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This paper analyzes two-handed grasping of a rigid object in the two-dimensional (2-D) space. The two hands under consideration have two fingers, and each finger is equipped with a tactile sensor. The hands are assumed to be respectively installed on two robotic manipulators capable of motion and force control.

In a separate paper [1], a necessary condition for proper grasping configurations of two two-fingered hands are obtained. A proper grasping configuration is a configuration of the two hands at the initial contact with the object guaranteeing that a stable grasp can be achieved. A detailed study of the properties of contacts with the two fingers of a hand, and a method to determine if a proper grasping configuration has been reached by the two hands, are provided.

Comments

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Two-Handed Grasping With Two-Fingered Hands

MS-CIS-90-83
Grasp Lab 242

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Abstract

This paper analyzes two-handed grasping of a rigid object in the two-dimensional (2-D) space. The two hands under consideration have two fingers, and each finger is equipped with a tactile sensor. The hands are assumed to be respectively installed on two robotic manipulators capable of motion and force control.

In a separate paper [1], a necessary condition for proper grasping configuration of two palms was established. This study explores the use of two-fingered hands in two-handed grasping under the same assumptions. Necessary conditions for proper grasping configurations of two two-fingered hands are obtained. A proper grasping configuration is a configuration of the two hands at the initial contact with the object guaranteeing that a stable grasp can be achieved. A detailed study of the properties of contacts with the two fingers of a hand, and a method to determine if a proper grasping configuration has been reached by the two hands, are provided.

1 Introduction

Automated grasping is essential in robotic manipulation. Before carrying out any task, the object involved in the task must be securely grasped.

In the field of grasp stability, important works have been performed assuming contacts with friction, as in [4, 5, 6, 7, 8], or frictionless grasps as in [9]. For grasps with friction, all the works cited assume knowledge of contact forces and location of them. These works deal with one-handed grasping, no works were found with an explicit treatment of two-handed grasps.

The analysis performed herein is focused to two-handed grasping, and is based on line geometry [2] and screw theory [3]. In a previous work [1], the problem of grasping with two palms was analyzed; the work presented in this paper uses a parallel approach to analyze the problem of grasping with two two-fingered hands. For grasping, two-fingered hands are better suited than single palms. By using two-fingered hands, the complexity of the analysis, and the capabilities of the system to get a stable grasp, are increased with respect to using single palms in the grasp.

This study does not require a precise definition of the contact location. The aim of the research performed has been to create simple grasping strategies, based on only tactile sensing, for objects placed in unstructured environments.

1.1 One-Handed vs. Two-Handed Grasping

Most of the works found in the area of grasp analysis deal with only one-handed grasping. In this paper, two-handed grasping is studied. One-handed grasping and two-handed grasping share the common characteristics of manipulating an object with multiple fingers, regardless of where the fingers belong to. The existing results on one-handed grasping, particularly conditions on grasping stability, can be applied to two-handed grasping.

However, two-handed grasping has some differences with respect to one-handed grasping. A robotic arm is a serial kinematic chain, and a hand is a parallel kinematic structure made out of a group of fingers (each finger itself being a serial kinematic chain). The individual properties of a "tree" formed by an arm and a hand become enhanced when it is associated to another structure for grasping and manipulating of an object. Two-handed grasping differs from one-handed grasping in that:

1. Kinematically, all the fingers are in two groups and the location of the fingers in one group relative to the object is independent of the location of the fingers in the other.

2. The need of high dexterity fingers for complex tasks, involving exertion of manipulating forces by the fingers, might be reduced by using two hands and concentrating all manipulating forces in the two arms (serial kinematic chains) instead of distributing them in the fingers of a single-hand robotic system able to perform similar tasks on a given object.

3. From the task point of view, while one hand is limited to grasp objects smaller than the opening of the hand, or objects with features such as handlers; two hands are capable of grasping a much larger class of objects including large, long, flexible and irregularly shaped objects.

4. The capability of balancing moments is increased since the length of the arms allows the location of the two hands on the object in such a way that the cancellation of moments require smaller forces.

Those differences warrant an independent investigation in order to fully utilize the potentials of two cooperative hands.

1.2 Organization of the Paper

This paper explores the capability of two hands to achieve a stable grasp of an object under the conditions stated in section 2. Section 3 states conventions used throughout this paper. Section 4 describes wrenches obtainable through contact at two points. Section 5 describes the relative positions attainable by the two fingers of a hand. Section 6 defines the friction cone of a two-fingered hand. Section 7 states the conditions for a proper grasping configuration with two two-fingered hands. In section 8, those conditions are applied for the particular case of hands with symmetric features. The possible extensions of the work presented are discussed in section 9. Appendix A gives a geometric description of the friction cone of two-fingered hands. Appendix B explains important characteristics of different configurations of the two fingers of a hand required for the analysis of grasping stability.

2 Problem

2.1 Scope and Assumptions

This study considers the problem of grasping a rigid object by two two-fingered hands. Its goal is to specify the conditions for guaranteeing the existence of a locally stable grasp of an object in the 2-D space. According to its definition, local stability means that no sliding between palms and object occurs in the grasp.

Objects are assumed to be placed within the workspace of the two-palm robotic system. No information about the shape or orientation of the objects is assumed.

The fingers considered in this study are flat surfaces, also called plates in the context of this work. Attached to the surface of each finger is a sensor to detect the presence of contacts with objects. It is assumed that contacts occur at the interior points of the plates. However, the specific location of contacts between the plates and the object is not assumed.

The kinematics of the hands and in general, of the robotic system, and all the parameters about the fingers such as their length, position and orientation are assumed to be known.

The strictest assumption of all is that the friction coefficient between the plates and the object is known. Although the friction coefficient is determined by the two contacting surfaces, attaching a high friction material such as sand paper on the plate surface, narrows the varying range of the unknown coefficient. Further, using the minimum value of friction coefficients between the plates and objects in the working environment results in conservative conditions. Lastly, if the domain of working environment is known *a priori*, a reasonable estimate of the friction coefficient can be obtained.

The study is restricted to the case of an object held between two hands, although its extensions to more hands in the system can be derived from this analysis. Outside-in and inside-out grasps [10] are considered. The results of this study are readily applicable to lightweight objects due to the fact that no information about the weight of the object is used. If the object is heavy additional considerations, such as fragility of the object and forces deliverable from the two-robot system, must be made. Those considerations are omitted in this paper.

2.2 Solution Strategy

Initially, contact at two single points, one at each finger will be analyzed.

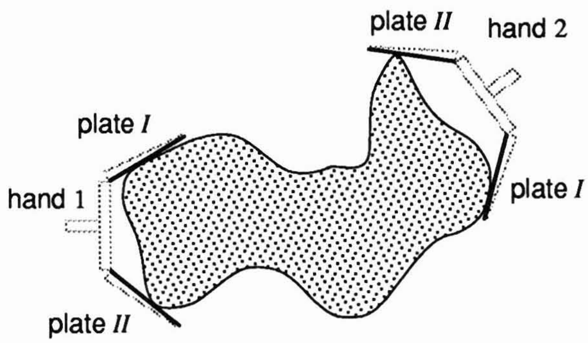
Contact at a finger consists of one or more point contacts. The analysis of contacts at a finger which itself is a plate, can be reduced to the analysis of the contact at an equivalent point on that plate [1]. Similarly, the analysis of contacts at two fingers can be reduced to the analysis of the contact at two equivalent points, one from each finger. General properties will be drawn from all combinations of two equivalent contact points obtained when two fingers contact an object. From them, the properties of contacts at the two fingers of a hand will be obtained.

For the analysis of a two-handed grasp, forces at each hand will be considered only in terms of the resultant wrench generated by contact forces on both fingers. A necessary condition to get a stable grasp of a massless object using two hands is that two opposite wrenches along the same line can be generated by the hands.

A grasping configuration is the configuration of the two hands when they make initial contact with the object. A *proper* grasping configuration is the grasping configuration guaranteeing that a stable grasp of the object can be achieved, no matter where in the fingers the contacts are being made.

In this study, it is assumed that if a stable grasp of a massless object can be achieved, by increasing the magnitude of the grasping forces, frictional forces can be made responsible for getting stability on grasping a same shape, but weighted object, with the same grasping configuration.

In the analysis performed here, characteristics of wrenches exertable by each hand are considered first. If wrenches that can be generated by the two hands cancel each other independently of the specific contact points at the two hands, a proper grasping configuration of the two hands has been reached, and a stable grasp of the object can be performed.



i, j denote hands; k, l denote plates of a hand;
m and n label extreme points of the plates.
$i = 1 \Rightarrow j = 2$; $i = 2 \Rightarrow j = 1$.
$k = I \Rightarrow m = 1, l = II, n = 2$; $k = II \Rightarrow m = 2, l = I, n = 1$.

Figure 1: Two Two-Fingered Hands Grasping an Object

3 Nomenclature

This paper deals with hands formed by two fingers, both fingers are straight plates. They will be named plate I and plate II. Throughout this work, subscripts k and l with $k \neq l$, are used to denote plates I and II indistinctly; subscripts m and n are applied to extreme points of plates k and l respectively; and subscripts i and j denote two different hands, i. e., $i, j = 1, 2, i \neq j$. Subscripts m and n have their values according to subscripts k and l, as follows:

$$m = \begin{cases} 1 & \text{if } k = I \\ 2 & \text{if } k = II \end{cases} ; \qquad n = \begin{cases} 1 & \text{if } k = II \\ 2 & \text{if } k = I \end{cases} \tag{1}$$

Figure 1 illustrates the nomenclature used for two two-plate hands grasping an object. The dotted lines in the figure are structural elements on which no contact occur.

For straight plate k of the hand considered, denote by $\mathbf{P}_k$ the set of contact points $\mathbf{p}_k$ on it. Define a plate coordinate frame $(\mathbf{x}_k, \mathbf{y}_k)$ located at $\mathbf{c}_k$, its center point, with $\mathbf{y}_k$ normal to the plate and pointing towards the object. The length of plate k is defined $2a_k$. μ_k is the friction coefficient of plate k. The angle $\theta_k = \arctan \mu_k$.

The extreme directions of the friction cone of any point $\mathbf{p}_k \in \mathbf{P}_k$, denoted by vectors $\mathbf{u}_k^*$ and $\mathbf{u}_k^{**}$, are given by

$$\mathbf{u}_k^e = \mathbf{R}_{(-1)^d \theta_k} \mathbf{y}_k; \quad e = *, **; \ d = \begin{cases} 1 \text{ for } e = *; \\ 2 \qquad e = ** \end{cases} \tag{2}$$

where $\mathbf{R}_\alpha$ represents the linear transformation that rotates a vector by an angle α.

A contact force $\mathbf{f}_{pk}$ applied at point $\mathbf{p}_k$, can be expressed in terms of its magnitude f_{pk}, and its direction $\mathbf{u}_k$, as $\mathbf{f}_{pk} = f_{pk}\mathbf{u}_k$

4 Two-Point Friction Cone

4.1 Two Contact Forces and Resultant Wrench

Two contact forces $\mathbf{f}_{pI}$ and $\mathbf{f}_{pII}$ applied at points $\mathbf{p}_I$ and $\mathbf{p}_{II}$ respectively, generate a wrench equal to the summation of the wrenches generated by the contact forces. If the resultant wrench is not a null vector, it may represent either a pure force ($\sum_k \mathbf{p}_k \times \mathbf{f}_{pk} = 0$), a pure moment ($\sum_k \mathbf{f}_{pk} = 0$), or a combination of a force and a moment ($\sum_k \mathbf{f}_{pk} \neq 0; \sum_k \mathbf{p}_k \times \mathbf{f}_{pk} \neq 0$).

4.2 Two-Point Friction Cone

The set of resultant wrenches generated by forces exerted at two contact points is larger than the simple addition of the sets of wrenches generated by each contact force. The characteristics of this set are fundamental to understand the grasping capabilities of a hand.

