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This paper introduces an operator M called the *mixed powerdomain* which generalizes the convex (Plotkin) powerdomain. The construction is based on the idea of representing partial information about a set of data items using a *pair* of sets, one representing partial information in the manner of the upper (Smyth) powerdomain and the other in the manner of the lower (Hoare) powerdomain where the components of such pairs are required to satisfy a consistency condition. This provides a richer family of meaningful partial descriptions than are available in the convex powerdomain and also makes it possible to include the *empty set* in a satisfactory way. The new construct is given a rigorous mathematical treatment like that which has been applied to the known powerdomains. It is proved that M is a continuous functor on bifinite domains which is left adjoint to the forgetful functor from a category of continuous structures called *mix algebras*. For a domain D with a coherent Scott topology, elements of $M D$ can be represented as pairs (U, V) where $U \subseteq D$ is a compact upper set, $V \subseteq D$ is a closed set and the downward closure of $U \cap V$ is equal to V . A Stone dual characterization of M is also provided.

Comments

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The Mixed Powerdomain

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**Revised
Feburary 1991**

THE MIXED POWERDOMAIN*

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February 12, 1991

Abstract

This paper introduces an operator \mathcal{M} called the *mixed powerdomain* which generalizes the convex (Plotkin) powerdomain. The construction is based on the idea of representing partial information about a set of data items using a *pair* of sets, one representing partial information in the manner of the upper (Smyth) powerdomain and the other in the manner of the lower (Hoare) powerdomain where the components of such pairs are required to satisfy a consistency condition. This provides a richer family of meaningful partial descriptions than are available in the convex powerdomain and also makes it possible to include the *empty set* in a satisfactory way. The new construct is given a rigorous mathematical treatment like that which has been applied to the known powerdomains. It is proved that \mathcal{M} is a continuous functor on bifinite domains which is left adjoint to the forgetful functor from a category of continuous structures called *mix algebras*. For a domain D with a coherent Scott topology, elements of $\mathcal{M}D$ can be represented as pairs (U, V) where $U \subseteq D$ is a compact upper set, $V \subseteq D$ is a closed set and the downward closure of $U \cap V$ is equal to V . A Stone dual characterization of \mathcal{M} is also provided.

1 Introduction.

A *powerdomain* is a “computable” analogue of the powerset operator. They were introduced in the 1970’s as a tool for providing semantics for programming languages with non-determinism. For such applications, the powerset operator was unsatisfactory for basically the same reasons that the full function space was unusable for the semantics of certain features of sequential programming languages (such as higher-order procedures and dynamic scoping). In the full powerset, there are *too many* sets and this causes problems for the solution of recursive domain equations. Hence, such applications call for a more parsimonious theory of subsets, based on a concept of non-deterministic computability.

The seminal work on powerdomains and their application in programming language semantics was G. Plotkin’s paper [Plo76] on what is often called the *Plotkin powerdomain*. Subsequent research by

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M. B. Smyth [Smy78] led to the discovery of two similar constructions often called the *Smyth* and *Hoare* powerdomains. These three powerdomains have been used widely in programming language theory, and they have also sparked a body of theoretical research into their properties and relationships to similar constructions in Mathematics. Smyth [Smy83] demonstrated a close connection between the Smyth and Hoare powerdomains and the concepts of upper and lower semi-continuity respectively. He also found that the Plotkin powerdomain was related to what is known as the *Vietoris construction* from topology. This research led Smyth to suggest the names for the three powerdomains which I will use below: upper (Smyth), lower (Hoare) and convex (Plotkin). The categorical significance of the powerdomains was demonstrated by Hennessy and Plotkin [HP79] who proved that each of the three can be seen as left adjoints to appropriate forgetful functors.

There has also been progress on understanding the powerdomains from the point of view of logic. G. Winskel [Win85] showed how each of the three powerdomains can be characterized using *modal* formulas under an interpretation in terms of non-deterministic computations. Substantial progress has also been made in clarifying the connections between logic and the powerdomains by utilizing concepts from Stone duality [Rob87, Vic89]. In particular, S. Abramsky has highlighted many useful connections between domains, topology and logic in his work on “domains in logical form” [Abr87a, Abr87b, Abr88] where he demonstrates in some detail how logics of programs can be derived from denotational semantics, including semantics utilizing the powerdomains.

My investigation in this paper is partly inspired by the work of researchers [BDW88, BO86, BJO89] in the area of databases who have investigated a construction known as *sandwiches* powerdomain. This construction is closely related to the other powerdomains and it is motivated by a desire to model partial information about sets in a more general way than the known structures permit. My purpose here is to introduce a similar construction which I call the *mixed* powerdomain. It maintains a kinship of spirit with the sandwiches powerdomain but enjoys some different properties. Based on an idea about testing properties of sets similar to those used in Plotkin’s original work on the convex powerdomain, R. Heckmann has independently defined the mixed powerdomain and studied many of its basic properties [Hec90b, Hec90a]. My own goal in this paper is to show how the mixed powerdomain fits cleanly into the mainstream of research on the other powerdomains as described above by carrying out an analysis of its order-theoretic, categorical and topological structure using the techniques which have been used for the classical operators. It will be assumed that the reader is familiar with basic results in domain theory and powerdomains as one finds in [Smy78], [SP82], [Sch86] and [GS90].

The second section of the paper attempts to provide some background on the information theoretic context of the mixed powerdomain by discussing ways in which sets of descriptions can be viewed as partial descriptions of sets of data items. The precise mathematical treatment of these ideas is the purpose of the introduction of the *mixed powerdomain* functor in the third section. This is characterized as the left adjoint of a forgetful function using *mix* algebras in the fourth section. The fifth and final section demonstrates an operator on coherent locales which corresponds to the mixed powerdomain.

2 Sets of properties as properties of sets.

Assume that we are given a collection X of *individuals* together with a family of predicates for describing properties of individuals. To focus our discussion and simplify matters somewhat, assume further that a property is *extensional* in the sense that it is uniquely determined by the individuals which possess it. This makes it possible to identify a property with the set of individuals which have the property. Assume furthermore, that any singleton individual $\{x\}$ is a property and the set X is itself a property. If we think of properties as *partial* descriptions of individuals, then a singleton $\{x\}$ is a *total* description of the individual x , whereas X is the most partial (uninformative) description because it is so vague that it is satisfied by all individuals.

It may help to work with a specific example. Consider the partially ordered set whose elements are records with fields `name`, `age`, `socsec` and `married?` having types given by the following expression:

```
{ name : { first : string,
          last  : string },
  age  : int,
  socsec : int,
  married? : bool }
```

Here is a sample record r_1 :

```
{ name = { first = "John",
             last  = "Smith" },
  age  = 28,
  socsec = 439048302,
  married? = true }
```

We will assume that records may have missing fields as in the following record r_2 :

```
{ name = { first = "John" },
  age  = 28 }
```

The intended interpretation of a basic type (such as `string`, `integer` or `bool`) is a flat domain where the only order relationship between two elements is $\perp \sqsubseteq x$ where \perp is the least element of the domain. A record is interpreted as a function from field labels into the disjoint sum of the interpretations of the types of its fields where each label is assigned a value in the appropriate type. The order on these functions is determined pointwise. A field is omitted in writing such a record (as the `socsec` field is omitted from r_2) just in case that field is assigned the value \perp .

Our *individuals* in this setting are records with all fields being non-bottom. So record r_1 may be viewed as the individual John Smith (*i.e.* the one whose social security number is 439048302) or, following the intuition offered above, it may be viewed as a *total* description of this individual. In short, individuals are the maximal records of our domain. On the other hand, r_2 is a *partial* description of an individual

who is 28 years old and has the name John. This partial description applies to the individual r_1 but it will also apply to many other individuals. Indeed, the partial description r_2 is uniquely determined by the set of individuals which can be obtained by filling in its blank fields. These records (with or without missing fields) form our space of *properties* in which a property applies to exactly the maximal elements which are above it in the partial order. Note that many partial descriptions of individuals are currently missing from our domain, such as the set of individuals whose social security number begins with a 4. So our descriptive capabilities are limited.

With this background I may now attempt to motivate the basic problem with which this paper is concerned. Given a family of individuals and partial descriptions of individuals, how should one construct a corresponding family of *sets of individuals* and partial descriptions of these sets? This is a basic question that one encounters in research in databases (which is the inspiration for the example above), but it is also a question that has arisen in the context of domain theory as applied to the semantics of non-determinism.

