} - u_n^{OS}). \tag{40}$$

Also, by (37) and (39):

$$(u_1^{sup} - u_1^{PD}) \leq \ldots \leq (u_n^{sup} - u_n^{PD}). \tag{41}$$

By (8), (40) and (41):

$$q_1 \left(\frac{u_1^{PD} - u_1^{OS}}{u_1^{sup} - u_1^{PD}} \right) \geq \ldots \geq q_n \left(\frac{u_n^{PD} - u_n^{OS}}{u_n^{sup} - u_n^{PD}} \right). \tag{42}$$

Now, for $i = 1,\ldots,n$, let

$$\delta_i = \frac{1}{1 + q_i \left(\frac{u_i^{PD} - u_i^{OS}}{u_i^{sup} - u_i^{PD}} \right)} \tag{43}$$

Note that the condition in (30) is equivalent to $\delta \geq \delta_i, i = 1,\ldots,n$. But by (42):

$$\delta_1 \leq \ldots \leq \delta_n. \tag{44}$$

So a necessary and sufficient condition for (30), or equivalently for (29), is $\delta \geq \delta_n$.

Thus, $\delta \geq \delta_n$ is a necessary and sufficient condition for the strategy profile corresponding to the above Nash reversion strategy to be a SPNE. Note that $u_i^{sup} - u_i^{PD} > 0 \; \forall i$

by (38). So by (43), $\delta_n < 1$ if and only if

$$u_n^{PD} > u_n^{OS}. \tag{45}$$

Thus, if (45) holds, then for δ large enough ($\delta \geq \delta_n$), the above Nash reversion strategy constitutes a SPNE. If $u_n^{PD} < u_n^{OS}$, then it does not constitute a SPNE. This is because, for primary n, the payoff under the one-shot game NE is higher than the pre-deviation payoff of the above Nash reversion strategy. So obviously, primary n will deviate in the first slot itself and set a price just below v.

Remark 1. *Note that the pre-deviation strategy profile in which every primary sets the maximum price of v can be interpreted as tacit collusion: if a primary i sees that other primaries are setting the maximum price and are not trying to undercut their competitors, then primary i also participates in the collusion and keeps setting a price of v in every slot. However, once at least one primary undercuts its competitors, the tacit collusion breaks down and primaries revert to the one-shot NE strategy.*

Remark 2. *Note that by* (33) *and* (39)*, the pre-deviation strategy profile in which all primaries set a price of v is most beneficial for primary* 1 *and least beneficial for primary n. This is intuitively the reason behind the fact that the condition ($\delta \geq \delta_n$) for the above Nash reversion strategy profile to constitute a SPNE is in terms of the parameter δ_n of primary n.*

3.5 Appendix

3.5.1 Proofs of results in Section 3.3.1

We first prove a series of lemmas and then deduce Properties 1 and 2 and Theorem 3 from them.

Lemma 4. *For $i = 1, \ldots, n$, $\psi_i(.)$ is continuous, except possibly at v. Also, at most one primary has a jump at v.*

Proof. Suppose $\psi_i(.)$ has a jump at a point x_0, $c < x_0 < v$. Then for some $\varepsilon > 0$, no primary $j \neq i$ chooses a price in $[x_0, x_0 + \varepsilon]$ because it can get a strictly higher payoff by choosing a price just below x_0 instead. This in turn implies that primary i gets a strictly higher payoff at the price $x_0 + \varepsilon$ than at x_0. So x_0 is not a best response for primary i, which contradicts the assumption that $\psi_i(.)$ has a jump at x_0. Thus, $\psi_i(.)$ is continuous at all $x < v$.

Now, suppose primary i has a jump at v. Then a primary $j \neq i$ gets a higher payoff at a price just below v than at v. So v is not a best response for primary j and he plays it with 0 probability. Thus, at most one primary has a jump at v. $\square$

Lemma 5. *For every $\varepsilon > 0$, there exist primaries m and j, $m \neq j$, such that $\psi_m(v - \varepsilon) < 1$ and $\psi_j(v - \varepsilon) < 1$.*

That is, at least two primaries play prices just below v with positive probability.

Proof. Suppose not. Fix i and let:

$$y = \inf\{x : \psi_l(x) = 1\ \forall l \neq i\}. \tag{46}$$

By definition of y, $\psi_l(x) = 1\ \forall l \neq i$ and $x > y$. Also, since $\psi_l(.)$ is a distribution function, it is right continuous [19]. So

$$\psi_l(y) = 1\ \forall l \neq i. \tag{47}$$

Suppose $y < v$. By (47):

$$P\{p_l \in (y, v]\} = 0,\ \forall l \neq i. \tag{48}$$

So every price $p_i \in (y, v)$ is dominated by $p_i = v$. Hence:

$$P\{p_i \in (y, v)\} = 0 \tag{49}$$

By (48) and (49):

$$P\{p_j \in (y, v)\} = 0,\ j = 1, \ldots, n. \tag{50}$$

By (46), $\forall \varepsilon > 0$, $\psi_l(y - \varepsilon) < 1$ for at least one primary $l \neq i$; otherwise the infimum in the RHS of (46) would be less than y. So this primary l plays prices just below y with positive probability. Now, if primary l sets a price $p_l < v$, he gets a payoff equal to the revenue, $(p_l - c)$, if bandwidth is sold, times the probability that bandwidth is sold. Also, by Lemma 4, $\psi_j(.), j = 1, \ldots, n$ are continuous at all prices below v. So by (50), a price p_l just below v yields a higher payoff than a price just below y. This is because, $p_l - c$ is lower by approximately $v - y$ for p_l just below y than for p_l just below v, but by (50) and continuity of $\psi_j(.), j = 1, \ldots, n$, the probability that bandwidth is sold for a price p_l just below y can be made arbitrarily close to the probability that bandwidth is sold for a price p_l just below v. This contradicts the assumption that primary l plays prices just below y with positive probability.

Thus, y in (46) equals v and hence at least one primary $j \neq i$ plays prices just below v with positive probability. The above arguments with j in place of i imply that at least one primary other than j plays prices just below v with positive probability. Thus, at least two primaries in $\{1,\ldots,n\}$ play prices just below v with positive probability. $\square$

Let $u_{i,max}$ and L_i be as defined in Section 3.3.1.

Lemma 6. *For $i = 1,\ldots,n$, L_i is a best response for primary i.*

Proof. By (9), either primary i has a jump at L_i or plays prices arbitrarily close to L_i and above it with positive probability.

Case (i): If primary i has a jump at L_i, then L_i is a best response for i because in a NE, no primary plays a price other than a best response with positive probability.

Case (ii): If primary i does not have a jump at L_i, then by (9), $\psi_i(L_i) = 0$. Since every primary selects a price in $[c,v]$, $\psi_i(v) = 1$. So $L_i < v$. So by Lemma 4, no primary among $\{1,\ldots,n\}\backslash i$ has a jump at L_i. Hence, primary i's payoff at a price above L_i and close enough to it is arbitrarily close to its payoff *at* L_i. But since primary i does not have a jump at L_i, by (9), he plays prices just above L_i with positive probability and they are best responses for him. So L_i is also a best response for primary i. $\square$

Lemma 7. *For some $c < \tilde{p} < v$, $L_1 = \ldots L_n = \tilde{p}$. Also, $u_{i,max} = (\tilde{p} - c)r$, $i = 1,\ldots,n$.*

That is, the lower endpoint of the support set of the price distribution of every primary is the same.

Proof. Let $L_{min} = \min\{L_m : m = 1, \ldots, n\}$, and $S_{min} = \{m : L_m = L_{min}\}$ be the set of primaries with the lowest endpoint. Let

$$k_{\min} = \min_k \{k : \gamma_k > 0\}.$$

Thus, $k_{\min}$ is the minimum number of secondaries at a location. Note that $k_{\min}$ will be 0 if $\gamma_0 > 0$, and $k_{\min} > 0$ otherwise. First, we show by contradiction that:

$$|S_{min}| \geq k_{\min} + 1. \tag{51}$$

Clearly, the above holds if $k_{\min} = 0$. We therefore show that it holds even otherwise. Suppose $|S_{min}| \leq k_{\min}$. If $L_{min} = v$, then all primaries play the price v w.p. 1, which does not constitute a NE by Lemma 4. So $L_{min} < v$ and again by Lemma 4, no primary has a jump at L_{min}. Also, by Lemma 6, L_{min} is a best response for the primaries in S_{min}. Let $\widehat{L} = \min\{L_m : m \notin S_{min}\}$ be the second lowest endpoint. Now, a primary $m \in S_{min}$ who has unused bandwidth can get a higher payoff at a price just below $\widehat{L}$ than at L_{min} because in both cases, since $|S_{min}| \leq k_{\min}$, primary m's bandwidth is sold w.p. 1; however, it gets a higher revenue at a price just below $\widehat{L}$ than at L_{min}. This contradicts the fact that L_{min} is a best response for primary m. Thus, (51) must hold.

Now, suppose $L_i < L_j$ for some i, j. By Lemma 6, L_j is a best response for primary j. Now, the expected payoff that primary j gets for $p_j = L_j$ is strictly less than the expected payoff that primary i would get if it set p_i to be just below L_j. This is because, if primaries i or j set a price of approximately L_j, then they see the same price distribution functions of the primaries other than i and j. But primary j may be undercut by primary i, since $L_i < L_j$, whereas primary i may not be undercut by primary j. Also, by (51), primary j's expected payoff is strictly lowered due to this undercutting by primary i. (Note that if $k_{\min} > 0$, undercutting by primary i would not lower primary j's probability of winning, and thereby the expected payoff, if a total of $\leq k_{\min} - 1$ primaries played prices below L_j with positive probability. This possibility is ruled out by (51). If $k_{\min} = 0$, $\gamma_0 > 0$. If in addition $\gamma_1 = 0$, and $S_{\min} = 1$, it is possible that only 1 primary (i.e., i) plays prices below L_j with positive probability. In this case, note that whenever at least 1 secondary is available, at least 2 secondaries are available (as $\gamma_1 = 0$), and hence undercutting by primary i does not lower primary j's probability of winning, and thereby the expected payoff. This possibility is ruled out by assumption (ii) on $\{\gamma_k\}$ in Section 5.2.1 since $\gamma_1 > 0$ if $\gamma_0 > 0$.) Hence, $u_{i,max} > u_{j,max}$.

Now, by Lemma 6, L_i is a best response for primary i. If primary j were to play price L_i, then it would get a payoff of $u_{i,max}$. This is because, when primary i plays price L_i, it gets payoff $u_{i,max}$. Since $L_j > L_i$, primary i is, w.p. 1, not undercut by primary j. If primary j were to set the price L_i, then w.p. 1, it would not be undercut by primary i. Also, the price distributions of the primaries other than i and j are exactly the same from the viewpoints of primaries i and j. Thus, primary j can strictly increase its payoff from $u_{j,max}$ to $u_{i,max}$ by playing price L_i, which contradicts the fact that L_j is a best response for him.

Thus, $L_i < L_j$ is not possible. By symmetry, $L_i > L_j$ is not possible. So $L_i = L_j$. Let $L_1 = \ldots = L_n = \tilde{p}$.

If $\tilde{p} = v$, then every primary plays the price v w.p. 1, which does not constitute a NE. So $\tilde{p} < v$. So by Lemma 4, no primary has a jump at $\tilde{p}$. Thus, since the lower endpoint of the support set of every primary is $\tilde{p}$, by (9), a price of $\tilde{p}$ is a best response for every primary i. Since no primary sets a price lower than $\tilde{p}$, a price of $\tilde{p}$ fetches a payoff of $\tilde{p} - c$ if $K \geq 1$ and a payoff of 0 if $K = 0$. So $u_{i,max} = (\tilde{p} - c)P(K \geq 1) = (\tilde{p} - c)r$, $i = 1, \ldots, n$. □

Let w_i be as defined in Section 3.3.1. Using (8), it can be easily shown that:

$$w_1 \leq w_2 \leq \ldots \leq w_n. \tag{52}$$

Lemma 8. $\tilde{p} = c + \frac{(1-w_1)(v-c)}{r}$.

Proof. If primary 1 sets the price $p_1 = v$, then it gets an expected payoff of at least $(v - c)(1 - w_1)$ because its bandwidth is sold at least in the event that $k - 1$ or fewer primaries out of $2, \ldots, n$ have unused bandwidth. So $u_{1,max} \geq (v - c)(1 - w_1)$. Since $u_{1,max} = (\tilde{p} - c)r$ by Lemma 7, we get:

$$\tilde{p} \geq c + \frac{(1 - w_1)(v - c)}{r}. \tag{53}$$

Now, by Lemma 5, at least two primaries, say m and j, play prices just below v with positive probability. By Lemma 4, at most one of them has a jump at v. So assume, WLOG, that no primary other than j has a jump at v. Then a price of $p_j = v$ is a best response for primary j and fetches a payoff of $u_{j,max} = (v - c)(1 - w_j) \leq$

$(v-c)(1-w_1)$, where the inequality follows from (52). Since $u_{j,max} = (\tilde{p}-c)r$ by Lemma 7, we get:

$$\tilde{p} \leq c + \frac{(1-w_1)(v-c)}{r}. \tag{54}$$

The result follows from (53) and (54). $\square$

Lemma 9. *Let $\tilde{p} \leq a < b \leq v$. Then at least two primaries play prices in (a,b) with positive probability.*

Proof. If $b = v$, then the claim is true by Lemma 5. If $a = \tilde{p}$, then the claim is true by Lemma 4 and Lemma 7, since $\tilde{p} < v$ is the lower endpoint of the support set of all primaries and no primary has a jump at $\tilde{p}$; hence all primaries play prices just above $\tilde{p}$ with positive probability.

Now, fix any a, b such that $\tilde{p} < a < b < v$. Let:

$$\underline{a} = \inf\{x \leq a : \psi_j(x) = \psi_j(a)\ \forall j = 1,\ldots,n\} \tag{55}$$

By Lemma 7, $\underline{a} > \tilde{p}$. Also, by definition of $\underline{a}$, $P\{p_j \in [\underline{a}, a]\} = 0\ \forall j = 1,\ldots,n$.

By definition of $\underline{a}$, at least one primary, say primary i, plays prices just below $\underline{a}$ with positive probability. (If not, then the infimum in (55) would be less than $\underline{a}$.) This implies that at least one primary $j \neq i$ plays prices in $(\underline{a}, b)$ with positive probability. (If not, then $p_i = b$ would yield a strictly higher payoff to primary i than prices just below $\underline{a}$.) Now, if primary j is the only primary among primaries $\{1,\ldots,n\}$ who play prices in $(\underline{a}, b)$ with positive probability, then $p_j = b$ yields a strictly higher payoff than $p_j \in (\underline{a}, b)$, which is a contradiction. So at least two primaries play prices in $(\underline{a}, b)$ with positive probability. But $P\{p_l \in [\underline{a}, a]\} = 0\ \forall l = 1, \ldots, n$ by definition of $\underline{a}$. Hence, at least two primaries play prices in (a, b) with positive probability. $\square$

Let $F_{-i}(x)$ be as defined in Section 3.3.2.

Lemma 10. *For a fixed $x \in (\tilde{p}, v]$, and primaries i and j, (i) $F_{-i}(x) = F_{-j}(x)$ iff $\phi_i(x) = \phi_j(x)$, (ii) $F_{-i}(x) < F_{-j}(x)$ iff $\phi_i(x) > \phi_j(x)$.*

Proof. Let $p'_{(l)}$ be the l'th smallest out of the pseudo-prices of the primaries *other than i and j*. Let $F_{-i,k}(x)$ be the probability that $p'_{-i} \leq x$ given that $K = k$. Clearly, $F_{-i,0}(x) = 1$ since $x > \tilde{p} \geq 0$, and $F_{-i,k}(x) = 0$ if $k > n-1$. We evaluate $F_{-i,k}(x)$ for $1 \leq k \leq n-1$. Conditioning on the event $\{p'_j \leq x\}$ and using the fact that $\{p'_l : l \neq i\}$ are independent, we get:

$$\begin{aligned}
& F_{-i,k}(x) \\
&= P\{k'\text{th smallest of } \{p'_l : l \neq i\} \leq x\} \\
&= P\{p'_j \leq x\} P\{p'_{(k-1)} \leq x\} + P\{p'_j > x\} P\{p'_{(k)} \leq x\} \\
&= \phi_j(x) P\{p'_{(k-1)} \leq x\} + (1 - \phi_j(x)) P\{p'_{(k)} \leq x\} \\
&= \phi_j(x)[P\{p'_{(k-1)} \leq x\} - P\{p'_{(k)} \leq x\}] \\
&\quad + P\{p'_{(k)} \leq x\} \qquad (56)
\end{aligned}$$

Similarly,

$$F_{-j,k}(x) = \phi_i(x)[P\{p'_{(k-1)} \leq x\} - P\{p'_{(k)} \leq x\}] + P\{p'_{(k)} \leq x\} \qquad (57)$$

By (56) and (57):

$$\begin{aligned} & F_{-i,k}(x) - F_{-j,k}(x) \\ = & \; (\phi_j(x) - \phi_i(x))[P\{p'_{(k-1)} \le x\} - P\{p'_{(k)} \le x\}] \\ = & \; (\phi_j(x) - \phi_i(x))\alpha_k \end{aligned} \tag{58}$$

where $\alpha_k = P\{p'_{(k-1)} \le x\} - P\{p'_{(k)} \le x\}$. Thus,

$$F_{-i}(x) - F_{-j}(x) = (\phi_j(x) - \phi_i(x)) \sum_{k=1}^{n-1} \alpha_k \gamma_k.$$

We will next show that $\alpha_k > 0$ for $1 \le k \le n-1$. Both parts of the result will then follow from the above.

Note that α_k equals the probability that exactly $(k-1)$ out of the pseudo-prices of the primaries other than i and j are $\le x$. Since $x > \tilde{p}$, all primaries play prices in $(\tilde{p}, x)$ with positive probability by Lemma 7. So:

$$\phi_l(x) = P\{p'_l \le x\} > 0, \; l = 1, \ldots, n. \tag{59}$$

Also,

$$\phi_l(x) \le \phi_l(v) = q_l < 1, \; l = 1, \ldots, n. \tag{60}$$

By (59) and (60):

$$0 < \phi_l(x) < 1, \; l = 1, \ldots, n. \tag{61}$$

Also, since $1 \le k \le n-1$, we have:

$$0 \le k-1 \le n-2. \tag{62}$$

Since α_k equals the probability of exactly $k-1$ successes out of $n-2$ independent Bernoulli events that have success probabilities $\{\phi_l(x) : l = 1, \ldots, n, l \neq i, j\}$, $\alpha_k > 0$ by (61) and (62). This completes the proof. □

Lemma 11. *(i)* $\phi_2(.), \ldots, \phi_n(.)$ *are continuous at* v. *(ii)* $\phi_1(.)$ *is continuous at* v *if* $q_1 = q_2$ *and has a jump of size* at most $q_1 - q_2$ *at* v *if* $q_1 > q_2$. *Also,*

$$\phi_1(v-) \geq q_2. \tag{63}$$

Proof. If no primary $i > 1$ has a jump at v, then primary 1 gets a payoff of $(v-c)(1-w_1)$, which equals $(\tilde{p}-c)r$ by Lemma 8, for a price p_1 just below v in the limit as $p_1 \to v-$. So if a primary $i \geq 2$ *has* a jump at v, primary 1 can get a payoff strictly greater than $(\tilde{p}-c)r$ by playing a price close enough to v. This contradicts the fact that $u_{1,max} = (\tilde{p}-c)r$ (see Lemma 7). Thus, no primary $i \geq 2$ has a jump at v and $\phi_2(.), \ldots, \phi_n(.)$ are continuous.

First, suppose $q_1 = q_2$. If primary 1 has a jump at v, then similar to the preceding paragraph, primary 2 can get a payoff strictly greater than $(\tilde{p}-c)r$ by playing a price just below v, which contradicts the fact that $u_{2,max} = (\tilde{p}-c)r$. So $\psi_1(.)$ is continuous.

Now suppose $q_1 > q_2$. First, suppose primary 1 has a jump of size *exactly* $q_1 - q_2$ at v. Then if primary 2 sets a price just below v, then the probability of being undercut by primary $j \in \{3, \ldots, n\}$ is approximately q_j. Also, since primary 1 has a jump of size $q_1 - q_2$ at v, the probability of being undercut by primary 1 is approximately $q_1 - (q_1 - q_2) = q_2$. So at a price just below v, primary 2 sees the same set of probabilities of being undercut by primaries other than itself as primary 1 would see if it set a price just below v. Hence, by the first paragraph of this proof, primary 2 gets a payoff of approximately $(\tilde{p}-c)r$ at a price just below v.

Hence, if primary 1 has a jump of size, not equal to, but greater than $q_1 - q_2$ at v, primary 2 gets a payoff of strictly greater than $(\tilde{p}-c)r$ at a price just below v. This contradicts the fact that $u_{2,max} = (\tilde{p}-c)r$.

Thus, primary 1 has a jump of at most size $q_1 - q_2$ at v. So $\phi_1(v) - \phi_1(v-) \leq q_1 - q_2$. This, along with $\phi_1(v) = q_1$, gives (63). $\square$

Lemma 12. *If $\tilde{p} \leq x < y < v$ and $\psi_i(x) = \psi_i(y)$ for some primary i, then $\psi_i(v-) = \psi_i(x)$.*

Thus, if $x \geq \tilde{p}$ is the left endpoint of an interval of constancy of $\psi_i(.)$ for some i, then to the right of x, the interval of constancy extends at least until v (there may be a jump at v).

Proof. Suppose not, *i.e.*:

$$\psi_i(v-) > \psi_i(x). \tag{64}$$

Let:

$$\bar{y} = \sup\{z \geq x : \psi_i(z) = \psi_i(x)\} \tag{65}$$

By (64), (65) and the fact that $\psi_i(.)$ is continuous below v (by Lemma 4), we get $\bar{y} < v$. So again by Lemma 4, no primary among $\{1,\ldots,n\}\backslash i$ has a jump at $\bar{y}$. Also, primary i uses prices just above $\bar{y}$ with positive probability (if not, the supremum in the RHS of

(65) would be $> \overline{y}$). So $\overline{y}$ is a best response for primary i and hence:

$$E\{u_i(\overline{y}, \psi_{-i})\} = (\overline{y} - c)(1 - F_{-i}(\overline{y})) = u_{i,max} = (\tilde{p} - c)r. \tag{66}$$

where the last equality follows from Lemma 7.

Now, by Lemma 9, there exists a primary $j \neq i$ who plays prices just below $\overline{y}$ with positive probability. Since no primary among $\{1, \ldots, n\} \backslash j$ has a jump at $\overline{y}$, $\overline{y}$ is a best response for primary j. Hence:

$$E\{u_j(\overline{y}, \psi_{-j})\} = (\overline{y} - c)(1 - F_j(\overline{y})) = u_{j,max} = (\tilde{p} - c)r. \tag{67}$$

By (66) and (67), $F_{-i}(\overline{y}) = F_{-j}(\overline{y})$. So by Lemma 10:

$$\phi_i(\overline{y}) = \phi_j(\overline{y}). \tag{68}$$

But since primary j plays prices just below $\overline{y}$ with positive probability, there exists $\varepsilon > 0$ such that $x < \overline{y} - \varepsilon$ and $\overline{y} - \varepsilon$ is a best response for primary j. So

$$\phi_j(\overline{y} - \varepsilon) < \phi_j(\overline{y}). \tag{69}$$

But by (65) and the continuity of $\phi_i(.)$ at $\overline{y}$:

$$\phi_i(\overline{y}) = \phi_i(\overline{y} - \varepsilon). \tag{70}$$

By (68), (69) and (70), $\phi_i(\overline{y} - \varepsilon) > \phi_j(\overline{y} - \varepsilon)$. So by Lemma 10:

$$F_{-j}(\overline{y} - \varepsilon) > F_{-i}(\overline{y} - \varepsilon)$$

This implies:

$$
\begin{aligned}
(\tilde{p}-c)r &= E\{u_j(\overline{y}-\varepsilon,\psi_{-j})\} \\
&= (\overline{y}-\varepsilon-c)(1-F_{-j}(\overline{y}-\varepsilon)) \\
&< (\overline{y}-\varepsilon-c)(1-F_{-i}(\overline{y}-\varepsilon)) \\
&= E\{u_i(\overline{y}-\varepsilon,\psi_{-i})\}
\end{aligned}
$$

which contradicts the fact that every primary gets a payoff of $(\tilde{p}-c)r$ at a best response in the NE. $\square$

Lemma 13. *Part 2 of Theorem 3 holds.*

Proof. We prove the result by induction. Let:

$$R_n = \inf\{x \geq \tilde{p} : \exists\, y > x \text{ and } i \text{ s.t. } \phi_i(y) = \phi_i(x)\} \tag{71}$$

Note that R_n is the smallest value $\geq \tilde{p}$ that is the left endpoint of an interval of constancy for some $\phi_i(.)$. For this i, $\phi_i(R_n) = \phi_i(y)$ for some $y > R_n$ [15]. We must have $R_n > \tilde{p}$. This is because, if $R_n = \tilde{p}$, then $\phi_i(y) = \phi_i(\tilde{p})$. But $\phi_i(\tilde{p}) = 0$, since $\tilde{p}$ is the lower endpoint of the support set of $\phi_i(.)$ by Lemma 7. So $\phi_i(y) = 0$, which implies that the lower endpoint of the support set of $\phi_i(.)$ is $\geq y > \tilde{p}$. This contradicts Lemma 7. Thus, $R_n > \tilde{p}$.

Now, by definition of R_n, all primaries play every sub-interval in $[\tilde{p}, R_n)$ with positive probability and hence every price $x \in [\tilde{p}, R_n)$ is a best response for every primary. So

[15] Note that $\phi_i(.)$ is a distribution function and hence is right continuous [19]. So $\phi_i(R_n+) = \phi_i(R_n)$.

similar to the derivation of (14), for $j \in \{1,\ldots,n\}$ and $x \in [\tilde{p}, R_n)$, $E\{u_j(x, \psi_{-j})\} = (x-c)(1-F_{-j}(x)) = (\tilde{p}-c)r$. Hence, $F_{-1}(x) = \ldots = F_{-n}(x)$ and by Lemma 10,

$$\phi_1(x) = \ldots = \phi_n(x) = \phi(x) \text{ (say)}, \ \tilde{p} \le x < R_n. \tag{72}$$

which proves (11) for $j = n$.

Case (i): Suppose $R_n = v$. Then $\phi_l(R_n) = q_l$, $l = 1,\ldots,n$ (since $\psi_l(v) = 1$), which proves (12).

Case (ii): Now suppose $R_n < v$. Then $\phi_j(.), j = 1,\ldots,n$ are continuous at R_n by Lemma 4. So by (72):

$$\phi_1(R_n) = \phi_2(R_n) = \ldots = \phi_n(R_n). \tag{73}$$

Since R_n is the left endpoint of an interval of constancy of $\phi_i(.)$, by Lemma 12:

$$\phi_i(R_n) = \phi_i(v-) = \phi_n(R_n) \le q_n \tag{74}$$

where the second equality follows from (73).

Now, suppose $i = 1$. Then by (63) and (74):

$$\phi_i(R_n) \ge q_2. \tag{75}$$

By (74), (75) and (8), $q_2 = q_3 = \ldots = q_n = \phi_i(R_n)$. Also, by (73), $\phi_j(R_n) = q_j$, $j = 2,\ldots,n$. So $\psi_j(R_n) = 1$, $j = 2,\ldots,n$. This implies, since $R_n < v$ by assumption, that at most one primary (primary 1) plays prices in the interval (R_n, v) with positive probability, which contradicts Lemma 5. Thus, $i \ne 1$.

So by Lemma 11, $\phi_i(.)$ is continuous at v and $\phi_i(v-) = \phi_i(v) = q_i$. So by (74):

$$\phi_i(R_n) = q_i. \tag{76}$$

By (73) and (76), $\phi_n(R_n) = q_i$. If $q_i > q_n$, then $\phi_n(R_n) > q_n$, which is a contradiction because $\phi_n(R_n) = q_n\psi_n(R_n) \leq q_n$. So $q_i \leq q_n$. Also, since $q_i \geq q_n$ by (8), $q_i = q_n$. So:

$$\phi_n(R_n) = q_n. \tag{77}$$

which proves (12) for $j = n$.

Now, as induction hypothesis, suppose there exist thresholds:

$$\tilde{p} < R_n \leq R_{n-1} \leq \ldots \leq R_{i+1} \leq v$$

such that for each $j \in \{i+1,\ldots,n\}$, $\phi_j(R_j) = q_j$,

$$\phi_1(x) = \ldots = \phi_j(x) = \phi(x),\ \tilde{p} \leq x < R_j, \tag{78}$$

and each of primaries $1,\ldots,j$ plays every sub-interval in $[\tilde{p}, R_j)$ with positive probability.

First, suppose $R_{i+1} < v$. Let:

$$\begin{aligned} R_i \quad = \quad & \inf\{x \geq R_{i+1} : \exists\, y > x \text{ and } j \in \{1,\ldots,i\} \\ & \text{s.t. } \phi_j(y) = \phi_j(x)\}. \end{aligned}$$

If $R_i = R_{i+1}$, then clearly by (78):

$$\phi_1(x) = \ldots = \phi_i(x) = \phi(x),\ \tilde{p} \leq x < R_i \tag{79}$$

which proves (11) for $j = i$. Also, similar to (77), it can be shown that $\phi_i(R_i) = q_i$, which proves (12) for $j = i$ and completes the inductive step. Now suppose $R_i > R_{i+1}$. Then similar to the proof of (72), it can be shown that:

$$\phi_1(x) = \ldots = \phi_i(x) = \phi(x),\ R_{i+1} \leq x < R_i. \tag{80}$$

By (78) and (80):

$$\phi_1(x) = \ldots = \phi_i(x) = \phi(x),\ \tilde{p} \leq x < R_i.$$

which proves (11) for $j = i$. Also, similar to the proof of (77), it can be shown that $\phi_i(R_i) = q_i$, which proves (12) for $j = i$. This completes the induction.

If $R_{i+1} = v$, then the induction is completed by simply setting $R_1 = \ldots = R_i = v$.

It remains to show that $R_1 = R_2 = v$. If $R_1 < v$, then no primary plays a price in (R_1, v), which contradicts Lemma 5. So $R_1 = v$. If $R_2 < v$, then only primary 1 plays prices in (R_2, v) with positive probability, which again contradicts Lemma 5. So $R_2 = v$. □

Lemma 11 showed that if $q_1 > q_2$, then $\phi_1(.)$ has a jump of size *at most* $q_1 - q_2$ at v. The following result shows that $\phi_1(.)$ has a jump of size *exactly* $q_1 - q_2$ at v.

Lemma 14. *If $q_1 > q_2$, then $\phi_1(.)$ has a jump of size $q_1 - q_2$ at v.*

Proof. By Lemma 13, $\phi_1(x) = \phi_2(x)$ for all $x < R_2 = v$. So:

$$\begin{aligned}
\phi_1(v-) &= \phi_2(v-) \\
&= \phi_2(v) \ \ \text{(since } \phi_2(.) \text{ is continuous by Lemma 11)} \\
&= q_2
\end{aligned}$$

Also, $\phi_1(v) = q_1\psi_1(v) = q_1$. So $\phi_1(v) - \phi_1(v-) = q_1 - q_2$. $\square$

Finally, (i) Property 1 follows from Lemmas 4, 11 and 14; (ii) Property 2 follows from Lemmas 7 and 8; (iii) Theorem 3 follows from Properties 1 and 2 and Lemma 13.

3.5.2 Proofs of results in Section 3.3.2

We verify that with R_i as in (19), $R_i \geq R_{i+1}$ as required by (10) in Theorem 3. Recall from Section 3.3.2.1 that $f_i(q_i)$ is the probability of K or more successes out of $n-1$ independent Bernoulli events, $i-1$ with success probability q_i and $n-i$ with $q_{i+1}, \ldots, q_n$. Also, $f_{i+1}(q_{i+1})$ is the probability of K or more successes out of $n-1$ Bernoulli events, $i-1$ with success probability q_{i+1} and $n-i$ with $q_{i+1}, \ldots, q_n$. Since $q_i \geq q_{i+1}$ by (8), it is easy to check that $f_i(q_i) \geq f_{i+1}(q_{i+1})$. So by (19), $R_i \geq R_{i+1}$, which is consistent with (10).

Proof of Lemma 2. First, let $f_i(.)$ be as defined in Section 3.3.2.1. To compute $f_i(y)$, for $i \in \{2, \ldots, n\}$, let $f_{i,k}(y)$ be the conditional probability given $K = k$, of K or more successes out of $n-1$ independent Bernoulli events, $(i-1)$ of which have the same success probability y and the remaining $(n-i)$ have success probabilities $q_{i+1}, \ldots, q_n$. Clearly,

$$f_i(y) = \sum_k f_{i,k}(y)\gamma_k.$$

Again, $f_{i,0}(y) = 1$ and $f_{i,k}(y) = 0$ if $k > n-1$.

Consider $1 \leq k \leq n-1$. For $l \in \{0, \ldots, n-i\}$, let $v_l^i(q_{i+1}, \ldots, q_n)$ be the probability of exactly l successes out of $n-i$ independent Bernoulli trials with success probabilities $q_{i+1},\dots,q_n$. Conditioning on the number of successes, say l, out of the $n-i$ trials with success probabilities $q_{i+1},\dots,q_n$, we get:

$$\begin{aligned} f_{i,k}(y) &= \sum_{l=k}^{n-i} v_l^i(q_{i+1},\dots,q_n) \\ &+ \sum_{l=0}^{\min(k-1,n-i)} v_l^i(q_{i+1},\dots,q_n)h_k(y), \end{aligned} \tag{81}$$

where $h_k(y)=\sum_{m=k-l}^{i-1}\begin{pmatrix} i-1 \\ m \end{pmatrix} y^m(1-y)^{i-1-m}$. Now, for l satisfying:

$$1 \leq k-l \leq i-1, \tag{82}$$

$h_k(y)$ is a strictly increasing function of y [70]. Also, it can be checked that $l=\min(k-1,n-i)$, which is one of the indices in the expression in (81), satisfies (82). So $f_{i,k}(y)$ is a strictly increasing function of y. Also, note that $f_{i,k}(.)$ is a continuous function. Thus, $f_i(y)$ is a strictly increasing and continuous function of y as well (since by assumptions on $\{\gamma_k\}$ $\gamma_k>0$ for some k between 1 and $n-1$).

Now, it can be checked from the definition of the function $f_i(.)$ that:

$$f_i(q_{i+1}) = f_{i+1}(q_{i+1}). \tag{83}$$

Also, replacing i with $i+1$ in (18), we get:

$$f_{i+1}(q_{i+1}) = g(R_{i+1}). \tag{84}$$

By (83) and (84), we get:

$$f_i(q_{i+1}) = g(R_{i+1}). \tag{85}$$

Now, as shown above, $f_i(y)$ is a continuous and strictly increasing function of y. So $f_i(.)$ is invertible. By (23), $\phi(.)$ is unique and is given by:

$$\phi(x) = f_i^{-1}(g(x)),\ R_{i+1} \le x < R_i. \tag{86}$$

Also, by (85) and (18), $f_i(q_{i+1}) = g(R_{i+1})$ and $f_i(q_i) = g(R_i)$. So $f_i(.)$ is a continuous one-to-one map from the compact set $[q_{i+1}, q_i]$ onto $[g(R_{i+1}), g(R_i)]$, and hence $f_i^{-1}(.)$ is continuous (see Theorem 4.17 in [58]). Also, $g(x)$ in (16) is continuous for all $x \in [\tilde{p}, v)$ since $x \ge \tilde{p} > c$. So from (86), $\phi(.)$ is a continuous function on $[R_{i+1}, R_i]$, since it is the composition of continuous functions f_i^{-1} and g (see Theorem 4.7 in [58]). Also, as shown above, $f_i(.)$ is strictly increasing; so $f_i^{-1}(.)$ is strictly increasing. Also, using $x \ge \tilde{p} > c$, it can be checked from (16) that $g'(x) > 0$; so $g(.)$ is strictly increasing. By (86), $\phi(.)$ is the composition of the strictly increasing functions $f_i^{-1}(.)$ and $g(.)$ and hence is strictly increasing on $[R_{i+1}, R_i]$. Also, by (11), (12), (18) and (86), $\phi(R_i) = f_i^{-1}(g(R_i)) = q_i$.

Thus, the function $\phi(.)$ is strictly increasing and continuous within each individual interval $[R_{i+1}, R_i]$; also, $\phi(R_i) = q_i$, $i = 2, \ldots, n$, and hence $\phi(.)$ is continuous at the endpoints $R_i, i = 2, \ldots, n$ of these intervals. So $\phi(.)$ is strictly increasing and continuous on $[\tilde{p}, v)$.

It remains to show that $\phi(\tilde{p}) = 0$. By definition of the function $f_i(.)$, $f_n(0) = 1 - r$. As shown above, $f_n(.)$ is one-to-one. So $f_n^{-1}(1 - r) = 0$. Also, by (16), $g(\tilde{p}) = 1 - r$ and by (10), $R_{n+1} = \tilde{p}$. Putting $i = n$ and $x = R_{n+1} = \tilde{p}$ in (86), we get $\phi(\tilde{p}) = f_n^{-1}(g(\tilde{p})) = f_n^{-1}(1 - r) = 0$. $\square$

Proof of Theorem 4. By Lemma 2 and equation (13), the functions $\phi_i(.)$, $i = 1,\ldots,n$ computed in Section 3.3.2 are continuous and non-decreasing on $[\tilde{p}, v]$; also, $\phi_i(\tilde{p}) = 0$ and $\phi_i(v) = q_i$. This is consistent with the fact that $\phi_i(.)$ is the d.f. of the pseudo-price p_i' and hence should be non-decreasing and right continuous [19], and $\phi_i(v) = q_i\psi_i(v) = q_i$ (see the beginning of Section 3.3).

Now, we have shown in Sections 3.3.1 and 3.3.2 that (13) is a necessary condition for the functions $\phi_i(.), i = 1,\ldots,n$ to constitute a NE. We now show sufficiency. Suppose for each $i \in \{1,\ldots,n\}$, primary i uses the strategy $\phi_i(.)$ in (13). Similar to the derivation of (14), the expected payoff that primary i gets at a price $x \in [\tilde{p}, v)$ is:

$$E\{u_i(x, \psi_{-i})\} = (x - c)(1 - F_{-i}(x)). \tag{87}$$

Now, for $x \in [\tilde{p}, R_i)$, by (10) and (13), $\phi_i(x) = \phi_1(x) = \phi(x)$, and hence by Lemma 10, $F_{-i}(x) = F_{-1}(x)$. Also note that $\phi(.)$ is the solution of (14), (22) and (23). By (14), (87) and the fact that $F_{-i}(x) = F_{-1}(x)$, for primary i, prices $x \in [\tilde{p}, R_i)$ fetch an expected payoff of $(\tilde{p} - c)r$.

Now let $x \in [R_i, v)$. Note that $R_i \le x < v = R_1$. So by (13), $\phi_i(x) = q_i$ and $\phi_1(x) = \phi(x) \ge \phi(R_i) = q_i$. So $\phi_1(x) \ge \phi_i(x)$. Hence, by Lemma 10, $F_{-1}(x) \le F_{-i}(x)$, which by (14) and (87) implies $E\{u_i(x, \psi_{-i})\} \le (\tilde{p} - c)r$.

Finally, note that a price below $\tilde{p}$ fetches a payoff of less than $(\tilde{p} - c)r$ for primary i. So each price in $[\tilde{p}, R_i)$ is a best response for primary i; also, by (13), it randomizes over prices only in this range under $\phi_i(.)$. So $\phi_i(.)$ is a best response. Thus, the functions $\phi_i(.), i = 1,\ldots,n$ constitute a NE. $\square$

3.5.3 Proofs of results in Section 3.3.3

Proof of Lemma 3. Since Z is the number of primaries who have unused bandwidth, and primary i has unused bandwidth with probability q_i, $E(Z) = \sum_{i=1}^{n} q_i$ and $var(Z) = \sum_{i=1}^{n} q_i(1-q_i)$.

We now prove the first part. Suppose $k_n \leq \sum_{i=2}^{n}(q_i - \varepsilon)$ for some $\varepsilon > 0$. Let the random variable Y_n be defined as:

$$Y_n = \begin{cases} K_n, & \text{if } Z \geq K_n \\ 0, & \text{else} \end{cases}$$

Then:

$$\begin{aligned}
& E\{\min(Z, K_n)\} \\
\geq\ & E(Y_n) \\
=\ & k_n P(Z \geq K_n) \\
=\ & k_n(1 - P(Z < K_n)) \\
=\ & k_n(1 - P(Z - K_n + k_n < k_n)) \\
\geq\ & k_n(1 - P(Z - K_n + k_n \leq \sum_{i=2}^{n}(q_i - \varepsilon))) \text{ (since } k_n \leq \sum_{i=2}^{n}(q_i - \varepsilon)) \\
\geq\ & k_n(1 - P(|Z - \sum_{i=1}^{n} q_i - K_n + k_n| \geq (n-1)\varepsilon)) \\
\geq\ & k_n\left(1 - 2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{n(1+\alpha)}\right)\right) \\
& \text{(by Hoeffding's inequality [26], since } E(Z) = \sum_{i=1}^{n} q_i \text{ and } E(K_n) = k_n) \qquad (88)
\end{aligned}$$

Now, let Z_1 be the number of primaries out of primaries $2, \ldots, n$ who have unused bandwidth. Note that $E(Z_1) = \sum_{i=2}^n q_i$ and $var(Z_1) = \sum_{i=2}^n q_i(1-q_i)$. We have:

$$
\begin{aligned}
1 - w_1 &= P(Z_1 < K_n) \\
&= P(Z_1 - K_n + k_n < k_n) \\
&\leq P(Z_1 - K_n + k_n \leq \sum_{i=2}^{n}(q_i - \varepsilon)) \text{ (since } k_n \leq \sum_{i=2}^{n}(q_i-\varepsilon)) \\
&\leq P(|Z_1 - \sum_{i=2}^{n} q_i - K_n + k_n| \geq (n-1)\varepsilon) \\
&\leq 2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{(n\alpha+n-1)}\right) \text{ (by Hoeffding's inequality [26])} \qquad (89)
\end{aligned}
$$

By (24), (88) and (89):

$$
\begin{aligned}
\eta &\leq \frac{2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{(n\alpha+n-1)}\right)\sum_{i=1}^n q_i}{k_n\left(1-2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{n(1+\alpha)}\right)\right)} \\
&\leq \frac{2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{(n\alpha+n-1)}\right)n}{\beta\left(1-2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{n(1+\alpha)}\right)\right)} \text{ (since } k_n \geq \beta \text{ and } q_i \leq 1\ \forall i) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}
$$

which proves the first part.

Now we prove the second part. Suppose $k_n \geq \sum_{i=2}^n (q_i + \varepsilon)$ for some $\varepsilon > 0$. Since $E\{\min(Z,K)\} \leq E(Z) = \sum_{i=1}^n q_i$, by (24):

$$
\begin{aligned}
\eta &\geq \frac{(1-w_1)\sum_{i=1}^n q_i}{\sum_{i=1}^n q_i} \\
&= 1 - w_1 \\
&= 1 - P(Z_1 \geq K_n) \\
&\geq 1 - 2\exp\left(\frac{-2(n-1)^2\varepsilon^2}{n\alpha+n-1}\right) \text{ (similar to the derivation of (89))} \\
&\to 1 \text{ as } n \to \infty
\end{aligned}
$$

which proves the second part. $\square$

Chapter 4

Spectrum Pricing Games with Random Valuations of Secondaries

4.1 Introduction

In the model introduced in Section 2.2, we stated that $p_i \leq v$ for every primary i for some constant v. This constant may either be a regulatory upper limit or the valuation of each secondary. So far, we have assumed that v is a constant and known to all the primaries. This would be the case in the first interpretation above, *i.e.* when v is a regulatory upper limit. However, in the second interpretation, the valuations of different secondaries may be different and unknown to the primaries. In this chapter, we study a generalized model in which the valuations of the secondaries are not constants, but random variables that can possibly take different values for different secondaries.

We describe the model in Section 4.2. For simplicity, we first analyze this model in Section 4.3 for the case where there is only one secondary and later generalize our results to an arbitrary number of secondaries in Section 4.4.

4.2 Model

Consider the model described in Section 2.2 with the following changes [16]. Instead of a common known valuation v for all the secondaries, let v_j, $j \in \{1,\dots,k\}$, denote the valuation of secondary j for 1 unit of bandwidth– secondary j does not buy bandwidth at a price greater than v_j. The valuations $v_1,\dots,v_k$ of the secondaries for 1 unit of bandwidth are independent and identically distributed (i.i.d.) random variables with distribution function (d.f.) $G(x) = P(v_j \leq x)$. We assume that $G(.)$ is continuous and $G(\underline{v}) = 0$, $G(\overline{v}) = 1$, where $c < \underline{v} < \overline{v}$. Thus, the valuation of each secondary lies in the range $[\underline{v},\overline{v}]$ w.p. 1. Note that in practice, the valuations of secondaries are upper bounded, and hence there always exists some finite upper bound $\overline{v}$. The assumption $\underline{v} > c$ means that a secondary's valuation is always greater than the cost that the seller incurs; so if trade occurs, then it is always profitable to both the buyer and the seller.

As before, we introduce the notion of a "pseudo-price". The pseudo-price of primary $i \in \{1,\dots,n\}$, denoted as p_i', is the price he selects if he has unused bandwidth and $p_i' = \overline{v} + 1$ otherwise [17].

[16] For simplicity, as in Chapter 2, we assume that the probability of having unused bandwidth is the same for each primary, and equals q.

[17] The choice $\overline{v} + 1$ is arbitrary. Any other value greater than $\overline{v}$ also works.

We formulate the above price competition as a game as in Section 2.3. Note that this is a symmetric game. Our goal is to explicitly compute a symmetric Nash Equilibrium (NE) and to show its uniqueness.

4.3 One Secondary

In this section, for simplicity, we find a symmetric NE and prove its uniqueness for the case in which there is only one secondary, *i.e.* $k = 1$. This secondary buys bandwidth from the primary who quotes the lowest price, provided this price is less than or equal to his valuation. In Section 4.4, we generalize our results to allow for multiple secondaries.

In Section 4.3.1, we will explicitly compute a symmetric NE and in Section 4.3.2 show that it is the unique symmetric NE.

4.3.1 Explicit Computation of Symmetric NE

Consider a symmetric NE under which every primary uses the strategy $\psi(.)$. The following lemma provides a necessary condition that $\psi(.)$ must satisfy.

Lemma 15. $\psi(.)$ *is continuous.*

Proof. Suppose, to reach a contradiction, that $\psi(.)$ has a jump at x_0. Fix an $i \in \{1, \ldots, n\}$. Since every primary other than primary i has a jump at x_0, for primary i, a price just below x_0 fetches a higher expected payoff than x_0. So x_0 is not a best response for primary i, which contradicts the fact that primary i uses $\psi(.)$ and hence has a jump at x_0. The result follows. $\square$

For a primary m, if

$$P(p'_j \leq x) = y,\ \forall j \in \{1, \dots, n\} \backslash m, \tag{90}$$

then let $f_x(y)$ be primary m's expected payoff if he sets the price $p_m = x$. Let

$$h(x) = f_x(q) \tag{91}$$

and

$$g(x) = f_x(0). \tag{92}$$

The following lemma provides an expression for $f_x(y)$:

Lemma 16.

$$f_x(y) = (x - c)(1 - G(x))(1 - y)^{n-1} \tag{93}$$

Proof. Suppose (90) holds. If primary m sets a price of x, he gets a payoff of $(x - c)$ if his bandwidth is sold and 0 otherwise. Also, his bandwidth is sold iff (i) the valuation of the secondary is x or more, which happens w.p. $1 - G(x)$, and (ii) no primary $j \in \{1, \dots, n\} \backslash m$ who has unused bandwidth sets a price lower than y, which happens w.p. $(1 - y)^{n-1}$ by (90). The result follows. $\square$

We now state some properties of $f_x(y)$, which are proved in the Appendix:

Lemma 17. *1. $f_x(y)$ is continuous in x and y.*

2. *For $x \leq \underline{v}$, $f_x(y)$ is a strictly increasing function of x for every fixed y. Also, $f_c(y) = 0$ for every fixed y.*

3. *$h(\underline{v}) = f_{\underline{v}}(q) > 0$. Also, $h(x) = 0$ for all $x \geq \overline{v}$.*

By (91) and part 1 of Lemma 17, $h(.)$ is a continuous function and hence has a maximizer on the compact set $[c, \overline{v}]$. Let $h_{max} = \max_{v \in [c,\overline{v}]} h(v)$ be the maximum value of $h(.)$ and

$$v_T = \inf\{v \in [c, \overline{v}] : h(v) = h_{max}\} \tag{94}$$

be the infimum of the set of maximizers of $h(.)$. Since $h(.)$ is continuous, by (94), v_T is itself a maximizer of $h(.)$ on $[c, \overline{v}]$. So $h(v_T) = h_{max}$. By part 2 of Lemma 17 and (91), $h(.)$ is strictly increasing on $[c, \underline{v}]$. Also, $h(\underline{v}) > 0$ and $h(x) = 0$ for all $x \geq \overline{v}$ by part 3 of Lemma 17. Since v_T is the smallest maximizer of $h(.)$ on $[c, \overline{v}]$:

$$\underline{v} \leq v_T < \overline{v}. \tag{95}$$

We will later show that the upper endpoint of the support set of $\psi(.)$ is v_T.