Definition 4.1 *The* two-point friction cone *is the set of points belonging to lines of action of wrenches exertable by the contact points.*

For a contact point $\mathbf{p}_k$, define wrench $\mathbf{F}_{pk}$ as $\mathbf{F}_{pk} = f_{pk}[\mathbf{u}_k^T, (\mathbf{p}_k \times \mathbf{u}_k)^T]^T$, where subscript pk denotes point $\mathbf{p}_k$ through which the line of action of wrench $\mathbf{F}_{pk}$ passes. Also, for $\mathbf{p}_k$, define extreme lines $\mathbf{U}^*_{pk}$ and $\mathbf{U}^{**}_{pk}$ as

$$\mathbf{U}^*_{pk} = [\mathbf{u}_k^{*T}, (\mathbf{p}_k \times \mathbf{u}_k^*)^T]^T \tag{3a}$$

$$\mathbf{U}^{**}_{pk} = [\mathbf{u}_k^{**T}, (\mathbf{p}_k \times \mathbf{u}_k^{**})^T]^T \tag{3b}$$

based on these lines, the set $\mathcal{U}_{pk}$ can be defined as

$$\mathcal{U}_{pk} = \left\{ \mathbf{U}_{pk} = \alpha(\lambda \mathbf{U}^*_{pk} + (1-\lambda)\mathbf{U}^{**}_{pk}) \;\middle|\; \alpha = \| \lambda \mathbf{u}_k^* + (1-\lambda)\mathbf{u}_k^{**} \|^{-1}; 0 \leq \lambda \leq 1 \right\} \tag{4}$$

$\mathbf{U}_{pk} = [\mathbf{u}_k^T, (\mathbf{p}_k \times \mathbf{u}_k)^T]^T$ is a line passing through $\mathbf{p}_k$ between or on lines $\mathbf{U}^*_{pk}$ and $\mathbf{U}^{**}_{pk}$. Then, wrenches with extreme directions exerted at $\mathbf{p}_k$ are given by $\mathbf{F}^e_{pk} = f_{pk}\mathbf{U}^e_{pk}$, with $e = *, **$; and the set of wrenches that can be exerted by point $\mathbf{p}_k$ is defined by

$$\mathcal{F}_{pk} = \{ \mathbf{F}_{pk} = f_{pk}\mathbf{U}_{pk} \mid \mathbf{U}_{pk} \in \mathcal{U}_{pk} \} \tag{5}$$

where $f_{pk} \geq 0$.

The set of wrenches that can be applied to the object when having contact with it simultaneously on points $\mathbf{p}_I$ and $\mathbf{p}_{II}$ is given by

$$\mathcal{W}_{pI,pII} = \{ \mathbf{W} = \mathbf{F}_{pI} + \mathbf{F}_{pII} \mid \mathbf{F}_{pI} \in \mathcal{F}_{pI}; \mathbf{F}_{pII} \in \mathcal{F}_{pII} \} \tag{6}$$

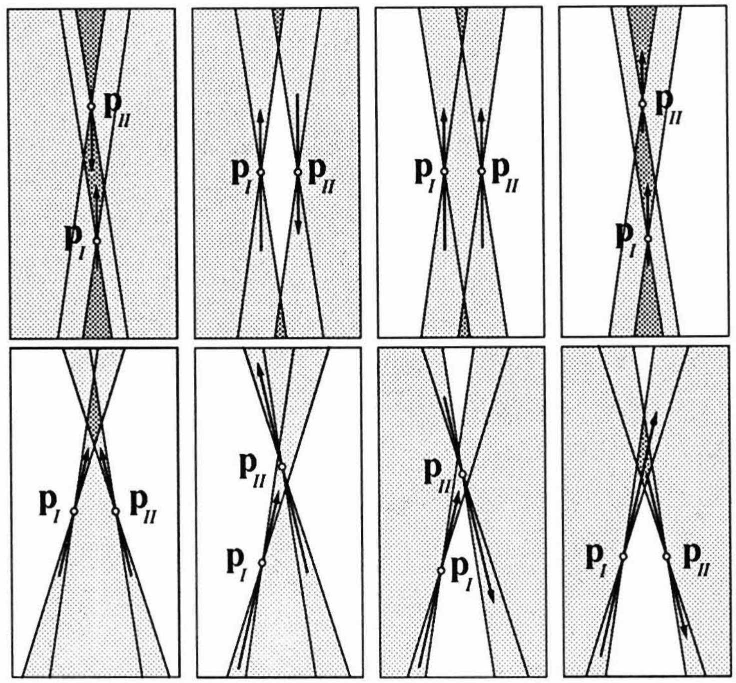


Figure 2: Two-Point Friction Cones

The two-point friction cone of the points $\mathbf{p}_I$ and $\mathbf{p}_{II}$, is the set of points $\mathcal{S}_{pI,pII}$ in the line of action of $\mathbf{W} \in \mathcal{W}_{pI,pII}$, which can be expressed as follows

$$\mathcal{S}_{pI,pII} = \left\{ \mathbf{s} \;\middle|\; f[\mathbf{u}^T, (\mathbf{s} \times \mathbf{u})^T]^T \in \mathcal{W}_{pI,pII} \right\} \tag{7}$$

Figure 2 shows two-point friction cones in different situations. The friction cones are indicated by both light and dark shadowed regions. The dark region represents the intersection of the cones of friction of the individual contact points, which is indeed part of the two-point friction cone. In the figure, the arrows represent directions inwards the object, normal to the contacts at each contact point. Each arrow is intersected at the contact point by two lines which delimit the cone of friction of each individual contact point.

If directions of all forces represented in Figure 2 are inverted for all cases, the friction cones are the same, and the directions of resultant forces exertable to an object through contact on those points are opposite to the resultant of the ones shown in it.

Remark 4.2.1 *Each point* $\mathbf{s} \in \mathcal{S}_{pI,pII}$ *is associated with a set* $\mathcal{U}_{s;pI,pII}$ *of directed lines passing through it.*

In this case, $\mathcal{U}_{s;pI,pII}$ is expressed in Plücker coordinates as

$$\mathcal{U}_{s;pI,pII} = \left\{ \mathbf{U} = [\mathbf{u}^T, (\mathbf{s} \times \mathbf{u})^T]^T \;\mid f\mathbf{U} \in \mathcal{W}_{pI,pII} \right\} \tag{8}$$

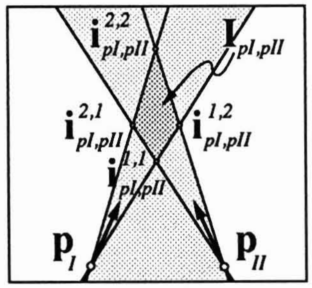


Figure 3: Intersection of Lines of Action of Wrenches Generated by Two Contact Points

i. e., the lines of action belonging to $\mathcal{U}_{s;pI,pII}$ go along wrenches $[\mathbf{f}^T, (\mathbf{s} \times \mathbf{f})^T]^T \in \mathcal{W}_{pI,pII}$. A set $\mathcal{W}_{s;pI,pII} \subset \mathcal{W}_{pI,pII}$ can be defined from $\mathcal{U}_{s;pI,pII}$

$$\mathcal{W}_{s;pI,pII} = \{\mathbf{W} = f\mathbf{U} \mid \mathbf{U} \in \mathcal{U}_{s;pI,pII}\} \tag{9}$$

Remark 4.2.2 *Definition 4.1 can be extended to multiple contacts. The friction cone of multiple contacts is the set of points belonging to lines of action of wrenches exertable by the contact points.*

4.3 Geometry of the Two-Point Friction Cone

The set of points included in the two-point friction cone can be defined on the basis of lines $\mathbf{U}^*_{pk}$ and $\mathbf{U}^{**}_{pk}$, $k = I, II$, as follows.

Define two lines $\mathbf{U}^1_{pI}$ and $\mathbf{U}^1_{pII}$ such that

$$\mathbf{u}^1_{pI} \cdot \mathbf{u}^1_{pII} = \min \mathbf{u}^e_{pI} \cdot \mathbf{u}^e_{pII}; \qquad e = *, ** \tag{10}$$

$\mathbf{U}^1_{pI}$ and $\mathbf{U}^1_{pII}$ are the most divergent extreme lines of two wrenches, each in sets $\mathcal{F}_{pI}$ and $\mathcal{F}_{pII}$ respectively. $\mathbf{U}^2_{pI}$ and $\mathbf{U}^2_{pII}$ are the two remaining extreme lines.

The two-point friction cone for points $\mathbf{p}_I$ and $\mathbf{p}_{II}$ is the set of points belonging to the lines expressed in line coordinates by

$$\alpha\mathbf{U}^1_{pI} + \beta\mathbf{U}^1_{pII} + \gamma\mathbf{U}^2_{pI} + \lambda\mathbf{U}^2_{pII}; \quad 0 \le \alpha, \beta, \gamma, \lambda \le 1; \alpha + \beta + \gamma + \lambda = 1 \tag{11}$$

Denote $\mathbf{I}_{pI,pII}$ the set of points where $\mathcal{F}_{pI}$ and $\mathcal{F}_{pII}$ intersect. Then the set of all wrenches in $\mathcal{W}_{pI,pII}$ pass through $\mathbf{I}_{pI,pII}$. Four extreme points on the set $\mathbf{I}_{pI,pII}$ can be found: $\mathbf{i}^{1,1}_{pI,pII}, \mathbf{i}^{1,2}_{pI,pII}, \mathbf{i}^{2,1}_{pI,pII}$, and $\mathbf{i}^{2,2}_{pI,pII}$ where $\mathbf{i}^{e,d}_{pI,pII}$ is the intersection between $\mathbf{U}^e_{pI}$ and $\mathbf{U}^d_{pII}$. Figure 3 shows these points at a two-point friction cone. All other cases also have their respective points $\mathbf{i}^{e,d}_{pI,pII}$. When $\mathbf{U}^e_{pI}$ and $\mathbf{U}^d_{pII}$ are parallel, point $\mathbf{i}^{e,d}_{pI,pII}$ is placed at infinity. The dark shadowed region of the cases shown in Figure 2 are also sets $\mathbf{I}_{pI,pII}$.

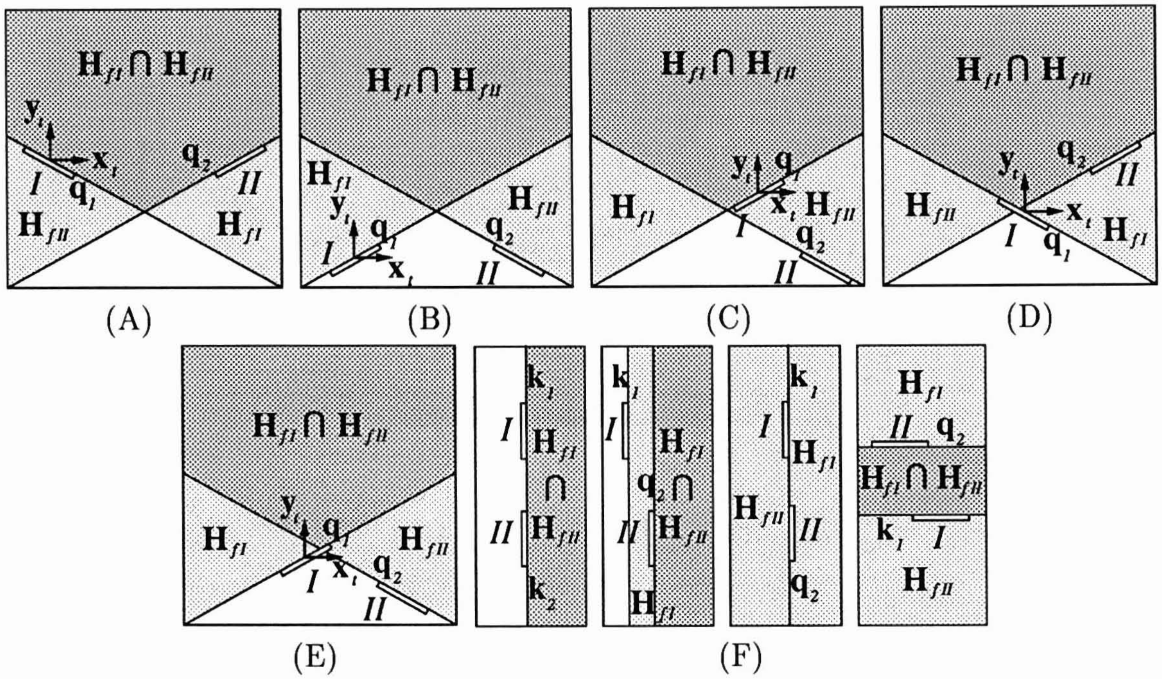


Figure 4: Relative Positions of Two Straight Plates

5 Position Description of Two-Fingered Hands

5.1 Front and Back of a Plate

The line on which all contact points of a straight plate are placed divide the space in two regions: the front of the plate, and the back of the plate. The front of the plate k is the half space $\mathbf{H}_{fk}$ containing points $\mathbf{p}_{fk}$ defined as

$$\mathbf{H}_{fk} = \{\mathbf{p}_{fk} \mid \mathbf{y}_k \cdot (\mathbf{p}_{fk} - \mathbf{p}_k) \geq 0\} \tag{12}$$

where $\mathbf{p}_k \in \mathbf{P}_k$ is a contact point on the plate k, and $\mathbf{y}_k$ is a unit vector normal to the plate k pointing inwards the object in contact with the plate. The back of plate k is the complement of its front.

