December 1989

The Mixed Powerdomain

Carl A. Gunter
University of Pennsylvania

University of Pennsylvania Department of Computer and Information Science Technical Report No. MS-CIS-89-77.

The Mixed Powerdomain

Abstract

This paper characterizes the powerdomain constructions which have been used in the semantics of programming languages in terms of formulas of first order logic under a preordering of provable implication. The goal is to reveal the basic logical significance of the powerdomain elements by casting them in the right setting. Such a treatment may contribute to a better understanding of their potential uses in areas which deal with concepts of sets and partial information such as databases and computational linguistics. This way of viewing powerdomain elements suggests a new form of powerdomain - called the *mixed* powerdomain - which expresses data in a different way from the well-known constructions from programming semantics. It is shown that the mixed powerdomain has many of the properties associated with the convex powerdomain such as the possiblity of solving recursive equations and a simple algebraic characterization.

Comments

University of Pennsylvania Department of Computer and Information Science Technical Report No. MS-CIS-89-77.

THE MIXED POWERDOMAIN

C.A. Gunter

MS-CIS-89-77
LOGIC & COMPUTATION 18

Department of Computer and Information Science
School of Engineering and Applied Science
University of Pennsylvania
Philadelphia, PA 19104

Revised
December 1989

ACKNOWLEDGEMENTS:
This research was supported in part by DARPA grant N00014-85-K-0018, NSF grants MCS-8219196-CER, IRI84-10413-AO2 and U.S. Army grants DAA29-84-K-0061, DAA29-84-9-0027.

THE MIXED POWERDOMAIN

Carl A. Gunter*

University of Pennsylvania
Department of Computer and Information Sciences
Philadelphia, PA 19104 U.S.A.

December 8, 1989

Abstract

This paper characterizes the powerdomain constructions which have been used in the semantics of programming languages in terms of formulas of first order logic under a preordering of provable implication. The goal is to reveal the basic logical significance of the powerdomain elements by casting them in the right setting. Such a treatment may contribute to a better understanding of their potential uses in areas which deal with concepts of sets and partial information such as databases and computational linguistics. This way of viewing powerdomain elements suggests a new form of powerdomain—called the *mixed* powerdomain—which expresses data in a different way from the well-known constructions from programming semantics. It is shown that the mixed powerdomain has many of the properties associated with the convex powerdomain such as the possiblity of solving recursive equations and a simple algebraic characterization.

1 Introduction.

A *powerdomain* is a "computable" analogue of the powerset operator. They were introduced in the 1970's as a tool for providing semantics for programming languages with non-determinism. For such applications, the powerset operator was unsatisfactory for basically the same reasons that the full function space was unusable for the semantics of certain features of sequential programming languages (such as higher-order procedures and dynamic scoping). In the full powerset, there are *too many* sets and this causes problems for the solution of recursive domain equations. Hence, such applications call for a more parsimonious theory of subsets, based on a concept of non-deterministic computability.

The study of powerdomains has revealed many interesting connections between the semantics of programming languages and traditional topics of mathematical research in topology and category theory. Moreover, there is a widening awareness of the *logical* properties of powerdomains. It is the goal of this paper to prove several results intended to deepen our understanding of the logic of powerdomains. It is demonstrated that each of the best known powerdomains can be characterized by considering appropriate

*Electronic mail address: gunter@cis.upenn.edu

families of first order propositions under the preordering of *provable implication*. These families provide a simple logical characterization of the information-theoretic content of the elements of the powerdomains. Such a view may suggest methods for relating the known theory of powerdomains to work on similar structures which are the subject of investigations in other areas such as databases, computational linguistics and artificial intelligence.

The seminal work on powerdomains and their application in programming language semantics was G. Plotkin's paper [Plo76] on what is often called the *Plotkin powerdomain*. Subsequent research by M. B. Smyth [Smy78] led to the discovery of two similar constructions often called the *Smyth* and *Hoare* powerdomains. These three powerdomains have been used widely in programming language theory, and they have also sparked a body of theoretical research into their properties and relationships to similar constructions in Mathematics. Smyth [Smy83] demonstrated a close connection between the Smyth and Hoare powerdomains and the concepts of upper and lower semi-continuity respectively. He also found that the Plotkin powerdomain was related to what is known as the *Vietoris construction* from topology. This research led Smyth to suggest the names for the three powerdomains which I will use below: upper (Smyth), lower (Hoare) and convex (Plotkin). The categorical significance of the powerdomains was demonstrated by Hennessy and Plotkin [HP79] who proved that each of the three can be seen as a left adjoints to appropriate forgetful functors.

There has also been progress on understanding the powerdomains from the point of view of logic. Results similar to those which will be proved below have been presented by G. Winskel [Win85], who showed how each of the three powerdomains can be characterized using *modal* formulas under an interpretation in terms of non-deterministic computations. Winskel's results have a slightly different intuition from the ones proved below since I will generally be viewing powerdomains as partially described *sets* rather than partially described *computations*. Abramsky has highlighted many useful connections between domains, topology and logic in his work on "domains in logical form" [Abr88, Abr87, Abr89] where he gives a thorough treatment of the logics of the upper, lower and convex powerdomains.

The paper is divided into nine sections. The powerdomains are defined in the second section and an extended example using sets of records is discussed. In the third section the intuitions about information discussed in the second section are characterized using first order logic. Theorems establishing a precise relationship for the upper and lower powerdomains are proved. In the fourth section, the convex powerdomain is also characterized in terms of first order logic and a new powerdomain, the *mixed* powerdomain, is defined. Relationships between the convex and mixed powerdomains and the *sandwich* powerdomain from database theory are discussed. The mixed powerdomain is also characterized with first order formulas. The fifth sections discusses domain-theoretic properties of the mixed powerdomain. It is shown that the mixed powerdomain can be used in recursive domain equations and comparisons are made with the other three powerdomains. In the sixth section it is shown that the mixed powerdomain (with emptyset) is characterized algebraically as the left ajoint to a forgetful functor from a category of algebras (called *mix* algebras) to the category of algebraic cpo's and continuous functions. Section seven discusses the mixed powerdomain as a relation on the product $\mathcal{U}D \times \mathcal{L}D$ of the upper and lower powerdomains. The eighth section shows how the Stone dual of the mixed powerdomain can be characterized using a form of modal logic. The ninth and final section offers conclusions.

The paper is written in a graded fashion. The first four sections have no prerequisites other than a slight knowledge of first order logic. The remaining sections require knowledge of domain theory and some basic category theory.

2 Sets of data.

This section begins by providing precise definitions for the upper, lower and convex powerdomains. As a guide to intuition, we will then look at several examples of sets from the the powerdomains of a simple datatype of records. Viewing things in such a concrete fashion aids one in seeing powerdomains as diverse theories of partially described sets and not just as a theories of the outcomes of non-deterministic computations.

Rather than follow the usual treatment which one can find in many places in the literature (see, for example, [Smy78] or [GS88]), I will reduce the domain-theoretic pre-requisites by working only with the action of the powerdomain operator on the *bases* of domains.[1] In this way, we may restrict our attention to the following simple class of directed graphs:

Definition: A *preorder* is a set A together with a binary relation $\gtrsim$ which is reflexive and transitive. We may write $y \lesssim x$ rather than $x \gtrsim y$. We write $x \approx y$ if $x \gtrsim y$ and $x \lesssim y$. ∎

A preorder is like a poset (partial order) except the anti-symmetry axiom need not hold. Intuitively, the elements of a preorder A may be thought of as propositions (of first order logic, say) under the preordering of provable implication. If we have propositions ϕ and ψ in A, then we may have $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$ without it being the case that ϕ and ψ are the same proposition (although their truth values must be the same). For this reason and another (more important) reason mentioned below, it is more convenient to work with preorders than posets.

Let $\langle A, \gtrsim \rangle$ be a preorder and suppose $\mathcal{P}_f^* A$ is the collection of non-empty finite subsets of A. We define three preorderings on $\mathcal{P}_f^* A$ as follows. Suppose $u, v \in \mathcal{P}_f^* A$, then

- $u \gtrsim^\sharp v$ iff for every $x \in u$ there is a $y \in v$ such that $x \gtrsim y$,

- $u \gtrsim^\flat v$ iff for every $y \in v$ there is a $x \in u$ such that $x \gtrsim y$,

- $u \gtrsim^\natural v$ iff $u \gtrsim^\sharp v$ and $u \gtrsim^\flat v$

It is easy to check that each of these relations is, in fact, a preordering. The preorder $\langle \mathcal{P}_f^* A, \gtrsim^\sharp \rangle$ is called the *upper powerdomain* of $\langle A, \gtrsim \rangle$ and it is denoted $\langle A^\sharp, \gtrsim^\sharp \rangle$ (or just $A^\sharp$ when the preordering is clear). The preorder $\langle \mathcal{P}_f^* A, \gtrsim^\flat \rangle$ is called the *lower powerdomain* of $\langle A, \gtrsim \rangle$ and it is denoted $\langle A^\flat, \gtrsim^\flat \rangle$. Finally, the preorder $\langle \mathcal{P}_f^* A, \gtrsim^\natural \rangle$ is called the *convex powerdomain* of $\langle A, \gtrsim \rangle$ and it is denoted $\langle A^\natural, \gtrsim^\natural \rangle$.

To get a few examples, let us look at the powerdomains of a simple preorder of records. Our records will have between zero and four fields. the available fields are `name`, `age`, `socsec` and `married?`.

[1] This way of doing things has been discussed in numerous references. The information systems of Scott [Sco82] are a popular tool; preorders and domains are discussed in some detail in [Gun87]. The technique has been carried much further in [Abr89].

The `age` and `socsec` fields may be filled with integers and the `married?` field may be filled with a boolean. The name field is a record with two fields: `first` and `second`. Each of these fields may be filled with a string. The type can be named by the following expression:

```
{ name = { first = string,
           last = string },
  age = int,
  socsec = int,
  married? = bool }
```

Here is a sample record r_1:

```
{ name = { first = "John",
          last = "Smith" },
  age = 28,
  socsec = 439048302,
  married? = true }
```

We will assume that records may have missing fields as in the following record r_2:

```
{ name = { first = "John" },
  age = 28 }
```

The record r_1 is *more informative* than r_2 because it provides more facts about the described individual "John". This concept of one record being more informative than another is basic to the discussion which follows. Records may have other relationships as well. In particular, there is an *inconsistency* between r_1, r_2 and the following record r_3:

```
{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2,
  married? = false }
```

We may model this collection of records and its associated information ordering as follows. First, we assume that we are given the types *string, int* and *bool* as *flat domains.* For example, the type of integers should contain the ordinary integers 1, -2, 0 and so on, together with a special *bottom* element $\bot$ which is intended to represent "no information". The ordering on these elements is given by taking $m \gtrsim n$ if and only if $n = \bot$ or $m = n$. For example, we *do not* have $28 \gtrsim 2$. This is what one would expect, after all; a record about a two year old John Smith is not less informative than a record about a 28 year old John Smith, these records are simply incompatible. The interpretation of strings is similar. The booleans are also a flat domain, but there are only three elements *true, false* and $\bot$. Now, the space of records is the product space

$$(\mathit{string} \times \mathit{string}) \times \mathit{int} \times \mathit{int} \times \mathit{bool}.$$

Of course, a record is interpreted in this space without regard to the order of its fields according to some convention perhaps (*e.g.* the first two strings are for the first and last names respectively; the first integer is the age and the second is the social security number). Missing record fields are interpreted as $\perp$. Records are ordered coordinate-wise. A pair of records r, r' is consistent if there is a record r'' such that $r'' \gtrsim r$ and $r'' \gtrsim r'$. Otherwise r and r' are inconsistent. Many of the sets in the powerdomain of our space of records will contain pairs of inconsistent records.

Our family of records is the raw material out of which we can build collections of data about some set of "real world entities". Some of our records probably make no real sense under any circumstances. For example:

```
{ name = { first = "John",
           last = "Smith" },
  age = 2,
  married? = true }
```

will probably not find its way into any useful database of records. There will also be pairs of records which are unlikely to be found together in the same database:

```
{ socsec = 229068403,
  age = 2 }

{ socsec = 229068403,
  age = 28 }
```

Moreover, most data items will be only partial descriptions (as is the case with most of the examples above). The question we need to answer is the following: how does a *set* of records provide a partial description of a set of real world entities?

Consider the following set s of records

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John" },
  age = 2 }
```

which might be the database [2] for a small nursery. When should we say of another set of records that it is *more informative* than the set of records above? Here is a first possibility s_1:

[2] I hope the reader will pardon my loose use of this term. It is not my intent to expound a serious theory of databases. The examples are meant to suggest the propositional consequences of the powerdomain orderings.

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  age = 2 }

{ name = { first = "Beth" }
  age = 3 }
```

This set seems more informative because it lists more of the children in the nursery and provides slightly more information about those who are enrolled (since we now have John's last name). In the lower powerdomain (pre)-ordering, $\gtrsim^\flat$, the set s_1 is greater (more informative) than s. But consider the following set s_2 of records:

```
{ name = { first = "Mary" },
  socsec = 439234970,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 429238406,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2 }
```

This seems more informative than s because it provides more information about the children in the class and eliminates the name of a child (Todd) who will not actually be attending. In the upper powerdomain ordering, $\gtrsim^\sharp$, the set s_2 is greater than s. However, it is *not* greater than s in the lower powerdomain ordering. Conversely s_2 is not greater than s in the upper powerdomain ordering.

These two alternative extensions should point out how the ordering of partial information suggests the intuitive significance of the set of records s. In the first case, under the lower ordering, s might be a list of children who have been enrolled in the nursery; more may enroll later. In the second case (under the upper ordering) s might be the list of all children who are on a waiting list; some children may drop off of the list but no new ones may enter (since the deadline for such entries has passed). In either case, a further refinement of the individual records through the addition of new fields results in a more informative set of records.