We now state another property of the function $f_x(y)$, which is proved in the Appendix.

Lemma 18. *For every fixed $x \in [c, v_T]$, $f_x(y)$ is a strictly decreasing function of y.*

Lemma 19. *There exists at least one $x \in (c, v_T)$ such that $g(x) = h(v_T)$. The minimum such x exists; let it be denoted by $\tilde{p}$. Then $g(x) < g(\tilde{p}) = h(v_T)$ $\forall c \leq x < \tilde{p}$.*

Proof. By (94) and part 3 of Lemma 17, $h(v_T) \geq h(\underline{v}) > 0$. Also, by (92) and part 2 of

Lemma 17:

$$g(c) = 0 < h(v_T). \tag{96}$$

By (91), (92) and Lemma 18:

$$h(v_T) = f_{v_T}(q) < f_{v_T}(0) = g(v_T) \tag{97}$$

By (96) and (97), $g(c) < h(v_T)$ and $g(v_T) > h(v_T)$. Also, $g(.)$ is continuous by (92) and part 1 of Lemma 17. So by the intermediate value theorem [58], there exists a solution of the equation $g(x) = h(v_T)$ in (c, v_T). The minimum such solution, say $\tilde{p}$, exists because $g(.)$ is continuous and hence the set $\{x : g(x) = h(v_T)\}$ is closed.

Now, suppose, to reach a contradiction, that $g(x') \geq h(v_T)$ for some $x' \in [c, \tilde{p})$. Then by (96) and the intermediate value theorem, there exists x'' such that $c \leq x'' \leq x' < \tilde{p}$ and $g(x'') = h(v_T)$. This contradicts the fact that $\tilde{p}$ is the smallest solution of $g(x) = h(v_T)$. Thus, $g(x) < h(v_T)$ for all $x < \tilde{p}$. □

By definition of $f_x(y)$ and by (92), if no primary in $\{1, \ldots, n\} \backslash i$ plays a price below x, then primary i gets a payoff of $g(.)$ at price x. It turns out that primaries do not play prices below $\tilde{p}$ and $\tilde{p}$ is a best response for every primary in the NE. So every primary gets a payoff of $g(\tilde{p})$ in the NE because when he plays a price of $\tilde{p}$, he is not undercut by the other primaries.

Let:

$$C = \{x \in [\tilde{p}, v_T] : g(x) \geq g(\tilde{p})\}. \tag{98}$$

Note that for a price in $[\tilde{p}, v_T] \backslash C$, primary i's payoff is less than the NE payoff and hence each primary plays prices in $[\tilde{p}, v_T] \backslash C$ with zero probability.

Lemma 20. *For every $x \in C$, there exists a unique $\gamma(x) \in [0,q]$ such that*

$$f_x(\gamma(x)) = g(\tilde{p}). \tag{99}$$

Also, $\gamma(\tilde{p}) = 0$ and $\gamma(v_T) = q$.

Proof. First, note that by (91) and Lemma 19:

$$f_{v_T}(q) = h(v_T) = g(\tilde{p}). \tag{100}$$

Now, fix an $x \in C$. By (92):

$$\begin{aligned} f_x(0) &= g(x) \\ &\geq g(\tilde{p}) \text{ (by (98), since } x \in C). \end{aligned} \tag{101}$$

Also, by (91):

$$\begin{aligned} f_x(q) &= h(x) \\ &\leq h(v_T) \text{ (since } v_T \text{ is the smallest} \\ &\quad \text{maximizer of } h(.) \text{ and } x \leq v_T) \\ &= g(\tilde{p}) \text{ (by (100))} \end{aligned} \tag{102}$$

By part 1 of Lemma 17, $f_x(y)$ is continuous in y. So by (101), (102) and the intermediate value theorem [58], the equation $f_x(y) = g(\tilde{p})$ has a solution $y = \gamma(x) \in [0,q]$. Also, by Lemma 18, this root is unique.

Now, by (92), $f_{\tilde{p}}(0) = g(\tilde{p})$. So $\gamma(\tilde{p}) = 0$. Also, by (100), $f_{v_T}(q) = g(\tilde{p})$. So $\gamma(v_T) = q$. □

Now, we state a general analytic fact, which is proved in the Appendix.

Fact 1. *Let $F(x,y)$ be any real-valued continuous function, where x and y are real, and $[a,b]$ be an interval such that for every $x \in [a,b]$, there exists a unique $y = \gamma(x)$ such that*

$$F(x,\gamma(x)) = \alpha, \tag{103}$$

where α is a constant. Then the function $\gamma(.)$ is continuous on $[a,b]$.

Now, let C be as in (98). Since $g(.)$ is continuous, C is closed. So C is the union of a set of disjoint closed intervals– let $C = \cup_{i\in\lambda} C_i$, where λ is some set of indices and $C_i = [a_i, b_i]$.

Fix an $i \in \lambda$. By Lemma 20, for every $x \in C_i$, there exists a unique $\gamma(x) \in [0,q]$ such that $f_x(\gamma(x)) = g(\tilde{p})$. By part 1 of Lemma 17, the function $f_x(y)$ is continuous in x and y. So by Fact 1, $\gamma(.)$ is continuous on C_i.

Thus, we have shown the following:

Lemma 21. *$\gamma(.)$ is continuous on each C_i, $i \in \lambda$.*

By definition of the function $f_x(y)$ and by (99), for every x, $\gamma(x)$ is a value such that if $P\{p'_j \le x\} = \gamma(x)$, $j \neq i$, then a price of $p_i = x$ fetches a payoff of exactly $g(\tilde{p})$, which is the payoff that every primary gets in the symmetric NE. This suggests $\gamma(.)$ as a candidate for the symmetric NE pseudo-price strategy d.f. But $\gamma(x)$ itself need not be a valid d.f. since it is not non-decreasing in general as shown in Fig. 4.1. So a natural idea is to consider the function:

$$\phi_{NE}(x) = \begin{cases} \max\{\gamma(y) : y \in C, y \leq x\}, & x \geq \tilde{p} \\ 0, & x < \tilde{p} \end{cases} \tag{104}$$

obtained by replacing the portions of decrease of $\gamma(.)$ by horizontal segments as illus-

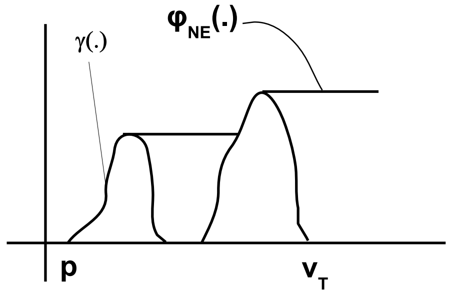

Figure 4.1: The figure shows $\phi_{NE}(.)$ and $\gamma(.)$ versus price.

trated in Fig. 4.1.

Theorem 5. *The strategy profile in which each primary uses the pseudo-price selection strategy $\phi_{NE}(.)$ is a NE.*

Proof. By (104), the function $\phi_{NE}(.)$ is non-decreasing on $[\tilde{p}, v_T]$. Also, by Lemma 21 and (104), it is continuous on $[\tilde{p}, v_T]$. By Lemma 20, $\gamma(x) \in [0,q]$ $\forall x \in C$. So by (104):

$$0 \leq \phi_{NE}(x) \leq q \ \ \forall x \tag{105}$$

Also, since $\gamma(\tilde{p}) = 0$ and $\gamma(v_T) = q$ (see Lemma 20), and by (104) and (105):

$$\phi_{NE}(x) = \begin{cases} 0, & x \leq \tilde{p} \\ q, & x \geq v_T \end{cases} \tag{106}$$

Thus, $\phi_{NE}(.)$ is a valid pseudo-price d.f. and its support set is a subset of $[\tilde{p}, v_T]$.

Suppose every primary uses the strategy $\phi_{NE}(.)$ to select his pseudo-price. By definition of $f_x(y)$ and the continuity of $\phi_{NE}(.)$, if primary 1 sets a price of $p_1 = x$, he gets an expected payoff of:

$$E\{u_1(x, \psi_{-1})\} = f_x(\phi_{NE}(x)). \tag{107}$$

By (104), $\phi_{NE}(x) \geq \gamma(x)$ for all $x \in [\tilde{p}, v_T]$.

Case (i): Suppose $x \in [\tilde{p}, v_T] \backslash C$. Then by (107):

$$\begin{aligned}
E\{u_1(x, \psi_{-1})\} &= f_x(\phi_{NE}(x)) \\
&\leq f_x(0) \text{ (by Lemma 18 and (105))} \\
&= g(x) \text{ (by (92))} \\
&< g(\tilde{p}) \text{ (since } x \in [\tilde{p}, v_T] \backslash C \\
&\quad \text{ and by (98))}
\end{aligned} \tag{108}$$

Case (ii): Suppose $x \in C$ and $\phi_{NE}(x) = \gamma(x)$. Then by (107) and (99), $E\{u_1(x, \psi_{-1})\} = f_x(\gamma(x)) = g(\tilde{p})$.

Case (iii): Now, suppose $x \in C$ and $\phi_{NE}(x) > \gamma(x)$. Then by (107), Lemma 18 and (99):

$$E\{u_1(x, \psi_{-1})\} < f_x(\gamma(x)) = g(\tilde{p}) \tag{109}$$

Also, x is part of an interval of constancy of $\phi_{NE}(x)$; so primaries play prices around x with 0 positive probability.

Case (iv): Suppose $x < \tilde{p}$. Then by (106), $\phi_{NE}(x) = 0$. So by (107),

$$\begin{aligned} E\{u_1(x, \psi_{-1})\} &= f_x(0) \\ &= g(x) \ \text{(by (92))} \\ &< g(\tilde{p}) \ \text{(by Lemma 19)} \end{aligned} \tag{110}$$

Case (v): Suppose $x \geq v_T$. Then by (106), $\phi_{NE}(x) = q$. So by (107),

$$\begin{aligned} E\{u_1(x, \psi_{-1})\} &= f_x(q) \\ &= h(x) \ \text{(by (91))} \qquad (111) \\ &\leq h(v_T) \ \text{(by (94))} \qquad (112) \\ &= g(\tilde{p}) \ \text{(by (100))} \qquad (113) \end{aligned}$$

Now, since $\phi_{NE}(.)$ is non-decreasing and continuous, it has alternating intervals of constancy and strict increase. Also, note that a primary who uses the d.f. $\phi_{NE}(.)$ to select his pseudo-price plays prices in the intervals of constancy with 0 probability and in the intervals of strict increase with positive probability. Now, by (106), the intervals $[c, \tilde{p}]$ and $[v_T, \overline{v}]$ (Cases (iv) and (v) respectively) are intervals of constancy of $\phi_{NE}(.)$. Also, it can be checked using (104) that the intervals which lie in the regions $[\tilde{p}, v_T] \backslash C$ and $\{x \in C : \phi_{NE}(x) > \gamma(x)\}$ (Cases (i) and (iii) respectively) are also regions of constancy. Thus, only intervals that lie in the region $\{x \in C : \phi_{NE}(x) = \gamma(x)\}$ (Case (ii)) can possibly be intervals of strict increase of $\phi_{NE}(.)$.

By Cases (i) to (v), primary 1 gets a payoff of at most $g(\tilde{p})$ at any price. Also, as shown in the previous paragraph, he can only play intervals in the region $\{x \in C :$

$\phi_{NE}(x) = \gamma(x)\}$ (Case (ii)) with positive probability. His expected payoff is $g(\tilde{p})$, the maximum possible, at a price in this region by Case (ii). Hence $\phi_{NE}(.)$ is a best response for primary 1. The result follows. $\square$

Note that in the proof of Theorem 5, we have shown the following:

Lemma 22. *In the symmetric NE in which every primary uses the strategy $\phi_{NE}(.)$, each primary gets an expected payoff of $g(\tilde{p})$.*

4.3.2 Uniqueness of Symmetric NE

Now, we show that the NE in Theorem 5 is the unique symmetric NE.

Let the functions $f_x(y)$, $h(.)$, $g(.)$, $\gamma(.)$ and $\phi_{NE}(.)$ be as in (93), (91), (92), Lemma 20 and (104) respectively. Also, let v_T, $\tilde{p}$ and the set C be as in (94), Lemma 19 and (98) respectively.

Consider a symmetric NE under which every primary uses the d.f. $\widehat{\psi}(.)$ to select the price, and let $\widehat{\phi}_{NE}(.) = q\widehat{\psi}(.)$ be the corresponding pseudo-price d.f.

Let v'_T be the upper endpoint of the support set of $\widehat{\psi}(.)$:

$$v'_T = \inf\{x : \widehat{\psi}(x) = 1\}. \tag{114}$$

Lemma 23. *$v'_T = v_T$. Also, v_T is a best response for each primary in the symmetric NE.*

Thus, the upper endpoint of the support set of $\widehat{\psi}(.)$ is v_T.

Proof. As in the proof of Lemma 15, $\widehat{\psi}(.)$ is continuous. Also, note that by (114), each primary plays prices in $[v'_T - \varepsilon, v'_T]$ with positive probability for every $\varepsilon > 0$. Hence, v'_T is a best response for each primary i.

To reach a contradiction, suppose $v'_T > v_T$. Then by (114), $\widehat{\psi}(v_T) < 1$ and hence

$$\widehat{\phi}_{NE}(v_T) < q. \tag{115}$$

Similar to the derivation of (107):

$$\begin{aligned}
E\{u_i(v_T, \widehat{\psi}_{-i})\} &= f_{v_T}(\widehat{\phi}_{NE}(v_T)) \\
&> f_{v_T}(q) \ \text{ (by (115) and Lemma 18)} \\
&= h(v_T) \ \text{ (by (91))} \\
&\geq h(v'_T) \ \text{ (by (94))} \qquad (116) \\
&= E\{u_i(v'_T, \widehat{\psi}_{-i})\} \qquad (117)
\end{aligned}$$

where (117) follows from (116) similar to the derivation of (111). Thus, $E\{u_i(v_T, \widehat{\psi}_{-i})\} > E\{u_i(v'_T, \widehat{\psi}_{-i})\}$, which contradicts the fact that v'_T is a best response. Thus, $v'_T > v_T$ is not possible

Now suppose $v'_T < v_T$. Then $\widehat{\psi}(v_T) = \widehat{\psi}(v'_T) = 1$ by (114); so $\widehat{\phi}_{NE}(v_T) = \widehat{\phi}_{NE}(v'_T) = q$. Similar to the derivation of (111):

$$\begin{aligned}
E\{u_i(v_T, \widehat{\psi}_{-i})\} &= h(v_T) \\
&> h(v'_T) \ \text{ (by (94))} \\
&= E\{u_i(v'_T, \widehat{\psi}_{-i})\}
\end{aligned}$$

which is again a contradiction. Thus, $v'_T < v_T$ is not possible and hence $v'_T = v_T$. $\square$

Recall that we have shown in Theorem 5 that $\phi_{NE}(.)$ constitutes a symmetric NE strategy. Now we show its uniqueness.

Theorem 6. $\phi_{NE}(.)$ *constitutes the unique symmetric NE strategy.*

Proof. Consider a symmetric NE in which every primary uses the strategy $\widehat{\phi}_{NE}(.)$. We will show that $\widehat{\phi}_{NE}(.) = \phi_{NE}(.)$.

As in the proof of Lemma 15, $\widehat{\phi}_{NE}(.)$ is continuous. Also, by Lemma 23, v_T is the upper endpoint of the support set of $\widehat{\phi}_{NE}(.)$ and is a best response for each primary i in the symmetric NE. Similar to the derivation of (107), the payoff that each primary i gets at price x in the NE is:

$$E\{u_i(x, \widehat{\psi}_{-i})\} = f_x(\widehat{\phi}_{NE}(x)) \tag{118}$$

Also, similar to the derivation of (111), the payoff that each primary i gets at price v_T is:

$$E\{u_i(v_T, \widehat{\psi}_{-i})\} = h(v_T) = g(\tilde{p}), \tag{119}$$

where the second equality follows from (100). Since v_T is a best response, each primary gets an expected payoff of $g(\tilde{p})$ in the NE.

Now, for a price $x < \tilde{p}$, by (118), primary i gets a payoff of:

$$\begin{aligned} E\{u_i(x, \widehat{\psi}_{-i})\} &= f_x(\widehat{\phi}_{NE}(x)) \\ &\leq f_x(0) \quad \text{(by Lemma 18)} && (120) \\ &< g(\tilde{p}) && (121) \end{aligned}$$

where (121) follows from (120) similar to the derivation of (110). Thus, primaries do not play prices below $\tilde{p}$ in the NE and hence $\widehat{\phi}_{NE}(\tilde{p}) = 0$.

Similar to the derivation of (108), it can be shown that for $x \in [\tilde{p}, v_T]\backslash C, E\{u_i(x, \widehat{\psi}_{-i})\} < g(\tilde{p})$ and hence x is not a best response. Thus, only prices in C can possibly be best responses.

If x_0 is a best response for primary i, then by (118):

$$E\{u_i(x_0, \widehat{\psi}_{-i})\} = f_{x_0}(\widehat{\phi}_{NE}(x_0)) = g(\tilde{p}), \tag{122}$$

By (122) and Lemma 20:

$$\widehat{\phi}_{NE}(x_0) = \gamma(x_0), \tag{123}$$

Now, since $\widehat{\phi}_{NE}(.)$ is continuous by Lemma 15, it consists of alternating intervals of strict increase and constancy. If $[a_s, b_s]$ is an interval of strict increase, then each $x \in [a_s, b_s]$ is a best response; so $\widehat{\phi}_{NE}(x) = \gamma(x)$ by (123). Thus,

$$\widehat{\phi}_{NE}(x) \leq \max\{y \leq x : \gamma(y)\} = \phi_{NE}(x), \; \forall x \in [a_s, b_s]. \tag{124}$$

where the equality follows by (104).

Now, let $[a_c, b_c]$ be a maximal interval of constancy of $\widehat{\phi}_{NE}(.)$ such that $\widehat{\phi}_{NE}(a_c) > 0$. Note that a_c is the right endpoint of an interval of strict increase [18]. So by continuity of $\widehat{\phi}_{NE}(.)$, a_c is a best response and hence $\widehat{\phi}_{NE}(a_c) = \gamma(a_c)$ by (123). So for all $x \in [a_c, b_c]$, $\widehat{\phi}_{NE}(x) = \widehat{\phi}_{NE}(a_c) = \gamma(a_c)$ Thus,

$$\widehat{\phi}_{NE}(x) \leq \max\{y \leq x : \gamma(y)\} = \phi_{NE}(x), \; \forall x \in [a_c, b_c]. \tag{125}$$

[18] Note that $\widehat{\phi}_{NE}(a_c) > 0$ implies that there exists an interval of strict increase of $\widehat{\phi}_{NE}(.)$ to the left of a_c.

where the equality follows by (104).

By (124) and (125):

$$\widehat{\phi}_{NE}(x) \leq \phi_{NE}(x) \ \forall x. \tag{126}$$

Now, it remains to show that $\widehat{\phi}_{NE}(x) \geq \phi_{NE}(x)$ for all x. To reach a contradiction, suppose $\widehat{\phi}_{NE}(x) < \phi_{NE}(x)$ for some x. Let:

$$x_l = \inf\{x : \widehat{\phi}_{NE}(x) < \phi_{NE}(x)\}. \tag{127}$$

Then for all $x < x_l$, $\widehat{\phi}_{NE}(x) = \phi_{NE}(x)$. So by continuity of $\widehat{\phi}_{NE}(.)$ and $\phi_{NE}(.)$,

$$\widehat{\phi}_{NE}(x_l) = \phi_{NE}(x_l). \tag{128}$$

Also, by (127), there exists an $x_0 = x_l + \varepsilon$, for some small $\varepsilon > 0$, such that:

$$\widehat{\phi}_{NE}(x_0) < \phi_{NE}(x_0). \tag{129}$$

and $[x_l, x_0]$ is an interval of strict increase of $\phi_{NE}(.)$. In particular, x_0 is a best response of primary 1 when the other primaries use $\phi_{NE}(.)$.

Now, by (118), the expected payoff of primary 1 for price $p_1 = x_0$ when other primaries play $\widehat{\phi}_{NE}(.)$ is:

$$\begin{aligned} f_{x_0}(\widehat{\phi}_{NE}(x_0)) &> f_{x_0}(\phi_{NE}(x_0)) \ \text{(by (129) and Lemma 18)} \\ &= g(\tilde{p}) \end{aligned} \tag{130}$$

where (130) follows from the fact that x_0 is a best response of primary 1 when the other primaries use $\phi_{NE}(.)$ and Lemma 22. This contradicts the fact that the maximum payoff that primary 1 can get when the other primaries use $\widehat{\phi}_{NE}(.)$ is $g(\tilde{p})$. Thus,

$$\widehat{\phi}_{NE}(x) \geq \phi_{NE}(x) \ \forall x \tag{131}$$

By (126) and (131), $\widehat{\phi}_{NE}(x) = \phi_{NE}(x) \ \forall x$ and the result follows. $\square$

4.4 Multiple Secondaries

In Section 4.3, we explicitly computed the symmetric NE and showed its uniqueness for the case of one secondary. We now generalize our results to multiple secondaries. Suppose there are k secondaries, where $k \geq 1$.

4.4.1 Primary Secondary Matching Scheme

Let $p'_{(1)} \leq p'_{(2)} \leq \ldots \leq p'_{(n)}$ be the pseudo-prices $p'_1, \ldots, p'_n$ of the primaries in increasing order. Also, let $v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(k)}$ be the valuations of the secondaries in decreasing order.

Note that since there are multiple secondaries with possibly different valuations, after the primaries reveal the prices they are willing to sell at and the secondaries reveal their valuations, there are in general different possible schemes for matching primaries with the secondaries who buy bandwidth from them. Let $\mathcal{A}$ be the set of all possible schemes of matching primaries with secondaries such that bandwidth is never bought from a primary if the bandwidth of a different primary who offers a lower pseudo-price remains unsold. Note that under every scheme in $\mathcal{A}$, the bandwidth of the primaries with the smallest i pseudo-prices $p'_{(1)}, \ldots, p'_{(i)}$ is sold, for some $i \in \{0, 1, \ldots, n\}$. Let W be the scheme in which the secondary with the highest valuation $v^{(1)}$ buys from the primary with the lowest price $p'_{(1)}$ (if $p'_{(1)} \leq v^{(1)}$), the secondary with the second-highest valuation $v^{(2)}$ buys from the primary with the second-lowest price $p'_{(2)}$ (if $p'_{(2)} \leq v^{(2)}$) and so on. Ties are broken at random.

For example, suppose $n = 4$, $k = 3$, the pseudo-prices of the primaries in increasing order are $p'_{(1)} = 1$, $p'_{(2)} = 2$, $p'_{(3)} = 3$, $p'_{(4)} = 4$ and the valuations of the secondaries in decreasing order are $v^{(1)} = 3.5$, $v^{(2)} = 2.5$, $v^{(3)} = 1.5$. For the scheme W, the following table shows the valuation of the secondary who buys bandwidth from each primary (a "-" indicates that the corresponding primary's bandwidth is unsold):

Primary's price	Secondary's valuation
1	3.5
2	2.5
3	-
4	-

Consider another scheme in $\mathcal{A}$ in which the secondary with the largest valuation buys bandwidth from the primary who offers the highest price that is below his valuation, the secondary with the second-largest valuation buys from the primary who offers the next highest price that is below his valuation and so on (ties are broken at random). The following table shows the matching of primaries and secondaries under this scheme for the above example:

Primary's price	Secondary's valuation
1	1.5
2	2.5
3	3.5
4	-

The above tables show that the second scheme is more "efficient" than the scheme W in the sense that more primaries sell their bandwidth. In fact, the following lemma shows that in this sense the scheme W is the worst-case or least efficient scheme in $\mathcal{A}$.

Lemma 24. *For any given set of pseudo-prices of the primaries and valuations of the secondaries, out of all the schemes in $\mathcal{A}$, the bandwidth of the fewest number of primaries is sold under the scheme W.*

Proof. Fix $p'_{(1)},\dots,p'_{(n)}$ and $v^{(1)},\dots,v^{(k)}$. Suppose, under the scheme W, the bandwidth of the primaries with psedo-prices $p'_{(1)},\dots,p'_{(i)}$ is sold. By definition of W, these i primaries sell their bandwidth to the secondaries with the i largest valuations $v^{(1)},\dots,v^{(i)}$ and the primary with pseudo-price $p'_{(i)}$ sells to the secondary with the smallest valuation $v^{(i)}$ out of these. Thus, $v^{(i)} \geq p'_{(i)}$ and hence:

$$v^{(j)} \geq p'_{(i)},\ j = 1,\dots,i. \tag{132}$$

Now, consider an arbitrary scheme $A \in \mathcal{A}$, and suppose, to reach a contradiction, that under A, only the bandwidth of the primaries with pseudo-prices $p'_{(1)},\dots,p'_{(i')}$ is sold for some $i' < i$. Hence, at most i' out of the secondaries with valuations $v^{(1)},\dots,v^{(i)}$ buy bandwidth under A and hence at least one of these secondaries does not buy bandwidth. However, by (132), the valuation of such a secondary is $\geq$ the pseudo-price $p_{(i'+1)}$, which contradicts the fact that the bandwidth of the primary with pseudo-price $p_{(i'+1)}$ remains unsold. $\square$

The following lemma is an immediate consequence of Lemma 24.

Lemma 25. *Out of all the schemes in $\mathcal{A}$ and for any given set of pseudo-price distributions of the primaries and distributions of the valuations of the secondaries, given that a primary i has unused bandwidth and sets price $p_i = x$, the probability that his bandwidth is sold, and hence his expected payoff, is minimized for the scheme W.*

We assume that primaries do not know the scheme that will be used to match the primaries and secondaries, and hence, each primary, so as to maximize his worst-case payoff, selects his price distribution assuming that the scheme W will be used.

4.4.2 Analysis

We now generalize the analysis in Section 4.3 to multiple secondaries. First, it is easy to see that Lemma 15 generalizes without change to the case of multiple secondaries. Now, recall from Section 4.2 that the valuations of the secondaries are i.i.d., and each has the d.f. $G(.)$. For $i = 1, \ldots, k$, let $G^{(i)}(.)$ be the d.f. of $v^{(i)}$. The following lemma provides some simple properties of the functions $G^{(i)}(.), i = 1, \ldots, k$.

Lemma 26. *$G^{(i)}(.), i = 1, \ldots, n$ are continuous. Also:*

$$G^{(i)}(x) = 0,\ x \leq \underline{v}; i = 1, \ldots, k \tag{133}$$

$$G^{(i)}(x) = 1,\ x \geq \overline{v}; i = 1, \ldots, k \tag{134}$$

$$G^{(1)}(x) \leq G^{(2)}(x) \leq \ldots \leq G^{(k)}(x), x \in [c, \overline{v}]. \tag{135}$$

Proof. The continuity of $G^{(i)}(.), i = 1, \ldots, n$ follows from the continuity of $G(.)$. Also, (133) and (134) follow from the fact that $P(\underline{v} \leq v_j \leq \overline{v}) = 1$ for every buyer j. Finally, we get (135) from the fact that $v^{(1)} \geq v^{(2)} \geq \ldots \geq v^{(k)}$. □

Let $f_x(y)$ be as defined just after (90) in Section 4.3 and $h(.)$ and $g(.)$ be as in (91) and (92) respectively. In Lemma 16, we derived an expression for $f_x(y)$ for the case of one secondary. The following lemma generalizes that expression to k secondaries.

Lemma 27.

$$f_x(y) = (x - c) \sum_{i=1}^{k} (1 - G^{(i)}(x)) \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} y^{i-1} (1-y)^{n-i} \tag{136}$$

Proof. Let Z be the number of primaries out of primaries $\{1, \ldots, n\} \backslash m$ for which the pseudo-price $p'_j \leq x$. By (90), the events $\{p'_j \leq x\}, j \in \{1, \ldots, n\} \backslash m$ are independent Bernoulli events with success probability y each. So:

$$P(Z = i - 1) = \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} y^{i-1} (1-y)^{n-i}. \tag{137}$$

Also, if $Z = i-1$ for some $i \in \{1,\ldots,k\}$, then primary m's bandwidth is sold iff i or more secondaries have valuations $\geq x$; the probability of the latter event is:

$$1 - G^{(i)}(x). \tag{138}$$

If $Z \geq k$, then primary m's bandwidth is not sold. Conditioning on Z and using (137) and (138), we get that the probability that primary m's bandwidth is sold given that he sets a price $p_m = x$ equals the summation in (136). This, combined with the fact that if primary m's bandwidth is sold at price $p_m = x$, then he gets a payoff of $x - c$, gives (136). $\square$

Let v_T be defined as in (94). The following lemma generalizes the properties of $f_x(y)$ that were shown for the case of one secondary.

Lemma 28. *The properties of $f_x(y)$ in Lemma 17 and Lemma 18 hold for the case of k secondaries.*

Now, the analysis in Section 4.3 after Lemma 18 does not use the expression for $f_x(y)$ and relies only on the properties of $f_x(y)$ in Lemmas 17 and 18. Since these properties go through for the case of k secondaries by Lemma 28, the analysis in Section 4.3 after Lemma 18 generalizes to the case of k secondaries. In particular, we define $\tilde{p}$, C, the function $\gamma(.)$ and the function $\phi_{NE}(.)$ as in Lemma 19, (98), Lemma 20 and (104) respectively. Theorems 5 and 6 generalize to the case of k secondaries and provide the unique symmetric NE strategy.

4.5 Discussion on Structure of Symmetric NE Strategy

When the valuations of all the secondaries are constant and equal, the symmetric NE strategy is contiguous (see Theorem 3 in Chapter 3). However, we now show by providing an example that when the valuations of the secondaries are random, the NE strategy $\phi_{NE}(.)$ can be non-contiguous, even when $k = 1$.

Let $c = 0$, $n = 2$, $k = 1$, $q = \frac{1}{6}$ and

$$1 - G(x) = \begin{cases} 1 - \frac{3x}{2}, & 0 \le x \le \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} < x \le \frac{3}{4} \\ 1 - x, & \frac{3}{4} < x \le 1 \end{cases}$$

By (92), (93) and using $c = 0$, we get $g(x) = x(1 - G(x))$ and hence:

$$g(x) = \begin{cases} x\left(1 - \frac{3x}{2}\right), & 0 \le x \le \frac{1}{2} \\ \frac{x}{4}, & \frac{1}{2} < x \le \frac{3}{4} \\ x(1 - x), & \frac{3}{4} < x \le 1 \end{cases}$$

The function $g(.)$ is plotted in Fig. 4.2. Also, it can be checked that $\tilde{p} = \frac{1}{4}$ and $v_T = \frac{3}{4}$. Fig. 4.2 plots the symmetric NE strategy $\phi_{NE}(.)$ and shows that it has an interval of constancy and hence is not contiguous. The reason the interval of constancy arises is as follows. Fig. 4.2 shows that within the interval $[\tilde{p}, v_T] = \left[\frac{1}{4}, \frac{3}{4}\right]$, there is a sub-interval (around the local minimum of $g(.)$ at $\frac{1}{2}$) in which $g(x) < g(\tilde{p})$. So with C as in (98), this sub-interval is not in C. Hence, each primary plays prices in this sub-interval with zero probability (see Case (i) in the proof of Theorem 5), which results in an interval of constancy in $\phi_{NE}(.)$.

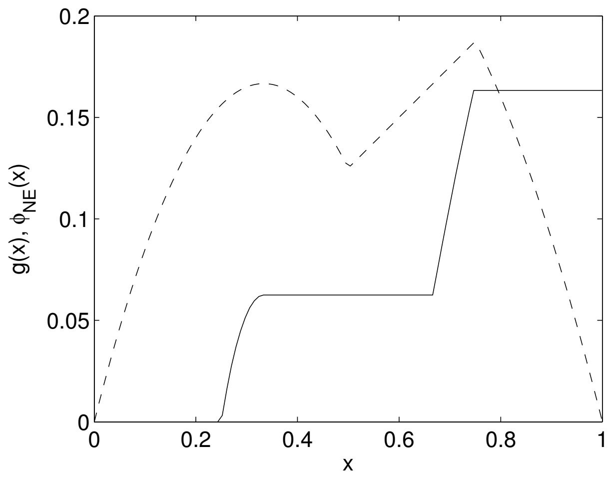

Figure 4.2: The figure plots $g(.)$ (dashed curve) and $\phi_{NE}(.)$ (solid curve) versus the price x for the example in Section 4.5.

4.6 Appendix

Proofs of Lemmas 17 and 18. Note that the expression for $f_x(y)$ in (93) is a special case with $k = 1$ of the expression for $f_x(y)$ in (136). Below, we directly prove Lemma 28, from which the proofs of Lemmas 17 and 18 follow. $\square$

Proof of Lemma 28. By Lemma 26, $G^{(i)}(.), i = 1, \dots, k$ are continuous. So by (136), it follows that $f_x(y)$ is continuous in x and y, which proves part 1 of Lemma 17 (for the case of k secondaries). By (133) and (136), for $x \leq \underline{v}$:

$$f_x(y) = (x - c) \sum_{i=1}^{k} \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} y^{i-1}(1-y)^{n-i} \tag{139}$$

from which part 2 of Lemma 17 follows. By (139) and the facts that $\underline{v} > c$ and $0 < q < 1$, it follows that $h(\underline{v}) = f_{\underline{v}}(q) > 0$. Also, by (134) and (136), $f_x(y) = 0$ for $x \geq \overline{v}$ and hence

$h(x) = f_x(q) = 0$ for $x \geq \overline{v}$. This proves part 3 of Lemma 17.

It remains to prove Lemma 18 (for the case of k secondaries). By (94) and part 3 of Lemma 17, $h(v_T) \geq h(\underline{v}) > 0$. Also, by (136) and since $h(v_T) = f_{v_T}(q)$ by (91), we get $1 - G^{(i)}(v_T) > 0$ for at least one value of i in $\{1, \ldots, k\}$. Since $1 - G^{(i)}(v_T) = P(v^{(i)} > v_T)$, it follows that $P(v_j > v_T) > 0$, $j = 1, \ldots, k$. So $1 - G^{(i)}(v_T) > 0$ for all $i = 1, \ldots, k$. But for each i, $1 - G^{(i)}(x)$ is a decreasing function of x. Hence:

$$1 - G^{(i)}(x) > 0,\ x \leq v_T, i = 1, \ldots, k. \tag{140}$$

Now, fix an arbitrary $x \in [c, v_T]$. Let $a_i = 1 - G^{(i)}(x), i = 1, \ldots, k$. By (135) and (140):

$$a_1 \geq a_2 \geq \ldots \geq a_k > 0 \tag{141}$$

Let $b_i(y) = \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} y^{i-1}(1-y)^{n-i}$. We have the following property from [70]:

Property 3. *For every $1 \leq j \leq n-1$, $\sum_{i=1}^{j} b_i(y)$ is a strictly decreasing function of y.*

Now, note that for $i \in \{1, \ldots, k\}$:

$$a_i = \sum_{j=i}^{k-1} (a_j - a_{j+1}) + a_k. \tag{142}$$

Now, by (136), $f_x(y) = (x-c)T$, where

$$\begin{aligned}
T &= \sum_{i=1}^{k} a_i b_i(y) \\
&= \sum_{i=1}^{k} \left\{ \sum_{j=i}^{k-1} (a_j - a_{j+1}) + a_k \right\} b_i(y) \quad \text{(by (142))} \\
&= a_k \sum_{i=1}^{k} b_i(y) + \sum_{i=1}^{k} \sum_{j=1}^{k-1} (a_j - a_{j+1}) b_i(y) I\{j \geq i\} \\
&= a_k \sum_{i=1}^{k} b_i(y) + \sum_{j=1}^{k-1} (a_j - a_{j+1}) \sum_{i=1}^{k} b_i(y) I\{j \geq i\} \\
&= a_k \sum_{i=1}^{k} b_i(y) + \sum_{j=1}^{k-1} (a_j - a_{j+1}) \left(\sum_{i=1}^{j} b_i(y) \right) \qquad (143)
\end{aligned}$$

By (141), each of the terms $a_j - a_{j+1}, j = 1, \ldots, k-1$ are nonnegative and $a_k > 0$; so by Property 3, the expression in (143) is strictly decreasing in y. So T, and hence $f_x(y) = (x-c)T$, is strictly decreasing in y for fixed x. $\square$

Proof of Fact 1. Let $\{x_n : n = 1, 2, 3, \ldots\}$ be any sequence such that $x_n \to x \in [a, b]$. It is sufficient to show that

$$\lim_{n\to\infty} \gamma(x_n) = \gamma(x). \qquad (144)$$

To show (144), consider the sequence

$$y_n = \gamma(x_n), n = 1, 2, 3, \ldots. \qquad (145)$$

Let $L = \liminf_{n\to\infty} y_n$. Then there exists a subsequence of the sequence $\{y_n\}$, say $\{y_{n_k}, k = 1, 2, 3, \ldots\}$, such that $y_{n_k} \to L$ as $k \to \infty$. By (103) and (145):

$$F(x_{n_k}, y_{n_k}) = \alpha, \; k = 1, 2, 3 \ldots \qquad (146)$$

So:

$$\lim_{k\to\infty} F(x_{n_k}, y_{n_k}) = \alpha.$$

By continuity of $F(.)$, and using $x_{n_k} \to x$ and $y_{n_k} \to L$:

$$F(x, L) = \alpha.$$

But since $y = \gamma(x)$ is the unique value that satisfies $F(x, y) = \alpha$, we get

$$L = \gamma(x). \tag{147}$$

Now, let $U = \limsup_{n\to\infty} y_n$. Similar to the proof of (147), we get:

$$U = \gamma(x). \tag{148}$$

By (145), (147) and (148), $\liminf_{n\to\infty} \gamma(x_n) = \gamma(x)$ and $\limsup_{n\to\infty} \gamma(x_n) = \gamma(x)$, from which (144) follows. This completes the proof. $\square$

Chapter 5

Spectrum Pricing Games with Spatial Reuse

In Chapters 3 and 4, we analyzed price competition in a setup where there are multiple primaries and secondaries *in a single location*. In this chapter, we analyze price competition under spatial reuse constraints.

5.1 Introduction

Radio spectrum is a commodity that allows *spatial reuse*: the same band can be simultaneously used at far-off locations without interference; on the other hand, simultaneous transmissions at neighboring locations on the same band interfere with each other. Thus, spatial reuse provides an *opportunity* to primaries to increase their profit by selling the same band to secondaries at different locations, which they can utilize subject to satisfying the interference *constraints*. So when multiple primaries own bandwidth in a large region, each needs to decide on a set of non-interfering locations within the region, which corresponds to an *independent set* in the *conflict graph* representing the region, at which to offer bandwidth. This is a source of strategic interaction among the primaries– each primary would like to select a maximum-sized independent set to offer bandwith at; but if a lot of primaries offer bandwidth at the same locations, there is intense competition at those locations. So a primary would have benefited by instead offering bandwidth at a smaller independent set and charging high prices at those locations.

In this chapter, we formulate the price competition scenario with spatial reuse as a game in which each primary needs to select (i) a set of locations at which to offer bandwidth and (ii) the price of bandwidth at each location. We first analyze the symmetric case $q_1 = \ldots = q_n = q$ for simplicity, which makes the game a symmetric game, and focus on symmetric NE. Our first contribution is to prove a *separation theorem* (Section 5.2.2), which states that in a symmetric NE, the price distributions used by the primaries at different nodes are uniquely determined once the independent set selection distributions are obtained. We therefore focus on computing the latter, which in turn provides the joint independent set and price selection strategies, by virtue of the results in Chapter 3 for the single location case.

We focus on a class of conflict graphs that we refer to as *mean valid graphs*. These are graphs whose node set can be partitioned into d disjoint maximal independent sets

$I_1, \ldots, I_d$, for some integer $d \geq 2$, and which satisfy another technical condition to be introduced later (in Section 5.3). As we show in Section 5.5.1, it turns out that the conflict graphs of a large number of topologies that arise in practice are mean valid. In particular, several lattice arrangements of nodes in two and three dimensions are mean valid, *e.g.*, a grid graph in two dimensions, such as that in part (b) of Fig. 5.2 or Fig. 5.3, which may be the conflict graph of shops in a shopping complex, the conflict graph of a cellular network with hexagonal cells (see Figs. 5.6 and 5.7), a grid graph in three dimensions, which may represent offices in a corporate building (see Fig. 5.5) etc.

We show that a mean valid graph has a *unique* symmetric NE; in this NE, each primary offers bandwidth only at some or all of the independent sets in $I_1, \ldots, I_d$ with positive probability and with 0 probability at every other independent set. These probabilities (and thereby the NE strategies) can be explicitly computed by solving a system of equations that we provide. The fact that primaries offer bandwidth with a positive probability at only a small number of independent sets is a surprising result, because in most graphs, including the examples in the previous paragraph, the number of independent sets is exponential in the number of nodes. Our characterization of the symmetric NE also reveals that when the probability q that a primary has unused bandwidth is small, primaries only offer bandwidth at the larger independent sets out of $I_1, \ldots, I_d$ and as q increases, primaries also start offering bandwidth at the smaller ones. This is because, for given prices, a larger independent set yields a larger revenue. However, as q increases, the price competition at the large independent sets becomes intense and drives down the prices and revenues at those independent sets. So primaries also offer bandwidth at the smaller independent sets.

This chapter is organized as follows. We describe our model in Section 5.2.1. We introduce mean valid graphs in Section 5.3 and provide several examples. In Section 5.4, we prove the theorem, discussed above, on characterization of the unique symmetric NE in mean valid graphs. In Section 5.5, we show that the conflict graphs of several topologies of practical interest as well as some other common types of graphs are mean valid. In Section 5.6, we show that the mean validity condition is a necessary condition for the existence of a symmetric NE of the above form when $d = 2$, and also find the symmetric NE and prove its uniqueness in a specific non mean valid graph. In Section 5.7, we generalize our results to the case in which $q_1, \ldots, q_n$ may not be equal and present numerical studies in Section 5.9.

We present some of the proofs in the main text and defer the rest until the appendix (Section 5.11).

5.2 Model and some Basic Results

5.2.1 Model

Suppose there are $n \geq 2$ primaries, each of whom owns a channel throughout a large region which is a geographically well-separated or separately administered area, such as a state or a country [19]. The channels owned by the primaries are all orthogonal to each other. In every slot, each primary independently either uses its channel throughout the region to satisfy its own subscriber demand, or does not use it anywhere in the region. A typical scenario where this happens is when primaries broadcast the same signal over the entire region, *e.g.*, if they are television broadcasters. Let $q \in (0,1)$ be the probability that a primary does *not* use its channel in a slot (to satisfy its subscriber demand). For simplicity, we assume that the probability q is the same for all primaries; we discuss the effect of relaxing this assumption in Section 5.7. Now, the region contains smaller parts, which we refer to as *locations*. For example, the large region may be a state, and the locations may be towns within it.

We assume that there are K_v secondaries at location v, where K_v is a random variable with probability mass function (p.m.f.) $Pr(K_v = k) = \gamma_k$. Also, the random variables K_v at different nodes v may be correlated. The primaries apriori know only the γ_ks, but not the values of K_v for any given location v. Also, the p.m.f. $\{\gamma_k\}$ satisfies the technical assumptions described in Section 3.2 and as before, let $r = P(K_v \geq 1) = 1 - \gamma_0$.

A primary who has unused bandwidth in a slot can lease it out to secondaries at a subset of the locations, provided this subset satisfies the *spatial reuse constraints*, which we describe next. The overall region can be represented by an undirected graph [71] $G = (V,E)$, where V is the set of nodes and E is the set of edges, called the *conflict graph*, in which each node represents a location, and there is an edge between two

[19]We assume that all the primaries own bandwidth in the same region.

nodes iff transmissions at the corresponding locations interfere with each other. Note that graphs have been widely used to model ad hoc networks, wherein wireless devices are modeled as nodes in an undirected graph, with mutually interfering nodes being connected by an edge [23], [66]. However, the concept of spatial reuse in our paper is more closely related to the corresponding notion in cellular networks, where cells are represented by nodes in an undirected graph, with interfering cells corresponding to neighbors in the graph [55]. Recall that an *independent set* [71] (I.S.) in a graph is a set of nodes such that there is no edge between any pair of nodes in the set. Now, a primary who is not using its channel must offer it at a set of mutually non-interfering locations, or equivalently, at an I.S. of nodes; otherwise secondaries[20] will not be able to successfully transmit simultaneously using the bandwidth they purchase, owing to mutual interference. Fig. 5.1 illustrates the model.

A primary i who offers bandwidth at an I.S. I, must also determine for each node $v \in I$, the access fee, $p_{i,v}$, to be charged to a secondary if the latter leases the bandwidth at node v. A primary incurs a cost of $c \geq 0$ per slot per node for leasing out bandwidth.

As in the single location case, we assume that $p_{i,v} \leq \nu$ for each primary i and each node v, for some constant $\nu > c$. This upper bound ν may either be a regulator-imposed limit or the valuation of each secondary for unit bandwidth. We assume that the primaries know this upper limit ν.

[20]Note that secondaries are usually customers or local providers, and purchase bandwidth for communication (and not television broadcasts). Thus, two secondaries can not use the same band simultaneously at interfering locations.

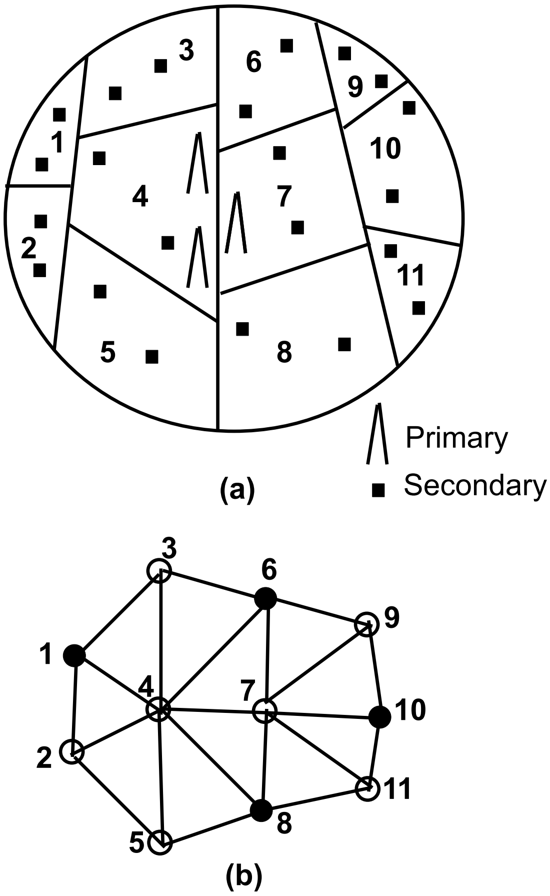

Figure 5.1: The figure illustrates the network model. Part (a) shows a region containing 11 locations. There are $n = 3$ primaries, and $k = 2$ secondaries in each location. Part (b) shows the conflict graph corresponding to the region in part (a). The darkened nodes constitute a maximal independent set.

Secondaries buy bandwidth from the primaries that offer the lowest price. More precisely, in a given slot, let Z_v be the number of primaries who offer unused bandwidth at a node v. Then, since there are K_v secondaries at node v, the bandwidth of the $\min(Z, K_v)$ primaries that offer the lowest prices is bought (ties are resolved at random) at the node. The utility of a primary i who offers bandwidth at an I.S. I and sets a price of $p_{i,v}$ at node $v \in I$ is given by $\sum(p_{i,v} - c)$, where the summation is over the nodes $v \in I$

at which primary i's bandwidth is bought. (The utility is 0 if bandwidth is not bought at any node).

Thus, each primary must jointly select an I.S. at which to offer bandwidth, and the prices to set at the nodes in it. Both the I.S. and price selection may be random. *Thus, a strategy, say ψ_i, of a primary i provides a probability mass function (p.m.f.) for selection among the I.S. and the price distribution it uses at each node (both selections contingent on having unused bandwidth).* Note that we allow a primary to use different (and arbitrary) price distributions for different nodes (and therefore allow, but do not require, the selection of different prices at different nodes), and arbitrary p.m.f. (i.e., discrete distributions) for selection among the different I.S.

