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Field Theory of Directed Percolation with Long-Range Spreading

Hans-Karl Janssen
Heinrich-Heine-Universitat Dusseldorf

Olaf Stenull
University of Pennsylvania, stenull@sas.upenn.edu

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Abstract

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Field theory of directed percolation with long-range spreading

Hans-Karl Janssen
Institut für Theoretische Physik III, Heinrich-Heine-Universität, 40225 Düsseldorf, Germany

Olaf Stenull
Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104, USA
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It is well established that the phase transition between survival and extinction in spreading models with short-range interactions is generically associated with the directed percolation (DP) universality class. In many realistic spreading processes, however, interactions are long ranged and well described by Lévy flights—i.e., by a probability distribution that decays in d dimensions with distance r as $r^{-d-\sigma}$. We employ the powerful methods of renormalized field theory to study DP with such long-range Lévy-flight spreading in some depth. Our results unambiguously corroborate earlier findings that there are four renormalization group fixed points corresponding to, respectively, short-range Gaussian, Lévy Gaussian, short-range, and Lévy DP and that there are four lines in the (σ,d) plane which separate the stability regions of these fixed points. When the stability line between short-range DP and Lévy DP is crossed, all critical exponents change continuously. We calculate the exponents describing Lévy DP to second order in an ε expansion, and we compare our analytical results to the results of existing numerical simulations. Furthermore, we calculate the leading logarithmic corrections for several dynamical observables.

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I. INTRODUCTION

The formation and the properties of random structures have been an exciting topic in statistical physics for many years. In the case that the formation of such structures obeys local rules, these processes can often be expressed in the language of epidemic spreading. It is well known that two special spreading processes referred to in this language, respectively, as simple epidemic with recovery or Gribov process [1,2] and epidemic with removal (general epidemic process) lead to random structures with the properties of percolation clusters: directed percolation [3–5] in the former case and isotropic percolation in the latter.

The Gribov process, also known in elementary particle physics as Reggeon field theory (RFT) [6–8], is a stochastic multiparticle process that describes the essential features of a vast number of growth phenomena of populations without exploitation of the environment near their extinction threshold. The transition between survival and extinction of the population (infected individuals) is a nonequilibrium continuous phase transition phenomenon and is characterized by universal scaling laws. The Gribov process belongs to the universality class of local growth processes with absorbing states [9–11] such as the contact process [12–14] and certain cellular automata [15,16], and it is relevant to a vast range of models in physics, chemistry, biology, and sociology [17]. As usual, we refer to this universality class as the directed percolation (DP) universality class. For recent reviews see [18,19].

A continuum description of DP in terms of a density $n(\mathbf{r},t)$ of infected individuals typically arises from a coarse-graining procedure in which a large number of microscopic degrees of freedom are averaged out. Their influence is simply modeled as a Gaussian noise term in a Langevin equation. The process has to respect the absorbing-state condition: $n(\mathbf{r},t)\equiv 0$ is always a stationary state. Then the minimal stochastic reaction-diffusion equation for the density $n(\mathbf{r},t)$ is constructed as [9]

$$\lambda^{-1}\partial_t n(\mathbf{r},t) = \nabla^2 n(\mathbf{r},t) - \left[\tau + \frac{g}{2}n(\mathbf{r},t)\right]n(\mathbf{r},t) + \zeta(\mathbf{r},t). \tag{1.1}$$

The Gaussian noise $\zeta(\mathbf{r},t)$ must also respect the absorbing-state condition, whence

$$\overline{\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')} = \lambda^{-1}g' n(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{1.2}$$

up to subleading contributions. The history of the process in space and time defines directed percolation clusters in a $(d+1)$-dimensional space. The minimal process defined by Eqs. (1.1) and (1.2) contains all the relevant terms needed for a proper field-theoretic description of DP.

In realistic situations the infection can be also long ranged. One may think, e.g., of an orchard where flying parasites contaminate the trees practically instantaneously in a widespread manner if the time scale of the flights of the parasites is much shorter than the mesoscopic time scale of the epidemic process itself. Thus, following a suggestion of Mollison [8], Grassberger [20] introduced a variation of the epidemic processes with an infection probability distribution $P(\mathbf{r})$ which decays with the distance r as a power law, $P(\mathbf{r})\sim r^{-d-\sigma}$. We will somewhat casually refer to such long-range infection as Lévy flights, although a true Lévy flight is defined via its Fourier transform as $\tilde{P}(\mathbf{q})\sim\exp(-bq^{\sigma})$ with $0<\sigma\leq 2$ (to ensure positiveness of the distribution).

In Fourier space and in a long-wavelength expansion, the Langevin equation (1.1) can be generalized to account for Lévy flights by a term proportional to $q^{\sigma}n(q,t)$. In the case of $2-\sigma\equiv 2\alpha>0$, the long-wavelength behavior is naively dominated by this new term. Grassberger calculated critical exponents in a one-loop calculation which were discontinuous in the limit $\alpha \to +0$, and therefore the applicability of the results was doubtful. In a former paper [21], we have shown by applying the Wilson momentum shell renormalization group that only two of the critical exponents are independent in long-range DP and that the critical exponents change continuously when the transition line between long-ranged and short-ranged spreading (with an $\alpha = \alpha_c < 0$) is crossed. We have also shown that

$$\sigma_c = 2(1 - \alpha_c) = d + z - \frac{2\beta}{\nu} = z - \eta \tag{1.3}$$

exactly, where β, ν, η, and z are the usual exponents of short-ranged DP in d (transversal) dimensions. These results have been confirmed numerically by Hinrichsen and Howard [22]. Note that $\eta_\lambda = z - 2 - \eta$, which corresponds to the Fisher exponent of equilibrium critical phenomena, is negative here [23]. For a recent review of DP with long-range interactions see [24]. In this paper we reconsider the problem using methods of renormalized field theory in conjunction with an expansion in ε and α.

The remainder of this paper is organized as follows: Section II reviews the field-theoretic formulation of DP with Lévy-flight spreading to set the stage, to provide background information, and to establish notation. Section II first reviews the short-range limit of this model and then discusses our field-theoretic analysis of the long-range limit. Section IV represents the main part of this paper. It treats in detail the hybrid model for $\alpha = O(\varepsilon)$. Section V builds up on the results of Sec. IV and presents results for the critical exponents and logarithmic corrections of various dynamical observables. Section VI contains some concluding remarks.

II. MODELING DP WITH LÉVY-FLIGHT SPREADING

To generalize the diffusional infection rate in the Langevin equation (1.1), we model spreading by writing

$$\partial_t n(\mathbf{r},t)|_{\text{inf}} = \int d^d r' P(\mathbf{r} - \mathbf{r}') n(\mathbf{r}',t). \tag{2.1}$$

As a particular model for the positive distribution $P(\mathbf{r})$ which contains all relevant properties, we use $P(\mathbf{r}) = P_{LR}(\mathbf{r}) + P_{SR}(\mathbf{r})$ with a short-range contribution $P_{SR}(\mathbf{r}) \propto \exp(-r^2/a^2)$ and a Lévy-flight part $P_{LR}(\mathbf{r}) \propto (r^2 + a^2)^{-(d+\sigma)/2}$. a denotes a microscopic length scale which, for simplicity, is assumed to be equal in both otherwise independent distributions. Fourier transformation leads to $\tilde{P}_{SR}(\mathbf{q}) \propto \exp[-(aq)^2/4]$ and $\tilde{P}_{LR}(\mathbf{q}) \propto K_{\sigma/2}(aq)$, where $K_{\sigma/2}$ is the modified Bessel function with index $\sigma/2$. Long-wavelength expansion leads to

$$\tilde{P}(\mathbf{0}) - \tilde{P}(\mathbf{q}) = A(aq)^2 + \frac{B}{2-\sigma}[(aq)^\sigma - (aq)^2] + O(q^4, q^{2+\sigma}), \tag{2.2}$$

with positive, nonsingular, σ-dependent constants A and B. $\tilde{P}(\mathbf{q})$ shows the two typical terms: diffusion $\sim (aq)^2$ and Lévy flights $\sim (aq)^\sigma$. It is IR stable; that is, $\tilde{P}(\mathbf{0}) - \tilde{P}(\mathbf{q})$ is positive in both regions $\sigma < 2$ and $\sigma > 2$ if $q \to 0$. Note the characteristic pole at $\sigma = 2$ that leads to a logarithmic contribution $\sim -(aq)^2 \ln(aq)$ to $\tilde{P}(\mathbf{q})$ for $\sigma \to 2$. In the following we use for $\tilde{P}(\mathbf{0}) - \tilde{P}(\mathbf{q})$ its long-wavelength approximation (2.2). Whereas $\tilde{P}(\mathbf{0}) - \tilde{P}(\mathbf{q})$ is always a positive real quantity, its long-wavelength approximation changes sign at a magnitude q_g of the momentum of order $aq_g = O(1)$ if $\sigma > 2$ or if $\sigma < 2$ and $A < B/(2-\sigma)$, which leads to a pole in the Green's function of the equation of motion. Of course, this pole is a nonphysical ghost; it arises when the long-wavelength approximation is used in a momentum regime where it is inapplicable. This happens in particular in dimensional regularization where integrations over internal momenta are extended to infinity. Hence, this method may become inconsistent if both types of q dependences, $(aq)^2$ and $(aq)^\sigma$, are used in common. We come back to this question in Sec. IV C. In contrast, Wilson's momentum shell renormalization procedure avoids this dangerous UV region $q = O(1/a)$ because all momenta are restricted to the sphere $q \leq O(\mu)$ with $\mu \ll 1/a$. This was the reason for using Wilson's renormalization group in our former publication [21]. Note also that dimensional regularization leads in the present problem like in other similar problems to the so-called triviality problem at the upper critical dimension; i.e., the dimensionally regularized theory misses logarithmic corrections and thus has to be viewed as an effective theory for low momenta.