Since a property of individuals represents a set of individuals, it is reasonable to think of a property of *sets* of individuals to be a *set* of sets of individuals. To form such properties, we may choose to use sets of properties. However, there is more than one way to interpret this idea. Suppose, for example, we are given the following set s of properties, intended to represent a property of a set of individuals:

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John" },
  age = 2 }
```

Which sets should we take this to represent? The different powerdomain orderings represent different views of how this set of records forms a partial description of a set of individuals. Under the lower powerdomain “philosophy” it is a partial description of any set which has two year olds with first names Mary, Todd and John. It may therefore describe a set of individuals which includes some other names besides these three. However, under the upper powerdomain philosophy, it describes sets of individuals for which all members are two year olds with first name Mary, Todd, or John. In this case a set having the property represented by s may not actually have anyone named Mary in it, although everyone else must be a Todd or John. The convex powerdomain philosophy combines these two senses of meaning to yield a description which is the conjunction of the properties just asserted for the upper and lower powerdomains.

The powerdomains will be defined rigorously in the next section, but to aid this more philosophical discussion, let us introduce a bit of temporary notation. Given a set X of individuals and a property p of individuals (that is, $p \subseteq X$), define

$$\Box p = \{q \mid q \text{ is a set of individuals such that } q \subseteq p\} \quad (1)$$

and

$$\Diamond p = \{q \mid q \text{ is a set of individuals such that } q \cap p \neq \emptyset\}. \quad (2)$$

Letting \vee and \wedge represent union and intersection of properties respectively, we can say that the “meaning” of the set $\{p_1, \dots, p_n\}$ (as a property of sets) is $(\Diamond p_1) \wedge \dots \wedge (\Diamond p_n)$ with respect to the lower powerdomain, whereas the “meaning” is $\Box(p_1 \vee \dots \vee p_n)$ with respect to the upper powerdomain. Hence, the property expressed by $\{p_1, \dots, p_n\}$ under the convex powerdomain is:

$$\Box(p_1 \vee \dots \vee p_n) \wedge (\Diamond p_1) \wedge \dots \wedge (\Diamond p_n).$$

Now, this matches with the kinds of expressions for powerdomain elements that one sees in the literature on logical characterizations of these operators such as [Win85, Abr88, Vic89]. However, it seems that there may be other reasonable ways to formulate properties besides those of the form above. For example, could we also consider properties of the following form:

$$\Box(p_1 \vee \dots \vee p_n) \wedge (\Diamond p_1) \wedge \dots \wedge (\Diamond p_k).$$

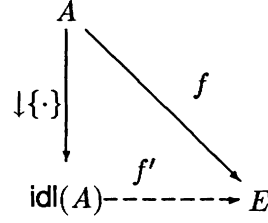
where $k \leq n$? This approach will allow us to have more partial descriptions, at the expense of needing a *pair* of sets $\{p_1, \dots, p_n\}$ and $\{p_1, \dots, p_k\}$ rather than the one set that we used before. The mixed powerdomain construction is based on this idea. Are there other possibilities as well?

A number of researchers have found themselves asking this question in the context of disparate objectives that drove interest in partial descriptions of sets. To approach this question, we must look carefully at the frameworks within which the known operators have been studied. The techniques developed in the above-mentioned references and others will not allow just anything—the constructions are expected to have many desirable properties. Moreover, the framework that I have used in this section to motivate the study which I will now pursue is wholly inadequate as it stands. In particular, the “extensionality” assumption which allows us to identify a property with the set of individuals satisfied by it is too restrictive. (There are also many technical problems that arise, such as the preservation of extensionality by various operators.) Fortunately, a host of techniques from topology, algebra and the theory of domains provide a way to rigorously develop the intuition behind the mixed powerdomain.

3 The mixed powerdomain functor.

A *pre-order* is a set A together with a binary relation \succeq which is reflexive and transitive. We often write $\preceq x$ rather than $x \succeq y$. We write $x \approx y$ if $x \succeq y$ and $x \preceq y$. Given a pre-order A and $u \subseteq A$, let $\downarrow u = \{y \in A \mid y \preceq x \text{ for some } x \in u\}$. A subset $u \subseteq A$ is an *ideal* if it is directed and $u = \downarrow u$. For a pre-order A , let $\text{idl}(A)$ be the poset of ideals on A , ordered by subset inclusion. Posets D which are isomorphic to such ideal completions will be called *domains* for the purposes of this discussion and they may be characterized as algebraic cpo’s. For a domain D , let KD be the basis of compact elements of D . Given a subset $M \subseteq D$ we write $\bigsqcup M$ for the least upper bound of M . A function $f : D \rightarrow E$ between domains is *continuous* if $f(\bigsqcup M) = \bigsqcup_{x \in M} f(x)$ for any directed $M \subseteq D$. The following lemma is a quite useful way to define continuous functions between domains:

Lemma 1 Let A be a pre-order and suppose E is a cpo. If $f : A \rightarrow E$ is monotone, then there is a unique continuous function f' which completes the following diagram



In particular, a continuous function $f : D \rightarrow E$ between domains D and E is uniquely determined by its restriction to KD . ■

Let $\langle A, \succeq \rangle$ be a pre-order and suppose $\mathcal{P}_f A$ is the collection of finite subsets of A . We define three pre-orderings on $\mathcal{P}_f A$ as follows. Suppose $u, v \in \mathcal{P}_f A$, then

- $u \succeq^\# v$ iff for every $x \in u$ there is a $y \in v$ such that $x \succeq y$,
- $u \succeq^b v$ iff for every $y \in v$ there is a $x \in u$ such that $x \succeq y$,
- $u \succeq^h v$ iff $u \succeq^\# v$ and $u \succeq^b v$

It is easy to check that each of these relations is, in fact, a pre-ordering. Given a pre-order A and a subset $u \subseteq A$, let

$$\begin{aligned}
 \uparrow u &= \{x \in A \mid x \succeq y \text{ for some } y \in u\} \\
 \downarrow u &= \{x \in A \mid x \preceq y \text{ for some } y \in u\}
 \end{aligned}$$

We can also characterize the first two orderings as follows:

- $u \succeq^\# v$ iff $u \subseteq \uparrow v$,
- $u \succeq^b v$ iff $\downarrow u \subseteq v$,

Now, define $A^\#$, A^b and A^h to be the pre-orders $\langle \mathcal{P}_f A, \succeq^\# \rangle$, $\langle \mathcal{P}_f A, \succeq^b \rangle$ and $\langle \mathcal{P}_f A, \succeq^h \rangle$ respectively. Given a domain D , define

- the *upper powerdomain* $UD = \text{idl}((KD)^\#)$
- the *lower powerdomain* $LD = \text{idl}((KD)^b)$
- the *convex powerdomain* $CD = \text{idl}((KD)^h)$