The vector $(\psi_1,\ldots,\psi_n)$ of strategies of the primaries is called a *strategy profile* [42]. Let $\psi_{-i}=(\psi_1,\ldots,\psi_{i-1},\psi_{i+1},\ldots,\psi_n)$ denote the vector of strategies of primaries other than i. Let $E\{u_i(\psi_i,\psi_{-i})\}$ denote the expected utility of primary i when it adopts strategy ψ_i and the other primaries adopt ψ_{-i}.

Now, let

$$w(q,n)=\sum_k \gamma_k \sum_{i=k}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} q^i(1-q)^{n-1-i} \tag{149}$$

This is the probability that K_v or more primaries out of a given set of $n-1$ primaries offer bandwidth at a given node v. Note that when the number of secondaries K_v is constant, *i.e.*, the p.m.f. $\{\gamma_k\}$ is concentrated at a single value, the above expression reduces to the expression in (2). We will later use the following fact [70]:

Lemma 29. *$w(q,n)$ is a strictly increasing function of q for fixed n.*

Let $\tilde{p}$ be as in Property 2 in Chapter 3. We showed in that chapter (see Section 3.3.4) that in the price competition game at a single location, there is a unique NE in which each primary randomizes over the prices in the range $[\tilde{p}, v]$ using a continuous distribution function (d.f.) $\psi(.)$. Also, under this symmetric NE, each primary receives an expected payoff of (see Property 2 in Chapter 3 and note that $w(q,n) = w_1$):

$$\tilde{p} - c = (\nu - c)(1 - w(q,n)) \tag{150}$$

5.2.2 A Separation Result

Recall that a strategy of a primary consists of a p.m.f. over I.S. *and* price distributions at individual nodes. We now provide a separation framework from which the price distributions at individual nodes in a symmetric NE follow once the I.S. selection p.m.f.s are determined.

Let $\mathscr{I}$ be the set of all I.S. in G. For convenience, we assume that the empty I.S. $I_\emptyset \in \mathscr{I}$ and we allow a primary to offer bandwith at $I_\emptyset$, *i.e.* to not offer bandwidth at any node, with some probability. Consider a symmetric strategy profile under which each primary offers bandwidth at I.S. $I \in \mathscr{I}$ w.p. $\beta(I)$, where:

$$\sum_{I \in \mathscr{I}} \beta(I) = 1. \tag{151}$$

The probability, say α_v, with which each primary offers bandwidth at a node $v \in V$

equals the sum of the probabilities associated with all the I.S. that contain the node, *i.e.*

$$\alpha_v = \sum_{I \in \mathscr{I} : v \in I} \beta(I) \tag{152}$$

Now, considering that each primary has unused bandwidth w.p. q, it offers it at node v w.p. $q\alpha_v$. *The price selection problem at each node v is now equivalent to that for the single location case, the difference being that each primary offers unused bandwidth w.p. $q\alpha_v$, instead of q, at node v.* Thus:

Lemma 30. *Suppose under a symmetric NE each primary selects node v w.p. α_v if it has unused bandwidth. Then under that NE the price distribution of each primary at node v is the d.f. $\psi(.)$ in the single location case, with $q\alpha_v$ in place of q.*

Thus, a symmetric NE strategy is completely specified once the I.S. selection p.m.f. $\{\beta(I) : I \in \mathscr{I}\}$ (which will in turn provide the α_vs via (152)) is obtained.

5.2.3 Node and I.S. Probabilities

Consider a symmetric NE where each primary uses the strategy ψ, under which it offers bandwidth at I.S. $I \in \mathscr{I}$ with some probability $\beta(I)$. The probability, α_v, with which each primary offers bandwidth at a node $v \in V$ is determined by the I.S. distribution $\{\beta(I) : I \in \mathscr{I}\}$ via (152).

Now, for simplicity, we normalize $\nu - c = 1$. With $w(q,n)$ as in (149), let:

$$W(\alpha) = (1 - w(q\alpha, n))(\nu - c) = (1 - w(q\alpha, n)). \tag{153}$$

By Lemma 30, and similar to (150) in the single location case, in a symmetric NE if primaries offer bandwidth at a node with probability α (and play the single-node NE strategy with $q\alpha$ in place of q at that node), then $W(\alpha)$ is the maximum expected payoff that each primary i can get at that node. It gets this payoff $W(\alpha)$ if it sets any price in the range $[\nu - w(q\alpha, n)(\nu - c), \nu]$ at that node. Under the above symmetric NE with strategy profile $(\psi, \dots, \psi)$, each primary offers bandwidth at node $v \in V$ w.p. α_v. So the expected payoff of each primary i is given by:

$$E\{u_i(\psi, \psi_{-i})\} = \sum_{v \in V} \alpha_v W(\alpha_v). \tag{154}$$

Now, in general, different I.S. distributions $\{\beta(I) : I \in \mathscr{I}\}$ can result in the same node distribution [21] $\{\alpha_v : v \in V\}$. However, by (154), the expected payoff of each primary in a symmetric NE is completely determined by the node distribution, *i.e.* it is the same under different I.S. distributions that correspond to the same node distribution. So if primary i knows the node distribution chosen by the other primaries, then it has sufficient information to choose its best response; it does not need to know their I.S. distribution in addition. Thus, the *game aspect* of the price competition, *i.e.* the strategic interaction between the primaries, is completely determined by the node distribution.

We now introduce a definition:

Definition 1 (Valid Distribution)**.** *An assignment* $\{\alpha_v : v \in V\}$ *of probabilities to the nodes is said to be a* valid *distribution if there exists a probability distribution* $\{\beta(I) : I \in \mathscr{I}\}$ *such that for each* $v \in V$, $\alpha_v = \sum_{I \in \mathscr{I} : v \in I} \beta(I)$.

Note that, given a valid distribution $\{\alpha_v : v \in V\}$, a corresponding I.S. distribution

[21] Although we refer to $\{\alpha_v : v \in V\}$ as a *distribution*, note that $\sum_{v \in V} \alpha_v$ need not equal 1 in general.

can be computed by solving the system of linear equations (152).

Thus, we can equivalently define the strategy of a primary in a symmetric NE as a node distribution $\{\alpha_v : v \in V\}$. So henceforth, we interchangeably speak of the strategy of a primary as either an I.S. distribution $\{\beta(I) : I \in \mathscr{I}\}$ (note that the price distribution follows by Lemma 30) or a node distribution $\{\alpha_v : v \in V\}$. Also, we say that the symmetric NE is unique if the node distribution $\{\alpha_v : v \in V\}$ is unique.

Remark 3. *In Theorem 4 of Chapter 3, we showed the uniqueness of the NE in the price competition game at a single location, even for the asymmetric case in which $q_1, \ldots, q_n$ are not equal. However, in presence of spatial reuse, there are multiple NE even in the symmetric case $q_1 = \ldots = q_n = q$. For example, suppose there are two nodes v_1 and v_2 connected by an edge, two primaries ($n = 2$) and one secondary with probability* 1 *at each node ($k = 1$). Then the strategy profiles in which primary* 1 *offers bandwidth at node v_1 and primary* 2 *at node v_2 w.p.* 1*, or vice versa, and both primaries set a price of* ν *w.p.* 1*, are NE, apart from the symmetric NE to be described in Theorem 8 below.*

5.3 Mean Valid Graphs

We now introduce a class of graphs, which we refer to as *mean valid graphs*. The motivation behind studying these graphs is that the conflict graphs of several topologies that commonly arise in practice are mean valid graphs. Also, as we show in the next section, these graphs have a unique symmetric NE, which can be explicitly computed and has a simple form.

5.3.1 Definition

Definition 2 (Mean Valid Graph)**.** *We refer to a graph $G = (V,E)$ as* mean valid *if it satisfies the following two conditions:*

1. *Its vertex set can be partitioned into d disjoint maximal [22] I.S. for some integer $d \geq 2$: $V = I_1 \cup I_2 \cup \ldots \cup I_d$, where I_j, $j \in \{1,\ldots,d\}$, is a maximal I.S. and $I_j \cap I_m = \emptyset$, $j \neq m$.*

   Let $|I_j| = M_j$,

   $$M_1 \geq M_2 \geq \ldots \geq M_d, \tag{155}$$

   and $I_j = \{a_{j,l} : l = 1,\ldots,M_j\}$.

2. *For every valid distribution [23] in which a primary offers bandwidth at node $a_{j,l}$ w.p. $\alpha_{j,l}$, $j = 1,\ldots,d$, $l = 1,\ldots,M_j$,*

   $$\sum_{j=1}^{d} \overline{\alpha}_j \leq 1, \tag{156}$$

   where

   $$\overline{\alpha}_j = \frac{\sum_{l=1}^{M_j} \alpha_{j,l}}{M_j}, j \in \{1,\ldots,d\}. \tag{157}$$

We now explain the two conditions in Definition 2. Recall that a graph $G = (V,E)$ is said to be d-partite if V can be partitioned into d disjoint I.S. $I_1,\ldots,I_d$ [71]. For example, when $d = 2$, G is a bipartite graph. The first condition in Definition 2 says

[22]Recall that an I.S. I is said to be maximal if $I \cup \{v\}$ is not an I.S. for all $v \in V$ [71].

[23]Note that we write $\alpha_{j,l}$ in place of $\alpha_{a_{j,l}}$ to simplify the notation.

that G is a d-partite graph and has the additional property that each of $I_1,\ldots,I_d$ is a *maximal* I.S.

Now we explain Condition 2 in Definition 2. Let $\{\alpha_{j,l} : j = 1,\ldots,d; l = 1,\ldots,M_j\}$ be an arbitrary valid distribution. Consider the distribution $\alpha'_{j,l} = \overline{\alpha}_j$, with $\overline{\alpha}_j$ as in (157), *i.e.* for each j and $l = 1,\ldots,M_j$, $\alpha'_{j,l}$ is set equal to the *mean* of $\alpha_{j,m}, m = 1,\ldots,M_j$. If (156) is true, then this distribution of means is a valid distribution because it corresponds to the I.S. distribution $\{\beta(I_j) = \overline{\alpha}_j, j = 1,\ldots,d, \beta(I_\emptyset) = 1 - \sum_{j=1}^{d} \overline{\alpha}_j; \beta(I) = 0, I \neq I_1,\ldots,I_d, I_\emptyset\}$. Thus, Condition 2 in Definition 2 says that in G, the distribution of means corresponding to every valid distribution is valid. As we will see in Section 5.4, this condition is the crux behind the fact that in the symmetric NE in a mean valid graph, each primary offers bandwidth with equal probabilities at all the nodes in I_j for every $j = 1,\ldots,d$.

5.3.2 Examples

Technical as Definition 2 may seem, it turns out that several conflict graphs that commonly arise in practice are mean valid. For example, consider the following graphs:

1. Let $\mathcal{G}_m$ denote a graph that is a linear arrangement of $m \geq 2$ nodes as shown in part (a) of Fig. 5.2, with an edge between each pair of adjacent nodes. As an example, this would be the conflict graph for locations along a highway or a row of roadside shops.

2. We consider two types of $m \times m$ grid graphs, denoted by $\mathcal{G}_{m,m}$ (see part (b) of

Fig. 5.2) and $\mathcal{H}_{m,m}$ (see Fig. 5.3). In both these graphs, m^2 nodes (locations) are arranged in a square grid. In $\mathcal{G}_{m,m}$, there is an edge only between each pair of adjacent nodes in the same row or column. In $\mathcal{H}_{m,m}$, in addition to these edges, there are also edges between nodes that are neighbors along a diagonal as shown in Fig. 5.3. For example, $\mathcal{G}_{m,m}$ or $\mathcal{H}_{m,m}$ may represent a shopping complex, with the nodes corresponding to the locations of shops with WiFi Access Points (AP) for Internet access. Depending on the proximity of the shops to each other and the transmission ranges of the APs, the conflict graph could be $\mathcal{G}_{m,m}$ or $\mathcal{H}_{m,m}$. $\mathcal{H}_{m,m}$ is also the conflict graph of a cellular network with square cells as shown in Fig. 5.4.

3. Let $\mathcal{T}_{m,m,m}$ be a three-dimensional grid graph (see Fig. 5.5), which may, for example, be the conflict graph for offices in a corporate building or rooms in a hotel.

4. The conflict graph (Fig. 5.7) of a cellular network with hexagonal cells (Fig. 5.6).

5. Consider a clique [24] of size e, where $e \geq 1$ is any integer. This is the conflict graph for any set of e locations that are close to each other.

All of the above are mean valid graphs:

Theorem 7. *The following graphs are mean valid, with d, the number of disjoint maximal I.S., indicated in each case:*

1. a clique of size $e \geq 1$ ($d = e$),

[24] Recall that a *clique* or a complete graph of size e is a graph with e nodes and an edge between every pair of nodes [71].

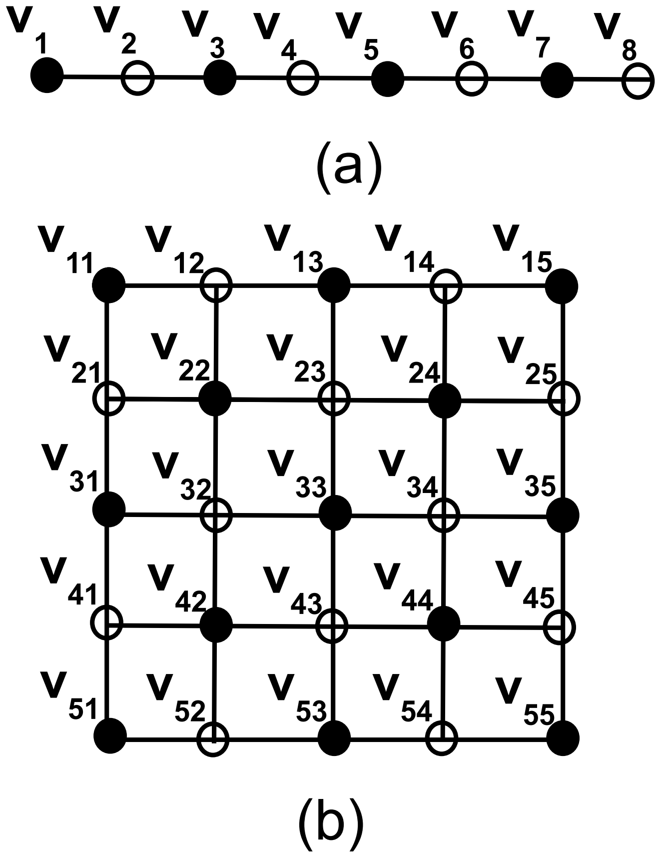

Figure 5.2: Part (a) shows a linear graph, $\mathcal{G}_m$, with $m = 8$ and part (b) shows a grid graph, $\mathcal{G}_{m,m}$, with $m = 5$. In both graphs, the darkened and un-darkened nodes constitute I_1 and I_2 respectively.

2. *a line graph* $\mathcal{G}_m$ *(*$d = 2$*),*

3. *a two-dimensional grid graph* $\mathcal{G}_{m,m}$ *(*$d = 2$*),*

4. *a two-dimensional grid graph* $\mathcal{H}_{m,m}$ *(*$d = 4$*),*

5. *a three-dimensional grid graph* $\mathcal{T}_{m,m,m}$ *(*$d = 8$*).*

6. *a cellular network with hexagonal cells, under Assumption 1 in Section 5.5.1 (*$d = 3$*).*

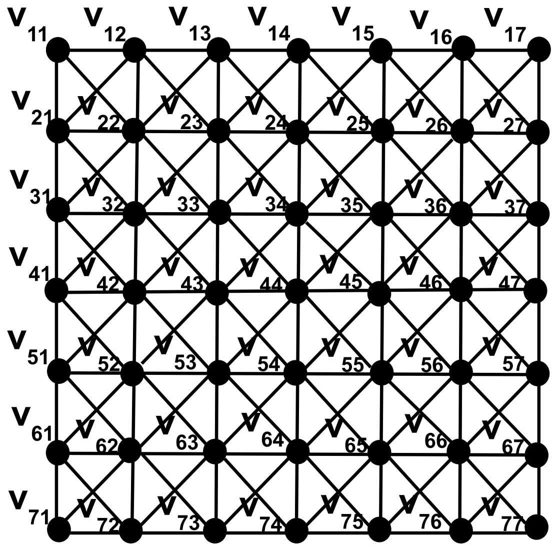

Figure 5.3: The figure shows a grid graph $\mathcal{H}_{m,m}$ with $m = 7$.

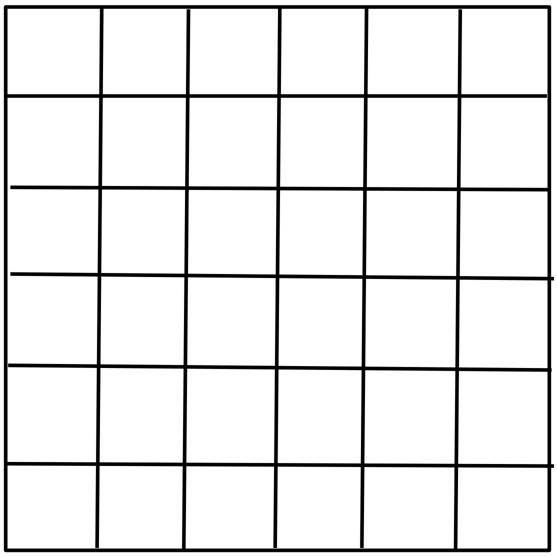

Figure 5.4: The figure shows a tiling of a plane with squares, *e.g.* cells in a cellular network. Transmissions at neighboring cells interfere with each other. The corresponding conflict graph is $\mathcal{H}_{6,6}$.

We defer the proof of Theorem 7 until Section 5.5.1. Also, as we show in Section 5.5.2, some other common classes of graphs, such as a star and a κ-regular bipartite

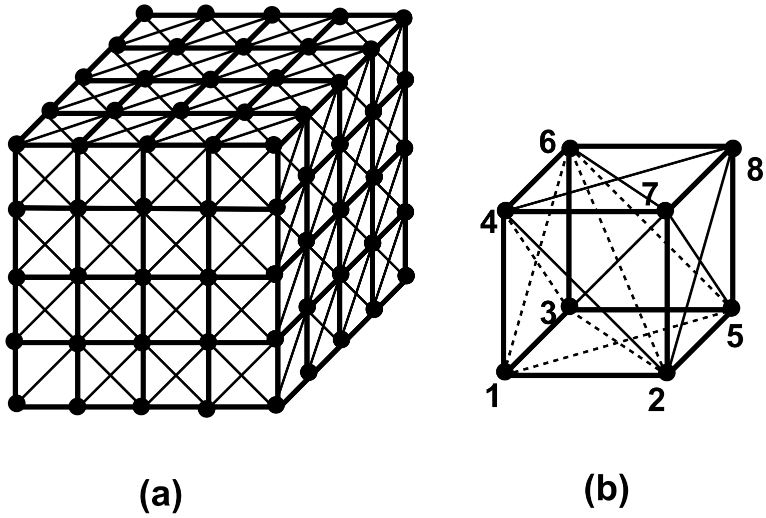

Figure 5.5: Part (a) shows a three-dimensional grid graph $\mathcal{T}_{m,m,m}$ for $m = 5$. It consists of periodic repetitions of the graph shown in part (b). Also, in part (b), the node labels show the I.S. $I_1, \ldots, I_8$ they are in, *i.e.* a node with the label j is part of the I.S. I_j, $j \in \{1, \ldots, 8\}$.

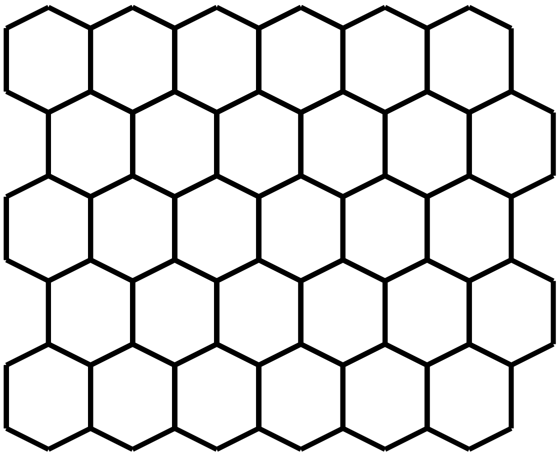

Figure 5.6: The figure shows a tiling of a plane with hexagons, *e.g.* cells in a cellular network. Transmissions at neighboring cells interfere with each other.

graph, are mean valid as well.

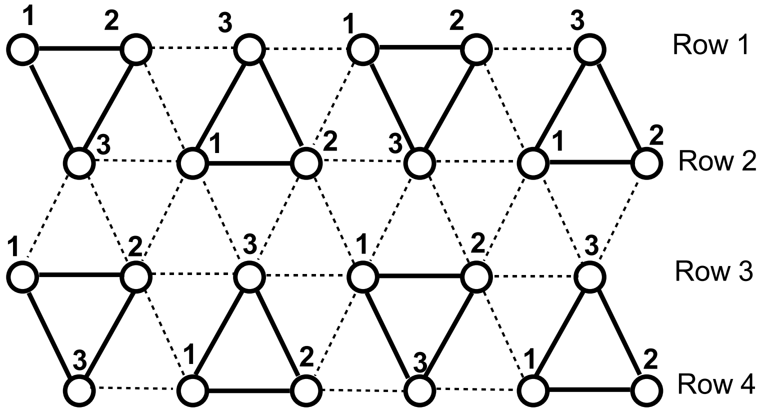

Figure 5.7: The figure shows the conflict graph of a hexagonal tiling of a plane. Both the solid and dotted edges are part of the graph. The nodes labelled j, $j \in \{1,2,3\}$, are in I.S. I_j. There are four rows of nodes. The figure also shows the construction of the graph from cliques of size 3 each, shown by the solid edges. The dotted edges are added later. Note that no edge is between two nodes in the same I.S., so the hypothesis of Lemma 38 is satisfied.

A graph obtained by considering the union of disjoint mean valid graphs, all of which correspond to the same integer d, and then adding some edges to get a connected graph, is a mean valid graph under some technical conditions [25], *e.g.*, the cellular networks in a group of neighboring towns or the WiFi networks in the departments of a university campus. Fig. 5.8 illustrates the latter example.

5.3.3 A Necessary and Sufficient Condition

We state a property of mean valid graphs for later use.

Lemma 31. *Let $G = (V,E)$ be a graph that satisfies Condition* 1 *in Definition 2. Sup-*

[25] These technical conditions are stated in Lemmas 38 and 39 in Section 5.5.1.

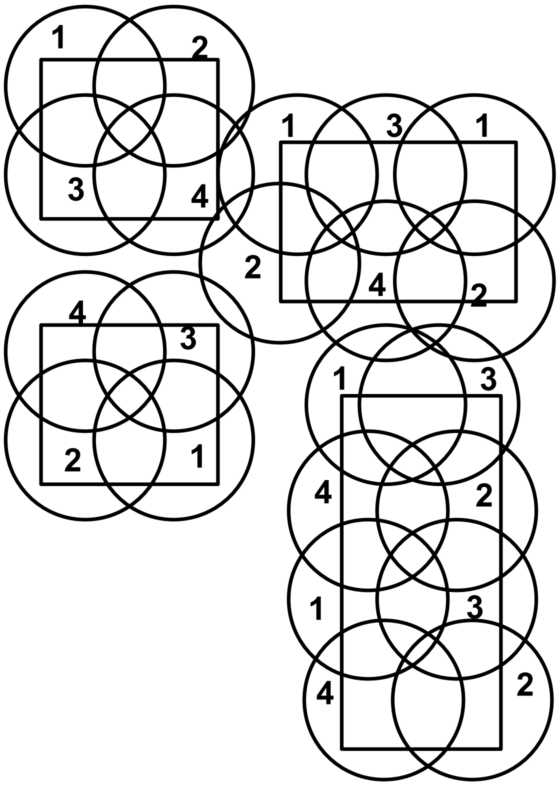

Figure 5.8: The rectangles represent departments in a university campus and the circles are the ranges of WiFi access points. The circles (nodes) in each rectangle constitute a grid graph $\mathscr{H}_{m,m}$, which is mean valid with $d = 4$ (see part 4 of Theorem 7). The overall graph is also mean valid with $d = 4$. With $I_j, j \in \{1,2,3,4\}$, being disjoint maximal I.S. as in Definition 2, the number in each circle indicates the I.S. to which the corresponding node belongs, *i.e.* nodes corresponding to the circles numbered $j \in \{1,2,3,4\}$ belong to I.S. I_j.

pose $I \in \mathscr{I}$ contains $m_j(I)$ nodes from I_j, $j = 1,\ldots,d$. G is mean valid if and only if:

$$\sum_{j=1}^{d} \frac{m_j(I)}{M_j} \leq 1 \ \forall I \in \mathscr{I} \tag{158}$$

5.4 Symmetric NE in Mean Valid Graphs

In this section, we show that a mean valid graph has a unique symmetric NE; in this NE, in the notation of Definition 2, primaries offer bandwidth at all the nodes in I_j, $j \in \{1,\ldots,d\}$, with the same probability t_j, *i.e.* $\alpha_{j,l} = t_j\ \forall l = 1,\ldots,M_j$, where $\{t_j : j = 1,\ldots,d\}$ is the unique solution of a set of equations that we provide.

Let G be a mean valid graph. Suppose there exists a symmetric NE in which each primary offers bandwidth at node $a_{j,l}$ w.p. $\alpha_{j,l}$, $j = 1,\ldots,d$, $l = 1,\ldots,M_j$, where $\{\alpha_{j,l}\}$ is a valid distribution. Let ψ denote this strategy. First, we will argue, by contradiction, that for each j, $\alpha_{j,l} = \overline{\alpha}_j\ \forall l = 1,\ldots,M_j$, where $\overline{\alpha}_j$ is given by (157). In the symmetric NE $(\psi,\ldots,\psi)$, by (153) and the discussion just after it, primary 1 gets an expected payoff of $W(\alpha_{j,l})$ at node $a_{j,l}$; also, by (154), its total expected payoff is:

$$E\{u_1(\psi,\psi_{-1})\} = \sum_{j=1}^{d}\sum_{l=1}^{M_j}\alpha_{j,l}W(\alpha_{j,l}) \tag{159}$$

Suppose $\alpha_{j,l}, l = 1,\ldots,M_j$ are not all equal for some j. By (153) and Lemma 29, $W(\alpha)$ is a strictly decreasing function of α; so primary 1 offers bandwidth with a high probability $\alpha_{j,l}$ at nodes $a_{j,l}$ at which it gets a low payoff $W(\alpha_{j,l})$. This seems to suggest that primary 1 could get a higher overall payoff by unilaterally switching to an alternative strategy, say ψ_0, under which it decreases (respectively, increases) the node probabilities at nodes that yield a low (respectively, high) payoff, if such a strategy ψ_0 were to exist. This would contradict the fact that the distribution $\{\alpha_{j,l} : j = 1,\ldots,d; l = 1,\ldots,M_j\}$ is primary 1's best response and thereby imply that $\alpha_{j,l}, l = 1,\ldots,M_j$ must be equal for every $j = 1, \ldots, d$.

The existence of such a strategy ψ_0 is guaranteed by Condition 2 in Definition 2– that (156) holds for every valid distribution. Let ψ_0 be a strategy under which primary 1 offers bandwidth at each node in I_j, $j \in \{1, \ldots, d\}$ w.p. $\overline{\alpha}_j$. Note that $\sum_{j=1}^{d} \overline{\alpha}_j \leq 1$ by (156); so ψ_0 is a valid distribution since it corresponds to the I.S. distribution $\beta(I_j) = \overline{\alpha}_j$, $j \in \{1, \ldots, d\}$, $\beta(I_\emptyset) = 1 - \sum_{j=1}^{d} \overline{\alpha}_j$, $\beta(I) = 0$, $I \neq I_1, \ldots, I_d, I_\emptyset$. By (154), the total expected payoff of primary 1 if it plays strategy ψ_0 is:

$$E\{u_1(\psi_0, \psi_{-1})\} = \sum_{j=1}^{d} \sum_{l=1}^{M_j} \overline{\alpha}_j W(\alpha_{j,l}) \tag{160}$$

By (159) and (160):

$$\begin{aligned} & E\{u_1(\psi, \psi_{-1})\} - E\{u_1(\psi_0, \psi_{-1})\} \\ = & \sum_{j=1}^{d} \left(\sum_{l=1}^{M_j} \alpha_{j,l} W(\alpha_{j,l}) - \overline{\alpha}_j \left(\sum_{l=1}^{M_j} W(\alpha_{j,l}) \right) \right) \end{aligned} \tag{161}$$

Now, we have the following algebraic fact, proved in Section 5.11.

Lemma 32. *Let $N \geq 2$ be an integer, $\alpha_1, \ldots, \alpha_N$ be real numbers and $\overline{\alpha} = \frac{\sum_{i=1}^{N} \alpha_i}{N}$. Let $f(x)$ be any strictly decreasing function of x. Then:*

$$\left(\sum_{i=1}^{N} \alpha_i f(\alpha_i)\right) \leq \overline{\alpha}\left(\sum_{i=1}^{N} f(\alpha_i)\right) \tag{162}$$

with equality iff $\alpha_1 = \ldots = \alpha_N = \overline{\alpha}$.

Intuitively, since $f(.)$ is strictly decreasing, in the LHS of (162), the terms in which $f(\alpha_i)$ is large are multiplied by small factors α_i and vice-versa; on the other hand, all terms $f(\alpha_i)$ on the RHS are multiplied by the same factor $\overline{\alpha}$. So the LHS is smaller.

Now, as mentioned above, $f(\alpha) = W(\alpha) = 1 - w(q\alpha, n)$ is a strictly decreasing function of α. So by Lemma 32, the expression in (161) is ≤ 0, with equality holding iff $\alpha_{j,1} = \ldots = \alpha_{j,M_j} = \overline{\alpha}_j$ for each $j \in \{1, \ldots, d\}$. But since ψ is a best response, $E\{u_1(\psi, \psi_{-1})\} \geq E\{u_1(\psi_0, \psi_{-1})\}$. So the expression in (161) must equal 0 and hence $\alpha_{j,1} = \ldots = \alpha_{j,M_j} = \overline{\alpha}_j$ for each $j \in \{1, \ldots, d\}$.

Now, suppose $\sum_{j=1}^{d} \overline{\alpha}_j < 1$. Then primary 1 can unilaterally offer bandwidth at each node in I_d with probability $1 - \sum_{j=1}^{d-1} \overline{\alpha}_j > \overline{\alpha}_d$ instead of $\overline{\alpha}_d$ and increase its payoff. This contradicts the fact that the distribution is a NE. So we must have $\sum_{j=1}^{d} \overline{\alpha}_j = 1$. Thus, we have shown the following:

Lemma 33. *In a mean valid graph, under every symmetric NE, each primary offers bandwidth at each node in I_j w.p. t_j, $j \in \{1, \ldots, d\}$, for some $t_j \geq 0$, $j = 1, \ldots, d$, where $\sum_{j=1}^{d} t_j = 1$.*

A typical way in which the node probability distribution $\alpha_{j,l} = t_j \ \forall l = 1, \ldots, M_j$, arises is via the I.S. distribution $\beta(I_j) = t_j, j = 1, \ldots, d; \beta(I) = 0 \ \forall I \neq I_1, \ldots, I_d$.

The following result provides necessary conditions for a distribution $\{t_j : j = 1, \ldots, d\}$ as in Lemma 33 to constitute a symmetric NE.

Lemma 34. *If a distribution $\{t_j : j = 1, \ldots, d\}$ as in Lemma 33 constitutes a symmetric NE, then $I_1, \ldots, I_{d'}$ are best responses and $I_{d'+1}, \ldots, I_d$ are not, for some integer $d' \in \{1, \ldots, d\}$. Also, each $I \in \mathscr{I}$ containing a node from I_j for some $j > d'$ is not a best response. Hence:*

$$t_j = 0, j > d'. \tag{163}$$

Intuitively, a primary prefers to offer bandwidth at a large I.S. because it gets some revenue at every node in the I.S. it selects and its total payoff is the sum of the revenues at the nodes of the I.S. Also, recall that by (155), $I_1,\ldots,I_d$ are in decreasing order of size. So a primary will (i) try to offer bandwidth only at the largest I.S. I_1, (ii) offer bandwidth at the next largest I.S. I_2 as well with some probability only if the competition at I_1 increases beyond a threshold, (iii) offer bandwidth at I_3 as well with some probability only if the competition at I_1 and I_2 increases beyond a certain threshold and so on. Hence, the set of best responses out of $I_1,\ldots,I_d$ is of the form $I_1,I_2,\ldots,I_{d'}$ for some $1 \leq d' \leq d$.

Now, if primary i offers bandwidth at I.S. $I' \in \mathscr{I}$, its overall expected payoff, denoted by $U_1(I')$, is the sum of the expected payoffs at the nodes in I', which, by (153) and the discussion just after it, is given by:

$$U_1(I') = \sum_{v \in I'} W(\alpha_v) = \sum_{v \in I'} (1 - w(q\alpha_v, n)). \tag{164}$$

Now, consider a symmetric NE with $\{t_j : j = 1,\ldots,d\}$ as in Lemma 33. By (164) and the fact that $|I_j| = M_j$, the payoff of primary 1 if it offers bandwidth at I_j is:

$$U_1(I_j) = M_j W(t_j) \tag{165}$$

By Lemma 34, $I_1,\ldots,I_{d'}$ are best responses and $I_{d'+1}$ is not. So:

$$U_1(I_1) = \ldots = U_1(I_{d'}) > U_1(I_{d'+1})$$

Substituting (165) into the above and using (163) and the fact that $W(0) = 1 - w(0,n) =$

$1-(1-r)=r$, we get:

$$M_1 W(t_1) = \ldots = M_{d'} W(t_{d'}) > M_{d'+1} r \tag{166}$$

Thus, we have shown the following:

Lemma 35. *A distribution $\{t_j : j = 1, \ldots, d\}$ as in Lemma 33 that constitutes a symmetric NE must satisfy* (163) *and* (166) *for some integer $d' \in \{1, \ldots, d\}$.*

Lemma 35 provides necessary conditions for a distribution $\{t_j : j = 1, \ldots, d\}$ to constitute a symmetric NE. The following lemma shows that these conditions are sufficient as well.

Lemma 36. *Let $1 \leq d' \leq d$ and $t_1, \ldots, t_d$ be a probability distribution such that* (163) *and* (166) *hold. Then the symmetric strategy profile in which every primary offers bandwidth at each node in I_j w.p. t_j, $j \in \{1, \ldots, d\}$, is a NE.*

The proof of Lemma 36 (see Section 5.11) is based on the fact that the graph, being mean valid, satisfies Condition 2 in Definition 2.

The following technical lemma shows the existence and uniqueness of a distribution $(t_1, \ldots, t_d)$ satisfying (163) and (166) for every value of q.

Lemma 37. *For every $q \in (0,1)$, there exists a unique integer $d' = d'(q)$ and a unique probability distribution $(t_1, \ldots, t_d)$ such that* (163) *and* (166) *hold. Also, $d'(q)$ is an increasing function of q and, for every value of q, $t_1 \geq t_2 \ldots \geq t_d$.*

Note that the fact that $d'(q)$ is an increasing function of q is consistent with the intuition that for small values of q, primaries tend to offer bandwidth at only the larger I.S.

out of $I_1, \dots, I_d$, and as q, and thereby the competition from other primaries increases, they also choose the smaller ones. Also, the fact that $t_1 \geq t_2 \dots \geq t_d$ for all q is consistent with the intuition that primaries offer bandwidth at the larger I.S. with a larger probability.

Finally, putting together the above discussion, we get the main result of this section:

Theorem 8. *In a mean valid graph, for every $q \in (0,1)$, there is a unique symmetric NE; in this NE, each primary offers bandwidth at every node in I_j, $j \in \{1, \dots, d\}$, w.p. t_j,* i.e. *$\alpha_{j,l} = t_j$, $l = 1, \dots, M_j$, where $(t_1, \dots, t_d)$ is the unique distribution satisfying* (163) *and* (166).

Proof. By Lemma 33, under every symmetric NE, each primary must offer bandwidth at all the nodes in I_j, $j \in \{1, \dots, d\}$, w.p. t_j for some probability distribution $(t_1, \dots, t_d)$. Also, by Lemma 35, (163) and (166) hold for this distribution. By Lemma 37, for a fixed value of $q \in (0,1)$, there exists a unique distribution $(t_1, \dots, t_d)$ satisfying (163) and (166). Finally, by Lemma 36, the strategy profile where each primary uses this distribution is a NE. The result follows. □

Thus, every mean valid graph has a unique symmetric NE, which can be explicitly computed by solving the system of equations (163) and (166). Note that this is a system of non-linear equations in the variables $t_1, \dots, t_d$ and d'. It can be solved using a standard solver for non-linear equations (e.g., *fsolve* in Matlab) in combination with a search procedure to find d'.

Example: Suppose there are $n = 2$ primaries and $k = 1$ secondary with probability 1. Consider a grid graph $\mathcal{H}_{m,m}$, which was introduced in Section 5.3.2, with $m = 7$ (see Fig. 5.3). By part 4 of Theorem 7, this is a mean valid graph and, in the notation of Definition 2, $d = 4$, the I.S. I_1, I_2, I_3 and I_4 are as described in Section 5.5.1, and $M_1 = 16$, $M_2 = M_3 = 12$, $M_4 = 9$. The symmetric NE is of the form described in Theorem 8 with $d'(q), t_1, t_2, t_3$ and t_4 for different $q \in (0,1)$ as follows:

1. For $0 < q < \frac{1}{4}$, $d' = 1$, $t_1 = 1$, $t_2 = t_3 = t_4 = 0$.
2. For $\frac{1}{4} \leq q < \frac{15}{16}$, $d' = 3$, $t_1 = \frac{1}{11}\left(3 + \frac{2}{q}\right)$, $t_2 = t_3 = \frac{1}{11}\left(4 - \frac{1}{q}\right)$ $t_4 = 0$.
3. For $\frac{15}{16} \leq q < 1$, $d' = 4$, $t_1 = \frac{1}{49}\left(9 + \frac{13}{q}\right)$, $t_2 = t_3 = \frac{1}{49}\left(\frac{1}{q} + 12\right)$ $t_4 = \frac{1}{49}\left(16 - \frac{15}{q}\right)$.

Note that, consistent with Theorem 8, $d'(q)$ is an increasing function of q and $t_1 \geq t_2 \geq t_3 \geq t_4$ for each value of q. In fact, for all q, $t_2 = t_3$, which is because I_2 and I_3 are of the same size. Fig. 5.9 plots t_1, t_2 and t_4 versus q. For small q, primaries offer bandwidth at the largest I.S. I_1 with probability 1; but as q increases, the competition at I_1 increases, inducing the primaries to shift probability mass from I_1 to the other I.S. So t_1 decreases in q. However, note that for all values of q, $t_1 \geq t_2 \geq t_4$ and t_4 is very small (less than 0.02).

Remark 4. *The unique symmetric NE need not be pure even with respect to the node selections, as the above example shows. However, this mixed choice is not really an artifact of mixed price choice. For instance, consider a scenario where all primaries must choose the same fixed price p_0 (perhaps the prices have been standardized because of*

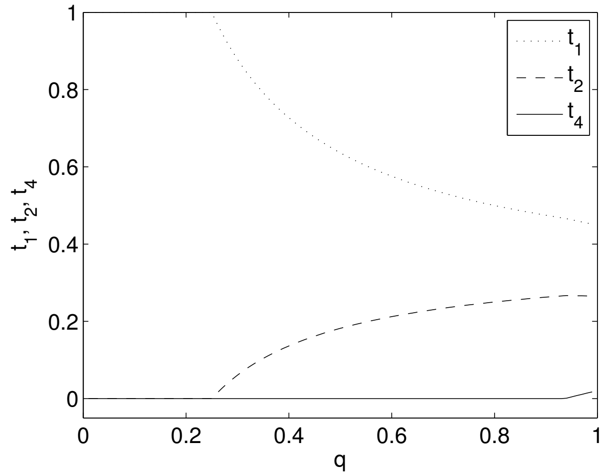

Figure 5.9: The figure shows the symmetric NE probabilities t_1, t_2 and t_4 for the example after Theorem 8.

government regulation). Suppose there are two nodes v_1 and v_2 connected by an edge, two primaries ($n = 2$) and one secondary with probability 1 *at each node ($k = 1$). Then it is easy to show that the strategy profile under which each primary offers bandwidth at v_1 and v_2 w.p.* $1/2$ *each constitutes the unique symmetric NE.*

The intuition behind randomization across different I.S. in a symmetric NE is that primaries would like to offer bandwidth at an I.S. at which other primaries do not offer *bandwidth with a high probability, whereas in a symmetric NE that is pure with respect to the node selection, all primaries offer bandwidth at the same I.S.*

5.5 Some Specific Mean Valid Graphs

Theorem 8 provides the form of the symmetric NE in mean valid graphs. So in this section, we identify some classes of mean valid graphs.

5.5.1 Topologies that Commonly Arise in Practice

We now prove Theorem 7.

The proof of part 1 of Theorem 7 is straightforward: let $\{v_1, \ldots, v_e\}$ be the nodes of the clique. $I_j = \{v_j\}, j = 1, \ldots, e$ are disjoint maximal I.S. whose union is V. Also, these are the only I.S. in the graph; so (158) holds and the clique is mean valid by Lemma 31.

Next, we prove some lemmas that we use to prove the other parts of Theorem 7.

Lemma 38. *Let $G = (V, E)$ be a mean valid graph, where $V = I_1 \cup \ldots \cup I_d$ and $I_1, \ldots, I_d$ are disjoint maximal I.S. Let $E' \supseteq E$ be any set such that no edge in E' is between two nodes in the same I.S. I_j, $j \in \{1, \ldots, d\}$. Then the graph $G' = (V, E')$ is mean valid.*

Thus, if a graph G is mean valid, then the graph G' obtained by adding edges in any fashion to G, while ensuring that I_j, $j = 1, \ldots, d$ continue to be I.S. in G', is a mean valid graph as well.

Lemma 39. *Suppose for each $i = 1, \ldots, N$, $G^i = (V^i, E^i)$ is a mean valid graph, where $V^i = I_1^i \cup \ldots \cup I_d^i$, $I_1^i, \ldots, I_d^i$ are disjoint maximal I.S., and $|I_j^i| = M_j^i$, $j = 1, \ldots, d$. Let $\mathbf{M}^i = (M_1^i, \ldots, M_d^i)$. If*

$$\mathbf{M}^i = c_i \mathbf{M}^0, \ i = 1, \ldots, N \tag{167}$$

for some vector $\mathbf{M}^0 = (M_1^0, \ldots, M_d^0)$ and positive scalars $c_1, \ldots, c_N$, then $G = (\cup_{i=1}^N V^i, \cup_{i=1}^N E^i)$ is mean valid.

Lemma 39 says that if $G^i, i = 1, \ldots, N$ are mean valid graphs, then their union G is a mean valid graph as well provided each of $G^i, i = 1, \ldots, N$ contains (i) the same number, d, of disjoint maximal I.S., and (ii) the same proportion of nodes in the d I.S. $I_1^i, \ldots, I_d^i$. Since the union graph G is a disconnected graph with N components, Lemma 39 is not useful by itself to prove that a graph is mean valid. But it can be effectively used in conjunction with Lemma 38 to combine a set of N mean valid graphs into a new connected mean valid graph by (i) first considering their union, which is a disconnected graph, (ii) and then adding some edges to it to make it connected.

A useful special case is when each of these N graphs G^i is a clique of size d (which is mean valid by Part 1 of Theorem 7) with vertex set $V^i = \{v_1^i, \ldots, v_d^i\}$. Note that these graphs satisfy the hypothesis of Lemma 39 with $I_j^i = \{v_j^i\}, M_j^i = 1, \forall i, j, \mathbf{M}^0 = (1, \ldots, 1)$ and $c_i = 1\ \forall i$. This special case can be used to prove the mean validity of several of the graphs mentioned in Theorem 7, as we explain below.

We introduce some notation for later use. For an integer $m \geq 1$, let m_e (respectively, m_o) denote the greatest even (respectively, odd) integer less than or equal to m.

We now prove part 2 of Theorem 7. Consider a linear graph $\mathcal{G}_m$ with node set $\{v_1, v_2, \ldots, v_m\}$ as shown in part (a) of Fig. 5.2. First, let m be even– say $m = 2N$. For $i = 1, \ldots, N$, let G^i be the clique of size 2 with the node set $V^i = \{v_{2i-1}, v_{2i}\}$ and the edge between the two nodes. In the notation of Lemma 39, let $I_1^i = \{v_{2i-1}\}$ and $I_2^i = \{v_{2i}\}$. By Lemma 39, $G = G^1 \cup G^2 \cup \ldots \cup G^N$ is a mean valid graph with $d = 2$ and the disjoint maximal I.S. $I_1 = \{v_1, v_3, v_5, \ldots v_{m_o}\}$ and $I_2 = \{v_2, v_4, v_6, \ldots, v_{m_e}\}$. We can obtain $\mathcal{G}_m$ by adding the edges (v_2, v_3), (v_4, v_5), $\ldots$, (v_{2N-2}, v_{2N-1}) to G as illustrated in part (a) of Fig. 5.10. Note that no edge is between two nodes in the same I.S. I_j, $j \in \{1,2\}$; so the hypothesis of Lemma 38 is satisfied. Hence, $\mathcal{G}_m$ is mean valid by Lemma 38. The proof of the fact that $\mathcal{G}_m$ is also mean valid for m odd is deferred until Section 5.11.

Now, we prove part 3 of Theorem 7. Consider $\mathcal{G}_{m,m}$, where m may be odd or even. Let v_{ij} be the node in the i'th row and j'th column $i,j \in \{1,\ldots,m\}$(see part (b) of Fig. 5.2). We start with a line graph $\mathcal{G}_{m^2}$, which is mean valid by part 2 of Theorem 7, and add some edges to obtain $\mathcal{G}_{m,m}$ as shown in Fig. 5.11. Specifically, let $\mathcal{G}_{m^2}$ be the line graph with the set of nodes $\{v_{1,1}$, $v_{1,2}$, $\ldots$, $v_{1,m}$, $v_{2,m}$, $v_{2,m-1}$, $\ldots$, $v_{2,1}$, $v_{3,1}$, $v_{3,2}$, $\ldots$, $v_{3,m}$, $v_{4,m}$, $v_{4,m-1}$, $\ldots\}$ and an edge between each pair of consecutive nodes in this order. $\mathcal{G}_{m^2}$ is mean valid with $d = 2$, and the disjoint maximal I.S. $I_1 = \{v_{11}$, v_{13}, $\ldots$, v_{1,m_o}, v_{22}, v_{24}, $\ldots$, v_{2,m_e}, v_{31}, v_{33}, $\ldots$, v_{3,m_o}, $\ldots\}$ and $I_2 = \{v_{12}$, v_{14}, $\ldots$, v_{1,m_e}, v_{21}, v_{23}, $\ldots$, v_{2,m_o}, v_{32}, v_{34}, $\ldots$, v_{3,m_e}, $\ldots\}$. $\mathcal{G}_{m,m}$ can be obtained from $\mathcal{G}_{m^2}$ by adding the remaining edges shown dotted in Fig. 5.11. Note that no edge is between the same I.S. $I_j, j = 1,2$. So $\mathcal{G}_{m,m}$ is mean valid by Lemma 38.

Next, we prove part 4 of Theorem 7. Consider $\mathcal{H}_{m,m}$ (see Fig. 5.3). As in $\mathcal{G}_{m,m}$, let v_{ij} be the node in the i'th row and j'th column. Let $d = 4$, $I_1 = \{v_{11}, v_{13}, v_{15}, \ldots, v_{1,m_o}$, $v_{31}, v_{33}, v_{35}, \ldots, v_{3,m_o}, \ldots\}$, $I_2 = \{v_{12}, v_{14}, v_{16}, \ldots, v_{1,m_e}, v_{32}, v_{34}, v_{36}, \ldots, v_{3,m_e}, \ldots\}$, $I_3 = \{v_{21}$, v_{23}, v_{25}, $\ldots$, v_{2,m_o}, v_{41}, v_{43}, v_{45}, $\ldots$, v_{4,m_o}, $\ldots\}$ and $I_4 = \{v_{22}$, v_{24}, v_{26}, $\ldots$, v_{2,m_e}, v_{42}, v_{44}, v_{46}, $\ldots$, v_{4,m_e}, $\ldots\}$ (see part (b) of Fig. 5.10). Note that I_1, I_2, I_3 and I_4 are disjoint maximal I.S. For $i,j \in \{1,\ldots,m-1\}$, let $C_{i,j}$ be the clique consisting of the nodes $\{v_{i,j}, v_{i,j+1}, v_{i+1,j}, v_{i+1,j+1}\}$ and the edges among them (see Fig. 5.17). First, let m be even. The proof that $\mathcal{H}_{m,m}$ is mean valid is similar to the above proof of mean validity of $\mathcal{G}_m$ with m even: we can obtain $\mathcal{H}_{m,m}$ by considering the union of the cliques $C_{i,j}$, $i, j \in \{1,3,5,\ldots,m-1\}$, which is a mean valid graph by Lemma 39, and then adding the remaining edges as illustrated in part (b) of Fig. 5.10. Note that no edge is between two nodes in the same I.S. I_j, $j \in \{1,2,3,4\}$; so the hypothesis of Lemma 38 is satisfied. Hence, $\mathcal{H}_{m,m}$ is mean valid by Lemma 38. The proof of the fact that $\mathcal{H}_{m,m}$ is also mean valid for m odd is deferred until Section 5.11.