It is important to note that powerdomains are only preorderings and not posets (*i.e.* partial orderings). If the record

```
{ name = { last = "Smith" }
  age = 2 }
```

is added to s_1, there is *no change* in the intended meaning of the set of records with respect to the lower preordering. In other words, if s_1' is the larger set, then $s_1 \gtrsim^\flat s_1'$ and also $s_1' \gtrsim^\flat s_1$. This is not true of the upper preordering. In that preordering, $s_1 \gtrsim^\sharp s_1'$, but $s_1' \not\gtrsim^\sharp s_1$. The following set of records

```
{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2 }

{ name = { first = "John" }
  age = 2 }
```

would not change, under either powerdomain ordering, if the following record were added:

```
{ name = { first = "John",
           last = "Smith" }
  age = 2 }
```

It may seem odd that we would allow in s_2 the possibility that a single record might split into two records as the record for John did. This seems more reasonable in other cases, however. For example, the singleton set of records containing only the record

```
{ age = 2 }
```

would indicate under the upper ordering that we are talking about a nursery of two year olds (whose names we do not yet know). In the lower ordering, this database would indicate only that there will be *some* two year old in the nursery (but there may also be some children of other ages). It is also possible for two data items to *merge* to form a new data item. For example, the following set of records:

```
{ name = { first = "Mary" } }

{ name = { first = "John" },
  age = 2 }
```

```
{ name = { last = "Smith" }
  socsec = 229068403 }

{ socsec = 429238406 }
```

is less descriptive (in either lower or upper ordering) than the set of records s_2 above.

We will look at some more examples of this kind when we get to the discussion of the convex ordering in a later section.

3 Powerdomains and logic.

Let us now try to relate the intuitions and preorderings discussed in the previous section to formulas of an appropriate logic. For this discussion first order predicate logic will be used because it is simple, well-known and seems to be sufficient for the job at hand. After some motivation, the upper and lower powerdomain operators on preorders will be precisely related to certain operations on collections of first order formulas.

In the examples provided in the previous section, we thought of sets of records as partial descriptions of sets of real world entities. However, one may dually think of a set of records as describing a *set* of "situations" compatible with the set of records. Each record can be treated as a predicate over a collection of *individuals*. For example, the record

```
{ name = { first = "John" } }
```

is satisfied by individuals whose first name is "John". More concretely, we might think of individuals as *total records* (*i.e.* records with all fields filled in) for the example of the previous section.[3] If we view things this way, can we think of sets of records as predicates too? First of all, we must ask what is being predicated by a set of records. The answer seems clear: sets of individuals. Hence, a set of records should be considered a predicate over sets of individuals or, put succinctly, a second order predicate.

This seems to justify a leap into second order logic for a description of powerdomains. We expect to find that the different powerdomain orderings give rise to different second order predicates. However, a *first* order formula may be considered a second order predicate if it contains a unary predicate symbol. Suppose we are given a distinguished unary predicate symbol W and a collection of predicate symbols U. In a given model, a formula like $U(x)$ might be asserting that x is a two year old. With this interpretation, a first order formula such as

$$\phi \equiv \forall x.\ W(x) \to U(x)$$

asserts that everyone in the interpretation of W is a two year old. Hence ϕ itself becomes a predicate of W. Of course, there will be many predicates defined by first order formulas in this way, but which of them (if any) correspond to the elements of the powerdomains?

[3] It will not always be intuitively reasonable to view things in this way, although it works well for the example at hand.

Let us attempt to work out an example similar to those in the previous section. Recall the set s of records:

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John" },
  age = 2 }
```

Let M, T and J be unary predicate symbols for having first name "Mary", "Todd" and "John" respectively. Under the lower powerdomain ordering, what is this collection of records telling us about the set of children in our hypothetical nursery? The first record of s seems to assert that *there is a child named "Mary" in the nursery.* If W is a predicate symbol which we are interpreting as the children in the nursery, this can be represented by the formula

$$\exists x.\ W(x) \wedge M(x)$$

which we may express more succinctly as $W \cap M \neq \emptyset$. Actually, the first record expresses a bit more than this. Let O be a predicate which is being interpreted as the set of all two year olds. Then the first record says: $W \cap M \cap O \neq \emptyset$. In summary, s corresponds to the following proposition:

$$\begin{aligned} &W \cap (M \cap O) \neq \emptyset \wedge \\ &W \cap (T \cap O) \neq \emptyset \wedge \\ &W \cap (J \cap O) \neq \emptyset \end{aligned}$$

As an exercise, the reader may express s_1 in this way and show that the resulting proposition implies the one above.

Now, what about the upper powerdomain ordering? Under this ordering, each record expresses a range of possibilities. The three records together assert that the children of the nursery (or those on its waiting list if that is preferred interpretation) are all named "Mary", "Todd" or "John". More specifically, a child on the waiting list must be a two year old "Mary", a two year old "Todd" or a two year old "John". However, this does not preclude the possibility that there is no "Todd" who is actually waiting for entry. If W is a new unary predicate symbol to be interpreted as the individuals in the nursery, then this assertion may be summarized as

$$\forall x.\ W(x) \rightarrow \theta \tag{1}$$

where θ is the disjunction

$$(M(x) \wedge O(x)) \vee (T(x) \wedge O(x)) \vee (J(x) \wedge O(x)).$$

The formula (1) may also be expressed with set-theoretic notation:

$$W \subseteq (M \cap O) \cup (T \cap O) \cup (J \cap O).$$

Again, the reader may find it instructive to express s_2 in this way and check that the resulting proposition implies this one.

It is tempting, at this point, to "think semantically" and try to view the powerdomains in terms of sets of individuals. This can be misleading, however. Given a predicate symbol U, let $[\![U]\!]$ be the interpretation of U in a fixed model. In particular, for the upper ordering, we may have

$$[\![U_1]\!] \cup \cdots \cup [\![U_m]\!] = [\![V_1]\!] \cup \cdots \cup [\![V_n]\!]$$

without it being the case that the $[\![U_i]\!] \subseteq [\![V_j]\!]$ or $[\![V_j]\!] \subseteq [\![U_i]\!]$ for *any* pair of predicate symbols U_i and V_j. It seems, therefore, that although the formulas

$$\phi \equiv W \subseteq U_1 \cup \cdots \cup U_n$$

and

$$\psi \equiv W \subseteq V_1 \cup \cdots \cup V_m$$

define the same family of predicates, this does not follow from the ordering under inclusion of the sets $[\![U]\!]$ for unary predicate symbols U of the language. For a *fixed* model, the interpretations of the predicates $\phi(W)$ and $\psi(W)$ may have more relationships than one can "obtain" from the ordering of the sets $[\![U]\!]$. One may place some *ad hoc* assumptions on the model to make things work out better. However, the treatment which I provide below uses *non-standard models* to hide this problem.

To crystalize this discussion by proving some theorems, it is necessary to be somewhat more formal about the ground rules. Some notation is helpful. Fix a first order language $\mathcal{L}$ of unary predicate symbols and a set T of formulas of the form $U \subseteq V$ where U and V are unary predicates in the language. Given a set of formulas Φ, the theory T *induces* a preordering on the formulas of Φ by provable implication. In other words, the induced preorder has, as its elements, formulas $\phi \in \Phi$ and it is preordered by taking $\phi \gtrsim \phi'$ iff $T \vdash \phi \to \phi'$. For the remainder of this paper, fix the theory T and assume that W is a new unary predicate symbol not in the language of T. It will simplify matters to assume that $U \subseteq V$ is in T whenever $T \vdash U \subseteq V$. Let A be the preorder which T induces on formulas of the form $U(x)$ where U is a unary predicate symbol of $\mathcal{L}$. Then we have the following:

Theorem 1 *The preorder which T induces on formulas of the form*

$$W \subseteq U_1 \cup \cdots \cup U_n$$

is exactly the upper powerdomain of A.

Proof: Suppose we are given formulas

$$\phi \equiv W \subseteq U_1 \cup \cdots \cup U_n$$
$$\psi \equiv W \subseteq V_1 \cup \cdots \cup V_m$$

It is not at all difficult to see that if, for each predicate U_i, there is predicate V_j such that $U_i \subseteq V_j$ is in the theory T, then

$$T \vdash \phi \to \psi.$$

What is less obvious is the fact that this is *the only way* such an implication can be proved. Suppose we know that $T \vdash \phi \rightarrow \psi$. By the Soundness Theorem for First Order Logic, we know that

$$T \models \phi \rightarrow \psi \tag{2}$$

Suppose that (2) holds, but there is a predicate U_i such that $U_i \subseteq V_j$ is not in T for any V_j. We demonstrate a contradiction. Define a model $\mathcal{A}$ of $T \cup \{\phi\}$ as follows. The universe of $\mathcal{A}$ is the set of predicate symbols of $\mathcal{L}$ (this does not include W). If U is a predicate symbol of $\mathcal{L}$, it is interpreted in $\mathcal{A}$ as the set of predicate symbols $V \in \mathcal{L}$ such that $U \subseteq V$ is in T. The predicate symbol W is interpreted as the set $\{U_1, \ldots, U_n\}$. Let $[\![U]\!]$ be our notation for the interpretation of a predicate symbol U. I claim that $\mathcal{A} \models T \cup \{\phi\}$. If $U \subseteq V$ is in T and $U' \in [\![U]\!]$, then $U' \subseteq U$ is in T so $U' \subseteq V$ is in T. Thus $U' \in [\![V]\!]$ and it follows that $[\![U]\!] \subseteq [\![V]\!]$ as desired. That $\mathcal{A} \models \phi$ follows immediately from the interpretation of W. On the other hand, I also claim that $\mathcal{A} \not\models \psi$. Since there is no V_j such that $U_i \subseteq V_j$ is in T, the element U_i is not in $[\![V_1]\!] \cup \cdots \cup [\![V_m]\!]$ and therefore $W \not\subseteq [\![V_1]\!] \cup \cdots \cup [\![V_m]\!]$. ▮

Theorem 2 *The preorder which T induces on formulas of the form*

$$(W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)$$

is exactly the lower powerdomain of A.

Proof: Define formulas

$$\phi' \equiv (W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)$$
$$\psi' \equiv (W \cap V_1 \neq \emptyset) \wedge \cdots \wedge (W \cap V_m \neq \emptyset)$$

If, for each V_j there is a predicate U_i such that $U_i \subseteq V_j$ is in T, then it is easy to show that

$$T \vdash \phi' \rightarrow \psi'$$

Conversely, if this holds then we also have

$$T \models \phi' \rightarrow \psi' \tag{3}$$

Suppose that (3) holds, but there is a predicate V_j such that $U_i \subseteq V_j$ is not in T for any U_i. I will demonstrate a contradiction. Let $\mathcal{A}$ be the model of T given in the proof of Theorem 1. Obviously $\mathcal{A} \models \phi'$. However, $[\![V_j]\!] \cap [\![W]\!]$ is the emptyset since there is no U_i in $[\![V_j]\!]$. ▮

4 Other powerdomains?

In this section I will look at a few more second order predicates such as the ones which were used to characterize the upper and lower powerdomains in the previous section. I begin by discussing the convex ordering and its information-theoretic significance using sets of records. A logical characterization of the convex powerdomain is then provided and a correspondence theorem similar to Theorems 1 and 2 will be given. I will then define a close relative of the *sandwich powerdomain* of Buneman, Davidson, Ohori and Watters [BDW88, BO86, BJO89] which has been used used for the semantics of databases.

Under the convex ordering, *none* of the three sets of records s, s_1, s_2 given earlier are related. The following set s_3 satisfies $s_3 \gtrsim^{\natural} s$:

```
{ name = { first = "Mary" },
  socsec = 4392349703,
  age = 2 }

{ name = { first = "Todd",
           last = "Smith" },
  socsec = 923799210,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 429238406,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2 }
```

Note that no new names were added in s_3 as we added the name "Beth" in s_1 (although the two John Smith's were disambiguated), and no names were removed from s as we removed "Todd" in s_2. On the other hand, the records of s_3 are considerably more specific than those in s. For example, if we assume that now two children have the same social security number, then no further refinement of s_3 will have more or less than four children. (However, sets with multiple names associated with the same social security number are permitted in the convex powerdomain.) As with the other powerdomains, it is easy to produce examples which show that the convex powerdomain of a poset may not satisfy anti-symmetry. The following can be proved by combining the proofs of Theorems 1 and 2:

Theorem 3 *The preorder which T induces on formulas of the form*

$$\begin{array}{l}(W \subseteq U_1 \cup \cdots \cup U_n) \wedge \\ (W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)\end{array}$$

is exactly the convex powerdomain of A. ▌

The convex powerdomain is generally considered to be more "natural" than the upper and lower powerdomains; this view is supported, for example, by the categorical characterizations of the three powerdomains [HP79, GS88] as well as considerations from the semantics of concurrency. However, when one views the three powerdomains from the standpoint of this paper, the convex powerdomain seems to entail a peculiar assumption. Each of the records in a database under the convex ordering must convey *both* upper and lower information; or, to put it another way, the upper and lower information conveyed by the database must be conveyed by the *same* set of predicates. We are permitted to use formulas of the form

$$\begin{aligned}&(W \subseteq U_1 \cup \cdots \cup U_n) \wedge \\ &(W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)\end{aligned} \tag{4}$$

but not formulas of the the more general form

$$\begin{aligned}&(W \subseteq U_1 \cup \cdots \cup U_m) \wedge \\ &(W \cap U_1' \neq \emptyset) \wedge \cdots \wedge (W \cap U_n' \neq \emptyset)\end{aligned} \tag{5}$$

While it makes perfectly good sense to make a restriction to formulas as in (4), it also seems reasonable, in some circumstances, *not* to make this restriction. The use of formulas such as those in (5) in the theory of databases has been discussed in several publications [BDW88, BO86, BJO89] using an operator known as the *sandwiches powerdomain*. Although questions about the categorical and topological significance of sandwiches are only beginning to be investigated, their information-theoretic significance and potential applications suggest interesting lines of investigation. I now define an operator which has a strong kinship to the sandwiches domain and demonstrate a logical characterization for it.

Definition: Let $\langle A, \gtrsim \rangle$ be a preorder. A *mix* (on A) is a pair $(u, v) \in \mathcal{P}^*_f A \times \mathcal{P}^*_f A$ such that $v \gtrsim^\sharp u$. We define the *mixed powerdomain* $\mathcal{M}^0 A$ to be the set of mixes on A under the pre-order given by taking $(u, v) \succsim (u', v')$ iff $u \gtrsim^\sharp u'$ and $v \gtrsim^\flat v'$. As with other preorders, we write $x \precsim y$ if $y \succsim x$. We also write $x \approx y$ if $x \precsim y$ and $x \succsim y$. ▮

As aside on uniformity of notation, we might have written $A^{(\sharp,\flat)}$ for $\mathcal{M}^0 A$ by analogy to the notations $A^\sharp$ and $A^\flat$. Similarly, $\gtrsim^{(\sharp,\flat)}$ would be another possible notation for $\succsim$. The reason for the superscript 0 on the symbol $\mathcal{M}$ will be explained in section 5.

The choice of preordering on the pairs $(u, v) \in \mathcal{M}^0 A$ is unsurprising. It is slightly less clear why only pairs (u, v) with $v \gtrsim^\sharp u$ are used. To understand this restriction and get a feeling for the mixed powerdomain, it is best to look at some examples. Rather than representing elements of the mixed powerdomain with a pair of sets of records it is convenient to write a set of records which are *tagged* to indicate whether they belong in the first or second coordinate of the pair. I will use a tag # for the records in the first coordinate (since this looks like the $\sharp$ sign) and a tag b for records in the second coordinate (since this looks like a $\flat$ sign). Forget, for the moment, about the condition that $v \gtrsim^\sharp u$ and consider the following set of (tagged) records t:

```
b{ name = { first = "Mary" } }

b{ name = { first = "Todd" } }

b{ name = { first = "John" } }

#{ age = 2 }
```

This is very similar in information content to the set of records s which were considered earlier. It describes a group of two year olds which must include a "Mary", a "Todd" and a "John". Here is another set of records t_1 similar to s_1:

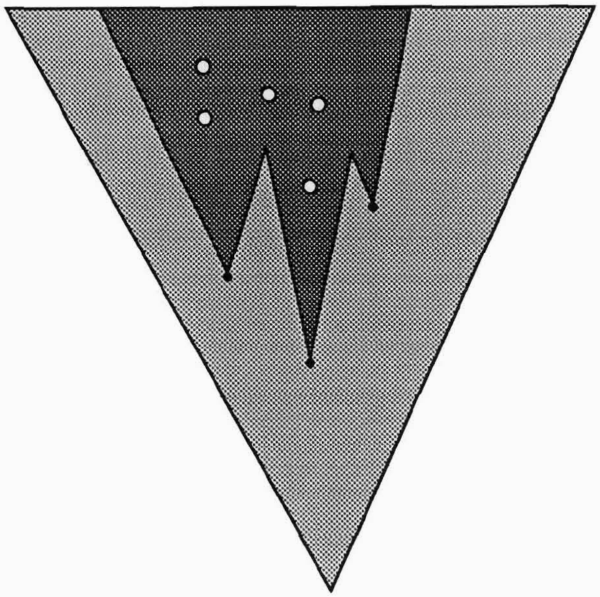


Figure 1: *A mixed powerdomain element* (u, v) *is illustrated above. The elements of the set* u *are indicated as closed circles (dots). They determine a shaded upper set within which the elements of* v *must lie. The elements of* v *are represented as open circles.*

```
b{ name = { first = "Mary" },
   age = 2 }

b{ name = { first = "Todd" },
   age = 2 }

b{ name = { first = "John",
            last = "Smith" },
   age = 2 }

b{ name = { first = "Beth" }
   age = 3 }

#{ age = 2 }

#{ age = 3 }
```

which allows that the nursery is now enrolling three year olds as well as two year olds. However, the following set of records is *nonsense:*

```
b{ name = { first = "Mary" },
   age = 2 }
```

```
b{ name = { first = "Todd" },
   age = 2 }

b{ name = { first = "John",
            last = "Smith" },
   age = 2 }

b{ name = { first = "Beth" }
   age = 3 }

#{ age = 2 }
```

because Beth is incorrectly recorded as a three year old or the new admissions policy has not be properly entered. In order for a set of mixed records such as these to make sense, it is essential that, for each ♭-record, there is a ♯-record which applies to it. Otherwise, the set of mixed records is "inconsistent." As another example, a dating service may have a database *d*:

```
b{ name = { first = "Sharon" },
   age = 26,
   married? = false}

b{ name = { first = "David" },
   age = 28,
   married? = false}

b{ name = { first = "Mabel" },
   age = 58,
   married? = false}

b{ name = { first = "Lee" },
   age = 55,
   married? = false}

#{ married? = false }
```

but trouble may arise from adding a record such as

```
b{ name = { first = "John" }
   age = 30,
   married? = true }
```

The sandwiches powerdomain is defined to include records like *t* above; *t* is not in the mixed powerdomain because the ♭-records are missing their age fields.

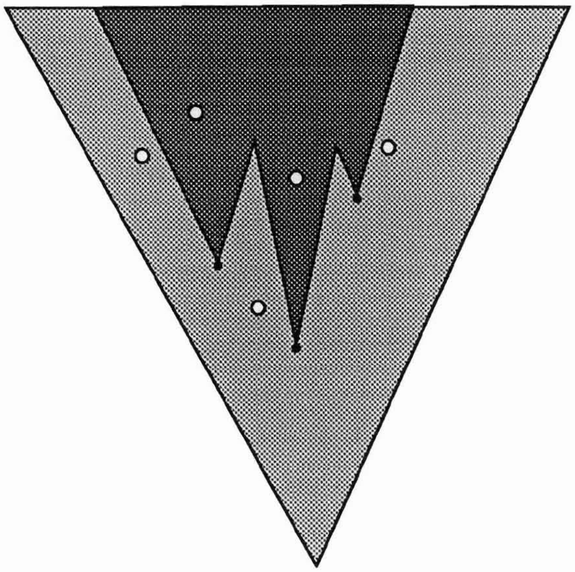


Figure 2: *A sandwich (u, v) is illustrated above. The elements of the set u are indicated as closed circles (dots). They determine a shaded upper set. The elements of v are represented as open circles; each element of v is required to have an upper bound in the shaded region.*

Definition: A *sandwich* is a pair

$$(u, v) \in \mathcal{P}_f^* A \times \mathcal{P}_f^* A$$

such that there is a set $w \in \mathcal{P}_f^* A$ such that $w \gtrsim^\sharp u$ and $w \gtrsim^\flat v$. The *sandwich powerdomain* of A is the set of sandwiches under the ordering $(u, v) \succsim (u', v')$ iff $u \gtrsim^\sharp u'$ and $v \gtrsim^\flat v'$

Obviously, any mix is a sandwich. Unfortunately, the logical interpretation of the sandwich powerdomain in the sense of this paper does not seem to be straight-forward.

To characterize the mixed powerdomain logically, it is necessary to generalize from formulas such as (4) to a set of formulas such as (5). It is easiest to do this for a subset of the mixes which is isomorphic to the mixed powerdomain. Suppose $(u, v) \in \mathcal{M}^0 A$. Since $u \gtrsim^\sharp v$, we have $u \cup v \cong u$. In particular, $(u \cup v, v)$ is a mix which is equivalent to (u, v). Since we are only really interested in the equivalence classes of mixes, we might therefore have included this condition our earlier definition of the mixed powerdomain. However, this would have complicated the examples slightly, since some elements would need to be listed twice.

Recall that T is a set of formulas of the form $U \subseteq V$ where U and V are unary predicates in a fixed first order language $\mathcal{L}$. A is the preorder which T induces on formulas of the form $U(x)$ where U is a unary predicate symbol of $\mathcal{L}$.

Theorem 4 *The preorder which T induces on formulas of the form*

$$(W \subseteq U_1 \cup \cdots \cup U_m \cup U_1' \cup \cdots \cup U_n') \wedge$$
$$(W \cap U_1' \neq \emptyset) \wedge \cdots \wedge (W \cap U_n' \neq \emptyset)$$

is isomorphic to the mixed powerdomain of A.

Proof: Suppose we have formulas

$$\begin{aligned}
\phi &\equiv W \subseteq U_1 \cup \cdots \cup U_m \cup U_1' \cup \cdots \cup U_n' \\
\phi' &\equiv (W \cap U_1' \neq \emptyset) \wedge \cdots \wedge (W \cap U_n' \neq \emptyset) \\
\psi &\equiv W \subseteq V_1 \cup \cdots \cup V_p \cup V_1' \cup \cdots \cup V_q' \\
\psi' &\equiv (W \cap V_1' \neq \emptyset) \wedge \cdots \wedge (W \cap V_q' \neq \emptyset)
\end{aligned}$$

and define

$$\begin{aligned}
\mathcal{U} &= \{U_1, \ldots, U_m\} \\
\mathcal{U}' &= \{U_1', \ldots, U_n'\} \\
\mathcal{V} &= \{V_1, \ldots, V_p\} \\
\mathcal{V}' &= \{V_1', \ldots, V_q'\}
\end{aligned}$$

We must show that

$$T \vdash (\phi \wedge \phi') \rightarrow (\psi \wedge \psi') \tag{6}$$

if and only if

1. for each predicate symbol $U \in \mathcal{U} \cup \mathcal{U}'$ there is a predicate symbol $V \in \mathcal{V} \cup \mathcal{V}'$ such that $U \subseteq V$ is in T, and

2. for each predicate symbol $V \in \mathcal{V}'$ there is a predicate symbol $U \in \mathcal{V}$ such that $U \subseteq V$ is in T, and

As with the earlier proofs of this kind, the harder part of the proof is showing that (6) implies items (1) and (2). As before, we utilize the Soundness Theorem to prove each of these items by contradiction. Define a model $\mathcal{A}$ of $T \cup \{\phi, \phi'\}$ as follows. The universe of $\mathcal{A}$ is the set of predicate symbols of $\mathcal{L}$ (this does not include the distinguished predicate symbol W). If U is a predicate symbol of $\mathcal{L}$, it is interpreted in $\mathcal{A}$ as the set of predicate symbols $V \in \mathcal{L}$ such that $U \subseteq V$ is in T. The predicate symbol W is interpreted as the set $\{U_1, \ldots, U_m, U_1', \ldots, U_n'\}$. That $\mathcal{A}$ is a model of $T \cup \{\phi, \phi'\}$ follows immediately from the definitions.

Now, suppose that (1) *fails*. Then there is some U such that

$$U \notin [\![V_1]\!] \cup \ldots \cup [\![V_p]\!] \cup [\![V_1']\!] \cup \ldots \cup [\![V_q']\!]$$

Since $U \in [\![W]\!]$, it follows that $[\![W]\!] \not\subseteq [\![V_1]\!] \cup \ldots \cup [\![V_p]\!] \cup [\![V_1']\!] \cup \ldots \cup [\![V_q']\!]$ and therefore $\mathcal{A}$ does not satisfy ψ. Suppose that (2) *fails*. Then there is some V_j' such that $U_i' \notin [\![V_j']\!]$ for each U_i'. To get the desired contradiction, we want to use a new model $\mathcal{A}'$ which is the same as $\mathcal{A}$ except $[\![W]\!] = U_1', \ldots, U_n'$. Clearly, $\mathcal{A}' \models T \cup \{\phi, \phi'\}$. But $[\![V_j']\!] \cap [\![W]\!] = \emptyset$ so $\mathcal{A}' \not\models \psi'$. ▮

5 The mixed powerdomain.

One is led to ask whether the mixed powerdomain, which has been motivated by a logical interpretation above, really enjoys the nice technical properties that the usual powerdomain operators do. The answer to this question is "yes" and it is the purpose of this section to provide a rigorous demonstration of this. I will try to keep the discussion as self-contained as possible, but the reader will need to know some domain theory in order to follow. Most of the necessary results can be found in [SP82], [Gun87] and [GS88].

The first step is to show how to define a continuous mixed powerdomain functor on the category of algebraic cpo's. This functor will take finite posets to finite posets, and hence it cuts down to a continuous functor on bifinite domains.[4] In the next section, we will look at algebraic cpo's that have a bottom element; this will be explicitly stated when it is needed.

Given a pre-order A and $x \subseteq A$, let $\downarrow x = \{b \in A \mid b \sqsubseteq a \text{ for some } a \in x\}$. A subset $x \subseteq A$ is an *ideal* if it is directed and $x = \downarrow x$. For a pre-order A, let $\mathsf{idl}(A)$ be the algebraic cpo of ideals on A, ordered by subset inclusion. For an algebraic cpo D, let KD be the basis of compact elements of D. The following lemma is a quite useful way to define functions between algebraic cpo's:

Lemma 5 *Let A be a preorder and suppose E is an algebraic cpo. If $f : A \to E$ is monotone, then there is a unique continuous function f' which completes the following diagram*

$$\begin{tikzcd}
A \arrow[d, "\downarrow(\cdot)"'] \arrow[dr, "f"] & \\
\mathsf{idl}(A) \arrow[r, dashed, "f'"] & E
\end{tikzcd}$$

In particular, a continuous function $f : D \to E$ between algebraic cpo's D and E is uniquely determined by its restriction to KD. ∎

Now, we define $\mathcal{M}D = \mathsf{idl}(\mathcal{M}^0(KD))$. If $f : D \to E$ is a continuous function between algebraic cpo's D and E, we define

$$\mathcal{M}(f)(x) = \{(u', v') \in \mathcal{M}^0(KE) \mid \exists (u, v) \in x.\ (f^*(u), f^*(v)) \succsim (u', v')\}$$

where $f^*(u)$ and $f^*(v)$ are the images of u and v respectively under the function f. We omit the proof that this makes sense and $\mathcal{M}(f)$ is a continuous function from $\mathcal{M}D$ into $\mathcal{M}E$. Suppose $g : E \to F$ is another continuous function.

$$\begin{aligned}
\mathcal{M}(g \circ f)(x) &= \{(u'', v'') \mid \exists (u, v) \in x.\ ((g \circ f)^*(u), (g \circ f)^*(v)) \succsim (u'', v'')\} \\
&= \{(u'', v'') \mid \exists (u, v) \in x.\ ((g \circ f)^*(u), (g \circ f)^*(v)) \succsim (u'', v'')\} \\
&= \{(u'', v'') \mid \exists (u, v) \in x.\ \exists (u', v'). \\
&\qquad\qquad (f^*(u), f^*(v)) \succsim (u', v') \text{ and } (g^*(u'), g^*(v')) \succsim (u, v)\} \\
&= \{(u'', v'') \mid \exists (u', v') \in \mathcal{M}(f)(x).\ ((g \circ f)^*(u), (g \circ f)^*(v)) \succsim (u'', v'')\} \\
&= \mathcal{M}(g)(\mathcal{M}(f)(x))
\end{aligned}$$

Since $\mathcal{M}(\mathsf{id}_D) = \mathsf{id}_{\mathcal{M}(D)}$, where id_D and $\mathsf{id}_{\mathcal{M}(D)}$ are the identity maps on D and $\mathcal{M}(D)$ respectively, it follows that $\mathcal{M}$ is an endofunctor on the category of algebraic cpo's with continuous functions. Indeed, we have the following

[4] Bifinite domains are called "profinite domains" in [Gun87].

Lemma 6 *If $f \leq g$, then $\mathcal{M}(f) \leq \mathcal{M}(g)$. If M is a directed subset of $D \to E$, then $\mathcal{M}(\bigvee M) = \bigvee \mathcal{M}(M)$.* ▮

Hence, the methods discussed in [SP82] can be used to show that $\mathcal{M}$ defines a continuous functor on algebraic cpo's (with continuous functions or embeddings).

If a pre-order A satisfies the ascending chain condition (*i.e.* A has no infinite chains $x_0 \lnsim x_1 \lnsim x_2 \lnsim \cdots$), then $\mathsf{idl}(A) \cong A$. In particular, $\mathcal{M}(R) \cong \mathcal{M}^0(R)$ for the domain R of records discussed in Section 2.

Now that the mixed powerdomain has been established as a continuous functor on bifinite domains, we may conclude that it can be used in any of the domain equations that are used with other powerdomains. This leaves a shopping list of questions about its technical similarity with the other operators:

1. Is the mixed powerdomain also closed on smaller categories, such as the bounded complete algebraic cpo's (Scott domains)?
2. Are the other powerdomains embedded in the mixed powerdomain?
3. Can the mixed powerdomain be used for the semantics of non-deterministic programming languages?
4. How is the mixed powerdomain related to the sandwiches powerdomain?
5. Do any of the topological and freeness theorems that hold for the upper, lower and convex powerdomains have an analog that holds for the mixed powerdomain?

I will provide at least a partial answer to each of these questions.

Notation: Given a pre-order A and a subset $u \subseteq A$, define

$$\uparrow u = \{x \in A \mid x \gtrsim y \text{ for some } y \in u\}$$
$$\downarrow u = \{x \in A \mid x \lesssim y \text{ for some } y \in u\} \ ▮$$

Proposition 7 *If A is a finite pre-order, then $\mathcal{M}A$ is isomorphic to the poset MA of pairs (u, v) such that u is an upper set, v is a lower set and $v = \downarrow(u \cap v)$ under the ordering $(u, v) \precsim (u', v')$ iff $u \subseteq u'$ and $v \subseteq v'$.*

Proof: Given a mix (u, v), the pair $f((u, v)) = (\uparrow u, \downarrow v)$ is an element of MA. To see this, suppose $x \in \downarrow v$, then there is a $y \in v$ such that $x \lesssim y$. Since $u \lesssim^\sharp v$, we know that $y \in \uparrow u$ so $x \in \downarrow(\uparrow u \cap \downarrow v)$. Since $\uparrow u \cap \downarrow v$ is clearly a subset of $\downarrow v$, it follows that $\downarrow(\uparrow u \cap \downarrow v) = \downarrow v$. Now, given $(u, v) \in MA$, the pair $g((u, v)) = (u, \uparrow u \cap \downarrow v)$ is a mix. It is straight-forward to check that $f \circ g$ and $f \circ g$ are both identities. ▮

Perhaps the most striking property of the mixed powerdomain is an anomaly which it shares with the convex powerdomain: *it does not preserve the property of bounded completeness*. For example, the poset displayed in Figure 3 is bounded complete, but its mixed powerdomain is not.[5] This fact, and the close similarity between the logical characterizations of the mixed and convex powerdomains, lead one to ask

[5] The example is due to Peter Buneman.

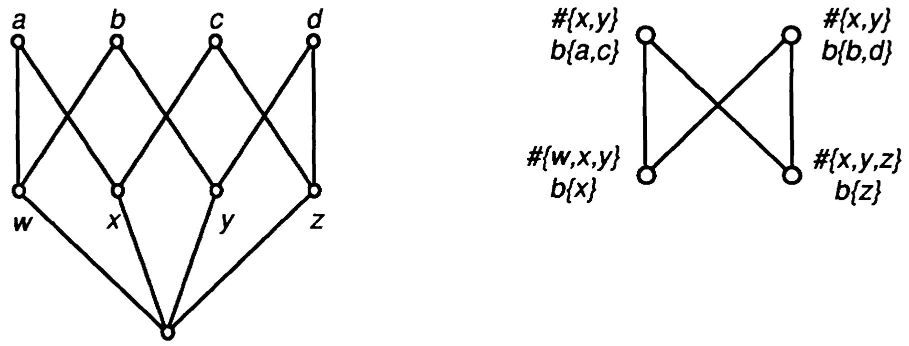


Figure 3: *The four elements indicated in the picture on the right show that the mixed powerdomain of the bounded complete domain pictured on the left is not bounded complete. The two mixes at the bottom have no least upper bound, since the two mixes indicated above them are minimal upper bounds.*

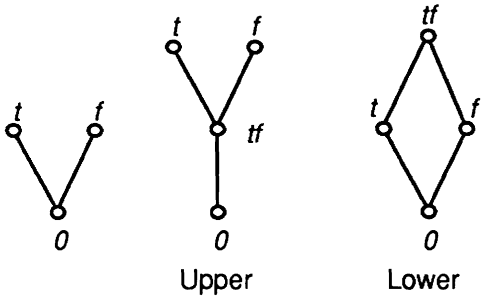
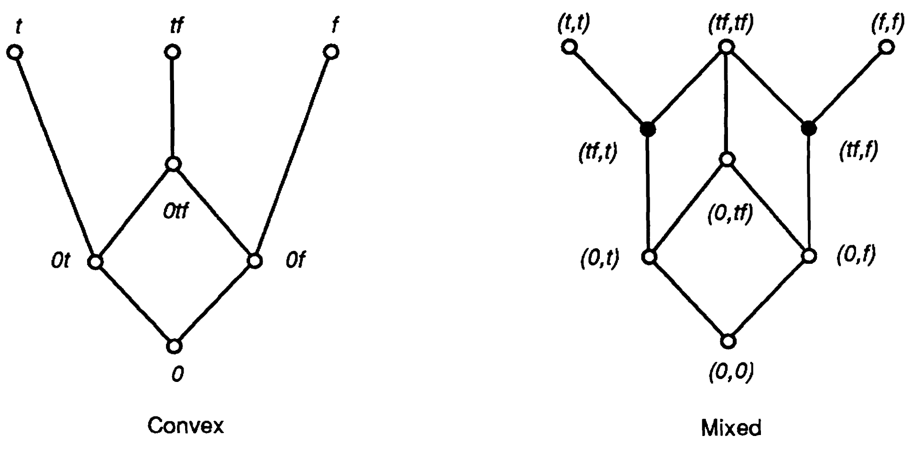


Figure 4: *Upper and lower powerdomains of the truth value cpo.*

whether these operators might actually be *isomorphic*. Given a preorder A, it is clear that there is a nice monotone map from $A^\natural$ into $\mathcal{M}^0 A$ defined by $u \mapsto (u, u)$. This map is an *order-embedding, i.e.* for each $u, v \in A^\natural$,

$$u \lesssim^\natural v \text{ iff } (u, u) \precsim (v, v).$$