Note, in particular, that \emptyset is included in the pre-orders on which the powerdomains are based. In the upper powerdomain, the principal ideal that it generates is the largest element of the powerdomain. In the lower powerdomain, on the other hand, it is the smallest. For the convex powerdomain, $\{\emptyset\}$ is itself an ideal which is incomparable to all others. In most references on powerdomains, the empty set is excluded from the definition.

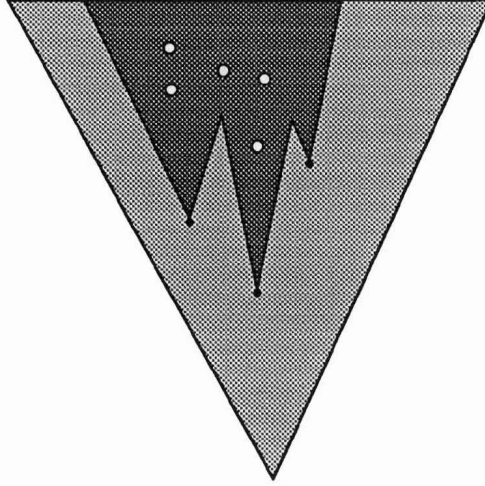


Figure 1: A mixed powerdomain element (u, v) is illustrated above. The elements of the set u are indicated as closed circles (dots). They determine a shaded upper set within which the elements of v must lie. The elements of v are represented as open circles.

Let $\langle A, \succeq \rangle$ be a pre-order. A *sandwich* is a pair $(u, v) \in \mathcal{P}_f A \times \mathcal{P}_f A$ such that there is some $w \in \mathcal{P}_f A$ with $w \succeq^{\sharp} u$ and $w \succeq^{\flat} v$. A *mix* (on A) is a pair $(u, v) \in \mathcal{P}_f A \times \mathcal{P}_f A$ such that $v \succeq^{\sharp} u$. Define a pre-ordering \succeq on sandwiches given by taking $(u, v) \succeq (u', v')$ iff $u \succeq^{\sharp} u'$ and $v \succeq^{\flat} v'$. Let us write $\mathcal{S}^0 A$ for sandwiches pre-ordered by \succeq and $\mathcal{M}^0 A$ for mixes pre-ordered by \succeq . Given a domain D , we define

- the *sandwich powerdomain* $SD = \text{idl}(\mathcal{S}^0(KD))$
- the *mixed powerdomain* $MD = \text{idl}(\mathcal{M}^0(KD))$

Evidently, any mix is also a sandwich. A pictorial representation of a mix and a sandwich appear in Figures 1 and 2 respectively. My discussion in this paper will focus on the mixed powerdomain.

If $f : D \rightarrow E$ is a continuous function between domains D and E , we define $\mathcal{M}(f) : \mathcal{M}(D) \rightarrow \mathcal{M}(E)$ by extending the definition on mixes

$$\mathcal{M}(f)(u, v) = \downarrow(f^*(u), f^*(v))$$

where $f^*(u)$ and $f^*(v)$ are the images of u and v respectively under the function f , to a continuous function between domains. It is easy to show that this defines \mathcal{M} as an endofunctor on the category of domains with continuous functions. Indeed, we have the following

Lemma 2 *If $f \sqsubseteq g$, then $\mathcal{M}(f) \sqsubseteq \mathcal{M}(g)$. If N is a directed subset of $D \rightarrow E$, then $\mathcal{M}(\bigsqcup N) = \bigsqcup \mathcal{M}(N)$. ■*

Hence, the methods discussed in [SP82] can be used to show that \mathcal{M} defines a continuous functor on domains (with continuous functions or embeddings).

In the case of finite posets, it is easy to see that mixes correspond to pairs of upward and downward closed subsets.

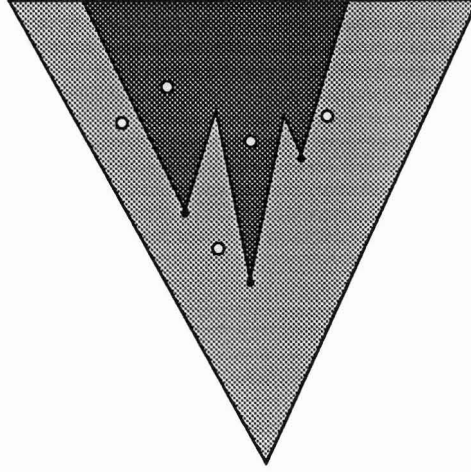


Figure 2: A sandwich (u, v) is illustrated above. The elements of the set u are indicated as closed circles (dots). They determine a shaded upper set. The elements of v are represented as open circles; each element of v is required to have an upper bound in the shaded region.

Proposition 3 *If A is a finite poset, then $\mathcal{M}A$ is isomorphic to the poset MA of pairs (u, v) such that u is an upper set, v is a lower set and $v = \downarrow(u \cap v)$ under the ordering $(u, v) \preceq (u', v')$ iff $u' \subseteq u$ and $v \subseteq v'$.*

Proof: Given a mix (u, v) , the pair $f((u, v)) = (\uparrow u, \downarrow v)$ is an element of MA . To see this, suppose $x \in \downarrow v$, then there is a $y \in v$ such that $x \preceq y$. Since $u \preceq^\# v$, we know that $y \in \uparrow u$ so $x \in \downarrow(\uparrow u \cap \downarrow v)$. Since $\uparrow u \cap \downarrow v$ is clearly a subset of $\downarrow v$, it follows that $\downarrow(\uparrow u \cap \downarrow v) = \downarrow v$. Now, given $(u, v) \in MA$, define $g(u, v)$ to be the principal ideal in $\mathcal{M}^0 A$ generated by $\downarrow(u, \uparrow u \cap \downarrow v)$. It is straight-forward to check that $f \circ g$ and $g \circ f$ are both identities. ■

This proposition may be generalized using the topological characterizations for the other powerdomains to obtain a characterization of the mixed powerdomain on domains with coherent Scott topology. Given a domain D , let ΩD denote its Scott topology. ΩD is said to be *coherent* if any finite intersection of compact open sets is again a compact (open) set. Order-theoretically, this corresponds to a condition on D sometimes called *property M* (see Proposition 5 in [GJ90]). On such domains a compact upper set is exactly the intersection of a filter basis of compact open subsets [HM81]; this fact can be used to characterize the upper powerdomain $\mathcal{U}D$ as the set of compact upper sets of D , ordered by superset inclusion (see [Smy78] or Section 11.2 of [Vic89]). On the other hand, the lower powerdomain $\mathcal{L}D$ is isomorphic to the poset of closed subsets of D ordered by subset inclusion. For the mixed powerdomain we have the following:

Theorem 4 *If D is a domain with coherent topology, then the mixed powerdomain of D is isomorphic to the set of pairs (U, V) where,*

1. U is a compact upper set
2. V is closed set

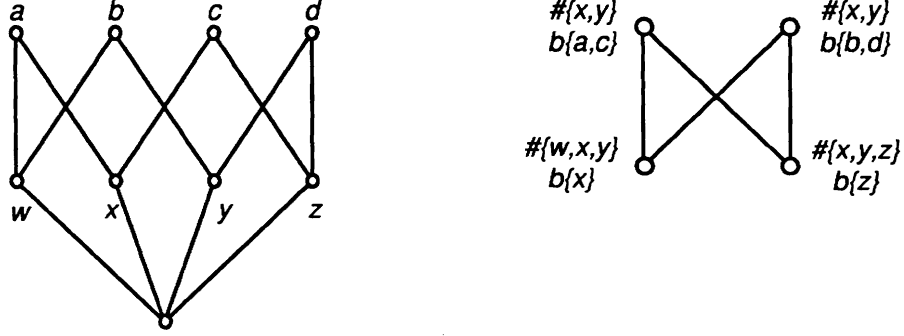


Figure 3: The four elements indicated in the picture on the right show that the mixed powerdomain of the bounded complete domain pictured on the left is not bounded complete. The two mixes at the bottom have no least upper bound, since the two mixes indicated above them are minimal upper bounds.

$$3. V = \downarrow(U \cap V)$$

under the ordering $(U, V) \succeq (U', V')$ if $U \subseteq U'$ and $V \supseteq V'$. ■

This Theorem, together with similar characterizations of the convex and sandwich powerdomains were discovered by Heckmann [Hec90a].

One noteworthy order-theoretic property of the mixed powerdomain is an anomaly which it shares with the convex powerdomain: *it does not preserve the property of bounded completeness*. For example, the poset displayed in Figure 3 is bounded complete, but its mixed powerdomain is not. Since \mathcal{M} is a continuous functor which sends finite posets to finite posets, it is guaranteed to send a *bifinite* domain to a bifinite domain (see [Gun87] or [GS90] for a discussion—they are called “profinite” domains in the former reference), so the closure properties of \mathcal{M} are quite similar to those of the convex powerdomain \mathcal{C} . The restriction of \mathcal{M} to the bifinite domains also allows us to apply Theorem 4 since a bifinite domain has a coherent Scott topology.