The proof of part 5 of Theorem 7 is similar to that of part 4: we outline the differences. For $i, j, l \in \{1,\ldots,m\}$, let v_{ijl} be the node in the i'th row, j'th column and l'th level (in the direction normal to the plane of the paper). The node set of $\mathcal{T}_{m,m,m}$ can be partitioned into 8 disjoint maximal I.S. $I_1,\ldots,I_8$ similar to $I_1,\ldots,I_4$ for $\mathcal{H}_{m,m}$ (see Fig. 5.5). Also, cliques C_{ijl}, $i, j, l \in \{1,\ldots,m-1\}$ of size 8 each can be defined similar to the cliques C_{ij} for $\mathcal{H}_{m,m}$. For m even, we can obtain $\mathcal{T}_{m,m,m}$ by considering the union of the cliques C_{ijl}, $i, j, l \in \{1,3,5,\ldots,m-1\}$ and then adding the remaining edges. The fact that $\mathcal{T}_{m,m,m}$ is mean valid then follows from Lemmas 39 and 38. The proof of the fact that $\mathcal{T}_{m,m,m}$ is also mean valid for m odd is outlined in Section 5.11.

We now prove part 6 of Theorem 7. Consider a cellular network as shown in Fig. 5.6, whose conflict graph is shown in Fig. 5.7. The nodes in the graph can be partitioned into three disjoint maximal I.S. I_1, I_2 and I_3 as shown in Fig. 5.7. We consider this conflict graph with the following assumption, which eliminates problems arising due to

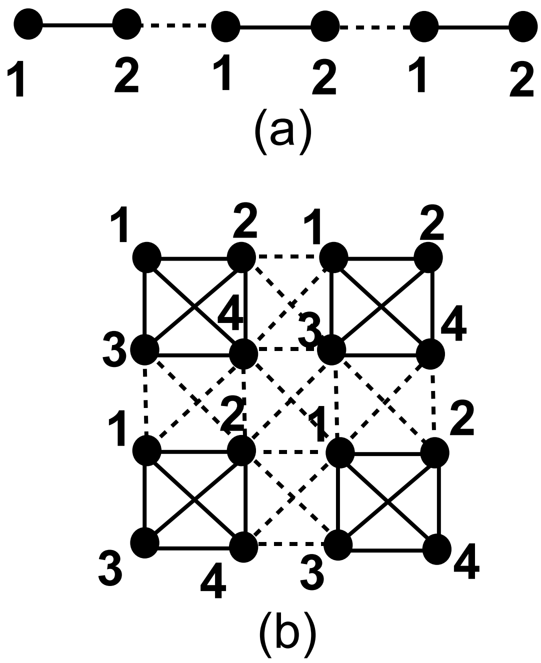

Figure 5.10: Part (a) (respectively, part (b)) shows the construction of $\mathcal{G}_6$ (respectively, $\mathcal{H}_{4,4}$) from 3 (respectively, 4) cliques of size 2 (respectively, 4) each. The solid edges constitute the cliques G^1, G^2, G^3 (respectively, $C_{1,1}$, $C_{1,3}$, $C_{3,1}$ and $C_{3,3}$) and the dotted edges are those that are added later. The numbers next to the nodes shows the I.S. they are in, *i.e.*, a node labeled j is in I.S. I_j, where $j \in \{1,2\}$ (respectively, $j \in \{1,2,3,4\}$). Note that no edge is between two nodes in the same I.S. I_j; so the hypothesis of Lemma 38 is satisfied.

boundary effects.

Assumption 1. *There are an even number, say* δ_1*, of rows of nodes, each containing* $3\delta_2$ *nodes, for some integer* $\delta_2 \geq 1$.

Under this assumption, as illustrated in Fig. 5.7, the graph can be obtained by considering the union of $\delta_1\delta_2$ disjoint cliques of size 3 each, which is a mean valid graph by Lemma 39, and then adding some edges. Note that no edge is between two nodes in

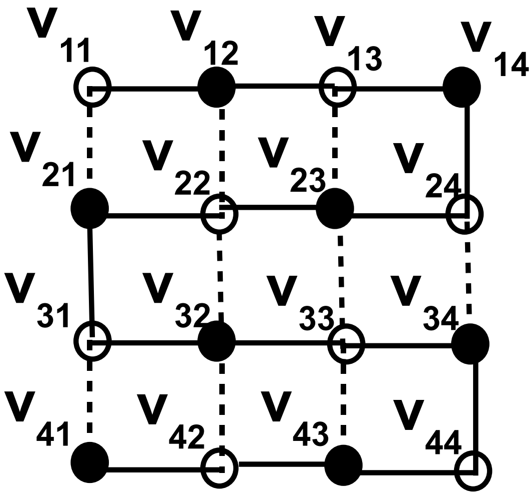

Figure 5.11: The figure shows the construction of the grid graph $\mathcal{G}_{m,m}$ from the line graph $\mathcal{G}_{m^2}$ for $m = 4$. The solid edges constitute $\mathcal{G}_{m^2}$ and the dotted edges are later added to obtain $\mathcal{G}_{m,m}$. The un-darkened and darkened nodes constitute I_1 and I_2 respectively in both $\mathcal{G}_{m^2}$ and $\mathcal{G}_{m,m}$. Note that no edge is between a node in I_1 and a node in I_2, so the hypothesis of Lemma 38 is satisfied.

the same I.S. $I_j, j \in \{1,2,3\}$ (see Fig. 5.7); so the hypothesis of Lemma 38 is satisfied. Hence, the graph is mean valid by Lemma 38.

Note that the above proof goes through if the graph can be partitioned into cliques of size 3 even if Assumption 1 is not satisfied. If the graph cannot be partitioned into cliques of size 3, then the analysis is more complicated because of boundary effects. We omit this analysis for brevity.

5.5.2 Some Other Classes of Mean Valid Graphs

In this subsection, we show that some other common classes of graphs are mean valid. We focus on connected bipartite graphs [71], which are of the form $G = (V, E)$ where

$V = A \cup B$ and every edge is between a node in A and a node in B. Without loss of generality, suppose $|A| \leq |B|$. In the notation of Definition 2, $d = 2$, $I_1 = B$ and $I_2 = A$. Also, a necessary condition for a node distribution $\{\alpha_i, i \in A; \gamma_j, j \in B\}$, under which bandwidth is offered at node $i \in A$ (respectively, $j \in B$) w.p. α_i (respectively, γ_j), to be valid is that

$$\alpha_i + \gamma_j \leq 1 \ \forall (i, j) \in E. \tag{168}$$

This is because, if $\alpha_i + \gamma_j > 1$ for some $(i, j) \in E$, then with a positive probability bandwidth would be offered at both nodes i and j, which are neighbors.

Recall that a κ-regular graph is one in which the degree of every node is κ [71].

Proposition 1. *A κ-regular bipartite graph is mean valid.*

Proof. Let $|A| = N$ and $|B| = M$. First, we show that $N = M$. Since κ edges are incident upon each node in A, $|E| = |A|\kappa = N\kappa$. Similarly, $|E| = M\kappa$. So $N = M$.

Now, let $\{\alpha_i, i \in A; \gamma_j, j \in B\}$ be a valid distribution. Adding (168) over all $(i, j) \in E$, we get:

$$\sum_{(i,j) \in E} (\alpha_i + \gamma_j) \leq |E| = N\kappa \tag{169}$$

But since exactly κ edges are incident on each node:

$$\sum_{(i,j) \in E} (\alpha_i + \gamma_j) = \kappa(\sum_{i \in A} \alpha_i + \sum_{j \in B} \gamma_j) \tag{170}$$

By (169) and (170),

$$\frac{\sum_{i \in A} \alpha_i}{N} + \frac{\sum_{j \in B} \gamma_j}{N} \leq 1$$

So Condition 2 in Definition 2 is satisfied and the graph is mean valid. $\square$

Recall that a star is a graph with a node a_1 called the center, nodes $b_1,\ldots,b_M$ called the leaves, and edges $(a_1,b_j), j=1,\ldots,M$ [71]. Note that this is a bipartite graph with edges only between the sets $A=\{a_1\}$ and $B=\{b_1,\ldots,b_M\}$.

Proposition 2. *A star is mean valid.*

Proof. Let $\{\alpha_1,\gamma_1,\ldots,\gamma_M\}$ be a valid distribution. By (168),

$$\alpha_1+\gamma_j \leq 1, j=1,\ldots,M$$

Adding these M inequalities and dividing by M gives $\alpha_1+\frac{\gamma_1+\ldots+\gamma_M}{M}\leq 1$. □

Now, note that every tree is a bipartite graph [71]. Given a tree, suppose the root constitutes layer 1, the children of the root constitute layer 2 and the children of all the nodes in layer i constitute layer $i+1$, $i=2,3,\ldots$. Not every tree is mean valid; a counterexample is presented in Section 5.6 (see Fig. 5.13). The following result provides a sufficient condition for a tree to be mean valid.

Proposition 3. *A tree in which every node in an odd layer has exactly* κ *children is mean valid.*

Proof. Let a tree T in which every node in an odd layer has κ children be given. Let N_j be the total number of nodes in the j'th layer of T and $N=\sum_{j\text{ odd}}N_j$.

Let $A_i=\{a_i\}$, $B_i=\{b_{i,1},\ldots,b_{i,\kappa}\}$ and G^i be a star with center a_i and κ leaves– the nodes in B_i. Note that each G^i is mean valid by Proposition 2. Also, $G^i, i=1,\ldots,N$, satisfy the hypothesis of Lemma 39 with $d=2$, $I_1^i=B_i$, $I_2^i=A_i$, $M_1^i=\kappa$, $M_2^i=1$ $\forall i$, $\mathbf{M}^0=(\kappa,1)$ and $c_i=1$ $\forall i$. So by Lemma 39, $G=G^1\cup G^2\cup\ldots\cup G^N$ is mean valid.

Now, we will obtain T by adding some edges to G as illustrated in Fig. 5.12. Let the center, a_1, of G^1 be the root of T. Note that its children are $b_{1,1}, \ldots, b_{1,\kappa}$, the leaves of G^1. For $j \in \{1, \ldots, \kappa\}$, suppose $b_{1,j}$ has l_j children. Join $b_{1,j}$ by an edge to each of the centers of l_j stars out of $G^2, \ldots, G^N$, using a different set of stars for each j. Thus, we have obtained the nodes in the first 4 layers of the tree and the edges connecting them. Suppose a node b in layer 4 has l' children. Join it to the centers of l' stars out of $G^1, \ldots, G^N$, which have not been used so far. Proceed in this manner to get the tree T. Note that there is no edge between two nodes in the same partition of the tree, which is a bipartite graph (see Fig. 5.12); so the hypothesis of Lemma 38 is satisfied. Hence, T is mean valid by Lemma 38. $\square$

5.6 Non Mean Valid Bipartite Graphs

We have shown in Theorem 8, that a mean valid graph has a unique symmetric NE that has a simple form– under this NE, for every $q \in (0,1)$, each primary offers bandwidth with the same probability t_j at all the nodes in each I.S. I_j, $j \in \{1, \ldots, d\}$. Thus, mean validity is a *sufficient* condition for an arbitrary graph to have a symmetric NE of the form in Theorem 8 for all values of q. The following result shows that for connected bipartite graphs, mean validity is also a *necessary* condition.

Theorem 9. *Let G be a connected bipartite graph that is not mean valid. If $w(q,n) > 1 - \frac{M_2}{M_1}$, then $\beta(I_1) = t_1$, $\beta(I_2) = t_2$, $\beta(I) = 0$ $\forall I \in \mathscr{I}, I \neq I_1, I_2$, is not a symmetric NE*

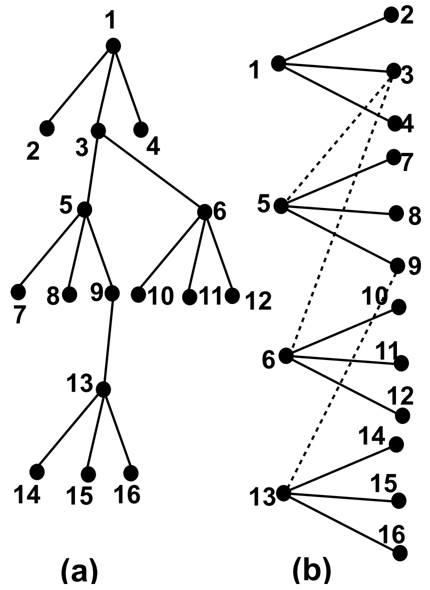

Figure 5.12: Part (a) shows a tree in which each node in an odd layer has exactly 3 children. Part (b) shows the construction of the tree. We start with stars with 3 leaves each, whose edges are shown in bold and then add some edges, shown dotted. Note that none of the dotted edges is between two nodes in the same partition of the bipartite graph, so the condition in Lemma 38 is satisfied.

for any value of t_1 *and* t_2*.*

Now, we provide an example of a non mean valid bipartite graph and find the symmetric NE and prove its uniqueness. The symmetric NE is not of the form in Theorem 8 for *any* value of $q \in (0,1)$.

Let the set of nodes be $A \cup B$, where $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, and let there be an edge between a_1 (respectively, b_1) and every edge in B (respectively, A) (see Fig. 5.13). The only maximal I.S. are $I_{ab} = \{a_2, a_3, b_2, b_3\}$, $I_a = A$ and $I_b = B$. The I.S. I_{ab} contains 2 nodes from each of A and B, *i.e.*, $m_1(I_{ab}) = m_2(I_{ab}) = 2$ in the notation

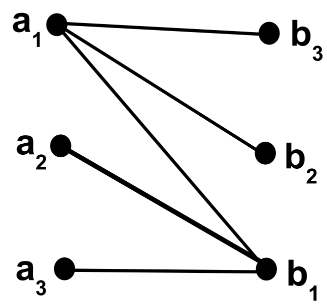

Figure 5.13: A non mean valid graph.

of Lemma 31. Also, $\frac{m_1(I_{ab})}{3} + \frac{m_2(I_{ab})}{3} > 1$; so (158) is not satisfied. Hence, this is not a mean valid graph by Lemma 31.

In every symmetric NE, $\beta(I_a) = t_a$, $\beta(I_b) = t_b$ and $\beta(I_{ab}) = 1 - t_a - t_b$ for some $0 \leq t_a, t_b \leq 1$.

Lemma 40. *If $w(q,n) > \frac{1}{2}$, then $f_1(x) = 2W(1-x) - W(x)$ has a unique root $t_1 \in [0,1]$. Also, $0 < t_1 < \frac{1}{2}$.*

The following theorem provides the symmetric NE in the above graph for each value of $q \in (0,1)$.

Theorem 10.

1. *If $w(q,n) \leq \frac{1}{2}$, then the symmetric strategy profile corresponding to $t_a = t_b = 0$ is the unique symmetric NE.*
2. *If $w(q,n) > \frac{1}{2}$, then the symmetric strategy profile in which $t_a = t_b = t_1$, the root of $f_1(.)$, is the unique symmetric NE.*

Note that I_{ab}, which contains 4 nodes, is the largest I.S. So for all values of q, primaries offer bandwidth with positive probability at the I.S. I_{ab} in the symmetric NE.

Since I_{ab} contains nodes from both A and B, the node probabilities at different nodes in A (and B) are different. Thus, the symmetric NE is not of the form in Theorem 8 for *any* value of q.

Again, since I_{ab} is the largest I.S., Theorem 10 shows, consistent with intuition, that for small values of q, primaries offer bandwith only at I_{ab} with positive probability; when q, and thereby the competition at I_{ab}, increases beyond a threshold, they also offer bandwidth at A and B with positive probability.

5.7 Asymmetric q

So far, we have analyzed the symmetric case $q_1 = \ldots = q_n = q$. Now we briefly outline how our results generalize to the general case where the q's may not be equal.

As noted in Remark 3, there are multiple NE in general in the spatial reuse setting. Hence, although the bandwidth availability probabilities $q_1, \ldots, q_n$ may be unequal, we focus on the special class of NE in which the I.S. selection probabilities $\{\beta(I) : I \in \mathscr{I}\}$ of each primary is the same. As before, let $\alpha_v = \sum_{v \in I} \beta(I)$ be the total probability with which a primary who has unused bandwidth offers it at node $v \in V$.

Since primary i has unused bandwidth w.p. q_i and offers it at node $v \in V$ w.p. α_v, he offers bandwidth at node $v \in V$ w.p. $q_i \alpha_v$. Let $w_i(\alpha_v)$ be the probability that K_v or more out of primaries $\{1, \ldots, n\} \backslash i$ offer it at node v.

Lemma 41. *$w_1(\alpha)$ is a strictly increasing function of α on $[0, 1]$.*

Next, we explain how the results for symmetric q generalize to the asymmetric q case. First, Lemma 30 readily generalizes to give:

Lemma 42. *Suppose under a NE in which the I.S. selection distribution* $\{\beta(I) : I \in \mathscr{I}\}$ *of each primary is the same (symmetric), each primary selects node* v *w.p.* α_v *if he has unused bandwidth. Then under that NE, the price distribution of primary* i *at node* v *is the price distribution* $\psi_i(.)$ *in the single location case with* $\alpha_v q_1, \ldots, \alpha_v q_n$ *in place of* $q_1, \ldots, q_n$.

Now, recall from Property 2 and Theorem 4 in Chapter 3 that when there are n primaries with bandwidth availability probabilities $q_1, \ldots, q_n$ at a single location, there is a unique NE; in this NE, each primary gets the same payoff, equal to $(\nu - c)(1 - w_1(1)) = 1 - w_1(1)$, where the equality follows from the fact that we have normalized $\nu - c$ to 1. Let

$$\overline{W}(\alpha) = 1 - w_1(\alpha). \tag{171}$$

By Lemma 42, and similar to Property 2 in Chapter 3 in the single location case, in a NE with symmetric I.S. distributions $\{\beta(I) : I \in \mathscr{I}\}$ of the primaries, if all primaries offer bandwidth at a node with probability α (and play the single-node NE strategy with $q_1\alpha, \ldots, q_n\alpha$ in place of $q_1, \ldots, q_n$ at that node), then $\overline{W}(\alpha)$ is the maximum expected payoff that each primary i can get at that node.

Now, we define the notions of valid distribution and mean valid graph as in the symmetric q case and the proofs that the specific topologies such as a line, grid graph, cellular network, star etc. are mean valid remain unchanged, since they are properties of the graphs and are independent of $q_1, \ldots, q_n$.

Consider a mean valid graph $G = (V, E)$, where $V = I_1 \cup \ldots \cup I_d$ and $I_1, \ldots, I_d$ are disjoint maximal I.S. Let $|I_j| = M_j$, $j = 1, \ldots, d$.

Theorem 8 generalizes to:

Theorem 11. *In a mean valid graph, for every $1 > q_1 \geq \ldots \geq q_n > 0$, there is a unique NE with symmetric I.S. distributions $\{\beta(I) : I \in \mathscr{I}\}$; in this NE, each primary offers bandwidth at every node in I_j, $j \in \{1, \ldots, d\}$, w.p. t_j,* i.e. $\alpha_{j,l} = t_j$, *$l = 1, \ldots, M_j$, where $(t_1, \ldots, t_d)$ is the unique distribution satisfying the equations:*

$$t_j = 0, j > d'. \tag{172}$$

and

$$M_1 \overline{W}(t_1) = \ldots = M_{d'} \overline{W}(t_{d'}) > M_{d'+1} r. \tag{173}$$

The proof of the above theorem changes from the symmetric q case only in that we now use the following lemma, which generalizes the corresponding lemma (Lemma 45) for the symmetric q case.

Lemma 43. *(i) For $0 < \alpha \leq 1$, $0 \leq \overline{W}(\alpha) \leq r$, (ii) $\overline{W}(0) = r$, and (iii) $\overline{W}(\alpha)$ is strictly decreasing in α.*

The proof of the above lemma follows from (171) and Lemma 41.

5.8 Threshold behavior

In this section, as in Section 5.7, we allow the bandwidth availability probabilities $q_1, \ldots, q_n$ of the primaries to be unequal. As in Section 3.3.3, we define the *efficiency*, η, of a NE as $\eta = \frac{R_{\text{NE}}}{R_{\text{OPT}}}$, where R_{NE} is the expected sum of payoffs of the n primaries at the NE and R_{OPT} is the maximum possible (optimal) expected sum of payoffs, attained when all primaries jointly select the independent sets and prices to maximize their aggregate revenue. Clearly, $\eta \leq 1$ quantifies the loss in aggregate revenue incurred owing to lack of cooperation among primaries. Also, since the NE in Section 5.7 is unique in the class of NE with symmetric I.S. distributions, η quantifies fundamental limits on the performance of NE in this category.

Let $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{q_n}{n} = q$ for some $q \in (0,1)$. Here, q represents the "average" bandwidth availability probability of the primaries. For simplicity, we assume that each secondary from a given pool independently seeks bandwidth, and let k_n be the expected number of secondaries at any given location[26]. Then, the NE structure exhibits interesting threshold behavior as $n \to \infty$; in particular, η switches from 1 to 0 depending on the relations between nq (availability) and k_n (demand).

Lemma 44. *Let*[27] $\overline{q}_n = \frac{q_1+\ldots+q_n}{n}$ *and let* $\tilde{p}_j$ *denote the common lower endpoint of the price distributions of the primaries who have unused bandwidth in the NE at nodes in*

[26]We allow (but do not require) the number (rather statistics) of the secondaries to scale with increase in n.

[27]For simplicity, we state this lemma under the assumption that $M_1, \ldots, M_d$ are distinct. In the Appendix, we provide the lemma with this assumption relaxed.

I.S. I_j (if they select I.S. I_j).

1. *If there exists an $\varepsilon > 0$ such that for all large n, $q < k_n/(n-1) - \varepsilon$, then $\eta \to 1$ as $n \to \infty$. Also, for all large n, $d' = 1$, $t_1 = 1, t_2 = t_3 = \ldots t_d = 0$, $\tilde{p}_1 \to \nu$.*

2. *Let $l < d$. If there exists an $\varepsilon > 0$ such that for all large n, $lk_n/(n-1) + \varepsilon < q < (l+1)k_n/(n-1) - \varepsilon$, then for all large n, $d' \geq l+1$, and $t_j \overline{q}_n \to k_n/(n-1)$ for all $j \leq l$.*

3. *If there exists an $\varepsilon > 0$ such that for all large n, $q > k_n d/(n-1) + \varepsilon$, then $\eta \to 0$ as $n \to \infty$. Also, for all large n, $d' = d$ and $\tilde{p}_j \to c$, $j = 1, \ldots, d$.*

Intuitively, if availability is less than demand, then owing to limited competition, primaries with available bandwidth select only the maximum-sized I.S. among $I_1, \ldots, I_d$, and choose prices in a neighborhood of ν. Thus, $\eta \to 1$, since no other strategy can enhance any primary's payoff. As availability increases, under NE, primaries diversify their choices among the I.S. $I_1, \ldots, I_d$ and are more likely to select low prices as well (the lower limits of the price distributions hover around c once availability exceeds demand), thereby drastically reducing the efficiency of the NE.

5.9 Numerical Studies

In this section, we describe numerical computations that are directed towards assessing the impact of price competition among the primaries on the aggregate revenue of the primaries and the affordability of spectrum for the secondaries. We consider the specific case of a grid graph $\mathcal{H}_{m,m}$, which was introduced in Section 5.3.2 (see Fig. 5.3). By part 4 of Theorem 7, this is a mean valid graph and, in the notation of Definition 2, $d = 4$ and the I.S. I_1, I_2, I_3 and I_4 are as described in Section 5.5.1. Throughout, we use the parameter values $\nu = 1$ and $c = 0$, and a constant number of secondaries k at each node. Also, $q_1, \ldots, q_n$ are uniformly spaced in $[q_L, q_H]$ for some parameters q_L and q_H. Let $q = \frac{q_L + q_H}{2}$ be the mean bandwidth availability probability of the primaries.

In $\mathcal{H}_{m,m}$, the NE is of the form in Theorem 8 and the plot on the left in Fig. 5.14 reveals, as expected, that price competition significantly reduces the aggregate revenue of the primaries under this NE relative to OPT, the optimal scheme in which the primaries collaborate to attain R_{OPT}, the maximum aggregate revenue of the primaries (Note that under OPT, the I.S. $I_1, \ldots, I_4$ are selected in order of size and all the primaries always select the highest price ν). Also, overall, the efficiency (η) decreases as q increases since the competition increases. The plot on the right in Fig. 5.14 shows that the trends are similar for a larger topology (larger m). The plot on the left in Fig. 5.15 shows that η improves as k increases. This is because, for small values of k, demand for bandwidth is scarce at each node. Under the NE, bandwidth is wasted at several nodes since $k + 1$ or more primaries offer bandwidth at those nodes, resulting in a shortage of bandwidth at other nodes. On the other hand, since all primaries cooperate in OPT, it judiciously supplies bandwidth precisely where it is needed. So OPT outperforms the NE by a large margin for small values of k. For large values of k, the demand is high and so is the tolerable margin of error in assigning the primaries to I.S.; and hence the performance of the NE improves relative to OPT. The plot on the right in Fig. 5.15 shows that η increases as m increases, which is because the four I.S. $I_1, \ldots, I_4$ become closer to each other in size as m increases and hence the loss in revenue resulting from choosing a smaller I.S. is lower.

Fig. 5.16 shows that under price competition, the expected price per unit of bandwidth is lower at the nodes in the larger I.S. This is because primaries prefer larger I.S. and hence the competition is more intense there, driving down the prices.

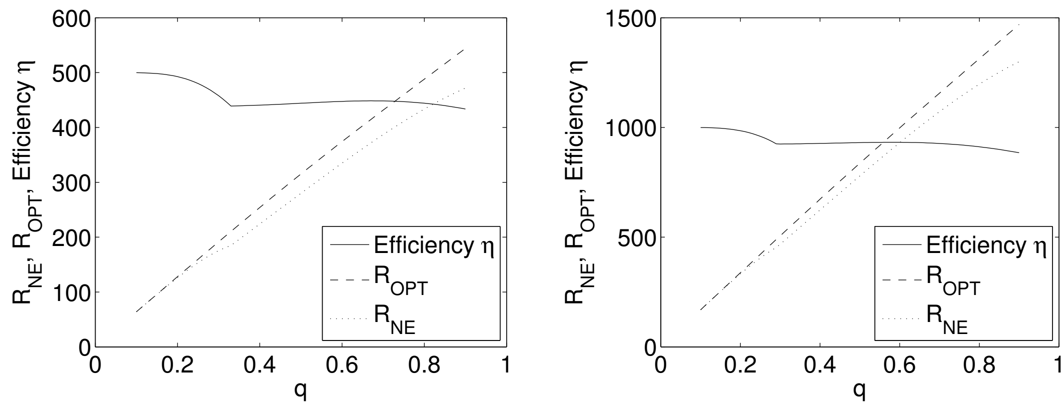

Figure 5.14: Both figures plot the aggregate revenues of the primaries, R_{NE} and R_{OPT}, under the NE and OPT respectively, and the efficiency of the NE, $\eta = \frac{R_{NE}}{R_{OPT}}$, versus q. In both figures, $n = 10$, $k = 5$ and $q_H - q_L = 0.2$ are used. Also, $m = 15$ (respectively, $m = 25$) for the figure on the left (respectively, right). η is scaled by a factor of 500 (respectively, 1000) in the figure on the left (respectively, right) in order to show it on the same figure as the other plots.

5.10 Conclusions, Discussion and Future Work

We analyzed price competition among multiple primaries in a CRN in the presence of

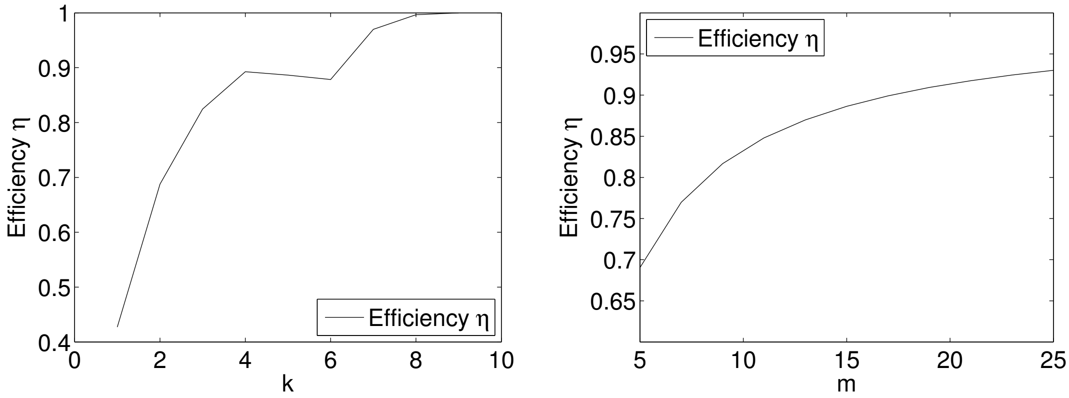

Figure 5.15: The figure on the left (respectively, right) plots the efficiency η of the NE versus k (respectively, m). For both figures, $n = 10$, $q_L = 0$ and $q_H = 1$ are used. Also, $m = 15$ for the figure on the left and $k = 5$ for the figure on the right.

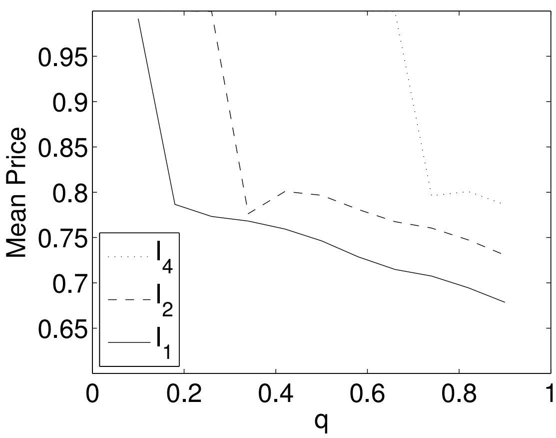

Figure 5.16: The figure shows the mean price of bandwidth, given that it is offered, at a (fixed) node in each of I_1, I_2 and I_4 under the NE vs q. Note that since $|I_3| = |I_2|$, the mean price of bandwidth at nodes in I_3 is the same as that at nodes in I_2. The parameter values used are $m = 15$, $n = 8$ and $k = 3$. Also, $q_H - q_L = 0.2$.

spatial reuse in the symmetric setting in which each primary has unused bandwidth with the same probability and in a class of graphs which we denote as mean-valid. We have proved that there exists a unique symmetric NE in this case, and have characterized this symmetric NE as a solution of a set of non-linear equations. Such equations can be easily solved even for large networks such as those consisting of 600 or more nodes and multiple (e.g., 10) primaries and secondaries. Our numerical computations reveal interesting insights regarding the efficiency of the NE and also the price and independent set selections of the primaries. We have also considered the asymmetric setting and investigated a special class of NE in which the independent set selection strategies of the primaries are symmetric.

It would be interesting to investigate whether the NE is stable to minor perturbations in the selections of the primaries. In this chapter, we characterized the NE in a special non mean valid graph. The characterization of the NE in other general (non mean valid) graphs both in the symmetric and the asymmetric settings remain open. We have also assumed that each primary knows the statistical distribution governing the bandwidth availabilities of other primaries and the number of secondaries at each node. Characterization of the NE when primaries have imperfect knowledge of the above, and seek to enhance their knowledge using learning strategies, remains open. Finally, we have only characterized the NE strategies in a one-shot game. Primaries may play this game repeatedly and may use their experience from previous slots and a learning algorithm to choose their strategy in the current slot. An investigation into whether the symmetric NE for the one-shot game constitutes a steady-state outcome of some natural learning algorithms in such a setting is an interesting direction for future research.

5.11 Appendix

Let $W(\alpha)$ be as in (153). We will use the following result throughout.

Lemma 45. *(i) For $0 < \alpha \leq 1$, $0 \leq W(\alpha) \leq r$, (ii) $W(0) = r$, and (iii) $W(\alpha)$ is strictly decreasing in α.*

Lemma 45 follows from (153), the fact that $w(0,n) = 1-r$ and Lemma 29.

5.11.1 Proofs of results in Section 5.3

Proof of Lemma 31. Suppose G is mean valid. Fix an $I \in \mathscr{I}$. Let

$$\mathbf{1}_I(a_{j,l}) = \begin{cases} 1, & \text{if } a_{j,l} \in I \\ 0, & \text{else} \end{cases}$$

Consider a distribution $\{\alpha_{j,l} : j = 1,\ldots,d; l = 1,\ldots,M_j\}$ in which bandwidth is offered at node $a_{j,l} \in I_j$ w.p. $\alpha_{j,l} = \mathbf{1}_I(a_{j,l})$. This is a valid distribution because it corresponds to the I.S. distribution $\{\beta(I) = 1, \beta(I') = 0 \forall I' \in \mathscr{I}, I' \neq I\}$. Also,

$$\sum_{l=1}^{M_j} \alpha_{j,l} = \sum_{l=1}^{M_j} \mathbf{1}_I(a_{j,l}) = m_j(I),\ \ j = 1,\ldots,d \tag{174}$$

Let $\overline{\alpha}_j$ be given by (157). Since the graph is mean valid, (156) holds. Substituting $\sum_{l=1}^{M_j} \alpha_{j,l} = m_j(I)$ from (174) into (156), we get (158).

To prove the converse, suppose (158) holds. Let $\{\alpha_{j,l} : j = 1,\ldots,d; l = 1,\ldots,M_j\}$ be a valid distribution. By definition, there exists a distribution $\{\beta(I) : I \in \mathscr{I}\}$ such that:

$$\alpha_{j,l} = \sum_{I \in \mathscr{I} : a_{j,l} \in I} \beta(I) \tag{175}$$

which can be written as:

$$\alpha_{j,l} = \sum_{I \in \mathscr{I}} \beta(I) \mathbf{1}_I(a_{j,l}) \tag{176}$$

Now,

$$\begin{aligned}
& \sum_{j=1}^{d} \left(\frac{\sum_{l=1}^{M_j} \alpha_{j,l}}{M_j} \right) \\
= \; & \sum_{j=1}^{d} \frac{1}{M_j} \left\{ \sum_{l=1}^{M_j} \sum_{I \in \mathscr{I}} \beta(I) \mathbf{1}_I(a_{j,l}) \right\} \text{ (by (176))} \\
= \; & \sum_{I \in \mathscr{I}} \beta(I) \left\{ \sum_{j=1}^{d} \frac{\sum_{l=1}^{M_j} \mathbf{1}_I(a_{j,l})}{M_j} \right\} \\
= \; & \sum_{I \in \mathscr{I}} \beta(I) \left\{ \sum_{j=1}^{d} \frac{m_j(I)}{M_j} \right\} \text{ (since } \sum_{l=1}^{M_j} \mathbf{1}_I(a_{j,l}) = m_j(I)) \\
\leq \; & 1 \text{ (by (158))}
\end{aligned}$$

So (156) holds and hence G is mean valid. $\square$

5.11.2 Proofs of results in Section 5.4

The following lemma is used in the proof of Lemma 32.

Lemma 46. *Let $N \geq 2$ be an integer and $\alpha_1, \ldots, \alpha_N, f_1, \ldots, f_N$ be real numbers. Then:*

$$N(\sum_{i=1}^{N} \alpha_i f_i) - (\sum_{i=1}^{N} \alpha_i)(\sum_{i=1}^{N} f_i) = \sum_{1 \leq i < j \leq N} (\alpha_j - \alpha_i)(f_j - f_i) \tag{177}$$

Proof. We prove the result by induction. For $N = 2$:

$$\begin{aligned}
LHS &= 2(\alpha_1 f_1 + \alpha_2 f_2) - (\alpha_1 + \alpha_2)(f_1 + f_2) \\
&= (\alpha_2 - \alpha_1)(f_2 - f_1) \\
&= RHS
\end{aligned}$$

Suppose the result is true for N. For $N+1$:

$$\begin{aligned}
LHS &= (N+1)(\sum_{i=1}^{N}\alpha_i f_i + \alpha_{N+1} f_{N+1}) - \\
&\quad (\sum_{i=1}^{N}\alpha_i + \alpha_{N+1})(\sum_{i=1}^{N} f_i + f_{N+1}) \\
&= \left\{ N(\sum_{i=1}^{N}\alpha_i f_i) - (\sum_{i=1}^{N}\alpha_i)(\sum_{i=1}^{N} f_i) \right\} \\
&\quad + N\alpha_{N+1} f_{N+1} + \sum_{i=1}^{N}\alpha_i f_i + \alpha_{N+1} f_{N+1} \\
&\quad - \alpha_{N+1}(\sum_{i=1}^{N} f_i) - (\sum_{i=1}^{N}\alpha_i) f_{N+1} - \alpha_{N+1} f_{N+1} \\
&= \sum_{1\le i<j\le N} (\alpha_j - \alpha_i)(f_j - f_i) \\
&+ \sum_{i=1}^{N} (\alpha_{N+1} f_{N+1} + \alpha_i f_i - \alpha_{N+1} f_i - \alpha_i f_{N+1}) \\
&\quad \text{(by induction hypothesis and collecting terms)} \\
&= RHS
\end{aligned}$$

The result follows by induction. ☐

Proof of Lemma 32. By symmetry, we can assume WLOG that $\alpha_1 \le \alpha_2 \ldots \le \alpha_N$. Since $f(.)$ is strictly decreasing, $f(\alpha_1) \ge f(\alpha_2) \ge \ldots \ge f(\alpha_N)$. Now:

$$\begin{aligned}
&(\sum_{i=1}^{N}\alpha_i f(\alpha_i)) - \overline{\alpha}(\sum_{i=1}^{N} f(\alpha_i)) \\
&\quad = \frac{1}{N}\left(N(\sum_{i=1}^{N}\alpha_i f(\alpha_i)) - (\sum_{i=1}^{N}\alpha_i)(\sum_{i=1}^{N} f(\alpha_i)) \right) \\
&\quad = \frac{1}{N}\sum_{1\le i<j\le N} (\alpha_j - \alpha_i)(f(\alpha_j) - f(\alpha_i)) \text{ (by (177))} \qquad (178)
\end{aligned}$$

For $i<j$, $\alpha_i \le \alpha_j$ and $f(\alpha_i) \ge f(\alpha_j)$. So each term in (178) is ≤ 0. Hence, the expression in (178) is 0 iff each term is 0, which happens iff $\alpha_1 = \ldots = \alpha_N = \overline{\alpha}$. ☐

Proof of Lemma 34. Let

$$\begin{aligned} U^* &= \max\{U_1(I_j) : j \in \{1,\dots,d\}\} \\ &= \max\{M_j W(t_j) : j \in \{1,\dots,d\}\} \text{ (by (165))} \end{aligned}$$

and $B = \{j \in \{1,\dots,d\} : M_j W(t_j) = U^*\}$. Note that B is the set of indices of the I.S. out of $I_1,\dots,I_d$ that yield the highest payoff and U^* is the value of that payoff.

By definition of B:

$$W(t_j) = \frac{U^*}{M_j}, \ \forall j \in B \tag{179}$$

$$W(t_j) < \frac{U^*}{M_j}, \ \forall j \notin B. \tag{180}$$

Let I be any I.S. containing $m_j(I)$ nodes from I_j, $j = 1,\dots,d$. By (164):

$$\begin{aligned} U_1(I) &= \sum_{j=1}^{d} m_j(I) W(t_j) \\ &\leq \sum_{j=1}^{d} m_j(I) \left(\frac{U^*}{M_j}\right) \text{ (by (179) and (180))} \qquad (181) \\ &\leq U^* \text{ (by (158))} \end{aligned}$$

So $\max_{I \in \mathscr{I}} U_1(I) \leq U^*$, and since $U_1(I_j) = U^*$, $j \in B$, each I_j, $j \in B$, is a best response. Now, for I as defined above, suppose $m_j(I) \geq 1$ for some $j \notin B$. Then the inequality in (181) is strict. So $U_1(I) < U^*$ and I is not a best response. Thus, each $I \in \mathscr{I}$ containing a node from I_j for some $j \notin B$ is not a best response. In particular, $\forall j \notin B$, I_j is not a best response and, since primaries offer bandwidth at I_j w.p. t_j in the above NE, $t_j = 0$ for all $j \notin B$.

It now suffices to show that $B = \{1,\dots,d'\}$ for some $1 \leq d' \leq d$. Suppose not. Then there exist $j, l \in \{1,\dots,d\}$ such that $j < l$, $j \notin B$ and $l \in B$. Since $j \notin B$, $t_j = 0$ by the previous paragraph. Now, by (164):

$$\begin{aligned} U_1(I_j) &= M_j W(t_j) \\ &= M_j r \text{ (by part (ii) of Lemma 45)} \\ &\geq M_l r \text{ (by (155), since } j < l) \\ &\geq M_l W(t_l) \text{ (by part (i) of Lemma 45)} \\ &= U^* \end{aligned}$$

So I_j is a best response, which is a contradiction since $j \notin B$. □

Proof of Lemma 36. Suppose primaries $2, \ldots, n$ use the strategy ψ, under which bandwidth is offered at the nodes in I_j w.p. $t_j, j = 1, \ldots, d$. By (163) and part (ii) of Lemma 45, $W(t_j) = r, j > d'$. So by (164), the payoff of primary 1 if it plays I.S. I_j, $j \in \{1, \ldots, d'\}$ (resp., $j \in \{d'+1, \ldots, d\}$) is $U_1(I_j) = M_j W(t_j)$ (resp., $U_1(I_j) = M_j r$). Hence, by (166) and (155), for some U^*,

$$U^* = U_1(I_1) = \ldots = U_1(I_{d'}) > U_1(I_{d'+1}) \geq \ldots \geq U_1(I_d).$$

The maximum payoff that primary 1 can get at a node $v \in I_j$, $j \in \{1, \ldots, d'\}$ equals

$$W(t_j) = \frac{U_1(I_j)}{M_j} = \frac{U^*}{M_j}. \tag{182}$$

Now, for $j > d'$, $M_j r = U_1(I_j) < U^*$. So the maximum payoff that primary 1 can get at a node $v \in I_j$, $j > d'$ is

$$r < \frac{U^*}{M_j}. \tag{183}$$

Now, let I be an I.S. containing $m_j(I)$ nodes from $I_j,\ j=1,\ldots,d$. By (182) and (183):

$$\begin{aligned} U_1(I) &\leq U^*\left(\sum_{j=1}^{d}\frac{m_j(I)}{M_j}\right) \qquad (184)\\ &\leq U^* \text{ (by (158))} \end{aligned}$$

Since $U_1(I_1)=\ldots=U_1(I_{d'})=U^*$, $I_1,\ldots,I_{d'}$ are best responses. Under the strategy ψ, primary 1 can only play $I_1,\ldots,I_{d'}$ with positive probability; hence, ψ is a best response.

□

Proof of Lemma 37. **Existence:** For convenience, let $M_{d+1}=0$. Fix $q\in(0,1)$. For $x\in[M_1W(1),M_1r]$ and $j\in\{1,\ldots,d\}$, if $M_jr\geq x$, then we show that the equation:

$$M_jW(t_j)=x \qquad (185)$$

has a unique solution $t_j(x)\in[0,1]$. Let $h(t_j)=M_jW(t_j)$. By part (ii) of Lemma 45, $h(0)=M_jr\geq x$. Also,

$$\begin{aligned} h(1) &= M_jW(1)\\ &\leq M_1W(1) \text{ (by (155))}\\ &\leq x \end{aligned}$$

Also, by (2) and (153), $h(t_j)$ is a continuous function of t_j. So by the intermediate value theorem [58], $h(t_j)=x$ has a solution in $[0,1]$. By part (iii) of Lemma 45, $h(t_j)$ is a strictly decreasing function of t_j; so this solution, say $t_j(x)$, is unique. For $x=M_jr$, $t_j=0$ satisfies (185) by part (ii) of Lemma 45. So $t_j(M_jr)=0$.

Since $h(t_j)$ is strictly decreasing on $0 \le t_j \le 1$, it is invertible. Also, since the inverse of a continuous function is continuous (see Theorem 4.17 in [58]), $h^{-1}(x)$ is continuous. But $x = h(t_j(x))$. So $t_j(x) = h^{-1}(x)$. Thus, $t_j(x)$ is continuous in x for $x \le M_j r$. For $x > M_j r$, define $t_j(x) = 0$. As shown above, $t_j(M_j r) = 0$. So $t_j(x)$ is continuous on $[M_1 W(1), M_1 r]$. Let,

$$T(x) = \sum_{j=1}^{d} t_j(x) \tag{186}$$

As shown above, $h(t_j)$ is strictly decreasing on $0 \le t_j \le 1$ for $j = 1, \ldots, d$. So $t_j(x) = h^{-1}(x)$ is strictly decreasing for $x \le M_j r$. Also, by definition, $t_j(x) = 0$ on $M_j r < x \le M_1 r$. So by (186), $T(x)$ is strictly decreasing on $[M_1 W(1), M_1 r]$ (note that $t_1(x)$ is strictly decreasing on $x \le M_1 r$). Also, $t_j(M_1 r) = 0$, $j = 1, \ldots, d$. So

$$T(M_1 r) = 0. \tag{187}$$

Now, for $j = 1$ and $x = M_1 W(1)$, $t_1 = 1$ satisfies (185). So $t_1(M_1 W(1)) = 1$ and hence, by (186):

$$T(M_1 W(1)) \ge 1. \tag{188}$$

Now, since each $t_j(x)$, $j = 1, \ldots, d$, is continuous on $[M_1 W(1), M_1 r]$, so is $T(x)$ by (186). Hence, by (187), (188) and the intermediate value theorem, the equation $T(x) = 1$ has a solution $x^* \in [M_1 W(1), M_1 r]$, which is unique because $T(x)$ is strictly decreasing. Let $d'(q) = \max\{j : M_j r \ge x^*\}$. By definition of $t_j(x)$, for $j = 1, \ldots, d'(q)$, $M_j W(t_j(x^*)) = x^*$ and for $j > d'(q)$, $M_j r < x^*$ and hence $t_j(x^*) = 0$. Thus, $(t_1(x^*), \ldots, t_d(x^*))$ satisfy (163) and (166). Also, by (186), $\sum_{j=1}^{d} t_j(x^*) = T(x^*) = 1$; so $(t_1(x^*), \ldots, t_d(x^*))$ is a probability distribution. The result follows.

Uniqueness: Fix q. We now show the uniqueness of $d'(q)$ and the distribution $(t_1,\ldots,t_d)$ satisfying (163) and (166). Assume, to reach a contradiction, that there exist $e,f \in \{1,\ldots,d\}$ and probability distributions $t=(t_1,\ldots,t_d)$ and $s=(s_1,\ldots,s_d)$ such that $t_j=0$ (respectively, $s_j=0$) for $j>e$ (respectively, $j>f$) and for some y and z:

$$y = M_1W(t_1) = \ldots = M_eW(t_e) > M_{e+1}r \tag{189}$$

$$z = M_1W(s_1) = \ldots = M_fW(s_f) > M_{f+1}r \tag{190}$$

First, suppose $e=f$. If $y=z$, then by (189) and (190), $M_jW(t_j)=M_jW(s_j)$, $j=1,\ldots,e$. By part (iii) of Lemma 45, $W(.)$ is a one-to-one function; so $t_j=s_j$, $j=1,\ldots,e$. Also, $t_j=s_j=0$, $j>e$. So $t=s$.

Now, suppose $z>y$. Then $M_jW(s_j)>M_jW(t_j)$, $j=1,\ldots,e$. So $W(s_j)>W(t_j)$, and by part (iii) of Lemma 45, $s_j<t_j$, $j=1,\ldots,e$. So $1=\sum_{j=1}^{e}s_j<\sum_{j=1}^{e}t_j=1$, which is a contradiction. Thus, $z>y$ is not possible. By symmetry, $z<y$ is also not possible.

Now, suppose $e<f$. Then by (189) and (190), $z=M_{e+1}W(s_{e+1})\leq M_{e+1}r<y$. So for $j\in\{1,\ldots,e\}$:

$$M_jW(s_j)=z<y=M_jW(t_j)$$

which implies $s_j>t_j$. So $\sum_{j=1}^{e}s_j>\sum_{j=1}^{e}t_j=1$, which is a contradiction. So $e<f$ is not possible. By symmetry, $e>f$ is also not possible. The result follows.