5.2 Relative Positions of the Two Plates of a Hand

When the lines in which the two straight plates of a hand lie are considered, two sets of relative orientations of the two plates can be found :

1. Lines along plates I and II are neither parallel lines nor the same line.
2. Lines along plates I and II are either parallel lines or the same line.

Dividing the 2-D space in two sets per plate k, front of plate k and back of plate k, further decomposition of all possible relative positions can be obtained. The two sets mentioned above can be decomposed into subsets.

Set 1 is made out of the following subsets:
A. Plate I is at the front of plate II, and vice versa.
B. Plate I is at the back of plate II, and vice versa.
C. Plate k is at the front of plate l, and plate l is at the back of plate k.
D. Plate k is at the front of plate l, and plate l touches both: the front and the back of plate k.
E. Plate k is at the back of plate l, and plate l touches both: the front and the back of plate k.
Set 2 can be renamed to be set F:
F. Plates I and II lie along parallel lines or along the same line.
Figure 4 shows finger configurations corresponding respectively to each subset of the two sets mentioned above.

5.3 Hand Coordinates

The hand coordinate system $(\mathbf{x}_t, \mathbf{y}_t)$, will be used to specify the notation for the extreme points of the two plates of the hand. This coordinate system has its origin at the center point of plate I, and its vector $\mathbf{y}_t$ is defined as follows

$$\mathbf{y}_t = \begin{cases} \frac{\mathbf{y}_I + \mathbf{y}_{II}}{\|\mathbf{y}_I + \mathbf{y}_{II}\|} & \text{if } \|\mathbf{y}_I + \mathbf{y}_{II}\| \neq 0 \\ \mathbf{y}_I & \text{otherwise} \end{cases} \tag{13}$$

5.4 Extreme Points of the Grasping Regions of the Hand

From the relative positions of the two plates of a hand, it is possible to name the extreme points of each plate in a convenient manner. The analysis of grasps will be performed on the basis of these extreme points.

Denote by $\mathbf{r}_k$ an extreme point of $\mathbf{P}_k$, $\mathbf{r}_k = [r_{kx}, r_{ky}]^T$ where r_{kx}, r_{ky} are represented in the hand coordinates.

One of the two extreme points $\mathbf{r}_k$, will be named $\mathbf{q}_m$, and the other will be named $\mathbf{k}_m$, according to the relative positions of the two plates of the hand as follows:
A. Define $\mathbf{q}_m$ $(m = 1, 2)$, such that $q_{my} = \min r_{ky}$.
B. Define $\mathbf{q}_m$ such that $q_{my} = \max r_{ky}$.
C. Define $\mathbf{q}_m$ such that $q_{my} = \max r_{ky}$.
D. Define $\mathbf{q}_m$ such that $q_{my} = \min r_{ky}$.
E. Define $\mathbf{q}_m$ such that $q_{my} = \max r_{ky}$.
F. For this configuration, $\mathbf{q}_m$ will be selected according to the kinematics of the two plates. The criterion to name $\mathbf{q}_m$ is based on keeping the same nomenclature of the configuration from which this has been achieved, the idea behind is to minimize the renaming of extreme points. If no other configuration between the two plates can be achieved, the nomenclature used will be as the corresponding case to the ones shown in Figure 4F. If any other configuration can also be achieved, the naming of the two extreme points will be either as shown

in the same figure, or its opposite way, i. e., all points shown as $\mathbf{k}_m$ must be renamed $\mathbf{q}_m$, and vice versa.

Most kinematic configurations of 2-D two-fingered hands do not require to redesignate points $\mathbf{q}_k$. The only cases in which a redesignation of these points becomes necessary involve certain relative positions achievable that can be known in advance. Therefore, an eventual renaming of extreme points doesn't require more than monitoring the kinematic state of the hand.

6 Friction Cone of a Two-Fingered Hand

As proved in [1], if a straight plate contacts a rigid object at more than one point, because of the individual contact forces exerted by it on the object have always the same direction, they can be substituted by an equivalent force applied to the object at some point on the plate. Its magnitude and position are specified in the resultant wrench of the contacts at the plate. A force exerted on the object by another plate of the same hand, increases the availability of positions and orientations of the resultant wrench.

In this problem, no information about exact contact location is assumed. The approach used here is based on the wrenches that can be applied by the hand regardless where the contact points actually are on the two plates forming the hand.

6.1 Two Straight Plates Instead of Two Points

Definition 6.1 *The* friction cone of a two-plate hand *is the intersection of all combinations of two-point friction cones, each point belonging to a different plate.*

Denoting the friction cone of a two-plate hand by $\boldsymbol{\mathcal{S}}$, according to Definition 6.1,

$$\boldsymbol{\mathcal{S}} = \bigcap_{\substack{\mathbf{p}_I \in \mathbf{P}_I \\ \mathbf{p}_{II} \in \mathbf{P}_{II}}} \boldsymbol{\mathcal{S}}_{pI,pII} \tag{14}$$

The friction cone of a hand formed by two straight plates is the set of points belonging to lines of action of wrenches that may be applied to an object when the two plates contact it, regardless of the specific contact points on the plates.

Remark 6.1.1 *Each point* $\mathbf{s} \in \boldsymbol{\mathcal{S}}$ *is associated with a set* $\boldsymbol{\mathcal{U}}_s$ *of directed lines passing through it.* $\boldsymbol{\mathcal{U}}_s$ *represents all possible lines of action of resultant wrenches passing through point* $\mathbf{s}$ *that could be generated when the plates apply contact forces.*

$\boldsymbol{\mathcal{U}}_s$ is expressed in Plücker coordinates as

$$\boldsymbol{\mathcal{U}}_s = \left\{ \mathbf{U} = [\mathbf{u}^T, (\mathbf{s} \times \mathbf{u})^T]^T \,|\mathbf{U} \in \cup_{pI,pII} \boldsymbol{\mathcal{U}}_{s;pI,pII}; \mathbf{p}_I \in \mathbf{P}_I; \mathbf{p}_{II} \in \mathbf{P}_{II} \right\} \tag{15}$$

An alternative way to describe the friction cone of a two-fingered hand is by considering only the two extreme points of each plate and their respective two-point friction cones.

There is a common subset of points which belongs to the two-point friction cones of any pair of points, one on each plate, of these extreme points. This subset is the friction cone of the hand.

6.2 Two-Fingered Hand Basic Configurations Description

Figure 5 illustrates the possibilities for the relative positions of two plates of a hand. A general description of them is given in this subsection. A detailed explanation of Figures 5 is provided in Appendix A.

Assume that $\theta_k < \pi/2, k = I, II$, i. e., there is no glue at the contact points, and contact forces are based solely on friction. The spatial configuration of a hand can be described in terms of the following basic configurations (modes) for two plates while making contact with an object:

i. The two plates are front to front, or back to back[1], forming an angle ρ with magnitude in the range of $\theta_I + \theta_{II} \geq \rho \geq -\theta_I - \theta_{II}$, and either: (a) plate k is completely included in the friction cone of each point of plate l, as shown on top of figure 5a; or (b) one or both plates are not completely included in the friction cone of points of the opposite plate, as shown in bottom of Figure 5a and Figure 5b.

ii. Plate I is at the front of plate II, forming an angle ρ between them, out of the range of $\theta_I + \theta_{II} \geq \Lambda \geq -\theta_I - \theta_{II}$; and either: (a) the intersections between the pairs of lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$, and $\mathbf{U}^2_{q1}$ and $\mathbf{U}^2_{q2}$, are at the front or at the back of both plates, this case is shown for two different versions in Figures 5c and 5d; or (b) the intersection between lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$ is at the front of plate I, and the intersection between lines $\mathbf{U}^2_{q1}$ and $\mathbf{U}^2_{q2}$ is at the back of plate I, as shown in figure 5e; or (c) the intersection between lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$ is at the back of plate I, and the intersection between lines $\mathbf{U}^2_{q1}$ and $\mathbf{U}^2_{q2}$ is at the front of plate I, as shown in figure 5f; or (d) the intersection between the pairs of lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$, and $\mathbf{U}^2_{q1}$ and $\mathbf{U}^2_{q2}$, are at the back of plate I, as in figure 5g.

iii. The angle ρ between the two straight plates in the hand is in the range:

$$\pi + \theta_1 + \theta_2 > \rho > \pi - \theta_1 - \theta_2 \tag{16}$$

and for point $\mathbf{q}_m \in \mathbf{P}_k$,

$$\mathbf{q}_m \in \boldsymbol{\mathcal{S}}_{pl} \tag{17}$$

where $\boldsymbol{\mathcal{S}}_{pl}$ denotes the set of points on lines $\mathbf{U}_{pl} \in \boldsymbol{\mathcal{U}}_{pl}$, $\boldsymbol{\mathcal{S}}_{pl}$ is the friction cone of a point $\mathbf{p}_l \in \mathbf{P}_l$. Figure 5h illustrates this mode.

In Figure 5, the shadowed region represents the friction cone of the hands in each of the modes considered. Note that only mode *i*a has a friction cone covering the whole 2-D space. The dark shadowed region on Figure 5 represents the points $\mathbf{i} \in \mathbf{I}_{q1k1} \cap \mathbf{I}_{q2k2}$

All these basic configurations described show symmetry in two senses: indexes I and II can be interchanged[2] and the friction cone of the configuration is the same; also, in-

[1] On Figure 5, the arrow attached to each plate points towards its front.

[2] Provided that the friction coefficients are the same for plates I and II.