Could this be an isomorphism? Let me attempt an intuitive answer to this question before actually providing a rigorous proof. Consider the *truth value cpo* T displayed in Figure 4. The upper and lower powerdomains of T are also displayed there with equivalent elements identified and representatives of the equivalence classes tagging the nodes. In the upper powerdomain, the set $\{t, f\}$ is a partial description with three possible "refinements". In the lower powerdomain, the set $\{t\}$ can be seen as a partial element with two possible refinements. The mixed powerdomain element $(\{t, f\}, \{t\})$ can be seen as a partial element which has three possible refinements, namely itself (which I'm counting as a refinement for now) and the total mixed elements $(\{t\}, \{t\})$ and $(\{t, f\}, \{t, f\})$. These last two elements are *total* descriptions of the sets $\{t\}$ and $\{t, f\}$. I claim that there is no counterpart to this partial description in the convex powerdomain. The only potential candidate is the set $\{0, t\}$, but the order-embedded element $(\{0, t\}, \{0, t\})$ is strictly less less than $(\{t, f\}, \{t\})$ in the mixed powerdomain and has the partial element $(0, \{t, f\})$ as a refinement, whereas this element is incomparable to $(\{t, f\}, \{t\})$. The reader may find it helpful to go through this discussion while looking at the Hasse diagrams of the convex and mixed



Convex

Mixed

Figure 5: *The convex and mixed powerdomains of the truth value cpo are not isomorphic. The order-embedded image of the convex powerdomain in the mixed powerdomain is indicated with open circles in the figure on the right. The two points not in the image of this order-embedding are indicated with closed circles.*

powerdomains of T which appear in Figure 5. In any case, it is clear from the pictures (which are my rigorous proof) that these posets are not isomorphic since the convex powerdomain has 7 elements whereas the mixed powerdomain has 9.

To use the mixed powerdomain for the semantics of programming languages, it is essential to define a collection of auxiliary functions such as those ordinarily associated with the powerset operation. There are two such operations which are of primary interest. The *mixed powerdomain union* is a function $\uplus : \mathcal{M}^0 A \times \mathcal{M}^0 A \to \mathcal{M}^0 A$. If $u = (u^\sharp, u^\flat)$ and $v = (v^\sharp, v^\flat)$ are elements of $\mathcal{M}^0 A$, their union is defined as follows:

$$u \uplus v = (u^\sharp \cup v^\sharp, u^\flat \cup v^\flat).$$

To see that this makes sense, we must first show that $u \uplus v$ is an element of the mixed powerdomain of A. Suppose $x \in u^\flat \cup v^\flat$. If $x \in u^\flat$, then there is an element $x' \in u^\sharp$ such that $x' \precsim x$ because $u^\sharp \precsim^\sharp u^\flat$. Since a similar fact holds for elements of $v^\flat$, it follows that $u \uplus v$ is indeed an element of $\mathcal{M}^0 A$. To see that the union is also monotone, suppose $v \succsim w$ for some $w \in \mathcal{M}^0 A$. To show that $u \uplus v \succsim u \uplus w$, we must show that

$$u^\sharp \cup v^\sharp \succsim^\sharp u^\sharp \cup w^\sharp \tag{7}$$

and

$$u^\flat \cup v^\flat \succsim^\flat u^\flat \cup w^\flat. \tag{8}$$

For the former inequation, suppose $x \in u^\sharp \cup w^\sharp$ We must show that there is an $x' \in u^\sharp \cup v^\sharp$ such that $x' \precsim x$. If $x \in u^\sharp$, then this is immediate since we can take $x' = x$. If $x \in w^\sharp$, then there is an $x' \in v^\sharp$ such that $x' \precsim x$ because of the mixed powerdomain ordering. This establishes inequation 7; a similar argument may be used to show inequation 8. To show the monotonicity of $\uplus$ in the other argument is also similar. The *mixed powerdomain singleton* is a function $\{\!|\cdot|\!\} : A \to \mathcal{M}^0 A$ given by taking

$$\{\!|x|\!\} = (\{x\}, \{x\})$$

for each $x \in A$. The proof that this is well defined and monotone is straight-forward. Of course, the union and singleton can be be defined as continuous functions

$$\begin{aligned} \uplus &: \mathcal{M}D \times \mathcal{M}D \to \mathcal{M}D \\ \{\!|\cdot|\!\} &: D \to \mathcal{M}D \end{aligned}$$

using Lemma 5 on the basis KD of D.

In some sense, the mixed powerdomain is "larger" than each of the upper, lower and convex powerdomains on those preorders A that have an element $\bot$ which satisfies $\bot \lesssim x$ for each $x \in A$. To see this, define $f : A^\sharp \to \mathcal{M}^0 A$ by $f(u) = (u, \{\bot\})$. This function extends to a continuous embedding from the upper powerdomain into the mixed powerdomain. Similarly, $g : A^\flat \to \mathcal{M}^0 A$ extends to a continuous embedding embedding from the upper powerdomain into the mixed powerdomain.

On the other hand, the mixed powerdomain of D may be viewed as a certain kind of open subset of the convex powerdomain of D. To see this, first recall the definition of the Scott topology on a cpo:

Definition: Let D be a cpo. A set $U \subseteq D$ is *Scott open* if

1. $U = \uparrow U$ and
2. whenever $M \subseteq D$ is directed and $\bigvee M \in U$, then $M \cap U \neq \emptyset$. ∎

The following is a basic fact about the poset of open subsets:

Lemma 8 *Let D be a algebraic cpo. Then the poset ΩD of Scott open subsets of D ordered by subset inclusion is an algebraic lattice such that $U \in KD$ iff there is a finite set $u \subseteq KD$ such that $U = \uparrow u$.* ∎

We can now express the desired embedding property:

Proposition 9 *There is an order-embedding from $\mathcal{M}D$ into the Scott open subsets of $\mathcal{C}D$.*

Proof: Let $f : \mathcal{M}^0(KD) \to K\Omega(\mathcal{C}D)$ be given by

$$f : (u, v) \mapsto \{w \mid (u, v) \precsim (w, w)\}$$

By Lemma 8, the value of f on a pair (u, v) is, in fact, a compact open subset because

$$f(u, v) = \uparrow\{w \mid v \subseteq w \subseteq u \cup v\}.$$

Moreover, f is clearly monotone. Now, suppose $f(u, v) = f(u', v')$. We must show that $(u, v) \approx (u', v')$. Since $(u \cup v, u \cup v) \in f(u', v')$, we must have $u' \lesssim^\sharp u \cup v$. But $u \cup v \lesssim^\sharp u$, so $u' \lesssim^\sharp u$. A similar argument shows that $u \lesssim^\sharp u'$. We also have $(v, v) \in f(u', v')$ so $v' \lesssim^\flat v$ and a similar argument for $v \lesssim^\flat v'$. By Lemma 5, the monotone function f extends to a unique continuous function $f' : \mathcal{M}D \to \Omega(\mathcal{C}D)$. The function f' will be an order-embedding since f is. ∎

6 Algebraic characterization of the mixed powerdomain.

One of the most challenging problems for new powerdomain constructions has been the discovery of the appropriate algebraic stuctures to capture the essential features of the new powerdomains. In the case of the convex powerdomain, the first intuitions—as described in [Plo76]—came from the semantics of parallel computation. Although [Plo76] describes all of the relevant algebraic operations, it was only later, in [HP79] that the convex powerdomain was characterized in terms of these operations together a simple set of equational axioms which they satisfy. At the same time, the upper and lower powerdomains were also thus characterized using the same algebraic signature but additional inequational axioms. Such characterizations are now treated as a standard element of the general methodology of semantics (see [Hen88] as an example of this). Unfortunately, no characterization of this kind has yet been found for the sandwiches powerdomain and this remains an open question. However, in this section, I will demonstrate an algebraic characterization of the mixed powerdomain in terms of a kind of structure called a *mix algebra.*

One problem which has arisen in the study of powerdomains for concurrency is how to derive a powerdomain which includes an *emptyset* element. This was missing from our earlier discussion where we always assumed that sets were non-empty. If we were to allow the emptyset as an element of the convex powerdomain, for example, we would have the problem that it is *unrelated* to other elements under the convex ordering. In particular, a powerdomain with emptyset would not have a least element. Given the importance of least elements for the solution of recursive equations, this straight-forward approach to adding an emptyset is unsatisfactory for the semantics of programming languages. The problem is generaly rectified by "adding the emepty set onto the side" of the convex powerdomain. So the emptyset element is related only to the least element. This approach seems acceptable in the sense that it makes a reasonable semantics possible, but it makes a mess of the algebraic characterization of the powerdomain. The simple problem is this, if we add an axiom which says that the least element is less than the empty element, the the least element is part of our signature and is therefore preserved by any of the homomorphisms which we construct. But this is not desirable since there may well be terms in the language whose intended interpretation is non-strict (*i.e.* does not send the least element of its domain to the least element of its range). In fact, it can be shown that this problem has no acceptable solution with the simplest signature and axioms; we sketch a proof of this impossibility later.

Another goal of this section is to show how the problems with the algebraic properties of powerdomains with emptyset can resolved by using the *mixed powerdomain with emptyset.* To anticipate the basic idea, consider the nature of the emptyset as a piece of partial information about a set. The information content of the emptyset as *upper* information is quite different from its significance as *lower* information. In the upper powerdomain ordering, the emptyset is totally descriptive—it means that the set being described has no members. On the other hand, in the lower powerdomain ordering, the emptyset is totally nondescriptive—it means that no element is known to be in the set being described. In the case that the underlying domain has a least element $\perp$, even the singleton set $\{\perp\}$ is more informative under the lower ordering than the emptyset. Now, in the mixed powerdomain with emptyset, the mix $(\{\perp\}, \emptyset)$ is the *least* element. Even the element $(\{\perp\}, \{\perp\})$ is more informative, since this latter element describes only *non-empty* sets! In the mixed powerdomain, the emptyset $(\emptyset, \emptyset)$ is a total (maximal) element which describes the unique set with no members.

A *mix algebra* (with unit) is a preorder N together with a monotone binary operation $* : N \times N \to N$, a monotone unary operation $\Box : N \to N$ and a constant $e \in N$ which satisfy the following nine axioms

1. associativity: $(r * s) * t \approx r * (s * t)$
2. commutativity: $r * s \approx s * r$
3. idempotence: $s * s \approx s$
4. unit: $e * s \approx s * e \approx s$
5. $\Box(s * r) \approx (\Box s) * (\Box r)$
6. $\Box\Box s \approx \Box s$
7. $s * (\Box s) \approx s$
8. $\Box s \lesssim s$
9. $s * (\Box r) \lesssim s$

A *homomorphism* between mix algebras M and N is a monotone function $f : M \to N$ such that $f(r{*}s) \approx (f(r)) * (f(s))$ and $f(\Box r) \approx \Box f(r)$ and $f(e) \approx e$. A *continuous* mix algebra is a mix algebra $\langle N, *, \Box, e\rangle$ where N is an algebraic cpo and $*$, $\Box$ are continuous. A homomorphism of continuous mix algebras is a continuous homomorphism of mix algebras.

As the reader is probably aware, the first four axioms are the axioms for a semi-lattice with unit. Given a mix algebra N, the binary operation $*$ on N induces a semi-lattice pre-ordering $\Subset$ given by $r \Subset s$ iff $r * s \approx s$. It is important not to confuse this subset ordering with the ordering $\lesssim$ of partial information since these orderings will rarely coincide. Note, in particular, that axiom (7) says that $\Box$ is a *closure operation* with respect to $\Subset$.

Definition: Let $\langle A, \gtrsim\rangle$ be a preorder. We define

$$\mathcal{M}^0_\emptyset A = \{(u,v) \in \mathcal{P}_f A \times \mathcal{P}_f A \mid v \gtrsim^\sharp u\}$$

and define $(u,v) \succsim (u',v')$ iff $u \gtrsim^\sharp u'$ and $v \gtrsim^\flat v'$. Let us refer to $\langle \mathcal{M}^0_\emptyset A, \succsim\rangle$ as the mixed powerdomain *with emptyset.* ▮