Monotonicity Now, we show that $d'(q)$ is an increasing function of q. Suppose not. Then there exist q and q' such that $q<q'$, $d'(q)=e$, $d'(q')=f$ and $e>f$. Hence, by (166) and (153), there exist probability distributions $(t_1,\ldots,t_d)$ and $(s_1,\ldots,s_d)$ such that for some y and z:

$$y = M_1(1 - w(qt_1, n)) = \ldots = M_e(1 - w(qt_e, n)) > M_{e+1}r \tag{191}$$

$$z = M_1(1 - w(q's_1, n)) = \ldots = M_f(1 - w(q's_f, n)) > M_{f+1}r \tag{192}$$

So

$$y = M_{f+1}(1 - w(qt_{f+1}, n)) \leq M_{f+1}r < z.$$

Hence, for $j = 1, \ldots, f$, $M_j(1-w(qt_j,n)) < M_j(1-w(q's_j,n))$. So $w(q's_j,n) < w(qt_j,n)$. By Lemma 29, $w(x,n)$ is strictly increasing in x. So $q's_j < qt_j$. Since $q < q'$, $t_j > s_j$. Thus, $\sum_{j=1}^{f} t_j > \sum_{j=1}^{f} s_j = 1$, which contradicts the fact that $(t_1, \ldots, t_d)$ is a probability distribution. The result follows.

Finally, we show that $t_1 \geq t_2 \ldots \geq t_d$. For $1 \leq i < j \leq d'(q)$, $M_i W(t_i) = M_j W(t_j)$ by (166). But $M_i \geq M_j$ by (155); so $W(t_i) \leq W(t_j)$ and hence, by part (iii) of Lemma 45, $t_i \geq t_j$. For $l > d'(q), t_l = 0$ by (163). The result follows. $\square$

5.11.3 Proofs of results in Section 5.5

Proof of Lemma 38. Since no edge in E' is between two nodes in the same I.S. I_j, it follows that in G', $I_1, \ldots, I_d$ are disjoint maximal I.S. whose union is V. Using the notation in Definition 2, let $\{\alpha_{j,l} : j = 1, \ldots, d; l = 1, \ldots, M_j\}$ be a valid distribution in G'. We will show that (156) holds. Then it will follow from Definition 2 that G' is mean valid.

Let $\mathscr{I}_{G'}$ (respectively, $\mathscr{I}_G$) be the set of I.S. in G' (respectively, G). Since $E \subset E'$, each I.S. in G' is an I.S. in G as well, *i.e.* $\mathscr{I}_{G'} \subset \mathscr{I}_G$.

Now, since the distribution $\{\alpha_{j,l}\}$ is valid in G', by definition, there exists a distribution $\{\beta'(I) : I \in \mathscr{I}_{G'}\}$ such that

$$\alpha_v = \sum_{I \in \mathscr{I}_{G'}: v \in I} \beta'(I) \ \forall v \in V. \tag{193}$$

Define a distribution on $\mathscr{I}_G$ as follows:

$$\beta(I) = \begin{cases} \beta'(I) & \text{if } I \in \mathscr{I}_{G'} \\ 0 & \text{if } I \in \mathscr{I}_G \setminus \mathscr{I}_{G'} \end{cases} \tag{194}$$

By (193) and (194):

$$\alpha_v = \sum_{I \in \mathscr{I}_G: v \in I} \beta(I) \ \forall v \in V. \tag{195}$$

So by definition, $\{\alpha_{i,j}\}$ is a valid distribution in G as well. Since G is mean valid, (156) holds, which completes the proof. $\square$

Proof of Lemma 39. First, note that $\{(I_j^1 \cup \ldots \cup I_j^N) : j = 1, \ldots, d\}$ are disjoint maximal I.S. in G; so the first condition in Definition 2 is satisfied.

Let $\{\alpha_{j,l}^i : j = 1, \ldots, d; l = 1, \ldots, M_j^i\}$ be a valid distribution in G^i. Since G^i is mean valid:

$$\sum_{j=1}^{d} \left(\frac{\sum_{l=1}^{M_j^i} \alpha_{j,l}^i}{M_j^i} \right) \leq 1, \ i = 1, \ldots, N \tag{196}$$

Now, it is given that:

$$M_j^i = c_i M_j^0, \ i = 1, \ldots, N; j = 1, \ldots, d \tag{197}$$

Adding (197) over $i = 1, \ldots, N$:

$$M_j^0(c_1 + \ldots + c_N) = M_j^1 + \ldots + M_j^N, \ j = 1, \ldots, d \tag{198}$$

Multiplying (196) by c_i, using (197) and adding over $i = 1, \ldots, N$, we get:

$$\sum_{i=1}^{N} \sum_{j=1}^{d} \left(\frac{\sum_{l=1}^{M_j^i} \alpha_{j,l}^i}{M_j^0} \right) \leq c_1 + \ldots + c_N$$

Dividing both sides by $c_1 + \ldots + c_N$ and using (198):

$$\sum_{j=1}^{d} \left(\frac{\sum_{i=1}^{N} \sum_{l=1}^{M_j^i} \alpha_{j,l}^i}{M_j^1 + \ldots + M_j^N} \right) \leq 1$$

So G satisfies the second condition in Definition 2 as well and hence is mean valid. $\square$

Proof of part 2 of Theorem 7. In Section 5.5.1, we showed that $\mathcal{G}_m$ is mean valid for even m. Now, let m be odd, say $m = 2N - 1$ for some integer $N \geq 2$. Consider a valid distribution $\{\alpha_i : i = 1, \ldots, 2N-1\}$, where α_i is the probability with which bandwidth is offered at node v_i. With I_1 and I_2 as defined in Section 5.5.1, note that $|I_1| = N$ and $|I_2| = N - 1$. Let

$$\overline{\alpha}^1 = \frac{\alpha_1 + \alpha_3 + \ldots + \alpha_{2N-1}}{N}$$

and

$$\overline{\alpha}^2 = \frac{\alpha_2 + \alpha_4 + \ldots + \alpha_{2N-2}}{N-1}$$

To show that Condition 2 in Definition 2 is satisfied, we need to show that $\overline{\alpha}^1 + \overline{\alpha}^2 \leq 1$, *i.e.*

$$\begin{aligned} &(N-1)(\alpha_1 + \alpha_3 + \ldots + \alpha_{2N-1}) \\ &+ \quad N(\alpha_2 + \alpha_4 + \ldots + \alpha_{2N-2}) \leq N(N-1) \end{aligned} \tag{199}$$

Since $\mathcal{G}_{2N-1}$ is a bipartite graph and the distribution $\{\alpha_i\}$ is valid, the necessary condition in (168) holds and in this case becomes:

$$\alpha_i + \alpha_{i+1} \leq 1,\ i = 1, 2, \ldots, 2N-2 \tag{200}$$

Now,

$$\begin{aligned}
&\text{LHS of (199)} \\
= \;& \{(N-1)(\alpha_1+\alpha_2)+(\alpha_2+\alpha_3)\} \\
& +\{(N-2)(\alpha_3+\alpha_4)+2(\alpha_4+\alpha_5)\} \\
& +\{(N-3)(\alpha_5+\alpha_6)+3(\alpha_6+\alpha_7)\} \\
& +\ldots \\
& +\{2(\alpha_{2N-5}+\alpha_{2N-4})+(N-2)(\alpha_{2N-4}+\alpha_{2N-3})\} \\
& +\{(\alpha_{2N-3}+\alpha_{2N-2})+(N-1)(\alpha_{2N-2}+\alpha_{2N-1})\} \\
\leq \;& \{(N-1)+1\}+\{(N-2)+2\}+\ldots \\
& +\{2+(N-2)\}+\{1+(N-1)\} \text{ (by (200))} \\
= \;& N(N-1)
\end{aligned}$$

which proves (199) and the result follows. □

Proof of part 4 of Theorem 7. In Section 5.5.1, we showed that $\mathcal{H}_{m,m}$ is mean valid for even m. Now, let m be odd. With I_1, I_2, I_3 and I_4 as defined in Section 5.5.1, it is easy to check that $|I_1| = \left(\frac{m+1}{2}\right)^2$, $|I_2| = \frac{m^2-1}{4}$, $|I_3| = \frac{m^2-1}{4}$ and $|I_4| = \left(\frac{m-1}{2}\right)^2$.

Consider a valid distribution $\{\alpha_z : z \in V\}$, where α_z is the probability with which bandwidth is offered at node z. We now show that the graph is mean valid by showing that (156) holds, which in this case becomes:

$$(m-1)^2(\sum_{z\in I_1}\alpha_z)+(m^2-1)(\sum_{z\in I_2}\alpha_z)+(m^2-1)(\sum_{z\in I_3}\alpha_z) \\ +(m+1)^2(\sum_{z\in I_4}\alpha_z)\leq\frac{(m^2-1)^2}{4}. \quad (201)$$

Consider cliques $C_{i,j}$, $i,j\in\{0,\ldots,m\}$. For $i,j\in\{1,\ldots,m-1\}$, $C_{i,j}$ is as defined in Section 5.5.1. For i or j (or both) equal to 0 or m, let $C_{i,j}$ be "dummy cliques", defined for convenience (see Fig. 5.17). For $i,j\in\{0,\ldots,m\}$:

$$\sum_{z\in C_{ij}}\alpha_z\leq 1, \quad (202)$$

because, if not, then bandwidth would be offered simultaneously at two or more of the nodes in C_{ij} (which are neighbors) with a positive probability. For $i\in\{0,\ldots,m\}$, let:

$$e_i=\begin{cases} m-i, & i \text{ odd} \\ i, & i \text{ even}\end{cases} \quad (203)$$

For $i,j\in\{0,\ldots,m\}$, let

$$f_{ij}=e_ie_j. \quad (204)$$

Note that by definition of the cliques $\{C_{i,j}\}$, node v_{ij} belongs to each of the cliques $C_{i-1,j-1}$, $C_{i-1,j}$, $C_{i,j-1}$ and $C_{i,j}$ as shown in Fig. 5.18. So multiplying (202) by f_{ij} and adding over $i,j\in\{0,1,\ldots,m\}$ gives:

$$\sum_{z\in V}g_z\alpha_z\leq g_0 \quad (205)$$

where,

$$g_{v_{ij}}=f_{i-1,j-1}+f_{i-1,j}+f_{i,j-1}+f_{ij} \quad (206)$$

and

$$
\begin{aligned}
g_0 &= \sum_{i=0}^{m}\sum_{j=0}^{m} f_{i,j} = \sum_{i=0}^{m}\sum_{j=0}^{m} e_i e_j = \left(\sum_{i=0}^{m} e_i\right)^2 \\
&= \left(\sum_{i=0,i \text{ odd}}^{m} (m-i) + \sum_{i=0,i \text{ even}}^{m} i\right)^2 = \frac{(m^2-1)^2}{4} \qquad (207)
\end{aligned}
$$

We will show below that

$$
g_z = \begin{cases} (m-1)^2, & z \in I_1 \\ (m^2-1), & z \in I_2 \text{ or } z \in I_3 \\ (m+1)^2, & z \in I_4 \end{cases} \qquad (208)
$$

Note that (201) follows from (205), (207) and (208), which shows that $\mathcal{H}_{m,m}$ is mean valid.

Now we show (208). By definition of the I.S. I_1, I_2, I_3 and I_4 (see Section 5.5.1), for $v_{ij} \in I_1$, i and j are odd, for $v_{ij} \in I_2$, i is odd and j is even, for $v_{ij} \in I_3$, i is even and j is odd and for $v_{ij} \in I_4$, i and j are even. So for $v_{ij} \in I_1$, by (203), (204) and (206):

$$
\begin{aligned}
g_{v_{ij}} &= (i-1)(j-1) + (i-1)(m-j) + (m-i)(j-1) \\
&\quad +(m-i)(m-j) \\
&= (m-1)^2
\end{aligned}
$$

Similarly, for $v_{ij} \in I_2$:

$$
\begin{aligned}
g_{v_{ij}} &= (i-1)(m-j+1) + (i-1)j + (m-i)(m-j+1) \\
&\quad +(m-i)j \\
&= m^2-1
\end{aligned}
$$

For $v_{ij} \in I_3$, $g_{v_{ij}} = m^2 - 1$ by symmetry with the case $v_{ij} \in I_2$. For $v_{ij} \in I_4$:

$$\begin{aligned} g_{v_{ij}} &= (m-i+1)(m-j+1) + (m-i+1)j \\ &\quad +i(m-j+1) + ij \\ &= (m+1)^2 \end{aligned}$$

Thus, we have shown (208), which completes the proof. □

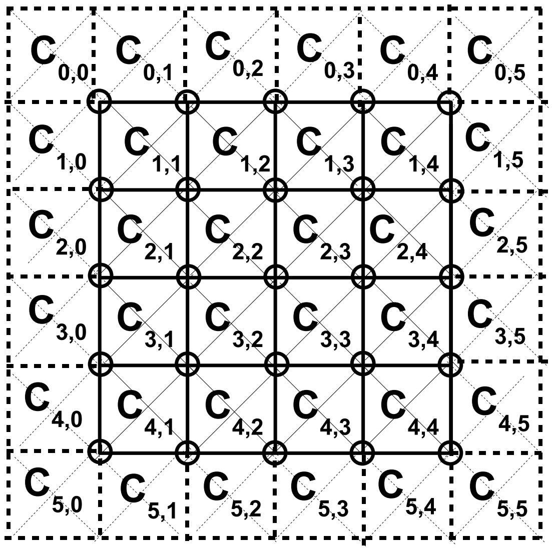

Figure 5.17: The figure shows the cliques in $\mathcal{H}_{5,5}$. The cliques with dotted outlines are the dummy cliques.

Proof of part 5 of Theorem 7. In Section 5.5.1, we considered the case m even. The proof of the fact that $\mathcal{T}_{m,m,m}$ is mean valid for m odd is similar to that for $\mathcal{H}_{m,m}$ with m odd; we outline the differences. We define the cliques C_{ijl}, $i, j, l \in \{0, 1, \ldots, m\}$, similar to C_{ij} for the case $\mathcal{H}_{m,m}$. Consider a valid distribution $\{\alpha_z : z \in V\}$. Then similar to

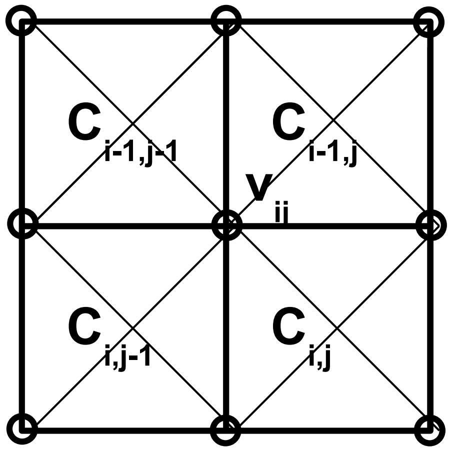

Figure 5.18: The node v_{ij} and the cliques $C_{i-1,j-1}$, $C_{i-1,j}$, $C_{i,j-1}$ and $C_{i,j}$.

(202), we get:

$$\sum_{z \in C_{ijl}} \alpha_z \leq 1 \tag{209}$$

Let e_i be as in (203) and $f_{ijl} = e_i e_j e_l$, $i, j, l \in \{0, \ldots, m\}$. Multiplying (209) by f_{ijl} and adding over $i, j, l \in \{0, 1, \ldots, m\}$, we get (205) for some numbers $\{g_z : z \in V\}$ and g_0. Now, node v_{ijl} is at the center of the cliques $C_{i-1,j-1,l-1}$, $C_{i-1,j-1,l}$, $C_{i-1,j,l-1}$, $C_{i-1,j,l}$, $C_{i,j-1,l-1}$, $C_{i,j-1,l}$, $C_{i,j,l-1}$, and $C_{i,j,l}$. Using this fact, $g_{v_{ijl}}$ for v_{ijl} in each of $I_1, \ldots, I_8$ can be computed similar to the derivation of (208). Also, g_0 can be calculated similar to (207). Substituting these values of $\{g_z : z \in V\}$ and g_0 into (205), we get (156) for $\mathcal{T}_{m,m,m}$ and thereby the mean validity follows from Definition 2. $\square$

5.11.4 Proofs of results in Section 5.6

Proof of Theorem 9. Suppose $\beta(I_1) = t_1$, $\beta(I_2) = t_2$, where $t_1 + t_2 = 1$ is a symmetric NE. By (164):

$$U_1(I_j) = M_j W(t_j), \; j = 1, 2. \tag{210}$$

First, suppose $t_1 = 0, t_2 = 1$. Since $\beta(I_2) = t_2 > 0$, I_2 is a best response. By (210) and part (ii) of Lemma 45, $U_1(I_1) = M_1$. Again, by (210), and since $W(1) < 1$ by part (i) of Lemma 45:

$$U_1(I_2) = M_2 W(1) < M_2 \leq M_1 = U_1(I_1)$$

which contradicts the fact that I_2 is a best response. So $t_1 = 0, t_2 = 1$ is not a symmetric NE.

Now, suppose $t_1 = 1, t_2 = 0$. Then I_1 is a best response. Similar to the previous paragraph, $U_1(I_1) = M_1 W(1)$ and $U_1(I_2) = M_2$. So by (153):

$$U_1(I_1) - U_1(I_2) = M_1 \left(1 - \frac{M_2}{M_1} - w(q,n)\right) < 0$$

since $w(q,n) > 1 - \frac{M_2}{M_1}$. This contradicts the fact that I_1 is a best response. Thus, $t_1 = 1, t_2 = 0$ is not a symmetric NE.

Suppose $0 < t_1, t_2 < 1$. Let $I \in \mathscr{I}$ be such that:

$$\frac{m_1(I)}{M_1} + \frac{m_2(I)}{M_2} > 1, \tag{211}$$

which exists by Lemma 31 since G is not mean valid. Since $\beta(I_1), \beta(I_2) > 0$, I_1 and I_2 are best responses. So $U_1(I_1) = U_1(I_2) = U^*$, where U^* is the maximum payoff of any

I.S. By (210):

$$W(t_j) = \frac{U^*}{M_j},\ j = 1,2 \tag{212}$$

Now, by (164):

$$\begin{aligned} U_1(I) &= m_1(I)W(t_1) + m_2(I)W(t_2) \\ &= U^*\left(\frac{m_1(I)}{M_1} + \frac{m_2(I)}{M_2}\right) \text{ (by (212))} \\ &> U^* \text{ (by (211))} \end{aligned}$$

which contradicts the fact that U^* is the maximum payoff of any I.S. □

Proof of Lemma 40. By (2) and (153), $W(.)$ is a continuous function. So $f_1(x)$ is continuous on $[0,1]$. Also, it can be shown that [15] the derivative of $w(x,n)$ with respect to x is given by:

$$w'(x,n) = (n-1)\begin{pmatrix} n-2 \\ k-1 \end{pmatrix} x^{k-1}(1-x)^{n-k-1}.$$

Note that $w'(x,n) > 0\ \forall x \in (0,1)$. So by (153), $W'(\alpha) < 0\ \forall \alpha \in (0,1)$. Hence, for $x \in (0,1)$:

$$f_1'(x) = -2W'(1-x) - W'(x) > 0.$$

So $f_1(x)$ is strictly increasing on $[0,1]$ [58].

Also, by (153), $f_1(0) = 2W(1) - W(0) = 1 - 2w(q,n) < 0$ since $w(q,n) > \frac{1}{2}$, and $f_1\left(\frac{1}{2}\right) = W\left(\frac{1}{2}\right) = 1 - w\left(\frac{q}{2},n\right) > 0$. So by the intermediate value theorem [58], $f_1(x)$ has a root $t_1 \in \left(0,\frac{1}{2}\right)$. Also, t_1 is the unique root in $[0,1]$ since $f_1(x)$ is strictly increasing.

□

5.11.4.1 Proof of Theorem 10

Consider a symmetric strategy profile under which each primary offers bandwidth at I_a (respectively, I_b) w.p. t_a (respectively, t_b) and at I_{ab} w.p. $1-t_a-t_b$. By (152), the corresponding node probabilities are $\alpha_{a_1}=t_a$, $\alpha_{a_2}=\alpha_{a_3}=1-t_b$, $\alpha_{b_1}=t_b$, $\alpha_{b_2}=\alpha_{b_3}=1-t_a$. So by (164), the total expected payoffs of primary 1 if it offers bandwidth at each of the three I.S. are:

$$U_1(I_{ab}) = 2W(1-t_b)+2W(1-t_a) \tag{213}$$

$$U_1(I_a) = 2W(1-t_b)+W(t_a) \tag{214}$$

$$U_1(I_b) = W(t_b)+2W(1-t_a) \tag{215}$$

Intuitively, since I_{ab} is the largest I.S., we expect that in a symmetric NE, primaries would not offer bandwidth at one or both of I_a and I_b without offering it at I_{ab}. The following result confirms this.

Lemma 47. *Let $q \in (0,1)$ be arbitrary. None of the following can hold in a symmetric NE: (i) $t_a = 1$, (ii) $t_b = 1$ (iii) $0 < t_a, t_b < 1$ and $t_a + t_b = 1$.*

Proof. First, suppose $t_a = 1$ in a symmetric NE. Since $t_a > 0$, I_a is a best response. Also, $t_b = 0$. So by (213), (214), (153) and the fact that $w(0,n) = 0$:

$$U_1(I_{ab}) - U_1(I_a) = 1 + w(q,n) > 0$$

So $U_1(I_{ab}) > U_1(I_a)$, which contradicts the fact that I_a is a best response. Thus, $t_a = 1$ is not possible. By symmetry, $t_b = 1$ is also not possible.

Now, suppose $0 < t_a, t_b < 1$ and $t_a + t_b = 1$. Since $t_a, t_b > 0$, I_a and I_b are best responses. So $U_1(I_a) = U_1(I_b)$. By (214), (215), (153) and the fact that $t_a + t_b = 1$, we get $w(qt_a, n) = w(qt_b, n)$. So by Lemma 29, $t_a = t_b = \frac{1}{2}$. Hence, by (213), (214) and (153):

$$U_1(I_{ab}) - U_1(I_a) = \left(1 - w\left(\frac{q}{2}, n\right)\right) > 0$$

which contradicts the fact that I_a is a best response. □

Now we are ready to prove Theorem 10.

Case 1: $w(q,n) \leq \frac{1}{2}$. Let t_a and t_b be arbitrary. By (213), (214) and (153):

$$\begin{aligned} & U_1(I_{ab}) - U_1(I_a) \\ = \;& 1 - 2w(q(1-t_a), n) + w(qt_a, n) \\ \geq \;& 1 - 2w(q,n) \text{ (by Lemma 29)} \\ \geq \;& 0 \left(\text{since } w(q,n) \leq \frac{1}{2}\right) \end{aligned} \tag{216}$$

Note that if $t_a > 0$, then the inequality in (216) is strict. So $U_1(I_{ab}) > U_1(I_a)$, which is a contradiction because $t_a > 0$ implies that I_a is a best response. Hence, $t_a = 0$. By symmetry, $t_b = 0$. If $t_a = t_b = 0$, then $U_1(I_{ab}) \geq U_1(I_a)$ and $U_1(I_{ab}) \geq U_1(I_b)$; so I_{ab} is a best response, which is consistent with the fact that it is played w.p. 1. Thus, $t_a = t_b = 0$ is the unique symmetric NE.

Case 2: $w(q,n) > \frac{1}{2}$. By Lemma 47, $t_a + t_b < 1$ for every symmetric NE and hence I_{ab} is a best response. Now, suppose $t_a = 0$. By (213), (214), (153) and the fact that $w(0,n) = 0$, we get $U_1(I_{ab}) - U_1(I_a) = 1 - 2w(q,n) < 0$ since $w(q,n) > \frac{1}{2}$. So

$U_1(I_a) > U_1(I_{ab})$, which contradicts the fact that I_{ab} is a best response. Hence, $t_a > 0$. By symmetry, $t_b > 0$.

Thus, all three of I_a, I_b and I_{ab} are best responses. So $U_1(I_{ab}) = U_1(I_a) = U_1(I_b)$. Substituting (213), (214) and (215), these are satisfied iff $t_a = t_b = t_1$, the root of $f_1(x) = 0$. This completes the proof.

5.11.5 Proofs of results in Section 5.7

Proof of Lemma 41. Let $0 \leq \alpha < \alpha' \leq 1$. It suffices to show that $w_1(\alpha) < w_1(\alpha')$.

Let $Y_i, i = 2, \ldots, n$ be independent Bernoulli random variables and let Y_i have mean $q_i\alpha$. Also, let $Z_i, i = 2, \ldots, n$ be independent Bernoulli random variables that are independent of $Y_i, i = 2, \ldots, n$ and let Z_i have mean $\frac{q_i\alpha' - q_i\alpha}{1 - q_i\alpha}$.

For $i = 2, \ldots, n$, let:

$$X_i = \begin{cases} 1, & \text{if } Y_i = 1 \text{ or } Z_i = 1 \text{ (or both)} \\ 0, & \text{else} \end{cases} \tag{217}$$

$$
\begin{aligned}
P(X_i = 1) &= P(\{Y_i = 1\} \cup (Z_i = 1)\}) \\
&= P(Y_i = 1) + P(Z_i = 1) \\
&\quad -P(\{Y_i = 1\} \cap \{Z_i = 1\}) \\
&= P(Y_i = 1) + P(Z_i = 1) - P(Y_i = 1)P(Z_i = 1) \\
&\quad \text{(since } Y_i \text{ and } Z_i \text{ are independent)} \\
&= q_i\alpha + \frac{q_i\alpha' - q_i\alpha}{1 - q_i\alpha} - (q_i\alpha)\left(\frac{q_i\alpha' - q_i\alpha}{1 - q_i\alpha}\right) \\
&= q_i\alpha'
\end{aligned}
$$

So X_i is Bernoulli with mean $q_i\alpha'$. Also, since $Y_i, i = 2, \ldots, n$ and $Z_i, i = 2, \ldots, n$ are independent, $X_i, i = 2, \ldots, n$ are independent.

But by (217),

$$
\{Y_i = 1\} \subset \{X_i = 1\},\ i = 1, \ldots, n \tag{218}
$$

Also,

$$
\begin{aligned}
P\{X_i = 1, Y_i = 0\} &= P(Z_i = 1, Y_i = 0) \\
&= P(Z_i = 1)P(Y_i = 0) \\
&= \left(\frac{q_i\alpha' - q_i\alpha}{1 - q_i\alpha}\right)(1 - q_i\alpha) \\
&= q_i\alpha' - q_i\alpha \\
&> 0 \qquad (219)
\end{aligned}
$$

By (218) and (219):

$$P(X_i = 1) > P(Y_i = 1). \tag{220}$$

Now, let $X = \sum_{i=2}^{n} X_i$ and $Y = \sum_{i=2}^{n} Y_i$. We interpret X_i (respectively, Y_i) as the indicator of the event that primary i offers bandwidth at a node v with node probability $\alpha_v = \alpha'$ (respectively, $\alpha_v = \alpha$). So X(respectively, Y) is the number of primaries who offer bandwidth at node v when $\alpha_v = \alpha'$ (respectively, $\alpha_v = \alpha$). By definition of the function $w_1(.)$, and conditioning on K_v, the number of secondaries at node v:

$$w_1(\alpha') = \sum_k P(X \geq k)P(K_v = k) = \sum_k \gamma_k P(X \geq k) \tag{221}$$

and

$$w_1(\alpha) = \sum_k \gamma_k P(Y \geq k). \tag{222}$$

By (220), (221), (222) and the facts $X = \sum_{i=2}^{n} X_i$, $Y = \sum_{i=2}^{n} Y_i$ and $\sum_{k=1}^{n-1} \gamma_k > 0$, it follows that $w_1(\alpha) < w_1(\alpha')$. $\square$

5.11.6 Proof of Lemma 44

In Lemmas 48, 49 and 50 below, we state and prove a generalization of Lemma 44 in which we relax the assumption that $M_1, \ldots, M_d$ are distinct.

Lemma 48. *Let $z = |\{i : M_i = M_1\}|$. If there exists an $\varepsilon > 0$ such that for all large n, $q < zk_n/(n-1) - \varepsilon$, then $\eta \to 1, \tilde{p}_j \to \nu,\ j = 1, \ldots, z$ as $n \to \infty$. Also, for all large n, $d' = z$, $t_1 = \ldots = t_z = 1/z, t_{z+1} = t_{z+2} = \ldots t_d = 0$.*

Proof. Note that for all large enough n, for each i, $\frac{\sum_{j=1}^{n} q_j - q_i}{z} < (n-1)q/z + (n-1)\varepsilon/2z$. Thus, if each primary selects an I.S. w.p. $1/z$, for a given primary with available bandwidth, the expected number of primaries among the rest minus the expected number of secondaries is less than $-(n-1)\varepsilon/2z$. Clearly, then, for each i, $w_i(1/z) \to 0$ as $n \to \infty$ (convergence is exponentially fast by Hoeffding's inequality [26]). Thus, $W(1/z) \to 1$ as $n \to \infty$. Thus, for all large enough n, $M_1 W(1/z) = M_2 W(1/z) = \ldots M_z W(1/z) > M_{z+1} r$. Thus, $(1/z, \ldots, 1/z, 0, \ldots, 0)$ satisfies the requisite equations for the symmetric NE I.S. selection p.m.f. The last part follows. For $j = 1, \ldots, z$, clearly $(\nu - c)(1 - w_1(1/z)) \le \tilde{p}_j - c \le \nu - c$. Thus, $\tilde{p}_j \to \nu$ as $n \to \infty$. Thus, the expected utility of any primary with available bandwidth converges to M_1, the maximum possible value, and the error decays exponentially with increase in n. Thus, $\eta \to 1$. □

Lemma 49. *Consider $l < d$. Let $l_{min} = \min\{i \le l : M_i = M_l\}$ and $l_{max} = \max\{i \ge l : M_i = M_l\}$. If there exists an $\varepsilon > 0$ such that for all large n, $lk_n/(n-1) + \varepsilon < \overline{q}_n < (l+1)k_n/(n-1) - \varepsilon$, then for all large n, $d'_n \ge \max(l+1, l_{max})$. Also, $t_{mn}\overline{q}_n \to k_n/(n-1)$ for $m = 1, \ldots, l_{min}-1$ and $t_{mn}\overline{q}_n \to \min\left(\frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max} - l_{min} + 1}, k_n/(n-1)\right)$ for $m = l_{min}, \ldots, l_{max}$.*

Proof. First let $d'_n \le l$. Then $t_{1n} \ge 1/d'_n \ge 1/l$. Thus, $t_{1n}\overline{q}_n \ge k_n/(n-1) + \varepsilon/l$. Thus, $W(t_{1n}) \to 0$ and $M_1 W(t_{1n}) \to 0$ as $n \to \infty$. Thus, $M_1 W(t_{1n}) < M_{l+1}$ for all large enough n (contradiction). Thus, $d'_n \ge l+1$. However, the fact that $d'_n \ge l$ implies that $d'_n \ge l_{max}$. To prove this, suppose not. Then $M_l W(t_l) > M_{l_{max}} r = M_l r$. So $W(t_l) > r$, which contradicts Lemma 43. So $d'_n \ge l_{max}$ and hence $d'_n \ge \max(l_{max}, l+1)$. Thus, the first part of the lemma holds.

Now, consider a $m \leq l_{max}$. Let there exist a $\delta > 0$ such that $t_{mn}\overline{q}_n > k_n/(n-1)+\delta$ for a certain subsequence $\{\overline{q}_n, k_n\}$. Then $W(t_{mn}) \to 0$ for that subsequence. Thus, $M_m W(t_{mn}) \to 0$ for that subsequence. Let $d'_n = d$ in a subsequence of the above subsequence. In this subsequence $t_{d'_n} \leq 1/d$, and thus $t_{d'_n}\overline{q}_n < k_n/(n-1)-\varepsilon$, $W(t_{d'_n}) \to 1$ and $M_{d'_n} W(t_{d'_n}) > 0$. Thus, $M_m W(t_{mn}) \neq M_{d'} W(t_{d'_n})$ for all large enough n (contradiction). Thus, $d'_n < d$ throughout the above subsequence. But then $M_m W(t_{mn}) < M_{d'_n+1} r$ for all large enough n (contradiction). Thus, no such subsequence exists. Thus, $\limsup t_{mn}\overline{q}_n \leq k_n/(n-1)$.

Now, for $m \in \{1, \ldots, l_{min}-1\}$, let there exist a $\delta > 0$ such that $t_{mn}\overline{q}_n < k_n/(n-1)-\delta$ for a certain subsequence $\{\overline{q}_n, k_n\}$. Then $W(t_{mn}) \to 1$ for that subsequence. Thus, in that subsequence, $M_m W(t_{mn}) > M_{l_{min}} r$ for all large enough n. Then for all large enough n, $d'_n \leq l_{min}-1 < l_{max}$ (contradiction). Thus, $\liminf t_{mn}\overline{q}_n \geq k_n/(n-1)$. Hence,

$$t_{mn}\overline{q}_n \to k_n/(n-1),\; m = 1, \ldots, l_{min}-1. \tag{223}$$

Now, let $m \in \{l_{min}, \ldots, l_{max}\}$. Since $M_{l_{min}} = \ldots = M_{l_{max}}$ and $M_{l_{min}} W(t_{l_{min}}) = \ldots = M_{l_{max}} W(t_{l_{max}})$, it follows that $t_{l_{min}} = \ldots = t_{l_{max}} = t_l$. Suppose for a subsequence, $t_l\overline{q}_n > \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1} + \delta$. This implies

$$(l_{max}-l_{min}+1)t_l + \left(\frac{1}{\overline{q}_n}\frac{(l_{min}-1)k_n}{n-1}\right) > \frac{\delta(l_{max}-l_{min}+1)}{\overline{q}_n} + 1$$

Taking limits as $n \to \infty$ on both sides and using (223) and the fact that $t_{l_{min}} = \ldots = t_{l_{max}} = t_l$, we get:

$$\sum_{m=l_{min}}^{l_{max}} t_m + \sum_{m=1}^{l_{min}-1} t_m > 1 + \frac{\delta(l_{max}-l_{min}+1)}{q}$$

which contradicts the fact that $(t_1,\ldots,t_d)$ is a probability distribution. Hence,

$$\limsup t_l \overline{q}_n \leq \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1}.$$

Now, we consider two cases.

Case (i):

$$\lim_{n\to\infty} \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1} \leq \lim_{n\to\infty} \frac{k_n}{n-1}. \tag{224}$$

Suppose there exists $\delta > 0$ such that for a subsequence t_{l_n}:

$$t_{l_n}\overline{q}_n < \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1} - \delta \tag{225}$$

For this subsequence, after accounting for the probability masses put on $I_1,\ldots,I_{l_{max}}$, there is still some left. So $d' \geq l_{max}+1$ for this subsequence. However, by (224) and (225):

$$t_{l_n}\overline{q}_n < \frac{k_n}{n-1} - \delta$$

for large enough n. So $W(t_{l_n}) \to 1$ for the subsequence. So in the subsequence, $M_l W(t_{l_n}) > M_{l_{max}+1} r$, which contradicts the fact that $d'_n \geq l_{max}+1$. Thus,

$$\liminf t_{l_n}\overline{q}_n \geq \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1} \tag{226}$$

and hence $t_{l_n}\overline{q}_n \to \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1}$.

Case (ii):

$$\lim_{n\to\infty} \frac{k_n}{n-1} < \lim_{n\to\infty} \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1} \tag{227}$$

Suppose

$$t_{l_n}\overline{q}_n < \frac{k_n}{n-1} - \delta \tag{228}$$

for a subsequence. Then

$$W(t_{l_n}) \to 1 \tag{229}$$

for that subsequence. Now, by (227) and (228):

$$t_{l_n}\overline{q}_n < \frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max} - l_{min} + 1}$$

for large enough n. So similar to Case (i), after accounting for the probability masses put on $I_1, \ldots, I_{l_{max}}$, there is still some left. So

$$d'_n \geq l_{max} + 1. \tag{230}$$

But by (229), $M_l W(t_{l_n}) > M_{l_{max}+1}$, which contradicts (230). Thus, in Case (ii), $t_{l_n}\overline{q}_n \to \frac{k_n}{n-1}$.

Hence, in both cases, $t_{mn}\overline{q}_n \to \min\left(\frac{\overline{q}_n - \frac{(l_{min}-1)k_n}{n-1}}{l_{max}-l_{min}+1}, k_n/(n-1)\right)$ and we are done. $\square$

Lemma 50. *If there exists an* $\varepsilon > 0$ *such that for all large n,* $q > k_n d/(n-1) + \varepsilon$*,* $\eta \to 0$ *as* $n \to \infty$*. Also, for all large n,* $d' = d$ *and* $\tilde{p}_j \to c$*,* $j = 1, \ldots, d$*.*

Proof. Clearly, $t_1 \geq 1/d$. Thus, $t_1 q \geq k_n/(n-1) + \varepsilon/d$. Now, for all large enough n, for each i, $\sum_{j=1}^{n} t_1 q_j - t_1 q_i > (n-1)t_1 q - t_1(n-1)\varepsilon/2$. Thus, if a given primary with available bandwidth selects I_1, then the expected number of other primaries he sees at a node there minus the expected number of secondaries is greater than $(n-1)\varepsilon/2$. Clearly, then for each i, $w_i(t_1) \to 1$ as $n \to \infty$ (convergence is exponentially fast by Hoeffding's inequality [26]). Thus, $W(t_1) \to 0$ and $M_1 W(t_1) \to 0$ as $n \to \infty$. Thus, $M_1 W(t_1) < M_d r$ for all large enough n. Thus, $d' = d$. So for $j = 1, \ldots, d$, $M_j W(t_j) =$

$M_1W(t_1) \to 0$ as $n \to \infty$ and hence $\tilde{p}_j \to c$. Thus, the second part of the lemma holds. Since $M_1W(t_1) \to 0$ as $n \to \infty$, expected utility of each primary approaches 0, and the approach is exponentially fast. Thus, the overall expected utility of all primaries approach 0. Clearly, the expected utility attained by OPT is bounded away from 0. The result follows. $\square$

Chapter 6

Dynamic Contract Trading in Spectrum Markets

In this chapter, as in Chapters 2 to 5, we focus on the two-step allocation scenario in which the regulator such as the FCC in the USA first allocates primary rights to operators on its channels, who then allocate unused portions on their channels to secondary users. In Chapters 2 to 5, we assumed that the number of players (primaries and secondaries) is small and hence that each player exerts considerable influence on the market prices. In this chapter, we consider the case in which there are a large number of players in the market and the price is determined by the market.

6.1 Introduction

We consider a spectrum market where the license holders (referred to as *primary providers* henceforth) can potentially sell to the *secondary providers* the spectrum they have licensed from the FCC but do not envision using in the near future. Primary providers may either be providers of TV broadcasts, or large providers of wireless service who operate nationwide. Secondary providers are relatively smaller, but larger in number, and can be geographically limited providers, whose access to spectrum occurs through the bandwidth (service) contracts that they buy from primary providers. Providers in both categories have their subscriber (TV or mobile communication subscriber) bases whom they need to serve using the spectrum they respectively license from the FCC or buy in the spectrum market. This spectrum market structure is motivated by, and closely resembles, secondary financial markets used for trading of financial instruments (such as stocks, bonds) among investment banks, hedge-funds etc. Like in secondary financial markets, we allow trading in spectrum markets, not only of the raw spectrum (bandwidth), but also of the different kinds of service *contracts* derived from the use of spectrum. A question that is key to the efficient operation of the spectrum market is how the players in the market – the *primary* and the *secondary* providers – should trade spectrum (bandwidth/service) contracts dynamically, based on time-varying demand patterns arising from their subscribers, to maximize their returns while satisfying their subscriber base. This is the central focus of this chapter.

We formulate and evaluate the solutions for the *spectrum contract trading* problem for the primary and the secondary providers. We consider two basic forms of contracts that are used for selling/buying spectral resources: i) *Guaranteed-bandwidth (Type-G)* contracts, and (ii) *Opportunistic-access (Type-O)* contracts. Under the Type-*G* contracts, a secondary provider purchases a guaranteed amount of bandwidth (in units of frequency bands or sub-bands) for a specified duration of time (typically a "long term") from a primary provider, and pays a fixed fee (either as a lump-sum or as a periodic payment through the duration of the contract) irrespective of how much it uses this bandwidth. If after selling the contract, the primary is unable to provide the promised bandwidth (this may for example happen when the primary is forced to use a band it has sold due to an unexpected rise in its subscriber demand), the primary financially compensates the secondary for contractual violation. On the other hand, Type-*O* contracts are short-term (one time unit in our model), and a secondary which buys a Type-*O* contract pays only for the amount of bandwidth it actually uses on the corresponding band. The primary does not provide any guarantee on a Type-*O* contract and may use the channel sold as a Type-*O* contract without incurring any penalty. Thus, a Type-*O* contract provides the secondary the right to use the channel if the primary is not using it.

The *spectrum contract trading* problem that we formulate and solve allows the primary (secondary) provider to dynamically adjust its *spectrum contract portfolio*, i.e, choose how much of each type of contract to sell (buy) at any time, so as to maximize (minimize) its profit (cost) subject to satisfying its own subscriber demand that varies with time, and given the current market prices of Type-*G* and Type-*O* contracts which also vary with time. The exact nature of the spectrum contract trading (selling/buying) question will depend on whether it is considered from the perspective of the primary provider (seller) or the secondary provider (buyer). We therefore separately address the *Primary's Spectrum Contract Trading (Primary-SCT)* problem (Section 6.2) and the *Secondary's Spectrum Contract Trading (Secondary-SCT)* problem (Section 6.3). We formulate each problem as a finite horizon stochastic dynamic program whose computation time is polynomial in the input size. We *prove* several structural properties of the optimum solutions. For example, we show that the optimal number of Type-*G* contracts, for both primary and secondary providers, are monotone (increasing or decreasing) functions of the subscribers' demands and the contract prices. These structural results provide more insight into the problems, and allow us to develop faster algorithms for solving the dynamic programs. Finally, using numerical evaluations, we investigate properties of the optimal solutions and demonstrate that the revenues they earn substantially outperform static spectrum portfolio optimization strategies that determine the portfolio based on the steady-state statistics of the contract price and subscriber demand processes (Section 6.4).

Although the spectrum contract trading problem has been motivated by analogues in financial markets, the actual questions posed and the techniques used to answer them turn out to be quite different owing to the nature of the specific commodity, that is RF spectrum, under consideration. First, both the primary and the secondary must decide their trading strategies considering their subscriber demand which changes with time. For example, a primary (or secondary) can not simply decide to sell (buy) a large number of Type-G contracts at any given time at which their market prices are high (low). This is because a primary will need to pay a hefty penalty if it can not deliver the promised bandwidth owing to an increase in its subscriber demand, and the secondary will need to pay for the contract even if it does not use the corresponding bands owing to a decrease in its subscriber demand. The portfolio optimization literature in finance does not usually address the demand satisfaction constraint. Next, spectrum usage must satisfy certain temporal and spatial constraints that are perhaps unique. Specifically, a frequency band can not be simultaneously successfully used at neighboring locations (without causing significant interference), but can be simultaneously successfully used at geographically disparate locations. Thus, the spectrum trading solution for the primary provider must also take into account spatial constraints for spectrum reuse, and therefore the computation of the optimal trading strategy requires a joint optimization across all locations. We prove a surprising *separation theorem* in this context: when the same signal is broadcast at all locations, the Primary-SCT problem can be solved separately for each location and the individual optimal solutions can subsequently be combined so as to optimally satisfy the global reuse constraints, and obtain the same revenue as the solution of a computationally prohibitive joint optimization across locations (Section 6.2).

6.2 The Primary's Spectrum Contract Trading (SCT) Problem

In this section we pose and address *Primary-SCT*, the spectrum contract trading question from a primary provider's perspective. We first formulate the problem when a primary provider owns channels in a single region (Section 6.2.1), solve it using a stochastic dynamic program (Section 6.2.2), and identify the structural properties of the optimal solution (Section 6.2.3). Later we formulate and solve the trading problem when the primary owns channels in multiple locations, considering the spatial reuse of channels across different locations (Section 6.2.4).

6.2.1 SCT in a single region

We now define the Primary-SCT problem for a primary provider that owns M frequency bands (channels) in a single region, which it sells as Type-G or Type-O contracts to secondary providers. We assume that each channel corresponds to one unit of bandwidth and at most one contract – either Type-G or Type-O – can stand leased on a channel at any time. We also assume that the market has *infinite liquidity*: there is a large number of buyers, and hence the primary provider can sell any or all of the channels it owns anytime and in any combination of Type-G and Type-O contracts.

We assume that time is slotted. Trading of bandwidth is done between primary and secondary providers separately in each of successive windows of duration T slots each. Henceforth, we focus on the optimization in a single window or time horizon of T time slots. At the beginning of each slot t, the primary determines the number of channels

$x_G(t)$ and $x_O(t)$ to be sold as Type-G and Type-O contracts respectively. A Type-G ("long term") contract that is sold at the beginning of any slot $t = 1, \ldots, T$ lasts till the end of the horizon. T therefore represents the maximum duration of a Type-G contract. Type-O contracts last for a single slot from the time they are negotiated.

The prices of both types of contracts (i.e, the prices at which they can be bought/sold in the spectrum market) vary randomly with time and are determined "by the market", possibly depending on the current supply-demand balance in the market and other factors. The "per-slot" market prices for Type-G and Type-O contracts at time t are denoted by $c_G(t)$ and $c_O(t)$ respectively. When a Type-G contract is sold at slot t, it remains active for $T-t+1$ slots (that is, until the end of the optimization horizon), and therefore fetches a revenue of $\alpha(T-t+1)c_G(t)$, where $\alpha(n)$ is a (deterministic) increasing function of n and captures the increase in value of a Type-G contract with the number of slots for which it remains active, *e.g.*, $\alpha(n) = n$. We assume that the process $\{c_G(t)\}$ (respectively, $\{c_O(t)\}$) constitutes a Discrete time Markov chain (DTMC) with a finite number of states and transition probability $H^G_{c,d}$ (respectively, $H^O_{c,d}$) from state c to d. For simplicity, we assume that the DTMCs $\{c_G(t)\}$ and $\{c_O(t)\}$ are independent of each other, although our results readily extend to the case when the joint process $\{c_G(t), c_O(t)\}$ is a DTMC.

Each primary provider is associated with a randomly time-varying *demand* process, $\{i(t)\}$ which corresponds to its subscriber demand (of TV channel subscribers or wireless service subscribers, for example) that it must satisfy. We assume that the process

$\{i(t)\}$ constitutes a DTMC with a finite number of states and transition probability Q_{ij} from state i to j, that is independent of the price process; each demand state corresponds to an integral amount of bandwidth consumption in subscriber demand.

We assume that the transition probabilities $\{H^G_{c,d}\}$, $\{H^O_{c,d}\}$ and $\{Q_{ij}\}$ are known to the primary provider. They can be estimated from the history of the price and demand processes.

The contract trading is done at the beginning of time slot t, and $(x_G(t), x_O(t))$ are determined after the market prices $c_G(t)$, $c_O(t)$ and demand levels $i(t)$ are known. Let $(a_G(t), x_O(t))$ denote the spectrum contract portfolio held by the primary during time slot t, *i.e.* the number of Type-G and Type-O contracts that stand leased. Since Type-G contracts last till the end of the time horizon, we have:

$$a_G(t) = \sum_{t' \leq t} x_G(t') \tag{231}$$

The bandwidth not leased as Type-G contracts or used to satisfy the demand is sold as Type-O contracts. Thus, at any time t:

$$x_O(t) = K(a_G(t), i(t)) := \max\{0, M - a_G(t) - i(t)\}. \tag{232}$$

However, for all slots, t, for which $a_G(t) + i(t) > M$, the primary will have to use channels already sold under Type-G contracts to satisfy its subscriber demand, due to unavailability of additional bandwidth. In this case, the primary incurs a penalty, $Y(a_G(t), i(t))$, for breaching Type-G contracts. The penalty is proportional to the number of such channels the provider uses for satisfying its subscriber demand. Thus,

$$Y(a_G(t),i(t)) \quad = \quad \beta \max\{0, a_G(t)+i(t)-M\}, \tag{233}$$

where β is the proportionality constant. We make the natural assumption that the penalty is hefty; in particular, β is greater than or equal to the maximum possible price of a Type-O contract.

