12-7-2010

Multifield Galileons and Higher Codimension Branes

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Suggested Citation:
Hinterbichler, K., Trodden, M. and Wesley, D. (2010). "Multifield Galileons and higher codimension branes." *Physical Review D*. **82**, 124018.

http://dx.doi.org/10.1103/PhysRevD.82.124018

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Abstract
In the decoupling limit, the Dvali-Gabadadze-Porrati model reduces to the theory of a scalar field π, with interactions including a specific cubic self-interaction—the Galileon term. This term, and its quartic and quintic generalizations, can be thought of as arising from a probe 3-brane in a five-dimensional bulk with Lovelock terms on the brane and in the bulk.We study multifield generalizations of the Galileon and extend this probe-brane view to higher codimensions. We derive an extremely restrictive theory of multiple Galileon fields, interacting through a quartic term controlled by a single coupling, and trace its origin to the induced brane terms coming from Lovelock invariants in the higher codimension bulk. We explore some properties of this theory, finding de Sitter like self-accelerating solutions. These solutions have ghosts if and only if the flat space theory does not have ghosts. Finally, we prove a general nonrenormalization theorem: multifield Galileons are not renormalized quantum mechanically to any loop in perturbation theory.

Disciplines
Physical Sciences and Mathematics | Physics

Comments
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Multifield Galileons and higher codimension branes

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(Received 30 August 2010; published 7 December 2010)

In the decoupling limit, the Dvali-Gabadadze-Porrati model reduces to the theory of a scalar field π, with interactions including a specific cubic self-interaction—the Galileon term. This term, and its quartic and quintic generalizations, can be thought of as arising from a probe 3-brane in a five-dimensional bulk with Lovelock terms on the brane and in the bulk. We study multifield generalizations of the Galileon and extend this probe-brane view to higher codimensions. We derive an extremely restrictive theory of multiple Galileon fields, interacting through a quartic term controlled by a single coupling, and trace its origin to the induced brane terms coming from Lovelock invariants in the higher codimension bulk. We explore some properties of this theory, finding de Sitter like self-accelerating solutions. These solutions have ghosts if and only if the flat space theory does not have ghosts. Finally, we prove a general nonrenormalization theorem: multifield Galileons are not renormalized quantum mechanically to any loop in perturbation theory.

DOI: 10.1103/PhysRevD.82.124018 PACS numbers: 04.50.Kd, 11.25.Uv

I. INTRODUCTION

A particularly fruitful way of extending both the standard models of particle physics and cosmology is the hypothesis of extra spatial dimensions beyond the three that manifest themselves in everyday physics. Historically, such ideas have provided a tantalizing possibility of unifying the basic forces through the geometry and topology of the extra-dimensional manifold and, in recent years, have been the basis for attempts to tackle the hierarchy problem. In this latter incarnation, a crucial insight has been the realization that different forces may operate in different dimensionalities by confining the standard model particles to a $3+1$-dimensional submanifold—the brane—while gravity probes the entire spacetime—the bulk—due to the equivalence principle. Such constructions allow, among other unusual features, for infinite extra dimensions, in contrast to the more usual compactified theories.

In the case of a single extra dimension, a further refinement was introduced in [1], where a separate induced gravity term was introduced on the brane. The resulting $4+1$-dimensional action

$$S = \frac{M_5^3}{2}\int d^5x\sqrt{-G}R[G] + \frac{M_4^2}{2}\int d^4x\sqrt{-g}R[g] \quad (1)$$

is known as the DGP (Dvali-Gabadadze-Porrati) model and yields a rich and dramatic phenomenology, with, for example, a branch of four-dimensional cosmological solutions, which self-accelerate at late times, and a set of predictions for upcoming missions, which will perform local tests of gravity.

It is possible to derive a four-dimensional effective action for the DGP model by integrating out the bulk. It has been claimed [2,3] that a decoupling limit for DGP exists, in which the four-dimentional effective action reduces to a theory of a single scalar π, representing the position of the brane in the extra dimension, with a cubic self-interaction term $\sim(\partial\pi)^2\Box\pi$ (though this claim is not without controversy, see for example [4]). This term has the properties that its field equations are second order (despite the fact that the Lagrangian is higher order), which is important for avoiding ghosts. It is also invariant (up to a total derivative) under the following *Galilean* transformation:

$$\pi(x) \rightarrow \pi(x) + c + b_\mu x^\mu, \quad (2)$$

with c and b_μ constants.

These properties are interesting in their own right, and terms that generalize the cubic DGP term studied (without considering a possible higher-dimensional origin) in [5] are referred to as *Galileons*. Requiring the invariance (2) forces the equations of motion to contain at least two derivatives acting on each field, and there exists a set of terms that lead to such a form with exactly two derivatives on each field (in fact, the absence of ghosts in a nonlinear regime demands that there be at most two derivatives on each field). These are the terms that were classified in [5] and take the schematic form

$$\mathcal{L}_n \sim \partial\pi\partial\pi(\partial^2\pi)^{n-2}, \quad (3)$$

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with suitable Lorentz contractions and dimensionful coefficients. In d spacetime dimensions there are d such terms, corresponding to $n = 2, \ldots, d+1$. The $n = 2$ term is just the usual kinetic term $(\partial\pi)^2$, the $n = 3$ case is the DGP term $(\partial\pi)^2\Box\pi$, and the higher terms generalize these.

These terms have appeared in various contexts apart from DGP; for example, the $n = 4, 5$ terms seem to appear in the decoupling limit of an interesting interacting theory of Lorentz invariant massive gravity [6]. They have been generalized to curved space [7,8], identified as possible ghost-free modifications of gravity and cosmology [5,9–13], and used to build alternatives to inflation [14] and dark energy [15,16].

Another remarkable fact, which we will prove for a more general multifield model in Sec. VI, is that the $\mathcal{L}_n$ terms above do not get renormalized upon loop corrections, so that their classical values can be trusted quantum mechanically. Also, from an effective field theory point of view, there can exist regimes in which only these Galileon terms are important.

It is natural to consider whether the successes of the DGP model can be extended and improved in models in which the bulk has higher codimension, and whether the drawbacks of the five-dimensional approach, such as the ghost problem in the accelerating branch, might be ameliorated in such a setting. Since our understanding of the complexities of the DGP model has arisen primarily through the development of a four-dimensional effective theory in a decoupling limit, one might hope to achieve a similar understanding of theories with larger codimension. This is the aim of this paper.

We do not consider the full higher codimension DGP or a decoupling limit thereof. Instead, we are interested in generalizing the Galileon actions to multiple fields and exploring the probe-brane-world view of these terms, extending the work of [17] on the single-field case. The theory which emerges from the brane construction in codimension N has an internal $SO(N)$ symmetry in addition to the Galilean symmetry. This is extremely restrictive, and in four dimensions it turns out that there is a single nonlinear term compatible with it. This makes for a fascinating four-dimensional field description: a scalar field theory with a single allowed coupling, which receives no quantum corrections.

II. SINGLE-FIELD GALILEONS AND GENERALIZATIONS

In codimension one, the decoupling limit of DGP consists of a four-dimensional effective theory of gravity coupled to a single scalar field π, representing the bending mode of the brane in the fifth dimension. The π field self-interaction includes a cubic self-interaction $\sim(\partial\pi)^2\Box\pi$, which has the following two properties:

(1) The field equations are second order,

(2) The terms are invariant up to a total derivative under the internal Galilean transformations $\pi \to \pi + c + b_\mu x^\mu$, where c, b_μ are arbitrary real constants.

In [5], this was generalized, and all possible Lagrangian terms for a single scalar with these two properties were classified in all dimensions. They are called Galileon terms, and there exists a single Galileon Lagrangian at each order in π, where "order" refers to the number of copies of π that appear in the term. For $n \geq 1$, the $(n+1)$th order Galileon Lagrangian is

$$\mathcal{L}_{n+1} = n\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \times (\partial_{\mu_1}\pi\partial_{\nu_1}\pi\partial_{\mu_2}\partial_{\nu_2}\pi\cdots\partial_{\mu_n}\partial_{\nu_n}\pi), \tag{4}$$

where

$$\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \equiv \frac{1}{n!}\sum_p(-1)^p\eta^{\mu_1 p(\nu_1)} \times \eta^{\mu_2 p(\nu_2)}\cdots\eta^{\mu_n p(\nu_n)}. \tag{5}$$

The sum in (5) is over all permutations of the ν indices, with $(-1)^p$ the sign of the permutation. The tensor (5) is antisymmetric in the μ indices, antisymmetric the ν indices, and symmetric under interchange of any μ, ν pair with any other. These Lagrangians are unique up to total derivatives and overall constants. Because of the antisymmetry requirement on η, only the first n of these Galileons are nontrivial in n dimensions. In addition, the tadpole term π is Galilean-invariant, and we therefore include it as the first-order Galileon.

Thus, at the first few orders, we have

$$\begin{aligned}
\mathcal{L}_1 &= \pi,\\
\mathcal{L}_2 &= [\pi^2],\\
\mathcal{L}_3 &= [\pi^2][\Pi] - [\pi^3],\\
\mathcal{L}_4 &= \frac{1}{2}[\pi^2][\Pi]^2 - [\pi^3][\Pi] + [\pi^4] - \frac{1}{2}[\pi^2][\Pi^2],\\
\mathcal{L}_5 &= \frac{1}{6}[\pi^2][\Pi]^3 - \frac{1}{2}[\pi^3][\Pi]^2 + [\pi^4][\Pi] - [\pi^5]\\
&\quad + \frac{1}{3}[\pi^2][\Pi^3] - \frac{1}{2}[\pi^2][\Pi][\Pi^2] + \frac{1}{2}[\pi^3][\Pi^2].
\end{aligned} \tag{6}$$

We have used the notation Π for the matrix of partials $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$, and $[\Pi^n] \equiv \mathrm{Tr}(\Pi^n)$, e.g. $[\Pi] = \Box\pi$, $[\Pi^2] = \partial_\mu\partial_\nu\pi\partial^\mu\partial^\nu\pi$, and $[\pi^n] \equiv \partial\pi\cdot\Pi^{n-2}\cdot\partial\pi$, i.e. $[\pi^2] = \partial_\mu\pi\partial^\mu\pi$, $[\pi^3] = \partial_\mu\pi\partial^\mu\partial^\nu\pi\partial_\nu\pi$. The above terms are the only ones which are nonvanishing in four dimensions. The second is the standard kinetic term for a scalar, while the third is the DGP π Lagrangian (up to a total derivative).