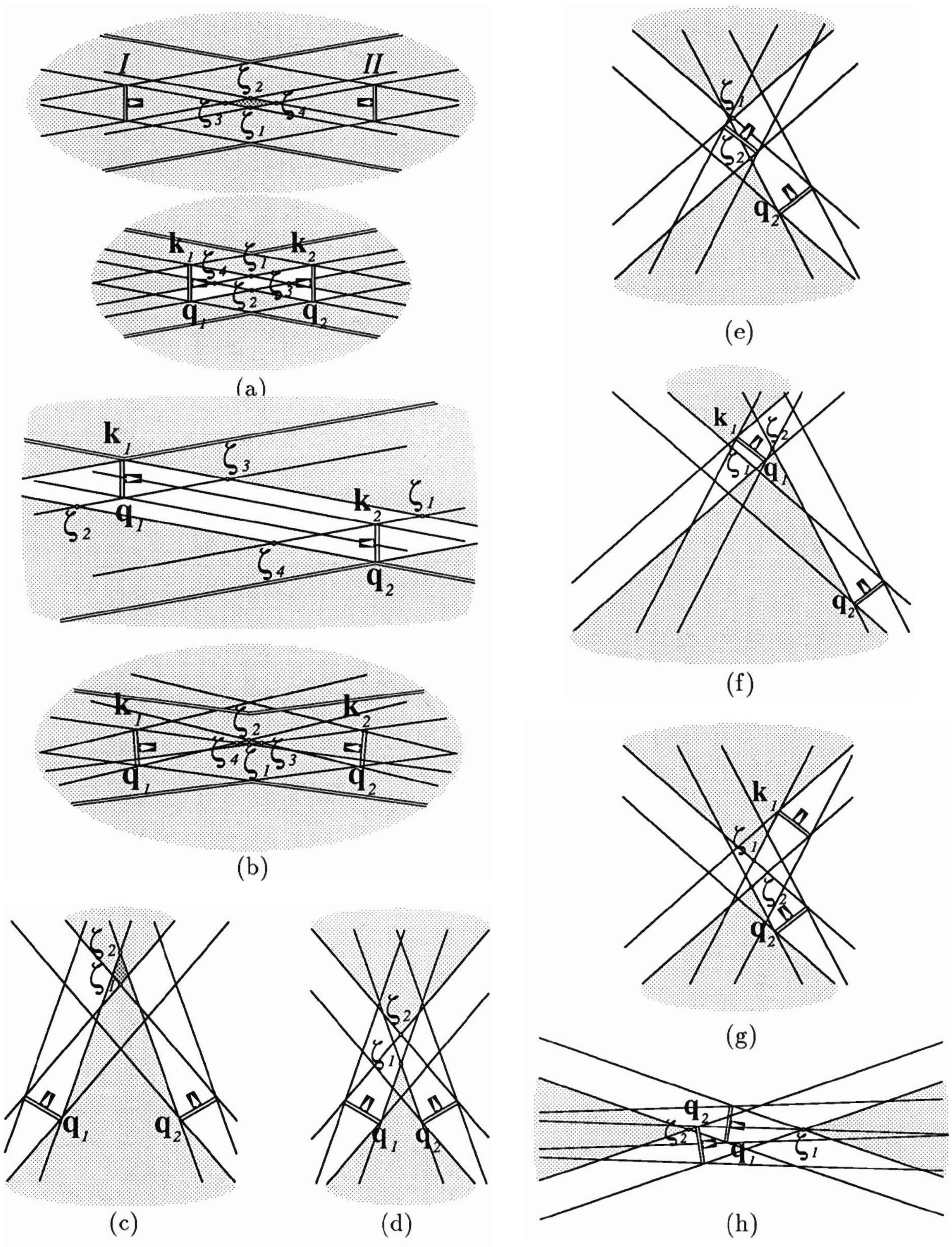


Figure 5: Friction Cones for Contacts with the Two Fingers of a Hand

terchanging the words "front" and "back" the friction cone of the resulting configuration remains the same.

Remark 6.2.1 *While the friction cone of a single plate was formed by the two convex cones described in [1], the friction cone of two plates represents a region larger than the addition of the friction cones of each single plate. Geometrically, this region can be represented as the addition of convex cones that may or may not overlap according to the configuration of the two plates.*

According to the reasoning explained in Appendix A, mode *iii* will not be considered for further analysis. In Appendix A, interchanging the words "back" and "front" in the definitions of the modes requires to change the sign of all directed lines $\mathbf{U}^{e}_{pd}$ used to define all modes[3]. This way the set of points in the friction cone of the hand remains the same for these and the originally proposed configurations.

For all cases, there always exist the possibility of having a point $\mathbf{i}^{e,d}_{pI,pII}$ placed at some point at infinity; i. e., $\mathbf{U}^{e}_{pI}$ and $\mathbf{U}^{d}_{pII}$, are parallel. A configuration like this, for mode *i*b, is illustrated in top of figure 5b.

The friction cone of a hand formed by plates I and II, is the subset of points reachable by a wrench generated by the hand, regardless on where on each plate the contact is made. As it might be expected, the friction cone of each single plate [1] is a subset of the friction cone of two plates for all modes. The uncertainty on the location of the contact on the two plates unables the determination of the specific set of wrenches associated with each point of the friction cone of two plates.

It is necessary to know if a wrench can be exerted on the object in the line of action of an opposite wrench exerted by the other hand (or exerted by two or more hands for multi-handed grasping), independently of the exact contact location at the plates.

7 Proper Grasping Configuration

Two hands in a proper grasping configuration, under the assumptions made in section 2, are able to generate two wrenches that balance each other, i. e., the two wrenches lie along the same line and have opposite directions.

In this section, the conditions for a proper grasping configuration with two hands will be developed. The subscripts used here follow the conventions established in sections 3 and 5.4. An extra subscript, either i or j, will be used to denote the hand.

[3]However, to be consistent with the nomenclature already set, for mode *ii*d, a renaming of the extreme points of each finger of a hand must be performed: interchanging the words "back" and "front" in the definitions implies interchanging all letters "k" and "q", and all subscripts "1" and "2". Anyhow, by doing so, the set of points remains the same.

7.1 Existence of Opposite Wrenches

A mathematical formulation of the requirement for a proper grasping configuration is developed in this subsection.

Consider a combination of two contact points on each hand, i. e., points $\mathbf{p}_{Ii}$, and $\mathbf{p}_{IIi}$ for hands $i = 1, 2$; and the sets $\mathcal{W}_{si;pI,pII}$ defined in Equation 9. In order for a pair of wrenches generated by that combination of points to be able to balance to each other, it is necessary that there exist points $\mathbf{s} \in \mathcal{S}_1 \cap \mathcal{S}_2$ on the line of action of wrenches $\mathbf{W}_{s1} \in \mathcal{W}_{s1;pI,pII}$ and $-\mathbf{W}_{s1} \in \mathcal{W}_{s2;pI,pII}$.

Let $\mathcal{W}^*_{s1;pI,pII} \subset \mathcal{W}_{s1;pI,pII}$ and $\mathcal{W}^*_{s2;pI,pII} \subset \mathcal{W}_{s2;pI,pII}$ be two sets of wrenches exertable by the combination of contact points on hands 1 and 2, which is defined by

$$\mathcal{W}^*_{si;pI,pII} = \{\mathbf{W}_{si} \mid \mathbf{W}_{si} = -\mathbf{W}_{sj}; \mathbf{W}_{si} \in \mathcal{W}_{si;pI,pII}; \mathbf{W}_{sj} \in \mathcal{W}_{sj;pI,pII}\} \tag{18}$$

To guarantee that two opposite wrenches can be generated by the two hands, it is required that for each combination of contact points in the hands, there exist points $\mathbf{s} \in \mathcal{S}_1 \cap \mathcal{S}_2$ for which $\mathcal{W}^*_{s1;pI,pII} \neq \emptyset$; i. e., each combination of points on one hand can generate wrenches counterbalancing wrenches generated by any combination of contact points on the other hand.

This condition can be similarly stated in terms of the lines of action of the wrenches generated by the hands. To do this, define $\mathcal{U}^*_{si;pI,pII}$

$$\mathcal{U}^*_{si;pI,pII} = \left\{\mathbf{U} \;\middle|\; f\mathbf{U} \in \mathcal{W}^*_{si;pI,pII}\right\} \tag{19}$$

The condition expressed above in terms of wrenches implies that there exist points $\mathbf{s} \in \mathcal{S}_1 \cap \mathcal{S}_2$ in lines $\mathbf{U}_{s1} \in \mathcal{U}^*_{s1;pI,pII}$ and $-\mathbf{U}_{s1} \in \mathcal{U}^*_{s2;pI,pII}$ for each combination of contact points $\mathbf{p}_{I1}$, $\mathbf{p}_{II1}$, $\mathbf{p}_{I2}$, and $\mathbf{p}_{II2}$. If $\mathcal{U}^*_{si;pI,pII} = \emptyset$, $\forall \mathbf{s} \in \mathcal{S}_1 \cap \mathcal{S}_2$, for a given combination of contact points, no wrenches that counterbalance to each other can be generated by the hands for that combination.

A procedure to test the condition stated above is to divide it into two steps: (1) verifying that the two hands are able to generate wrenches with opposite orientation; and (2) verifying that two wrenches with opposite orientation are on the same line.

7.2 The Orientation Condition

Here, a method to verify that the two hands are able to generate wrenches with opposite orientation is shown. Note that a hand in mode ia, or in mode ib, can generate a resultant wrench having any orientation.

Denote by ρ_i, the angle between the plates of hand i. Define

$$\theta_i^1 = \frac{|\pi - \rho_i| + \theta_{1i} + \theta_{2i}}{2} \tag{20}$$

which is half the angle between the two extreme directions, $\mathbf{U}^1_{pIi}$ and $\mathbf{U}^1_{pIIi}$ of points $\mathbf{p}_{Ii}$ and $\mathbf{p}_{IIi}$ on plates I and II, of hand i.

Denote by $\mathbf{U}_i^{c1}$ a line that, with respect to both lines $\mathbf{U}_{pIi}^1$ and $\mathbf{U}_{pIIi}^1$, has an angle whose absolute value is equal to θ_i^1. Assuming that $\theta_{Ii} = \theta_{IIi}$, i. e., the friction coefficient of the two plates are the same; then, $\theta_i^1 = \frac{|\pi - \rho_i|}{2} + \theta_{ki}$; and, line $\mathbf{U}_i^{c1}$ is parallel to y axis of hand i if $\rho_i \neq 0$, and is orthogonal if $\rho_i = 0$.

7.2.1 Exerting Parallel Forces with Opposite Direction with a Hand

The interest on a hand with the capability of exerting opposite forces is because this feature makes possible to exert wrenches that lie far from the hand, with a line of action not touching any of the hand contact points. Therefore, a balance for a bigger set of wrenches on the opposite hand can be provided.

Consider hand i in either mode ia or ib. Only in those modes can the hand exert opposite forces to the object. Name $\mathbf{U}_{m,n;i}^{k,q}$ the line passing through points $\mathbf{k}_{mi}$ and $\mathbf{q}_{ni}$ with direction going from $\mathbf{k}_{mi}$ to $\mathbf{q}_{ni}$. Under this notation, $\mathbf{U}_{n,m;i}^{q,k} = -\mathbf{U}_{m,n;i}^{k,q}$. The angle between lines $\mathbf{U}_i^{c1}$ and $\mathbf{U}_{m,n;i}^{k,q}$ will be denoted by $\gamma_{m,n;i}^{k,q}$. According to the notation established in 5.4, $|\gamma_{m,n;i}^{k,q}| > \frac{\pi}{2}$. Note that $\gamma_{m,n;i}^{q,k} = \pi + \gamma_{n,m;i}^{k,q}$.

The orientation of the wrench generated by opposite forces depends on the contact points. In this study, only wrenches that can be generated by opposite forces independently of contact points at the plates of the hand will be considered.

The orientation of a wrench that can be generated by opposite forces exerted at any combination of contact points in the hand, can be expressed by a vector $\mathbf{u}$, defined as follows:

1. If hand i is in mode ia,

$$\mathbf{u} = \pm \begin{cases} \alpha \left(\lambda \mathbf{u}_{kmi}^1 + (1-\lambda)\mathbf{u}_{m,n;i}^{k,q}\right) \\ \beta \left(\epsilon \mathbf{u}_{n,m;i}^{k,q} - (1-\epsilon)\mathbf{u}_{kmi}^1\right) \end{cases} \tag{21}$$

2. If hand i is in mode ib, and a resultant wrench can be generated from forces exerted at any combination of contact points in the hand; i. e., if hand i is in mode ib and

   (a) $|\gamma_{m,n;i}^{k,q}| - \frac{\pi}{2} \geq \theta_{ki}$,

$$\mathbf{u} = \pm\alpha \left(\lambda \mathbf{u}_{kmi}^1 + (1-\lambda)\mathbf{u}_{kmi}^2\right) \tag{22}$$

   (b) or $|\gamma_{m,n;i}^{k,q}| - \frac{\pi}{2} < \theta_{ki}$, but $|\gamma_{m,n;i}^{k,q}| < \theta_i^1$

$$\mathbf{u} = \pm\alpha \left(\lambda \mathbf{u}_{kmi}^1 + (1-\lambda)\mathbf{u}_{m,n;i}^{k,q}\right) \tag{23}$$

where $0 \leq \lambda, \epsilon \leq 1$; and α, β are normalizing factors.

If hand i is in mode ib, pure moments either clockwise or counterclockwise can be generated; whereas in mode ia, moments in both directions can be generated.

7.2.2 Orientation Condition

All wrenches in the friction cone of points $\mathbf{p}_{Ii}$ and $\mathbf{p}_{IIi}$ have an orientation that can be expressed as a linear combination of the orientation of the lines with extreme directions, $\mathbf{U}^1_{pIi}$, $\mathbf{U}^1_{pIIi}$ $\mathbf{U}^2_{pIi}$ and $\mathbf{U}^2_{pIIi}$, as expressed by equation 11.