The Primary-SCT problem then is to choose the primary's trading strategy $((x_G(t), x_O(t))$, $t = 1, \ldots T$*, so as to maximize its expected revenue, expressed as*

$$\boldsymbol{E}\left(\sum_{t=1}^{T}(\alpha(T-t+1)c_G(t)x_G(t)+c_O(t)x_O(t)-Y(a_G(t),i(t)))\right), \tag{234}$$

subject to relations (231)-(233). The optimum strategy must be causal in that for each $t \in \{1, \ldots T\}$*,* $(x_G(t), x_O(t))$ *must be chosen by time* t*. Note that at time* t*,* $\{i(t'), c_G(t'), c_O(t') : t' = 1, \ldots, t\}$ *are known, but* $\{i(t'), c_G(t'), c_O(t') : t' = t+1, \ldots, T\}$ *are* not known *to the primary provider. From (231) and (232),* $x_O(t)$ *is a function of* $\{x_G(t') : t' = 1, \ldots, t\}$ *and the current demand* $i(t)$*. Therefore, the Primary-SCT problem as defined above reduces to finding the optimal* $(x_G(t), t = 1, \ldots, T)$*.*

Note that the revenue function in (234) ignores any revenue earned from the primary's subscribers. Since the subscriber demand process $i(t)$ is unaffected by the trading decisions, such revenue adds a constant offset to the revenue in (234), and therefore does not influence the optimal spectrum trading decisions.

Generalizations:

1) For a Type-O contract, the secondary provider pays the primary only for the amount

of bandwidth it uses. Thus, the expected revenue earned by a primary on selling such a contract equals the secondary's expected usage of such a channel times the market price of such a contract. We can incorporate this by considering the revenue from a Type-O contract in slot t as $\kappa c_O(t)$, where κ is the secondary's expected usage of such a channel. The formulation and the results extend to this case.

2) Our formulation and results can be extended to consider the case that $i(t)$ is only an estimate of the demand in slot t, and the estimation error in each slot is an independent, identically distributed random variable whose distribution is known to the primary. Then, $x_O(t)$ must be selected so that $M - x_O(t) - a_G(t)$ is greater than or equal to the actual demand with a desired probability. Thus, $x_O(t)$ will be a function, $K(a_G(t), i(t))$, of $(a_G(t), i(t))$, which may be different from that in (232), but can nevertheless be determined from the knowledge of the distribution of the estimation error. Also, in this case, the lack of exact knowledge of the demand will force the primary to use part or whole of the bandwidth it has sold as Type-O contracts to satisfy its demand. This will not incur any penalty for the primary owing to the nature of the contract, but will reduce the secondary's expected usage κ of each channel sold as a Type-O contract, and thereby reduce the expected amount $\kappa c_O(t)$ the secondary pays the primary for each such channel.

3) For clarity of exposition, we assumed integral demands $i(t)$. However, in practice, the demands may be fractional. For example, when a set of subscribers intermittently access the Internet on a channel, a fraction of the bandwidth on a channel is used every

slot. In this case, a Type-G or Type-O contract may be sold on the channel (while incurring a penalty proportional to the fraction used on the channel for the former). All our results apply without change in this case.

6.2.2 Polynomial-time optimal trading

We show that the Primary-SCT problem defined in Section 6.2.1 can be solved as a stochastic dynamic program (SDP) [54]. A *policy* [54] is a rule, which specifies the decisions ($x_G(t)$ and $x_O(t)$) at each slot t, as a function of the demands and prices and past decisions. Now, since the demand and prices are Markovian, the statistics of the future evolution of the system from slot t onwards are completely determined by the vector $(a_G(t-1), i(t), c_G(t), c_O(t))$, which we call the *state* at slot t, and the primary's decisions $\{x_G(t') : t' = t, \ldots, T\}$ under the policy being used. Now, in general, a policy may determine $(x_G(t), x_O(t))$ at slot t based on all past states and actions. However, a well-known result (Theorem 4.4.2 in [54]) shows that there exists an optimal policy which specifies the optimal $x_G(t)$ at any slot t only as a (deterministic) function of the current state and t [28]. We next compute such an optimal policy by solving a SDP.

For a given t, let $n = T - t + 1$ be the number of slots remaining until the end of the horizon, and $V_n(a, i, c_G, c_O)$ denote the maximum possible revenue from the remaining n slots, under any policy, when the current state is $(a_G(t-1), i(t), c_G(t), c_O(t)) =$

[28] Such a policy is called a *deterministic Markov* policy [54].

(a,i,c_G,c_O). In particular, note that $V_T(0,i,c_G,c_O)$ is the maximum possible value of the expected revenue in (234) under any policy when $i(1) = i$, $c_G(1) = c_G$ and $c_O(1) = c_O$. The function $V_n(.)$ is called the *value function* [54]. We have:

$$V_n(a,i,c_G,c_O) = \max_{0 \le x \le M-a} W_n(a,i,c_G,c_O,x), \tag{235}$$

$$\text{where } W_n(a,i,c_G,c_O,x) = \alpha(n)c_G x + J(x+a,i,c_O)$$

$$+\sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij} V_{n-1}(a+x,j,d_G,d_O), \text{ and} \tag{236}$$

$$J(a_G(t),i(t),c_O(t)) = c_O(t)K(a_G(t),i(t)) - Y(a_G(t),i(t)), \tag{237}$$

and the maximum in (235) is over integer values of x in $[0,M-a]$. Equation (235) is called *Bellman's optimality equation* [54] and holds because, by definition of $V_{n-1}(.)$, $W_n(a,i,c_G,c_O,x)$ defined by (236) is the maximum possible expected revenue when n slots remain until the end of the horizon and $x_G(t) = x$ is chosen. Note that the first two terms in (236) account for the revenue earned in slot t from the sale of Type-G and Type-O contracts minus the penalty paid. The last term in (236) is the maximum expected revenue from slot $t+1$ onwards. The summations over d_G, d_O and j take the expectation of the revenue over the prices of Type-G and Type-O contracts and the demand respectively in slot $t+1$. We get (235) by taking the maximum over all permissible values of x. Denote the (largest) x that maximizes $W_n(a,i,c_G,c_O,x)$ by $x^*_n(a,i,c_G,c_O)$. The function $x^*_n(.)$ provides the optimal solution to the Primary-SCT problem.

Now, the value function and optimal policy can be found from (235) using *backward induction* [54], which proceeds as follows. Note that $V_0(.) = 0$. Thus, $W_1(.)$ can be computed using (236), and $V_1(.)$ and $x_1^*(.)$ using (235), and similarly, $W_2(.), V_2(.), x_2^*(.), \ldots W_n(.), V_n(.), x_n^*(.$ can be successively computed. This backward induction consumes $O((N_G N_O M^2)^2 T)$ time, where N_G (N_O) is the number of states in the Markov Chain $\{c_G(t)\}$ ($\{c_O(t)\}$)– the computation time is therefore polynomial in the input size.

Remark 5. *Note that we consider a finite horizon formulation. An alternative would be to consider an infinite horizon formulation, in which a Type-G contract is valid for T slots from the time of sale (instead of until the end of horizon). But in this case, at a given slot, the state would include $(y_1^G(t), \ldots, y_T^G(t))$, where $y_j^G(t)$ is the number of Type-G contracts that are valid for j slots more. Thus, the size of the state space is $O(M^T)$, which is exponential in T. Hence, we do not consider an infinite horizon formulation.*

6.2.3 Properties of the optimal solution

We analytically prove a number of structural properties of the optimal policy, which provide insight into the nature of the optimal solution. Our results are quite general in that they hold not only for the $K(.), Y(.)$ functions defined in (232), (233), but also for any functions that satisfy the following properties (which are of course satisfied by those in (232), (233)). This loose requirement allows our results to extend to the generalizations described at the end of Section 6.2.1.

Property 4. *$K(a,i)$ decreases in a and $Y(a,i)$ increases in a for each i. Hence, by (237), for each i and c_O, $J(a,i,c_O)$ decreases in a.*

Property 5. *The $K(.), Y(.)$ functions are such that $J(a,i,c_O)$ is concave* [29] *in a for fixed i, c_O.*

Property 6. *The $K(.), Y(.)$ functions are such that, for each a, $J(a,i,c_O) - J(a+1,i,c_O)$ is an increasing function of i.*

We next state a technical assumption on the statistics of the demand and price processes that we need for our proofs.

Assumption 2. *If X_i is the demand in the next slot given that the present demand is i, or, if X_i is the price of a Type-G (respectively, Type-O) contract in the next slot given that the present price is i, then for $i \leq i'$, $X_i \leq_{st} X_{i'}$ (X_i is* stochastically smaller *[57] than $X_{i'}$),* i.e.*, for each $b \in R$, $Pr(X_i > b) \leq Pr(X_{i'} > b)$.*

Intuitively, this assumption says that the primary's demand and the prices do not fluctuate very rapidly, and the demand (or price) in the next slot is more likely to be high when the current demand (or price) is high as opposed to when the current demand (or price) is low.

We are now ready to state the structural properties of the optimum trading policy. We defer the proofs of these properties until Section 6.5.1.

The first property identifies the relation between $x_n^*(a,i,c_G,c_O)$ and a:

Theorem 12. *For each n, i, c_G, c_O,*

$$x_n^*(a+1,i,c_G,c_O) = \max(x_n^*(a,i,c_G,c_O) - 1, 0). \tag{238}$$

[29] A function $f(k)$ with domain being a subset of the integers is concave [7] if $f(k+2) - f(k+1) \leq f(k+1) - f(k)$ for all k [57]. If the inequality is reversed, $f(.)$ is convex.

Intuitively, this theorem suggests that for each n, i, c_G, c_O, there exists an optimal portfolio level of Type-G contracts, $a_G^*(t)$, such that if $a_G(t-1) = a$, then $x_G(t)$ should be chosen so as to make $a_G(t) = a_G^*(t)$. That is, the optimal $x_G(t) = a_G^*(t) - a$ (if the latter is non-negative).

Also, due to Theorem 12, for each n, i, c_G and c_O, it is sufficient to find $x_n^*(a, i, c_G, c_O)$ only for $a = 0$ while performing backward induction, and $x_n^*(a, i, c_G, c_O)$ for other a can be deduced using (238). This reduces the overall computation time by a factor of M: the optimal policy can now be computed in $O((N_G N_O)^2 M^3 T)$ time.

The next two results identify the nature of the dependence between $x_n^*(a, i, c_G, c_O)$ and the demand i and prices c_G, c_O.

Theorem 13. *For each n, a, c_G and c_O, $x_n^*(a, i, c_G, c_O)$ is monotone decreasing in i.*

Theorem 13 confirms the intuition that when the primary's demand is high, it should sell fewer Type-G contracts so as to reserve bandwidth to meet its demand and vice versa. At the same time, note that this result is not obvious– when the demand is lower, more free bandwidth is available, which can be sold as Type-G *or as Type-O contracts*. Clearly, the number of Type-G versus Type-O contracts sold would influence the states reached in the future and the revenue earned. Theorem 13 asserts that the primary should sell at least as many Type-G contracts as before (that is, as for the high demand state), while possibly also increasing the number of Type-O contracts to sell.

Theorem 14. *$x_n^*(a, i, c_G, c_O)$ is monotone increasing in c_G for fixed n, a, i, c_O and monotone decreasing in c_O for fixed n, a, i, c_G.*

Theorem 14 confirms the intuition that the primary should preferentially sell the type of contract (G or O) with a "high" price.

Remark 6. *Theorems* 2 *and* 3 *can be used to speed up the computation of the optimal policy using the* monotone backward induction *algorithm [54]. Similarly, in Theorem 21 (in Section 6.5.1), we prove that the value function is concave in a for fixed* n, i, c_G, c_O, *which can be used to speed up the computation of* $x_n^*(.)$ *from the value function since the maximizer in (235) can be found in* $O(\log M)$ *time using a binary search like algorithm [25]. In both cases, the worst case asymptotic running time remains the same, although substantial savings in computation can be obtained in practice.*

6.2.4 SCT across multiple locations

We now consider spectrum contract trading across multiple locations from a primary provider's point of view. Wireless transmissions suffer from the fundamental limitation that the same channel can not be successfully used for simultaneous transmissions at neighboring locations, but can support simultaneous transmissions at geographically disparate locations. Thus, a primary provider can not trade contracts in the same channel at neighboring locations, but can do so at far off locations. Hence, the spectrum contract trading problem at different locations is inherently coupled, and must be optimized jointly. We now extend the problem formulation to consider the case of multiple locations, taking into account possible interference relationships between adjacent regions.

We model the overall region under consideration using an undirected graph $\mathcal{G}$ with the set of nodes S. Each node represents a certain area at some location in the overall region. There is an edge between two nodes if and only if transmissions at the corresponding locations on the same channel interfere with each other. A primary provider owns M channels throughout the region. At any time slot, at a given node, on each channel (a) either a Type-G contract can be sold, (b) a Type-O contract can be sold or (c) no contract can be sold, subject to the constraint that at no point in time, a contract can stand leased at neighbors on the same channel. That is, on each channel, the set of nodes at which a contract stands leased constitutes an independent set [71].

A primary provider needs to satisfy its subscriber demand which is also subject to certain reuse constraints. We consider the case where the subscribers of a primary provider require broadcast transmissions. This, for example, happens when the primary is a TV transmitter that broadcasts signals across all locations over different channels. At any given slot t, the primary needs to broadcast over a certain number, say $i(t)$, channels which randomly varies with time depending on subscriber demands. Whenever the primary broadcasts on a channel, the broadcast reaches all nodes, and thus the channel can not be used by the secondaries at any node. Hence, if the primary has sold a Type-G contract on the channel at any node it incurs a penalty of β at the node. Thus, at slot t, $i(t)$ represents the primary's demand at all nodes. Note that the set of nodes at which the primary uses a given channel for demand satisfaction does not constitute an independent set (as opposed to the set of nodes at which contracts stand leased). Also, the primary's usage status on any given channel at any given time (*i.e.*, whether or not the primary is using the channel for subscriber demand satisfaction) is the same across all nodes.

The durations of Type-G and Type-O contracts are as described in Section 6.2.1. We assume that at any slot t, Type-G (respectively, Type-O) contracts have equal prices $c_G(t)$ (respectively, $c_O(t)$) at all nodes. The processes $(i(t), c_G(t), c_O(t))$ evolve as per independent DTMCs as stated in Section 6.2.1.

The spectrum contract trading problem across multiple locations for a primary (Primary-SCTM) is to optimally choose at each slot t, the type of contract to sell (if any) at each location on each channel so as to maximize the total expected revenue from all nodes over a finite horizon of T slots.

Theorem 15. *Primary-SCTM is NP-Hard.*

The proof is deferred until Section 6.5.2.

We now characterize the optimal solution of the Primary-SCTM problem.

Lemma 51. *Consider the class of policies $\mathcal{F}$, such that a policy $f \in \mathcal{F}$ operates as follows. At the beginning of the horizon, it finds a maximum independent set, $I(S)$, in $\mathcal{G}$. Then, in each slot, it sells contracts only at nodes in $I(S)$. There exists a policy in $\mathcal{F}$ that optimally solves the Primary-SCTM problem.*

The proof is deferred until Section 6.5.2.

We refer to a policy in $\mathcal{F}$, which at each node in $I(S)$, sells contracts according to the optimal solution of the Primary-SCT problem with demand and price processes $\{i(t), c_G(t), c_O(t)\}$ as a *Separation Policy*.

Theorem 16 (Separation Theorem)**.** *A Separation Policy optimally solves the Primary-SCTM problem.*

Proof of Theorem 16. By Lemma 51, we can restrict our search for an optimal policy to the policies in $\mathcal{F}$. Now, the total revenue of a policy in $\mathcal{F}$ is the sum of the revenues at the nodes in $I(S)$. Clearly, the total revenue is maximized if the stochastic dynamic program for the single node case is executed at each node. Note that this solution satisfies the interference constraints since $I(S)$ is an independent set. $\square$

Note that the optimum solution at any node can be computed in polynomial time using the SDP presented in Section 6.2.1. However, computation of a maximum size independent set is an NP-hard problem [35]. This computation therefore seems to be the basis of the NP-hardness of Primary-SCTM. Also, the following theorem, which is a direct consequence of Theorem 16, shows that Primary-SCTM can be approximated in polynomial time within a factor of μ if the maximum independent set problem can be approximated in polynomial time within a factor of μ.

Theorem 17 (Approximate Separation Theorem)**.** *Consider a μ-separation policy that differs from a separation policy in that it sells contracts as per the single node optimum solution, at each node of an independent set whose size is at least $\frac{1}{\mu}$ times that of a maximum independent set. This policy's expected revenue is at least $\frac{1}{\mu}$ times the optimal* *expected revenue.*

However, in a graph with N nodes, the maximum size independent set problem can not in general be approximated to within a factor of $O(N^{\varepsilon})$ for some $\varepsilon > 0$ in polynomial time unless $P = NP$ [3]. Nevertheless, *polynomial time approximation algorithms* (PTAS) i.e., algorithms that compute an independent set whose size is within $(1-\varepsilon)$ of the maximum size independent set, for any given $\varepsilon > 0$, using a computation time of $O(N^{1/\varepsilon})$ are known in important special cases, e.g., when the degree of each node is upper-bounded [4] (this happens in our case when the number of locations each location interferes with is upper-bounded). Thus, in view of Theorem 17, for any given $\varepsilon > 0$, the Primary-SCTM problem can be approximated within a factor of $1-\varepsilon$ using a computation time of $O(N^{1/\varepsilon})$ in such graphs.

6.3 Secondary's Spectrum Contract Trading Problem

In this section we pose and address *Secondary-SCT*, the spectrum contract trading question from a secondary provider's (buyer's) perspective. First note that the Secondary-SCT problem need not consider the interference constraints for channels since the secondary provider buys the spectrum bands that are offered in the market (presumably in a manner that satisfies the reuse constraints), and also because they are usually localized (i.e., operate in small regions). Thus, the secondary's spectrum trading decisions in different regions can be separately optimized. So henceforth in this section, we restrict ourselves to the case of a single location.

6.3.1 Formulation

We consider an arbitrary secondary provider that is interested in buying contracts in the secondary spectrum market. Our assumptions regarding the optimization horizon T, the durations of Type-G and Type-O contracts and their price processes $(c_G(t), c_O(t))$ remain the same as in Section 6.2.1. Let $\tilde{i}(t)$ denote the subscriber demand of the provider at time t– it is a DTMC similar to $\{i(t)\}$ in Section 6.2.1, but with transition probabilities P_{ij} in place of Q_{ij}.

The secondary decides the number of Type-G and Type-O contracts it will buy (from primary providers) at slot t, $(\tilde{x}_G(t), \tilde{x}_O(t))$, after it learns the market prices $c_G(t)$ and $c_O(t)$ and the demand level $\tilde{i}(t)$ at t. We continue to assume that the market has infinite liquidity, which now implies that the market has a lot of sellers (i.e., primary providers), and hence the secondary can buy as many contracts of any type by paying their market price. Let $(\tilde{a}_G(t), \tilde{x}_O(t))$ denote the spectrum contract portfolio held by the secondary during slot t, where $\tilde{a}_G(t)$ denotes the number of Type-G contracts that the secondary has leased out until time t. Then we have

$$\tilde{a}_G(t) \;=\; \sum_{t' \leq t} \tilde{x}_G(t'). \tag{239}$$

The secondary provider's spectrum trading goal is to meet its time-varying subscriber demand in every time slot at the minimum cost, by choosing an appropriate portfolio of Type-G and Type-O contracts, $\{(\tilde{a}_G(t), \tilde{x}_O(t))\}$, adjusted dynamically.

Note that there are uncertainties on how much bandwidth the secondary actually ends up getting from each contract at a time t during its duration, since a Type-O contract only allows the secondary the right to use the channel when the owner (primary) is not using it, and there is a non-zero probability of contract violation for a Type-G contract by the primary due to its subscriber demand level plus the number of Type-G contracts sold exceeding its total owned spectrum (see the Primary-SCT formulation in Section 6.2). Due to this, the subscriber demand $\tilde{i}(t)$ can be met only in statistical terms, e.g., in expectation, or with a certain probability, by any spectrum contract portfolio. (We assume that statistics on such contract violations are available (possibly from historical data) to the buyers, and can be incorporated in the corresponding contract trading decision.) We generalize this notion by associating with each value of subscriber demand δ, a *demand satisfaction set* $\mathcal{F}_\delta$ within which a spectrum contract portfolio $(\tilde{a}_G, \tilde{x}_O)$ must lie for meeting the demand level δ satisfactorily. A portfolio $(\tilde{a}_G(t), \tilde{x}_O(t))$ is said to be *demand-satisfactory* at time t if it can meet the demand level at time t satisfactorily, i.e., if $(\tilde{a}_G(t), \tilde{x}_O(t)) \in \mathcal{F}_{\tilde{i}(t)}$.

Thus, the Secondary-SCT problem is to minimize the expected contract trading cost subject to the spectrum contract portfolio being demand-satisfactory at all times t. The objective is thus to minimize

$$\boldsymbol{E}\left(\sum_{t=1}^{T}\left(\alpha(T-t+1)c_G(t)\tilde{x}_G(t)+c_O(t)\tilde{x}_O(t)\right)\right), \tag{240}$$

subject to (239) and

$$(\tilde{a}_G(t), \tilde{x}_O(t)) \quad \in \quad \mathcal{F}_{\tilde{i}(t)},\ \forall t, \tag{241}$$

and such that for each $t \in \{1, \ldots T\}$, $(\tilde{x}_G(t), \tilde{x}_O(t))$ must be chosen by time t. Note that *at time t,* $\{\tilde{i}(t'), c_G(t'), c_O(t') : t' = 1, \ldots, t\}$ *are known, but* $\{\tilde{i}(t'), c_G(t'), c_O(t') : t' = t+1, \ldots, T\}$ *are* not *known.*

We assume that the sets $\mathcal{F}_\delta$ for different δ are given. Typically, we will have $\mathcal{F}_{\delta'} \subseteq \mathcal{F}_\delta$ for $\delta \leq \delta'$. Also, we make the natural assumption that if $(\tilde{a}_G, \tilde{x}_O) \in \mathcal{F}_\delta$ for some δ, then $(\tilde{a}_G, \tilde{x}'_O) \in \mathcal{F}_\delta\ \forall \tilde{x}'_O \geq \tilde{x}_O$. Accordingly, let $L(\tilde{a}_G(t), \tilde{i}(t))$ be the minimum number of Type-O contracts $\tilde{x}_O$ required for a portfolio $(\tilde{a}_G(t), \tilde{x}_O)$ to be in $\mathcal{F}_{\tilde{i}(t)}$, for a given $(\tilde{a}_G(t), \tilde{i}(t))$. It is easy to see that for a given $(\tilde{a}_G(t), \tilde{i}(t))$, it is optimal to select $\tilde{x}_O = L(\tilde{a}_G(t), \tilde{i}(t))$ (not more).

For example, suppose the secondary seeks to meet the current demand level in expectation. Due to the uncertain amount of bandwidth available on Type-G and Type-O contracts, suppose the expected amount of bandwidth obtained from a Type-G contract is γ $(0 < \gamma \leq 1)$. Also, η Type-O contracts are required, on average, to meet one unit of demand, where η is a positive integer. For simplicity, assume that the product $\gamma\eta$ is an integer. Then:

$$L(\tilde{a}_G(t), \tilde{i}(t)) = \max\left\{\eta(\tilde{i}(t) - \gamma\tilde{a}_G(t)), 0\right\} \tag{242}$$

Remarks: 1) Note that in (240), we do not consider the revenue earned from the penalties paid by the primary due to Type-G contract violations. Such penalties lead to a net decrease in the price of a Type-G contract, and their effects can be incorporated by considering the price process of Type-G contracts as $\{\tilde{c}_G(t)\}$, where $\tilde{c}_G(t) = c_G(t) - \kappa(t)$, where $\kappa(t)$ is i.i.d and independent of $\{c_G(t)\}$. Subsequent formulations and analysis do not change owing to the above modification.

2) Like for the Primary-SCT problem, our results can be extended to the case where the secondary knows only an estimate of $\tilde{i}(t)$ at the beginning of time slot t.

3) Like for the Primary-SCT problem, the cost function in (240) ignores any revenue earned from the secondary's subscribers. Since the subscriber demand process $\tilde{i}(t)$ is unaffected by the trading decisions, such revenue adds a constant offset to the cost in (240), and therefore does not influence the optimal spectrum trading decisions.

6.3.2 Analysis

We formulate the secondary's problem as a stochastic dynamic program (SDP) and prove a number of structural properties of the optimal solution. The formulation and analysis are very similar to that for the primary; hence we only provide a brief outline.

Let $(\tilde{a}_G(t-1), \tilde{i}(t), c_G(t), c_O(t))$ be the state at the beginning of slot t, $n = T - t + 1$ and $V_n(a, i, c_G, c_O)$ denote the value function, *i.e.*, the minimum possible cost over the remaining slots, starting from slot t. In particular, note that $V_T(0, i, c_G, c_O)$ is the minimum possible value of the expected cost in (240) under any policy when $\tilde{i}(1) = i$, $c_G(1) = c_G$ and $c_O(1) = c_O$. Then the optimality equation is given by:

$$V_n(a, i, c_G, c_O) = \min_x W_n(a, i, c_G, c_O, x) \tag{243}$$

where

$$W_n(a, i, c_G, c_O, x) = \alpha(n) c_G x + c_O L(x + a, i)$$
$$+ \sum_{d_G} \sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j P_{ij} V_{n-1}(a + x, j, d_G, d_O) \tag{244}$$

and the minimum in (243) is over nonnegative integer values of x. Denote the (smallest) x that minimizes $W_n(a,i,c_G,c_O,x)$ by $\tilde{x}_n^*(a,i,c_G,c_O)$. The value function and optimal policy can be found from (243) using *backward induction* [54] in $O((N_G N_O D^2)^2 T)$ time, where D is the number of states in the Markov Chain $\{\tilde{i}(t)\}$.

We now identify the structure of the optimal trading strategy $\{\tilde{x}_n^*(a,i,c_G,c_O), n = 1,\ldots,T\}$ for the following properties of the $L(.)$ function, which are analogous to Properties 4, 5 and 6 of the $J(.)$ function for the Primary-SCT problem. (i) For each i, $L(a,i)$ decreases in a, (ii) $L(a,i)$ is convex in a for fixed i, (iii) For each a, $L(a,i) - L(a+1,i)$ is an increasing function of i. It can be checked that these properties are true for the function $L(.)$ in (242). We also assume that the price and demand processes satisfy Assumption 2.

We have the following structural results, which closely parallel Theorems 12 to 14. The proofs are similar to those of Theorems 12 to 14, and hence omitted.

Theorem 18. *For each n,i,c_G,c_O, $\tilde{x}_n^*(a+1,i,c_G,c_O) = \max\left(\tilde{x}_n^*(a,i,c_G,c_O) - 1, 0\right)$.*

Theorem 19. *For each n, a, c_G and c_O, $\tilde{x}_n^*(i,a,c_G,c_O)$ is monotone increasing in i.*

Theorem 20. *$\tilde{x}_n^*(a,i,c_G,c_O)$ is monotone decreasing in c_G for fixed n,a,i,c_O and monotone increasing in c_O for fixed n,a,i,c_G.*

6.4 Numerical Studies

We next study the properties of the optimal trading strategy using numerical investigations, and explore how the expected revenue varies as a function of key system parameters. Due to the similarity in the results for Primary-SCT and Secondary-SCT, we only present our results for the former. We consider $M = 20$ channels, penalty parameter $\beta = 3.0$ and a birth-death demand process with 21 states and integral state values $\{0, 1, \ldots, 20\}$. The price process c_G (c_O) is again a birth-death process that varies between 1.0 and 4.0 (1.0 and 2.0, respectively) with a total of 10 uniformly-spaced states. For both the demand and price processes, we assume that the forward and backward transition probabilities equal p (a parameter).

In Theorems 13 and 14, we have established the monotonicity properties of the optimal solution $x_n^*(a, i, c_G, c_O)$ with respect to the demand level i and prices c_G, c_O. Recall that $n = T - t + 1$ at slot t, and represents the duration of a Type-G contract made at slot t. Now, our numerical evaluations suggest that the optimal solution $x_n^*(.)$ is decreasing in n, and when n is close to T, $x_n^*(.)$ is zero (see Figure 6.1). Thus, the primary prefers Type-G contracts towards the end of the optimization horizon, and Type-O towards the beginning. This is because when n is close to T, Type-G contracts are very long-term, and hence likely to incur hefty penalties since demand and prices may be difficult to predict long-term.

The two plots in Figure 6.2 show the variation in the primary's average (expected) revenue per slot with respect to p and T. For these results, the initial state for the demand and price processes are chosen according to the steady state distributions of these processes. The average revenue obtained from the optimal dynamic trading strategy is compared with that of an optimal *static* strategy. In the latter strategy, the number of Type-G contracts is chosen only once (optimally, based on the steady state distribution of the demand and price processes), at the very beginning of the time horizon; the number of Type-O contracts made is adjusted dynamically to the amount of "free bandwidth" available at any slot (i.e., the number of channels minus the sum of the demand and Type-G contracts made). We observe that the average revenue for the optimal static strategy is invariant to changes in p or T – this happens because the initial states for the demand and price processes follow their steady state distributions, which in our case is uniform and does not depend on p or T. We observe that the optimal dynamic contract trading strategy significantly outperforms the optimal static strategy, demonstrating the benefits of dynamic choice of the number of Type-G contracts. Note that if the static strategy buys a Type-G contract, it must buy one that is really long-term (i.e., one that lasts for the entire T slots), whereas the dynamic strategy can choose the duration of Type-G contracts it buys by deciding when they are purchased, based on its demand and prices of the contracts that evolve dynamically. The figures also show that the primary's average revenue per slot under dynamic choice increases with an increase in p and T (for the same value of the other parameters). Note that a larger p (respectively, larger T) implies larger temporal variation in the prices (respectively, a longer optimization horizon), giving the primary more opportunities in which the price of a Type-G con-

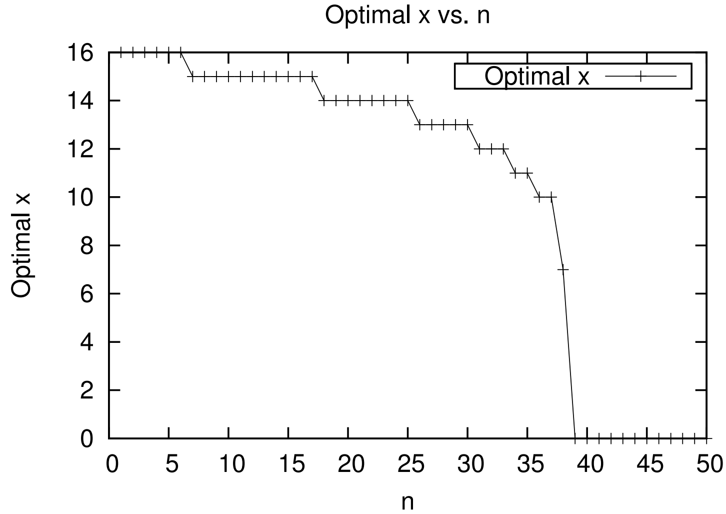

Figure 6.1: $x_n^*(a, i, c_G, c_O)$ versus n for $a = 0$, $i = 4$, $c_G = 2.0, c_O = 1.0$ and $T = 50$.

tract is high and the primary can "lock in" a good price for a contract. From the bottom plot in Figure 6.2, we also observe that the average per-slot revenue shows diminishing returns as T increases, and appears to stabilize eventually (at a faster rate for a larger p). This is intuitive since the revenue earned per unit time is upper bounded, and also because very long-term Type-G contracts offer small returns.

6.5 Appendix

6.5.1 Proofs of results in Section 6.2.3

Notation: Let R denote the set of real numbers.

Let X_i be as in Assumption 2. Recall that Q_{ij}, H_{ij}^G and H_{ij}^O are the transition probabilities of the demand and the prices of Type-G and Type-O contracts respectively. So,

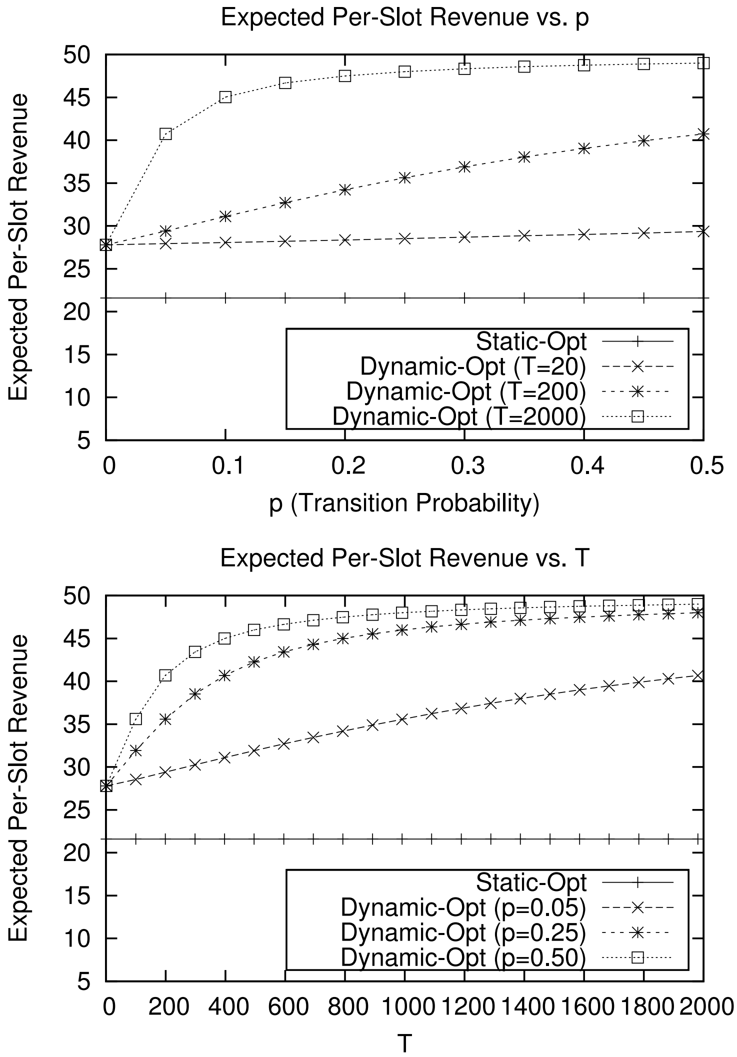

Figure 6.2: The top plot shows the average per-slot revenue vs transition probability p. The bottom plot shows the average per-slot revenue vs time horizon T.

if X_i represents the demand, price of a Type-G contract or price of a Type-O contract respectively in the next slot given that the present demand, price of a Type-G contract or price of a Type-O contract equals i, then for a function $f(.)$, $E(f(X_i))$ equals $\sum_j Q_{ij} f(j)$, $\sum_j H^G_{ij} f(j)$ and $\sum_j H^O_{ij} f(j)$ respectively. The assumption $X_i \leq_{st} X_{i'}$ for $i \leq i'$

in Assumption 2 is equivalent to the following condition [57]:

Condition 1. *For every increasing function $f(i)$,*

$$E(f(X_i)) \leq E(f(X_{i'})) \ \forall i \leq i'$$

i.e., $\sum_j Q_{ij} f(j)$, $\sum_j H^G_{ij} f(j)$ *and* $\sum_j H^O_{ij} f(j)$ *are increasing functions of i.*

Note that in the summations in Condition 1, as well as in those in the rest of this section, the summation is over all possible states of the respective Markov Chain.

6.5.1.1 Proof of Theorem 12

We first prove that the value function is concave in a (Theorem 21). Then, using Theorem 21, we prove Theorem 12. We start with a simple lemma, which is used in the proof of Theorem 21.

Lemma 52. *For fixed i, c_G, c_O, $V_n(a,i,c_G,c_O)$ decreases in a.*

Proof. We prove the result by induction. Let $V_0(a,i,c_G,c_O) = 0$. Then the claim is true for $n = 0$. Suppose $V_{n-1}(a,i,c_G,c_O)$ decreases in a for each i, c_G, c_O. Now, let $a_1 \geq 1$ and $x^*_n(a_1,i,c_G,c_O) = x_1$ for some x_1. Then, by (235):

$$V_n(a_1,i,c_G,c_O) = W_n(a_1,i,c_G,c_O,x_1) \tag{245}$$

Now,

$$
\begin{aligned}
& V_n(a_1-1,i,c_G,c_O) \\
\geq\ & W_n(a_1-1,i,c_G,c_O,x_1) \text{ (by (235))} \\
=\ & \alpha(n)c_Gx_1+J(x_1+a_1-1,i,c_O) \\
& +\sum_{d_G}\sum_{d_O}H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}V_{n-1}(a_1+x_1-1,j,d_G,d_O) \\
\geq\ & \alpha(n)c_Gx_1+J(x_1+a_1,i,c_O) \\
& +\sum_{d_G}\sum_{d_O}H^G_{c_Gd_G}H^O_{c_Od_O}\sum_j Q_{ij}V_{n-1}(a_1+x_1,j,d_G,d_O) \\
& \text{(by induction hypothesis and Property 4)} \\
=\ & W_n(a_1,i,c_G,c_O,x_1) \\
=\ & V_n(a_1,i,c_G,c_O) \ \text{ (by (245))}
\end{aligned}
$$

The result follows. $\square$

Theorem 21. *For each n, $V_n(a,i,c_G,c_O)$ is concave in a for fixed i, c_G, c_O.*

Proof. We prove the result by induction. $V_0(a,i,c_G,c_O)$ is concave in a since it is equal to 0. Suppose $V_{n-1}(a,i,c_G,c_O)$ is concave in a for fixed i,c_G,c_O. Recall that $V_{n-1}(a,i,c_G,c_O)$ is defined for integer values of a. Now, for fixed i, c_G and c_O, define $\tilde{V}_{n-1}(a,i,c_G,c_O)$ for a real as the function obtained by linearly interpolating $V_{n-1}(a,i,c_G,c_O)$ between each pair of adjacent integers a_0 and a_0+1. Similarly, define $\tilde{J}(a,i,c_O)$.

Now, $J(x+a,i,c_O)$ (respectively, $V_{n-1}(x+a,i,c_G,c_O)$) is concave decreasing in $x+a$ for fixed i,c_O (respectively, for fixed i,c_G,c_O) by Properties 4 and 5 (respectively, by Lemma 52 and induction hypothesis). Hence, we get:

Property 7. *$\tilde{J}(x+a,i,c_O)$ (respectively, $\tilde{V}_{n-1}(x+a,i,c_G,c_O)$) is concave decreasing in $x+a$ for fixed i,c_O (respectively, for fixed i,c_G,c_O).*

Now, consider the function

$$\tilde{W}_n(a,i,c_G,c_O,x) = \alpha(n)c_G x + \tilde{J}(x+a,i,c_O)$$
$$+\sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}\tilde{V}_{n-1}(a+x,j,d_G,d_O) \tag{246}$$

as a function of the two real variables a,x, *i.e.* the vector (a,x).

Recall the following property of composition of functions [7]:

Property 8. *Let $h: R \to R$, $g: R^k \to R$, where $k \geq 1$ and R^k denotes the k-dimensional Euclidean space. Let $f: R^k \to R$ be defined by $f(\mathbf{v}) = h(g(\mathbf{v}))$. If $h(.)$ is concave and decreasing, and $g(\mathbf{v})$ is convex in $\mathbf{v}$, then $f(\mathbf{v})$ is concave in $\mathbf{v}$.*

By the fact that $a+x$ is linear and hence [7] convex in (a,x), Property 7 and Property 8, it follows that $\tilde{J}(x+a,i,c_O)$ (respectively, $\tilde{V}_{n-1}(a+x,j,d_G,d_O)$) is concave in (a,x) for fixed i,c_O (respectively, for fixed j,d_G,d_O). Also, x is clearly concave in (a,x). Hence, $\tilde{W}_n(a,i,c_G,c_O,x)$ being a nonnegative weighted linear combination of these functions, is concave in (a,x) for fixed i,c_G,c_O.

Now, define:

$$\tilde{V}_n(a,i,c_G,c_O) = \sup_{x\in R, 0\leq x\leq M-a} \tilde{W}_n(a,i,c_G,c_O,x) \tag{247}$$

Note that $\{x : x \in R, 0 \leq x \leq M-a\}$ is a non-empty convex set. Recall the following property [7]:

Property 9. *If $f(a,x)$ is concave in (a,x) and C is a convex nonempty set, then the function*

$$g(a) = \sup_{x \in C} f(a,x)$$

is concave in a, provided $g(a) < \infty$ for some a.

Now, $\tilde{V}_n(a,i,c_G,c_O) < \infty$ (since the costs of Type-G and Type-O contracts are upper bounded). So by (247), Property 9 and the fact that $\tilde{W}_n(.)$ is concave in (a,x), $\tilde{V}_n(a,i,c_G,c_O)$ is concave in a for fixed i,c_G,c_O.

Now, we will show that $V_n(a,i,c_G,c_O) = \tilde{V}_n(a,i,c_G,c_O)$ for a integer, which will imply that $V_n(a,i,c_G,c_O)$ is concave.

Fix i,c_G,c_O and an integer a. Note that by (235) and (247) and since $\tilde{W}_n(.) = W_n(.)$ at integer a and x, $V_n(a,i,c_G,c_O)$ is the maximum of $\tilde{W}_n(a,i,c_G,c_O,x)$ over integer x, whereas $\tilde{V}_n(a,i,c_G,c_O)$ is the supremum over real x in the range $[0, M-a]$. Hence, to prove that $V_n(a,i,c_G,c_O) = \tilde{V}_n(a,i,c_G,c_O)$, it will suffice to show that the supremum over real x occurs at integer x.

Now, by the definition of the functions $\tilde{J}(.)$ and $\tilde{V}_{n-1}(.)$, $f(x) = \tilde{W}_n(a,i,c_G,c_O,x)$ is continuous and piecewise linear in x, with breakpoints at the integers. Also, note that the endpoints of the domain of $f(x)$, viz. 0 and $M-a$ are integers that are contained in the domain. As a result, it can be checked that the maximum of $f(x)$ must occur at an integer. This completes the proof. $\square$

Note that $W_n(a,i,c_G,c_O,x)$ is concave in (a,x) and $V_n(a,i,c_G,c_O)$ is the maximum of $W_n(.)$ over a *non-convex* set, namely the set of integers in $[0, M-a]$. This makes the above proof more involved, since had the maximum been over a convex set, the concavity of $V_n(a,i,c_G,c_O)$ would have simply followed from Property 9.

We are now ready to prove Theorem 12.

Proof of Theorem 12. From (236), we have:

$$W_n(a,i,c_G,c_O,x) = W_n(a+1,i,c_G,c_O,x-1) + \alpha(n)c_G, \ \forall x \geq 1 \tag{248}$$

Now, by optimality of $x_n^*(a,i,c_G,c_O)$:

$$W_n(a,i,c_G,c_O,x_n^*(a,i,c_G,c_O)) \geq W_n(a,i,c_G,c_O,x) \ \forall x \geq 1 \tag{249}$$

If $x_n^*(a,i,c_G,c_O) \geq 1$, then from (248) and (249) and some algebra, we get:

$$W_n(a+1,i,c_G,c_O,x_n^*(a,i,c_G,c_O)-1) \geq W_n(a+1,i,c_G,c_O,x-1) \ \forall x \geq 1$$

which shows that $x_n^*(a+1,i,c_G,c_O) = x_n^*(a,i,c_G,c_O) - 1$ if $x_n^*(a,i,c_G,c_O) \geq 1$.

Now, suppose $x_n^*(a,i,c_G,c_O) = 0$. By Theorem 21 and Property 5, since $V_{n-1}(a+x,j,d_G,d_O)$ and $J(x+a,i,c_O)$ are concave in x for fixed a,j,d_G,d_O,i,c_O, it follows from (236) that $W_n(a,i,c_G,c_O,x)$ is concave in x. For $x \geq 2$, we have:

$$\begin{aligned}
& W_n(a+1,i,c_G,c_O,x-1) - W_n(a+1,i,c_G,c_O,0) \\
= \;& W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,1) \text{ (by (248))} \\
\leq \;& W_n(a,i,c_G,c_O,x-1) - W_n(a,i,c_G,c_O,0) \\
& \text{(by concavity)} \\
\leq \;& 0 \text{ (since } x_n^*(a,i,c_G,c_O) = 0)
\end{aligned}$$

which shows that $x_n^*(a+1, i, c_G, c_O) = 0$. $\square$

6.5.1.2 Proofs of Theorems 13 and 14

The proofs of Theorems 13 and 14 are based on the concepts of submodularity and supermodularity, which we briefly review. Let $I \subseteq R$ and $X \subseteq R$ be two sets. A function $g(i,x) : I \times X \to R$ is called *supermodular* [54] if for $i^+ \geq i^-$ in I and $x^+ \geq x^-$ in X,

$$g(i^+, x^+) + g(i^-, x^-) \geq g(i^+, x^-) + g(i^-, x^+)$$

If the inequality is reversed, g is called *submodular* [54].

We will require the following key result [54].

Theorem 22. *If $g(i,x)$ is supermodular (submodular) on $I \times X$, then the (largest) maximizer of $g(i,x)$ for a given i:*

$$f(i) = \max\{x' : x' \in \underset{x}{\operatorname{argmax}}\, g(i,x)\}$$

is increasing (decreasing) in i.

To prove Theorem 13, we show that $W_n(a, i, c_G, c_O, x)$ is submodular in (i,x). The monotonicity of $x_n^*(a, i, c_G, c_O)$ in i then follows from Theorem 22. First, we prove some lemmas.

The following lemma provides a necessary and sufficient condition for submodularity.

Lemma 53. *Let $g(i,x)$ be a function with domain being integer values of x and real values of i. $g(i,x)$ is submodular in (i,x) if and only if $g(i,x) - g(i,x+1)$ is an increasing* *function of i for all x.*

Proof. The necessity directly follows from the definition of submodularity. We now prove sufficiency. Suppose $g(i,y) - g(i,y+1)$ is an increasing function of i for all y. For an integer $z > 0$:

$$g(i,x) - g(i,x+z) = [g(i,x) - g(i,x+1)] + \ldots + [g(i,x+z-1) - g(i,x+z)]$$

So $g(i,x) - g(i,x+z)$, being the sum of increasing functions, is increasing in i.

Hence, for $x^- < x^+$, $g(i,x^-) - g(i,x^+)$ is increasing in i. So for $i^- < i^+$:

$$g(i^-,x^-) - g(i^-,x^+) \leq g(i^+,x^-) - g(i^+,x^+)$$

Hence, $g(i,x)$ is submodular in (i,x) by definition. □

For $m \geq 1$, define [30]

$$i_n^m(a,c_G,c_O) = \max\{i : x_n^*(a,i,c_G,c_O) \geq m\}. \tag{250}$$

Lemma 54. *If $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i, then*

$$i_n^1(a,c_G,c_O) \geq i_n^2(a,c_G,c_O) \geq \ldots \geq i_n^{M-a}(a,c_G,c_O)$$

Also, $x_n^*(a,i,c_G,c_O) = m$ if and only if $i_n^m(a,c_G,c_O) \geq i > i_n^{m+1}(a,c_G,c_O)$.

Proof. The result follows by definition of $i_n^m(.)$. □

The next lemma establishes a sufficient condition for monotonicity of $x_n^*(i,a,c_G,c_O)$.

[30] If $x_n^*(a,i,c_G,c_O) < m \; \forall i$, then let $i_n^m(a,c_G,c_O)$ be equal to the smallest demand state.

Lemma 55. *Fix n. Suppose $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each a, d_G and d_O. Then $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i for each a,c_G and c_O.*

It is important to note that the lemma requires $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ to be increasing in *j for a fixed n*, and asserts that $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in *i for that n.*

Proof. By (236):

$$
\begin{aligned}
&W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,x+1) \\
&= -\alpha(n)c_G + [J(a+x,i,c_O) - J(a+x+1,i,c_O)] \\
&+ \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}\,(V_{n-1}(a+x,j,d_G,d_O) \\
&\qquad - V_{n-1}(a+x+1,j,d_G,d_O))
\end{aligned}
$$

The first term on the right hand side is constant, the second term is increasing in i by Property 6 and the third term is increasing in i since $V_{n-1}(a+x,j,d_G,d_O) - V_{n-1}(a+x+1,j,d_G,d_O)$ is increasing in j and by Condition 1.

So $W_n(a,i,c_G,c_O,x) - W_n(a,i,c_G,c_O,x+1)$ is increasing in i. Hence, by Lemma 53, $W_n(a,i,c_G,c_O,x)$ is submodular in (i,x) and so by Theorem 22, $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i. □

The next lemma is a simple consequence of (238).

Lemma 56. *Fix n. If $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i for each a,c_G,c_O, then $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ for $m = 1,2,\ldots$.*

Proof. Fix c_G and c_O, and let $m \geq 1$. Separately with a and with $a+1$, start with $i = M$ (the highest demand state) and keep decreasing it to the next lower state, one at a time. By (238), the maximum i at which $x_n^*(a,i,c_G,c_O) \geq m+1$ is precisely the maximum i at which $x_n^*(a+1,i,c_G,c_O) \geq m$. So $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ by definition of $i_n^m(.)$. □

Lemma 57. *For each n, $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is an increasing function of i for each a,c_G,c_O.*

Proof. We prove the claim by induction. Since $V_0(a,i,c_G,c_O) \equiv 0$, the claim is true for $n = 0$.