The stochastic equation of motion of the DP process with the Lévy-flight spreading and short-range diffusion can be written as

$$\lambda^{-1}\partial_t n(\mathbf{r},t) = [\nabla^2 - c(-\nabla^2)^{1-\alpha}] n(\mathbf{r},t) - \left[\tau + \frac{1}{2} g n(\mathbf{r},t)\right] n(\mathbf{r},t) + \zeta(\mathbf{r},t), \tag{2.3}$$

where we set $\sigma = 2(1-\alpha)$. Here the Lévy term on the right-hand side is defined in Fourier space as $(-\nabla^2)^{1-\alpha} n(\mathbf{r},t) = \int_q q^{2(1-\alpha)} n(\mathbf{q},t) \exp(i\mathbf{q}\cdot\mathbf{r})$. In order to develop a renormalized field theory, it is useful to recast the Langevin equation (2.3) as a dynamic response functional [25,26]

$$\mathcal{J}[\tilde{s},s] = \int d^d r dt \lambda \tilde{s} \left\{ \lambda^{-1}\partial_t + [\tau - \nabla^2 + c(-\nabla^2)^{1-\alpha}] + \frac{g}{2}(s - \tilde{s}) \right\} s, \tag{2.4}$$

where $s(\mathbf{r},t) \sim n(\mathbf{r},t)$ is the rescaled density which ensures that $g' = g$ and for which the time inversion symmetry $s(\mathbf{r},t) \leftrightarrow -\tilde{s}(\mathbf{r},-t)$ (rapidity reversal in RFT) holds. $\tilde{s}(\mathbf{r},t)$ is a response field that describes the response when a local particle source $h(\mathbf{r},t) \geq 0$ is added to the Langevin equation (2.3). At the level of the dynamic response functional, this source leads to an additional term $\int d^d r dt h(\mathbf{r},t)\tilde{s}(\mathbf{r},t)$ in Eq. (2.4). Having the dynamic response functional, correlation and response functions can be computed as functional averages (path integrals) of monomials of s and $\tilde{s}$ with weight $\exp\{-\mathcal{J}\}$. Throughout this paper, functional integrals are interpreted in the sense of the so-called prepoint discretization that sets the step function $\theta(t)$ equal to zero for $t=0$ [27]. We stress that the usual short-ranged DP model is recovered from the general expression of $\mathcal{J}$ simply by setting $c=0$, or $c=1$ with $\alpha=0$.

As a first step towards the renormalization group (RG) analysis of this model, we discuss its canonical scaling behavior. Introducing the usual inverse length scale μ, we readily find $\tilde{s}\sim s\sim\mu^{d/2}$. For $\alpha>0$, the long-range Lévy term $\sim(-\nabla^2)^{1-\alpha}$ naively dominates the usual diffusion term $\sim\nabla^2$. Hence, we may neglect the latter for $\alpha>0$, and we redefine (by rescaling of some parameters) $c=1$. This produces an inverse time scale $\lambda\mu^{\sigma}$, and $\tau\sim\mu^{\sigma}$ for the scaling of the control parameter. Moreover, we obtain $g^2\sim\mu^{\bar{\varepsilon}}$, where $\bar{\varepsilon}=2\sigma-d=\varepsilon-4\alpha$ (we will reserve the symbol ε for the short-range case—i.e., $\varepsilon=4-d$). The naive dimension of the coupling constant g allows us to identify the upper critical dimension $d_c(\alpha)=4(1-\alpha)=2\sigma$. This boundary separates trivial (mean-field or Gaussian) from nontrivial long-range behavior if $\alpha>0$. Of course, the boundary $\alpha=0$, $d>4$ separates the regions with trivial long-range and trivial short-range DP, and the boundary $\alpha<0$, $d=4$ separates trivial and nontrivial short-range DP.

III. SHORT-RANGE AND LONG-RANGE MODELS

We now turn to perturbation theory. In this section, we will first briefly review the short-range model obtained for $c=0$, which has been discussed previously in many places (see [19] and the references cited therein). Then we will treat, also briefly, the long-range model obtained for $c\to\infty$. As usual in dynamical field theory, we focus on those correlation and response functions

$$G_{N\tilde{N}}=\langle[s]^N[\tilde{s}]^{\tilde{N}}\rangle \tag{3.1}$$

that require renormalization due to the presence of UV divergences in Feynman diagrams as well as the corresponding one-particle irreducible (1PI) vertex functions with $\tilde{N}$ (N) external $\tilde{s}$ (s) legs, $\Gamma_{\tilde{N}N}$. For background on the methods of renormalization theory, we refer to [28].

A. Short-range model: $c=0$

We first review ordinary DP which is modeled by $\mathcal{J}$ as given in Eq. (2.4) with $c=0$ [9,10]. The upper critical dimension is $d_c(0)=4$. Straightforward dimensional analysis shows that there are three superficially divergent vertex functions: $\Gamma_{1,1}$, $\Gamma_{1,2}=-\Gamma_{2,1}$, where the last relation follows from time inversion symmetry. In the following, we use the symbol $\circ$ over a character to denote bare (unrenormalized) couplings and we use the following renormalization scheme to cure the model of its UV divergences:

$$\mathring{s}=Z^{1/2}s,\quad \mathring{\tilde{s}}=Z^{1/2}\tilde{s},\quad \mathring{\lambda}=Z^{-1}Z_\lambda\lambda,$$

$$\mathring{\tau}=Z_\lambda^{-1}Z_\tau\tau+\mathring{\tau}_c,\quad \mathring{g}^2=G_\varepsilon^{-1}Z^{-1}Z_\lambda^{-2}Z_u u\mu^\varepsilon, \tag{3.2}$$

where $G_\varepsilon=\Gamma(1+\varepsilon/2)/(4\pi)^{d/2}$ is a convenient amplitude, u represents the dimensionless coupling constant, and the control parameter τ is zero at the critical point. In dimensional regularization, the critical bare value of the control parameter $\mathring{\tau}_c$ is of the form

$$\mathring{\tau}_c=\mathring{g}^{4/\varepsilon}S(\varepsilon), \tag{3.3}$$

where the Symanzik function $S(\varepsilon)$ has simple IR poles at each $\varepsilon=2/k$ with $k=1,2,\ldots$. Hence, $\mathring{\tau}_c$ is not a perturbational quantity and is formally zero in the ε expansion. Note, however, that minimal renormalization—i.e., dimensional regularization in conjunction with minimal subtraction—does not imply the ε expansion [29]. The renormalization factors $Z_{\ldots}$ are functions of u and have in minimal renormalization the expansions

$$Z_{\ldots}=1+\sum_{n=1}^{\infty}\frac{Y_{\ldots}^{(n)}(u)}{\varepsilon^n},\qquad Y_{\ldots}^{(n)}(u)=\sum_{l=n}^{\infty}\frac{Y_{\ldots,l}^{(n)}}{l}u^l, \tag{3.4}$$

where the $Z_{\ldots}$ are determined in such a way that the perturbation expansions of renormalized quantities are free of singularities if ε goes to zero. They are given to second order by [9,10,19]

$$Z=1+\frac{u}{4\varepsilon}+\left(\frac{7}{\varepsilon}-3+\frac{9}{2}\ln\frac{4}{3}\right)\frac{u^2}{32\varepsilon}+O(u^3),$$

$$Z_\lambda=1+\frac{u}{8\varepsilon}+\left(\frac{13}{\varepsilon}-\frac{31}{4}+\frac{35}{2}\ln\frac{4}{3}\right)\frac{u^2}{128\varepsilon}+O(u^3),$$

$$Z_\tau=1+\frac{u}{2\varepsilon}+\left(\frac{1}{\varepsilon}-\frac{5}{16}\right)\frac{u^2}{2\varepsilon}+O(u^3),$$

$$Z_u=1+\frac{2u}{\varepsilon}+\left(\frac{7}{\varepsilon}-\frac{7}{4}\right)\frac{u^2}{2\varepsilon}+O(u^3). \tag{3.5}$$

A renormalization group equation (RGE) for the model can be derived in a routine fashion by exploiting the fact that the unrenormalized response and correlation functions have to be independent of the inverse length scale μ introduced by renormalization. This reasoning leads straightforwardly to the RGE

$$\left[\mathcal{D}+\frac{N+\tilde{N}}{2}\gamma\right]G_{N,\tilde{N}}=0, \tag{3.6}$$

with an RGE differential operator $\mathcal{D}=\mu\partial/\partial\mu|_{\text{bare}}$ (the $|_{\text{bare}}$ indicates that bare quantities are kept fixed while taking the derivates) given by

$$\mathcal{D}=\mu\frac{\partial}{\partial\mu}+\zeta\lambda\frac{\partial}{\partial\lambda}+\kappa\tau\frac{\partial}{\partial\tau}+\beta_u\frac{\partial}{\partial u}. \tag{3.7}$$