The equations of motion derived from (4) are

$$\mathcal{E}_{n+1} \equiv \frac{\delta \mathcal{L}_{n+1}}{\delta \pi} = -n(n+1)\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \times (\partial_{\mu_1}\partial_{\nu_1}\pi\partial_{\mu_2}\partial_{\nu_2}\pi\cdots\partial_{\mu_n}\partial_{\nu_n}\pi) = 0 \quad (7)$$

and are second order, as advertised.[1]

The first few orders of the equations of motion are

$$\mathcal{E}_1 = 1, \quad (14)$$

$$\mathcal{E}_2 = -2[\Pi], \quad (15)$$

$$\mathcal{E}_3 = -3([\Pi]^2 - [\Pi^2]), \quad (16)$$

$$\mathcal{E}_4 = -2([\Pi]^3 + 2[\Pi^3] - 3[\Pi][\Pi^2]), \quad (17)$$

$$\mathcal{E}_5 = -\frac{5}{6}([\Pi]^4 - 6[\Pi^4] + 8[\Pi][\Pi^3] - 6[\Pi]^2[\Pi^2] + 3[\Pi^2]^2). \quad (18)$$

By adding a total derivative, and by using the following identity for the η symbol in $\mathcal{L}_{n+1}$

$$\eta^{\mu_1\nu_1\ldots\mu_n\nu_n} = \frac{1}{n}(\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2\ldots\mu_n\nu_n} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1\mu_2\nu_3\ldots\mu_n\nu_n} + \cdots + (-1)^n\eta^{\mu_1\nu_n}\eta^{\mu_2\nu_1\ldots\mu_n\nu_{n-1}}), \quad (19)$$

the Galileon Lagrangians can be brought into a (sometimes more useful) different form, which illustrates that the $(n+1)$th order Lagrangian is just $(\partial\pi)^2$ times the nth order equations of motion

$$\mathcal{L}_{n+1} = -\frac{n+1}{2n(n-1)}(\partial\pi)^2\mathcal{E}_n - \frac{n-1}{2}\partial_{\mu_1} \times [(\partial\pi)^2\eta^{\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}\partial_{\nu_1} \times \pi\partial_{\mu_2}\partial_{\nu_2}\pi\cdots\partial_{\mu_{n-1}}\partial_{\nu_{n-1}}\pi]. \quad (20)$$

From the simplified form (20), we can see that $\mathcal{L}_3$, for example, takes the usual Galileon form $(\partial\pi)^2\Box\pi$.

These Galileon actions can be generalized to the multifield case, where there is a multiplet π^I of fields.[2] The action in this case can be written

$$\mathcal{L}_{n+1} = S_{I_1I_2\cdots I_{n+1}}\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \times (\pi^{I_{n+1}}\partial_{\mu_1}\partial_{\nu_1}\pi^{I_1}\partial_{\mu_2}\partial_{\nu_2}\pi^{I_2}\cdots\partial_{\mu_n}\partial_{\nu_n}\pi^{I_n}), \quad (21)$$

with $S_{I_1I_2\cdots I_{n+1}}$ a symmetric constant tensor. This is invariant under individual Galilean transformations for each field $\pi^I \to \pi^I + c^I + b^I_\mu x^\mu$, and the equations of motion are second order

$$\mathcal{E}_I \equiv \frac{\delta\mathcal{L}}{\delta\pi^I} = (n+1)S_{II_1I_2\cdots I_n}\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \times (\partial_{\mu_1}\partial_{\nu_1}\pi^{I_1}\partial_{\mu_2}\partial_{\nu_2}\pi^{I_2}\cdots\partial_{\mu_n}\partial_{\nu_n}\pi^{I_n}). \quad (22)$$

The theory containing these Galilean-invariant operators is not renormalizable, i.e. it is an effective field theory with a cutoff Λ, above which some UV completion is required. As was mentioned in the introduction, the $\mathcal{L}_n$ terms above do not get renormalized upon loop corrections, so that their classical values can be trusted quantum mechanically (see Sec. VI). The structure of the one-loop effective action (in 3 + 1 dimensions) is, schematically[3] [3],

$$\Gamma \sim \sum_m\left[\Lambda^4 + \Lambda^2\partial^2 + \partial^4\log\left(\frac{\partial^2}{\Lambda^2}\right)\right]\left(\frac{\partial\partial\pi}{\Lambda^3}\right)^m. \quad (23)$$

One should consider quantum effects within the effective theory, since there are other operators of the same dimension that might compete with the Galileon terms. However, there can exist interesting regimes where

[1]Beyond their second-order nature, these Lagrangians possess a number of other interesting properties. Under the shift symmetry $\pi \to \pi + \epsilon$, the Noether current is

$$j^\mu_{n+1} = n(n+1)\eta^{\mu\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}(\partial_{\nu_1}\pi\partial_{\mu_2}\partial_{\nu_2}\pi\cdots\partial_{\mu_n}\partial_{\nu_n}\pi). \quad (8)$$

Shift symmetry implies that the equations of motion are equivalent to the conservation of this current

$$\mathcal{E}_{n+1} = -\partial_\mu j^\mu_{n+1}. \quad (9)$$

However, the Noether current itself can also be written as a derivative

$$j^\mu_{n+1} = \partial_\nu j^{\mu\nu}_{n+1}, \quad (10)$$

where there are many possibilities for j^μ_{n+1}, two examples of which are

$$j^{\mu\nu}_{n+1} = n(n+1)\eta^{\mu\nu\mu_2\nu_2\cdots\mu_n\nu_n}(\pi\partial_{\mu_2}\partial_{\nu_2}\pi\cdots\partial_{\mu_n}\partial_{\nu_n}\pi), \quad (11)$$

$$j^{\mu\nu}_{n+1} = -n(n+1)\eta^{\mu\nu\mu_2\nu_2\cdots\mu_n\nu_n} \times (\partial_{\mu_2}\pi\partial_{\nu_2}\pi\partial_{\mu_3}\partial_{\nu_3}\pi\cdots\partial_{\mu_n}\partial_{\nu_n}\pi). \quad (12)$$

Thus the equations of motion can in fact be written as a double total derivative

$$\mathcal{E}_{n+1} = -\partial_\mu\partial_\nu j^{\mu\nu}_{n+1}. \quad (13)$$

[2]As we put the finishing touches to this paper, several preprints appeared which also discuss generalizations to the Galileons [18–20].

[3]Strictly speaking, quantum effects calculable solely within the effective theory are only those associated with log-divergences. Power divergences are regularization dependent and depend upon some UV completion or matching condition. In dimensional regularization with minimal subtraction, they do not even show up, corresponding to making a special and optimistic assumption about the UV completion, i.e. that power-law divergences are precisely cancelled somehow by the UV contributions. However, it is important to stress that the conclusions about the Galileon Lagrangian are true even in the presence of generic power divergences, i.e. even with a generic UV completion.

nonlinearities from the Galileon terms are important, yet quantum effects from terms such as (23) are under control. From the tree-level action containing only the Galileon terms (4), and where all dimensionful couplings carry the scale Λ as appropriate for an effective field theory with cutoff Λ, we see that the strength of classical nonlinearities is measured by

$$\alpha_{\rm cl} \equiv \frac{\partial\partial\pi}{\Lambda^3}, \tag{24}$$

in the sense that the nth order Galileon interaction $\mathcal{L}_n$ is roughly $\alpha_{\rm cl}^{n-2}$ times the kinetic energy for π. On the other hand, by factoring out two powers of π from the effective action

$$\Gamma \sim \sum_{m'} [\alpha_q + \alpha_q^2 + \alpha_q^3 \log\alpha_q]\partial\pi\partial\pi\left(\frac{\partial\partial\pi}{\Lambda^3}\right)^{m'}, \tag{25}$$

it is clear that the quantity suppressing quantum effects relative to classical ones is

$$\alpha_q \equiv \frac{\partial^2}{\Lambda^2}. \tag{26}$$

This separation of scales allows for the existence of regimes in which there exist classical field configurations with nonlinearities of order one $\alpha_{\rm cl} = \partial\partial\pi/\Lambda^3 \sim 1$, and yet which nevertheless satisfy $\alpha_q \ll 1$, so that quantum effects are under control. Thus, it can be possible to study nonlinear classical solutions involving all the Galileon terms and still trust these solutions in light of quantum corrections.[4]

An example of such a configuration can be seen in the theory with only the cubic Galileon term (setting the others to zero is a technically natural choice, since they are not renormalized) coupled to the trace of the stress tensor of matter T,

$$\mathcal{L} = -3(\partial\pi)^2 - \frac{1}{\Lambda^3}(\partial\pi)^2\Box\pi + \frac{1}{M_{\rm Pl}}\pi T. \tag{27}$$

Here $M_{\rm Pl}$ is a mass scale controlling the strength of the coupling to matter (in applications to modified gravity, it is the Planck mass).

Consider the static spherically symmetric solution, $\pi(r)$ around a point source of mass M, $T \sim M\delta^3(r)$ [3]. The solution transitions, at the distance scale $R_V \equiv \frac{1}{\Lambda}\left(\frac{M}{M_{\rm Pl}}\right)^{1/3}$, between a linear and nonlinear regime

$$\pi(r) \sim \begin{cases} \Lambda^3 R_V^2 \left(\frac{r}{R_V}\right)^{1/2} & r \ll R_V, \\ \Lambda^3 R_V^2 \left(\frac{R_V}{r}\right) & r \gg R_V \end{cases}. \tag{28}$$

Assuming $M \gg M_{\rm Pl}$ so that $R_V \gg \frac{1}{\Lambda}$, we can identify three distinct regimes: Far from the source, at distances $r \gg R_V$, we have $\alpha_{\rm cl} \sim (\frac{R_V}{r})^3 \ll 1$ and $\alpha_q \sim \frac{1}{(r\Lambda)^2} \ll 1$, so quantum corrections are under control, but also the interesting classical nonlinearities of the cubic term are unimportant. Close to the source $r \ll \frac{1}{\Lambda}$, we have $\alpha_{\rm cl} \sim (\frac{R_V}{r})^{3/2} \gg 1$ and $\alpha_q \sim \frac{1}{(r\Lambda)^2} \gg 1$. Here, interesting nonlinear effects are important, but quantum effects are not under control, and any attempt to extract physics would require a UV completion. There is, however, an intermediate range $\frac{1}{\Lambda} \ll r \ll R_V$, in which $\alpha_{\rm cl} \sim (\frac{R_V}{r})^{3/2} \gg 1$ and $\alpha_q \sim \frac{1}{(r\Lambda)^2} \ll 1$ so that interesting nonlinear effects are important, while quantum effects are under control.