Given a domain D, define $\Box : \mathcal{M}^0_\emptyset(KD) \to \mathcal{M}^0_\emptyset(KD)$ as $\Box : (u,v) \mapsto (u,\emptyset)$. We show that $\langle KD, \uplus, \Box, (\emptyset,\emptyset)\rangle$ is a mix algebra. Axioms (1)-(4) are immediate consequences of the definition of $\uplus$. To prove (5), let (u,v) and (u',v') be elements of $\mathcal{M}^0_\emptyset(KD)$. Then $\Box((u,v) \uplus (u',v')) \approx \Box(u \cup u', v \cup v') \approx (u \cup u', \emptyset) \approx (u,\emptyset) \uplus (u',\emptyset) \approx \Box(u,v) \uplus \Box(u',v')$. Axiom (6) is immediate from the definition of $\Box$. For axiom (7), $(u,v) \uplus (\Box(u,v)) \approx (u \cup u, v \cup \emptyset) \approx (u,v)$. To see axiom (8), note that $\emptyset \lesssim^\flat v$ for any v. For axiom (9), $(u,v) \uplus (\Box(u',v')) \approx (u \cup u', v) \lesssim (u,v)$ since $u \cup u' \lesssim^\sharp u$.

From this proof that $\langle KD, \uplus, \Box, (\emptyset,\emptyset)\rangle$ is a mix algebra, it follows that $\langle D, \uplus, \Box, e\rangle$ is a continuous mix algebra, where $\uplus$ and $\Box$ are the unique continuous extensions of the corresponding operations on KD and e is the principal ideal generated by $(\emptyset,\emptyset)$.

Theorem 10 *Let A be a preorder. Suppose N is a mix algebra. For any monotone $f : A \to N$, there is a unique homomorphism $f^+ : \mathcal{M}^0_\emptyset A \to N$ which completes the following diagram:*

$$\begin{tikzcd}
A \arrow[d, "\{\!|\cdot|\!\}"'] \arrow[dr, "f"] & \\
\mathcal{M}^0_\emptyset A \arrow[r, dashed, "f^+"] & N
\end{tikzcd}$$

Proof: First of all, note that if (w, v) is a mix, then it is equivalent to a mix of the form $(u \cup v, v)$ where u and v are disjoint. If $u = \{a_1, \ldots, a_n\}$ and $v = \{b_1, \ldots, b_m\}$, then

$$(u, v) = \Box\{\!|a_1|\!\} \uplus \cdots \uplus \Box\{\!|a_n|\!\} \uplus \{\!|b_1|\!\} \uplus \ldots \uplus \{\!|b_m|\!\}.$$

Hence, if a homomorphism f^+ which completes the diagram exists, then

$$f^+(u \cup v, v) = \Box f(a_1) * \ldots * \Box f(a_n) * f(b_1) * \ldots * f(b_m).$$

Degenerate cases are defined as follows:

$$\begin{aligned}
&f^+(v, v) = f(b_1) * \ldots * f(b_m)\\
&f^+(u, \emptyset) = \Box f(a_1) * \ldots * \Box f(a_n)\\
&f^+(\emptyset, \emptyset) = e
\end{aligned}$$

We must show that f^+ is monotone and that it is a homomorphism.

First let us show that f^+ is indeed monotone. Suppose that $(u \cup v, v)$ and $(u' \cup v', v')$ are mixes where $u \cap v = \emptyset$ and $u' \cap v' = \emptyset$ and u, v, u', v' are all non-empty. Suppose that $u \cup v \lesssim^\sharp u' \cup v'$ and $v \lesssim^\flat v'$. Also define

$$\begin{aligned}
u &= \{a_1, \ldots, a_n\}\\
v &= \{b_1, \ldots, b_l\}\\
u' &= \{a'_1, \ldots, a'_k\}\\
v' &= \{b'_1, \ldots, b'_m\}
\end{aligned}$$

Then

$$\begin{aligned}
f^+(u \cup v, v) &= \Box f(a_1) * \cdots * \Box f(a_n) * f(b_1) * \cdots * f(b_l) \\
&\precsim \Box f(a_1) * \cdots * \Box f(a_n) * \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * f(b_1) * \cdots * f(b_l) && \text{by (7)} \\
&\precsim \Box f(a_1) * \cdots * \Box f(a_n) * \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * \Box f(b'_1) * \cdots * \Box f(b'_m) \\
&\quad * f(b_1) * \cdots * f(b_l) && \text{since } u \cup v \precsim^\sharp u' \cup v' \\
&\precsim \Box f(a_1) * \cdots * \Box f(a_n) * \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * f(b'_1) * \cdots * f(b'_m) \\
&\quad * f(b_1) * \cdots * f(b_l) && \text{by (8)} \\
&\precsim \Box f(a_1) * \cdots * \Box f(a_n) * \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * f(b'_1) * \cdots * f(b'_m) && \text{since } v \precsim^\flat v' \\
&\precsim \Box f(a'_1) * \cdots * \Box f(a'_k) * f(b'_1) * \cdots * f(b'_m) && \text{by (9)} \\
&= f^+(u' \cup v', v')
\end{aligned}$$

Since the emptyset plays a very special role in this construction, I will now write out the proof of monotonicy for the degenerate cases where at least one of u, v, u', v' is empty. Suppose $v = \{b_1, \ldots, b_l\} \neq \emptyset$ and $(u \cup v, v) \precsim (u' \cup v', v')$ as above. Then $v' \neq \emptyset$; say $v' = \{b'_1, \ldots, b'_m\}$. If $u = \emptyset$ and $u' = \{a'_1, \ldots, a'_k\} \neq \emptyset$, then

$$\begin{aligned}
f^+(v, v) &= \Box f(b_1) * \cdots * \Box f(b_l) * f(b_1) * \cdots * f(b_l) \\
&\precsim \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * \Box f(b'_1) * \cdots * \Box f(b'_m) \\
&\quad * f(b_1) * \cdots * f(b_l) && \text{since } v \precsim^\sharp u' \cup v' \\
&\precsim \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * f(b'_1) * \cdots * f(b'_m) \\
&\quad * f(b_1) * \cdots * f(b_l) && \text{by (8)} \\
&\precsim \Box f(b_1) * \cdots * \Box f(b_l) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * f(b'_1) * \cdots * f(b'_m) && \text{since } v \precsim^\flat v' \\
&\precsim \Box f(a'_1) * \cdots * \Box f(a'_k) * f(b'_1) * \cdots * f(b'_m) && \text{by (9)} \\
&= f^+(u' \cup v', v')
\end{aligned}$$

If $u \neq \emptyset$ and $u' = \emptyset$, then the argument is quite similar. Suppose v, v' are as above, but $u = u' = \emptyset$.

Then

$$\begin{aligned}
f^+(v,v) &= \Box f(b_1) * \cdots * \Box f(b_l) * f(b_1) * \cdots * f(b_l) \\
&\lesssim \Box f(b_1) * \cdots * \Box f(b_l) * f(b_1) * \cdots * f(b_l) \\
&\quad * \Box f(b'_1) * \cdots * \Box f(b'_m) && \text{since } v \lesssim^{\sharp} v' \\
&\lesssim \Box f(b_1) * \cdots * \Box f(b_l) * f(b_1) * \cdots * f(b_l) \\
&\quad * \Box f(b'_1) * \cdots * \Box f(b'_m) * f(b'_1) * \cdots * f(b'_m) && \text{by (8)} \\
&\lesssim f(b_1) * \cdots * f(b_l) \\
&\quad * \Box f(b'_1) * \cdots * \Box f(b'_m) * f(b'_1) * \cdots * f(b'_m) && \text{by (9)} \\
&\lesssim \Box f(b_1) * \cdots * \Box f(b'_m) * f(b'_1) * \cdots * f(b'_m) && \text{since } v \lesssim^{\flat} v' \\
&= f^+(v',v')
\end{aligned}$$

Now suppose $v = u' = v' = \emptyset$ and $u \neq \emptyset$, then $f^+(u,\emptyset) = \Box f(a_1) * \cdots * \Box f(a_n) * e \lesssim e = f^+(\emptyset,\emptyset)$ by (4) and (9). Suppose $v = \emptyset$ but u, u', v' are non-empty. Then

$$\begin{aligned}
f^+(u,\emptyset) &= \Box f(a_1) * \cdots * \Box f(a_n) \\
&\lesssim \Box f(a_1) * \cdots * \Box f(a_n) \\
&\quad * \Box f(a'_1) * \cdots * \Box f(a'_k) * \Box f(b'_1) * \cdots * \Box f(b'_m) && \text{since } u \lesssim^{\sharp} u' \cup v' \\
&\approx f^+(u' \cup v', v') && \text{by (8) and (9)}
\end{aligned}$$

The cases where $v' = \emptyset$ or $u' = \emptyset$ are very similar. If $u = v = \emptyset$, then $u = v' = \emptyset$ so there is nothing to prove in this case. This completes all of the degenerate cases so we may conclude that f^+ is indeed monotone.

To prove that f^+ is a homomorphism, suppose $(u \cup v, v)$ is a mix as above. Then

$$\begin{aligned}
\Box f^+(u \cup v, v) &= \Box(\Box f(a_1) * \cdots * \Box f(a_n) * f(b_1) * \cdots * f(b_l)) \\
&\approx \Box\Box f(a_1) * \cdots * \Box\Box f(a_n) * \Box f(b_1) * \cdots * \Box f(b_l) && \text{by (5)} \\
&\approx \Box f(a_1) * \cdots * \Box f(a_n) * \Box f(b_1) * \cdots * \Box f(b_l) && \text{by (6)} \\
&= f^+(u \cup v, \emptyset) \\
&= f^+(\Box(u \cup v, v))
\end{aligned}$$

For $u = v = \emptyset$, note that $\Box e \approx e * \Box e \approx e$ by (7) and (4). The other cases are all very similar to the proof above. ▮

Corollary 11 *Let D be an algebraic cpo. Suppose N is a continuous mix algebra. For any continuous $f : D \to N$, there is a unique homomorphism $f^+ : \mathcal{M}D \to N$ which completes the following diagram:*

$$\begin{tikzcd}
D \arrow[d, "\{\!|\cdot|\!\}"'] \arrow[dr, "f"] & \\
\mathcal{M}D \arrow[r, dashed, "f^+"] & N
\end{tikzcd}$$

Proof: Let f_0 be the restriction of f to KD. By Theorem 10, there is a homomorphism $f_0^+ : \mathcal{M}^0_\emptyset D \to N$ of mix algebras such that $f_0^+ \circ \{\!\!|\cdot|\!\!\} = f_0$. By Lemma 5, this homomorphism has a unique extension to a continuous function $f^+ : MD \to N$ which satisfies the desired diagram. This map will be a homomorphism. ▮

The reader familiar with category theory will naturally recognize that Corollary 11 can be restated as follows: *The forgetful functor from the category of continuous mix algebras and continuous homomorphisms to the category of algebraic cpo's and continuous functions is adjoint to the mixed powerdomain functor.* Several other results such as this are known for powerdomains. The most interesing of these is Theorem 12 below.

Definition: A *continuous semi-lattice* is an algebraic cpo N together with a binary operation $*$ which satisfies the axioms (1)–(3) for mix algebras. A *homomorphism* of continuous semi-lattices M, N is a continuous function $h : M \to N$ such that $h(r * s) \approx h(r) * h(s)$. ▮

The following Theorem is proved in [HP79]:

Theorem 12 (Hennessy and Plotkin) *The convex powerdomain is left adjoint to the forgetful functor from the category of continuous semi-lattices (with bottom) and homomorphisms to the category of algebraic cpo's (with bottom) and continuous functions.* ▮

Definition: A continuous semi-lattice *with unit* is a continuous semi-lattice with a constant e which satisfies axiom (4) for mix algebras. A homomorphism of continuous semi-lattices M, N with unit is a homomorphism of continuous semi-lattices which sends the unit of M to the unit of N. ▮

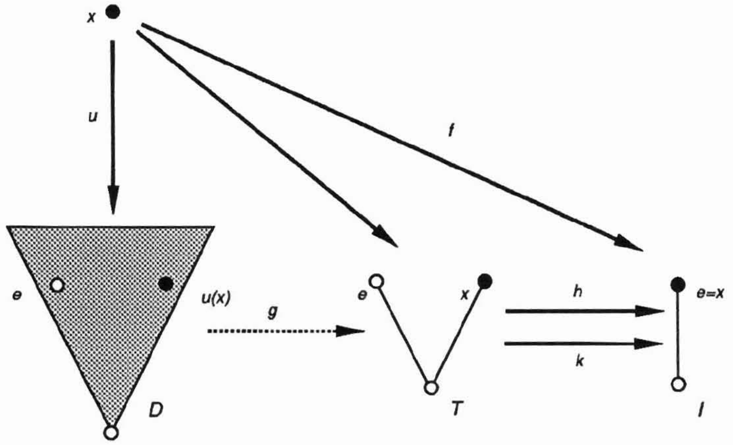
The following proposition appears as an exercise in [Plo82] (see Exercise 101 on page 52 of the chapter *Nondeterminism and Parallelism*):

Proposition 13 (Plotkin) *There is* no *left adjoint to the forgetful functor from continuous semi-lattices with unit and bottom to that of algebraic cpo's with bottom.* ▮

Proof: Suppose that there is a left adjoint to the forgetful functor and let D be the free continuous semi-lattice with unit generated by the poset $\{x\}$ with one element x. Let I be the semi-lattice with unit that has two elements $\bot, x$ with $\bot \lesssim x$ and $\bot * x = \bot$. Let f be the map from $\{x\}$ to I which sends x to $\bot$. We demonstrate a contradiction by showing that for no map u is there a unique homomorphism f^+ which completes the diagram

$$\begin{tikzcd}
\{x\} \arrow[d, "u"'] \arrow[dr, "f"] & \\
D \arrow[r, dashed, "f^+"] & I
\end{tikzcd}$$

Now, let T be the poset with three distinct elements $e, x, \bot$ with $\bot \precsim e$ and $\bot \precsim x$. This poset can be given the structure of a semi-lattice with unit e by defining $x * \bot = \bot$. Since D is freely generated by $\{x\}$, there is a homomorphism $g : D \to T$ which sends the image of x under u to $x \in T$. If e is the unit of D, then $g(e) = e$. Since g is monotone, this means $u(x)$ is incomparable to e in D and, consequently, $g(\bot) = \bot$. Now, consider the map $h : T \to I$ which sends the elements of T constantly to $e = x$ and the map $k : T \to I$ which sends $\bot$ to $\bot$. The situation can be pictured as follows:



Both of these maps are homomorphisms so we must have $f^+ = h \circ g = k \circ g$. But this is clearly false, since the two compositions are not equal. ▌

Proposition 14 *The mixed powerdomain is left adjoint to the forgetful functor from the category of continuous mix algebras with bottom to the category of algebraic cpo's with bottom.*

Proof: This is immediate from Corollary 11 since the mixed powerdomain of a algebraic cpo D has a least element given as the principal ideal generated by $(\{\bot\}, \emptyset)$. ▌

The big union function is the unique homomorphism $\biguplus$ which completes the following diagram

$$\begin{tikzcd}
\mathcal{M}D \arrow[d, "\{\!|\cdot|\!\}"'] \arrow[dr, "\mathsf{id}"] & \\
\mathcal{M}\mathcal{M}D \arrow[r, dashed, "\biguplus"] & \mathcal{M}D
\end{tikzcd}$$

where id is the identity function. It can be calculated as follows. Suppose $(U \cup V, V) \in \mathcal{M}^0_\emptyset(\mathcal{M}^0_\emptyset D)$ where

$$U = \{(u_1 \cup v_1, v_1), \ldots, (u_n \cup v_n, v_n)\}$$
$$V = \{(u'_1 \cup v'_1, v'_1), \ldots, (u'_m \cup v'_m, v'_m)\}$$

and $U \cap V = \emptyset$ and $u_i \cap v_i = \emptyset$ for $1 \leq i \leq n$ and $u'_j \cap v'_j = \emptyset$ for $1 \leq j \leq m$. Then

$$\begin{aligned}
\biguplus(U \cup V, V) &= \Box(u_1 \cup v_1, v_1) \uplus \cdots \uplus \Box(u_n \cup v_n, v_n) \uplus (u'_1 \cup v'_1, v'_1) \uplus \cdots \uplus (u'_m \cup v'_m, v'_m) \\
&\approx (u_1 \cup v_1, \emptyset) \uplus \cdots \uplus (u_n \cup v_n, \emptyset) \uplus (u'_1 \cup v'_1, v'_1) \uplus \cdots \uplus (u'_m \cup v'_m, v'_m) \\
&\approx (\textstyle\bigcup_{i \leq n} u_i \cup v_i \cup \bigcup_{j \leq m} u'_j \cup v'_j, \quad \bigcup_{j \leq m} v'_j)
\end{aligned}$$

The degenerate cases can be expressed similarly.

7 Mixes as a relation.

It is tempting to wonder how the mixed powerdomain of a domain D is related to the product $\mathcal{U}D \times \mathcal{L}D$ of its upper and lower powerdomains. It seems that it must be order isomorphic to some subset of this product, can we characterize the subset simply? What closure properties does it have (if any)? This section discusses the answers to these questions.