The close similarity between the convex and mixed powerdomains leads one to ask whether these operators might actually be *isomorphic*. Given a pre-order A , it is clear that there is a nice monotone map from A^\flat into $\mathcal{M}^0 A$ defined by $u \mapsto (u, u)$. This map is an *order-embedding*, i.e. for each $u, v \in A^\flat$, $u \preceq^\flat v$ if, and only if, $(u, u) \preceq (v, v)$. Could this be an isomorphism? The answer is obvious when the empty set is present (since its behavior in the convex and mixed powerdomains is very different). It is also easy to see that this map is not an isomorphism for the powerdomains without empty set by inspecting the respective powerdomains of the truth value cpo pictured in Figure 4. The upper and lower powerdomains (without empty set) of T are also displayed there with equivalent elements identified and representatives of the equivalence classes tagging the nodes. It is clear from the pictures that these posets are not isomorphic since the convex powerdomain has 7 elements whereas the mixed powerdomain has 9.

To use the mixed powerdomain for the semantics of programming languages, it is essential to define a collection of auxiliary functions such as those ordinarily associated with the powerset operation. There are two such operations which are of primary interest. The mixed powerdomain *union* is a function

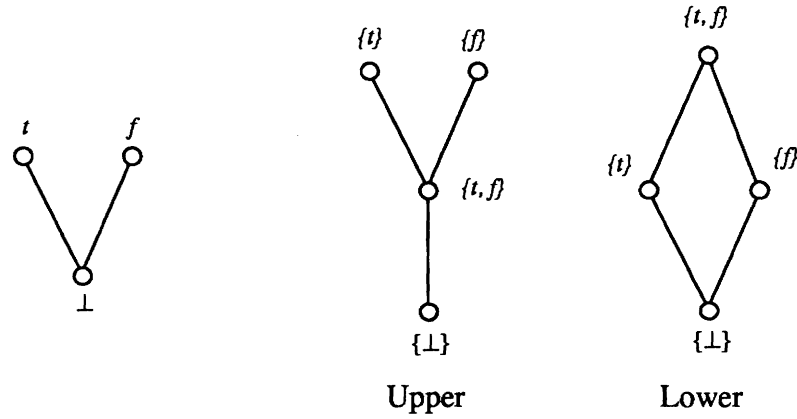


Figure 4: *Upper and lower powerdomains (without emptyset) for the truth value cpo.*

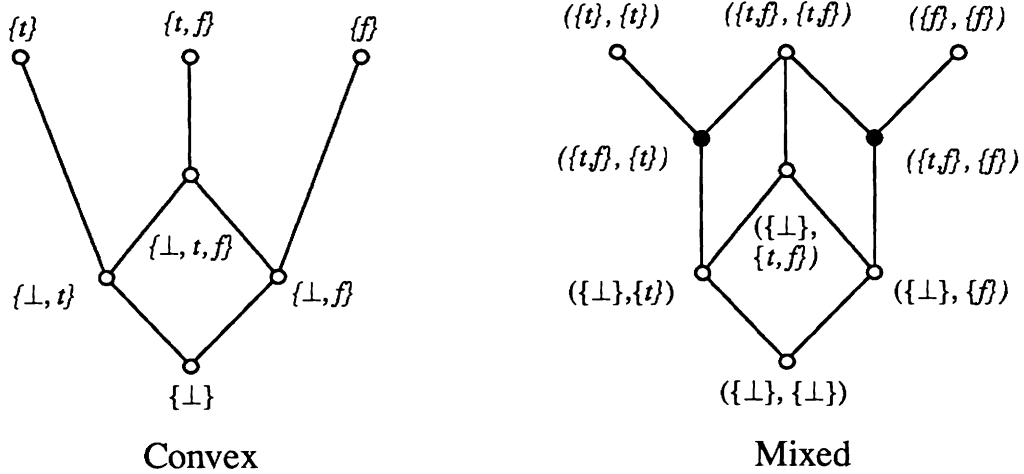


Figure 5: *The convex and mixed powerdomains of the truth value cpo are not isomorphic. The order-embedded image of the convex powerdomain (without empty set) in the mixed powerdomain (without empty set) is indicated with open circles in the figure on the right. The two points not in the image of this order-embedding are indicated with closed circles.*

$\sqcup : \mathcal{M}D \times \mathcal{M}D \rightarrow \mathcal{M}D$. If $u = (u^\sharp, u^\flat)$ and $v = (v^\sharp, v^\flat)$ are mixes over KD , their union is defined as follows:

$$u \sqcup v = (u^\sharp \cup v^\sharp, u^\flat \cup v^\flat).$$

We show that $u \sqcup v$ is a mix over KD . Suppose $x \in u^\flat \cup v^\flat$. If $x \in u^\flat$, then there is an element $x' \in u^\sharp$ such that $x' \preceq x$ because $u^\sharp \preceq^\sharp u^\flat$. Since a similar fact holds for elements of v^\flat , it follows that $u \sqcup v$ is indeed an element of $\mathcal{M}^0(KD)$. To see that the union is also monotone, suppose $v \succeq w$ for some $w \in \mathcal{M}^0(KD)$. To show that $u \sqcup v \succeq u \sqcup w$, we must show that

$$u^\sharp \cup v^\sharp \succeq^\sharp u^\sharp \cup w^\sharp \tag{3}$$

and

$$u^\flat \cup v^\flat \succeq^\flat u^\flat \cup w^\flat. \tag{4}$$

For the former inequation, suppose $x \in u^\sharp \cup w^\sharp$. We must show that there is an $x' \in u^\sharp \cup v^\sharp$ such that $x' \preceq x$. If $x \in u^\sharp$, then this is immediate since we can take $x' = x$. If $x \in w^\sharp$, then there is an $x' \in v^\sharp$ such that $x' \preceq x$ because of the mixed powerdomain ordering. This establishes inequation 3; a similar argument may be used to show inequation 4. Proof of monotonicity of \sqcup in the other argument is similar. This defines a monotone function from $\mathcal{M}^0(KD) \times \mathcal{M}^0(KD)$ into $\mathcal{M}^0(KD)$. The desired union function \sqcup is obtained by extension to a continuous map on the mixed powerdomain using Lemma 1. Because this way of using the Lemma is so common it is convenient to leave its application implicit in defining a function or proving an equational property of a function.

The mixed powerdomain *singleton* is a function $\{\cdot\} : KD \rightarrow \mathcal{M}^0D$ given by $\{x\} = (\{x\}, \{x\})$. (Of course, this extends to a function $\{\cdot\} : D \rightarrow \mathcal{M}D$.) Finally, the principal ideal generated by the pair (\emptyset, \emptyset) represents the *empty set* in the mixed powerdomain; it is a unit for the binary operation \sqcup .

In a sense, the mixed powerdomain is “larger” than each of the upper, lower and convex powerdomains on certain domains. It was mentioned above that the convex powerdomain can be order-embedded in the mixed powerdomain. For the upper and lower powerdomains, there is more that can be said. First of all, define a map $\square : \mathcal{M}^0(KD) \rightarrow \mathcal{M}^0(KD)$ by $\square(u, v) = (u, \emptyset)$. The corestriction of this function can be viewed as a continuous projection (surjective right adjoint) from the mixed powerdomain onto the upper powerdomain. An injective left adjoint (which is usually called an *embedding* in domain theory) is an order-embedding, so the lower powerdomain is order-embedded in the mixed powerdomain. Now, when a domain D has a coherent Scott topology ΩD , it also has a finite set of minimal elements r . The singleton $\{r\}$ is an ideal with respect to the ordering \sqsubseteq^\sharp so $\mathcal{U}D$ has this singleton as its least element. For such domains, define a map $\diamond : \mathcal{M}^0(KD) \rightarrow \mathcal{M}^0(KD)$ given by $\diamond(u, v) = (r, v)$. The corestriction of this function can be viewed as a continuous projection from the mixed powerdomain onto the lower powerdomain.

4 Algebraic characterization of the mixed powerdomain.

One of the most challenging problems for new powerdomain constructions has been the discovery of the appropriate signatures and equations (or inequations) to capture their fundamental algebraic structure.

In the case of the convex powerdomain, the first intuitions—as described in [Plo76]—came from the semantics of parallel computation. Although [Plo76] describes all of the relevant algebraic operations, it was only later, in [HP79] that the convex powerdomain was characterized in terms of these operations together with a simple set of equational axioms which they satisfy. At the same time, the upper and lower powerdomains were also thus characterized using the same algebraic signature but additional inequational axioms. Such characterizations are now treated as a standard element of the general methodology of semantics (see [Hen88] as an example of this). Unfortunately, no characterization of this kind has yet been found for the sandwiches powerdomain and this remains an open question. In this section, I will demonstrate an algebraic characterization of the mixed powerdomain in terms of a category of structures called *mix algebras*. These algebras, and their freeness property, were independently noted by Heckmann [Hec90b] who also proposed a characterization of the sandwiches powerdomain using a theory involving *partial* operations.