The RG functions result from the finite logarithmic derivatives of the renormalization factors,

$$\gamma_{\ldots}=\left.\mu\frac{\partial\ln Z_{\ldots}}{\partial\mu}\right|_{\text{bare}}=-u\frac{\partial}{\partial u}Y_{\ldots}^{(1)}(u)=-\sum_{l=1}^{n}Y_{\ldots,l}^{(n)}u^l, \tag{3.8}$$

as

$$\beta_u = \mu \frac{\partial u}{\partial \mu}\bigg|_{\text{bare}} = (-\varepsilon + 2\gamma_\lambda + \gamma - \gamma_u)u,$$

$$\zeta = \mu \frac{\partial \ln \lambda}{\partial \mu}\bigg|_{\text{bare}} = \gamma - \gamma_\lambda,$$

$$\kappa = \mu \frac{\partial \ln \tau}{\partial \mu}\bigg|_{\text{bare}} = \gamma_\lambda - \gamma_\tau. \tag{3.9}$$

Their perturbation expansions are

$$\gamma(u) = -\frac{u}{4} + \left(2 - 3\ln\frac{4}{3}\right)\frac{3u^2}{32} + O(u^3),$$

$$\zeta(u) = -\frac{u}{8} + \left(17 - 2\ln\frac{4}{3}\right)\frac{u^2}{256} + O(u^3),$$

$$\kappa(u) = \frac{3u}{8} - \left(7 + 10\ln\frac{4}{3}\right)\frac{7u^2}{256} + O(u^3),$$

$$\beta_u(u) = \left[-\varepsilon + \frac{3u}{2} - \left(169 + 106\ln\frac{4}{3}\right)\frac{u^2}{128} + O(u^3)\right]u. \tag{3.10}$$

The asymptotic solution of the RGE, Eq. (3.6), leads to the stable fixed point $u=u_*$ with u_* given by

$$u_* = u_*^{DP}(\varepsilon) = \frac{2\varepsilon}{3}\left[1 + \left(169 + 106\ln\frac{4}{3}\right)\frac{\varepsilon}{288} + O(\varepsilon^2)\right], \tag{3.11}$$

as the stable solution of $\beta_u(u_*)=0$, and to the scaling form

$$G_{N,\tilde{N}}(\{\mathbf{r},t\},\tau) = l^{(N+\tilde{N})(d+\eta_{SR})/2} G_{N,\tilde{N}}(\{l\mathbf{r}, l^{z_{SR}}t\}, l^{-1/\nu_{SR}}\tau) \tag{3.12}$$

of the response and correlation functions, with the three independent critical exponents

$$\eta_{SR} = \gamma(u_*), \quad 1/\nu_{SR} = 2 - \kappa(u_*), \quad z_{SR} = 2 + \zeta(u_*). \tag{3.13}$$

These are the very well-known critical exponent for short-range DP. To second order in ε expansion, they are given by

$$\eta_{SR} = -\frac{\varepsilon}{6}\left[1 + \left(\frac{25}{288} + \frac{161}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$

$$z_{SR} = 2 - \frac{\varepsilon}{12}\left[1 + \left(\frac{67}{288} + \frac{59}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right],$$

$$\nu_{SR} = \frac{1}{2} + \frac{\varepsilon}{16}\left[1 + \left(\frac{107}{288} - \frac{17}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2)\right]. \tag{3.14}$$

B. Long-range model: $c \to \infty$

As we have shown in [21] by using Wilson's momentum shell renormalization group and as we discussed in the Introduction, the discontinuity of short-range and long-range critical exponents at $\alpha=0$ is spurious and can be remedied. In renormalized field theory, the key is to recognize [30] that there is a region of small $\alpha=O(\varepsilon)$, where a careful analysis of the RG flow reveals a smooth connection between the $\alpha<0$, $\alpha=O(\varepsilon)$, and $\alpha>0$ regions. Here, we analyze the last case, which belongs to the true long-range region. The case $\alpha=O(\varepsilon)$, being a "hybrid" between short-range and long-range models, will be deferred to the next subsection.

We recall from our discussion at the end of Sec. II that the upper critical dimension for $\alpha>0$ is $d_c(\alpha)=4(1-\alpha)=2\sigma$, and we define $\bar{\varepsilon}=2\sigma-d$, to be distinguished from $\varepsilon=4-d$. Considering the response functional $\mathcal{J}$, Eq. (2.4) with $c\neq 0$, we see that the operator $\tilde{s}\nabla^2 s$ is superficially irrelevant compared to $\tilde{s}(-\nabla^2)^{1-\alpha}s$ and may be dropped formally in the limit $c\to\infty$. This limit is feasible after the rescaling $\lambda\to\lambda/c$, $\tau\to\tau c$, and $g\to gc$. The canonical dimensions of the fields do not change compared to the short-range model. However, the inverse time scale changes to $\lambda\mu^\sigma$ and the canonical dimensions of the remaining parameters are $\tau\sim\mu^\sigma$ and $g\sim\mu^{\bar{\varepsilon}/2}$. As above, $\Gamma_{1,1}$ and $\Gamma_{1,2}=-\Gamma_{2,1}$ are superficially divergent for $\bar{\varepsilon}\to 0$. Moreover, all divergent contributions to any vertex function are polynomial in the momenta, so that the operator $\tilde{s}(-\nabla^2)^{1-\alpha}s$ needs no counterterm. Hence, we use the renormalization scheme

$$\mathring{s} = \bar{Z}^{1/2}s, \quad \mathring{\tilde{s}} = \bar{Z}^{1/2}\tilde{s}, \quad \mathring{\lambda} = \bar{Z}^{-1}\lambda,$$

$$\mathring{\tau} = \bar{Z}_\tau\tau + \mathring{\tau}_c, \quad \mathring{g}^2 = A_{\bar{\varepsilon}}^{-1}\bar{Z}^{-1}\bar{Z}_u u\mu^{\bar{\varepsilon}}, \tag{3.15}$$

which produces the renormalized response functional

$$\mathcal{J}_{\text{LR}} = \int d^d r dt \lambda\tilde{s}\left\{\lambda^{-1}\bar{Z}\partial_t + [\bar{Z}_\tau\tau + (-\nabla^2)^{1-\alpha}] + \bar{Z}_u^{1/2}\frac{g}{2}(s-\tilde{s})\right\}s. \tag{3.16}$$

Here and in the following we use an overbar to distinguish the renormalization factors of the long-range and hybrid models from those of the short-range model. $A_{\bar{\varepsilon}}$ is a suitable amplitude whose precise definition will be given later. Here, in minimal renormalization, $\mathring{\tau}_c=\mathring{g}^{2\sigma/\bar{\varepsilon}}\bar{S}(\bar{\varepsilon})$ with an appropriate Symanzik function $\bar{S}(\bar{\varepsilon})$ having simple poles at $\bar{\varepsilon}=\sigma/k$ with $k=1,2,\ldots$. Note by comparing Eq. (3.2) with Eq. (3.15) that the renormalization schemes for the short-range and long-range models are of the same form except for the renormalization factor of the kinetic coefficient, which now is $\bar{Z}_\lambda=1$. Here, we have the expansions

$$\bar{Z}_{\ldots} = 1 + \sum_{n=1}^{\infty}\frac{\bar{Y}_{\ldots}^{(n)}(u)}{\bar{\varepsilon}^n}, \quad \bar{Y}_{\ldots}^{(n)}(u) = \sum_{n=1}^{\infty}\frac{\bar{Y}_{\ldots,l}^{(n)}}{l}u^l. \tag{3.17}$$

Of course, the functions $\bar{Y}_{\ldots}^{(n)}(u)$ are different from the functions $Y_{\ldots}^{(n)}(u)$ and have to be determined by perturbation theory. Nevertheless, the RG functions for the long-range case can be transcribed from the short-range case, Eqs. (3.6)–(3.8), simply by decorating each RG function with an overbar and setting $\bar{\gamma}_\lambda=0$—i.e.,

$$\bar{\beta}= \mu\frac{\partial u}{\partial\mu}\bigg|_{\text{bare}} =(-\bar{\varepsilon}+\bar{\gamma}-\bar{\gamma}_u)u,$$

$$\bar{\zeta}= \mu\frac{\partial \ln\lambda}{\partial\mu}\bigg|_{\text{bare}} =\bar{\gamma}, \quad \bar{\kappa}= \mu\frac{\partial \ln\tau}{\partial\mu}\bigg|_{\text{bare}} =-\bar{\gamma}_\tau. \tag{3.18}$$

The scaling form of the response and correlation functions follows from the RGE as

$$G_{N,\tilde{N}}(\{\mathbf{r},t\},\tau)=l^{(N+\tilde{N})(d+\eta_{lR})/2}G_{N,\tilde{N}}(\{l\mathbf{r},l^{z_{LR}}t\},l^{-1/\nu_{LR}}\tau), \tag{3.19}$$

with the η_{LR} and ν_{LR} given by

$$\eta_{LR}=\bar{\gamma}(u_*),\quad 1/\nu_{LR}=\sigma-\bar{\kappa}(u_*). \tag{3.20}$$

The third exponent z_{LR} is related to these exponents by the exact relation

$$z_{LR}=\sigma+\eta_{LR}. \tag{3.21}$$

Hence, we have only two independent critical exponents in the long-range case—viz. η_{LR} and ν_{LR}.