An analogous situation is familiar from general relativity. In that case, the relevant field is the canonically normalized metric perturbation $g_{\mu\nu} \sim \eta_{\mu\nu} + \frac{1}{M_{\rm Pl}}h_{\mu\nu}$. The action consists of a linear kinetic term $\sim\partial^2 h^2$ and an infinite number of nonlinear terms of the form $\partial^2 h^n$ with $n \geq 3$, which sum up into the Einstein-Hilbert action $\sim M_{\rm Pl}^2\sqrt{-g}R$. Diffeomorphism invariance ensures that the relative coefficients of these nonlinear terms are not renormalized, so their classical forms can be trusted. The measure of nonlinearity in this case is $\alpha_{\rm cl} \sim h/M_{\rm Pl}$, with nonlinear operators suppressed relative to the kinetic terms by powers of this factor. Quantum effects are expected to generate higher curvature terms, for example, $\sqrt{-g}R^2$, $\frac{1}{M_{\rm Pl}^2}\sqrt{-g}R^3$, which will generate higher-derivative operators of the form $\partial^m h^n$ with $m \geq 4$. These are suppressed relative to classical operators by powers of the factor $\alpha_q \sim \frac{\partial}{M_{\rm Pl}}$. The analogous spherically symmetric static solution is $h_{\mu\nu} \sim \frac{M}{M_{\rm Pl} r}$, where $M \gg M_{\rm Pl}$ is the total mass of the solution, so that $\alpha_{\rm cl} \sim \frac{M}{M_{\rm Pl}^2 r}$. Therefore, for $r \gg R_S \equiv \frac{M}{M_{\rm Pl}^2}$ (such as in the solar system), classical nonlinearities are unimportant, whereas for $r \ll R_S$ (such as inside and near the horizon of a black hole), they dominate. Since $\alpha_q \sim \frac{1}{M_{\rm Pl} r}$, quantum effects are negligible for $r \gg \frac{1}{M_{\rm Pl}}$ but become important near and below the Planck length. Thus, the black hole horizon is the interesting middle regime, where classical nonlinearities are large and produce important effects, which can be trusted in light of quantum corrections. These nonlinear, quantum-controlled regimes are where interesting models of inflation, cosmology, modified gravity, etc. employing these Galileon actions should be placed.

III. BRANE ORIGINS OF GALILEAN INVARIANCE

The internal Galilean symmetry $\pi \to \pi + c + b_\mu x^\mu$ of the theories we have discussed above can be thought of as inherited from symmetries of a probe brane floating in a higher-dimensional flat bulk, in a small field limit [17]. To see this, consider a 3-brane (3 + 1 spacetime dimensions) embedded in five-dimensional Minkowski space. Let the

[4]In fact, for even larger nonlinearities $\partial\partial\pi/\Lambda^3 \gg 1$, quantum fluctuations receive a correspondingly larger kinetic term from the expansion of the nonlinear terms about the nontrivial background, thus effectively becoming weakly coupled and suppressing loop corrections even further [3].

bulk coordinates be X^A, ranging over five dimensions, and let the brane coordinates be x^μ, ranging over 4 dimensions. The bulk metric is flat η_{AB}, and the embedding of the brane into the bulk is given by embedding functions $X^A(x)$, which are the dynamical degrees of freedom.

We require the action to be invariant under Poincaré transformations of the bulk

$$\delta_P X^A = \omega^A{}_B X^B + \epsilon^A, \tag{29}$$

where ϵ^A and antisymmetric $\omega^A{}_B$ are the infinitesimal parameters of the bulk translations and Lorentz transformations, respectively. We also require the action to be gauge invariant under reparametrizations of the brane

$$\delta_g X^A = \xi^\mu \partial_\mu X^A, \tag{30}$$

where $\xi^\mu(x)$ is the gauge parameter.

We may use this gauge freedom to fix a unitary gauge

$$X^\mu(x) = x^\mu, \qquad X^5(x) \equiv \pi(x), \tag{31}$$

where the index set A has been separated into μ along the brane and X^5 transverse to the brane. Now,

$$\delta_P X^\mu = \omega^\mu{}_\nu x^\nu + \omega^\mu{}_5 \pi + \epsilon^\mu, \tag{32}$$

and so the Poincaré transformations (29) do not preserve this gauge. However, the gauge may be restored by making a gauge transformation $\delta_g X^\mu = \xi^\nu \partial_\nu x^\mu = \xi^\mu$ with the choice

$$\xi^\mu = -\omega^\mu{}_\nu x^\nu - \omega^\mu{}_5 \pi - \epsilon^\mu. \tag{33}$$

Thus, the combined transformation $\delta_{P'} = \delta_P + \delta_g$ leaves the gauge fixing intact and is a symmetry of the gauge fixed action. Its action on the remaining field π is

$$\begin{aligned}\delta_{P'}\pi = {}& -\omega^\mu{}_\nu x^\nu \partial_\mu \pi - \epsilon^\mu \partial_\mu \pi + \omega^5{}_\mu x^\mu \\ & - \omega^\mu{}_5 \pi \partial_\mu \pi + \epsilon^5.\end{aligned} \tag{34}$$

The first two terms correspond to unbroken four-dimensional Poincaré invariance, the second two terms correspond to the broken boosts (which will become the Galilean symmetry for small π), and the fifth term is the shift symmetry corresponding to the broken translations in the fifth direction.

In total, the group $ISO(1,4)$ is broken to $ISO(1,3)$. Renaming $\omega^5{}_\mu \equiv \omega_\mu$ and $\epsilon^5 \equiv \epsilon$, we obtain the internal relativistic invariance under which π transforms like a Goldstone boson

$$\delta_{P'}\pi = \omega_\mu x^\mu - \omega^\mu \pi \partial_\mu \pi + \epsilon. \tag{35}$$

This is the relativistic version of the internal Galilean invariance we have been considering. It is the symmetry of theories describing the motion of a brane in a flat bulk, such as DBI. The nonrelativistic limit corresponds to taking the small π limit, and in this limit the relativistic invariance reduces to the nonrelativistic Galilean invariance

$$\delta_{P'}\pi = \omega_\mu x^\mu + \epsilon. \tag{36}$$

This codimension one construction immediately suggests a generalization. Consider codimension greater than one, so that there will be more than one π field. Let the bulk coordinates be X^A, ranging over D dimensions, and let the brane coordinates be x^μ, ranging over d dimensions, so that the codimension is $N = D - d$. The relevant action will still be invariant under the Poincaré transformations (29) and the gauge reparameterization symmetries (30), and we may use this gauge freedom to fix a unitary gauge

$$X^\mu(x) = x^\mu, \qquad X^I(x) \equiv \pi^I(x), \tag{37}$$

where the I part of the index A represents directions transverse to the brane. Once again the Poincaré transformations (29) do not preserve this gauge, since

$$\delta_P X^\mu = \omega^\mu{}_\nu x^\nu + \omega^\mu{}_I \pi^I + \epsilon^\mu, \tag{38}$$

but the gauge can be restored by making a gauge transformation, $\delta_g X^\mu = \xi^\nu \partial_\nu x^\mu = \xi^\mu$, with the choice

$$\xi^\mu = -\omega^\mu{}_\nu x^\nu - \omega^\mu{}_I \pi^I - \epsilon^\mu. \tag{39}$$

Thus, the combined transformation $\delta_{P'} = \delta_P + \delta_g$ leaves the gauge fixing intact and is a symmetry of the gauge fixed action. Its action on the remaining fields π^I is

$$\begin{aligned}\delta_{P'}\pi^I = {}& -\omega^\mu{}_\nu x^\nu \partial_\mu \pi^I - \epsilon^\mu \partial_\mu \pi^I + \omega^I{}_\mu x^\mu \\ & - \omega^\mu{}_J \pi^J \partial_\mu \pi^I + \epsilon^I + \omega^I{}_J \pi^J.\end{aligned} \tag{40}$$

The first five terms are obvious generalizations of those in (34), while the last term is new to codimension greater than one and corresponds to the unbroken $SO(N)$ symmetry in the transverse directions. In total, the group $ISO(1, D-1)$ is broken to $ISO(1, d-1) \times SO(N)$.

Taking the small π^I limit, we find the extended nonrelativistic internal Galilean invariance under which the π^I transform:

$$\delta_{P'}\pi^I = \omega^I{}_\mu x^\mu + \epsilon^I + \omega^I{}_J \pi^J. \tag{41}$$

This consists of a Galilean invariance acting on each of the π^I as in (21), and, importantly as we shall see, an extra internal $SO(N)$ rotation symmetry under which the π's transform as a vector.

To obtain the multifield actions invariant under (41), we must choose the tensor S in (21), so that it is invariant under $SO(N)$ rotations acting on all its indices. Equivalently, we must contract up the $I, J, \ldots$ indices on the fields with each other using δ_{IJ}, the only $SO(N)$ invariant tensor (contracting with the epsilon tensor would give a vanishing action). This simple fact immediately rules out all the Lagrangians with an odd number of π fields, including the DGP cubic term. For an even number of π fields, there are naively two different contractions we can make. On the one hand, we may contract together the two π's appearing with single derivatives and then the remaining π's in any way (the symmetry of $\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n}$ under interchange of $\mu\nu$ pairs with each other makes these all equivalent). On the other hand, we may contract each of the single derivative π's with a double derivative π. By integrating by parts one of the double derivatives in one of the contractions $\partial\pi^I\partial\partial\pi_I$, it is straightforward to show that this second method of contracting the indices is actually equivalent to the first, up to a total derivative. Thus the unique multifield Galileon can be written

$$\begin{aligned}\mathcal{L}_{n+1} = n\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} &\times (\partial_{\mu_1}\pi^{I_1}\partial_{\nu_1}\pi_{I_1}\partial_{\mu_2}\partial_{\nu_2}\pi^{I_2}\partial_{\mu_3}\partial_{\nu_3}\pi_{I_2}\cdots \\ &\times \partial_{\mu_{n-1}}\partial_{\nu_{n-1}}\pi^{I_{n-1}}\partial_{\mu_n}\partial_{\nu_n}\pi_{I_{n-1}}). \end{aligned} \tag{42}$$

In four dimensions, there are now therefore only two possible terms: the kinetic term and a fourth order interaction term[5]

$$\begin{aligned}\mathcal{L}_2 &= \partial_\mu\pi^I\partial^\mu\pi_I, \\ \mathcal{L}_4 &= \partial_\mu\pi^I\partial_\nu\pi_I(\partial^\mu\partial_\rho\pi^J\partial^\nu\partial^\rho\pi_J - \partial^\mu\partial^\nu\pi^J\Box\pi_J) \\ &\quad + \frac{1}{2}\partial_\mu\pi^I\partial^\mu\pi_I(\Box\pi^J\Box\pi_J - \partial_\nu\partial_\rho\pi^J\partial^\nu\partial^\rho\pi_J). \end{aligned} \tag{43}$$

In particular, it is important to note that both the cubic and quintic terms are absent.

This represents an intriguing four-dimensional scalar field theory: there is a single possible interaction term and thus a single free coupling constant (as in, for example, Yang-Mills theory). Of course there are other possible terms compatible with the symmetries, namely, those which contain two derivatives on every field, and where the field indices are contracted. However, the quartic term above is the only one with six derivatives and four fields. All other Galilean-invariant terms have at least two derivatives per field. Thus, as argued in the introduction, there can exist regimes in which the above quartic term is the only one which is important. Furthermore, as will be shown in Sec. VI, this term is not renormalized to any order in perturbation theory, so classical calculations in these interesting regimes are in fact exact.