Suppose the statement is true for $n-1$, *i.e.*, $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each a,d_G,d_O. Then by Lemma 55, $x_n^*(a,i,c_G,c_O)$ is monotone decreasing in i. Hence, by Lemma 56, $i_n^{m+1}(a,c_G,c_O) = i_n^m(a+1,c_G,c_O)$ for $m = 1,2,\ldots$.

Now, we show that $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is an increasing function of i. Fix a,c_G and c_O. We have the following cases:

Case 1: $i > i_n^1(a,c_G,c_O)$

By Lemma 54 and Lemma 56:

$$i > i_n^1(a,c_G,c_O) \geq i_n^2(a,c_G,c_O) = i_n^1(a+1,c_G,c_O)$$

So by Lemma 54, $x_n^*(a,i,c_G,c_O) = x_n^*(a+1,i,c_G,c_O) = 0$. Hence, by (235) and (236):

$$\begin{aligned}
& V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O) \\
= \;& W_n(a,i,c_G,c_O,0) - W_n(a+1,i,c_G,c_O,0) \\
= \;& (J(a,i,c_O) - J(a+1,i,c_O)) \\
+ \;& \sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) \\
& -V_{n-1}(a+1,j,d_G,d_O)) \qquad (251)
\end{aligned}$$

Case 2: $i_n^m(a,c_G,c_O) \geq i > i_n^{m+1}(a,c_G,c_O)$, where $m \geq 1$.

By Lemma 54, $x_n^*(a,i,c_G,c_O) = m$ and hence by Theorem 12, $x_n^*(a+1,i,c_G,c_O) = m-1$. So by (235) and (236) and some cancellation of terms, we get:

$$\begin{aligned}
& V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O) \\
= \;& W_n(a,i,c_G,c_O,m) - W_n(a+1,i,c_G,c_O,m-1) \\
= \;& \alpha(n)c_G \qquad (252)
\end{aligned}$$

By (251) and (252), $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$

$$= \begin{cases} \alpha(n)c_G & \text{if } i \leq i_n^1(a,c_G,c_O), \\ (J(a,i,c_O) - J(a+1,i,c_O)) & \\ +\sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} \sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) & \\ -V_{n-1}(a+1,j,d_G,d_O)) & \text{if } i > i_n^1(a,c_G,c_O). \end{cases}$$

The expression for $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ for $i > i_n^1(a,c_G,c_O)$ is an increasing function of i by Property 6, induction hypothesis and Condition 1. Thus, to show that $V_n(a,i,c_G,c_O) - V_n(a+1,i,c_G,c_O)$ is increasing in i, it is sufficient to show that for

$i > i_n^1(a, c_G, c_O)$:

$$
\begin{aligned}
&(J(a,i,c_O) - J(a+1,i,c_O)) \\
+\sum_{d_G}\sum_{d_O} H^G_{c_G d_G} H^O_{c_O d_O} &\sum_j Q_{ij}(V_{n-1}(a,j,d_G,d_O) \\
&-V_{n-1}(a+1,j,d_G,d_O)) \geq \alpha(n)c_G
\end{aligned} \tag{253}
$$

By (236), (253) is equivalent to $W_n(a,i,c_G,c_O,0) \geq W_n(a,i,c_G,c_O,1)$, which is true because $x_n^*(a,i,c_G,c_O) = 0$ for $i > i_n^1(a,c_G,c_O)$. The result follows. □

From the above lemmas, we get the desired monotonicity of $x_n^*(i,a,c_G,c_O)$.

Proof of Theorem 13. Fix n, a, c_G and c_O. By Lemma 57, $V_{n-1}(a,j,d_G,d_O) - V_{n-1}(a+1,j,d_G,d_O)$ is an increasing function of j for each d_G, d_O. The result follows by Lemma 55. □

Proof of Theorem 14. The proof is very similar to the proof of Theorem 13 and hence omitted. □

6.5.2 Proofs of results in Section 6.2.4

Proof of Theorem 15. We show that the Maximum Independent Set (MIS) problem is a special case of Primary-SCTM. Consider the following special case of Primary-SCTM: $M = 1$, $T = 1$. At each node, the primary's demand is always 0, and the prices of Type G and O contracts are fixed, equal to $\frac{1}{2}$ and 1 respectively. Thus, it is optimal never to sell a type G contract.

The Primary-SCTM problem reduces to that of finding a maximum independent set of nodes (at which to sell Type O contracts). The result follows, since the MIS problem is NP-Hard [35]. $\square$

Proof of Lemma 51. Let $N^t_{e,j}$ be the number of Type-j contracts ($j \in \{G, O\}$) sold by a policy P in slot t on channel e. We make the following key observations:

(1) The revenue of any policy depends only on the *number* of Type-G and Type-O contracts it sells on each channel, in each slot, independent of *which* nodes it sells them at. That is, the revenue of the policy P is completely determined by:

$$\{N^t_{e,G}, N^t_{e,O} : e = 1, \ldots, M; t = 1, \ldots, T\}$$

This follows from the fact that on each channel, the prices of both types of contracts and the usage status (*i.e.*, whether or not the primary is using the channel for subscriber demand satisfaction) are the same at all nodes.

(2) For every policy, on each channel, at any time, the total number of Type-G and Type-O contracts currently leased is at most equal to $|I(S)|$.

That is, for the above policy P, for every slot t:

$$\sum_{\tau=1}^{t} N^{\tau}_{e,G} + N^t_{e,O} \leq |I(S)|,\ e = 1, \ldots, M \tag{254}$$

This follows from the fact that $I(S)$ is a maximum independent set.

Now, let P be an optimal policy. Consider a policy $f \in \mathcal{F}$, which initially finds a maximum independent set $I(S)$. Also, whenever P sells a contract, f sells the same type of contract on the same channel at a node in $I(S)$ at which no contract has been sold on this channel. More precisely, number the nodes in $I(S)$ from 1 to $|I(S)|$. In slot t, on channel e, policy f sells Type-G contracts at the nodes $\sum_{\tau=1}^{t-1} N_{e,G}^{\tau} + 1$ to $\sum_{\tau=1}^{t} N_{e,G}^{\tau}$ and Type-O contracts at the nodes $\sum_{\tau=1}^{t} N_{e,G}^{\tau} + 1$ to $\sum_{\tau=1}^{t} N_{e,G}^{\tau} + N_{e,O}^{t}$. It can be checked that on each channel e, (a) for policy f, two or more contracts never stand leased at the same node and (b) by (254), in each slot t, f finds enough nodes in $I(S)$ to sell contracts at.

Now, by observation (1), the revenue of f is the same as that of P, and therefore f is optimal. $\square$

Chapter 7

A Spectrum Auction Framework for Joint Access Allocation to Primaries and Secondaries

7.1 Introduction

Recall from Section 1.1 that there are two possibilities for spectrum allocation in CRNs– one-step and two-step allocation. In Chapters 2 to 6, we focussed on the two-step allocation scenario. In this chapter, we consider the one-step allocation scenario; recall that in this scenario, a regulator such as the Federal Communications Commission (FCC) in the United States jointly allocates the rights to be the primary and the secondary networks on its channels in a single allocation.

Different networks may attach different value to being primary and secondary. A network may wish to mainly transmit delay-sensitive traffic like voice or video. Such a network will attach a high value to the rights to be primary. On the other hand, a network may be mainly interested in transmitting delay-insensitive traffic like email or file transfer. Such a network would not need primary rights and would prefer secondary rights since the latter would be priced lower than the former. Also, a network whose traffic is a mixture of delay-sensitive and delay-insensitive traffic would want primary rights on some channels and secondary rights on some channels.

Auctions are suitable for selling the rights to be primary and secondary on the channels. Since the regulator need not know the values that bidders attach to primary and secondary rights, auctions provide a mechanism for the regulator to get a higher revenue than that obtainable through static pricing [22]. Auctions are also beneficial for the bidders since in general they assign goods to the bidders who value them most [22]. FCC has been conducting spectrum auctions since 1994 to allocate licenses for radio spectrum [1] (however, so far, no auctions have been conducted for cognitive radio networks).

In this chapter, we develop a comprehensive auction framework for the one-step allocation scenario, using which a regulator can simultaneously allocate the rights to be primary and secondary on the channels. One-step allocation may be more desirable than its two-step counterpart in certain cases. For example, one-step allocation gives a greater degree of control to the regulator. In particular, it allows the regulator to choose a "socially beneficial" channel allocation that maximizes the social welfare. Note that in one-step allocation, a network can bid for, and can even be granted, primary and secondary access to more than one or even all channels. Also, the allocation resulting from two-step allocation may indeed turn out to be that for one-step allocation but only when it is the most socially beneficial allocation.

We consider a scenario in which the regulator conducts an auction to sell the rights to be primary and secondary networks on a set of channels. Networks can bid for these rights based on their utilities and traffic demands. The regulator uses these bids to solve the *access allocation* problem, *i.e.*, the problem of deciding which networks will be the primary and secondary networks on each channel. The goal of the regulator may be either to maximize its revenue or to maximize the social welfare of the bidding networks. Now, networks can have utilities or valuations that are functions of the number of channels on which they get primary and secondary rights, how many and which other networks they share these channels with etc. The number of valuations of a network may be large and an exponential amount of space may be required to express a bid for each valuation. So we design bidding languages, that is, compact formats for networks to express bids for their valuations. For different bidding languages, we design algorithms for the access allocation problem.

This chapter is organized as follows. We describe the system model in Section 7.2. In Section 7.3, we describe how the bidding languages and algorithms that we design in the paper can be used to maximize the auctioneer's revenue or to maximize social welfare. In Section 7.4, we describe a model in which the bids of a network depend on which other networks it shares a channel with. In Section 7.4.1, we design an optimal algorithm for the access allocation problem for a simple case with only one secondary network on each channel. We show the intractability (NP-Competeness of the access allocation problem or exponential size of bids) of the extensions of this simple case in Section 7.4.2. In Section 7.5, we consider the case in which the bids of a network are independent of which networks it shares a channel with and provide an optimal dynamic programming algorithm for the access allocation problem in Section 7.6. The algorithm is polynomial-time when the number of possible cardinalities of the set of secondary networks on a channel is upper-bounded. In Section 7.7, we describe a bidding language that can be used for the independent bids case for an arbitrary number of cardinalities of the set of secondary networks on a channel and provide a greedy 2-approximation algorithm for the access allocation problem. In Section 7.8, using simulations, we show that the above approximation algorithm in fact performs optimally in a variety of scenarios.

7.2 System Model

We consider a scenario in which there are M identical orthogonal channels in a region. A regulator conducts an auction to sell the rights to be the primary and secondary networks on the channels. N bidders participate in the auction. Each bidder is an independent network of multiple wireless nodes. Each bidding network submits bids to the regulator and based on the bids, the latter allocates the rights to be the primary and secondary networks on the channels.

A primary network on a channel must have priortized access to the channel. If two or more independent networks were to be the primary networks on a single channel, then the access of each one of them would be constrained by the transmissions of the other primary networks, which would transmit at the same priority level. To avoid this, we assume that there is exactly one primary network on each channel. However, we allow multiple networks to have secondary rights on a channel.

We assume that all the secondary networks on a channel have equal rights on the channel. This is because complicated multiple access protocols [5] would be required to grant access at different priority levels to different secondary networks on a channel (with all of them getting lower priority access than the primary network). On the other hand, simple multiple access protocols would suffice if all secondary networks have equal rights on the channel.

Now, since a primary network has priortized access on a channel, the average delay of its traffic is low. On the other hand, the average delay of a secondary network's traffic is high. Hence, primary rights (respectively secondary rights) are suitable for communicating delay-sensitive (respectively delay-insensitive) traffic. We assume that each network has two kinds of traffic: (a) delay-sensitive traffic like voice, video etc. and (b) delay-insensitive or elastic traffic like email, file-transfer etc. A network uses its primary rights to transmit its delay-sensitive traffic and its secondary rights to transmit its elastic traffic.

A single network i may be both the primary network and one of the secondary networks on a channel. In this case, we assume that it transmits its delay-sensitive traffic as a primary network, *i.e.*, with high-priority, and when it does not have any delay-sensitive traffic to transmit, it transmits its elastic traffic as a secondary network. Also, the other secondary networks on the channel can transmit whenever network i is not transmitting its delay-sensitive traffic.

Let K be the set of all possible ways in which the M channels can be allocated to the N bidders. For example, consider the simple case in which $M = 3$, $N = 9$ and there can be at most four secondary networks on a channel. An example of an allocation of the channels is one in which network 1 becomes the primary network on channels 1 and 2, network 2 becomes primary on channel 3, network 3 becomes the sole secondary network on channel 1, networks 4 and 5 become secondary networks on channel 2, networks 1, 4, 6 and 7 become secondary networks on channel 3 and networks 8 and 9 do not become primary or secondary networks on any channel.

Let $x_i(k)$ be network i's *valuation* or *utility* from the channel allocation $k \in K$, *i.e.*, the value that it *conjectures or expects* to derive from the allocation k *when it submits the bids*. Note that since network i will share channels with other networks in the allocation k, the *actual* utility that network i will derive from an allocation k after the networks start using the allocated channels depends on the transmission patterns of the other networks that are not completely known to network i when it submits the bids.

So a network can only submit bids based on the expected utilities $x_i(k)$, which reflect its expectations about the actual utilities that it will eventually get. Henceforth, we use the terms valuation or utility for $x_i(k)$, but they should be understood to mean the conjectured utility or valuation of network i for the channel allocation k.

The valuations $x_i(.)$ of network i for the allocations in K depend on its traffic demands, *i.e.*, the volumes of delay-sensitive and elastic traffic that it wants to transmit. Now, for given traffic demands, the valuation of a network i for a channel allocation $k \in K$ may depend upon the number of channels on which network i has primary and secondary rights in the allocation k, how many and which other networks have rights on each of the channels on which network i has primary or secondary rights etc. For example, a network that wants to transmit a lot of delay-sensitive traffic will ascribe a high valuation to an allocation in which it is primary on several channels. Note that network i may have the same valuation for different allocations $k \in K$.

Network i's net utility is of the form:

$$u_i(k, \tau_i, x_i) = x_i(k) - \tau_i \tag{255}$$

where τ_i is the payment that network i makes to the auctioneer. The auctioneer determines the channel allocation and the payment τ_i that each network i makes to the auctioneer. The *social welfare* of an allocation k is defined to be the quantity:

$$\sum_{i=1}^{N} x_i(k)$$

Thus, the social welfare is the sum of utilities of all bidders from the allocation k.

Now, there could be two goals for designing the auction: revenue maximization and maximizing social welfare. In the first goal, based on its valuations, each network submits a set of bids to the auctioneer. Let $z_i(k)$ be the bid of network i for the allocation $k \in K$, *i.e.*, the amount of money it is willing to pay if the allocation $k \in K$ is chosen. Let k^* be the channel allocation that maximizes the revenue of the auctioneer, given the bids $z_i(.)$ for bidders $1, \ldots, N$. That is, k^* satisfies:

$$\sum_{i=1}^{N} z_i(k^*) \geq \sum_{i=1}^{N} z_i(k) \quad \forall k \in K \tag{256}$$

In the second goal of maximizing social welfare, $z_i(.)$ are not the bids of the networks, but have a different interpretation: they are the *declared valuations* of the networks (explained in Section 7.3). In this case, the channel allocation that maximizes the social welfare of the N networks can again be found by finding the k^* satisfying equation (256).

For both goals, the *access allocation* problem is to determine the channel allocation k^* satisfying (256). Depending on the interpretation of $z_i(.)$, this allocation k^* either maximizes the auctioneer's revenue or the social welfare of the N networks.

Now, the set K of possible channel allocations may be exponential in size. Hence, the total number of different valuations of network i may be exponential in general. However, it is not computationally tractable to communicate a bid for each valuation in this large set. So we introduce *bidding languages* for the auction models that we consider. A bidding language [13] is a format to compactly encode the bid information of a bidder. When there are an exponential number of valuations, a bidding language expresses the bids approximately, not exactly.

We now remark on some implementation issues: (i) One way in which the regulator can implement the auction is by deploying a central controller in the region, which would periodically collect bids that are sent by the bidding networks over a common control channel, compute the channel allocation and payments and send them to the bidders over the control channel. (ii) The frequency at which auctions are conducted is determined by the following tradeoff: the higher the frequency, the more responsive is the channel allocation to changes in traffic demands and higher is the spectrum utilization, but the overhead is also higher. Hence, the inverval between successive auctions is chosen to be as small as possible while ensuring that the overhead is below an acceptable limit.

7.3 Solution Framework

As stated earlier, an auction could be designed for two different objectives. In our context, the first objective is to choose the channel allocation that maximizes the regulator's revenue for a given set of bids of the bidders. This can be done by choosing the allocation k^* satisfying (256) when $z_i(k)$ is the bid of network i for the channel allocation k.

The second possible objective for the auction could be to achieve efficiency, that is, to choose the allocation that maximizes social welfare. To this end, each bidder is asked to declare its valuation function $x_i(.)$. With an abuse of notation, let $z_i(k)$ denote the *declared* valuation of network i for the allocation k, which may be different from $x_i(k)$ if bidder i believes that falsely declaring its valuations will improve its net utility. *Truth-telling* is said to be a *weakly-dominant strategy* [42] for network i, if for any possible declarations of networks other than i, the net utility of network i is maximized when it sets $z_i(k) = x_i(k)\ \forall k \in K$. It follows from the revelation principle [42] that to maximize social welfare, it is sufficient to consider mechanisms in which the payments τ_i are chosen such that for each bidder i, truth-telling is a weakly dominant strategy. Such a mechanism is called *incentive compatible*.

To date, the Vickrey-Clarke-Groves (VCG) mechanism [42] is the only known general incentive compatible mechanism that can be used to maximize social welfare. Under this mechanism, given the declared valuation functions $z_i(.)$ of the bidders, the allocation k^* satisfying (256) is chosen. Let k^*_{-i} be the allocation which would have maximized the social welfare if network i did not participate in the auction. That is, k^*_{-i} satisfies:

$$\sum_{j=1, j\neq i}^{N} z_j(k^*_{-i}) \geq \sum_{j=1, j\neq i}^{N} z_j(k) \quad \forall k \in K \tag{257}$$

Under the VCG mechanism, the payment made by network i to the auctioneer is given by:

$$\tau_i = \sum_{j=1, j\neq i}^{N} z_j(k^*_{-i}) - \sum_{j=1, j\neq i}^{N} z_j(k^*) \tag{258}$$

The key to implementing the VCG mechanism is to find the allocations k^* and k^*_{-i}, $i = 1, \ldots, N$. Now, k^* can be found using an algorithm for the access allocation problem (256) and k^*_{-i} can be found by running the same algorithm on the set of bidders

$\{1,\ldots,N\}\backslash i$.

Now, in general, the set of different valuations of a bidding network is exponential in size. First, we consider the special case when the number of different valuations of each bidding network is of poynomial space complexity. But K can still be exponential in size. This is because a bidder may have the same valuation for two or more allocations in K. Even in this case, it is sometimes computationally intractable to devise an algorithm to find the optimal allocation k^* satisfying (256), possibly because this is NP-hard, but instead an approximation algorithm for the access allocation problem can be devised. In this case, if the payments are chosen according to the VCG formula (258) with sub-optimal allocations instead of k^* and k^*_{-i}, then truth-telling is no longer a weakly dominant strategy for the bidders. To address this problem, Nisan and Ronen [45] devised the *second-chance mechanism* under which, the auctioneer publishes the sub-optimal algorithm that it will use for the access allocation problem. Each bidder submits its (declared) valuations $z_i(.)$ and a so-called *appeal function* (see [45]) to the auctioneer. Each bidder optimizes the valuations and the appeal functions to submit so as to maximize its own utility. The auctioneer specifies a time limit by which the valuations and appeal functions must be submitted. The auctioneer uses the sub-optimal algorithm for the access allocation problem to find the channel allocation using the submitted valuations *and appeal functions*. The VCG formula (258) is used to determine the payment that each bidder will make. Now, the *strategic knowledge* of a bidder i is a function that for a set of valuations submitted by the other bidders, gives the valuation that bidder i must declare so as to get the maximum utility. It is shown in [45] that when there is a bound on the time each bidder i can take to compute its strategic knowledge and when the time limit allowed to each bidder to compute the valuations and appeal functions to submit is at least as much as this bound, then truthfully declaring the valuation function is a dominant strategy for each bidder under the second-chance mechanism. Moreover, the social welfare attained by the second-chance mechanism is at least as good as the social welfare of the sub-optimal algorithm used for the access allocation problem.

Now, in some cases, the set of valuations of a bidder takes an exponential amount of space and hence bidders have to use incomplete bidding languages (see Section 7.2) to convey their valuations. In this case as well, incentive compatibility does not hold if the VCG formula (258) is used for payments. As a solution to this problem, Ronen [56] devised the *extended second-chance mechanism*. In these mechanisms, each bidder submits a description of its set of valuations in some bidding language, an appeal function, and an *oracle* [56], which is a program that can be queried by the auctioneer for the bidder's valuation. The auctioneer determines an allocation based on the above submitted quantities using a (possibly sub-optimal) algorithm for the access allocation problem. It is shown in [56] that under reasonable assumptions (see [56]), truth-telling is a dominant strategy for the bidders under the extended second-chance mechanism.

Note that in addition to incentive compatibility, the VCG, second-chance and extended second-chance mechanisms have the desirable property of *individual rational-*

ity [42], *i.e.*, bidders get a non-negative utility when they participate in the auction.

In this paper, we propose several spectrum auction models and design bidding languages and algorithms for the access allocation problem. These can be used for the objective of maximizing the revenue of the auctioneer or for maximizing the social welfare of the bidders in conjunction with the VCG, second-chance or extended second-chance mechanism, as appropriate. In particular, in Section 7.4.1, we describe an auction model that allows networks to completely express their bids under certain assumptions (Assumptions 3 and 4). We provide a polynomial-time algorithm that finds the optimal solution in the access allocation problem. This algorithm can be used to maximize the auctioneer's revenue or, in conjunction with the VCG mechanism, to maximize the social welfare of the bidders. In the auction model in Section 7.5, we provide a bidding language that allows bidders to completely express their bids when they have no knowledge of the channel usage behavior (defined in Section 7.4) on a channel of the other bidders and approximately express their bids when they have this knowledge. Section 7.6 provides a polynomial-time algorithm to optimally solve the access allocation problem for the model in Section 7.5 when the number of cardinalities of the set of secondary networks on a channel is upper-bounded. When bidders have no knowledge of the channel usage behavior of other bidders, this algorithm can be used to maximize the auctioneer's revenue or to maximize social welfare in conjunction with the VCG mechanism. When bidders have this knowledge, the algorithm can be used to maximize the auctioneer's revenue or in conjunction with the extended second-chance mechanism to maximize social welfare. Finally, in the auction model in Section 7.7, we provide a bidding language and a 2-approximation algorithm for the access allocation problem that is polynomial-time for an arbitrary number of cardinalities of the set of secondary networks on a channel. This algorithm can be used to approximate the maximum revenue of the auctioneer or in conjunction with the extended second-chance mechanism to approximate the maximum social welfare.

For notational convenience, throughout the paper, we assume that $z_i(.)$ are the bids expressed by bidder i and view the access allocation problem as the problem of maximizing the revenue of the auctioneer. However, our framework applies without change to the problem of maximizing social welfare.

7.4 Auction with Dependent Bids

A primary or secondary network on a channel shares the channel with other networks and hence its actual utility from the channel depends on the transmissions of those networks. A network may have some knowledge or beliefs about the typical transmission patterns of the other bidding networks. For example, the agency owning the network may conduct a survey on the typical transmission patterns of the other networks in its region or, if auctions are periodically conducted to allocate spectrum in the region, the agency may gain this knowledge about the networks with whom it shared channels previously. Thus, the conjectured utilities and hence the bids of a network would depend on which networks it will share different channels with.

7.4.1 Basic Model

In the basic model with dependent bids, we consider the model described in Section 7.2 with the following additional assumptions.

Assumption 3. *There is only one secondary network on each channel.*

Assumption 4. *Each network can be either the primary or the secondary network on only one channel.*

We explore the effect of relaxing either of these assumptions in Section 7.4.2. We assume that $N \geq 2M$, so that all M channels can be allocated.

A secondary network on a channel can use the channel whenever the primary network is not using it. So the throughput and delay of the secondary network on the channel depends on the *channel usage behavior* of the primary on the channel, *i.e.*, on the rate of its transmissions on the channel and how these transmissions are spread over time. On the other hand, the primary network on a channel has priortized access to the channel. That is, when the secondary network wants to transmit on the channel, it senses the channel and can transmit only if it finds that the primary network is not transmitting. However, due to the imperfect nature of sensing, the secondary network will sometimes transmit while the primary network is transmitting, resulting in a collision. Hence the primary network's utility depends on the channel usage behavior of the secondary network on the channel. Thus, the actual utility of a primary or secondary network depends on which network it shares a channel with. As explained above, a network may in general have certain beliefs about the channel usage behavior of other networks and hence may wish to express bids dependent on the network with whom it shares the channel. To model this, let

$$z_i^p(j),\ j \in \{1,\ldots,N\}\backslash\{i\}$$

be the bid of network i for the case when it is the primary network on a channel and network j is the secondary network on the channel. Similarly, let

$$z_i^s(j),\ j \in \{1,\ldots,N\}\backslash\{i\}$$

be the bid of network i for the case when it is the secondary network on a channel and network j is the primary network.

Let

$$k = \{(i_1, j_1),\ldots,(i_M, j_M)\}$$

be an allocation of the M channels to a set of networks. k is a set of M orderered pairs (i_t, j_t) such that network i_t is the primary network on channel t and network j_t is the secondary network on channel t. Note that the revenue of the allocation k is:

$$\sum_{t=1}^{M}(z_{i_t}^p(j_t) + z_{j_t}^s(i_t))$$

We describe an algorithm for determining k^*, the allocation that maximizes the revenue, by reduction to a maximum weight matching problem in a graph. Let G be a weighted undirected graph with N nodes, one node corresponding to each network. G is a complete graph, *i.e.*, between every pair of nodes, there is an edge. Let the weight of the edge joining nodes i and j be

$$w_{ij} = \max(z_i^p(j) + z_j^s(i), z_j^p(i) + z_i^s(j)) \tag{259}$$

Note that the weights are nonnegative real numbers. The interpretation of the weights w_{ij} is as follows. If network i (respectively, network j) is the primary network on a channel and network j (respectively, network i) is the secondary network, then the sum of the amounts paid by networks i and j is $z_i^p(j) + z_j^s(i)$ (respectively, $z_i^s(j) + z_j^p(i)$). So w_{ij}, the greater of these two quantities, is the maximum sum of payments of networks i and j if they are the two networks on the same channel.

A *matching* $\mathcal{M}$ in a graph is defined to be a subset of the edges such that no two edges in the subset share a common node. The weight of a matching is the sum of the weights of its edges.

The following algorithm finds the channel allocation k^* that maximizes the revenue:

STEP1: In graph G, find a matching $\mathcal{M}_M^*$ of maximum weight among matchings with exactly M edges [31] (we say how later).

STEP2: Let $e_1, \ldots, e_M$ be the M edges in the matching $\mathcal{M}_M^*$. Let e_t^1 and e_t^2 be the two endpoints of edge e_t. The allocation k^* is chosen such that for $t = 1, \ldots, M$, networks e_t^1 and e_t^2 become the two networks (primary and secondary) on channel t. If

$$z_{e_t^1}^p(e_t^2) + z_{e_t^2}^s(e_t^1) \geq z_{e_t^2}^p(e_t^1) + z_{e_t^1}^s(e_t^2)$$

then network e_t^1 becomes the primary network on channel t and network e_t^2 becomes

[31] Note that there exists a matching with exactly M edges since there are $N \geq 2M$ nodes and G is a complete graph.

the secondary network, otherwise network e_t^2 becomes the primary network on channel t and network e_t^1 becomes the secondary network.

Theorem 23. *The allocation k^* found from the matching $\mathcal{M}_M^*$ in the above algorithm is the one that maximizes the revenue.*

Proof. There is a many-to-one correspondence between the set of channel allocations and the set of matchings with exactly M edges. (It is many-to-one since the allocations obtained from any allocation by swapping the roles of the primary and secondary networks on one or more channels correspond to the same matching). From the interpretation of the weight of an edge given above, it follows that the weight of a matching $\mathcal{M}_M$ has the maximum revenue among the revenues of the channel allocations that correspond to it. Therefore, the weight of the maximum weight matching $\mathcal{M}_M^*$ equals the maximum revenue among the revenues of all the channel allocations. Also, note that Step 2 of the above algorithm ensures that we select the channel allocation k^*, whose revenue is the same as the weight of $\mathcal{M}_M^*$. It follows that the allocation k^* found from the matching $\mathcal{M}_M^*$ is the one that maximizes the revenue. $\square$

Now, it remains to show how to find the matching $\mathcal{M}_M^*$. Edmonds [17] gave a polynomial-time algorithm for finding the maximum weight matching (with any number of edges) in a graph. However, we are interested in a maximum weight matching among matchings with M edges, which cannot be directly obtained by Edmonds' algorithm. It can be obtained in $O(M^4 + M^2N^2)$ time [32] using White's modification [72],

[32] Recall that a function $f(n)$ is said to be $O(g(n))$ if there exist positive constants c and n_0 such that

[73] to Edmonds' algorithm.

7.4.2 Intractability of Extensions

We now explore the effect of relaxing either one of Assumptions 3 and 4. Suppose Assumption 3 is relaxed and Assumption 4 is retained. That is, we assume that each network can be the primary or a secondary network on only one channel. However, there can be multiple secondary networks on a channel. We show that even if there are two secondary networks on a channel, the problem of finding a channel allocation that maximizes the revenue is NP-complete.

Let $z_i^p(j_1,\ldots,j_{v-1})$ be the bid of network i for the case in which it is primary on a channel and networks $j_1,\ldots,j_{v-1}$ are secondary. Let $z_{j_1}^s(i,j_2,\ldots,j_{v-1})$ be the bid of network j_1 for the case in which network i is the primary and networks $j_1,\ldots,j_{v-1}$ are the secondary networks. Also, let z_i^p be the bid of network i for the case in which it is primary on a channel with no secondary on the channel. We now define the r-Network Dependent Bid Access Allocation Problem (r-DBA).

Definition 3 (The r-DBA Problem)**.** *Suppose M channels are to be allocated to N bidders such that on each channel, one network is primary and at most $r-1$ networks are secondary, where r is a fixed positive integer. Each bidder can be a primary or secondary network on at most one channel and the bids of networks are as given above. Find the allocation that maximizes the revenue.*

$f(n) \leq cg(n)$ for all $n \geq n_0$ [11].

We show that r-DBA is NP-Complete. To this end, we first show that a simpler version of r-DBA, which we call the Exactly r-Network Dependent Bid Access Allocation Problem (r-EDBA) is NP-Complete. The r-EDBA problem is defined in the same way as r-DBA, except that on each channel, *exactly* $r-1$ networks are secondary, instead of *at most* $r-1$ networks.

Note that if in an instance of r-EDBA, $N < rM$, then there is no channel allocation with r networks on each channel. In this case, we define the optimal revenue of the r-EDBA instance to be $-\infty$.

The decision version of r-DBA or r-EDBA is as follows: given a bound D, is there a channel allocation such that the revenue under the allocation is at least D? We next show that (the decision version of) 3-EDBA is NP-Complete.

Lemma 58. *3-EDBA is NP-Complete.*

Proof. Given an allocation of the M channels, we can verify in polynomial time whether the revenue under the allocation is at least D. This shows that 3-EDBA is in the class NP.

Next, we show that the 3-Dimensional Matching problem (3DM), which is known to be NP-complete [35], is polynomial-time reducible to 3-EDBA, *i.e.*, 3DM $\leq_p$ 3-EDBA. An instance of 3DM is as follows [35]: Given disjoint sets A, B, C of m elements each and a set T of ordered triples of the form (a,b,c), where $a \in A$, $b \in B$ and $c \in C$, does there exist a set of m triples in T so that each element of $A \cup B \cup C$ is contained in exactly one of these triples?

From this instance of 3DM, we construct an instance of 3-EDBA as follows. Let there be $M = m$ channels and $3m$ networks– one network corresponding to each element of $A \cup B \cup C$. We now design the bids, which will complete the construction. For every set $\{i, j, l\}$ of three networks such that (i, j, l) (or one of its permutations (j, l, i), (l, j, i) etc.) is a triple in T, define all of the following bids to be equal to $\frac{1}{3}$: $z_i^p(j,l)$, $z_j^p(i,l)$, $z_l^p(i,j)$, $z_i^s(j,l)$, $z_i^s(l,j)$, $z_j^s(i,l)$, $z_j^s(l,i)$, $z_l^s(i,j)$, $z_l^s(j,i)$. For every set $\{i, j, l\}$ of three networks such that no permutation of (i, j, l) is a triple in T, let all of the above bids be equal to $\frac{1}{6}$. In this 3-EDBA problem, we ask: is there a channel allocation of the m channels with revenue of at least $D = m$? We claim that the answer is yes if and only if the answer in the original 3DM problem is yes.

To prove sufficiency, suppose there exists a subset $T' \subseteq T$ of m triples such that each element of $A \cup B \cup C$ is contained in exactly one of these triples. Let

$$T' = \{(a_t, b_t, c_t) : t = 1, \ldots, m\}$$

Then allocate the m channels such that network a_t is the primary network and networks b_t and c_t are the secondary networks on channel t, $t = 1, \ldots, m$. The revenue of this allocation is:

$$\begin{aligned} & \sum_{t=1}^{m} \{z_{a_t}^p(b_t, c_t) + z_{b_t}^s(a_t, c_t) + z_{c_t}^s(a_t, b_t)\} \\ = & \sum_{t=1}^{m} \left\{ \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right\} \\ = & \; m \end{aligned}$$

Hence, the answer in the 3-EDBA problem is yes.

Conversely, suppose there exists an allocation of the m channels with revenue of at least m. In this allocation, let a_t be the primary and b_t and c_t be the secondary networks on channel t, $t = 1, \ldots, m$. If (a_t, b_t, c_t) or its permutation is a triple in T, then the sum of payments of networks a_t, b_t and c_t is 1, else it is $\left\{\frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right\} = \frac{1}{2}$. Since there are m channels and the revenue of the allocation is at least m, it follows that the revenue is exactly m and that for each t, (a_t, b_t, c_t) or one of its permutations is a triple in T. Moreover, since each network can be the primary or a secondary network on only one channel, it follows that each of the $3m$ networks is a primary or secondary network on exactly one channel. Hence, the m triples in T corresponding to (a_t, b_t, c_t) or its permutation for $t = 1, \ldots, m$ are such that each element of $A \cup B \cup C$ is contained in exactly one of the triples. So the answer to the 3DM problem is yes. This shows that 3DM $\leq_p$ 3-EDBA and hence that 3-EDBA is NP-Complete. $\square$

By an analogous reduction from r-Dimensional Matching, it can be shown that r-EDBA is NP-Complete for fixed $r > 3$. Note that for $r > 3$, r-Dimensional Matching is NP-Complete, which follows from a trivial reduction from 3-Dimensional Matching.

Now we show that for any fixed $r \geq 3$, (the decision version of) r-DBA is NP-Complete by a reduction from r-EDBA.

Theorem 24. *For $r \geq 3$, r-DBA is NP-Complete.*

Proof. Clearly, r-DBA is in the class NP.

Now we show that r-EDBA $\leq_p$ r-DBA. From any instance of r-EDBA with given

M, N, D and bid functions $z_i^p(.)$ and $z_i^s(.)$, we construct an instance of r-DBA as follows. The number of channels, number of networks and the bound on revenue are the same as in the original r-EDBA instance (M, N and D respectively). The bids of network i are given by:

$$\tilde{z}_i^p(j_1,\ldots,j_{v-1}) = \begin{cases} z_i^p(j_1,\ldots,j_{v-1}) & \text{if } v = r \\ 0 & \text{if } 2 \le v < r \end{cases}$$

$$\tilde{z}_i^s(j_1,\ldots,j_{v-1}) = \begin{cases} z_i^s(j_1,\ldots,j_{v-1}) & \text{if } v = r \\ 0 & \text{if } 2 \le v < r \end{cases}$$

and

$$\tilde{z}_i^p = 0$$

Recall that if $N < rM$, then there is no channel allocation in the r-EDBA instance with exactly r networks on each channel. Hence, the answer to the decision version is negative. Thus, let $N \ge rM$. We now show that there exists a channel allocation with revenue at least D in the r-EDBA instance if and only if there exists one such in the r-DBA instance. If there is a channel allocation with revenue at least D in the r-EDBA instance, then by construction of the bids in the r-DBA instance, the revenue of that channel allocation is the same in the r-DBA instance and hence at least D.

Conversely, suppose there is a channel allocation k with revenue at least D in the r-DBA problem. From this channel allocation, construct a channel allocation k' for the r-EDBA instance as follows: if there are $r-1$ secondary networks on a channel in k, let the primary and secondary networks on the channel be the same in k'. From the construction of bids in the r-DBA instance, it follows that the sum of payments of the networks on this channel in k' is the same as that in k. If there are $v-1$ secondary networks on a channel l in k, where $v < r$, then on channel l in k', let the same v networks be primary and secondary and in addition, let $r-v$ more networks be secondaries, which were not primary or secondary on any channel in k. Such networks exist since $N \geq rM$. By the construction of the bids in the r-DBA instance, the sum of payments of the networks on channel l in k is 0, whereas that in k' is at least 0. Thus, the revenue of allocation k' is at least as much as the revenue of channel k and hence is at least D.

This shows that r-EDBA $\leq_p$ r-DBA. Since r-EDBA is NP-Complete as shown above, it follows that r-DBA is NP-Complete. $\square$

Note that in the r-DBA problem, if r is unbounded, then each bidder i would have to submit an exponential number of bids $z_i^p(j_1, \ldots, j_{v-1})$ and $z_i^s(j_1, \ldots, j_{v-1})$.

Now, suppose we relax Assumption 4 and retain Assumption 3. Then each network can become a primary or secondary network on up to M channels. As explained above, the utility of a network from the primary or secondary rights on a given channel depends upon the channel usage behavior of the network it shares the channel with. However, the channel usage behavior of this network on the channel may in turn depend upon the number of channels on which it has primary and secondary rights and the channel usage behavior of the networks it shares those channels with and so on. Thus, in general, the utility of a network may depend upon which networks are the primary and secondary networks on each channel. The number of possible ways of choosing the primary and secondary networks on the M channels is clearly exponential. Thus, relaxing Assumption 4 would require each network to express an exponential number of bids in the auction with dependent bids, which is computationally intractable.

7.5 Auction with Independent Bids

In Section 7.4, we noted that when networks have some knowledge of the channel usage behavior of other networks, they would like to express bids dependent on which networks they will share channels with. However, it is quite possible in some scenarios that networks have no knowledge of the channel usage behavior of the other bidding networks. In this case, their conjectures about the utility that they will actually get from a channel allocation would be based only on the number of channels on which they will get primary and secondary rights and the *number of* other networks they will share these channels with in the allocation and would be independent of *which* other networks they will share channels with. Thus, they would submit bids, based on these conjectured utilities, that are independent of which networks share different channels with them.

Moreover, in Section 7.4.2, we showed that bids of exponential size are needed in the auction with dependent bids when Assumptions 3 and 4 are relaxed. This motivates the idea that even when networks have some knowledge of the channel usage behavior of the other networks, we can obtain a compact bidding language, that is, a means for networks to approximately convey their bids, by imposing the restriction that the bids of a network be independent of which other networks it shares different channels with. We study the auction resulting from imposing this restriction in this section.

7.5.1 Model

Consider the model in Section 7.2 with the following additions. On each channel, one network can be the primary network and m_1, m_2, …, $m_{(n-1)}$ or m_n networks can be the secondary networks, where $1 \leq m_1 < m_2 < \ldots < m_n$. Note that n is the number of possible cardinalities of the set of secondary networks on a channel.

When the results of the auction are declared, let $n_{i,0}$ be the number of channels on which bidder i is the primary network. Let $n_{i,j}, j = 1, \ldots, n$ be the number of channels on which bidder i is a secondary network along with $m_j - 1$ other secondary networks.

Suppose there are m_j secondary networks on a channel. Recall from Section 7.2 that each of these m_j secondary networks have equal rights on the channel. The share of each of these networks in the secondary rights on the channel is called a *secondary part of type j*. Also, the channel is said to be divided into m_j secondary parts of type j. Similarly, since exactly one network becomes a primary network on a channel, if a network is the primary network on l channels, we say that it is allocated *l primary parts*. Also, we refer to the throughput received by a network as a secondary network as its *secondary throughput*.

In general, network i's utility may depend not only on the total expected secondary throughput that it gets, but also on the distribution of this secondary throughput over the M channels. For example, it may get the same expected secondary throughput (a) if it is the secondary network on two channels with one other secondary network on each and (b) if it is the sole secondary network on one channel. But it may prefer one of these scenarios over the other. This is because a network has to sense different channels on which it has secondary rights for ongoing transmissions and also communicate on them. There may be costs due to delays for switching the antennas of the network's nodes between different channels. To take into account this possibility, in this section, we assume that the utility of network i depends not just on the expected secondary throughput (and the number of primary parts) it receives, but on the vector $(n_{i,0}, n_{i,1}, \ldots, n_{i,n})$. We allow bidder i to submit bids as a function of this vector.

Each bidder i submits the following bid vector to the auctioneer:

$$\{z_i(n_{i,0}, n_{i,1}, \ldots n_{i,n}) : 0 \leq n_{i,0}, n_{i,1}, \ldots n_{i,n} \leq M,$$

$$n_{i,1} + n_{i,2} + \ldots + n_{i,n} \leq M; n_{i,j} \text{ integer}, j = 0, 1, \ldots n\}$$

where $z_i(n_{i,0}, n_{i,1}, \ldots n_{i,n})$ is network i's bid for becoming the primary network on $n_{i,0}$ channels and becoming a secondary network on $n_{i,j}$ channels along with $m_j - 1$ other secondary networks, for $j = 1, 2, \ldots n$.

7.5.2 Feasible Allocation

We say that an allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$ is *feasible* if it is possible to assign to networks, the rights to be primary and secondary on each of the M channels such that network i, $i = 1, \ldots, N$ is allocated $n_{i,0}$ primary parts and $n_{i,j}$ secondary parts of type j for $j = 1, \ldots, n$. The following lemma describes necessary and sufficient conditions for an allocation to be feasible.

Lemma 59. *An allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$ is feasible if and only if $n_{i,0}, n_{i,1} \ldots n_{i,n}$ for $i = 1, \ldots, N$ are integers such that for some nonnegative integers $M_j, j = 1, \ldots n$ satisfying $M_1 + \ldots + M_n = M$:*

$$0 \le n_{i,0} \le M, i = 1, \ldots, N \tag{260}$$

$$\sum_{i=1}^{N} n_{i,0} = M \tag{261}$$

$$0 \le n_{i,j} \le M_j, i = 1, \ldots, N; j = 1, \ldots, n \tag{262}$$

$$\sum_{i=1}^{N} n_{i,j} = m_j M_j, j = 1, \ldots, n \tag{263}$$

Note that the integer M_j in the above lemma corresponds to the number of channels that are divided into m_j secondary parts of type j. We assume that the number of bidders is at least m_1 so that a feasible allocation exists.

Proof. The necessity of all conditions is obvious. Now we show sufficiency. Suppose all the above conditions are satisfied. We construct a feasible allocation. Allocate $n_{i,0}$ primary parts to network i for $i = 1, \ldots, N$. Since $\sum_{i=1}^{N} n_{i,0} = M$, each primary part is allocated exactly once. Now consider the M_j channels divided into m_j secondary parts. Label the m_j secondary parts of type j on each of these channels from 1 to m_j. Also, label the M_j channels from 1 to M_j. Now consider the following order of the $m_j M_j$ secondary parts of type j: secondary part 1 of channel 1, part 1 of channel 2, ..., part 1 of channel M_j, part 2 of channel 1, part 2 of channel 2, ..., part 2 of channel M_j,

$\ldots$, part m_j of channel 1, part m_j of channel 2, $\ldots$, part m_j of channel M_j. Now, with secondary parts in the above order, first allocate $n_{1,j}$ secondary parts to network 1, then $n_{2,j}$ parts to network 2, $\ldots$, then $n_{N,j}$ parts to network N. Since $\sum_{i=1}^{N} n_{i,j} = m_j M_j$, in this way it is possible to allocate each secondary part of type j exactly once. Also, since $n_{i,j} \leq M_j \forall i$, it is clear that no network is assigned two or more secondary parts on the same channel. Hence the allocation is feasible. $\square$

From a feasible allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$, it is easy to construct a consistent specification of the primary and secondary networks on each channel. Hence, the access allocation problem reduces to finding a feasible allocation $\{n_{i,j} : i = 1, \ldots, N; j = 0, \ldots, n\}$ that maximizes the auctioneer's revenue given the submitted bid vectors $z_i(.)$.

Let

$$k = \{n_{i,j} : i = 1, \ldots N; j = 0, \ldots, n\} \tag{264}$$

denote a feasible allocation. Let K be the set of all feasible allocations.

7.6 Optimal Solution of Access Allocation Problem

In this section, we present an algorithm for optimally solving the access allocation problem for the auction described in Section 7.5. The algorithm is polynomial-time when n, the number of possible cardinalities of the set of secondary networks on a channel, is fixed (and m_n is allowed to grow with the problem size). This special case can be useful in practice because even with small n, flexibility in channel allocation can be achieved by choosing $m_1, \ldots, m_n$ judiciously. For example, with $n = 3$, we can choose $m_1 = 1$, $m_2 = 4$ and $m_3 = 8$. In this case, large chunks of secondary throughput can be allocated to a network by having it the sole secondary network on several channels and small chunks can be allocated to networks by having 4 or 8 networks share a channel.

7.6.1 Algorithm Description

A dynamic programming algorithm is given in [67] and [13] for the winner determination problem in a combinatorial auction with multiple units of a fixed number of different types of objects. We generalize the algorithm in [67], [13] in two directions: (a) the objects in a combinatorial auction are indivisible, whereas we need to decide into how many secondary parts to divide each channel and (b) in our auction, the allocation has to be feasible according to the conditions in Lemma 59.

We first summarize the algorithm. Given the bids $z_i(.)$, our goal is to find the allocation k^* that maximizes revenue. For each set of nonnegative integers $M_1, \ldots, M_n$ such that $M_1 + \ldots + M_n = M$, a dynamic programming algorithm is used to find out the maximum revenue and the maximizing channel allocation when M_j channels are divided into m_j secondary parts, $j = 1, \ldots, n$. Then we maximize over all sets of $M_1, \ldots, M_n$ to find the optimal set $M_1^*, \ldots, M_n^*$.

We now give the details of the algorithm. Fix $M_1, \ldots, M_n$ such that $M_1 + \ldots + M_n = M$. Let $T(j_0, j_1, \ldots j_n, i)$ denote the maximum possible revenue from all participating networks when j_0 primary parts and j_t secondary parts of type t, $t = 1,\ldots,n$, are to be allocated and networks 1, ..., i are participating in the auction. More precisely, let $K(j_0, j_1,\ldots, j_n, i)$ be the set of allocations $k_i = \{n_{v,t} : v = 1,\ldots,i; t = 0,\ldots,n\}$ satisfying the following conditions, which parallel the conditions in Lemma 59:

$$0 \leq n_{v,0} \leq M, v = 1,\ldots,i \tag{265}$$

$$\sum_{v=1}^{i} n_{v,0} = j_0 \tag{266}$$

$$0 \leq n_{v,t} \leq M_t, v = 1,\ldots,i; t = 1,\ldots,n \tag{267}$$

$$\sum_{v=1}^{i} n_{v,t} = j_t, t = 1,\ldots,n \tag{268}$$

Then:

$$T(j_0, j_1,\ldots j_n, i) = \max\{\sum_{v=1}^{i} z_v(n_{v,0}, n_{v,1},\ldots, n_{v,n}) : k_i \in K(j_0, j_1,\ldots, j_n, i)\}$$

Thus, $T(M, m_1M_1,\ldots m_nM_n, N)$ is the maximum revenue from networks $1,\ldots,N$ when M_j channels are divided into m_j secondary parts of type j, for $j = 1,\ldots,n$. We now give a dynamic programming algorithm to find $T(M, m_1M_1,\ldots m_nM_n, N)$.