Another problem which has arisen in the study of powerdomains for concurrency is how to derive a powerdomain which includes an *empty set* element. When we allow the empty set as an element of the convex powerdomain, we have the problem that it is *unrelated* to other elements under the convex ordering. In particular, a convex powerdomain with empty set does not have a least element. Given the importance of least elements for the solution of recursive equations, this straight-forward approach to adding an empty set is unsatisfactory for the semantics of programming languages. The problem can be partly rectified by “adding the empty set onto the side” of the convex powerdomain as proposed in [Abr87a]. So the empty set element is related only to the least element. This approach seems acceptable in the sense that it makes a reasonable semantics possible, but it makes a mess of the algebraic characterization of the powerdomain. The simple problem is this, if we add an axiom which says that the least element is less than the empty element, then the least element is part of our signature and is therefore preserved by any of the homomorphisms which we construct. But this is not desirable since there may well be terms in the language whose intended interpretation is non-strict (*i.e.* does not send the least element of its domain to the least element of its range). In fact, it can be shown that this problem has no acceptable solution with the simplest signature and axioms; we offer a proof of this impossibility in this section.

Thus, one goal of this section is to show how the problems with the algebraic properties of powerdomains with empty set can be resolved by using the mixed powerdomain. To anticipate the basic idea, consider the nature of the empty set as a piece of partial information about a set. The information content of the empty set as *upper* information is quite different from its significance as *lower* information. In the upper powerdomain ordering, the empty set is totally descriptive—it means that the set being described has no members. On the other hand, in the lower powerdomain ordering, the empty set is totally nondescriptive—it means that no element is known to be in the set being described. In the case that the underlying domain has a least element \perp , even the singleton set $\{\perp\}$ is more informative under the lower ordering than the empty set. Now, in the mixed powerdomain with empty set, the mix $(\{\perp\}, \emptyset)$ is the *least* element. Even the element $(\{\perp\}, \{\perp\})$ is more informative, since this latter element describes only *non-empty* sets! In the mixed powerdomain, the empty set (\emptyset, \emptyset) is a total (maximal) element which describes the unique set with no members.

A *mix algebra* is a partial order N, \sqsubseteq together with a monotone binary operation $* : N \times N \rightarrow N$, a

monotone unary operation $\square : N \rightarrow N$ and a constant $e \in N$ which satisfy the following nine axioms

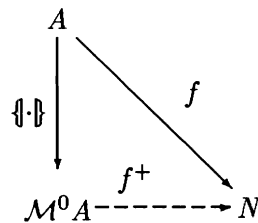
1. associativity: $(r * s) * t = r * (s * t)$
2. commutativity: $r * s = s * r$
3. idempotence: $s * s = s$
4. unit: $e * s = s * e = s$
5. $\square(s * r) = (\square s) * (\square r)$
6. $\square \square s = \square s$
7. $s * (\square s) = s$
8. $\square s \sqsubseteq s$
9. $s * (\square r) \sqsubseteq s$

A *homomorphism* between mix algebras M and N is a monotone function $f : M \rightarrow N$ such that $f(r * s) = (f(r)) * (f(s))$ and $f(\square r) = \square f(r)$ and $f(e) = e$. A *continuous* mix algebra is a mix algebra $\langle N, *, \square, e \rangle$ where N is a domain and the functions $*$ and \square are continuous. A homomorphism of continuous mix algebras is a continuous homomorphism of mix algebras.

Of course, the first four axioms are the axioms for a semi-lattice with unit. Given a mix algebra N , the binary operation $*$ on N induces a semi-lattice ordering \in given by $r \in s$ iff $r * s = s$. It is important not to confuse this subset ordering with the ordering \sqsubseteq of partial information since these orderings will rarely coincide. Note, in particular, that axiom (7) says that \square is a *kernel operator* with respect to \in , i.e. $\square s \in s$.

Given a domain D , recall the definition of the map $\square : \mathcal{M}^0(KD) \rightarrow \mathcal{M}^0(KD)$ by $\square : (u, v) \mapsto (u, \emptyset)$. We show that $\langle \mathcal{M}D, \sqcup, \square, (\emptyset, \emptyset) \rangle$ is a continuous mix algebra by demonstrating that the desired equations are satisfied by the actions of the operators on its basis $\mathcal{M}^0(KD)$. Axioms (1)-(4) are immediate consequences of the definition of \sqcup . To prove (5), let (u, v) and (u', v') be elements of $\mathcal{M}^0(KD)$. Then $\square((u, v) \sqcup (u', v')) \approx \square(u \sqcup u', v \sqcup v') \approx (u \sqcup u', \emptyset) \approx (u, \emptyset) \sqcup (u', \emptyset) \approx \square(u, v) \sqcup \square(u', v')$. Axiom (6) is immediate from the definition of \square . For axiom (7), $(u, v) \sqcup (\square(u, v)) \approx (u \sqcup u, v \sqcup \emptyset) \approx (u, v)$. To see axiom (8), note that $\emptyset \sqsubseteq^b v$ for any v . For axiom (9), $(u, v) \sqcup (\square(u', v')) \approx (u \sqcup u', v) \preceq (u, v)$ since $u \sqcup u' \sqsubseteq^b u$.

Theorem 5 *Let A be a pre-order and suppose N is a mix algebra. For any monotone $f : A \rightarrow N$, there is a unique homomorphism $f^+ : \mathcal{M}^0 A \rightarrow N$ which completes the following diagram:*



Proof: First of all, note that if (w, v) is a mix, then it is equivalent to a mix of the form $(u \cup v, v)$ where u and v are disjoint. If $u = \{a_1, \dots, a_n\}$ and $v = \{b_1, \dots, b_m\}$, then

$$(u \cup v, v) = \square \{a_1\} \cup \dots \cup \square \{a_n\} \cup \{b_1\} \cup \dots \cup \{b_m\}.$$

Hence, if a homomorphism f^+ which completes the diagram exists, then

$$f^+(u \cup v, v) = \square f(a_1) * \dots * \square f(a_n) * f(b_1) * \dots * f(b_m).$$

Of course, we set $f^+(\emptyset, \emptyset) = e$. This proves the uniqueness of f^+ given its existence. To show existence, we must prove that f^+ is monotone and that it is a homomorphism.

To see that f^+ is monotone, suppose that $(u \cup v, v)$ and $(u' \cup v', v')$ are mixes where $u \cap v = \emptyset$ and $u' \cap v' = \emptyset$. Suppose that $u \cup v \lesssim^{\#} u' \cup v'$ and $v \lesssim^b v'$. Also define

$$\begin{aligned} u &= \{a_1, \dots, a_n\} \\ v &= \{b_1, \dots, b_l\} \\ u' &= \{a'_1, \dots, a'_k\} \\ v' &= \{b'_1, \dots, b'_m\} \end{aligned}$$

Then

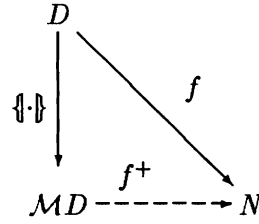
$$\begin{aligned} f^+(u \cup v, v) &= \square f(a_1) * \dots * \square f(a_n) * f(b_1) * \dots * f(b_l) \\ &= \square f(a_1) * \dots * \square f(a_n) * \square f(b_1) * \dots * \square f(b_l) \\ &\quad * f(b_1) * \dots * f(b_l) && \text{by (7)} \\ &\sqsubseteq \square f(a_1) * \dots * \square f(a_n) * \square f(b_1) * \dots * \square f(b_l) \\ &\quad * \square f(a'_1) * \dots * \square f(a'_k) * \square f(b'_1) * \dots * \square f(b'_m) \\ &\quad * f(b_1) * \dots * f(b_l) && \text{since } u \cup v \lesssim^{\#} u' \cup v' \\ &\sqsubseteq \square f(a_1) * \dots * \square f(a_n) * \square f(b_1) * \dots * \square f(b_l) \\ &\quad * \square f(a'_1) * \dots * \square f(a'_k) * f(b'_1) * \dots * f(b'_m) \\ &\quad * f(b_1) * \dots * f(b_l) && \text{by (8)} \\ &\sqsubseteq \square f(a_1) * \dots * \square f(a_n) * \square f(b_1) * \dots * \square f(b_l) \\ &\quad * \square f(a'_1) * \dots * \square f(a'_k) * f(b'_1) * \dots * f(b'_m) && \text{since } v \lesssim^b v' \\ &\sqsubseteq \square f(a'_1) * \dots * \square f(a'_k) * f(b'_1) * \dots * f(b'_m) && \text{by (9)} \\ &= f^+(u' \cup v', v') \end{aligned}$$