To fully specify the theory, it is necessary to couple the π fields to matter. The simple linear coupling $\pi^I T$, where $T \equiv \eta_{\mu\nu}T^{\mu\nu}$ is the trace of the energy-momentum tensor, used in [20], does not respect the $SO(N)$ symmetry of the multi-Galileon Lagrangian. There are, of course, many other couplings that do respect this symmetry. The simplest of these is $\pi^I\pi_I T$, but this has its own drawback, namely, that it does not respect the Galilean symmetry. To leading order in an expansion in π^I, a coupling that respects both the internal $SO(N)$ symmetry and the Galilean symmetry is given by

$$\partial_\mu\pi^I\partial_\nu\pi_I T^{\mu\nu}_{\text{flat}}, \tag{44}$$

where $T^{\mu\nu}_{\text{flat}}$ is the energy-momentum tensor computed using the flat four-dimensional metric $\eta_{\mu\nu}$. Indeed such a coupling will naturally emerge from a minimal coupling $\mathcal{L}_{\text{matter}}(g_{\mu\nu}, \psi)$ to brane matter ψ.

These terms will be important in discussing the phenomenology of multi-Galileon theories, but we shall not need to discuss them further in this paper, except for a brief comment when we treat quantum corrections in Sec. VI.

IV. HIGHER CO-DIMENSION BRANES AND ACTIONS

In this section, we show how to construct Galilean and internally relativistic invariant scalar field actions from the higher-dimensional probe-brane prescription. This was done in [17] for the codimension one case, and here we extend that approach to higher codimension.

In the codimension 1 case, to obtain an action invariant under the Galilean symmetry (36), we need only construct an action for the embedding of a brane $X^A(x)$, which is invariant under the reparametrizations (30) and the Poincaré transformations (29). The reparametrizations force the action to be a diffeomorphism scalar constructed out of the induced metric $g_{\mu\nu} \equiv \frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu}G_{AB}(X)$, where G_{AB} is the bulk metric as a function of the embedding variables X^A. Poincaré invariance then requires the bulk metric to be the flat Minkowski metric $G_{AB}(X) = \eta_{AB}$. Fixing the gauge $X^\mu(x) = x^\mu$ then fixes the induced metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi. \tag{45}$$

Any action which is a diffeomorphism scalar, evaluated on this metric, will yield an action for π having the internal Poincaré invariance (36), in addition to the usual four-dimensional spacetime Poincaré invariance. The ingredients available to construct such an action are the metric $g_{\mu\nu}$, the covariant derivative ∇_μ compatible with the induced metric, the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$ corresponding to this derivative, and the extrinsic curvature $K_{\mu\nu}$ of the embedding. Thus, the most general action is

$$S = \int d^4x\sqrt{-g}F(g_{\mu\nu}, \nabla_\mu, R^\rho{}_{\sigma\mu\nu}, K_{\mu\nu})|_{g_{\mu\nu}=\eta_{\mu\nu}+\partial_\mu\pi\partial_\nu\pi}. \tag{46}$$

For example, the DBI action arises from

$$\int d^4x\sqrt{-g} \rightarrow \int d^4x\sqrt{1+(\partial\pi)^2}. \tag{47}$$

To recover a Galilean-invariant action, with the symmetry (36), we have only to take the small π limit. For example, the DBI action above yields the kinetic term $\mathcal{L}_2$ in this limit. The DGP cubic term comes from the

[5]As we were completing the draft of this paper, we received [19], where these exact terms are also considered.

action $\sim\sqrt{-g}g^{\mu\nu}K_{\mu\nu}$. Note that this in this construction the brane is merely a probe brane and no decoupling limit is taken, which is fundamentally different from what occurs in the decoupling limit of DGP (for the effect of higher order curvature terms in DGP, see, for example, [21]).

To generalize this prescription to higher codimension, we must now consider diffeomorphism scalars constructed from the induced metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \pi^I \partial_\nu \pi_I. \tag{48}$$

A much more difficult question concerns the ingredients from which to construct the action, i.e. the geometric quantities associated with a higher codimension brane. We review the details of how to identify these in Appendix A. The main difference from the codimension one case is that the extrinsic curvature now carries an extra index $K^i_{\mu\nu}$. The i index runs over the number of codimensions and is associated with an orthonormal basis in the normal bundle to the hypersurface. In addition, the covariant derivative ∇_μ has a connection $\beta^i_{\mu j}$ that acts on the i index. For example, the covariant derivative of the extrinsic curvature reads

$$\nabla_\rho K^i_{\mu\nu} = \partial_\rho K^i_{\mu\nu} - \Gamma^\sigma_{\rho\mu} K^i_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} K^i_{\mu\sigma} + \beta^i_{\rho j} K^j_{\mu\nu}. \tag{49}$$

The connection $\beta^i_{\mu j}$ is antisymmetric in its i, j indices and so is a new feature appearing in codimensions ≥ 2; it vanishes in codimension one. It has an associated curvature $R^i_{j\mu\nu}$. Therefore, an action of the form

$$S = \int d^4x \sqrt{-g} F(g_{\mu\nu}, \nabla_\mu, R^i{}_{j\mu\nu}, R^\rho{}_{\sigma\mu\nu}, K^i_{\mu\nu}) \times |_{g_{\mu\nu}=\eta_{\mu\nu}+\partial_\mu\pi^I\partial_\nu\pi_I} \tag{50}$$

will have the required relativistic symmetry (40), and its small field limit will have the Galilean invariance (41).

A. Brane quantities

To evaluate the action (50), it is necessary to know how to express the various geometric quantities in terms of the π^I.

The tangent vectors to the brane are

$$e^A{}_\mu = \frac{\partial X^A}{\partial x^\mu} = \begin{cases} \delta^\nu_\mu & A = \nu, \\ \partial_\mu \pi^I & A = I, \end{cases} \tag{51}$$

and the induced metric is

$$g_{\mu\nu} = e^A{}_\mu e^B_\nu \eta_{AB} = \eta_{\mu\nu} + \partial_\mu \pi^I \partial_\nu \pi_I, \tag{52}$$

where the I index is raised and lowered with δ_{IJ}. The inverse metric can then be written as a power series

$$g^{\mu\nu} = \eta^{\mu\nu} - \partial^\mu \pi^I \partial^\nu \pi_I + \mathcal{O}(\pi^4). \tag{53}$$

To find the (orthonormal) normal vectors n^A_i (the index i takes the same values as I, but it is the orthonormal frame index, whereas I is the transverse coordinate index), we solve the defining equations

$$e^A{}_\mu n^B_i \eta_{AB} = 0, \qquad n^A{}_i n^B_j \eta_{AB} = \delta_{ij}. \tag{54}$$

The first equation tells us that

$$n_{Ai} = \begin{cases} -n_{Ii}\partial_\mu \pi^I & A = \mu, \\ n_{Ii} & A = I, \end{cases} \tag{55}$$

where n_{Ii} are the as yet undetermined $A = I$ components of n_{Ai}. The second equation of (54) then gives

$$\delta_{ij} = n^I{}_i n^J{}_j (\partial_\mu \pi_I \partial^\mu \pi_J + \delta_{IJ}). \tag{56}$$

Thus, the n^I_i must be chosen to be vielbeins of the transverse "metric" $g_{IJ} \equiv \partial_\mu \pi_I \partial^\mu \pi_J + \delta_{IJ}$. The ambiguity in this choice due to local $O(N)$ transformations reflects the freedom to change orthonormal basis in the normal space of the brane. The vielbeins summed over their Lorentz indices i, j give the inverse of the metric to g_{IJ}, which expanded in powers of π gives

$$n^I{}_i n^J{}_j \delta^{ij} = \delta^{IJ} - \partial_\mu \pi^I \partial^\mu \pi^J + \mathcal{O}(\pi^4). \tag{57}$$

The metric determinant can be expanded as

$$\sqrt{-g} = 1 + \frac{1}{2}\partial_\mu \pi^I \partial^\mu \pi_I + \mathcal{O}(\pi^4), \tag{58}$$

and the extrinsic curvature is

$$\begin{aligned} K_{i\mu\nu} &= e^A{}_\mu e^B{}_\nu \nabla_A n_{Bi} \\ &= e^B{}_\nu \partial_\mu n_{Bi} \\ &= \partial_\mu n_{\nu i} + \partial_\nu \pi^I \partial_\mu n_{Ii} \\ &= -\partial_\mu (n_{Ii} \partial_\nu \pi^I) + \partial_\nu \pi^I \partial_\mu n_{Ii} \\ &= -n_{Ii} \partial_\mu \partial_\nu \pi^I. \end{aligned} \tag{59}$$

Finally, the twist connection is

$$\begin{aligned} \beta_{\mu ij} &= n^B{}_i e^A{}_\mu \nabla_A n_{Bj} \\ &= n^B{}_i \partial_\mu n_{Bj} \\ &= n^\nu{}_i \partial_\mu n_{\nu j} + n^I{}_i \partial_\mu n_{Ij} \\ &= \partial^\nu \pi^I n_{Ii} \partial_\mu (\partial_\nu \pi^J n_{Jj}) + n^I{}_i \partial_\mu n_{Ij} \\ &= (\delta^{IJ} + \partial_\nu \pi^I \partial^\nu \pi^J) n_{Ii} \partial_\mu n_{Jj} + n_{Ii} n_{Jj} \partial^\nu \pi^I \partial_\mu \partial_\nu \pi^J. \end{aligned} \tag{60}$$

The action (50) is an $SO(N)$ scalar and so will not depend on how the $\eta^I{}_i$ are chosen.

B. Lovelock terms and the probe-brane prescription

A general choice for the action (50) will not lead to scalar field equations that are second order. One of the key insights of de Rham and Tolley [17] is that the actions that do lead to second-order equations are precisely those that are related to Lovelock invariants. It is well-known that the possible extensions of Einstein gravity which remain second order are given by the famous Lovelock terms [22].

These terms are combinations of powers of the Riemann tensor, which are dimensional continuations of characteristic classes. We summarize some properties of these terms in Appendix B. The problem of finding extensions of the π Lagrangian which possess second-order equations of motion is therefore equivalent to the problem of finding extensions of higher-dimensional Einstein gravity which have second-order equations of motion.

In the presence of lower-dimensional hypersurfaces or branes, Lovelock gravity in the bulk must be supplemented by terms which depend on the intrinsic and extrinsic geometry of the brane. These additional surface terms are required in order to ensure that the variational problem of the combined brane/bulk system is well posed [23]. The variation of the surface term precisely cancels the higher-derivative variations on the surface which would otherwise appear in the equations of motion. For the case of Einstein gravity, these considerations lead one to supplement the Einstein-Hilbert Lagrangian by the Gibbons-Hawking-York boundary term [24,25]

$$S = \int_M \mathrm{d}^4x\sqrt{-g}R + 2\int \mathrm{d}^3y\sqrt{-h}K, \tag{61}$$

where x, y are the bulk and brane coordinates, respectively, R is the Ricci scalar of the bulk metric g, and K is the trace of the extrinsic curvature of the induced metric h on the brane.

The addition of Gibbons-Hawking-York boundary terms is closely related to the issue of matching conditions for the bulk metric. When there are distributional sources of stress energy supported on the brane, the extrinsic curvatures on either side of the brane must be related to the brane stress energy in a specific way. This relationship can be derived by supplementing (61) by an action for the brane matter, then varying with respect to the bulk and induced metrics.