After this general discussion, we now turn to our actual perturbation calculation. The propagator of the theory reads $G(\mathbf{q},t)=\theta(t)\exp[-\lambda(\tau+|\mathbf{q}|^\sigma)t]$ in wave-vector–time representation. Note that the prepoint discretization mandates that we have to use $\theta(0)=0$ throughout [27]. Note also that the propagator is free of the ghost problem; i.e., the exponent is always negative. The one-loop self-energy with frequency ω and wave vector $\mathbf{q}$ as the first contribution to the vertex function $\Gamma_{1,1}$ reads

$$\Sigma(\mathbf{q},\omega)=\frac{\lambda g^2}{2}\int_{\mathbf{p}}\frac{1}{i\omega/\lambda+2\tau+|\mathbf{p}+\mathbf{q}/2|^\sigma+|\mathbf{p}-\mathbf{q}/2|^\sigma}. \tag{3.22}$$

To calculate this self-energy, it is useful to expand in $\mathbf{q}$ and ω and to use the identity

$$2\pi^{-d/2}\Gamma(1+d/2)\int d^dpf(|\mathbf{p}|^\sigma)$$
$$=2\pi^{-d/\sigma}\Gamma(1+d/\sigma)\int d^{2d/\sigma}pf(|\mathbf{p}|^2), \tag{3.23}$$

which leads the calculation of the primitive divergent one-loop diagrams back to integrals of the usual known short-range type. We obtain

$$\Sigma(\mathbf{q},\omega)=-\frac{g^2}{4\bar{\varepsilon}}\tau^{-\bar{\varepsilon}/\sigma}A_{\bar{\varepsilon}}\left\{i\omega+\frac{2\sigma}{(\sigma-\bar{\varepsilon})}\lambda\tau\right\}+\text{finite}, \tag{3.24}$$

where we have displayed only the pole terms in $\bar{\varepsilon}$. The pole at $\bar{\varepsilon}=\sigma$ is an IR pole and can be removed by introducing a new mass parameter $m\sim\mu$ instead of τ via introducing $m^{-\sigma}=\partial\ln\Gamma_{1,1}(\omega=0,q,\tau)/\,\partial q^\sigma|_{q=0}$ (keeping in mind that it is an IR poles, one may also simply ignore it in the $\bar{\varepsilon}$ expansion). The amplitude $A_{\bar{\varepsilon}}$ is defined by

$$A_{\bar{\varepsilon}}=\frac{\Gamma(2-\bar{\varepsilon}/\sigma)\Gamma(1+\bar{\varepsilon}/\sigma)}{\Gamma(\sigma-\bar{\varepsilon}/2)(4\pi)^{d/2}}. \tag{3.25}$$

Note that $A_{\bar{\varepsilon}}$ becomes G_ε if $\sigma\to 2$. Expanding $(\mu^\sigma/\tau)^{\bar{\varepsilon}/\sigma}/(1-\bar{\varepsilon}/\sigma)$ in $\bar{\varepsilon}$ and using the renormalization scheme, Eq. (3.15), we arrive at the singular part of the vertex function Γ_{11} in one-loop approximation:

$$\Gamma_{1,1}(\mathbf{q},\omega)=(\bar{Z}i\omega+\bar{Z}_\tau\lambda\tau)-\frac{u}{\bar{\varepsilon}}\left(\frac{i\omega}{4}+\frac{\lambda\tau}{2}\right)+\cdots. \tag{3.26}$$

As announced above, there is no singular term proportional to $|\mathbf{q}|^\sigma$. Using the same techniques, we get

$$\Gamma_{1,2}=-\Gamma_{2,1}=\lambda g\left(\bar{Z}_u^{1/2}-\frac{u}{\bar{\varepsilon}}\right) \tag{3.27}$$

for the singular part of the other superficially divergent vertex functions. We read off the one-loop renormalizations of the long-range model:

$$\bar{Z}=1+\frac{u}{4\bar{\varepsilon}}+O(u^2),\quad \bar{Z}_\tau=1+\frac{u}{2\bar{\varepsilon}}+O(u^2),$$

$$\bar{Z}_u=1+\frac{2u}{\bar{\varepsilon}}+O(u^2). \tag{3.28}$$

The RG functions here have the expansions

$$\bar{\zeta}=\bar{\gamma}=-\frac{1}{4}u+O(u^2),\quad \bar{\kappa}=\frac{1}{2}u+O(u^2),$$

$$\bar{\beta}=\left(-\bar{\varepsilon}+\frac{7}{4}u+O(u^2)\right)u, \tag{3.29}$$

instead of those given in Eq. (3.18) for the short-range model. The stable fixed point is $u_*=u_*^{LR}=4\bar{\varepsilon}/7+O(\bar{\varepsilon}^2)$, and the expansions to first order of the long-range critical exponents are

$$\eta_{LR}=-\frac{\bar{\varepsilon}}{7}+O(\bar{\varepsilon}^2),\quad 1/\nu_{LR}=\sigma-\frac{2\bar{\varepsilon}}{7}+O(\bar{\varepsilon}^2), \tag{3.30}$$

which should be compared to the short-range (SR) exponents given in Eq. (3.20). As has to be the case, our one-loop results (3.30) are in perfect agreement with the one-loop results for the long-range (LR) exponents derived in [21] by Wilson's method.

IV. HYBRID MODEL: $\alpha=O(\varepsilon)$

Here, we turn to the analysis of the key region in (d,α) space—namely, $\alpha=O(\varepsilon)$. The naive $\alpha\to 0$ limit of the long-range model presupposes $\varepsilon\ll\alpha$ and hence fails to resolve the crossover between the SR and LR models which occurs for

$\alpha=O(\varepsilon)$. For both α and ε small, we follow the work of Honkonen and Nalimov [30].

A. Renormalization

Our starting point is the renormalized response functional

$$\mathcal{J}[\tilde{s},s]=\int dt d^d r \bar{\lambda}\tilde{s}\Big\{\bar{Z}\bar{\lambda}^{-1}\partial_t+\bar{Z}_u^{1/2}\frac{\bar{g}}{2}(s-\tilde{s}) + [\bar{Z}_\tau\bar{\tau}-\bar{Z}_\lambda\nabla^2+c(-\nabla^2)^{1-\alpha}]\Big\}s, \qquad (4.1)$$

where we use the renormalization scheme

$$\mathring{s}=\bar{Z}^{1/2}s,\quad \mathring{\tilde{s}}=\bar{Z}^{1/2}\tilde{s},\quad \mathring{\lambda}=\bar{Z}^{-1}\bar{Z}_\lambda\bar{\lambda},$$

$$\mathring{\tau}=\bar{Z}_\lambda^{-1}\bar{Z}_\tau\bar{\tau}+\mathring{\tau}_c,\quad \mathring{g}^2=G_\varepsilon^{-1}\bar{Z}_\lambda^{-1}\bar{Z}^{-1}\bar{Z}_u\bar{u}\mu^\varepsilon,$$

$$\mathring{c}=\bar{Z}_\lambda^{-1}w\mu^{2\alpha}, \qquad (4.2)$$

and the abbreviations $\bar{g}=\sqrt{\bar{u}}\mu^{\varepsilon/2}$ and $c=w\mu^{2\alpha}$. As before, the term $\tilde{s}(-\nabla^2)^{1-\alpha}s$ does not need a counterterm as long as $\alpha \neq 0$.

Using the approach by Honkonen and Nalimov, we construct our renormalization factors $\bar{Z}_{\ldots}$ by generalizing the minimal renormalzation program that led to the expansions (3.4) and (3.17). Here, the renormalization factors are now functions of $\bar{u}$ and w, and they contain poles of all linear combinations $\delta_{l,k}=l\varepsilon+2k\alpha$ with $l=1,2,\ldots$ and $k=0,1,2,\ldots$:

$$\bar{Z}_{\ldots}=1+\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\frac{\bar{Y}^{(1)}_{\ldots;l,k}}{l\varepsilon+2k\alpha}w^k\bar{u}^l+O(\delta^{-2}), \qquad (4.3)$$

where the coefficients $\bar{Y}^{(1)}_{\ldots;l,k}$ can be chosen such that they are independent of ε and α if both are of the same order. To explain this, let us consider the primitive divergence of an irreducible diagram consisting of V vertices, P propagators, L independent loops, and E external amputated legs. By definition, a primitive divergence is a divergence that arises if all the Ld inner momentum integrations tend uniformly to infinity. In primitive diagrams, they are the only UV divergences. In nonprimitive diagrams, they are the divergences that remain after all divergences of the renormalization parts of the diagrams are tamed by lower-order counterterms. After time integrations over the $(V-1)$ time segments each between vertices, any diagram has the qualitative form

$$I=\int\frac{(d^dp)^L}{(m^2+p^2+cp^\sigma)^{V-1}}\bar{g}^V \sim\sum_{k=0}^{\infty}\binom{1-V}{k}\int\frac{(cp^{\sigma-2})^k(d^dp)^L}{(m^2+p^2)^{V-1}}\bar{g}^V \sim\sum_{k=0}^{\infty}\binom{1-V}{k}\int\frac{(d^dp)^L}{(m^2+p^2)^{V-1+k\alpha}}c^k\bar{g}^V \sim\sum_{k=0}^{\infty}C_{L,V,k}\Lambda^{\Delta(L,V,k)}c^k\bar{g}^V. \qquad (4.4)$$

Here, the mass m^2 is a linear combination of the control parameter τ and frequencies ω and serves as an IR regulator. Λ is a momentum cutoff, and $\Delta(L,V,k)=dL-2(V-1)-2k\alpha$ is the degree of primitive divergence of the diagram. Using the topological relations $P=L+(V-1)$ and $3V=2P+E$, hence $V=(E-2)+2L$, we obtain

$$\Delta(L,V,k)=2(3-E)-(L\varepsilon+2k\alpha). \qquad (4.5)$$

The first part $2(3-E)$ is the superficial divergence of the diagram with E legs. The second part $-(L\varepsilon+2k\alpha)$ denotes the combination which is converted to a simple pole in dimensional regularization. This pole term must be eliminated by a counterterm if both ε and α become small quantities. Thus, the overall form of the renormalization factor is as given in Eq. (4.3). Up to now, the constants $\bar{Y}^{(1)}_{\ldots;l,k}$ may still be functions of ε and α. However, if α/ε is finite in the limit $\varepsilon\to 0$, we can neglect this dependence in the sense of the minimal renormalization. Hence, we can apply this Honkonen-Nalimov scheme only if $\alpha=O(\varepsilon)$.