To prove that f^+ is a homomorphism, suppose $(u \cup v, v)$ is a mix as above. Then

$$\begin{aligned} \square f^+(u \cup v, v) &= \square (\square f(a_1) * \dots * \square f(a_n) * f(b_1) * \dots * f(b_l)) \\ &= \square \square f(a_1) * \dots * \square \square f(a_n) * \square f(b_1) * \dots * \square f(b_l) && \text{by (5)} \\ &= \square f(a_1) * \dots * \square f(a_n) * \square f(b_1) * \dots * \square f(b_l) && \text{by (6)} \\ &= f^+(u \cup v, \emptyset) \\ &= f^+(\square(u \cup v, v)) \end{aligned}$$

For $u = v = \emptyset$, note that $\square e = e * \square e = e$ by (7) and (4). ■

Corollary 6 *Let D be a domain. Suppose N is a continuous mix algebra. For any continuous $f : D \rightarrow N$, there is a unique homomorphism $f^+ : \mathcal{M}D \rightarrow N$ which completes the following diagram:*



Proof: Let f_0 be the restriction of f to KD . By Theorem 5, there is a homomorphism $f_0^+ : \mathcal{M}^0 D \rightarrow N$ of mix algebras such that $f_0^+ \circ \mathbb{I} = f_0$. By Lemma 1, this homomorphism has a unique extension to a continuous function $f^+ : \mathcal{M}D \rightarrow N$ which satisfies the desired diagram. This map will be a homomorphism. ■

Corollary 6 can be restated as follows: *The mixed powerdomain is left adjoint to the forgetful functor from the category of continuous mix algebras and continuous homomorphisms to the category of domains and continuous functions.* Several other results such as this are known for powerdomains. The most interesting of these is Theorem 7 below.

Definition: A *continuous semi-lattice* is a domain N together with a binary operation $*$ which satisfies the axioms (1)–(3) for mix algebras. A *homomorphism* of continuous semi-lattices M, N is a continuous function $h : M \rightarrow N$ such that $h(r * s) = h(r) * h(s)$. ■

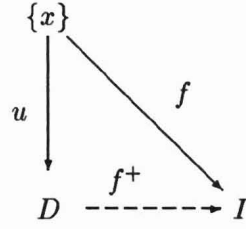
The following Theorem is proved in [HP79]:

Theorem 7 (Hennessy and Plotkin) *The convex powerdomain is left adjoint to the forgetful functor from the category of continuous semi-lattices (with bottom) and homomorphisms to the category of domains (with bottom) and continuous functions.* ■

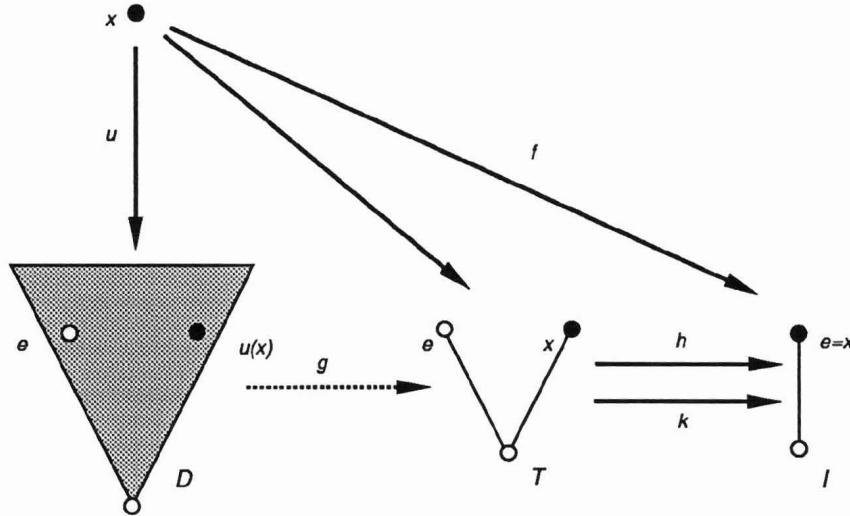
Definition: A continuous semi-lattice *with unit* is a continuous semi-lattice with a constant e which satisfies axiom (4) for mix algebras. A homomorphism of continuous semi-lattices M, N with unit is a homomorphism of continuous semi-lattices which sends the unit of M to the unit of N . ■

Proposition 8 (Plotkin) *There is no left adjoint to the forgetful functor from continuous semi-lattices with unit and bottom to that of domains with bottom.* ■

Proof: Suppose that there is a left adjoint to the forgetful functor and let D be the free continuous semi-lattice with unit generated by the poset $\{x\}$ with one element x . Let I be the semi-lattice with unit that has two elements \perp, x with $\perp \sqsubseteq x$ and $\perp * x = \perp$. Let f be the map from $\{x\}$ to I which sends x to $x \in I$. We demonstrate a contradiction by showing that for no map u is there a unique homomorphism f^+ which completes the diagram



Now, let T be the poset with three distinct elements e, x, \perp with $\perp \sqsubseteq e$ and $\perp \sqsubseteq x$. This poset can be given the structure of a semi-lattice with unit e by defining $x * \perp = \perp$. Since D is freely generated by $\{x\}$, there is a homomorphism $g : D \rightarrow T$ which sends the image of x under u to $x \in T$. If e is the unit of D , then $g(e) = e$. Since g is monotone, this means $u(x)$ is incomparable to e in D and, consequently, $g(\perp) = \perp$. Now, consider the map $h : T \rightarrow I$ which sends the elements of T constantly to $e = x$ and the map $k : T \rightarrow I$ which sends \perp to \perp . The situation can be pictured as follows:



Both of these maps are homomorphisms so we must have $f^+ = h \circ g = k \circ g$ by the uniqueness of f^+ . But this is nonsense, since the two compositions are not equal. ■

Proposition 9 *The mixed powerdomain is left adjoint to the forgetful functor from the category of continuous mix algebras with bottom to the category of domains with bottom.*

Proof: This is immediate from Corollary 6 since the mixed powerdomain of a domain D has a least element given as the principal ideal generated by $(\{\perp\}, \emptyset)$. ■

The way in which the mixed powerdomain is able to treat the empty set algebraically may make it useful for various applications in programming language semantics. Also, the extra descriptive power provided by the operator \Box , could make mix algebras useful in specifying non-determinism (this idea is explored in [Gun90]).

5 Normal forms and Stone duality.

In this section we produce a theory which is capable of rigorously capturing the intuitions that were mentioned in the second section of this paper. We replace the notion of a “property” as it was discussed before with the concept of a Scott open subset of the domain and utilize the techniques of Stone duality theory to formalize the operators \square and \diamond and the mixed powerdomain construction itself in those terms. Most of the basic ideas we need below are already present in the basic literature on Stone duality (e.g. [Joh82, Vic89]). For the specific case of operators on domains, Stone duality properties have been studied by a number of individuals. A thorough exploration appears in recent work of Abramsky [Abr87a, Abr87b, Abr88]. The goal of this section is to show how the theory in [Abr88] applies to the mixed powerdomain. In particular, a concept of the “logic of properties” of a domain D is given and we seek to demonstrate an operator which takes the “logic of properties” of D and produces the corresponding “logic of properties” of its mixed powerdomain $\mathcal{M}D$. The idea is a variation on the techniques which deal with the convex powerdomain—the mixed powerdomain basically arises from dropping one of the axioms! The results in this section were conjectured by Samson Abramsky and Steve Vickers and the proofs follow closely those which appear in [Abr88]. *In this section, the results apply to the mixed (and convex) powerdomain without empty set.* This restriction makes it more straight-forward to use Abramsky’s results. Similar results can no doubt be derived for the case with empty set.

Definition: A *coherent algebraic prelocale* is a structure $A = \langle |A|, \lesssim, 0, \vee, 1, \wedge, P \rangle$ with universe $|A|$ such that

- \lesssim is a binary relation on $|A|$
- 0 and 1 are elements of $|A|$
- \vee and \wedge are binary operators on $|A|$
- P is a unary predicate

such that the following axioms and rules are satisfied

$$\begin{aligned}
 (d1) \quad & 0 \lesssim a \quad \frac{a \lesssim b \quad b \lesssim c}{a \lesssim c} \\
 (d2) \quad & a \lesssim a \quad \frac{a \lesssim b \quad a \lesssim c}{a \lesssim b \wedge c} \quad a \wedge b \lesssim a \quad a \wedge b \lesssim b \\