Similarly, boundary terms (Myers terms) for the Lovelock invariants must be added [26,27]. The prescription of [17] is as follows: the d-dimensional single-field Galileon terms with an even number N of π's are obtained from the $(N-2)$th Lovelock term on the brane, constructed from the brane metric (see Appendix B for numbering convention of the Lovelock terms). The terms with an odd number N of π's are obtained from the boundary term of the $(N-1)$th $d+1$ dimensional bulk Lovelock term. For instance, in $d=4$, the kinetic term with two π's is obtained from $\sqrt{-g}$ on the brane; the cubic π term is obtained from the Gibbons-Hawking-York term $\sqrt{-g}K$; the quartic term is obtained from $\sqrt{-g}R$; and the quintic term arises from the boundary term of the bulk Gauss-Bonnet invariant. There are no further nontrivial Lovelock terms for $d=4$ in either the brane or the bulk, corresponding to the fact that there are no further nontrivial Galileon terms.

Our goal is to build upon this prescription and extend it to higher codimension. For this, we need the corresponding higher-codimension boundary terms induced by the bulk Lovelock invariants. These were studied by Charmousis and Zegers [28], who found that, despite the freedom to specify a fairly general bulk gravitational theory and number of extra dimensions, the resulting four-dimensional terms are surprisingly constrained, corresponding to the fact that the multi-Galileon action is essentially unique.

The summary of brane terms claimed in [28], for a brane of dimension $d=4$, is as follows:

(i) If the codimension N is odd and $N \neq 3$, one obtains the dimensional continuation of the Gibbons-Hawking-York and Myers terms, with the extrinsic curvature replaced by a distinguished normal component of $K^i_{\mu\nu}$. When $N=3$, there are additional terms involving the extrinsic curvature, and the boundary term is not the dimensional continuation of the Myers term.

(ii) If N is even (see also [29]),

If $N=2$, then the boundary terms include only a brane cosmological constant and the following term

$$\mathcal{L}_{N=2} = \sqrt{-g}(R[g] - (K^i)^2 + K^i_{\mu\nu}K_i^{\mu\nu}). \tag{62}$$

If $N>2$, the boundary term includes only a brane cosmological constant and an induced Einstein-Hilbert term.

In what follows, we will restrict to the even codimension case, since it is unclear to us how the normal components in the odd terms are to be interpreted.

C. Recovering the multifield Galileon

As we saw in the previous subsection, the unique brane action in four dimensions for even codimension ≥ 4 is

$$S = \int d^4x\sqrt{-g}(-a_2 + a_4R). \tag{63}$$

The Galileon action is obtained by substituting $g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\pi^I\partial_\nu\pi_I$ and expanding each term to lowest nontrivial order in π. The cosmological constant term yields an $\mathcal{O}(\pi^2)$ piece, and the Einstein-Hilbert term yields an $\mathcal{O}(\pi^4)$ piece. Up to total derivatives, we have[6]

[6]A nice way to expand the Einstein-Hilbert term is to think in terms of a metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu} = \partial_\mu\pi^I\partial_\nu\pi_I$, as in weak-field studies of general relativity. Then fourth order in π is second order in $h_{\mu\nu}$, but the second order in $h_{\mu\nu}$ is just the familiar Lagrangian for a massless graviton

$$\begin{aligned}\frac{1}{2}\delta^2(\sqrt{-g}R) = &-\frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \frac{1}{2}\partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} \\ &-\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial_\lambda h\partial^\lambda h \\ &+ (\text{total derivative}).\end{aligned}$$

Evaluating this on $h_{\mu\nu} = \partial_\mu\pi^I\partial_\nu\pi_I$ gives (apart from the total derivative) the coefficient of a_4 in (64).

$$S = \int d^4x \Big[-a_2 \frac{1}{2} \partial_\mu \pi^I \partial^\mu \pi_I + a_4 \partial_\mu \pi^I \partial_\nu \pi^J \times (\partial_\lambda \partial^\mu \pi_J \partial^\lambda \partial^\nu \pi_I - \partial^\mu \partial^\nu \pi_I \Box \pi_J) \Big]. \tag{64}$$

Again, by adding a total derivative, we can see that the a_4 term is proportional to the fourth order term (44), so we recover the four-dimensional multifield Galileon model

$$S = \int d^4x \Big[-\frac{1}{2} a_2 \mathcal{L}_2 + \frac{1}{2} a_4 \mathcal{L}_4 \Big]. \tag{65}$$

The equations of motion are

$$\frac{\delta S}{\delta \pi^I} = a_2 \Box \pi_I + a_4 [\Box \pi_I (\partial_\mu \partial_\nu \pi_J \partial^\mu \partial^\nu \pi^J - \Box \pi^J \Box \pi_J) + 2 \partial_\mu \partial_\nu \pi_I (\partial^\mu \partial^\nu \pi_J \Box \pi^J - \partial^\mu \partial_\lambda \pi_J \partial^\nu \partial^\lambda \pi^J)]. \tag{66}$$

For codimension two, there is the additional K^2 part to the boundary term. This cancels the contribution from the Ricci scalar and thus yields nothing new. Therefore, (64) is the unique multi-Galileon term in four dimensions and any even codimension. Keeping all orders in π would lead to a relativistically invariant action, a multifield generalization of DBI with second-order equations.

V. DE SITTER SOLUTIONS OF THE UNIQUE FOURTH ORDER ACTION

While the main aim of this paper is a derivation of the unique multi-Galileon action and its origin in the geometry of braneworlds in codimension greater than one, it is worth exploring the simplest properties of the resulting theories. Perhaps the most straightforward question to ask concerns the nature of maximally symmetric solutions to the equations of motion. If the Galileon were being used to describe a modification to gravity, the interest would be in scalar field profiles that correspond to a gravitational de Sitter background solution. As was argued in [3,5] for the single-field Galileons, these profiles take the form $\sim x^\mu x_\mu$ at short distances, where x^μ is the spacetime coordinate. In fact, this is easy to see geometrically; a de Sitter 3-brane can be embedded in five-dimensional Minkowski space via the equation $X^A X_A = \mathcal{R}^2$, where $\mathcal{R}$ is the radius of the de Sitter space. Thus, taking $x^\mu = X^\mu$ as the brane coordinates and $y = X^5$ as the transverse coordinate, the π profile is

$$\pi \sim y = \sqrt{\mathcal{R}^2 - x^\mu x_\mu} \approx \frac{-1}{2\mathcal{R}} x^\mu x_\mu + \text{constant}, \tag{67}$$

where we have expanded for short distances. The constant can be ignored due to the shift symmetry of π.

Thus we consider the ansatz

$$\pi^I = \Lambda^I x^\mu x_\mu, \tag{68}$$

where Λ^I are constants. This corresponds to a de Sitter brane bending along some general transverse direction. It is easy to see that (66) then yields the condition

$$a_2 \Lambda^I - 24 a_4 \Lambda^I \Lambda^2 = 0, \tag{69}$$

where $\Lambda^2 \equiv \Lambda^I \Lambda_I$. A nontrivial solution requires setting

$$\Lambda^2 = \frac{a_2}{24 a_4} \tag{70}$$

and exists if and only if a_2 and a_4 have the same sign.

To study the stability of these solutions, we expand the field in fluctuations about the de Sitter solution, setting $\pi^I = \Lambda^I x^\mu x_\mu + \delta\pi^I$. The part of the action quadratic in fluctuations reads

$$\mathcal{L}_{\mathcal{O}(\delta\pi^2)} = 48 a_4 \Lambda_I \Lambda_J \partial_\mu \pi^I \partial^\mu \pi^J. \tag{71}$$

Since $\Lambda_I \Lambda_J$ is a matrix of rank 1, only one of the π fields propagates on this background. No new degrees of freedom appear (contrary to the situation, for example, in massive gravity, where a sixth degree of freedom appears around nontrivial backgrounds). This is a general feature of Galileon-type theories—the second-order property of the equations guarantee that no new degrees of freedom propagate around nontrivial backgrounds.

However, since $\Lambda_I \Lambda_J$ is a positive matrix, our degree of freedom is a ghost if $a_4 > 0$, signaling that this solution is unstable.[7] If $a_2 > 0$, so that there is no ghost around flat space, then we must have $a_4 > 0$ for a nontrivial de Sitter solution to exist, and hence there will be a ghost around the de Sitter solution. If we choose $a_4 < 0$ to avoid the ghost around de Sitter, then we necessarily have $a_2 < 0$, and the ghost reappears around flat space.

VI. QUANTUM PROPERTIES AND NONRENORMALIZATION

One of the most interesting properties of the Galileon actions is their stability under quantum corrections (discussed for the special case of a single-field cubic term in [2]). In this section, we show that, in any theory with Galilean symmetry on each field, the general multifield scalar Galileon term receives no quantum corrections, to any order in perturbation theory, in any number of dimensions.

Consider an effective field theory for scalars π^I invariant under individual Galilean transformations $\pi^I \to \pi^I + c^I + b^I_\mu x^\mu$ (in this section we remain more general and do not impose any additional internal symmetries among the π fields). The classical action may contain the general multifield scalar Galileon terms (21),

$$\mathcal{L}_{n+1} \sim S_{I_1 I_2 \cdots I_{n+1}} \eta^{\mu_1 \nu_1 \mu_2 \nu_2 \cdots \mu_n \nu_n} \times (\pi^{I_{n+1}} \partial_{\mu_1} \partial_{\nu_1} \pi^{I_1} \partial_{\mu_2} \partial_{\nu_2} \pi^{I_2} \cdots \partial_{\mu_n} \partial_{\nu_n} \pi^{I_n}), \tag{72}$$

with $S_{I_1 I_2 \cdots I_{n+1}}$ a symmetric constant tensor. These are the only terms that yield second-order equations of motion and are the only n-field terms that contain $2n - 2$ derivatives.

[7] Note that we use the $(-, +, +, +)$ metric convention.

There are no terms with n fields that contain fewer than $2n - 2$ derivatives, but there are plenty of possible Galilean-invariant terms with $\geq 2n$ derivatives (i.e. any term with two or more derivatives on each π), and we also allow for the presence of these terms in the classical action.

Consider quantum corrections by calculating the quantum effective action for the classical field $\Gamma(\pi^c)$ expanded about the expectation value $\langle\pi\rangle = 0$,

$$\Gamma(\pi^c) = \Gamma^{(2)}\pi^c\pi^c + \Gamma^{(3)}\pi^c\pi^c\pi^c + \cdots. \tag{73}$$

The term $\Gamma^{(n)}$ is calculated in momentum space by summing all $1PI$ diagrams with n external π lines. The position space action is obtained by expanding in powers of the external momenta and then replacing the momenta with derivatives. $\Gamma^{(n)}$ thus contains all terms with n fields and any number of derivatives, the number of derivatives being the power of external momenta in the expansion of the n-point $1PI$ diagram.