Next, we calculate the logarithmic derivatives of the renormalization factors. Note that in minimal renormalization the only terms of the β functions, $\beta_{\bar{u}}=\mu\partial\bar{u}/\partial\mu|_{\text{bare}}$ and $\beta_w=\mu\partial w/\partial\mu|_{\text{bare}}$, which contain ε and α explicitly come from the μ factors making $\mathring{g}^2$ and $\mathring{c}$ dimensionless [cf. Eq. (4.2)]—i.e., $\beta_{\bar{u}}=-\varepsilon\bar{u}+\cdots$ and $\beta_w=-2\alpha w+\cdots$. Thus, we obtain from Eq. (4.3)

$$\bar{\gamma}_{\ldots}=\mu\frac{\partial\ln\bar{Z}_{\ldots}}{\partial\mu}\bigg|_{\text{bare}} =-\left(\varepsilon\bar{u}\frac{\partial}{\partial\bar{u}}+2\alpha\frac{\partial}{\partial w}\right)\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\frac{\bar{Y}^{(1)}_{\ldots;l,k}}{l\varepsilon+2k\alpha}w^k\bar{u}^l =-\sum_{l=1}^{\infty}\sum_{k=0}^{\infty}\bar{Y}^{(1)}_{\ldots;l,k}w^k\bar{u}^l, \qquad (4.6)$$

which should be compared to Eq. (3.8) of the short-range case.

Now we can the relate the functions $\bar{\gamma}_{\ldots}(\bar{u},w)$ to the original DP functions $\gamma_{\ldots}(u)$, Eq. (3.8), pertaining to the short-range case. To this end, we consider the hybrid response functional and its renormalizations for $\alpha=0$:

$$\mathcal{J}[\tilde{s},s]|_{\alpha=0}=\int dt d^d r\bar{\lambda}\tilde{s}\Big\{\bar{Z}\bar{\lambda}^{-1}\partial_t+\bar{Z}_\tau\bar{\tau}-(\bar{Z}_\lambda+w)\nabla^2 +\bar{Z}_u^{1/2}\frac{\bar{g}}{2}(s-\tilde{s})\Big\}s. \qquad (4.7)$$

This functional takes on the form of the original short-range DP response functional,

$$\mathcal{J}[\tilde{s},s]=\int dt d^d r\lambda\tilde{s}\Big\{Z\lambda^{-1}\partial_t+Z_\tau\tau-Z_\lambda\nabla^2+Z_u^{1/2}\frac{g}{2}(s-\tilde{s})\Big\}s, \qquad (4.8)$$

if we identify the parameters

$$\lambda=(1+w)\bar{\lambda},\quad \tau=(1+w)^{-1}\bar{\tau},\quad u=(1+w)^{-2}\bar{u} \quad (4.9)$$

and the renormalization factors

$$Z(u)=\bar{Z}(\bar{u},w),\quad (1+w)Z_\lambda(u)=\bar{Z}_\lambda(\bar{u},w)+w,$$

$$Z_\tau(u)=\bar{Z}_\tau(\bar{u},w),\quad Z_u(u)=\bar{Z}_u(\bar{u},w). \quad (4.10)$$

The last identifications lead, by comparison of Eq. (4.3) with Eq. (3.4), to the relations

$$\sum_{k=0}^{\infty}\bar{Y}^{(1)}_{\ldots;l,k}w^k=Y^{(1)}_{\ldots,l}(1+w)^{-2l}, \quad (4.11)$$

in the case of Z, Z_τ, and Z_u, and to

$$\sum_{k=0}^{\infty}\bar{Y}^{(1)}_{\lambda;l,k}w^k=Y^{(1)}_{\lambda,l}(1+w)^{1-2l}, \quad (4.12)$$

in the case of Z_λ. Collecting, we obtain for the logarithmic derivatives, Eq. (4.6),

$$\bar{\gamma}(\bar{u},w)=\gamma(u),$$

$$\bar{\gamma}_\tau(\bar{u},w)=\gamma_\tau(u),$$

$$\bar{\gamma}_u(\bar{u},w)=\gamma_u(u),$$

$$\bar{\gamma}_\lambda(\bar{u},w)=(1+w)\gamma_\lambda(u). \quad (4.13)$$

Using the renormalizations (4.2), the RG functions become

$$\bar{\beta}_{\bar{u}}=\left.\mu\frac{\partial\bar{u}}{\partial\mu}\right|_{\text{bare}}=(-\bar{\varepsilon}+2\bar{\gamma}_\lambda+\bar{\gamma}-\bar{\gamma}_u)\bar{u},$$

$$\bar{\beta}_w=\left.\mu\frac{\partial w}{\partial\mu}\right|_{\text{bare}}=(-2\alpha+\bar{\gamma}_\lambda)w,$$

$$\bar{\zeta}=\left.\mu\frac{\partial\ln\bar{\lambda}}{\partial\mu}\right|_{\text{bare}}=\bar{\gamma}-\bar{\gamma}_\lambda,$$

$$\bar{\kappa}=\left.\mu\frac{\partial\ln\bar{\tau}}{\partial\mu}\right|_{\text{bare}}=\bar{\gamma}_\lambda-\bar{\gamma}_\tau. \quad (4.14)$$

However, Eqs. (4.9) and (4.13) suggest that it is more appropriate to use instead of $\bar{\lambda}$, $\bar{\tau}$, $\bar{u}$, and w the parameters λ, τ, and u, defined by Eq. (4.9), and

$$v=\frac{2\alpha w}{1+w}, \quad (4.15)$$

as the parameters of the theory. The response functional then becomes

$$\mathcal{J}[\tilde{s},s]=\int dtd^dr\lambda\tilde{s}\Big\{Z\lambda^{-1}\partial_t+Z_\tau\tau-Z_\lambda\nabla^2+\frac{v}{2\alpha}[\mu^{2\alpha}(-\nabla^2)^{1-\alpha}+\nabla^2]+Z_u^{1/2}\frac{g}{2}(s-\tilde{s})\Big\}s. \quad (4.16)$$

Note that $\mathcal{J}$ coincides up to a rescaling with $\mathcal{J}_{LR}$, Eq. (3.16), if $v=2\alpha$. Note also that the gradient terms of this response functional take the same form as proposed for the long-wavelength or gradient expansion of the general long-range spreading, Eq. (2.2). Hence, we expect the same difficulties concerning the positivity of the propagator for momenta $q>\mu$ and $\sigma>2$ if $v>2\alpha$.

The RG functions for the new variables are easily derived from Eq. (4.14) by using Eqs. (4.13):

$$\beta_u=\left.\mu\frac{\partial u}{\partial\mu}\right|_{\text{bare}}=(-\varepsilon+2v+2\gamma_\lambda+\gamma-\gamma_u)u,$$

$$\beta_v=\left.\mu\frac{\partial v}{\partial\mu}\right|_{\text{bare}}=(-2\alpha+v+\gamma_\lambda)v,$$

$$\zeta=\left.\mu\frac{\partial\ln\lambda}{\partial\mu}\right|_{\text{bare}}=\gamma-\gamma_\lambda-v,$$

$$\kappa=\left.\mu\frac{\partial\ln\tau}{\partial\mu}\right|_{\text{bare}}=\gamma_\lambda-\gamma_\tau+v. \quad (4.17)$$

In comparison with the short-range RG functions, Eq. (3.9), the new Gell-Mann-Low function β_v as well as the functions β_u, ζ, and κ have only an additive contribution of the variable v. Note that we have to consider both u and v as being parameters of order $\varepsilon\sim\alpha$. In terms of the new variables, the RGE is still of the form given in Eq. (3.6); however, here

$$\mathcal{D}=\mu\frac{\partial}{\partial\mu}+\zeta\lambda\frac{\partial}{\partial\lambda}+\kappa\tau\frac{\partial}{\partial\tau}+\beta_u\frac{\partial}{\partial u}+\beta_v\frac{\partial}{\partial v} \quad (4.18)$$

has to be inserted as the RG differential operator.

B. Asymptotic scaling regions in the (ε,α) expansion

In this subsection we analyze the different scaling regions in a (d,σ) diagram using the results on the hybrid model derived in Sec. IV A. The asymptotic scaling of response and correlation functions is governed by the various fixed points of the renormalization group. To find stable fixed points of the RGE and the corresponding scaling behavior, we have to find solutions of the equations $\beta_u=\beta_v=0$ with the Gell-Mann-Low functions β_u and β_v as given in Eqs. (4.17). The different fixed points can be classified by setting to zero the different factors of these functions.

There is the trivial fixed point $u_*=v_*=0$. To find its stability conditions we determine the eigenvalues of its stability matrix

$$\underline{\underline{\beta}}_* = \begin{pmatrix} \partial\beta_u/\partial u|_* & \partial\beta_u/\partial v|_* \\ \partial\beta_v/\partial u|_* & \partial\beta_v/\partial v|_* \end{pmatrix} = \begin{pmatrix} -\varepsilon & 0 \\ 0 & -2\alpha \end{pmatrix}. \quad (4.19)$$

The eigenvalues of this matrix, $\omega_1=-\varepsilon$ and $\omega_2=-2\alpha$, are positive for $d>4$ and $\sigma>2$; i.e., we retrieve the mean-field region of short-range DP.