Therefore, for two hands to be able to exert counterbalancing wrenches, the possibility of creating wrenches with opposite orientation must exist. This, translated to relative angles between lines $\mathbf{U}^{c1}_i$ and $\mathbf{U}^{c1}_j$, imply the following condition:

- If neither hand is in mode ia nor ib, the angle ϑ between lines $\mathbf{U}^{c1}_i$ and $\mathbf{U}^{c1}_j$, must be restricted to the interval:

$$\pi - \theta^1_i - \theta^1_j \leq \vartheta \leq \pi + \theta^1_i + \theta^1_j \tag{24}$$

- If parallel forces with opposite direction are being exerted by hand i; and hand j is neither in mode ia nor mode ib, to have a resultant wrench with an orientation independent of the contact points, the angle ϑ between lines $\mathbf{U}^{c1}_i$ and $\mathbf{U}^{c1}_j$ must be restricted to the following interval

  1. For mode ia

$$\begin{array}{rc} & \pi + \theta^1_i - 2\theta_{ki} + \rho_i - \theta^1_j \leq \vartheta \leq \gamma^{k,q}_{2,1;i} + \theta^1_j \\ \text{or} & \pi + \gamma^{k,q}_{1,2;i} - \theta^1_j \leq \vartheta \leq \pi + \theta^1_i + \theta^1_j \end{array} \tag{25}$$

  2. For mode ib under the condition that $\min(|\gamma^{k,q}_{m,n;i}|) > \theta^1_i - 2\theta_{ki} + \rho_i$. Define $\theta^2_i = \max\left((\theta^1_i - 2\theta_{ki}), \min(|\gamma^{k,q}_{m,n;i}|)\right)$,

$$\pi + \theta^2_i - \theta^1_j \leq \vartheta \leq \pi + \theta^1_i + \theta^1_j \tag{26}$$

No restrictions on orientation apply if one of the hands is in either mode ia or ib and no parallel forces with opposite direction are exerted by it, or when the hands are in one or the other of modes ia, and ib, even if both hands are applying opposite forces.

The orientation condition is always satisfied by any proper grasping configuration. If, for a grasping configuration, the orientation condition is not true, no points $\mathbf{s} \in \mathcal{S}_1 \cap \mathcal{S}_2$ such that $\mathcal{W}^*_{s1} \neq \emptyset$ exist. Therefore, the grasping configuration is not proper.

7.3 Two Hands in a Proper Grasping Configuration

7.3.1 The Position Condition

Two opposite wrenches can be generated only if the orientation condition has been fulfilled. However, even if the orientation condition is true, the capability of exerting two opposite wrenches is not assured for all possible combinations of two points on each hand. An additional condition, namely the position condition, must be fulfilled. The position condition states that every combination of contact points at the two hands must be able to generate two wrenches with opposite orientation acting on the same line.

This section analyzes the position condition.

Two basic issues are explored here: (1) the characteristics of wrenches generable by a hand; and (2) how, per every combination of point contacts, at least a wrench generable at the opposite hand can always be counterbalanced.

7.3.2 Intersection Region for Wrenches Generable by a Hand

As it was seen in section 4.3, all wrenches generated by contact forces exerted on two points $\mathbf{p}_I$ and $\mathbf{p}_{II}$ pass through set $\mathbf{I}_{pI,pII}$. Then, the condition of having two opposite wrenches along the same line of action can be established in terms of sets $\mathbf{I}_{pI,pII}$. This is, wrenches generated on hand i must be able to cross sets $\mathbf{I}_{pIj,pIIj}\,\forall\,\mathbf{p}_{kj} \in \mathbf{P}_{kj}$, for $i, j = 1, 2$.

To use a compact notation, define $\mathcal{I}_i$ as

$$\mathcal{I}_i = \{\mathbf{i} \mid \mathbf{i} \in \mathbf{I}_{pIi,pIIi}, \mathbf{p}_{Ii} \in \mathbf{P}_{Ii}, \mathbf{p}_{IIi} \in \mathbf{P}_{IIi}\} \tag{27}$$

All the lines $\mathbf{U} \in \mathcal{U}_{si} \forall \mathbf{s} \in \mathcal{S}_i$, pass through points $\mathbf{i} \in \mathcal{I}_i$.

Also, define $\mathcal{I}_i^*$ as the minimum size convex set of points containing (1) at least one point from each set of points $\mathbf{I}_{u,v}$, where $u = \mathbf{k}_{1i}, \mathbf{q}_{1i}; v = \mathbf{k}_{2i}, \mathbf{q}_{2i}$, if the hand is in any mode except ib; or (2) all the points in the space not included in the friction cone of the two-plate hand, if the hand is in mode ib. For the first case, $\mathcal{I}_i^* \subset \mathcal{I}_i$.

To illustrate the definition of sets $\mathcal{I}_i^*$, Figure 5 can be used. The case shown in top of Figure 5a corresponds to infinitely many sets $\mathcal{I}_i^*$, each of them is just one point in the darkest shadowed region. In all remaining cases, $\mathcal{I}_i^*$ is a single convex set of points.

Except for mode ib, the definition of the set $\mathcal{I}_i^*$ is determined by the two possibilities for the sets $\mathbf{I}_{u,v}$ in hand i: they either have or do not have a common region. Denote by $\mathcal{T}$, the common region shared by all $\mathbf{I}_{u,v}$, $\mathcal{T} = \bigcap_{u,v} \mathbf{I}_{u,v}$. The characteristics of the $\mathcal{T}$ are analyzed in Appendix B. For mode ib, the focus will rather be on the characteristics of the set $\mathcal{I}_i^*$, some situations for this mode are also shown in Appendix B.

7.3.3 A Sufficient Condition

Note that a grasp with a hand in mode ia is proper, since that hand can generate a wrench in any line of action and with any direction. The remaining modes do not have this special capability, therefore, to prove a proper grasping configuration with hands in those modes deeper analysis is required.

If wrenches that can be generated by any combination of contact points at hand i are able to cross the whole set $\mathcal{I}_j$, for $i, j = 1, 2$; then, wrenches with opposite direction acting along infinitely many lines can be generated by the two hands.

Based on the previous reasoning, a sufficient condition for a proper grasping configuration is that all points $\mathcal{I}_i$ lie in the cone of friction of the hand j, i. e., $\mathcal{I}_i \in \mathcal{S}_j$; for $i = 1, 2$. This way, the existence of lines $\mathbf{U}_{si}$ such that $\mathbf{U}_{si} = -\mathbf{U}_{sj}$ is guaranteed for points $\mathbf{s} \in \mathcal{S}_i \cap \mathcal{S}_j$, for all combinations of contact points $\mathbf{p}_{I1}$, $\mathbf{p}_{II1}$, $\mathbf{p}_{I2}$, and $\mathbf{p}_{II2}$.

A less restrictive sufficient condition for a proper grasping configuration, applicable to grasps having hand i in either mode ia, or mode ib, is that all points $\mathbf{i} \in \mathcal{I}_j$ belong to lines of action $\mathbf{U}_{si} \in \mathcal{U}^*_{si}$. This means that for at least hand i, all points in $\mathcal{I}_j$ belong to lines of action of wrenches $\mathbf{W}_{si} \in \mathcal{W}^*_{si}$. This way, per every combination of contacts at the two plates on each hand, there are wrenches generable on one hand able to counterbalance wrenches generable at the other hand.

These conditions are sufficient but not necessary. If fulfilled, more than one choice of generable wrenches would be available per combination of contacts at the hands to get a balanced system. A sufficient and necessary condition is that at least a wrench generable per every combination of contacts at the two plates on each hand, is able to counterbalance at least a wrench generable at the other hand.

7.3.4 Relaxing the Sufficient Condition

An effort to reduce the demands of the sufficient condition expressed in the previous subsection will be made here.

Define the set $\mathbf{T}_j$, $\mathbf{T}_j \subset \mathbf{S}_j$, the set for which all $\mathbf{t} \in \mathbf{T}_j$ have wrenches $\mathcal{W}_{tj;pIpII}$ (set of wrenches generated by hand j, passing through point $\mathbf{t}$), crossing through $\mathcal{I}^*_j$, for all combinations of contact points $\mathbf{p}_{kj} \in \mathbf{P}_{kj}; k = I, II; j = 1, 2$. Define set $\mathbf{R}_j$ to be the complement of $\mathbf{T}_j$ with respect to $\mathbf{S}_j$.

For two fingers in hand j intended to generate a wrench counterbalancing the one generated at hand i, a set $\mathcal{W}_{si}$ crossing a single set $\mathcal{I}^*_j \subset \mathcal{I}_j$ will be considered.

Requiring that only points in a single set $\mathcal{I}^*_j$ instead of points in the whole set $\mathcal{I}_j$ be completely crossed by $\mathcal{W}_{ti}$ (the set of wrenches generated by hand i, passing through point $\mathbf{t}$) is a sufficient condition to guarantee that there is at least a wrench generable per every combination of contacts at the two plates on each hand able to counterbalance at least a wrench generable at the other hand.

This way, the size of the set of points considered is reduced. The set of points $\mathcal{I}^*_j$ includes at least one point touched by a generable wrench per combination of two contacts, one at each finger of the hand.

However, for points $\mathbf{r} \in \mathbf{R}_j$, the simple substitution of $\mathcal{I}_j$ by $\mathcal{I}^*_j$ as the set of points to be completely crossed by the set $\mathcal{W}_{ri}$, does not guarantee the existence of at least a wrench generated per every combination of contact points, able to counterbalance at least a wrench generable at the other hand.

To have a wrench $\mathbf{W} \in \mathcal{W}_{rj}$ such that $-\mathbf{W} \in \mathcal{W}_{ri}$ exists, it is required that $\mathcal{W}_{ri}$ covers $\mathcal{I}^*_j$, and a linear combination of the points $\mathbf{p}_I$ and $\mathbf{p}_{II}$ for which $\mathcal{W}_{rj;pIpII}$ does not cross $\mathcal{I}^*_j$. This means that there also must exist $\mathbf{W} \in \mathcal{W}_{ri}$ passing through:

1. $\mathbf{k}_1$, $\mathbf{q}_1$, $\mathbf{k}_2$, and $\mathbf{q}_2$; for mode ib.
2. a point in segment $\overline{\mathbf{q}_1\mathbf{q}_2}$; for mode iia.
3. all points in $\mathbf{P}_I$, or a point in segment $\overline{\mathbf{q}_1\mathbf{q}_2}$; for modes iib and iic.
4. at least $\mathbf{k}_1$ or $\mathbf{q}_2$ for mode iid.

This way, at least a wrench generable per every combination of contacts at the two plates on each hand, is able to counterbalance at least a wrench generable at the other hand.

Note that for each mode, all points $\mathbf{r} \in \mathbf{R}_j$ belong to friction cones of the extreme points of plates of a hand required to be covered by the friction cone of the opposite hand. Rename that extreme point whose friction cone includes the point $\mathbf{r}$ in consideration; call that point $\mathbf{d}_{mj}$ instead of $\mathbf{k}_{mj}$ or $\mathbf{q}_{mj}$.

In conclusion, satisfying the condition that $\mathcal{W}^*_{ri}$ covers both, the set $\mathcal{I}^*_j$, and the extreme point $\mathbf{d}_{mj}$, for $i = 1, 2$, implies that a proper grasping configuration has been achieved by the two hands.

7.3.5 Counterbalancing Wrenches

According with the results obtained so far, if hand i is in mode ia, hand i can always generate a wrench opposite to a wrench resultant from forces exerted at the plates of hand j.

If hand j is in mode ib, and hand i is not in mode ia, for hand j to be able to generate a wrench opposite to a wrench resultant from forces exerted at the plates of hand i, it is required that, firstly, the condition established in section 7.3.4 is true for hand j; and secondly, that hand j can generate a moment with opposite direction than the one generated by hand i.