To show that the terms (72) do not receive quantum corrections, we argue that all n-point diagrams, constructed with vertices drawn from the classical action, contain at least $2n$ powers of the external momenta. To do this, we show that each external line contributes at least two powers of the external momenta.

Focus on any given vertex connected to external lines, as depicted in Fig. 1. If the external lines hit only the $\partial\partial\pi$ pieces (this encompasses the case where the vertex is drawn from non-Galileon terms, i.e. terms with at least two derivatives on every π), then the vertex will contribute two powers of momentum for each external line. The other possibility is that one of the external lines hits the undifferentiated π in a vertex of the form (72). Suppose there are m external lines, then the contraction looks like

$$\begin{aligned}\mathcal{L}_{n+1} \sim\ & S_{I_1 I_2\cdots I_{n+1}}\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \\ & \times(\pi_{\text{ext}}^{I_{n+1}}\partial_{\mu_1}\partial_{\nu_1}\pi_{\text{ext}}^{I_1}\cdots\partial_{\mu_{m-1}}\partial_{\nu_{m-1}} \\ & \times\pi_{\text{ext}}^{I_{m-1}}\partial_{\mu_m}\partial_{\nu_m}\pi_{\text{int}}^{I_m}\cdots\partial_{\mu_n}\partial_{\nu_n}\pi_{\text{int}}^{I_n}).\end{aligned} \tag{74}$$

Using the antisymmetry of η, we may write the part containing π_{int} as a double total derivative

$$\begin{aligned}\mathcal{L}_{n+1} \sim\ & S_{I_1 I_2\cdots I_{n+1}}\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_n\nu_n} \\ & \times(\pi_{\text{ext}}^{I_{n+1}}\partial_{\mu_1}\partial_{\nu_1}\pi_{\text{ext}}^{I_1}\cdots\partial_{\mu_{m-1}}\partial_{\nu_{m-1}}\pi_{\text{ext}}^{I_{m-1}}\partial_{\mu_m}\partial_{\nu_m} \\ & \times[\pi_{\text{int}}^{I_m}\cdots\partial_{\mu_n}\partial_{\nu_n}\pi_{\text{int}}^{I_n}]).\end{aligned} \tag{75}$$

The Feynman rule for this contraction therefore contains two factors of the sum of the internal momenta $\sum p_{\text{int}}$. By momentum conservation at each vertex, we can trade these for the external momenta $-\sum p_{\text{ext}}$. This adds two powers of p_{ext} to the count, making up for the undifferentiated π and bringing the total to $2n$.

This means that the expansion of the n-point diagram in powers of external momenta must start at order $\geq 2n$, so the terms of the form (72), which have $2n - 2$ derivatives,

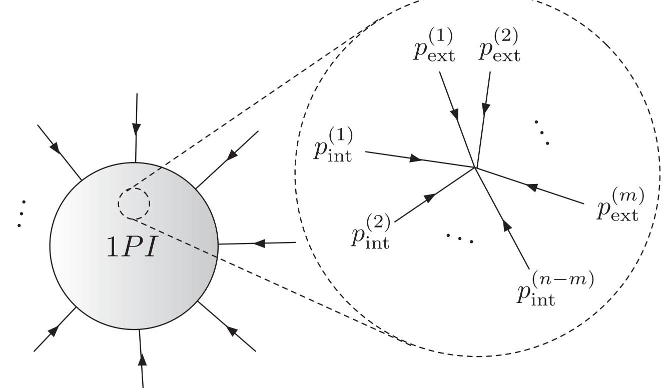

FIG. 1. A general Feynman diagram and vertex potentially contributing to quantum corrections to the Galileon terms. As we prove, such corrections vanish in these theories.

cannot receive new contributions. This holds at all loops in perturbation theory and regardless of any other terms of the form $(\partial\partial\pi)^{\text{power}}$ that are present in the classical action. Note that the kinetic term is of the form (72), so there is no wave function renormalization in these theories.

This nonrenormalization theorem is not a consequence of a symmetry of the theories. In quantum field theory, we are used to seeing terms vanish or stay naturally small because of symmetry, but here the terms (72) are compatible with the symmetries and yet still do not receive quantum corrections. The situation is more analogous to that in supersymmetric theories, where superpotentials do not receive quantum corrections even though they are compatible with supersymmetry. In the supersymmetric case there is an underlying reason, namely, holomorphy of the superpotential. Here, the reason seems to be that the Galileon terms just do not contain sufficient numbers of derivatives, yet still manage to be Galilean invariant.

These conclusions may be changed when couplings to matter, as mentioned in Sec. III, are included. However, any corrections to the Galileon terms must be proportional to the π-matter coupling and thus must go to zero as these couplings do. In particular, in applications to modified gravity, couplings to matter will typically be Planck-suppressed.

VII. CONCLUSIONS

Brane-world models with induced gravity have been extensively studied in codimension one. The relevant action contains a nonlinear cubic interaction which yields interesting cosmological phenomenology and strict constraints from local tests of gravity. In this paper, we have systematically extended this idea to higher codimension and have explored the origin of the allowed terms and the symmetry group under which they transform, in the geometric terms arising in the action for the brane in the higher-dimensional space. The relevant terms are generalizations of those obtained in [17] and are related to the bulk Lovelock terms and their associated boundary actions.

The existence of more than one extra spatial dimension allows for multiple brane bending modes and correspondingly the four-dimensional effective theory contains multiple Galileon fields. Interestingly, the residual symmetry group of this theory contains an internal $SO(N)$ subgroup that forbids nonlinear interactions with odd numbers of Galileon fields. Thus, the usual Galileon term does not remain in higher codimension. Instead what results is a highly constrained theory with a single coupling constant, governing the strength of a unique nonlinear quartic derivative interaction. We have further proved a general nonrenormalization theorem, which demonstrates that in any number of codimensions, the resulting Galileon theory contains only terms that receive no quantum corrections at any loop in perturbation theory.

Multi-Galileon theories in principle possess a rich and interesting phenomenology. While not the main thrust of this paper, we have initiated such a study by considering the simplest example of maximally symmetric backgrounds. For suitable choices of signs of the coupling constants, we have demonstrated the existence of a de Sitter background and have explored the stability of the theory around it. The result is a generalization of the familiar DGP case of a ghost in the accelerating branch. More precisely, we demonstrate that when the de Sitter solution exists, then it is possible for either it or the flat space solution to be ghost-free but not both. The implications of this result for self-accelerating cosmologies from multi-Galileon theories remain to be seen.

ACKNOWLEDGMENTS

The authors are grateful to Melinda Andrews, Christos Charmousis, Lam Hui, Justin Khoury, and Alberto Nicolis for discussions. This work is supported in part by NASA ATP Grant No. NNX08AH27G, NSF Grant No. PHY-0930521, and by Department of Energy Grant No. DE-FG05-95ER40893-A020. M. T. is also supported by the Fay R. and Eugene L. Langberg chair.

APPENDIX A: MATHEMATICS OF HIGHER CO-DIMENSION HYPERSURFACES

Here we describe the formalism necessary to deal with submanifolds of higher codimension. The geometric setup is shown in Fig. 2.

1. Submanifolds and adapted basis

Let $\mathcal{M}$ be a manifold of dimension D, with coordinates X^A. We describe an d-dimensional submanifold $\mathcal{N}$ of $\mathcal{M}$ as the locus of zeros of $N \equiv D - d$ functions

$$\phi^I(X) = 0, \qquad I = 1 \ldots N. \tag{A1}$$

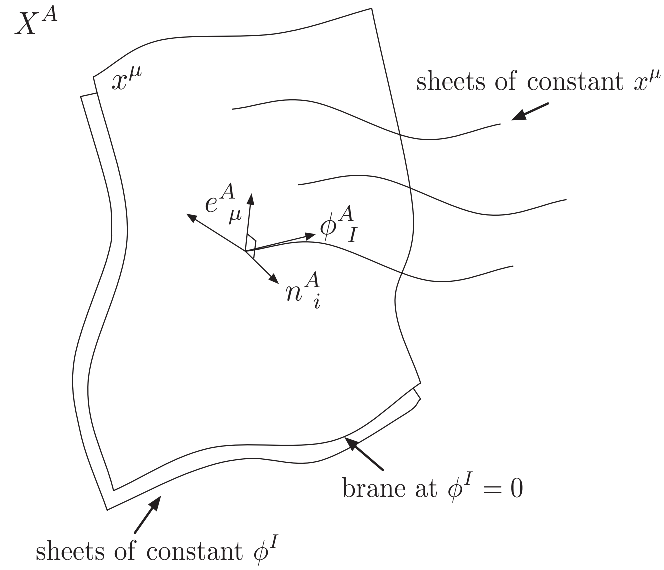

FIG. 2. The geometric setup for a higher codimension brane.

The level sets of ϕ^I give a foliation of $\mathcal{M}$ into a family of d-dimensional submanifolds, of which $\mathcal{N}$ is a member. The submanifolds have codimension N.

We now describe a new set of coordinates on $\mathcal{M}$, adapted to the foliation. First, set up coordinates x^μ, $\mu = 1 \ldots d$, on $\mathcal{N}$. Now set up functions $x^\mu(X)$, which are independent of the $\phi^I(X)$ and each other, and whose values on $\mathcal{N}$ coincide with the coordinates x on $\mathcal{N}$. The level sets of the $x^\mu(X)$ will define a congruence of curves intersecting all the submanifolds. We use this congruence to assign coordinates on all the other submanifolds from those on $\mathcal{N}$, so that the coordinates are given by x^μ. The x^μ along with the ϕ^I now form a new coordinate system on $\mathcal{M}$. We have a transformation from these new coordinates to the old coordinates X^A,

$$X^A(x^\mu, \phi^I), \qquad \phi^I(X^A), \qquad x^\mu(X^A). \tag{A2}$$

The basis vectors of this new coordinate system are

$$\phi^A_{\ I} = \frac{\partial X^A}{\partial \phi^I}, \qquad e^A_{\ \mu} = \frac{\partial X^A}{\partial x^\mu}. \tag{A3}$$

The basis one forms are

$$\phi^I_A = \frac{\partial \phi^I}{\partial X^A}, \qquad \tilde{e}_A^{\ \mu} = \frac{\partial x^\mu}{\partial X^A}. \tag{A4}$$

(We have put a tilde on $\tilde{e}_A^{\ \mu}$ because later we will introduce a metric and use normal vectors in place of $\phi^A_{\ I}$, so the dual basis will have to change, at which point we will use $e_A^{\ \mu}$.)