Another fixed point, the trivial long-range fixed point, is given by $u_*=0$, $v_*=2\alpha$. Its stability matrix reads

$$\underline{\underline{\beta}}_* = \begin{pmatrix} 4\alpha-\varepsilon & 0 \\ 2\alpha\gamma'_{\lambda*} & 2\alpha \end{pmatrix}, \quad (4.20)$$

where the stroke at γ'_λ denotes the derivative with respect to u. The eigenvalues $\omega_1=4\alpha-\varepsilon$ and $\omega_2=2\alpha$ are positive for $d>4(1-\alpha)=2\sigma$, $\sigma<2$, which marks the region of stability of the trivial long-range fixed point.

Next, we come to the fixed point $v_*=0$, $u_*>0$, as the solution of $2\gamma_{\lambda*}+\gamma_*-\gamma_{u*}=\varepsilon$. Of course, this is the fixed point of the normal short-range DP with $u_*=u_*^{DP}$. The stability matrix is

$$\underline{\underline{\beta}}_* = \begin{pmatrix} (2\gamma'_{\lambda*}+\gamma'_*-\gamma'_{u*})u_* & 2u_* \\ 0 & \gamma_{\lambda*}-2\alpha \end{pmatrix}. \quad (4.21)$$

The first eigenvalue of this matrix, $\omega_1=(2\gamma'_{\lambda*}+\gamma'_*-\gamma'_{u*})u_* =\varepsilon+O(\varepsilon^2)$, shows the stability range of nontrivial short-range DP: $d<4$. Using $\gamma_{\lambda_*}=\gamma_\lambda(u_*^{DP})=\eta^{SR}+2-z^{SR}$, we find the stability condition against long-range spreading:

$$\sigma > z^{SR} - \eta^{SR}. \quad (4.22)$$

Now, we come to the interesting LR region where $u_* >0$ and $v_*\neq 0$. In this domain, the stable fixed points of (4.17) are solutions of the fixed-point equations

$$v_* = 2\alpha - \gamma_{\lambda*} > 0, \quad \bar{\varepsilon} = \varepsilon - 4\alpha = \gamma_* - \gamma_{u*}. \quad (4.23)$$

Using the γ functions which follow from Eq. (3.5) and which have been utilized in Eq. (3.10), we find the fixed point

$$u_*^{LR} = \frac{4\bar{\varepsilon}}{7}\left[1 + \left(50 + 9\ln\frac{4}{3}\right)\frac{\bar{\varepsilon}}{98} + O(\bar{\varepsilon}^2, \alpha\bar{\varepsilon})\right],$$

$$v_*^{LR} = 2\alpha + \frac{\bar{\varepsilon}}{14}\left[1 - \left(17 - 526\ln\frac{4}{3}\right)\frac{\bar{\varepsilon}}{392} + O(\bar{\varepsilon}^2, \alpha\bar{\varepsilon})\right]. \quad (4.24)$$

The critical exponents in the LR region are found from Eq. (4.17) as $\eta_{LR}=\gamma(u_*^{LR})$ and $1/\nu_{LR}=2-\kappa(u_*^{LR})$ with the expansions

$$\eta_{LR} = -\frac{\bar{\varepsilon}}{7}[1 + c_\eta(\alpha)\bar{\varepsilon} + O(\bar{\varepsilon}^2)],$$

$$1/\nu_{LR} = \sigma - \frac{2\bar{\varepsilon}}{7}[1 + c_\nu(\alpha)\bar{\varepsilon} + O(\bar{\varepsilon}^2)], \quad (4.25)$$

where

$$c_\eta(\alpha) = \left(\frac{4}{49} + \frac{36}{49}\ln\frac{4}{3}\right) + O(\alpha),$$

$$c_\nu(\alpha) = \left(\frac{15}{98} + \frac{9}{98}\ln\frac{4}{3}\right) + O(\alpha). \quad (4.26)$$

Note that we have already encountered the first-order contributions in Eq. (3.30) above. Of course, z_{LR} follows from the exact relation (3.21). The stability matrix for this fixed point reads

$$\underline{\underline{\beta}}_* = \begin{pmatrix} Au_* & 2u_* \\ \gamma'_{\lambda*}v_* & v_* \end{pmatrix}, \quad (4.27)$$

with $A=(2\gamma'_{\lambda*}+\gamma'_*-\gamma'_{u*})=3/2+O(\bar{\varepsilon})>0$. The two eigenvalues are given by

$$\omega_\pm = \left(\frac{Au_*+v_*}{2}\right) \pm \sqrt{\left(\frac{Au_*+v_*}{2}\right)^2 - (A-2\gamma'_{\lambda*})u_*v_*}. \quad (4.28)$$

They are positive as long as $(A-2\gamma'_{\lambda*})u_*v_*=(\gamma'_*-\gamma'_{u*})u_*v_* =[49/16+O(\bar{\varepsilon})]\bar{\varepsilon}v_*>0$. This condition leads to $\bar{\varepsilon}>0$ and $v_*=2\alpha-\gamma_{\lambda*}>0$. The long-range fixed point loses its stability if the line $2\alpha=\gamma_{\lambda*}$ is reached. At that point, all critical exponents change over continuously to the usual short-range DP exponents as can be easily seen from Eq. (4.17). Hence, the stability boundary of the long-range Lévy-flight exponent $\sigma=2(1-\alpha)$ is given by

$$\sigma = \sigma_c = z_{SR} - \eta_{SR} = 2 - \frac{\varepsilon}{12}\left[1 + \left(-\frac{17}{288} + \frac{263}{144}\ln\frac{4}{3}\right)\varepsilon + O(\varepsilon^2, \alpha\varepsilon)\right], \quad (4.29)$$

which is less than 2. This fact is astonishing because for σ lower than 2 but greater than σ_c, the long-range part q^σ is naively irrelevant in comparison to the normal diffusional part q^2. However, in an interacting theory it is not the free propagator, but rather the response function $\chi(\omega,\mathbf{q};\tau) =\Gamma_{1,1}(\omega,\mathbf{q};\tau)^{-1}=q^{\eta-z}f(\omega/q^z,\tau/q^{1/\nu})$, which is the deciding quantity. Hence, one has to compare q^σ with $q^{z_{SR}-\eta_{SR}}$ to find out which is leading for $q\to 0$. The other stability boundary is approached if u_* goes to zero and is given by $\bar{\varepsilon}=\varepsilon-4\alpha =0$. This boundary coincides with the value found above: $d =2\sigma$. At this line the exponents cross over to their long-range mean-field values.

To summarize our findings regarding the scaling regions: all boundaries between the four scaling regions—namely, short-ranged DP and long-ranged DP, as well their two mean-field counterparts—are generally given by the four lines in a (d,σ) diagram where one or two of the fixed point values u_* and v_* vanish. There is no room for other stability lines as some authors argued [31].

C. Landau's ghost

As we have remarked at several points, the propagator becomes problematic for higher momenta. Consider the q-dependent part of the inverse renormalized propagator of the hybrid theory,

$$G(\mathbf{q},0,0)^{-1} = \lambda q^2\left[1 + \frac{v}{2\alpha}[(q/\mu)^{-2\alpha} - 1]\right], \qquad (4.30)$$

where $v \geqslant 0$ to ensure stability (positivity) for $q \to 0$ for both signs of α. The part proportional to v changes sign and leads to a loss of stability for negative α at a momentum q_g given by

$$\ln(q_g/\mu) = \frac{1}{2|\alpha|}\ln(1 + 2|\alpha|/v) \sim 1/v \qquad (4.31)$$

for small α. This ghost reminds us of Landau's ghost in quantum electrodynamics. The momentum of the ghost goes exponentially to infinity if $v \to 0$. This ghost even arises for positive α if $v > 2\alpha$. The fixed point of the hybrid theory always belongs to this region. The correct interpretation is the following: our asymptotic theory is just an effective theory in the sense that it can only be used in the low-momentum limit in a perturbation expansion with v as an expansion parameter. Therefore, the second part of Eq. (4.30) must be considered as a perturbation.

Let us demonstrate this in some detail for the inverse response function $\Gamma_{1,1}$ as an example. To this end, we work to one-loop order using u, v, ε, and α as first-order quantities. The zeroth order is

$$\Gamma_{1,1}^{(0)}(\mathbf{q},\omega=0,\tau=0) = \lambda q^2. \qquad (4.32)$$

Using renormalized perturbation theory, adding both first-order terms, and neglecting higher-order terms, we obtain

$$\begin{aligned}\Gamma_{1,1}(\mathbf{q},0,0) &= \lambda q^2\left[1 + \frac{v}{2\alpha}\left(\left(\frac{q}{\mu}\right)^{-2\alpha} - 1\right) + \frac{u}{8}\ln\left(\frac{q}{2\mu}\right)\right] + \cdots\\ &= \lambda q^2\left[1 - v\ln\left(\frac{q}{\mu}\right) + \frac{u}{8}\ln\left(\frac{q}{2\mu}\right)\right] + \cdots\\ &= \left(1 - \frac{u}{8}\ln 2\right)\lambda q^2\left[1 + \left(\frac{u}{8} - v\right)\ln\left(\frac{q}{\mu}\right)\right] + \cdots .\end{aligned} \qquad (4.33)$$

Now we use the fixed point result $v_* = 2\alpha - \gamma_{\lambda *}$, with $\gamma_\lambda = -u/8$, to get

$$\Gamma_{1,1}(\mathbf{q},\omega=0,\tau=0) = \left(1 - \frac{u_*}{8}\ln 2\right)\lambda q^2\left[1 - 2\alpha\ln\left(\frac{q}{\mu}\right)\right] + \cdots \sim \lambda q^{2(1-\alpha)}. \qquad (4.34)$$

As expected, the RG proves to be the systematic tool to resum all logarithms to yield the correct critical exponent.