 (d3) \quad & a \lesssim 1 \quad \frac{a \lesssim c \quad b \lesssim c}{a \vee b \lesssim c} \quad a \lesssim a \vee b \quad b \lesssim a \vee b \\
 (d4) \quad & a \wedge (b \vee c) \lesssim (a \wedge b) \vee (a \wedge c) \\
 (p1) \quad & \frac{P(a) \quad a \approx b}{P(b)} \\
 (p2) \quad & \text{If } P(a) \text{ and } a \lesssim \bigvee_{i \in I} b_i \text{ then } \exists i \in I. a \lesssim b_i \\
 (p3) \quad & \forall a \in |A|. \exists b_1, \dots, b_n \in P(A). a \approx \bigvee_{i=1}^n b_i
 \end{aligned}$$

where $a \approx b$ iff $a \lesssim b$ and $b \lesssim a$. ■

Axiom (p2) says that elements satisfying the predicate P are coprimes:

Definition: An element x of a lattice L is said to be a *coprime* if, whenever $x \leq \bigvee_{i=0}^n x_i$, there is some i such that $x \leq x_i$. ■

Theorem 10 *If D is a bifinite domain, then the structure*

$$\langle K(\Omega D), \subseteq, \emptyset, \cup, D, \cap, \{\uparrow x \mid x \in KD\} \rangle$$

is a prelocale. ■

I will write $K(\Omega D)$ for this structure as well as for the compact elements of ΩD and refer to it as *the prelocale determined by D* . The name comes from the fact that the ideal completion of this structure is isomorphic to the locale ΩD . As often happens, it is convenient here to work with the basis of an algebraic cpo (such as ΩD for domains D) rather than with the full poset. This makes it possible to use a finitary signature and avoid conditions about directed joins. The following is a basic fact which characterizes the compact open subsets $K(\Omega D)$:

Lemma 11 *Let D be a domain. Then the poset ΩD of Scott open subsets of D ordered by subset inclusion is an algebraic lattice such that $U \in K(\Omega D)$ iff there is a finite set $u \subseteq KD$ such that $U = \uparrow u$.* ■

From this, the proof of Theorem 10 is not difficult using the fact that bifinite domains have a coherent Scott topology. Although I am restricting myself to bifinite domains here, it is worth noting that many of the results below could probably be proved even for spaces that are not coherent using techniques such as those discussed in [Rob86].

Our goal can now be described as follows. Let D be a bifinite domain and suppose $A = K(\Omega D)$ is the prelocale which it determines. We will define an operator $\bar{\mathcal{M}}$ on prelocales such that $\bar{\mathcal{M}}A$ is the prelocale determined by the mixed powerdomain of D . This will be done, by defining a carrier $|\bar{\mathcal{M}}A|$ and establishing a set of axioms and rules for the order relation \preceq and primality relation P on this carrier. To this end, we will use the following:

Notation: For a set S , the *prelocalic expressions over S* are defined as follows. Any element of S is a prelocalic expression over S . Constants 0 and 1 are prelocalic expressions over S . If ϕ and ψ are prelocalic expressions over S then so are $\phi \vee \psi$ and $\phi \wedge \psi$. ■

Let A be a given prelocale. The carrier $|\bar{\mathcal{M}}A|$ is defined to be the set of prelocalic expressions over $S \cup T$ where

$$S = \{\Box \phi \mid \phi \text{ is a prelocalic expression over } |A|\}$$

$$T = \{\Diamond \phi \mid \phi \text{ is a prelocalic expression over } |A|\}$$

Define \preceq and P to be the least relations over $|\bar{\mathcal{M}}A|$ which satisfy (d1)–(d4), (p1) and the following axioms:

$$\begin{aligned} (\Box - \wedge) \quad & \Box(a \wedge b) \approx \Box a \wedge \Box b \\ (\Diamond - \vee) \quad & \Diamond(a \vee b) \approx \Diamond a \vee \Diamond b \\ (\Diamond - \wedge) \quad & \Box a \wedge \Diamond b \preceq \Diamond(a \wedge b) \\ (\Box - 0) \quad & \Box 0 \approx 0 \\ (\Diamond - 1) \quad & \Diamond 1 \approx 1 \end{aligned}$$

and rules

$$\begin{aligned}
(\Box - \lesssim) \quad & \frac{a \lesssim b}{\Box a \lesssim \Box b} & (\Diamond - \lesssim) \quad & \frac{a \lesssim b}{\Diamond a \lesssim \Diamond b} \\
(P - \Box - \Diamond) \quad & \frac{\{P(a_i) \mid i \in I\}}{P(\Box \bigvee_{i \in I} a_i \wedge \bigwedge_{j \in J} \Diamond a_j)}
\end{aligned}$$

where $\emptyset \neq J \subseteq I$. Adding another axiom

$$(\Box - \vee) \quad \Box(a \vee b) \lesssim \Box a \vee \Diamond b$$

to the axioms above and altering $(P - \Box - \Diamond)$ by requiring $I = J$ yields the analogous construction for the convex powerdomain. Omitting $(\Box - \vee)$ weakens the facts that one can prove about normal forms in the calculus and thus generalizes from the convex to the mixed powerdomain.

Let D be a bifinite domain. We define a semantic function $\llbracket \cdot \rrbracket$ which assigns to each prelocalic expression in $|\bar{\mathcal{M}}(K(\Omega D))|$ a subset of the mixed powerdomain of D as follows:

$$\begin{aligned}
\llbracket \Box a \rrbracket &= \{x \in \mathcal{M}D \mid \exists (u, v) \in x. u \subseteq a\} \\
\llbracket \Diamond a \rrbracket &= \{x \in \mathcal{M}D \mid \exists (u, v) \in x. v \cap a \neq \emptyset\} \\
\llbracket \phi \vee \psi \rrbracket &= \llbracket \phi \rrbracket \cup \llbracket \psi \rrbracket \\
\llbracket \phi \wedge \psi \rrbracket &= \llbracket \phi \rrbracket \cap \llbracket \psi \rrbracket \\
\llbracket 0 \rrbracket &= \emptyset \\
\llbracket 1 \rrbracket &= \mathcal{M}D
\end{aligned}$$

Note the similarity between the definitions of \Box and \Diamond above and the meanings attached to them using equations 1 and 2 in the intuitive discussion in the second section.

Proposition 12 *Suppose D is a bifinite domain. For each $\phi \in |\bar{\mathcal{M}}(K(\Omega D))|$, we have $\llbracket \phi \rrbracket \in K(\Omega(\mathcal{M}D))$.*

Proof: The proof goes by induction on the structure of ϕ . Suppose $a \in K(\Omega D)$. We must show that $\llbracket \Box a \rrbracket$ and $\llbracket \Diamond a \rrbracket$ are compact open subsets of $\mathcal{M}D$. By Lemma 11, there is a finite subset $u = \{x_1, \dots, x_n\} \subseteq KD$ such that $a = \uparrow u$.

- **Claim:** $x \in \Box a$ iff $(u, \{x_i\}) \in x$ for some $i \leq n$. (\Rightarrow) If $x \in \Box a$, then there is a mix $(u', v') \in x$ such that $u' \subseteq a$. In other words, $u \lesssim^{\#} u'$. If $p \in v'$, then there is some $x_i \in u$ such that $x_i \lesssim p$ since $v' \gtrsim^{\#} u' \gtrsim^{\#} u$. Hence $(u, \{x_i\}) \lesssim (u', v')$ and therefore $(u, \{x_i\}) \in x$. The converse (\Leftarrow) follows immediately from the definitions. That $\llbracket \Box a \rrbracket$ is a compact open subset of $\mathcal{M}D$ now follows from the claim and Lemma 11.
- Let w be the (finite) set of minimal elements of D . Clearly $w \subseteq KD$. **Claim:** $x \in \Diamond a$ iff $(w, \{x_i\}) \in x$ for some $i \leq n$. (\Rightarrow) If $x \in \Diamond a$, then there is a mix $(u', v') \in x$ such that $v' \cap a \neq \emptyset$. Hence there is some $p \in v'$ and $x_i \in u$ such that $x_i \lesssim p \in a$. Thus $(w, \{x_i\}) \lesssim (u', v')$ and therefore $(w, \{x_i\}) \in x$. The converse (\Leftarrow) follows immediately from the definitions. That $\llbracket \Diamond a \rrbracket$ is a compact open subset of $\mathcal{M}D$ now follows from the claim and Lemma 11.