They satisfy duality and completeness relations

$$\begin{aligned} &\phi^A_{\ I}\phi_A^{\ J} = \delta^J_I, \qquad e^A_{\ \mu}\tilde{e}_A^{\ \nu} = \delta^\nu_\mu, \\ &\phi^A_{\ I}\tilde{e}_A^{\ \mu} = e^A_{\ \mu}\phi_A^{\ I} = 0. \end{aligned} \tag{A5}$$

$$\phi^A_{\ I}\phi_B^{\ I} + e^A_{\ \mu}\tilde{e}_B^{\ \mu} = \delta^A_{\ B}. \tag{A6}$$

2. Metric

Now suppose there is a bulk metric G_{AB}. The metric can have any signature, but we demand that the foliation be non-null. There is now a well defined normal subspace of the tangent space of $\mathcal{M}$ at each point, which may be different from the subspace defined by the congruence, which is spanned by $\phi^A{}_I$. We set up a basis consisting of N orthonormal normal vectors $n^A{}_i$, as well as the $e^A{}_\mu$, which are not required to be orthonormal among themselves.

$$G_{AB}n^A{}_i n^B{}_j = \eta_{ij}, \qquad G_{AB}e^A{}_a n^B{}_j = 0. \tag{A7}$$

Here η_{ij} is the N-dimensional flat Minkowski or Euclidean metric carrying whatever signature the transverse space has. We define the associated dual forms $e_A{}^\mu$, $n_A{}^i$, at each point

$$\begin{aligned} n^A{}_i n_A{}^j &= \delta^j_i, \qquad e^A{}_\nu e_A{}^\mu = \delta_\nu{}^\mu, \\ n^A{}_i e_A{}^\mu &= e^A{}_\mu n_A{}^i = 0. \end{aligned} \tag{A8}$$

$$n^A{}_i n_B{}^i + e^A{}_\mu e_B{}^\mu = \delta^A{}_B. \tag{A9}$$

This choice of basis is unique up to local orthogonal rotations in the normal space.

3. Parallel and normal tensors

First, we consider tensors which are parallel to the submanifold $\mathcal{N}$. A vector V^A is parallel if it admits the decomposition $V^A = V^\mu e^A{}_\mu$. A form V_A is parallel if it admits the decomposition $V_A = V_\mu e_A{}^\mu$. (Notice that, unlike a vector, the notion of a form being parallel depends on the dual basis, will change if the dual basis is changed, and hence depends on the metric.) Similarly, a general tensor $T^{AB\ldots}{}_{C\ldots}$ is parallel if it admits an analogous decomposition

$$T^{AB\ldots}{}_{C\ldots} = A^{\mu\nu\ldots}{}_{\rho\ldots} e^A{}_\mu e^B{}_\nu e_C{}^\rho \cdots. \tag{A10}$$

There is a bijective relation between tensors on the submanifold $\mathcal{N}$ (really a N-parameter family of tensors, one on each surface, parametrized by ϕ^I) and parallel tensors in the bulk. Given a parallel bulk tensor $T^{AB\ldots}{}_{C\ldots}$, it corresponds to the submanifold tensor $A^{\mu\nu\ldots}{}_{\rho\ldots}$ and vice versa.

Define the projection tensor

$$P^A{}_B \equiv \delta^A_B - n^A{}_i n_B{}^i. \tag{A11}$$

It projects the tangent space of $\mathcal{M}$ onto the tangent space of $\mathcal{N}$, along the subspace spanned by $n^A{}_i$. It satisfies

$$P^A{}_C P^C{}_B = P^A{}_B, \tag{A12}$$

$$P^A{}_B e^B{}_\mu = e^A{}_\mu, \qquad P^A{}_B n^B{}_i = 0, \tag{A13}$$

$$P^A{}_B e_A{}^\mu = e_B{}^\mu, \qquad P^A{}_B n_A{}^i = 0. \tag{A14}$$

Given any bulk tensor, $T^{AB\ldots}{}_{C\ldots}$, we can make a parallel tensor by projecting it along all its indices

$$T^{\parallel AB\ldots}{}_{C\ldots} \equiv P^A{}_D P^B{}_E P^F{}_C \cdots T^{DE\ldots}{}_{F\ldots}. \tag{A15}$$

A tensor is parallel if and only if it is equal to its projection.

We have the relation

$$e^A{}_\mu e_B{}^\mu = P^A{}_B. \tag{A16}$$

Projecting the metric gives the induced metric h_{AB} on the hypersurfaces, whose intrinsic components we denote $g_{\mu\nu}$,

$$\begin{aligned} h_{AB} &= P^C{}_A P^D{}_B G_{CD} = g_{\mu\nu} e_A{}^\mu e_B{}^\nu, \\ g_{\mu\nu} &= e^A{}_\mu e^B{}_\nu h_{AB} = e^A{}_\mu e^B{}_\nu g_{AB}. \end{aligned} \tag{A17}$$

We raise and lower bulk indices $A, B, \ldots$ with G_{AB} and its inverse G^{AB}, and we raise and lower submanifold indices $\mu, \nu, \ldots$ with $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. We raise and lower perpendicular indices $i, j, \ldots$ using η_{ij} and its inverse η^{ij}. In particular, we have

$$g^{\mu\nu} G_{AB} e^B{}_\nu = e_A{}^\mu, \qquad \eta^{ij} G_{AB} n^B{}_j = n_A{}^i, \tag{A18}$$

$$g_{\mu\nu} G^{AB} e_B{}^\nu = e^A{}_\mu, \qquad \eta_{ij} G^{AB} n_B{}^j = n^A{}_i. \tag{A19}$$

as well as

$$G_{AC} P^C{}_B = h_{AB}, \qquad G^{AC} P^B{}_C = h^{AB}. \tag{A20}$$

We next consider tensors which are normal to the submanifolds. A vector V^A is normal if it admits the decomposition $V^A = V^i n^A{}_i$. A form V_A is normal if it admits the decomposition $V_A = V_i n_A{}^i$. Similarly, a general tensor $T^{AB\ldots}{}_{C\ldots}$ is normal if it admits an analogous decomposition

$$T^{AB\ldots}{}_{C\ldots} = A^{ij\ldots}{}_{k\ldots} n^A{}_i n^B{}_j n_C{}^k \cdots. \tag{A21}$$

Define another projection tensor

$$P^A_{\perp B} \equiv \delta^A_B - e^A{}_\mu e_B{}^\mu. \tag{A22}$$

It projects the tangent space of $\mathcal{M}$ onto the normal space of $\mathcal{N}$, along the tangent space. It satisfies

$$P^A_{\perp C} P^C_{\perp B} = P^A_{\perp B}, \tag{A23}$$

$$P^A_{\perp B} n^B{}_i = n^A{}_i, \qquad P^A_{\perp B} e^B{}_\mu = 0 \tag{A24}$$

$$P^A_{\perp B} n_A{}^i = n_B{}^i, \qquad P^A_{\perp B} e_A{}^\mu = 0. \tag{A25}$$

Given any bulk tensor, e.g. $T^{AB\ldots}{}_{C\ldots}$, we can make a normal tensor by projecting it

$$T^{\perp AB\ldots}{}_{C\ldots} = P^A_{\perp D} P^B_{\perp E} P^F_{\perp C} \cdots T^{DE\ldots}{}_{F\ldots}. \tag{A26}$$

A tensor is normal if and only if it is equal to its normal projection.

We have the relations

$$n^A{}_i n_B{}^i = P^A_{\perp B}. \tag{A27}$$

$$P^A_{\perp C}P^C{}_B = P^A{}_C P^C_{\perp B} = 0. \tag{A28}$$

$$P^A_{\perp B} + P^A{}_B = \delta^A{}_B. \tag{A29}$$

We may also define mixed tensors, with some indices tangent and others normal. Such a tensor $T^{A\cdots}{}_{B\cdots}{}^{C\cdots}{}_{D\cdots}$, where the first group of indices $A\cdots, B\cdots$ are to be tangent and the second group $C\cdots, D\cdots$ are to be normal, is one that admits the decomposition

$$T^{A\cdots}{}_{B\cdots}{}^{C\cdots}{}_{D\cdots} = T^{\mu\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{j\cdots}e^A{}_\mu\cdots e_B{}^\nu\cdots n^C{}_i\cdots n_D{}^j\cdots \tag{A30}$$

A general tensor can always be decomposed into parallel, normal, and mixed components. For example, a general (1, 1) tensor $T^A{}_B$ can be written

$$T^A{}_B = T^\mu{}_\nu e^A{}_\mu e_B{}^\nu + T^\mu{}_i e^A{}_\mu n_B{}^i + T^i{}_\mu n^A{}_i e_B{}^\mu + T^i{}_j n^A{}_i n_B{}^j. \tag{A31}$$

4. Induced connections

Consider now the covariant derivatives of a vector in the parallel directions. This is a quantity which is well defined on the brane itself, i.e. the vector need only be defined on the brane. Starting from the covariant derivatives of a parallel vector in the parallel directions, we may expand the result into tangent and normal directions via the Gauss-Weingarten relation

$$e^B{}_\mu \nabla_B e^A{}_\nu = \Gamma^\rho_{\mu\nu}e^A{}_\rho - K^i_{\mu\nu}n^A{}_i. \tag{A32}$$

Here $\Gamma^\rho_{\mu\nu}$ and $K^i_{\mu\nu}$ are defined as the expansion coefficients, equal to

$$\Gamma^\rho_{\mu\nu} = e_A{}^\rho e^B{}_\mu \nabla_B e^A{}_\nu, \tag{A33}$$

$$K^i_{\mu\nu} = -n_A{}^i e^B{}_\mu \nabla_B e^A{}_\nu. \tag{A34}$$

It is straightforward to show that $\Gamma^\rho_{\mu\nu}$ transforms as a connection under changes in the brane coordinates x^μ, and it is in fact precisely the Levi-Civita connection of the induced metric $g_{\mu\nu}$,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{A35}$$

The quantity $K^i_{\mu\nu}$ transforms as a tensor in its $\mu\nu$ indices under changes in the brane coordinates and as a vector in its i index under orthogonal changes in the frame $n^A{}_i$. It is called the *extrinsic curvature*. We can also write it as

$$K^i_{\mu\nu} \equiv \nabla_B n_A{}^i e^B{}_\mu e^A{}_\nu, \tag{A36}$$

by using the relation $\nabla_B(n_A{}^i e^A{}_\nu) = 0$. The extrinsic curvature is symmetric

$$K^i_{\mu\nu} = K^i_{\nu\mu}, \tag{A37}$$

which can be easily shown by noting that the basis vectors have zero lie bracket, hence $e^B{}_\nu \nabla_B e^A{}_\mu = e^B{}_\mu \nabla_B e^A{}_\nu$. We also have

$$K^i_{\mu\nu} = \nabla_{(A}n_{B)}{}^i e^A{}_\mu e^B{}_\nu = \frac{1}{2}e^A{}_\mu e^B{}_\nu \mathcal{L}_{n_i}G_{AB}. \tag{A38}$$

Its trace is given by

$$K^i = g^{\mu\nu}K^i_{\mu\nu} = \nabla_A n^{Ai}. \tag{A39}$$

Note that in higher codimension, the extrinsic curvature gains another index i. There is one extrinsic curvature component for each normal direction.