The following expression is used for finding the values of $T(j_0, j_1,\ldots j_n, 1)$.

$$T(j_0, j_1,\ldots j_n, 1) = \begin{cases} z_1(j_0, j_1,\ldots, j_n) \\ \text{ if } j_0 \leq M, j_t \leq M_t, t = 1,\ldots,n \\ -\infty \text{ otherwise} \end{cases} \tag{269}$$

The reason the above equation holds is as follows. Since there is only one network (network 1), the only possibility is to allocate all parts to network 1. But if $j_0 > M$, then $n_{1,0} > M$, which violates condition (265). Similarly, if $j_t > M_t$, then $n_{1,t} > M_t$, which violates condition (267). Hence if $j_0 > M$ or $j_t > M_t$, then $T(.)$ is set to $-\infty$.

The following recursion is used for finding the values of $T(j_0, j_1, \ldots j_n, i)$ for $i \geq 2$.

$$\begin{aligned} T(j_0, j_1, \ldots, j_n, i) = \max(\\ T(j_0 - l_0, j_1 - l_1, \ldots, j_n - l_n, i-1) + z_i(l_0, l_1, \ldots, l_n) : \\ l_0 \in \{0, 1, \ldots, \min(j_0, M)\}, l_v \in \{0, 1, \ldots, \min(j_v, M_v)\}, \\ v = 1, \ldots, n) \end{aligned} \tag{270}$$

In the above recursion, if l_0 primary parts and l_v secondary parts of type v, $v = 1, \ldots, n$ are allocated to network i, then it is willing to pay $z_i(l_0, \ldots, l_n)$ and the maximum revenue obtainable from networks $1, \ldots, i-1$ for the remaining parts is by definition $T(j_0 - l_0, j_1 - l_1, \ldots, j_n - l_n, i-1)$. Moreover, $l_v \leq j_v$ since j_v secondary parts of type v are available and $l_v \leq M_v$ by (267). So $l_v \leq \min(j_v, M_v)$ for $v = 1, \ldots, n$ and similarly $l_0 \leq \min(j_0, M)$. Equation (270) follows by maximizing the revenue from networks $1, \ldots, i-1$ over all possible values of $l_0, l_1, \ldots, l_n$.

A feasible channel allocation that achieves the maximum revenue $T(M, m_1 M_1, \ldots m_n M_n, N)$ for the fixed values $M_1, \ldots, M_n$ considered, can be found from the array $T(.)$ by repeatedly finding the $l_0, l_1, \ldots, l_n$ that achieve the maximum in the right side of (270).

For all sets $M_1, \ldots, M_n$ such that $M_1 + \ldots + M_n = M$, $T(M, m_1 M_1, \ldots m_n M_n, N)$ and the revenue maximizing allocation are found as explained above. Then the optimal set

$(M_1^*,\ldots,M_n^*)$ is found as follows:

$$(M_1^*,\ldots,M_n^*) = \underset{M_1+\ldots+M_n=M}{\operatorname{argmax}} T(M, m_1M_1, \ldots m_nM_n, N) \qquad (271)$$

The revenue maximizing allocation with $M_1 = M_1^*$, ..., $M_n = M_n^*$ is the one that maximizes revenue over all channel allocations.

7.6.2 Running Time

The maximum in (270) is taken over $O(MM_1M_2\ldots M_n)$ values. Moreover, $T(j_0, j_1, \ldots, j_n, i)$ is calculated for i from 1 to N, j_0 from 0 to M, j_1 from 0 to m_1M_1, ..., j_n from 0 to m_nM_n, that is (since $m_j = O(m_n)$ for $j = 1,\ldots,n$), for $O(MM_1M_2\ldots M_n m_n^n N)$ values. Finally, this process is carried out for all $M_1,\ldots,M_n$ such that $M_1+\ldots+M_n = M$. Hence, the time to compute k^* is:

$$\begin{aligned}
& \sum_{M_1+\ldots M_n=M} O((MM_1\ldots M_n)^2 m_n^n N) \\
\leq \; & \sum_{M_1=0}^{M} \cdots \sum_{M_n=0}^{M} O((MM_1\ldots M_n)^2 m_n^n N) \\
= \; & O(M^{3n+2} m_n^n N)
\end{aligned}$$

Thus, the running time is $O(M^{3n+2}m_n^n N)$, which is polynomial for fixed n.

7.6.3 Space Complexity

Each network i submits its bid $z_i(n_{i,0}, n_{i,1}, \ldots, n_{i,n})$ for $n_{i,0} \in \{0,1,\ldots,M\}$ and all sets $n_{i,1},\ldots,n_{i,n}$ satisfying $\sum_{j=1}^{n} n_{i,j} \leq M$. There are $O(M^{n+1})$ such bids. Summing over the N networks, the storage requirement for bids is $O(M^{n+1}N)$.

To find the revenue maximizing allocation for a fixed set $M_1, \ldots, M_n$, we need to store the array $T(j_0, j_1, \ldots, j_n, i)$, $j_0 \in \{0, 1, \ldots, M\}$, $j_t \in \{0, 1, \ldots, m_t M_t\}$, $t = 1, \ldots, n$, $i = 1, \ldots, N$. This requires $O(M^{n+1} m_n^n N)$ amount of storage. Once the allocation has been found, only the allocation and the value of $T(M, m_1 M_1, \ldots m_n M_n, N)$ can be stored, which require $O(nN)$ and $O(1)$ storage respectively, and the rest of the array $T(j_0, j_1, \ldots, j_n, i)$ can be discarded.

We need to store the revenue maximizing allocation and the value $T(M, m_1 M_1, \ldots m_n M_n, N)$ for all sets $(M_1, \ldots, M_n)$ such that $M_1 + \ldots + M_n = M$. The number of such sets is $O(M^n)$. So the storage required is $O(M^n N n)$.

Thus, the maximum amount of storage required at any given time during the entire algorithm to compute k^* is $O(M^{n+1} m_n^n N)$.

7.7 A Greedy 2-Approximation Algorithm

The scheme described in Section 7.6 is computationally tractable for fixed n, the number of possible cardinalities of the set of secondary networks on a channel. However, if n is allowed to grow, the set of bids of a network is exponential in size as shown in Section 7.6.3 and hence the scheme is computationally intractable. In this section, we first provide a compact bidding language for the case with large n. We conjecture that under this bidding language, the access allocation problem is NP-hard. We give a basis for this conjecture in Section 7.9. We provide a polynomial-time algorithm that approximates the maximum revenue of the auctioneer within a factor of 2.

We describe the bidding language in Section 7.7.1. In Section 7.7.2, we introduce residual bid functions, a concept used in the approximation algorithm. We describe the algorithm in Section 7.7.3 and prove that it achieves an approximation ratio of 2 in Section 7.7.4. Finally, in Section 7.7.5, we describe an efficient implementation of the algorithm.

7.7.1 Bidding Language

Consider the model in Section 7.5 with the following changes. Let the bandwidth of each of the M channels be W bps. We assume that the primary network on a channel uses the channel for an expected fraction of time α, where $0 < \alpha < 1$. When auctions are repeated periodically to assign spectrum, α can be estimated based on long-term measurements of the primary networks' channel usage. Alternatively, it can be estimated via simulations. Since secondary networks can use the channel whenever the primary is not using it, an expected bandwidth of $W(1-\alpha)$ is available on a channel for the secondary networks. So when m_j secondary networks share a channel, each one of them can get an expected secondary throughput of $\frac{W(1-\alpha)}{m_j}$ on the channel. [33]

In this section, we allow a network to express bids as a function of the number of channels $n_{i,0}$ on which it is primary and its total expected secondary throughput T_i^s on

[33]Note that an expected bandwidth of at least $\frac{W(1-\alpha)}{m_j}$ is available to each of the m_j secondary networks. If some of them do not use this full bandwidth, then more than $\frac{W(1-\alpha)}{m_j}$ is available to the other networks.

all M channels. Note that:

$$T_i^s = \sum_{j=1}^{n} \frac{n_{i,j} W(1-\alpha)}{m_j} \tag{272}$$

In the sequel, for brevity, we simply say secondary throughput instead of expected secondary throughput. Moreover, we assume that the utility, and hence the bid $z_i(n_{i,0}, T_i^s)$, of each network i when it is primary on $n_{i,0}$ channels and has T_i^s units of secondary throughput, is separable, *i.e.*, of the form:

$$z_i(n_{i,0}, T_i^s) = w_i(n_{i,0}) + y_i(T_i^s) \tag{273}$$

where $w_i(n_{i,0})$ is its bid for being primary on $n_{i,0}$ channels and $y_i(T_i^s)$ is its bid for T_i^s units of throughput as a secondary network. This assumption is a good approximation since networks transmit different kinds of traffic (delay-sensitive and elastic respectively) as a primary and secondary network.

Under this assumption, the access allocation problem separates out into two independent problems– allocating the primary parts and allocating the secondary parts. The problem of allocating the primary parts can be optimally solved in $O(M^2N)$ time using the dynamic programming algorithm in Section 7.6 with $n = 0$. In this section, we focus on giving a 2-approximation algorithm for the problem of allocating the secondary parts. In the rest of the section, "revenue" refers to the auctioneer's revenue from selling the secondary rights on the M channels.

Assume that $y_i(.)$ is a concave increasing function for each network i. We use piecewise linear concave functions to compactly represent the bid functions of the networks. They can be used to closely approximate arbitrary concave functions [6] and have been previously used in the context of spectrum auctions in [22]. Each network i specifies its bid for at most P different levels of secondary throughput, for a positive integer P. More precisely, let $P_i \leq P$ be a positive integer and let:

$$0 = q_{i,1} < q_{i,2} < \ldots < q_{i,P_i} \tag{274}$$

For $v = 1, \ldots, P_i$, network i specifies $y_i(q_{i,v})$, which is its bid for $q_{i,v}$ units of secondary throughput. Network i's bid for q units of secondary throughput, where $q_{i,v} < q < q_{i,v+1}$ is found by linear interpolation:

$$y_i(q) = y_i(q_{i,v}) + \left(\frac{y_i(q_{i,v+1}) - y_i(q_{i,v})}{q_{i,v+1} - q_{i,v}} \right) (q - q_{i,v}) \tag{275}$$

Note that $q_{i,1}, \ldots, q_{i,P_i}$ are the breakpoints of the piecewise linear function $y_i(.)$.

We assume that for each network i, $q_{i,1} = 0$, that $y_i(q_{i,1}) = y_i(0) = 0$ and that

$$q_{i,P_i} \geq MW(1 - \alpha). \tag{276}$$

Since $MW(1 - \alpha)$ is the total secondary throughput available on the M channels, the second assumption means that network i's bid for any amount of secondary throughput on the M channels can be found by linear interpolation.

7.7.2 Residual Bid Functions

Our algorithm uses the following concept.

Definition 4. *Let $\tilde{q} \geq 0$. The $\tilde{q}$-residual bid function of network i is the function $\tilde{y}_i(.)$ given by:*

$$\tilde{y}_i(q) = y_i(\tilde{q} + q) - y_i(\tilde{q}) \tag{277}$$

We will sometimes say, “the residual bid function after accounting for $\tilde{q}$” instead of the $\tilde{q}$-residual bid function. Informally, once network i has been allocated $\tilde{q}$ units of secondary throughput, $\tilde{y}_i(.)$ acts as its bid function for allocations of additional secondary throughput. The following lemma gives some simple properties about the $\tilde{q}$-residual bid function.

Lemma 60. *Let $\tilde{y}_i(q)$ be the $\tilde{q}$-residual bid function of network i for some $\tilde{q} \geq 0$. Then*

1. $\tilde{y}_i(q) \leq y_i(q)\ \forall q \geq 0$.

2. *$\tilde{y}_i(q)$ is a piecewise-linear concave increasing function of q.*

Proof.

$$y_i(q+\tilde{q}) \leq y_i(q) + y_i(\tilde{q}),\ \forall q \geq 0$$

by concavity of $y_i(.)$. Hence,

$$\tilde{y}_i(q) = y_i(q+\tilde{q}) - y_i(\tilde{q}) \leq y_i(q)\ \forall q \geq 0$$

which proves part 1.

Now, $y_i(q)$ is piecewise-linear, concave and increasing by assumption. Thus, $y_i(q+\tilde{q})$ is a piecewise-linear, concave and increasing function of q as well. Part 2 follows by (277). ☐

The significance of the $\tilde{q}$-residual bid function is given by the following lemma.

Lemma 61. *Suppose the bid function of network i is $y_i(.)$ and it is successively allocated secondary throughputs of $q_1, q_2, \ldots, q_f$. Let $y_i^v(.)$ denote the $(q_1 + \ldots + q_v)$-*

residual bid function of network i, for $v = 1, \dots, f$. Then

$$y_i(q_1 + \dots + q_f) = y_i(q_1) + y_i^1(q_2) + \dots + y_i^{f-1}(q_f) \tag{278}$$

Proof. By definition:

$$y_i^1(q_2) = y_i(q_1 + q_2) - y_i(q_1)$$

which implies that:

$$y_i(q_1 + q_2) = y_i(q_1) + y_i^1(q_2) \tag{279}$$

Similarly,

$$\begin{aligned} y_i(q_1 + q_2 + q_3) &= y_i(q_1 + q_2) + y_i^2(q_3) && (280) \\ &= y_i(q_1) + y_i^1(q_2) + y_i^2(q_3) && (281) \end{aligned}$$

where the second step follows from (279). Similarly proceeding for f steps, we get the desired result (278). □

Thus, the significance of the residual bid function is that if a network i is successively allocated chunks q_1, ..., q_f of secondary throughput (*e.g.* by successive steps of an algorithm), then we can keep track of its residual bid function after every allocation so that the extra money that network i is willing to pay for the v'th allocation q_v is simply $y_i^{v-1}(q_v)$. Moreover, this tracking can be done using the update rule in part 1 of the following lemma to calculate $y_i^{v+1}(.)$ from $y_i^v(.)$.

Lemma 62. *Let $\tilde{y}_i(.)$ and $y_i^+(.)$ be the $\tilde{q}$-residual bid function and $(\tilde{q} + \widehat{q})$-residual bid function of network i respectively. Then*

1. $y_i^+(q) = \tilde{y}_i(q+\widehat{q}) - \tilde{y}_i(\widehat{q})\ \forall q \geq 0$

2. $y_i^+(q) \leq \tilde{y}_i(q)\ \forall q \geq 0$.

Proof.

$$\begin{aligned} & \tilde{y}_i(q+\widehat{q}) - \tilde{y}_i(\widehat{q}) \\ = \; & (y_i(q+\widehat{q}+\tilde{q}) - y_i(\tilde{q})) - (y_i(\widehat{q}+\tilde{q}) - y_i(\tilde{q})) \\ = \; & y_i(q+\widehat{q}+\tilde{q}) - y_i(\widehat{q}+\tilde{q}) \\ = \; & y_i^+(q) \end{aligned}$$

Hence, $y_i^+(.)$ is the $\widehat{q}$-residual bid function corresponding to the bid function $\tilde{y}_i(.)$. So by Lemma 60, $y_i^+(q) \leq \tilde{y}_i(q)\ \forall q \geq 0$. □

7.7.3 Algorithm Description

We now describe the greedy 2-approximation algorithm. The algorithm determines S_l^G, the set of secondary networks on channel l for $l = 1, \ldots, M$. Denote by $q_{i,l}^G$, the amount of secondary throughput allocated by the greedy algorithm to network i in the l'th channel. Since each network in S_l^G equally shares the secondary throughput on channel l, we have:

$$q_{i,l}^G = \begin{cases} \frac{W(1-\alpha)}{|S_l^G|} & \text{if } i \in S_l^G \\ 0 & \text{else} \end{cases} \tag{282}$$

Let $y_i^l(.)$ be the $(q_{i,1}^G + \ldots + q_{i,l}^G)$-residual bid function of network i, that is, its residual bid function after accounting for the secondary throughput allocated to it in channels 1 to l. Note that for each network i, $y_i^0(.)$ is the bid function $y_i(.)$.

The greedy algorithm successively determines S_l^G, for $l = 1, \ldots, M$, one channel at a time. Suppose the algorithm has determined $S_1^G, S_2^G, \ldots, S_{l-1}^G$ and for each network i, has found the residual bid function $y_i^{l-1}(.)$. It assigns channel l using the following steps:

STEP1: For $j = 1, \ldots, n$, find the maximum increase in revenue R_j^l obtainable from channel l by dividing the channel into m_j secondary parts using the following rule. Sort the set of numbers $y_i^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right), i = 1, \ldots, N$ into decreasing order. Let $y_{(v)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$ denote the v'th largest element. Then R_j^l is given by:

$$R_j^l = \sum_{v=1}^{m_j} y_{(v)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$$

STEP2: Find the maximum among $R_1^l, \ldots, R_n^l$. Suppose R_j^l is the maximum. Then divide the l'th channel into m_j secondary parts. On the l'th channel, the m_j networks with the m_j largest values $y_{(1)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right), \ldots, y_{(m_j)}^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$, which were determined in STEP1, become secondary networks. This determines S_l^G.

STEP3: For each $i \in S_l^G$, find the function $y_i^l(.)$ from its bid function $y_i(.)$ and $q_{i,1}^G, \ldots, q_{i,l}^G$. Note that $q_{i,l}^G$ is given by (282).

Comments on Algorithm:

1. Once channels $1, \ldots, l-1$ have been allocated, steps 1 and 2 allocate channel l so as to obtain the maximum possible increase in revenue over the revenue from channels $1, \ldots, l-1$. This property will be crucial in proving the approximation

ratio of 2.

2. In Step 3, for conceptual clarity, we have not presented the most efficient implementation. Specifically, the function $y_i^l(.)$ need not be computed from scratch. It can be found iteratively from $y_i^{l-1}(.)$ using the update rule in part 1 of Lemma 62. See Section 7.7.5 for details.

7.7.4 Approximation Ratio

Theorem 25. *Let R^* be the maximum possible revenue under any allocation of the rights to be secondary networks on the M channels and let R^G be that achieved by the above greedy algorithm. Then $R^G \geq \frac{R^*}{2}$.*

Proof. Let R^l be the increase in revenue obtained by the greedy algorithm from allocating the l'th channel. By part 2 of Lemma 62:

$$y_i^l(q) \leq y_i^{l-1}(q) \ \forall q \geq 0 \tag{283}$$

From the discussion after Lemma 61, it follows that after channels $1,\ldots,l$ were allocated, the extra money network i was willing to pay for its share in channel $(l+1)$ is $y_i^l(q_{i,l+1}^G)$. Moreover, if the greedy algorithm were to allocate the l'th channel to the same set of networks, S_{l+1}^G, to whom it actually allocated the $(l+1)$'st channel, then:

1. each network in S_{l+1}^G would have received on the l'th channel, a throughput of $\frac{W(1-\alpha)}{|S_{l+1}^G|}$, which equals $q_{i,l+1}^G$ by (282),

2. after channels $1,\ldots,l-1$ were allocated, the extra money network i would be willing to pay for its share in channel l would have been $y_i^{l-1}(q_{i,l+1}^G)$ and hence by (283),

3. the increase in revenue from the l'th channel would have been at least R^{l+1}.

But the actual increase in revenue from the l'th channel, R^l, is by definition of the greedy rule, the maximum possible from allocating the l'th channel. Hence $R^l \geq R^{l+1}$. Thus, we get:

$$R^1 \geq R^2 \geq \ldots \geq R^M.$$

Since $R^G = R^1 + \ldots + R^M$, we get:

$$R^M \leq \frac{R^G}{M} \tag{284}$$

Now, let q_i^* be the total secondary throughput allocated by the optimal algorithm to network i and q_i^G be that allocated by the greedy algorithm. Also, let S_l^* be the set of secondary networks on the l'th channel, $l = 1,\ldots,M$, in the optimal allocation. Next, we will upper bound $R^* - R^G$, the excess revenue of the optimal allocation over the greedy allocation. To this end, for each network i, we account for its payment for $\max(q_i^* - q_i^G, 0)$, the excess secondary throughput if any, of the optimal allocation over the greedy algorithm's allocation, by accounting for its payments for the chunks $q_{i,l}^e, l = 1,\ldots,M$. Here, $q_{i,l}^e$ is the contribution of channel l to the excess $\max(q_i^* - q_i^G, 0)$, once the contributions of channels $1, \ldots, l-1$ have been accounted for and is given by:

$$q_{i,l}^e = \min\Bigg(\frac{W(1-\alpha)}{|S_l^*|},$$

$$\max(q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,l-1}^e, 0)\Bigg),\ i \in S_l^* \tag{285}$$

$$q_{i,l}^e = 0,\ i \notin S_l^* \tag{286}$$

We motivate the expressions above. The second term in the min in (285) is equal to the as yet unaccounted for excess, if any, obtained by subtracting the contributions $q_{i,1}^e, \ldots, q_{i,l-1}^e$ of channels $1, \ldots, l-1$ from the total excess throughput $\max(q_i^* - q_i^G, 0)$. Also, since channel l is shared by $|S_l^*|$ networks, $q_{i,l}^e \leq \frac{W(1-\alpha)}{|S_l^*|}$. Hence, $q_{i,l}^e$ is the minimum of the two terms in (285).

From (285) and (286), it can be shown using a simple, yet tedious, case by case analysis that:

$$q_i^* - q_i^G \leq \sum_{l=1}^{M} q_{i,l}^e,\ i = 1, \ldots, N \tag{287}$$

We relegate the proof of (287) to Section 7.10.

Let $y_{i,l}^e(.)$ be the $(q_i^G + q_{i,1}^e + \ldots + q_{i,l}^e)$-residual bid function of network i.

Now,

$$
\begin{aligned}
R^* &= \sum_{i=1}^{N} y_i(q_i^*) \\
&\leq \sum_{i=1}^{N} y_i(q_i^G + \sum_{l=1}^{M} q_{i,l}^e) \\
&\quad \text{(by (287) and since } y_i(.) \text{ is increasing)} \\
&= \sum_{i=1}^{N} \left(y_i(q_i^G) + \sum_{l=1}^{M} y_{i,l-1}^e(q_{i,l}^e) \right) \text{ (by Lemma 61)} \\
&= R^G + \sum_{l=1}^{M} \sum_{i=1}^{N} y_{i,l-1}^e(q_{i,l}^e) \\
&= R^G + \sum_{l=1}^{M} \sum_{i \in S_l^*} y_{i,l-1}^e(q_{i,l}^e) \qquad (288)
\end{aligned}
$$

where the last step follows since $q_{i,l}^e = 0$ if $i \notin S_l^*$ by (286) and since $y_{i,l-1}^e(0) = 0$.

Now, by the definitions of $y_i^{M-1}(.)$ and $y_{i,l-1}^e(.)$, part 2 of Lemma 62 and (288), we get:

$$
R^* - R^G \leq \sum_{l=1}^{M} \left(\sum_{i \in S_l^*} y_i^{M-1}(q_{i,l}^e) \right) \qquad (289)
$$

Now, $q_{i,l}^e \leq \frac{W(1-\alpha)}{|S_l^*|}$ by (285) and (286), and since $y_i^{M-1}(.)$ is increasing by part 2 of Lemma 60, we get the following inequality from (289):

$$
R^* - R^G \leq \sum_{l=1}^{M} \left(\sum_{i \in S_l^*} y_i^{M-1} \left(\frac{W(1-\alpha)}{|S_l^*|} \right) \right) \qquad (290)
$$

Now, we have:

$$
\sum_{i \in S_l^*} y_i^{M-1} \left(\frac{W(1-\alpha)}{|S_l^*|} \right) \leq R^M \qquad (291)
$$

because when the greedy algorithm was about to allocate channel M, the increase in revenue it would have got from the channel if it allocated the channel to the $|S_l^*|$ networks in the set S_l^* is equal to the expression on the left hand side of (291) (refer to Lemma 61 and the discussion immediately following it). This expression is at most R^M, since the greedy algorithm allocates the M'th channel so as to maximize the increase in revenue from it.

By (290) and (291), we get:

$$\begin{aligned} R^* - R^G &\leq \sum_{l=1}^{M} R^M \\ &\leq R^G \text{ (from (284))} \end{aligned}$$

The result follows.

□

7.7.5 Efficient Implementation

We now describe an efficient implementation of the greedy algorithm.

We first discuss how to store the function $y_i^l(.)$ so that $y_i^l(q)$ can be found for any q in $O(\log P)$ time. Recall from part 2 of Lemma 60 that $y_i^l(.)$ is piecewise linear. Similar to the representation of the bid function $y_i(.)$, $y_i^l(.)$ is stored by storing its value at P_i^l values $q_{i,1}^l$, …, $q_{i,P_i^l}^l$, which are the breakpoints of the piecewise linear function $y_i^l(.)$. Also, $y_i^l(q)$, where $q_{i,v}^l < q < q_{i,v+1}^l$ is found by linear interpolation similar to (275):

$$y_i^l(q) = y_i^l(q_{i,v}^l) + \left(\frac{y_i^l(q_{i,v+1}^l) - y_i^l(q_{i,v}^l)}{q_{i,v+1}^l - q_{i,v}^l} \right) (q - q_{i,v}^l) \tag{292}$$

Now, the numbers $q_{i,1}^l, \ldots, q_{i,P_i^l}^l$ and the numbers $y_i^l(q_{i,1}^l), \ldots, y_i^l(q_{i,P_i^l}^l)$ can be stored in two sorted arrays, so that for any v, $q_{i,v}^l$ and $y_i^l(q_{i,v}^l)$ can be accessed in constant time.

Also, since $P_i^l \leq P_i \leq P$ (see the last step in the steps below), given any q, we can find v such that $q_{i,v}^l \leq q < q_{i,v+1}^l$ by binary search [11] in $O(\log P)$ time. Once this v is found, we can find $y_i^l(q)$ in constant time using (292).

Suppose the algorithm has allocated the first $l-1$ channels and hence has computed $y_i^{l-1}(.)$ and $q_{i,v}^{l-1}, v = 1, \ldots, P_i^{l-1}$. Also, suppose the l'th channel has been divided into m_j secondary parts. While allocating channel l, in Step 3 , $y_i^l(.)$ can be found as follows from $y_i^{l-1}(.)$ using the update rule in part 1 of Lemma 62. For network i, first find out v such that:

$$q_{i,v}^{l-1} \leq \frac{W(1-\alpha)}{m_j} < q_{i,v+1}^{l-1}.$$

Then find $y_i^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$ using equation (292). Next, perform the following steps:

$q_{i,1}^l = 0$

$y_i^l(q_{i,1}^l) = 0$

for $t = 2, 3, \ldots, P_i^{l-1} - v + 1$ **do**

  $q_{i,t}^l = q_{i,t+v-1}^{l-1} - \frac{W(1-\alpha)}{m_j}$

  $y_i^l(q_{i,t}^l) = y_i^{l-1}(q_{i,t+v-1}^{l-1}) - y_i^{l-1}\left(\frac{W(1-\alpha)}{m_j}\right)$

end for

$P_i^l = P_i^{l-1} - v + 1$

The second statement in the **for** loop implements the update rule in part 1 of Lemma 62. Also, it can be checked that the first statement in the **for** loop appropriately sets the breakpoints of the function $y_i^l(.)$.

It can be shown that the running time of the greedy algorithm is $O(nMN\log NP + MPm_n)$ when the above implementation is used.

7.8 Simulations

In Section 7.7.4, we proved that the greedy approximation algorithm achieves an approximation ratio of 2. In this section, we show via simulations, that in fact, for a variety of scenarios, the greedy algorithm achieves the optimal revenue.

In all our simulations, we used the values $n = 2$, $m_1 = 1$, $m_2 = 4$ and $W(1-\alpha) =$ 4 units. First, we simulated the case in which the bid function of every network is different and is a piecewise linear approximation of a quadratic function. Let C_{min}, C_{max} and MAX be parameters such that $C_{max} > C_{min} > 0$ and $MAX > 0$. Consider the following quadratic function:

$$\widehat{y}_i(q) = c_i \left(1 - \frac{(q - MAX)^2}{(MAX)^2}\right), \; i = 1, \ldots, N \tag{293}$$

The bid function $y_i(q)$ of network i is chosen to be a piecewise-linear approximation of the above function, where the parameters c_i, $i = 1, \ldots, N$ are uniformly spaced in the interval $[C_{min}, C_{max}]$:

$$c_i = C_{min} + (i-1)\frac{(C_{max} - C_{min})}{N-1}, \; i = 1, \ldots, N \tag{294}$$

With these bid functions, we found the revenue using the greedy approximation algorithm and the optimal revenue using the dynamic programming algorithm in Section 7.6. We used small values for n and M since the running time of the dynamic programming algorithm grows rapidly with these parameters (see Section 7.6.2). For different values of the parameters N, C_{min}, C_{max} and MAX, we evaluated the revenues of the greedy algorithm and the optimal revenue for M varying from 5 to 60 and found that the greedy algorithm achieves the optimal revenue.

Next, we considered the case in which there are two classes of networks and the bid function of each network in the same class is the same. The bid functions $y_i(q)$ of networks $i = 1,\ldots,N_1$ and of networks $i = N_1+1,\ldots,N$ are piecewise linear approximations of the following exponential functions respectively:

$$\widehat{y}_i(q) = B_1(1-\exp(-a_1 q)),\ i = 1,\ldots,N_1 \tag{295}$$

$$\widehat{y}_i(q) = B_2(1-\exp(-a_2 q)),\ i = N_1+1,\ldots,N \tag{296}$$

where a_1, a_2, B_1, B_2 and N_1 are parameters. For different values of these parameters, we evaluated the revenues of the greedy algorithm and the optimal revenue for M varying from 5 to 60 and found that the greedy algorithm achieves the optimal revenue.

Thus, although the worst-case approximation ratio of the greedy algorithm is 2, in a variety of scenarios, it achieves the optimal revenue.

Nevertheless, we could construct some pathological examples in which the greedy algorithm achieves a revenue equal to $\frac{5}{6}$ times the optimal revenue and is therefore strictly sub-optimal. We now describe one such example. Let $M = 2$, $N = 3$, $n = 2$, $m_1 = 2$, $m_2 = 3$ and $W(1-\alpha) = 6$. The bid function of network i, $i \in \{1,2,3\}$ is given by:

$$y_i(q) = \begin{cases} q\beta_i & \text{if } q \le 4 \\ 4\beta_i & \text{if } q > 4 \end{cases} \tag{297}$$

where $\beta_1 = \beta_2 = 1$, $\beta_3 = 1 - \varepsilon$ and ε is a small positive constant. It can be checked that the greedy algorithm assigns channel 1 to networks 1 and 2 and channel 2 to networks 1, 2 and 3 and achieves a revenue of $R^G = (10 - 2\varepsilon)$. The optimal algorithm assigns each one of channels 1 and 2 to networks 1, 2 and 3 and achieves a revenue of $R^* = (12 - 4\varepsilon)$. Note that $\frac{R^G}{R^*}$ equals $\frac{5}{6}$ in the limit as ε tends to 0.

In summary, the greedy algorithm is sub-optimal only for pathological input instances, and is optimal for a large variety of "well-behaved" inputs; thus, it performs well in practice.

7.9 Future Work

We now describe some directions for future research. We conjecture that the access allocation problem described in Section 7.7.1 is NP-hard. Our conjecture is motivated by the facts that (a) the bid function of each network can be an *arbitrary real-valued* function satisfying the conditions in Section 7.7.1, (b) the number of secondary networks on each channel can be selected from a possibly large set $\{m_1, \ldots, m_n\}$ and (c) the set of secondary networks on each channel can be an arbitrary subset of the set of all N networks. The proof of the conjecture remains an open problem for future research.

Also, we considered the case when the M channels are identical. The extension to non-identical channels remains an open problem.

When the auctioneer's objective is to maximize its revenue, note that the algorithms that we designed for the access allocation problem can be used to maximize the auctioneer's revenue *given the bids* $z_i(.)$ *of the bidders*. To compute its bid, a bidder i may use different strategies, which it thinks will maximize its net utility in (255). For example, when auctions are conducted periodically, a bidder may compute its bid based on its knowledge of the outcomes of previous auctions. An open problem is the design of allocation strategies for the auctioneer and bidding strategies for the bidders when each player chooses its strategies based on the outcomes of previous auctions in order to influence the other players to act to its own advantage.

7.10 Appendix

Proof of (287)

By (285) and (286), for each channel l, one of the following cases must hold true for each network i:

Case 1: If $i \notin S_l^*$, then $q_{i,l}^e = 0$.

Case 2: Else, if $\frac{W(1-\alpha)}{|S_l^*|} \leq q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,l-1}^e$ then $q_{i,l}^e = \frac{W(1-\alpha)}{|S_l^*|}$.

Case 3: Else, if $q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,l-1}^e < 0$, then $q_{i,l}^e = 0$.

Case 4: Else, $q_{i,l}^e = q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,l-1}^e$.

Fix a network i. If **Case 3** holds for some channel, then let v be the first such channel.

Then, $q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,v-1}^e < 0$, $q_{i,v}^e = 0$ and by (285) and (286), $q_{i,v+1}^e = \ldots = q_{i,M}^e = 0$. Hence:

$$q_i^* - q_i^G < \sum_{l=1}^{M} q_{i,l}^e$$

and (287) is satisfied.

Now, if **Case 4** holds for some channel, then let v be the first such channel. Then $q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,v-1}^e - q_{i,v}^e = 0$ and by (285) and (286), $q_{i,v+1}^e = \ldots = q_{i,M}^e = 0$. Hence:

$$q_i^* - q_i^G = \sum_{l=1}^{M} q_{i,l}^e$$

and (287) is satisfied.

Otherwise, neither of **Case 3** and **Case 4** holds for any channel. Then, for each channel, one of **Case 1** and **Case 2** holds and we get that for $l = 1, \ldots, M$:

$$q_{i,l}^e = \begin{cases} \frac{W(1-\alpha)}{|S_l^*|} & \text{if } i \in S_l^* \\ 0 & \text{else} \end{cases} \tag{298}$$

Let $l = M'$ be the last channel for which $i \in S_l^*$. Then by the above equation, $q_{i,M'}^e = \frac{W(1-\alpha)}{|S_{M'}^*|}$ and by (285) with $l = M'$:

$$q_i^* - q_i^G - q_{i,1}^e - \ldots - q_{i,M'-1}^e \geq \frac{W(1-\alpha)}{|S_{M'}^*|} = q_{i,M'}^e \tag{299}$$

By definition of M', $q_{i,M'+1}^e = \ldots = q_{i,M}^e = 0$. So if equality holds in (299), then (287) is satisfied with equality. If there is a strict inequality in (299), then:

$$q_i^* > q_{i,1}^e + \ldots + q_{i,M}^e$$

which is a contradiction because

$$q_i^* = q_{i,1}^e + \ldots + q_{i,M}^e$$

since $q_{i,l}^e$ given by (298) is precisely the amount of secondary throughput allocated by the optimal algorithm to network i on channel l. Hence, there cannot be a strict inequality in (299).

Thus, (287) is satisfied in all cases.

Chapter 8

Conclusions and Future Work

We investigated the economics of spectrum allocation in CRNs in this dissertation. We considered both the one-step allocation scheme, in which the regulator allocates spectrum simultaneously to the primaries and the secondaries in a single allocation and the two-step allocation scheme in which the regulator first allocates spectrum to primary users, who then separately allocate unused portions on their channels to secondary users.

Chapters 2 to 6 focussed on the two-step allocation scenario. In particular, Chapters 2 to 5 analyzed the case in which there are a small number of primaries and secondaries using the framework of game theory. In Chapter 3, we found the Nash equilibrium (NE) and proved its uniqueness for price competition among multiple primaries when all the players are located in a single location. The structure of the NE provides several interesting insights into the behavior of the primaries; in particular, it suggests that we can expect to see randomization in the setting of the prices by the primaries. As we mentioned earlier, this can be interpreted as the primaries holding sales to attract secondaries. In Chapter 4, we generalized our model to allow for random valuations of secondaries, found the symmetric NE and showed its uniqueness. We showed that the symmetric NE strategy can be non-contiguous, unlike in the case of constant valuations of secondaries. In Chapter 5, we analyzed price competition among primaries in the presence of spatial reuse. We showed that in a fairly general class of graphs, called mean valid graphs, which covers several conflict graphs that commonly arise in practice, there exists a unique NE in the class of NE in which all the primaries use the same distribution for selecting the independent set to offer bandwidth at. Also, this NE has a simple form and we provided a system of equations which can be solved to explicitly compute the NE.

We have mainly studied the game in a single slot. As future work, it will be interesting to analyze the case where primaries interact over multiple slots and employ learning strategies to adapt their behavior in a slot based on their experience from past slots. Another open question is whether there exist new equilibria if we allow for correlated equilibria. Analysis of the game in which groups of primaries collude and coordinate their prices is another open problem for future research.

In Chapter 6, we considered a spectrum market with a large number of primaries and secondaries and analyzed the problem of optimal dynamic selection of a portfolio of different types of spectrum contracts using the framework of stochastic dynamic programming. We proved several interesting structural properties of the optimal solution that provide insight and can be used to speed up the computation of the optimal solution.

In Chapter 7, we designed an auction framework for the one-step allocation scenario– we devised several bidding languages that the primaries and secondaries can use to compactly express their bids and polynomial-time algorithms that the regulator can use to solve the access allocation problem.

Bibliography

[1] *FCC Auctions*. Available at: http://wireless.fcc.gov/auctions/.

[2] I. F. Akyildiz, W.-Y. Lee, M. C. Vuran, and S. Mohanty. Next generation/dynamic spectrum access/cognitive radio wireless networks: A survey. *Computer Networks Journal (Elsevier*, 50:2127–2159, 2006.

[3] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and hardness of approximation problems. In *Proc. of FOCS*, pages 14–23, Oct. 1992.

[4] P. Berman and M. Furer. Approximating maximum independent set in bounded degree graphs. In *Proc. of Symposium on Discrete Algorithms*, pages 365–371, 1994.

[5] D. Bertsekas and R. Gallager. *Data Networks*. Prentice Hall, second edition edition, 1992.

[6] D. Bertsimas and J. Tsitsiklis. *Introduction to Linear Optimization*. Athena-Scientific, 1997.

[7] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[8] C.E Caicedo and M.B.H Weiss. Spectrum trading: An analysis of implementation issues. In *Proceedings of New Frontiers in Dynamic Spectrum Access Networks, DySPAN 2007*, pages 579–584, 2007.

[9] S. Chawla and T. Roughgarden. Bertrand competition in networks. In *In Symposium on Algorithmic Game Theory*, May 2008.

[10] S.-F. Cheng, D.M. Reeves, Y. Vorobeychik, and M.P. Wellman. Notes on equilibria in symmetric games. In *In AAMAS-04 Workshop on Game-Theoretic and Decision-Theoretic Agents,*, 2004.

[11] T. Cormen, C. Leiserson, R. Rivest, and C. Stein. *Introduction to Algorithms*. MIT Press, second edition edition, 2001.

[12] C. Courcoubetis and R. Weber. *Pricing Communication Networks*. John Wiley & Sons, Ltd., 2003.

[13] P. Cramton, Y. Shoham, and R. Steinberg. *Combinatorial Auctions*. MIT Press, 2006.

[14] A.A. Daoud, M. Alanyali, and D. Starobinski. Secondary pricing of spectrum in cellular cdma networks. In *Proceedings of New Frontiers in Dynamic Spectrum Access Networks, DySPAN 2007*, pages 535–542, 2007.

[15] H.A. David and H.N. Nagaraja. *Order Statistics*. Wiley, third edition edition, 2003.

[16] M. H. A. Davis, M. A. H. Dempster, S. P. Sethi, and D. Vermes. Optimal capacity expansion under uncertainty. *Advances in Applied Probability*, 19(1):156–176, 1987.

[17] J. Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards*, 69B:125–130, 1965.

[18] R Etkin, A Parekh, and D Tse. Spectrum sharing for unlicensed bands. *IEEE Journal on Selected Areas in Communications*, 25(3):517–528, April 2007.

[19] B.S. Everitt. *The Cambridge Dictionary of Statistics*. Cambridge University Press, 3rd ed. edition, 2006.

[20] G. R. Faulhaber. The question of spectrum: Technology, management, and regime change. *Journal on Telecommunications & High Technology Law*, 4(123), 2005.

[21] FCC Spectrum Policy Task Force. *Report of the Spectrum Efficiency Working Group*, Nov. 2002. Available at: http://www.fcc.gov/sptf/reports.html.

[22] S. Gandhi, C. Buragohain, L. Cao, H. Zheng, and S. Suri. Towards real-time dynamic spectrum auctions. *Computer Networks*, 52(4):879–897, 2008.

[23] P. Gupta and P.R. Kumar. The capacity of wireless networks. *IEEE TIT*, Vol. 46(2), March 2000.

[24] M.M. Halldorsson, J. Y. Halpern, L. Li, and V.S. Mirrokni. On spectrum sharing games. In *ACM Symposium on Principles of Distributed Computing*, pages 107–114, 2004.

[25] K. Hinderer. On the structure of solutions of stochastic dynamic programs. In *Proc. of the Seventh Conference on Probability Theory*, 1982.

[26] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, March 1963.

[27] J. Huang, R. Berry, and M. Honig. Auction mechanisms for distributed spectrum sharing. In *In Proc. of 42nd Allerton Conference*, 2004.

[28] J. Mitola III. Cognitive radio for flexible mobile multimedia communication. In *Proc. IEEE International Workshop on Mobile Multimedia Communications (MoMuC)*, pages 3–10, Nov. 1999.

[29] J. Mitola III. *Cognitive radio: an integrated agent architecture for software defined radio*. Ph.d thesis, KTH Royal Institute of Technology, 2000.

[30] O. Ileri, D. Samardzija, T. Sizer, and N. B. Mandayam. Demand responsive pricing and competitive spectrum allocation via a spectrum server. In *In Proc. of IEEE DySPAN*, pages 194–202, 2005.

[31] M. Janssen and E. Rasmusen. Bertrand competition under uncertainty. *Journal of Industrial Economics*, 50(1):11–21, 2002.

[32] Z. Ji and K.J. Ray Liu. Belief-assisted pricing for dynamic spectrum allocation in wireless networks with selfish users. In *In Proc. of IEEE SECON*, 2006.

[33] Z. Ji and K.J.R Liu. Collusion-resistant dynamic spectrum allocation for wireless networks via pricing. In *Proceedings of New Frontiers in Dynamic Spectrum Access Networks, DySPAN 2007*, pages 187–190, 2007.

[34] S. Kimmel. *Bertrand Competition Without Completely Certain Production*. U.S. Department of Justice, Antitrust Division, 2002.

[35] J. Kleinberg and E. Tardos. *Algorithm Design*. Addison Wesley, 2005.

[36] D.M. Kreps and J.A. Scheinkman. Quantity precommitment and bertrand competition yield cournot outcomes. *Bell Journal of Economics*, 14:326–337, 1983.

[37] R.H. Lasseter. Microgrids and distributed generation. *Journal of Energy Engineering*, 133:144–149, 2007.

[38] W. Lehr and J. Crowcroft. Managing shared access to a spectrum commons. In *Proceedings of the First IEEE Symposium on New Frontiers in Dynamic Spectrum Access Networks*, 2005.

[39] H. Luss. Operations research and capacity expansion problems: A survey. *Operations Research*, 30(5):907–947, 1982.

[40] P. Maille and B. Tuffin. Analysis of price competition in a slotted resource allocation game. In *In Proc. of Infocom*, 2008.

[41] P. Maille and B. Tuffin. Price war with partial spectrum sharing for competitive wireless service providers. In *In Proc. of IEEE Globecom*, Dec. 2009.

[42] A. Mas-Colell, M. Whinston, and J. Green. *Microeconomic Theory*. Oxford University Press, 1995.

[43] R. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press, 1997.

[44] N. Nie and C. Comaniciu. Adaptive channel allocation spectrum etiquette for cognitive radio networks. *Mobile Networks and Applications*, 11(6):779–797, 2006.

[45] N. Nisan and A. Ronen. Computationally feasible vcg mechanisms. In *Proc. of ACM Conference on Electronic Commerce*, 2000.

[46] D. Niyato and E. Hossain. Competitive pricing for spectrum sharing in cognitive radio networks: Dynamic game, inefficiency of nash equilibrium, and collusion. *IEEE JSAC*, Vol. 26(1), 2008.

[47] D. Niyato, E. Hossain, and Z. Han. Dynamics of multiple-seller and multiple-buyer spectrum trading in cognitive radio networks: A game-theoretic modeling approach. *IEEE TMC*, Vol. 8(8):1009–1022, Aug. 2009.

[48] M.J. Osborne and C. Pitchik. Price competition in a capacity-constrained duopoly. *Journal of Economic Theory*, 38(2):238–260, 1986.

[49] J. Peha. Spectrum management policy options. *IEEE Commun. Surv.*, 1998.

[50] J. Peha. Approaches to spectrum sharing. *IEEE Commun. Magazine*, pages 10–12, 2005.

[51] J. Peha. Emerging technology and spectrum policy reform. In *In Proc. of ITU Workshop on Market Mechanisms for Spectrum Management*, Jan. 2007.

[52] J. M. Peha and S. Panichpapiboon. Real-time secondary markets for spectrum. *Telecommunications Policy*, 28(7–8):603–618, 2004.

[53] C. Peng, H. Zheng, and B.Y. Zhao. Utilization and fairness in spectrum assignment for opportunistic spectrum access. *Mobile Networks and Applications*, 11(4):555–576, Aug. 2006.

[54] M. Puterman. *Markov Decision Processes*. Wiley, 1994.

[55] T.S. Rappaport. *Wireless Communications: Principles and Practices*. Prentice Hall, 2nd ed. edition, 2002.

[56] A. Ronen. Mechanism design with incomplete languages. In *Proc. of ACM Conference on Electronic Commerce*, 2001.

[57] S. Ross. *Introduction to Stochastic Dynamic Programming*. Academic Press, 1983.

[58] W. Rudin. *Principles of Mathematical Analysis*. Mc-Graw Hill, third edition edition, 1976.

[59] A. Sahasrabudhe and K. Kar. Bandwidth allocation games under budget and access constraints. In *Proceedings of CISS*, March 2008.

[60] H. Scarf. The optimality of (s,s) policies in the dynamic inventory problem. In J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in Social Sciences*. Stanford University Press, 1960.

[61] S. Sengupta and M. Chatterjee. Sequential and concurrent auction mechanisms for dynamic spectrum access. In *In Proc. of CROWNCOM*, 2007.

[62] S. Sengupta and M. Chatterjee. An economic framework for dynamic spectrum access and service pricing. *IEEE/ ACM TON*, 17(4), Aug. 2009.

[63] S.P. Sethi and F. Cheng. Optimality of (s,s) policies in inventory models with markovian demand. *Operations Research*, 45(6):931–939, 1997.

[64] S. Subramani, T. Basar, S. Armour, D. Kaleshi, and Z. Fan. Non-cooperative equilibrium solutions for spectrum access in distributed cognitive radio networks. In *Proceedings of New Frontiers in Dynamic Spectrum Access Networks, DySPAN 2008*, pages 1–5, 2008.

[65] A.P Subramanian and H. Gupta. Fast spectrum allocation in coordinated dynamic spectrum access based cellular networks. In *Proceedings of New Frontiers in Dynamic Spectrum Access Networks, DySPAN 2007*, pages 320–330, 2007.

[66] L. Tassiulas and A. Ephremides. Stability properties of constrained queuing systems and scheduling for maximum throughput in multihop radio networks. *IEEE TAC*, Vol. 37(12):1936–1949, Dec. 1992.

[67] M. Tennenholtz. Some tractable combinatorial auctions. In *Proc. of National Conference on Artificial Intelligence (AAAI)*, 2000.

[68] T.M. Valletti. Spectrum trading. *Telecommunications Policy*, 25:655–670, 2001.

[69] H.R. Varian. A model of sales. *American Economic Review*, 70:651–659, 1980.

[70] J. E. Walsh. Existence of every possible distribution for any sample order statistic. *Statistical Papers*, 10(3), Sept. 1969.

[71] D. West. *Introduction to Graph Theory*. Prentice Hall, second edition edition, 2000.

[72] L. White. *A Parametric Study of Matchings and Coverings in Weighted Graphs*. Ph.d. dissertation, University of Michigan, Ann Arbor, 1967.

[73] L. White. An efficient algorithm for maximum k-matching in weighted graphs. In *Proc. of 12th Allerton Conference on Circuits and Systems Theory*, 1974.

[74] Y. Xing, R. Chandramouli, and C. Cordeiro. Price dynamics in competitive agile spectrum access markets. *IEEE JSAC*, Vol. 25(3), April 2007.

[75] Q. Zhao and B. Sadler. A survey of dynamic spectrum access. *IEEE Signal Processing Magazine*, 24(3):79–89, 2007.

[76] X. Zhou, S. Gandhi, S. Suri, and H. Zheng. ebay in the sky: Strategy-proof wireless spectrum auctions. In *In Proc. of MobiCom*, 2008.

[77] X. Zhou and H. Zheng. Trust: A general framework for truthful double spectrum auctions. In *In Proc. of Infocom*, April 2009.