A verification strategy for the second requirement is as follows. Take the extreme points of hand j with the maximum difference in hand j coordinates on (1) ordinate value, if $\rho_j \neq 0$; (2) abscissa value, if $\rho_j = 0$. Name those points $\mathbf{a}_{mj}$ and $\mathbf{a}_{nj}$. Use as a reference $\mathbf{a}_{mj}$. Define $\mathbf{v}_i$, a scaled linear combination of vectors $\mathbf{u}^1_{pIi}$ and $\mathbf{u}^1_{pIIi}$. Determine if there exists a $\mathbf{v}_i$ placed at point[4] $\mathbf{q}_{1i}$ such that its moment with respect to $\mathbf{a}_{mj}$ is opposite to the moment of any linear combination of vectors $\mathbf{u}^1_{amj}$ and $-\mathbf{u}^1_{anj}$ placed at point $\mathbf{a}_{nj}$; i. e., determine if there exist two vectors $\mathbf{v}_i = \alpha\left(\lambda \mathbf{u}^1_{pIi} + (1-\lambda)\mathbf{u}^1_{pIIi}\right)$, and $\mathbf{v}_j = \epsilon \mathbf{u}^1_{amj} - (1-\epsilon)\mathbf{u}^1_{anj}$ with $0 \leq \lambda, \epsilon \leq 1$, and $\alpha > 0$, such that

$$(\mathbf{a}_{nj} - \mathbf{a}_{mj}) \times \mathbf{v}_j + (\mathbf{q}_{1i} - \mathbf{a}_{mj}) \times \mathbf{v}_i = 0 \qquad (28)$$

If not, the grasping configuration is not proper. Generally speaking, it is expected that two hands, both in mode ib are in a proper grasping configuration.

For all remaining cases, the conditions established in section 7.3.4 are required.

7.4 Testing a Grasping Configuration

A proper grasping configuration of two two-plate hands, implies the existence of at least one common line, through which opposite wrenches can act, per every combination of contacts at the plates of both hands; i. e., a wrench $\mathbf{W}_i$ can be found for each possible combination of contact points $\mathbf{p}_{Ii}$ and $\mathbf{p}_{IIi}$, able to counterbalance the wrench $\mathbf{W}_j$ resultant from wrenches $\mathbf{F}_{pIj}$ and $\mathbf{F}_{pIIj}$, exerted by hand j.

[4] Any point related to hand i, and required to be in the friction cone of hand j, as stated in section 7.3.4, can be used. $\mathbf{q}_{1i}$ is just an easy one to pick.

According to the analysis developed, the requirements for a proper grasping configuration are that either, (I) at least one of the hands is in mode ia; or, (II) for both hands

1. The orientation condition, stated in subsection 7.2, is fulfilled.

2. The friction cone of hand i (for $i = 1, 2$), made out of two plates, covers:

   (a) At least one entire set $\mathcal{I}_j^*$,

   (b) If all sets $\mathcal{I}_i^*$ have an extreme point lying inside the friction cone of the point $\mathbf{d}_{mj}$, the point $\mathbf{d}_{mj}$ of the respective cone must also be covered;

or finally, (III) if hand i is in mode ib with a configuration according to item 2 in section 7.2.1, and it is the only hand fulfilling items 2a, and 2b of the previous case, the conditions stated in section 7.3.5 are required to be satisfied.

When the previous conditions are true, the direction of a wrench generated by hand i, can be opposite to a wrench generated by hand j, for any combination of contact points at the two plates of the hands.

After the orientation condition has been satisfied, the number of points required to be checked to prove a proper grasping configuration, is considerably small (between 2 and 4 per hand).

Hands with Symmetric Features

One particular case for hands made out of two plates is hands with identical plates in a symmetric kinematic configuration, and with the same friction coefficient for the plates. In this case, there are just two degrees of freedom for the whole hand: one determines the distance between the plates; and the other one determines the angle formed by them.

There is a symmetry axis where the intersection of all $\mathbf{U}^e_{rI}$ and $\mathbf{U}^e_{rII}$, for $e = 1, 2; \mathbf{r}_k = \mathbf{c}_k + \lambda \mathbf{y}_k; -1 \leq \lambda \leq 1; k = I, II$, occur for each given value of λ. Here, $\mathbf{y}_k$ represents the basis vector of plate k coordinates normal to the plate surface. The friction cone is also symmetrical along the same axis; and the configuration of the hand can only be on modes ia, ib or iia. Line $\mathbf{U}^{c1}_i$ is parallel to the symmetry axis of hand i.

If one hand is in mode ia the grasping configuration is proper. If both hands are in mode ib a pair of opposite wrenches generated by the two hands can always be found, then, the grasping configuration is also proper. For the remaining cases, the angle ϑ between lines $\mathbf{U}^{c1}_1$ and $\mathbf{U}^{c1}_2$, must be restricted to the interval expressed in Inequality 24.

To illustrate the application of the position condition in a grasp using two hands with symmetric features, consider two hands in mode iia. Two possibilities exist for the hands: (a) hand i is in such a configuration that more than one different set $\mathcal{I}_i^*$ can be found for it ($\bigcap_{u,v} \mathbf{I}_{u,v} \neq \emptyset$); and (b) hand i is in such a configuration that only a unique set $\mathcal{I}_i^*$ can be found for it ($\bigcap_{u,v} \mathbf{I}_{u,v} = \emptyset$).

To guarantee that hand i can balance, through the grasped object, forces exerted by hand j, no matter where on the hands the forces are being applied, it is necessary that:

(i) the angle ϑ is in the interval expressed in Inequality 24; and (ii) for case (a), at least one point of each segment $\overline{\mathbf{i}^{1,2}_{k2,q1}\mathbf{i}^{1,2}_{k1,q2}}$, and $\overline{\mathbf{q}_{1i}\mathbf{q}_{2i}}$, of hand i, are inside of the friction cone of hand j; and for case (b) all points of the segment $\overline{\mathbf{i}^{1,2}_{k2,q1}\mathbf{i}^{1,2}_{k1,q2}}$, and at least one point of the segment $\overline{\mathbf{q}_{1i}\mathbf{q}_{2i}}$ of hand i, are inside of the friction cone of hand j. Condition (ii) must be true for all hands i in modes iia; if no hand is in mode iia, it must be true for at least one hand in mode ib.

9 Comments and Discussion

The results obtained so far, can be easily modified to include the case of a grasp using two different hands: one with a single plate and the other with two plates.

It is also possible to extend the scope of this analysis to the multi-agent grasping, including whole arm grasping. For whole arm grasping, an outline of the approach that could be used is described as follows. The contact surface can be divided in regions. Providing each region with the sensors considered for this analysis, the results obtained in this work can be applied. Two or more plates contacting the object can be associated in one "hand". The resultant friction cone of that hand would be the addition of the sets of points of each combination of two plates contacting the object.

For the application of the results shown here, it is required to detect if the contact is being made at the edge of the plate. For a contact at the edge of a plate, the object's surface would be the one which would be determining the normal to the contact, and the reasoning developed here would not be true for this situation.

In the problems treated so far, it has been assumed that the objects to be grasped are weightless. To be able to extend the results obtained to a weighted object, a capability of detecting moments at the contact surfaces or of having force measurements at the joints that can be clearly attributed to the weight of the object being grasped is necessary.

In the grasping process, before trying to lift or manipulate the object, the emphasis has been given to exerting forces on the object that can counterbalance to each other, and static equilibrium can be achieved.

When static equilibrium has been guaranteed, it is possible to find the room that, according with the scale of the contact forces applied to the object, could be used to overcome the weight of the object. The capability of balancing the weight of the object could rely heavily on friction. Friction forces are proportional to normal forces. Then, to have large friction forces, we require large normal forces. Here, the fragility of the object becomes an issue which is not considered in the present study.

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A Friction Cones for Each Mode

This section describes how the geometry of the friction cone of a hand can be expressed in terms of convex cones.

The focal point of a plate k considered here, is defined as the point $\boldsymbol{\zeta}_k$ in front of plate k in which lines $\mathbf{U}^d_{kk}$ and $\mathbf{U}^e_{qk}$ intersect, with $d, e = 1, 2$; $d \neq e$ [1]. The friction cone of a two-fingered hand is shown as a shaded area in Figure 5. It can be defined in terms of the convex cones $\mathbf{S}_r$, as the set of points $\boldsymbol{\mathcal{S}}$ which is, for each case, as follows.

Mode i.

a. The friction cones of the two plates intersect, having the focal point $\boldsymbol{\zeta}_k$ inside of the friction cone of plate l, where $l, k = I, II$; $l \neq k$; therefore the friction cone of the two plates covers the whole space; i. e.

$$\mathcal{S} = \Re^2 \tag{29}$$

$\mathcal{S}$ can be represented as

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \cup \mathbf{S}_4 \tag{30}$$

where

$$\begin{aligned}
\mathbf{S}_1 &= \boldsymbol{\zeta}_1 - \psi_1 \mathbf{u}_{k1}^1 - \psi_2 \mathbf{u}_{k2}^1 \\
\mathbf{S}_2 &= \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}_{q1}^2 - \psi_4 \mathbf{u}_{q2}^2 \\
\mathbf{S}_3 &= \boldsymbol{\zeta}_3 + \psi_5 \mathbf{u}_{k1}^1 + \psi_6 \mathbf{u}_{q1}^2 \\
\mathbf{S}_4 &= \boldsymbol{\zeta}_4 + \psi_7 \mathbf{u}_{k2}^1 + \psi_8 \mathbf{u}_{q2}^2
\end{aligned}$$

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}_{k1}^1$ and $\mathbf{U}_{k2}^1$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}_{q1}^2$ and $\mathbf{U}_{q2}^2$, $\boldsymbol{\zeta}_3$ is the intersection point between lines $\mathbf{U}_{k1}^1$ and $\mathbf{U}_{q1}^2$, and $\boldsymbol{\zeta}_4$ is the intersection point between lines $\mathbf{U}_{k2}^1$ and $\mathbf{U}_{q2}^2$, $(\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_I, \boldsymbol{\zeta}_4 = \boldsymbol{\zeta}_{II})$; and $\psi_i \geq 0; i = 1, \cdots, 8$.

This case is shown in top of figure 5a, the darker area represents $\mathbf{S}_1 \cap \mathbf{S}_2 \cap \mathbf{S}_3 \cap \mathbf{S}_4$. Resultant wrenches passing through this region can be generated regardless to where the contact on the two plates are. It can be treated similarly to the case of two contact points with opposite forces acting on them, and each point lying inside the friction cone of the other.

If opposite forces are being exerted by the plates, depending on the direction of them, they can exert on the grasped object pure moments in both directions: clockwise and counterclockwise; for all possible combination of contact points in both plates. If forces exerted are collinear, no pure moment can be exerted.

b. The friction cone of the two plates doesn't cover the whole space since there is the possibility that two contact points, each on one different plate, don't lie in the friction cone of each other.

The friction cone of two plates for this case is illustrated in the bottom of figure 5a and in figure 5b. The shaded area shows the friction cone of the two plates assuming that opposite forces in parallel lines of action are not being exerted on the object. A strip between the forces exerted must be excluded from the friction cone. The limits of this region are shown for different cases in Figures 5a and 5b, by a double line.

For this case, it is important to note that if one of the plates is touched by the friction cone of one or more points on the other plate, it may or may not be possible to exert pure moments in both directions: clockwise and counterclockwise; and given the uncertainty on the location of contacts, it will not be possible to figure out in advance in which direction a pure moment could be generated. If no friction cone of any point on each plate, touches the other plate, pure moments in only one direction can be generated.