This procedure holds for all $\alpha = O(\varepsilon)$ irrespective of the sign as long as the fixed point with $v_* > 0$ is stable. Otherwise, $v_* = 0$, $u_* = 2\varepsilon/3$, and one gets $\Gamma_{1,1} \sim \lambda q^{2-\eta_{SR}}$ with $\eta_{SR} = -\varepsilon/12$—i.e., the known behavior for the short-range case. We once more point out that the second term (the Lévy-flight contribution) has to be handled as a perturbation to the desired order and not as a part of the unperturbed propagator. Also, one has to interpret the special case $\alpha = 0$ as a relevant logarithmic perturbation $\sim vq^2\ln(\mu/q)$ of q^2. Only if v is strictly zero is the short-range case recovered. Thus, one cannot expect a continuous behavior at $\alpha = 0$ comparing the short-range versus the Lévy-flight directed percolation.

V. CRITICAL BEHAVIOR OF DYNAMIC OBSERVABLES IN LÉVY-FLIGHT DIRECTED PERCOLATION

In this section we will harvest some of our previous results to calculate scaling forms and logarithmic correction for those dynamic quantities in long-ranged DP that are most suitable from the vantage point of numerical simulations [24]. Two key observables with respect to simulations are the density of infected individuals, $\rho(t) = \langle s(\mathbf{r},t)\rangle_{\rho 0}$ for $t > 0$ if the initial state at time $t = 0$ is prepared with a homogeneous initial density ρ_0, and the response function $\chi(\mathbf{r},t) = \langle s(\mathbf{r},t)\tilde{s}(\mathbf{0},0)\rangle$, which yields the density of infected individuals after the epidemic is initialized by a pointlike source at $t = 0$ and $\mathbf{r} = 0$.

A. Scaling properties

The scaling properties of the density of infected individuals and the response function follow from the RGE (3.6) taken at the long-range fixed point of Eq. (4.17) and by identifying $\chi(\mathbf{r},t)$ and $\rho(t)$ with the Green's functions $G_{1,1}(\mathbf{r},t;\tau)$ and $G_{1,0}(\mathbf{r},t;\tau,\rho_0)$, respectively. The initial density ρ_0 is introduced into the response functional via a (bare) source $\mathring{h}(\mathbf{r},t) = h\hat{\rho}_0\delta(t)$ with renormalization $\mathring{\rho}_0 = Z^{-1/2}\rho_0$ [19], which leads to an additional derivative term $\frac{1}{2}\gamma\rho_0\partial/\partial\rho_0$ in the RGE. We obtain the scaling forms

$$\begin{aligned}\rho(t) &= t^{-\delta}S_\rho(\tau t^{1/z\nu}, \rho_0 t^{\delta+\theta}),\\ \chi(\mathbf{r},t) &= t^{-2\delta}S_\chi(\mathbf{r}/t^{1/z}; \tau t^{1/z\nu}),\end{aligned} \qquad (5.1)$$

where the $S_{\ldots}$ are appropriate scaling functions. We drop in this section all the subscripts at the critical exponents because we are interested in the long-range case only. Hence, $\eta = \eta_{LR}$ and $1/\nu = 1/\nu_{LR}$ [see Eqs. (4.25)] and $z = \sigma + \eta$, $\delta = (d+\eta)/2z$, $\theta = -\eta/z$. The expansions of the latter two are

$$\begin{aligned}\delta &= 1 - \frac{3\bar{\varepsilon}}{7\sigma}[1 + c_\delta(\alpha)\bar{\varepsilon} + O(\bar{\varepsilon}^2)],\\ \theta &= \frac{\bar{\varepsilon}}{7\sigma}[1 + c_\theta(\alpha)\bar{\varepsilon} + O(\bar{\varepsilon}^2)],\end{aligned} \qquad (5.2)$$

where

$$c_\delta(\alpha) = \left(\frac{17}{294} - \frac{6}{49}\ln\frac{4}{3}\right) + O(\alpha),$$

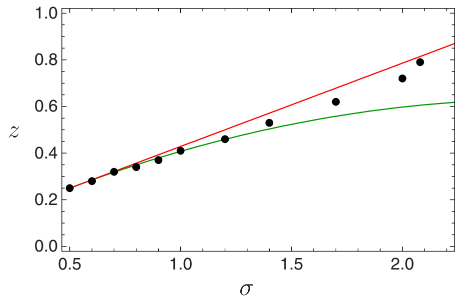

FIG. 1. (Color online) The exponent z as a function of σ for $d=1$. The data points stem from simulations by data by Hinrichsen [18]. The red (upper) and the green (lower) curve correspond to our one-loop and two-loop results, respectively.

$$c_\theta(\alpha) = \left(\frac{23}{196} + \frac{36}{49}\ln\frac{4}{3}\right) + O(\alpha). \tag{5.3}$$

At the critical point $\tau=0$, Eqs. (5.1) show that the mean-square radius of spreading from the origin scales as $R^2(t) \sim t^{2/z}$ and the average number of infected individuals $N(t) = \int d^d r \chi(\mathbf{r},t) \sim t^\theta$. Starting with a homogeneous finite value ρ_0, the critical density first increases in a universal time regime with the same exponent as $\rho(t) \sim \rho_0 t^\theta$. Then, after some crossover time, it decreases as $\rho(t) \sim t^{-\delta}$. If one starts with a full lattice of infected sites corresponding to an infinite initial value ρ_0, only the last scaling behavior is seen. Because of asymptotic time-reflection invariance of DP (duality symmetry), this behavior characterizes also the survival probability [32] $P(t) \sim t^{-\delta}$.

To first order all exponents are identical to the LR exponents derived from Eqs. (3.30), of course. The full α content of the functions $c_{\ldots}(\alpha)$ in Eqs. (4.25) and (5.2) must be calculated from the long-range model, Eq. (3.16). Hence, one has to be careful when applying the expansions to $O(\bar{\varepsilon}^2)$ for $\alpha=0$, e.g., in $d=1$ where $\alpha=(3-\bar{\varepsilon})/4$. In Figs. 1–3, we compare our results for the exponents z, δ, and θ to numerical results for $d=1$ by Hinrichsen [18]. For this comparison, we use the first-order expansions in $\bar{\varepsilon}$ (red curves), which are exact in α, and the second-order expansions in $\bar{\varepsilon}$ where we neglect the α-dependent parts of the $c_{\ldots}(\alpha)$ (green and blue curves). The green curves show our second-order result for z and results obtained for δ and θ by using without further

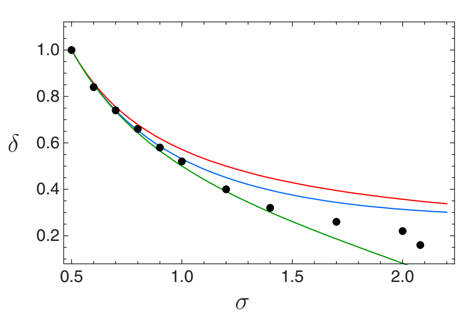

FIG. 2. (Color online) The exponent δ as a function of σ for $d=1$. The red (upper) curve corresponds to our one-loop results. The green (lower) and blue (middle) curves correspond to two versions of our two-loop result (see main text).

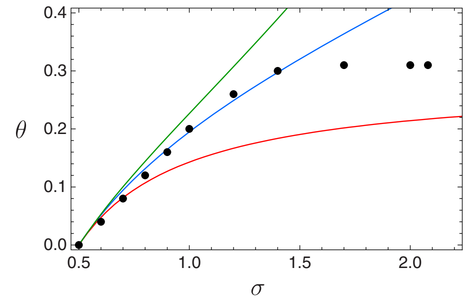

FIG. 3. (Color online) The exponent θ as a function of σ for $d=1$. The color coding of the curves is the same as in Fig. 2.

expansion the scaling relations relating δ and θ to z and η. The blue curves stem from using these scaling relations and then properly expanding δ and θ to second order in $\bar{\varepsilon}$. For σ in the range from 1/2 to roughly 1, the numerical data and the analytic results agree remarkably well. For larger σ, the agreement suffers, but is well within the expectations for the methods used here.