If $\phi \equiv \psi \vee \theta$ or $\phi \equiv \emptyset$, then the result is immediate since finite unions of compact sets are compact and the emptyset is compact. Since D is bifinite, $\mathcal{M}D$ is also bifinite, so it is coherent. This means

that intersections of pairs of compact sets are compact and $\mathcal{M}D$ is compact, so this covers the remaining cases for ϕ . ■

Lemma 13 *Let D be a domain. An element $U \in \Omega D$ is coprime in the algebraic lattice of open subsets if and only if $U = \uparrow x$ for some $x \in KD$. ■*

For the remainder of this section, let D be a fixed bifinite domain.

Theorem 14 *The axioms and rules on $|\bar{\mathcal{M}}(K(\Omega D))|$ are sound with respect to the interpretation $\llbracket \cdot \rrbracket$. That is*

1. *If $\phi \preceq \psi$, then $\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket$.*
2. *If $P(\phi)$, then $\llbracket \phi \rrbracket$ is a coprime in $\Omega \mathcal{M}D$.*

Proof: The proof of (1) is straight-forward. I will write out the most interesting case, the $(\Diamond - \wedge)$ axiom. Suppose $x \in \llbracket \Box a \wedge \Diamond b \rrbracket = \llbracket \Box a \rrbracket \cap \llbracket \Diamond b \rrbracket$, then there is a mix $(u, v) \in x$ such that $u \subseteq a$ and a mix $(u', v') \in x$ such that $v' \cap b \neq \emptyset$. Since x is directed, there is a mix $(u'', v'') \in x$ such that $(u, v), (u', v') \preceq (u'', v'')$. Now $u \sqsubseteq^\# u'' \sqsubseteq^\# v''$ and $u \subseteq a$ so $v'' \subseteq a$. Also $v' \sqsubseteq^b v''$ and $v' \cap b \neq \emptyset$ implies $v'' \cap b \neq \emptyset$. From this it follows that $v'' \cap (a \cap b) = v'' \cap b \neq \emptyset$ so $x \in \llbracket \Diamond(a \wedge b) \rrbracket$.

To prove (2), suppose that a_i is a coprime element of ΩD for each i in a finite non-empty indexing set I . Following axiom $(P - \Box - \Diamond)$, we must show that for any finite non-empty subset $J \subseteq I$, the set

$$W = \llbracket \Box \bigvee_{i \in I} a_i \rrbracket \cap \llbracket \bigwedge_{j \in J} \Diamond a_j \rrbracket$$

is a coprime. By Lemma 13, there are compact elements x_i such that $a_i = \uparrow x_i$ for each $i \in I$. Let $u = \{x_i \mid i \in I\}$ and $v = \{x_i \mid i \in J\}$. Since $J \subseteq I$, the pair (u, v) is a mix. We show that $\uparrow(u, v) = W$. Let $U = \bigcup_{i \in I} a_i$ and suppose $(u, v) \preceq (u', v')$. Since $u \preceq^\# u'$, we must have $u' \subseteq U$ and hence $(u', v') \in \llbracket \Box \bigvee_{i \in I} a_i \rrbracket$. Since $v \cap \llbracket \Diamond a_j \rrbracket \neq \emptyset$ for each a_j with $j \in J$, and $v' \succeq^b v$, we must have $v' \cap \llbracket \Diamond a_j \rrbracket \neq \emptyset$ for each $j \in J$, so $(u, v) \in \llbracket \bigwedge_{j \in J} \Diamond a_j \rrbracket$. Thus $(u', v') \in W$. Suppose on the other hand that $(u', v') \in W$. Then $(u', v') \in \llbracket \Box \bigvee_{i \in I} a_i \rrbracket$ so $u' \subseteq U$, so $u' \succeq^\# u$. Since $(u', v') \in \llbracket \bigwedge_{j \in J} \Diamond a_j \rrbracket$ as well, $v' \cap a_j \neq \emptyset$ for each $j \in J$. This means that for each x_j , there is some element $x'_j \in v'$ such that $x_j \preceq x'_j$. But this just means that $v' \succeq^b v$. Hence $(u', v') \in \uparrow(u, v)$ as desired. The fact that W is coprime now follows from Lemma 13. ■

Lemma 15 *Suppose $\phi, \psi \in |\bar{\mathcal{M}}(K(\Omega D))|$. If $P(\phi)$ and $P(\psi)$ and $\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket$, then $\phi \preceq \psi$.*

Proof: Suppose $J \subseteq I$ and $L \subseteq K$ are finite non-empty indexing sets and $\{a_i \mid i \in I\}$ and $\{b_k \mid k \in K\}$ are sets of coprimes such that

$$\begin{aligned} \phi &\approx \Box \bigvee_{i \in I} a_i \wedge \bigwedge_{j \in J} \Diamond a_j \\ \psi &\approx \Box \bigvee_{k \in K} b_k \wedge \bigwedge_{l \in L} \Diamond b_l \end{aligned}$$

Say $a_i = \uparrow x_i$ and $b_k = \uparrow y_k$ for each $i \in I$ and $k \in K$. Let $u = \{x_i \mid i \in I\}$ and $u' = \{x_j \mid j \in J\}$ and $v = \{y_k \mid k \in K\}$ and $v' = \{y_l \mid l \in L\}$. Then

$$\begin{aligned}
\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket &\Rightarrow \uparrow(u, u') \subseteq \uparrow(v, v') \\
&\Rightarrow (v, v') \preceq (u, u') \\
&\Rightarrow v \preceq^{\#} u \text{ and } v' \preceq^b u' \\
&\Rightarrow \forall i \in I \exists k \in K. y_i \preceq x_k \text{ and } \forall l \in L \exists j \in J. y_j \preceq x_l \\
&\Rightarrow \forall i \in I \exists k \in K. a_i \subseteq b_k \text{ and } \forall l \in L \exists j \in J. a_j \subseteq b_l \\
&\Rightarrow \square \bigvee_{i \in I} a_i \preceq \square \bigvee_{j \in J} b_j \text{ and } \bigwedge_{j \in J} \Diamond a_j \preceq \bigwedge_{l \in L} \Diamond b_l \\
&\Rightarrow \phi \preceq \psi \quad \blacksquare
\end{aligned}$$

The proof of this lemma is basically contained in the proof of Proposition 3.4.8 in [Abr88]:

Lemma 16 *For every $a \in |\bar{\mathcal{M}}(K(\Omega D))|$, there are coprimes $b_1, \dots, b_n \in P$ such that $a \approx \bigvee_{i=1}^n b_i$. \blacksquare*

Theorem 17 *For all $\phi, \psi \in |\bar{\mathcal{M}}(K(\Omega D))|$, $\phi \preceq \psi$ iff $\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket$.*

Proof: (\Rightarrow) is part of the Theorem 14. To prove (\Leftarrow) , we begin by using Lemma 16 to deduce the existence of finite sets of coprimes $\{\phi_i \mid i \in I\}$ and $\{\psi_j \mid j \in J\}$ such that

$$\phi = \bigvee_{i \in I} \phi_i \quad \text{and} \quad \psi = \bigvee_{j \in J} \psi_j.$$

We may now make the following deductions:

$$\begin{aligned}
\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket &\Rightarrow \bigcup_{i \in I} \llbracket \phi_i \rrbracket \subseteq \bigcup_{j \in J} \llbracket \psi_j \rrbracket \\
&\Rightarrow \forall i \in I \exists j \in J. \llbracket \phi_i \rrbracket \subseteq \llbracket \psi_j \rrbracket \quad \text{by Theorem 14} \\
&\Rightarrow \forall i \in I \exists j \in J. \phi_i \preceq \psi_j \quad \text{by Lemma 15} \\
&\Rightarrow \bigvee_{i \in I} \phi \subseteq \bigvee_{j \in J} \psi \quad \blacksquare
\end{aligned}$$

Theorem 18 *If D is a bifinite domain, then $\bar{\mathcal{M}}(K(\Omega D)) \cong K(\Omega(\mathcal{M}D))$.*

Proof: The map $\llbracket \cdot \rrbracket$ is order preserving and reflecting by Lemma 17. To see that it is also a surjection, and hence an isomorphism, suppose $U \in K\Omega(\mathcal{M}D)$. By Lemma 11, there are elements $x_1, \dots, x_n \in K\mathcal{M}D$ such that $U = \bigcup_{i=1}^n \uparrow x_i$. By the proof of Theorem 14 we know that, for each i , there is some $\phi_i \in |\bar{\mathcal{M}}(K\Omega(D))|$ such that $\llbracket \phi_i \rrbracket = \uparrow x_i$. Hence $\llbracket \bigvee_{i=1}^n \phi_i \rrbracket = \bigcup_{i=1}^n \llbracket \phi_i \rrbracket = U$. \blacksquare

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