Next consider the covariant derivatives of a normal vector in the parallel directions and expand the result into normal and tangent directions

$$e^B{}_\mu \nabla_B n^A{}_i = \beta^j_{\mu i}n^A{}_j + K_{i\mu}{}^\nu e^A{}_\nu. \tag{A40}$$

Here $\beta^j_{\mu i}$ and $K_{i\mu}{}^\nu$ are defined as the expansion coefficients, equal to

$$\beta^j_{\mu i} = n_A{}^j e^B{}_\mu \nabla_B n^A{}_i, \tag{A41}$$

$$K_{i\mu}{}^\nu = e_A{}^\nu e^B{}_\mu \nabla_B n^A{}_i. \tag{A42}$$

The $K_{i\mu}{}^\nu$ are again the extrinsic curvature, with indices raised and lowered as shown.

The $\beta^j_{\mu i}$ transform as a connection under orthogonal changes in the frame $n^A{}_i$. It is called the *twist connection* and is the metric connection on the normal bundle, metric compatibility being expressed as the antisymmetry relation

$$\beta^k_{\mu j}\eta_{ki} = -\beta^k_{\mu i}\eta_{kj}. \tag{A43}$$

The twist connection vanishes identically in codimension one, so it is an essentially higher codimension object.

Using the connection on the tangent bundle $\Gamma^\rho_{\mu\nu}$ and the connection $\beta^j_{\mu i}$ on the normal bundle, we can define covariant derivatives D_μ. Acting on a general mixed tensor $T^{\mu\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{j\cdots}$,

$$\begin{aligned} D_\rho T^{\mu\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{j\cdots} = {}& \partial_\rho T^{\mu\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{j\cdots} + \Gamma^\mu_{\rho\sigma}T^{\sigma\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{j\cdots} + \cdots \\ & - \Gamma^\sigma_{\rho\nu}T^{\mu\cdots}{}_{\sigma\cdots}{}^{i\cdots}{}_{j\cdots} - \cdots \\ & + \beta^i_{\rho k}T^{\mu\cdots}{}_{\nu\cdots}{}^{k\cdots}{}_{j\cdots} + \cdots \\ & - \beta^k_{\rho j}T^{\mu\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{k\cdots} - \cdots. \end{aligned} \tag{A44}$$

The covariant derivative $D_\rho T^{\mu\cdots}{}_{\nu\cdots}{}^{i\cdots}{}_{j\cdots}$ transforms as a tensor, in the manner indicated by its indices.

5. Curvatures

By commutating the covariant derivatives, we arrive at curvature tensors

$$[D_\mu, D_\nu]T^{\rho\cdots}{}_{\sigma\cdots}{}^{i\cdots}{}_{j\cdots} = {}^{(d)}R^\rho{}_{\lambda\mu\nu}T^{\lambda\cdots}{}_{\sigma\cdots}{}^{i\cdots}{}_{j\cdots} + \cdots - {}^{(d)}R^\lambda{}_{\sigma\mu\nu}T^{\rho\cdots}{}_{\lambda\cdots}{}^{i\cdots}{}_{j\cdots} - \cdots + {}^{(\perp)}R^i{}_{k\mu\nu}T^{\rho\cdots}{}_{\sigma\cdots}{}^{k\cdots}{}_{j\cdots} + \cdots - {}^{(\perp)}R^k{}_{j\mu\nu}T^{\rho\cdots}{}_{\sigma\cdots}{}^{i\cdots}{}_{k\cdots} - \cdots, \tag{A45}$$

where the curvatures are defined as

$${}^{(d)}R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{A46}$$

$${}^{(\perp)}R^i{}_{j\mu\nu} = \partial_\mu\beta^i_{\nu j} - \partial_\nu\beta^i_{\mu j} + \beta^i_{\mu k}\beta^k_{\nu j} - \beta^i_{\nu k}\beta^k_{\mu j}. \tag{A47}$$

These are antisymmetric in their first two indices and in their last two indices and transform as tensors.

The bulk curvature components, which can be determined from data localized solely on the brane, can be written in terms of brane quantities. The relations are the Gauss, Codazzi, and Ricci equations, respectively,

$$R_{ABCD}e^C{}_\mu e^D{}_\nu e^A{}_\rho e^B{}_\sigma = {}^{(d)}R_{\rho\sigma\mu\nu} + K^i_{\mu\sigma}K_{i\nu\rho} - K^i_{\nu\sigma}K_{i\mu\rho} \tag{A48}$$

$$R_{ABCD}e^C{}_\mu e^D{}_\nu e^B{}_\rho n^{Ai} = D_\nu K^i_{\mu\rho} - D_\mu K^i_{\nu\rho} \tag{A49}$$

$$R_{ABCD}e^C{}_\mu e^D{}_\nu n^A{}_j n^B{}_i = {}^{(\perp)}R_{ji\mu\nu} + K_{i\mu}{}^\rho K_{j\nu\rho} - K_{i\nu}{}^\rho K_{j\mu\rho}. \tag{A50}$$

The final equation only appears in codimension >1. Recall that in these expressions the covariant derivative must also act on $i, j\cdots$ indices, via the connection $\beta^i_{\mu j}$.

APPENDIX B: LOVELOCK TERMS

Let the dimension be D. For even $N \geq 2$, define

$$\mathcal{L}^{(N)} = \frac{1}{2^{N/2}}N!\delta^{\mu_1\mu_2\ldots\mu_{N-1}\mu_N}_{\nu_1\nu_2\ldots\nu_{N-1}\nu_N}R_{\mu_1\mu_2}{}^{\nu_1\nu_2} \times R_{\mu_3\mu_4}{}^{\nu_3\nu_4}\cdots R_{\mu_{N-1}\mu_N}{}^{\nu_{N-1}\nu_N}. \tag{B1}$$

The delta symbol is defined as

$$\delta^{\mu_1\mu_2\ldots\mu_{n-1}\mu_n}_{\nu_1\nu_2\ldots\nu_{n-1}\nu_n} \equiv \delta^{[\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_{n-1}}_{\nu_{n-1}}\delta^{\mu_n]}_{\nu_n} = \frac{1}{n!}\begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_n} \\ \vdots & \ddots & \vdots \\ \delta^{\mu_n}_{\nu_1} & \cdots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}. \tag{B2}$$

It is antisymmetric in the μ's, antisymmetric in the ν's, and symmetric under the interchange of any μ, ν pair with another. For $n \geq m$ it satisfies the identity

$$\delta^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}\delta^{\nu_1\ldots\nu_m}_{\mu_1\ldots\mu_m} = \frac{(n-m)!}{n!}\left[\prod_{i=1}^m(D-(n-i))\right]\delta^{\mu_{m+1}\ldots\mu_n}_{\nu_{m+1}\ldots\nu_n}, \tag{B3}$$

as well as identities obtained by expanding out the determinant above in minors, such as the following

$$\delta^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = \frac{1}{n}(\delta^{\mu_1}_{\nu_1}\delta^{\mu_2\ldots\mu_n}_{\nu_2\ldots\nu_n} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2\ldots\mu_n}_{\nu_1\nu_3\ldots\nu_n} + \cdots + (-1)^n\delta^{\mu_1}_{\nu_n}\delta^{\mu_2\ldots\mu_n}_{\nu_1\ldots\nu_{n-1}}) = \frac{1}{n}(\delta^{\mu_1}_{\nu_1}\delta^{\mu_2\ldots\mu_n}_{\nu_2\ldots\nu_n} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1\mu_3\ldots\mu_n}_{\nu_2\ldots\nu_n} + \cdots + (-1)^n\delta^{\mu_n}_{\nu_1}\delta^{\mu_1\ldots\mu_{n-1}}_{\nu_2\ldots\nu_n}). \tag{B4}$$

The term $\mathcal{L}^{(N)}$ vanishes identically for $N < D$ (with D even or odd). For D even, the integral over a compact oriented Riemannian manifold gives the Euler characteristic

$$\chi(M) = \frac{1}{(4\pi)^{D/2}(\frac{D}{2})!}\int d^Dx\sqrt{|g|}\mathcal{L}^{(D)}. \tag{B5}$$

In particular, this integral does not depend on the metric. Therefore, for any background metric its variation with respect to the metric must vanish, and thus the integrand must be a total derivative $\sqrt{|g|}\mathcal{L}^{(D)} = \partial_\mu(\text{something})^\mu$.

The first few terms are

$$\mathcal{L}^{(0)} = 1, \qquad \mathcal{L}^{(2)} = R, \qquad \mathcal{L}^{(4)} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}. \tag{B6}$$

[1] G. R. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. B **485**, 208 (2000).
[2] M. A. Luty, M. Porrati, and R. Rattazzi, J. High Energy Phys. 09 (2003) 029.
[3] A. Nicolis and R. Rattazzi, J. High Energy Phys. 06 (2004) 059.
[4] G. Gabadadze and A. Iglesias, Phys. Lett. B **639**, 88 (2006).
[5] A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D **79**, 064036 (2009).
[6] C. de Rham and G. Gabadadze, Phys. Lett. B **693** 334 (2010).
[7] C. Deffayet, G. Esposito-Farese, and A. Vikman, Phys. Rev. D **79**, 084003 (2009).
[8] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D **80**, 064015 (2009).

[9] F. P. Silva and K. Koyama, Phys. Rev. D **80**, 121301 (2009).
[10] T. Kobayashi, Phys. Rev. D **81**, 103533 (2010).
[11] A. De Felice and S. Tsujikawa, Phys. Rev. Lett. **105**, 111301 (2010).
[12] N. Chow and J. Khoury, Phys. Rev. D **80**, 024037 (2009).
[13] N. Agarwal, R. Bean, J. Khoury, and M. Trodden, Phys. Rev. D **81**, 084020 (2010).
[14] P. Creminelli, A. Nicolis, and E. Trincherini, arXiv:1007.0027.
[15] R. Gannouji and M. Sami, Phys. Rev. D **82**, 024011 (2010).
[16] C. Deffayet, O. Pujolas, I. Sawicki, and A. Vikman, J. Cosmol. Astropart. Phys. 10 (2010) 026arXiv:1008.0048.
[17] C. de Rham and A. J. Tolley, J. Cosmol. Astropart. Phys. 05 (2010) 015.
[18] C. Deffayet, S. Deser, and G. Esposito-Farese, Phys. Rev. D **82**, 061501 (2010).
[19] A. Padilla, P. M. Saffin, and S.-Y. Zhou, arXiv:1008.0745.
[20] A. Padilla, P. M. Saffin, and S.-Y. Zhou, arXiv:1007.5424.
[21] M. Cadoni and P. Pani, Phys. Lett. B **674**, 308 (2009).
[22] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).
[23] E. Dyer and K. Hinterbichler, Phys. Rev. D **79**, 024028 (2009).
[24] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[25] J. W. York, Jr., Phys. Rev. Lett. **28**, 1082 (1972).
[26] R. C. Myers, Phys. Rev. D **36**, 392 (1987).
[27] O. Miskovic and R. Olea, J. High Energy Phys. 10 (2007) 028.
[28] C. Charmousis and R. Zegers, J. High Energy Phys. 08 (2005) 075.
[29] C. Charmousis and R. Zegers, Phys. Rev. D **72**, 064005 (2005).