Let us extend our notation for the mixed powerdomain to the other powerdomains, by writing $\mathcal{U}D$, $\mathcal{L}D$, and $\mathcal{C}D$ for $\mathsf{idl}((KD)^\sharp)$, $\mathsf{idl}((KD)^\flat)$ and $\mathsf{idl}((KD)^\natural)$ respectively. Of course, given a pre-order A, we could also use the notations $\mathcal{U}^0 A$, $\mathcal{L}^0 A$ and $\mathcal{C}^0 A$ for $A^\sharp$, $A^\flat$ and $A^\natural$ respectively, but I will not do so in this paper.[6]

For any algebraic cpo D, there is an evident order-embedding $\iota : \mathcal{M}^0(KD) \to (KD)^\sharp \times (KD)^\flat$ given by the inclusion map. By Lemma 5, this map extends to an order-embedding from the mixed powerdomain of D into the product $\mathcal{U}D \times \mathcal{L}D$ of the upper and lower powerdomains. What do ideals in this order-embedded image look like? They are pairs (r, s) which are "generated" by mixes (u, v) such that $u \in r$ and $v \in s$. This is a binary relation $\bowtie$ on $\mathcal{U}D \times \mathcal{L}D$ which can be described succinctly as follows:

Definition: For an algebraic cpo D, define a binary relation $\bowtie$ on $\mathcal{U}D \times \mathcal{L}D$ by $r \bowtie s$ iff for every $u \in r$ and $v \in s$, there is a $v' \in s$ such that $u \lesssim^\sharp v'$ and $v \lesssim^\flat v'$. ▮

Recall the following definitions for the singleton and union operations on the upper and lower powerdomains:

$$
\begin{aligned}
\{\!|x|\!\}^\sharp &= \{u \mid \exists a \in u.\ a \in x\} \\
\{\!|x|\!\}^\flat &= \{u \mid \forall a \in u.\ a \in x\} \\
r \uplus^\sharp s &= \{w \mid u \cup v \gtrsim^\sharp w \text{ for some } u \in r \text{ and } v \in s\} \\
r \uplus^\flat s &= \{w \mid u \cup v \gtrsim^\flat w \text{ for some } u \in r \text{ and } v \in s\}
\end{aligned}
$$

In general, I will put in the superscripts only to emphasize the types of the operations—usually the type will be clear from the context.

Theorem 15 *Let D and E be a algebraic cpo's. For any continuous $f : D \to \mathcal{U}E$, the map $\hat{f} : \mathcal{U}D \to \mathcal{U}E$ given by*

$$\hat{f}(r) = \bigsqcup \{f(a_1) \uplus^\sharp \cdots \uplus^\sharp f(a_n) \mid \{a_1, \ldots, a_n\} \in r\}$$

is the unique continuous, $\uplus^\sharp$-preserving function which completes the following diagram:

[6] Warning: this is at variance with the notation in [GS88] where, for example, $D^\sharp$ is used for $\mathcal{U}D$. The depth in which notation is being used in this paper demands a more discriminating use of symbols for the various operators than was needed in [GS88].

$$\begin{tikzcd}
D \arrow[d, "\{\!|\cdot|\!\}"'] \arrow[dr, "f"] & \\
\mathcal{U}D \arrow[r, dashed, "\hat{f}"] & \mathcal{U}E
\end{tikzcd}$$

Theorem 16 *Let D and E be a algebraic cpo's. For any continuous $f : D \to \mathcal{L}E$, the map $\check{f} : \mathcal{L}D \to \mathcal{L}E$ given by*

$$\check{f}(r) = \bigcup\{f(a_1) \uplus^\flat \cdots \uplus^\flat f(a_n) \mid \{a_1, \ldots, a_n\} \in r\}$$

is the unique continuous, $\uplus^\flat$-preserving function which completes the following diagram:

$$\begin{tikzcd}
D \arrow[d, "\{\!|\cdot|\!\}"'] \arrow[dr, "f"] & \\
\mathcal{L}D \arrow[r, dashed, "\check{f}"] & \mathcal{L}E
\end{tikzcd}$$

Proposition 17 *Let D and E be an algebraic cpos. Then*

1. $\{\!|x|\!\}^\sharp \bowtie \{\!|x|\!\}^\flat$ *for each* $x \in D$.

2. *if* $r, r' \in \mathcal{U}D$ *and* $s, s' \in \mathcal{L}D$ *and* $r \bowtie r'$ *and* $s \bowtie s'$, *then* $r \uplus^\sharp r' \bowtie s \uplus^\flat s'$.

3. *if* $f : D \to \mathcal{U}E$ *and* $g : D \to \mathcal{L}E$ *satisfy* $f(x) \bowtie g(x)$ *for each* $x \in D$, *then* $\hat{f}(r) \bowtie \check{g}(s)$ *whenever* $r \bowtie s$. ▌

Proof: 1. Let $w = (u \cap x) \cup v$. Since $w \subseteq x$ and x is an ideal, there is an $a \in x$ such that $b \lesssim a$ for each $b \in w$. Now $\{a\} \in \{\!|x|\!\}^\flat$ and $u \lesssim^\sharp \{a\}$ and $v \lesssim^\flat \{a\}$.

2. Suppose $u \in r \uplus^\sharp r'$ and $v \in s \uplus^\flat s'$. By the definitions of the union operations in the repective powerdomains, there are elements $p \in r$, $p' \in r'$, $q \in s$, and $q' \in s'$ such that $u \lesssim^\sharp p \cup p'$ and $v \lesssim^\flat q \cup q'$. Since $r \bowtie s$, there is a $w \in s$ such that $p \lesssim^\sharp w$ and $q \lesssim^\flat w$. Similarly, since $r' \bowtie s'$, there is a $w' \in s'$ such that $p' \lesssim^\sharp w'$ and $q' \lesssim^\flat w'$. Now, $p \cup p' \lesssim^\sharp w \cup w'$ and $q \cup q' \lesssim^\flat w \cup w'$ so $u \lesssim^\sharp w \cup w'$ and $v \lesssim^\flat w \cup w'$. Since $w \cup w' \in s \uplus s'$, we are done.

3. Let $u \in \hat{f}(r)$ and $v \in \check{g}(s)$. By the definitions of $\hat{f}$ and $\check{g}$, there are sets $u' = \{a_1, \ldots, a_n\} \in r$ and $v' = \{b_1, \ldots, b_m\} \in s$ such that

$$u \in f(a_1) \uplus^\sharp \ldots \uplus^\sharp f(a)n)$$
$$v \in g(b_1) \uplus^\flat \ldots \uplus^\flat g(b_m)$$

Since $r \bowtie s$, there is a $w = \{c_1, \dots, c_l\} \in s$ such that $u' \lesssim^\sharp w$ and $v' \lesssim^\flat w$. Thus $u \in r' = f(c_1) \uplus^\sharp \dots \uplus^\sharp f(c_l)$ and $v \in s' = g(c_1) \uplus^\flat \dots \uplus^\flat g(c_l)$. But $f(c_i) \bowtie g(c_i)$ for each $i \leq l$, so $r' \bowtie s'$ by part (2) above. Now $u \in r'$ and $v \in s'$ so there is some $w' \in s'$ such that $u \lesssim^\sharp w'$ and $v \lesssim^\flat w'$. But $s' \subseteq \check{g}(s)$ so we are done. ▮

8 Normal forms and Stone duality.

An approach to the logical characterization of powerdomains—different in several regards from the one which was proposed earlier in this paper—utilizes the fact that domains have a topological structure. If we view an open subset of a domain as a "property", then the fact that a domain has a T_0 (Scott) topology tells us that each element is uniquely determined by its properties. Now, the opens of a topological space form a complete Heyting algebra so they support a logic of their own and any operator on domains lifts uniquely to a corresponding operator on an appropriate class of Heyting algebras (*viz.* those which correspond to the Scott topologies of domains). A characterization of this related operator on (appropriate) Heyting algebras therefore represents a very reasonable candidate for a logical characterization of the operator in question.

Many of the basic ideas we need below are already present in the basic literature on Stone duality (*e.g.* [Joh82]). For the specific case of of operators on domains, Stone duality properties have been studied by a number of individuals. A thorough exploration appears in recent work of Abramsky [Abr88, Abr87, Abr89]. The goal of this section is to show how the theory in [Abr89] applies to the mixed powerdomain. The idea is a quite simple variation on the techniques which deal with the convex powerdomain—the mixed powerdomain arises from simply dropping one of the axioms! The results in this section were predicted by Samson Abramsky and Steve Vickers and the proofs follow closely those which appear in [Abr89].

Definition: A *coherent algebraic prelocale* is a structure $A = \langle |A|, \lesssim, 0, \vee, 1, \wedge, P \rangle$ where

- $|A|$ is a countable set
- $\lesssim$ is a binary relation on $|A|$
- 0 and 1 are elements of $|A|$
- $\vee$ and $\wedge$ are binary relations on $|A|$
- P is a unary predicate

such that the following axioms and rules are satisfied

$$\begin{array}{lllll}
(d1) & a \precsim a & \dfrac{a \precsim b \quad b \precsim c}{a \precsim c} & & \\
(d2) & 0 \precsim a & \dfrac{a \precsim c \quad b \precsim c}{a \wedge b \precsim c} & a \precsim a \wedge b & b \precsim a \wedge b \\
(d3) & a \precsim 1 & \dfrac{a \precsim b \quad a \precsim c}{a \precsim b \vee c} & a \vee b \precsim a & a \vee b \precsim b \\
(d4) & \multicolumn{4}{l}{a \wedge (b \vee c) \precsim (a \wedge b) \vee (a \wedge c)} \\
(p1) & \multicolumn{4}{l}{\dfrac{P(a) \quad a \approx b}{P(b)}} \\
(p2) & \multicolumn{4}{l}{\text{If } P(a) \text{ and } a \precsim \bigvee_{i \in I} b_i \text{ then } \exists i \in I.\ a \precsim b_i} \\
(p3) & \multicolumn{4}{l}{\forall a \in |A|.\ \exists b_1, \ldots, b_n \in P(A).\ a \approx \bigvee_{i=1}^{n} b_i}
\end{array}$$

where $a \approx b$ iff $a \precsim b$ and $b \precsim a$. ∎

Axiom (p2) says that elements satisfying the predicate P are primes:

Definition: An element x of a lattice L is said to be a *prime* if, whenever $x \leq \bigvee_{i=0}^{n} x_i$, there is some i such that $x \leq x_i$. ∎

The following is part of the proof of Theorem 3.2.6 in [Abr89]:

Theorem 18 *If D is a bifinite domain, then the structure*

$$\mathbf{Ploc}(D) = \langle K\Omega(D),\ \subseteq,\ \emptyset,\ \cup,\ D,\ \cap,\ \{\uparrow x \mid x \in KD\}\rangle$$

is a prelocale. ∎

Let us refer to $\mathbf{Ploc}(D)$ as *the prelocale determined by D*. Recall from Lemma 8, that an element of $K\Omega(D)$ is the upper set $\uparrow u$ of a finite set u of compact elements of D.

Our goal can now be described as follows. Let D be a bifinite domain and suppose $A = \mathbf{Ploc}(D)$ is the prelocale which it determines. We will define an operator $\mathbf{M}$ on prelocales such that $\mathbf{M}(A)$ is the prelocale determined by the mixed powerdomain of D. This will be done, by defining a carrier $|\mathbf{M}(A)|$ and establishing a set of axioms and rules for the order relation $\precsim$ and primality relation P on this carrier. To this end, we will use the following:

Notation: For a set S, the *prelocalic expressions over S* are defined as follows. Any element of S is a prelocalic expression over S. Constants 0 and 1 are prelocalic expressions over S. If ϕ and ψ are prelocalic expressions over S then so are $\phi \vee \psi$ and $\phi \wedge \psi$. ∎

Let A be a given prelocale. The carrier $|\mathbf{M}(A)|$ is defined to be the set of prelocalic expressions over $S \cup T$ where

$$S = \{\Box\phi \mid \phi \text{ is a prelocalic expression over } |A|\}$$
$$T = \{\Diamond\phi \mid \phi \text{ is a prelocalic expression over } |A|\}$$

Define $\precsim$ and P to be the least relations over $|\mathbf{M}(A)|$ which satisfy (d1)–(d4), (p1) and the following axioms:

$$\begin{array}{ll}
(\Box - \wedge) & \Box(a \wedge b) \approx \Box a \wedge \Box b \\
(\Diamond - \vee) & \Diamond(a \vee b) \approx \Diamond a \vee \Diamond b \\
(\Diamond - \wedge) & \Box a \wedge \Diamond b \precsim \Diamond(a \wedge b) \\
(\Box - 0) & \Box 0 \approx 0 \\
(\Diamond - 1) & \Diamond 1 \approx 1
\end{array}$$

and rules

$$(\Box\!-\precsim)\quad \frac{a \precsim b}{\Box a \precsim \Box b} \qquad\qquad (\Diamond\!-\precsim)\quad \frac{a \precsim b}{\Diamond a \precsim \Diamond b}$$

$$(P - \Box - \Diamond)\quad \frac{\{P(a_i) \mid i \in I\}}{P(\Box \bigvee_{i \in I} a_i \wedge \bigwedge_{j \in J} \Diamond a_j)}$$

where $\emptyset \neq J \subseteq I$. Adding another axiom

$$(\Box - \vee) \qquad \Box(a \vee b) \precsim \Box a \vee \Diamond b$$

to the axioms above yields the analogous construction for the convex powerdomain. Omitting this axiom weakens the facts that one can prove about normal forms in the calculus and thus generalizes from the convex to the mixed powerdomain.

Let D be a bifinite domain and let $A = \mathbf{Ploc}(D)$. We define a semantic function $[\![\cdot]\!]$ which assigns to each prelocalic expression in $|\mathbf{M}(A)|$ a subset of the mixed powerdomain of D as follows:

$$\begin{array}{l}
[\![\Box a]\!] = \{x \in \mathbf{M}(D) \mid \exists (u, v) \in x.\ u \subseteq a\} \\
[\![\Diamond a]\!] = \{x \in \mathbf{M}(D) \mid \exists (u, v) \in x.\ v \cap a \neq \emptyset\} \\
[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!] \\
[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!] \\
[\![0]\!] = \emptyset \\
[\![1]\!] = \mathcal{M}(D)
\end{array}$$

Proposition 19 *For each $\phi \in |\mathbf{M}(A)|$, we have $[\![\phi]\!] \in K\Omega\mathcal{M}D$.*

Proof: The proof goes by induction on the structure of ϕ. The interesting cases are $\Box a$ and $\Diamond a$ where $a = \uparrow u$ for a finite set $u = \{x_1, \ldots, x_n\} \subseteq KD$.

- Claim: $x \in \Box a$ iff $(u, \{x_i\}) \in x$ for some $i \leq n$. ($\Rightarrow$) If $x \in \Box a$, then there is a mix $(u', v') \in x$ such that $u' \subseteq a$. In other words, $u \precsim^\natural u'$. If $p \in v'$, then there is some $x_i \in u$ such that $x_i \precsim p$ since $v' \succsim^\natural u' \succsim^\natural v'$. Hence $(u, \{x_i\}) \precsim (u', v')$ and therefore $(u, \{x_i\}) \in x$. The converse ($\Leftarrow$) follows immediately from the definitions. That $[\![\Box a]\!]$ is a compact open subset of $\mathcal{M}D$ now follows from the claim and Lemma 8.