The friction cone for this case is given by the set

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_3 \cup \mathbf{S}_4 \cup \mathbf{S}_{k1} \cup \mathbf{S}_{k2} \cup \mathbf{S}_{q1} \cup \mathbf{S}_{q2} \tag{31}$$

where

$$\mathbf{S}_1 = \boldsymbol{\zeta}_1 - \psi_1 \mathbf{u}^1_{k1} - \psi_2 \mathbf{u}^1_{k2}$$
$$\mathbf{S}_2 = \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}^2_{q1} - \psi_4 \mathbf{u}^2_{q2}$$
$$\mathbf{S}_3 = \boldsymbol{\zeta}_3 + \psi_5 \mathbf{u}^1_{k1} + \psi_6 \mathbf{u}^2_{q1}$$
$$\mathbf{S}_4 = \boldsymbol{\zeta}_4 + \psi_7 \mathbf{u}^1_{k2} + \psi_8 \mathbf{u}^2_{q2}$$
$$\mathbf{S}_{k1} = \begin{cases} \mathbf{k}_1 - \psi_9 \mathbf{u}^1_{k1} - \psi_{10} \mathbf{u}^2_{k1} & \text{if } (\boldsymbol{\zeta}_1 - \mathbf{k}_1) \cdot \mathbf{y}_I \geq 0 \\ \emptyset & \text{otherwise} \end{cases}$$
$$\mathbf{S}_{k2} = \begin{cases} \mathbf{k}_2 - \psi_{11} \mathbf{u}^1_{k2} - \psi_{12} \mathbf{u}^2_{k2} & \text{if } (\boldsymbol{\zeta}_1 - \mathbf{k}_2) \cdot \mathbf{y}_{II} \geq 0 \\ \emptyset & \text{otherwise} \end{cases}$$
$$\mathbf{S}_{q1} = \begin{cases} \mathbf{q}_1 - \psi_{13} \mathbf{u}^1_{q1} - \psi_{14} \mathbf{u}^2_{q1} & \text{if } (\boldsymbol{\zeta}_2 - \mathbf{q}_1) \cdot \mathbf{y}_I \geq 0 \\ \emptyset & \text{otherwise} \end{cases}$$
$$\mathbf{S}_{q2} = \begin{cases} \mathbf{q}_2 - \psi_{15} \mathbf{u}^1_{q2} - \psi_{16} \mathbf{u}^2_{q2} & \text{if } (\boldsymbol{\zeta}_2 - \mathbf{q}_2) \cdot \mathbf{y}_{II} \geq 0 \\ \emptyset & \text{otherwise} \end{cases}$$

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}^2_{q1}$ and $\mathbf{U}^2_{q2}$, $\boldsymbol{\zeta}_3$ is the intersection point between lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^2_{q1}$, and $\boldsymbol{\zeta}_4$ is the intersection point between lines $\mathbf{U}^1_{k2}$ and $\mathbf{U}^2_{q2}$, $(\boldsymbol{\zeta}_3 = \boldsymbol{\zeta}_I, \boldsymbol{\zeta}_4 = \boldsymbol{\zeta}_{II})$; and $\psi_i \geq 0; i = 1, \cdots, 16$.

Mode *ii*.

a. The friction cone of the two plates in this configuration is given by the set

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_{q1} \cup \mathbf{S}_{q2} \tag{32}$$

where

$$\mathbf{S}_1 = \boldsymbol{\zeta}_1 + \psi_1 \mathbf{u}^1_{k1} + \psi_2 \mathbf{u}^1_{k2}$$
$$\mathbf{S}_2 = \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}^2_{q1} - \psi_4 \mathbf{u}^2_{q2}$$
$$\mathbf{S}_{q1} = \mathbf{q}_1 - \psi_5 \mathbf{u}^1_{q1} - \psi_6 \mathbf{u}^2_{q1}$$
$$\mathbf{S}_{q2} = \mathbf{q}_2 - \psi_7 \mathbf{u}^1_{q2} - \psi_8 \mathbf{u}^2_{q2}$$

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}^2_{q1}$ and $\mathbf{U}^2_{q2}$; and $\psi_i \geq 0; i = 1, \cdots, 8$. On Figure 5c, the darker area represents $\mathbf{S}_1 \cap \mathbf{S}_2$. Resultant wrenches passing through this region can be generated regardless to where the contact on the two plates are.

b. Plate I is in a region not covered by the friction cone of point $\mathbf{k}_2$, but it is at least partially covered by the friction cone of a point on plate II. The friction cone of the two plates in this configuration is given by the set

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_{q2} \tag{33}$$

where

$$\mathbf{S}_1 = \boldsymbol{\zeta}_1 + \psi_1 \mathbf{u}^1_{k1} + \psi_2 \mathbf{u}^1_{k2}$$

$$\mathbf{S}_2 = \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}_{q1}^1 - \psi_4 \mathbf{u}_{q2}^2$$

$$\mathbf{S}_{q2} = \mathbf{q}_2 - \psi_5 \mathbf{u}_{q2}^1 - \psi_6 \mathbf{u}_{q2}^2$$

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}_{k1}^1$ and $\mathbf{U}_{k2}^1$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}_{q1}^1$ and $\mathbf{U}_{q2}^2$; and $\psi_i \geq 0; i = 1, \cdots, 6$.

c. Plate I is in a region covered by both, the friction cone of point $\mathbf{k}_2$, and the friction cone of point $\mathbf{q}_2$. The friction cone of the two plates in this configuration is given by the set

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_{k1} \cup \mathbf{S}_{q1} \cup \mathbf{S}_{q2} \tag{34}$$

where

$$\mathbf{S}_1 = \boldsymbol{\zeta}_1 + \psi_1 \mathbf{u}_{k1}^2 + \psi_2 \mathbf{u}_{k2}^1$$

$$\mathbf{S}_2 = \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}_{q1}^2 - \psi_4 \mathbf{u}_{q2}^2$$

$$\mathbf{S}_{k1} = \mathbf{k}_1 + \psi_5 \mathbf{u}_{k1}^1 + \psi_6 \mathbf{u}_{k1}^2$$

$$\mathbf{S}_{q1} = \mathbf{q}_1 - \psi_7 \mathbf{u}_{q1}^1 - \psi_8 \mathbf{u}_{q1}^2$$

$$\mathbf{S}_{q2} = \mathbf{q}_2 - \psi_9 \mathbf{u}_{q2}^1 - \psi_{10} \mathbf{u}_{q2}^2$$

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}_{k2}^1$ and $\mathbf{U}_{k1}^2$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}_{q1}^2$ and $\mathbf{U}_{q2}^2$; and $\psi_i \geq 0; i = 1, \cdots, 10$.

d. In this case, plate I is in a region not covered by the friction cone of point $\mathbf{q}_2$. The friction cone of the two plates in this configuration is given by the set

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \cup \mathbf{S}_{k1} \cup \mathbf{S}_{q2} \tag{35}$$

where

$$\mathbf{S}_1 = \boldsymbol{\zeta}_1 + \psi_1 \mathbf{u}_{k1}^2 + \psi_2 \mathbf{u}_{k2}^1$$

$$\mathbf{S}_2 = \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}_{q1}^1 - \psi_4 \mathbf{u}_{q2}^2$$

$$\mathbf{S}_{k1} = \mathbf{k}_1 + \psi_5 \mathbf{u}_{k1}^1 + \psi_6 \mathbf{u}_{k1}^2$$

$$\mathbf{S}_{q2} = \mathbf{q}_2 - \psi_7 \mathbf{u}_{q2}^1 - \psi_8 \mathbf{u}_{q2}^2$$

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}_{k1}^2$ and $\mathbf{U}_{k2}^1$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}_{q1}^1$ and $\mathbf{U}_{q2}^2$; and $\psi_i \geq 0; i = 1, \cdots, 8$.

Mode *iii*.

The friction cone of some points of each of the plates touch points of the other plate. The two-plate friction cone for this mode is given by the set

$$\mathcal{S} = \mathbf{S}_1 \cup \mathbf{S}_2 \tag{36}$$

where

$$\mathbf{S}_1 = \boldsymbol{\zeta}_1 + \psi_1 \mathbf{u}_{k1}^1 + \psi_2 \mathbf{u}_{k2}^1$$

$$\mathbf{S}_2 = \boldsymbol{\zeta}_2 - \psi_3 \mathbf{u}_{q1}^1 - \psi_4 \mathbf{u}_{q2}^1$$

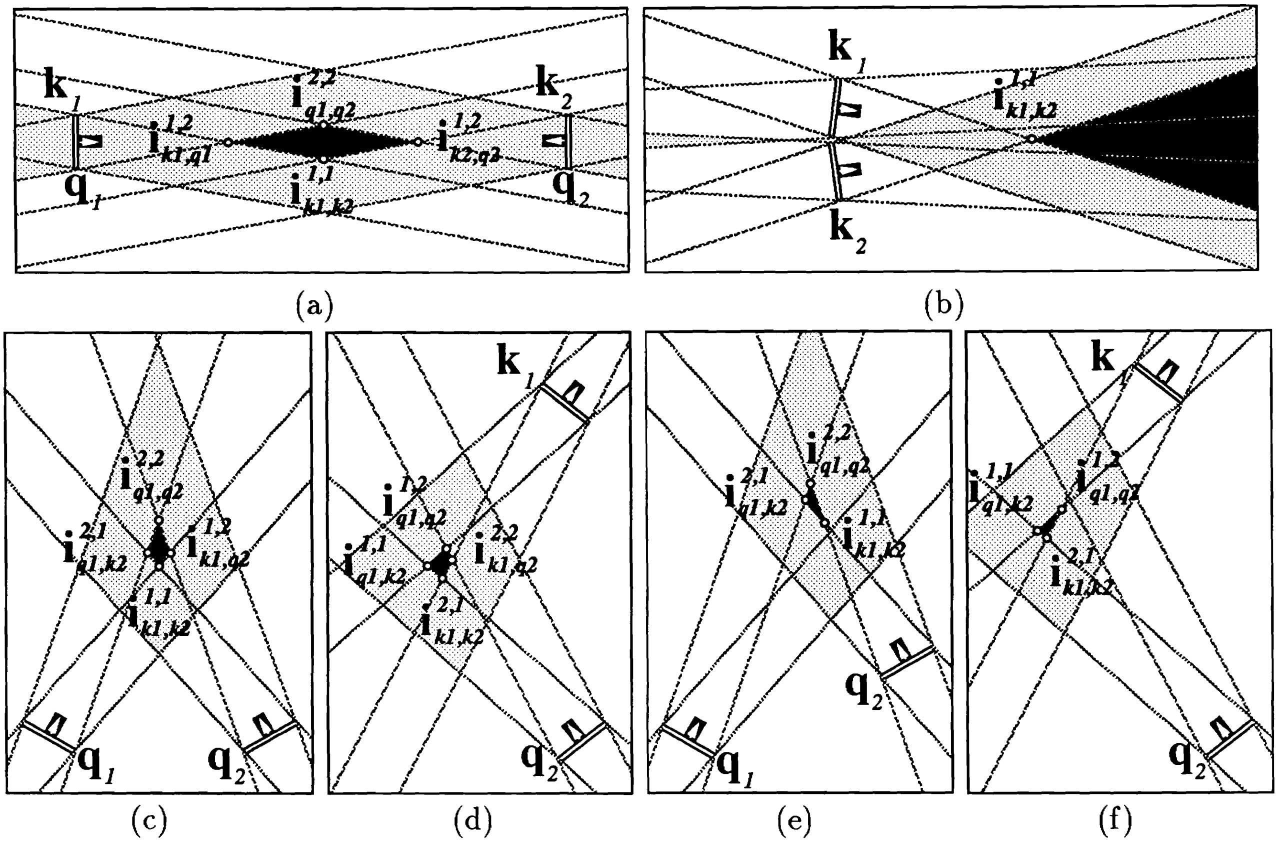


Figure 6: The Sets of Points $\mathbf{I}_{u,v}$ Intersect All at a Region

$\boldsymbol{\zeta}_1$ is the intersection point between lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{k2}$, $\boldsymbol{\zeta}_2$ is the intersection point between lines $\mathbf{U}^1_{q1}$ and $\mathbf{U}^1_{q2}$; and $\psi_i \geq 0; i = 1, \cdots, 4$. As it can be seen in Figure 5h the friction cone of this mode has the same geometry as the one of a single plate with extreme points placed at the intersection of lines $\mathbf{U}^1_{k1}$ and $\mathbf{U}^1_{q2}$, and $\mathbf{U}^1_{q1}$ and $\mathbf{U}^1_{k2}$, respectively, and with extreme directions parallel to lines $\mathbf{U}^1_{pI}$ and $\mathbf{U}^1_{pII}$. Therefore, this mode is not considered in the analysis performed in this work. To analyze its characteristics, make the substitution suggested above and consider it as a single plate cone of friction.

B Characteristics of $\mathcal{T}$, and of $\mathcal{I}^*$

B.1 Case: Not mode *i*b, and $\mathcal{T} \neq \emptyset$

When all sets of points $\mathbf{I}_{u,v}$ intersect, they intersect in the region $\mathcal{T}$ where friction cones of plates I and II intersect. $\mathcal{I}^*$ is reduced to be only one point on region $\mathcal{T}$.