B. Logarithmic corrections

Above the boundary between the genuine and the trivial long-range regions ($d>2\sigma$, $\sigma<2$), the coupling constant g tends to zero under the RG. However, g represents a dangerously irrelevant variable here, since it scales various observables, and setting $g=0$ rigorously leads either to zero or infinity for relevant quantities. Due to its twofold nature as both a relevant scaling variable and an irrelevant loop-expansion generating parameter, g has to be treated very carefully. To set the stage for such a treatment, let us briefly review a few fundamentals of dynamic field theory. In broad terms, one attempts to determine the cumulant-generating functional defined by the functional integral

$$\mathcal{W}_{\mathrm{LR}}[H,\tilde{H}] = \ln \int \mathcal{D}[\tilde{s},s]\exp[-\mathcal{J}_{\mathrm{LR}}[\tilde{s},s] + (H,s) + (\tilde{H},\tilde{s})]. \tag{5.4}$$

Functional derivatives with respect to the sources H and $\tilde{H}$ define the Green's functions. The generating functional for the vertex functions $\Gamma_{\mathrm{LR}}[\tilde{s},s]$, the dynamic free energy, is related to the cumulant-generating functional via the Legendre transformation

$$\Gamma_{\mathrm{LR}}[\tilde{s},s] + \mathcal{W}_{\mathrm{LR}}[H,\tilde{H}] = (H,s) + (\tilde{H},\tilde{s}), \tag{5.5}$$

with $s=\delta\mathcal{W}_{\mathrm{LR}}/\delta H$ and $\tilde{s}=\delta\mathcal{W}_{\mathrm{LR}}/\delta\tilde{H}$, and vice versa. In terms of Γ_{LR}, the twofold nature of g is lucidly exposed by writing [33]

$$\Gamma_{\mathrm{LR}}[\tilde{s},s;\tau,g] = g^{-2}\Phi_{\mathrm{LR}}[g\tilde{s},gs;\tau,u]. \tag{5.6}$$

The expansion of the functional $\Phi_{\mathrm{LR}}[g\tilde{s},gs;\tau,u]$ into a series with respect to u yields the loop expansion. The zeroth term $g^{-2}\Phi_{\mathrm{LR}}[g\tilde{s},gs;\tau,0]$ is just the response functional $\mathcal{J}_{\mathrm{LR}}$, Eq. (3.16), itself. The scaling form of the generating functional for the cumulants that corresponds to Eq. (5.6) reads

$$\mathcal{W}_{\mathrm{LR}}[H,\tilde{H};\tau,g] = g^{-2}\Omega_{\mathrm{LR}}[gH,g\tilde{H};\tau,u]. \tag{5.7}$$

To leading order in the logarithmic corrections, we may neglect the dependence of Ω and Φ on u. Functional derivation leads to the Green's functions

$$G_{N,\tilde{N}}(\{\mathbf{r},t\},\tau;u) \simeq u^{-1+(N+\tilde{N})/2}T_{N,\tilde{N}}(\{\mathbf{r},t\},\tau,u^{1/2}\rho_0), \tag{5.8}$$

where $T_{N,\tilde{N}}$ are the contributions of loopless trees consisting of $N+\tilde{N}-1$ propagators and $N+\tilde{N}-2$ vertices.

The characteristic equations that follow from the RG functions, Eqs. (3.29), with $\bar{\mu}(\ell)=\ell\mu$ and $d=2\sigma$, are to lowest order given by

$$\ell\frac{d\bar{u}(\ell)}{d\ell} = \bar{\beta}(\bar{u}(\ell)) = \frac{7}{4}\bar{u}(\ell)^2,$$

$$\frac{d\ln X(\ell)}{d\ln\ell} = \bar{\gamma}(\bar{u}(\ell)) = -\frac{\bar{u}(\ell)}{4},$$

$$\frac{d\ln X_\lambda(\ell)}{d\ln\ell} = \bar{\zeta}(\bar{u}(\ell)) = -\frac{\bar{u}(\ell)}{4},$$

$$\frac{d\ln X_\tau(\ell)}{d\ln\ell} = \bar{\kappa}(\bar{u}(\ell)) = \frac{\bar{u}(\ell)}{2}, \tag{5.9}$$

Solving these equations, we obtain, asymptotically for $\ell \ll 1$,

$$\bar{u}(\ell) \sim |\ln\ell|^{-1}, \quad X(\ell) \sim |\ln\ell|^{1/7},$$

$$X_\lambda(\ell) \sim |\ln\ell|^{1/7}, \quad X_\tau(\ell) \sim |\ln\ell|^{-2/7}. \tag{5.10}$$

Hence, we get

$$\begin{aligned} G_{N,\tilde{N}}(\{\mathbf{r},t\},\tau;u) &\simeq \bar{u}(\ell)^{-1}[\bar{u}(\ell)\ell^{2\sigma}X(\ell)]^{(N+\tilde{N})/2}T_{N,\tilde{N}}(\{\ell\mathbf{r},\ell^{\sigma}X_\lambda(\ell)t\},\ell^{-\sigma}X_\tau(\ell)\tau,\bar{u}(\ell)^{1/2}\ell^{-\sigma}X(\ell)^{1/2}\rho_0) \\ &\sim |\ln\ell|[\ell^{\sigma}|\ln\ell|^{-3/7}]^{N+\tilde{N}}T_{N,\tilde{N}}(\{\ell\mathbf{r},\ell^{\sigma}|\ln\ell|^{1/7}t\},\ell^{-\sigma}|\ln\ell|^{-2/7}\tau,\ell^{-\sigma}|\ln\ell|^{-3/7}\rho_0) \end{aligned} \tag{5.11}$$

as solutions of the entire RGE. Choosing either $\ell^{\sigma}|\ln\ell|^{1/7}t \sim 1$ or $\ell^{-\sigma}|\ln\ell|^{-2/7}\tau \sim 1$, we deduce that at the critical point

$$\rho(t) \sim \rho_0(\ln t)^{1/7} \tag{5.12}$$

in the initial-time region,

$$\rho(t) \sim P(t) \sim t^{-1}(\ln t)^{3/7} \tag{5.13}$$

in the late-time region, and

$$R^2(t) \sim [t(\ln t)^{1/7}]^{2/\sigma}. \tag{5.14}$$

VI. CONCLUDING REMARKS

In summary, we studied DP with Lévy-flight spreading by using the powerful methods of renormalized field theory. Our work confirms the previously known RG fixed-point structure including their stability regions and the fact that the critical exponents change continuously in the crossover between short-range DP and Lévy DP. We calculated the critical exponents for Lévy DP, which have hitherto been known to first order, to second order in an expansion in ε and α. These results agree well with the existing numerical simulations for $d=1$. In addition, we calculated the leading logarithmic corrections for several dynamical observables that are typically measured in simulations.

We hope that our work stimulates further interest in long-range DP. It would be interesting to see further simulation results, e.g., for the critical exponents for $d>1$ and for logarithmic corrections. Also, it would be interesting to have analytical and numerical results for other universal quantities such as scaling functions and amplitudes. In a forthcoming paper we will apply the same methods to the long-range GEP—that is, to dynamic isotropic percolation [33].

ACKNOWLEDGMENT

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[1] P. Grassberger and K. Sundermeyer, Phys. Lett. **77B**, 220 (1978).
[2] P. Grassberger and A. de La Torre, Ann. Phys. (N.Y.) **122**, 373 (1979).
[3] S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957).
[4] J. L. Cardy and R. L. Sugar, J. Phys. A **13**, L423 (1980).
[5] S. P. Obukhov, Physica A **101**, 145 (1980).
[6] V. N. Gribov, Zh. Eksp. Teor. Fiz. **53**, 654 (1967) [Sov. Phys. JETP **26**, 414 (1968)].
[7] V. N. Gribov and A. A. Migdal, Zh. Eksp. Teor. Fiz. **55**, 1498 (1968) [Sov. Phys. JETP **28**, 784 (1969)].

[8] O. Mollison, J. R. Stat. Soc. Ser. B (Methodol.) **39**, 283 (1977).
[9] H. K. Janssen, Z. Phys. B: Condens. Matter **42**, 151 (1981).
[10] H. K. Janssen, J. Stat. Phys. **103**, 801 (2001).
[11] P. Grassberger, Z. Phys. B: Condens. Matter **47**, 365 (1982).
[12] T. E. Harris, Ann. Probab. **2**, 969 (1974).
[13] T. M. Liggett, *Interacting Particle Systems* (Springer, New York, 1985).
[14] I. Jensen and R. Dickman, Physica A **203**, 175 (1994).
[15] W. Kinzel, in *Percolation Structures and Processes*, edited by G. Deutsch, R. Zallen, and J. Adler, (Hilger, Bristol, 1983).
[16] W. Kinzel, Z. Phys. B: Condens. Matter **58**, 229 (1985).
[17] N. T. J. Bailey, *The Mathematical Theory of Infectious Diseases* (Griffin, London, 1975).
[18] A comprehensive recent overview over directed percolation is given in H. Hinrichsen, Adv. Phys. **49**, 815 (2000).
[19] For a recent review on the field theory approach to percolation processes see H. K. Janssen and U. C. Täuber, Ann. Phys. (N.Y.) **315**, 147 (2005).
[20] P. Grassberger, in *Fractals in Physics*, edited by L. Pietronero and E. Tosatti (Elsevier, New York, 1986).
[21] H. K. Janssen, K. Oerding, F. van Wijland, and H. J. Hilhorst, Eur. Phys. J. B **7**, 137 (1999).
[22] H. Hinrichsen and M. Howard, Eur. Phys. J. B **7**, 635 (1999).
[23] H. K. Janssen, Phys. Rev. E **58**, R2673 (1998).
[24] H. Hinrichsen, J. Stat. Mech.: Theory Exp. (2007) P07066.
[25] H. K. Janssen, Z. Phys. B **23**, 377 (1976); R. Bausch, H. K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976); H. K. Janssen, in *Dynamical Critical Phenomena and Related Topics*, edited by C. P. Enz, Lecture Notes in Physics Vol. 104 (Springer, Heidelberg, 1979).
[26] C. De Dominicis, J. Phys. (Paris), Colloq. **37**, C1-247 (1976); C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).
[27] H. K. Janssen, in *From Phase Transitions to Chaos*, edited by G. Györgyi, I. Kondor, L. Sasvári, and T. Tél (World Scientific, Singapore, 1992).
[28] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th revised ed. (Clarendon, Oxford, 2002).
[29] R. Schloms and V. Dohm, Nucl. Phys. B **328**, 639 (1989).
[30] J. Honkonen and M. Yu. Nalimov, J. Phys. A **22**, 751 (1989).
[31] N. V. Antonov (private communication).
[32] H. K. Janssen, J. Phys.: Condens. Matter **17**, S1973 (2005).
[33] H. K. Janssen and O. Stenull, Phys. Rev. E **69**, 016125 (2004).