- Let w be the (finite) set of minimal elements of D. Claim: $x \in \Diamond a$ iff $(w, \{x_i\}) \in x$ for some $i \leq n$. ($\Rightarrow$) If $x \in \Diamond a$, then there is a mix $(u', v') \in x$ such that $v' \cap a \neq \emptyset$. Hence there is some $p \in v'$ and $x_i \in u$ such that $x_i \precsim p$. Thus $(w, \{x_i\}) \precsim (u', v')$ and therefore $(\{\bot\}, u) \in x$. The converse ($\Leftarrow$) follows immediately from the definitions. That $[\![\Diamond a]\!]$ is a compact open subset of $\mathcal{M}D$ now follows from the claim and Lemma 8. ▌

Lemma 20 *Let D be an algebraic cpo. An element $U \in \Omega D$ is prime in the algebraic lattice of open subsets if and only if $U = \uparrow x$ for some $x \in KD$.* ▮

Theorem 21 *The axioms and rules on $|\mathbf{M}(A)|$ are sound with respect to the interpretation $[\![\cdot]\!]$. That is*

1. *If $\phi \precsim \psi$, then $[\![\phi]\!] \subseteq [\![\psi]\!]$.*
2. *If $P(\phi)$, then $[\![\phi]\!]$ is a prime in $\Omega \mathcal{M} D$.*

Proof: The proof of (1) is straight-forward. I will write out the most interesting case, the $(\Diamond - \wedge)$ axiom. Suppose $x \in [\![\Box a \wedge \Diamond b]\!] = [\![\Box a]\!] \cap [\![\Diamond b]\!]$, then there is a mix $(u, v) \in x$ such that $u \subseteq a$ and a mix $(u', v') \in x$ such that $v' \cap b \neq \emptyset$. Since x is directed, there is a mix $(u'', v'') \in x$ such that $(u, v), (u', v') \precsim (u'', v'')$. Now $u'' \subseteq a \cap b$ since $u, u' \precsim^{\sharp} u''$. Thus $v'' \subseteq a \cap b$ and $v'' \cap (a \cap b) \neq \emptyset$ since v'' is non-empty. Therefore $x \in [\![\Diamond a \wedge b]\!]$.

Suppose that a_i is a prime element of ΩD for each i in a finite non-empty indexing set I. Following axiom $(P - \Box - \Diamond)$, we must show that for any finite non-empty subset $J \subseteq I$, the set

$$W = [\![\Box \bigvee_{i \in I} a_i]\!] \cap [\![\bigwedge_{j \in J} \Diamond a_j]\!]$$

is a prime. By Lemma 20, there are compact elements x_i such that $a_i = \uparrow x_i$ for each $i \in I$. Let $u = \{x_i \mid i \in I\}$ and $v = \{x_i \mid j \in J\}$. Since $J \subseteq I$, the pair (u, v) is a mix. We show that $\uparrow(u, v) = W$. Let $U = \bigcup_{i \in I} a_i$ and suppose $(u, v) \precsim (u', v')$. Since $u \precsim^{\sharp} u'$, we must have $u' \subseteq U$ and hence $(u', v') \in [\![\Box \bigvee_{i \in I} a_i]\!]$. Since $v \cap [\![\Diamond a_j]\!] \neq \emptyset$ for each a_j with $j \in J$, and $v' \succsim^{\flat} v$, we must have $v' \cap [\![\Diamond a_j]\!] \neq \emptyset$ for each $j \in J$, so $(u, v) \in [\bigwedge_{j \in J} \Diamond a_j]\!]$. Thus $(u', v') \in W$. Suppose on the other hand that $(u', v') \in W$. Then $(u', v') \in [\![\Box \bigvee_{i \in I} a_i]\!]$ so $u' \subseteq U$, so $u' \succsim^{\sharp} u$. Since $(u', v') \in [\![\bigwedge_{j \in J} \Diamond a_j]\!]$ as well, $v' \cap a_j \neq \emptyset$ for each $j \in J$. This means that for each x_j, there is some element $x'_j \in v'$ such that $x_j \lesssim x'_j$. But this just means that $v' \succsim^{\flat} v$. Hence $(u', v') \in \uparrow(u, v)$ as desired. The fact that W is prime now follows from Lemma 20. ▮

Lemma 22 *Suppose $\phi, \psi \in |\mathbf{M}(A)|$. If $P(\phi)$ and $P(\psi)$ and $[\![\phi]\!] \subseteq [\![\psi]\!]$, then $\phi \precsim \psi$.*

Proof: Suppose $J \subseteq I$ and $L \subseteq K$ are finite non-empty indexing sets and $\{a_i \mid i \in I\}$ and $\{b_k \mid k \in K\}$ are sets of primes such that

$$\phi \approx \Box \bigvee_{i \in I} a_i \wedge \bigwedge_{j \in J} \Diamond a_j$$
$$\psi \approx \Box \bigvee_{k \in K} b_k \wedge \bigwedge_{l \in L} \Diamond b_l$$

Say $a_i = \uparrow x_i$ and $b_k = \uparrow y_k$ for each $i \in I$ and $k \in K$. Let $u = \{x_i \mid i \in I\}$ and $u' = \{x_j \mid j \in J\}$ and $v = \{y_k \mid k \in K\}$ and $v' = \{y_l \mid l \in L\}$. Then

$$\begin{array}{rcl} [\![\phi]\!] \subseteq [\![\psi]\!] & \Rightarrow & \uparrow(u, u') \subseteq \uparrow(v, v') \\ & \Rightarrow & (v, v') \precsim (u, u') \\ & \Rightarrow & v \precsim^{\sharp} u \text{ and } v' \precsim^{\flat} u' \\ & \Rightarrow & \forall i \in I \; \exists k \in K.\; y_i \lesssim x_k \text{ and } \forall l \in L \; \exists j \in J.\; y_j \lesssim x_l \\ & \Rightarrow & \forall i \in I \; \exists k \in K.\; a_i \subseteq b_k \text{ and } \forall l \in L \; \exists j \in J.\; a_j \subseteq b_l \\ & \Rightarrow & \Box \bigvee_{i \in I} a_i \lesssim \Box \bigvee_{j \in J} \text{ and } \bigwedge_{j \in J} \Diamond a_j \lesssim \bigwedge_{l \in L} \Diamond b_l \\ & \Rightarrow & \phi \precsim \psi \quad ▮ \end{array}$$

The proof of this lemma is basically contained in the proof of Proposition 3.4.8 in [Abr89]:

Lemma 23 *For every $a \in |\mathbf{M}(A)|$, there are primes $b_1, \ldots, b_n \in P$ such that $a \eqsim \bigvee_{i=1}^{n} b_i$.* ∎

Theorem 24 *For all $\phi, \psi \in |\mathbf{M}(A)|, \phi \precsim \psi$ iff $\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket$.*

Proof: ($\Rightarrow$) is part of the Theorem 21. To prove ($\Leftarrow$), we begin by using Lemma 23 to deduce the existence of finite sets of primes $\{\phi_i \mid i \in I\}$ and $\{\psi_j \mid j \in J\}$ such that

$$\phi = \bigvee_{i \in I} \phi_i \qquad \text{and} \qquad \psi = \bigvee_{j \in J} \phi_j.$$

We may now make the following deductions:

$$\begin{array}{rll}
\llbracket \phi \rrbracket \subseteq \llbracket \psi \rrbracket & \Rightarrow & \bigcup_{i \in I} \llbracket \phi \rrbracket \subseteq \bigcup_{j \in J} \llbracket \psi \rrbracket \\
& \Rightarrow & \forall i \in I\ \exists j \in J.\ \llbracket \phi_i \rrbracket \subseteq \llbracket \psi_j \rrbracket \quad \text{by Theorem 21} \\
& \Rightarrow & \forall i \in I\ \exists j \in J.\ \phi_i \precsim \psi_j \qquad \text{by Lemma 22} \\
& \Rightarrow & \bigvee_{i \in I} \phi \subseteq \bigvee_{j \in J} \psi \qquad \blacksquare
\end{array}$$

Theorem 25 *If D is a bifinite domain, then* $\mathbf{M}(\mathbf{Ploc}(D)) \cong \mathbf{Ploc}(\mathcal{M}(D))$.

Proof: The map $\llbracket \cdot \rrbracket$ is order preserving and reflecting by Lemma 24. To see that it is also a surjection, and hence and isomorphism, suppose $U \in K\Omega(\mathcal{M}(D))$. By Lemma 8, there are mixes $x_1, \ldots, x_n \in K\mathcal{M}(D)$ such that $U = \bigcup_{i=1}^{n} \uparrow x_i$. It is easy to see from the proof of Theorem 21 that, for each i, there is some $\phi_i \in |\mathbf{M}(\mathbf{Ploc}(D))|$ such that $\llbracket \phi \rrbracket_i = \uparrow x_i$. Hence $\llbracket \bigvee_{i=1}^{n} \phi \rrbracket = \bigcup_{i=1}^{n} \llbracket \phi_i \rrbracket = U$. ∎

9 Conclusion.

This paper began with a general exposition, illustrated by examples, of the intuitive logical significance of powerdomain elements. These intuitions were then captured by showing how an element of a powerdomain can be viewed as a monadic second order predicate. The form of these predicates suggested a simple generalization leading to a new structure which I have called the *mixed powerdomain*. This new operator was then generalized to an operator on a significant class of domains. It was shown to be distinct from any of the structures currently being studied and shown to have many of the basic properties which make other powerdomains suitable for the semantics of programs. To further understand the mixed powerdomain it was characterized in two further ways:

- algebraically: by demonstrating that it was left adjoint to a forgetful functor on a class of what I have called *mix algebras* and
- topologically: by constructing a Stone dual for the mixed powerdomain.

Both of these treatments provide perspective on the relationship of the mixed powerdomain to other powerdomains which have also been characterized in these ways.

I would like to urge that these results demonstrate that there are other interesting operators which modify and generalize known domain-theoretic constructions. With all of the machinery which we have developed for studying such constructions, it is possible to take an interesting theory of partial information and derive an elegant and thorough mathematical theory by allowing oneself to be guided by the variety of ideas for characterization and development.

To some extent my discussion in this paper has run counter to the conventional wisdom that one should start with a programming application and base the development of new structures on the needs of solving the problem in question. For example, the first paper on powerdomains (Plotkin's [Plo76]) developed the operator to model a specific programming construct. To some extent I have done this, since some of the inspiration for this paper derives from work on generalizing relational databases (as discussed in [BDW88, BO86, BJO89]) and I have also discussed elsewhere the potential use of the mixed powerdomain in the specification of bounded non-determinism [Gun90]. Nevertheless, it is my feeling that a well-designed mathematical treatment of an interesting new construction which captures some of the central issues about the structuring of partial information is likely to find, or even create, some of its own applications.

10 Acknowledgements.

The work on this paper owes a debt to a number of individuals who inspired the research and guided the technical developments. I would like to single out Peter Buneman for his role in getting me to look closely at new powerdomain constructs and Dana Scott for helping me obtain an intuitive perspective on the information-theoretic significance of the powerdomains. I also thank Samson Abramsky, Susan Davidson, Mike Main, Mike Mislove, Prakash Panangaden, Steve Vickers and Aaron Watters for encouragement and numerous technical contributions.

The research on this paper was supported in part by U.S. Army Research Office Grant DAAG29-84-K-0061 and ONR contract number N00014-88-K-0557.

References

[Abr87] S. Abramsky. Domains in logical form. In D. Gries, editor, *Symposium on Logic in Computer Science*, pages 47–53, IEEE Computer Society Press, Ithaca, New York, June 1987.

[Abr88] S. A. Abramsky. *Domain Theory and the Logic of Observable Properties*. PhD thesis, Queens College, 1988.

[Abr89] S. A. Abramsky. Domain theory in logical form. *Annals of pure and applied logic*, 1989. To appear.

[BDW88] P. Buneman, S. Davidson, and A. Watters. A semantics for complex objects and approximate queries. In *Principles of Database Systems*, ACM, March 1988.

[BJO89] P. Buneman, A. Jung, and A. Ohori. Using powerdomains to generalize relational databases. *Theoretical Computer Science*, 1989. To appear.

[BO86] P. Buneman and A. Ohori. A domain theoretic approach to higher-order relations. In *International Conference on Database Theory*, Springer-Verlag, 1986.

[GS88] C. A. Gunter and D. S. Scott. Semantic domains. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages ??–??, North Holland, To appear in 1988.

[Gun87] C. A. Gunter. Universal profinite domains. *Information and Computation*, 72:1–30, 1987.

[Gun90] C. A. Gunter. Relating partial and total correctness interpretations of non-deterministic programs. In P. Hudak, editor, *Principles of Programming Languages*, pages ??–??, ACM, 1990.

[Hen88] M. Hennessy. *Algebraic Theory of Processes*. MIT Press, 1988.

[HP79] M. Hennessy and G. D. Plotkin. Full abstraction for a simple parallel programming language. In J. Bečvář, editor, *Mathematical Foundations of Computer Science*, pages 108–120, *Lecture Notes in Computer Science vol. 74,* Springer, 1979.

[Joh82] P. Johnstone. *Stone Spaces*. Volume 3 of *Studies in Advanced Mathematics*, Cambridge University Press, 1982.

[Plo76] G. D. Plotkin. A powerdomain construction. *SIAM Journal of Computing*, 5:452–487, 1976.

[Plo82] G. D. Plotkin. The category of complete partial orders: a tool for making meanings. Postgraduate lecture notes, Department of Computer Science, Edinburgh University, 1982.

[Sco82] D. S. Scott. Domains for denotational semantics. In M. Nielsen and E. M. Schmidt, editors, *International Colloquium on Automata, Languages and Programs*, pages 577–613, *Lecture Notes in Computer Science vol. 140,* Springer, 1982.

[Smy78] M. Smyth. Power domains. *Journal of Computer System Sciences*, 16:23–36, 1978.

[Smy83] M. Smyth. Power domains and predicate transformers: a topological view. In J. Diaz, editor, *International Colloquium on Automata, Languages and Programs*, pages 662–676, *Lecture Notes in Computer Science vol. 154,* Springer, 1983.

[SP82] M. Smyth and G. D. Plotkin. The category-theoretic solution of re¡cursive domain equations. *SIAM Journal of Computing*, 11:761–783, 1982.

[Win85] G. Winskel. On powerdomains and modality. *Theoretical Computer Science*, 36:127–137, 1985.