A configuration for which $\mathcal{T} \neq \emptyset$ can be easily described. It can be characterized by its extreme points. To obtain them, define $\boldsymbol{\xi}_k = \boldsymbol{\zeta}_k, \boldsymbol{\zeta'}_k =$ focal points of plate k, $\mathcal{S}_l =$ friction cone of plate l.

The case in which all the sets of points $\mathbf{I}_{u,v}$ intersect at region $\mathcal{T} \neq \emptyset$ happens:

- If the configuration of the two plates fit one of these cases:

  1. Mode ia, this is, when
  $$\frac{\mathbf{y}_k \cdot (\boldsymbol{\xi}_l - \boldsymbol{\xi}_k)}{||\boldsymbol{\xi}_l - \boldsymbol{\xi}_k||} \leq \cos\theta_k \tag{37}$$
  for either $\boldsymbol{\xi}_k = \boldsymbol{\zeta}_k$ and $\boldsymbol{\xi}_l = \boldsymbol{\zeta}_l$, or $\boldsymbol{\xi}_k = \boldsymbol{\zeta}'_k$ and $\boldsymbol{\xi}_l = \boldsymbol{\zeta}'_l$. $k, l = I, II; k \neq l$. This case is illustrated in Figure 6a.

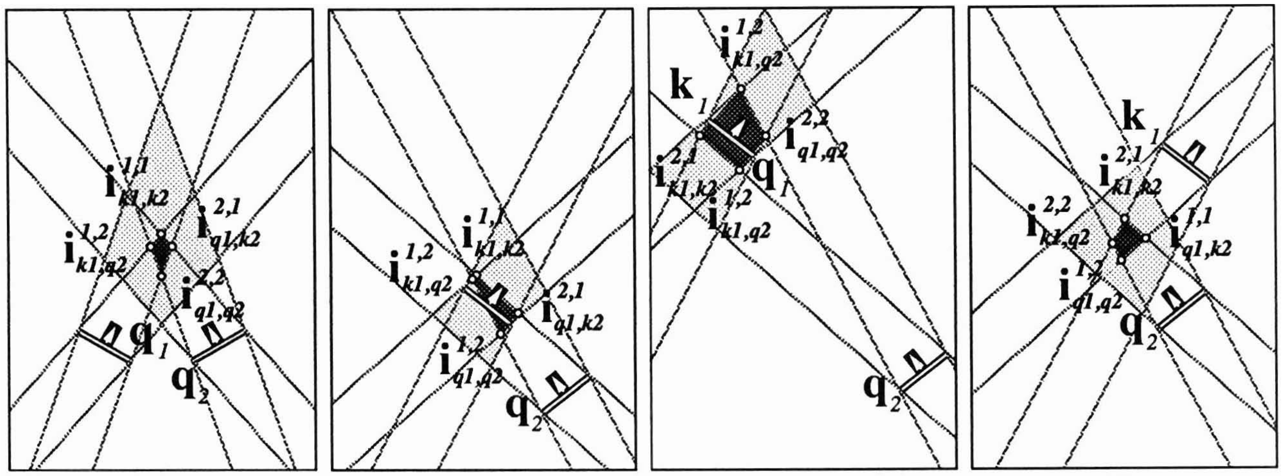
  2. Mode iia or iid; $\boldsymbol{\xi}_k$ is out of $\mathcal{S}_l$; and either
  (a) the angle ρ between the two straight plates in the hand is in the range:
  $$\pi + \theta_1 + \theta_2 > \rho > \pi - \theta_1 - \theta_2 \tag{38}$$
  or
  (b) after solving for ψ_k in
  $$\mathbf{c}_I + \psi_I \mathbf{y}_I = \mathbf{c}_{II} + \psi_{II} \mathbf{y}_{II} \tag{39}$$
  where $\mathbf{c}_k$ is the center of straight plate k, $\mathbf{y}_k$ is its unit normal vector, and ψ_k is the value that makes the equality true; it is found that ψ_I and ψ_{II} are of the same sign, and $|\psi_k| \geq b_k$, where $b_k = a_k/\mu_k$, with a_k and μ_k is defined as in section 3. Figure 6b shows two plates in mode iia in a configuration like the one described first. The configuration described after that is illustrated for modes iia and iid respectively in Figures 6c and 6d.

  3. Mode iia, or iid; and either $\boldsymbol{\zeta}_k$, or $\boldsymbol{\zeta}'_k$ is inside of the friction cone $\mathcal{S}_l$; i. e., if in either mode iia or mode iid, equation 39 yields $|\psi_k| > b_k$ and $|\psi_l| < b_l$, and
  $$\frac{\mathbf{y}_k \cdot (\boldsymbol{\xi}_l - \boldsymbol{\xi}_k)}{||\boldsymbol{\xi}_l - \boldsymbol{\xi}_k||} \leq \cos\theta_k \tag{40}$$
  where defining $r = k, l$; $\boldsymbol{\xi}_r = \boldsymbol{\zeta}_r$ if $\psi_r > 0$, $\boldsymbol{\xi}_r = \boldsymbol{\zeta}'_r$ if $\psi_r < 0$ and, $\boldsymbol{\zeta}_r$, $\boldsymbol{\zeta}'_r$ are the focal points of straight plate r. Figures 6e and 6f respectively, illustrate this case for modes iia, and iid.

- Then, the intersection region $\mathcal{T}$ is bounded by the lines

  1. $\mathbf{U}^1_{k1}$, $\mathbf{U}^1_{k2}$, $\mathbf{U}^2_{q1}$, and $\mathbf{U}^2_{q2}$ for configuration ia.
  2. $\mathbf{U}^1_{k1}$, $\mathbf{U}^1_{k2}$, $\mathbf{U}^2_{q1}$, and $\mathbf{U}^2_{q2}$ for configuration iia; and along lines $\mathbf{U}^2_{k1}$, $\mathbf{U}^1_{k2}$, $\mathbf{U}^1_{q1}$, and $\mathbf{U}^2_{q2}$ for configuration iid.
  3. $\mathbf{U}^1_{km}$, $\mathbf{U}^2_{qm}$, and $\mathbf{U}^2_{qn}$ for configurations iia and iid; where m, and n are defined as in condition of Equation 1.

- Therefore, the extreme points of the respective regions $\mathcal{T}$ are,



(a) (b) (c) (d)

Figure 7: The Sets of Points $\mathbf{I}_{u,v}$ Do Not Intersect

1. $\mathbf{i}^{1,1}_{k1,k2}$, $\mathbf{i}^{1,2}_{k1,q1}$, $\mathbf{i}^{1,2}_{k2,q2}$, and $\mathbf{i}^{2,2}_{q1,q2}$.
2. For mode *ii*a: $\mathbf{i}^{1,1}_{k1,k2}$, $\mathbf{i}^{1,2}_{k1,q2}$, $\mathbf{i}^{2,1}_{q1,k2}$, and $\mathbf{i}^{2,2}_{q1,q2}$; and for mode *ii*d: $\mathbf{i}^{2,1}_{k1,k2}$, $\mathbf{i}^{2,2}_{k1,q2}$, $\mathbf{i}^{1,1}_{q1,k2}$, and $\mathbf{i}^{1,2}_{q1,q2}$.
3. $\mathbf{i}^{1,2}_{km,qm}$, $\mathbf{i}^{1,2}_{km,qn}$, and $\mathbf{i}^{2,2}_{qm,qn}$.

The region $\mathcal{T}$ is the set of points $\mathbf{t}$ that can be expressed as a linear combination of the extreme points given above for each of the possible cases. Each point in $\mathcal{T}$ is by itself a set $\mathcal{I}^*$.

B.2 Case: Not mode *i*b, and $\mathcal{T} = \emptyset$

The set $\mathcal{I}^*$ is defined in this case as the convex set bounded by the extreme points:
For mode *ii*a: $\mathbf{i}^{1,1}_{k1,k2}$, $\mathbf{i}^{1,2}_{k1,q2}$, $\mathbf{i}^{2,1}_{q1,k2}$, and $\mathbf{i}^{2,2}_{q1,q2}$.
For mode *ii*b: $\mathbf{i}^{1,1}_{k1,k2}$, $\mathbf{i}^{1,2}_{k1,q2}$, $\mathbf{i}^{2,1}_{q1,k2}$, and $\mathbf{i}^{1,2}_{q1,q2}$.
For mode *ii*c: $\mathbf{i}^{2,1}_{k1,k2}$, $\mathbf{i}^{1,2}_{k1,q2}$, $\mathbf{i}^{1,1}_{q1,k2}$, and $\mathbf{i}^{2,2}_{q1,q2}$.
For mode *ii*d: $\mathbf{i}^{2,1}_{k1,k2}$, $\mathbf{i}^{2,2}_{k1,q2}$, $\mathbf{i}^{1,1}_{q1,k2}$, and $\mathbf{i}^{1,2}_{q1,q2}$

B.3 $\mathcal{I}^*$ for mode *i*b

For mode *i*b there are different possible definitions for $\mathcal{I}^*$, since no generalization can be made. Each case must be considered separately. Some examples of $\mathcal{I}^*$, are illustrated in Figure 8. In the figure, the lightest shadowed region represents the portion of $\mathcal{I}$ not intersecting with $\mathcal{I}^*$, the remaining shadowed regions represent $\mathcal{I}^*$ with the lighter of them being $\mathcal{I}^* \cap \mathcal{I}$.

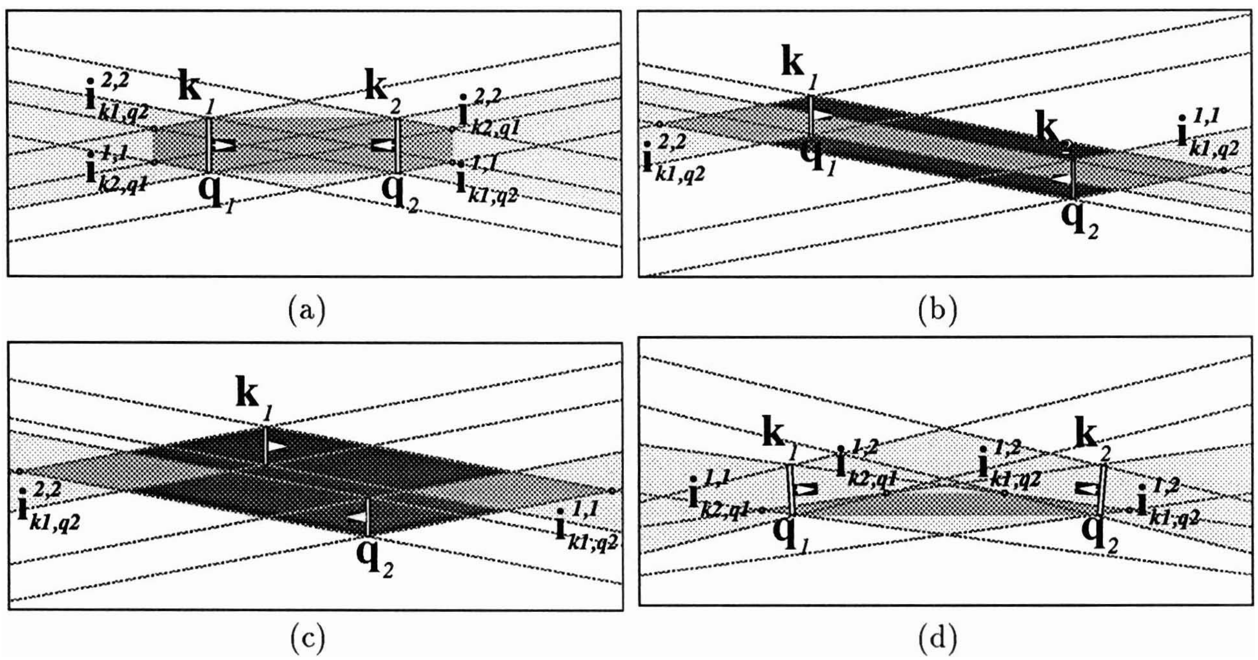


Figure 8: $\mathcal{I}^*$ for mode ib