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## Spherical Diffusion for Surface Smoothing and Denoising

### Abstract

Data defined on spherical domains occurs in various applications, such as surface modeling, omnidirectional imaging, and the analysis of keypoints in volumetric data. The theory of spherical signals lacks important concepts like the Gaussian function, which is permanently used in planar image processing. We propose a definition of a spherical Gaussian function as the Green's function of the spherical diffusion process. This allows to introduce a linear scale space on the sphere. We apply this new filter to the smoothing of 3D object surfaces.

### Comments

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# Spherical Diffusion\*

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## Abstract

Data defined on spherical domains occurs in various applications, such as surface modeling, omnidirectional imaging, and the analysis of keypoints in volumetric data. The theory of spherical signals lacks important concepts like the Gaussian function, which is permanently used in planar image processing. We propose a definition of a spherical Gaussian function as the Green's function of the spherical diffusion process. This allows to introduce a linear scale space on the sphere. We apply this new filter to the smoothing of 3D object surfaces.

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\* For Kostas Daniilidis on the 30th of December 2001.

## 1 Introduction

Different ways have been proposed to define a Gaussian function on the sphere. One possibility is to define a spherical Gaussian function using the spherical coordinates  $(\varphi, \vartheta)$  instead of the  $(x, y)$  coordinates. This has the undesired effect, that a mere translation of the function in the  $(\varphi, \vartheta)$ -plane leads to a distortion of the filter on the sphere. Furthermore, an isotropic Gauss function on the  $(\varphi, \vartheta)$ -plane is generally elongated on the sphere [11]. Another way is to stereographically project a planar Gauss function centered at the origin to the sphere and translate it to the desired center by rotation on the sphere [1, 2]. This overcomes the aforementioned problems. However, the problem remains that a convolution of two Gaussians on the sphere is *not* a spherical Gaussian! Thus, this definition cannot be used in order to define a spherical scale space.

In image processing linear scale-space has been defined in [10] and, as Weickert et al. point out [9], already in [8]. The linear scale space of an image is defined as the set of solutions of a linear diffusion equation with the original image as initial condition. It turns out that this set can be created by convolving the image with Gaussian functions of different scales. The Gaussian function is the Green's function of the linear diffusion equation.

In this paper we propose the analogous approach on the sphere. We solve the linear spherical diffusion equation and define its Green's function as the spherical Gaussian function. It will be shown that the spherical convolution of two such defined Gaussians is again a Gaussian, such that a spherical scale space can be build upon this definition.

In the following section we recap mathematical preliminaries related to spherical harmonics, which will be used for the solution of the spherical diffusion equation, and convolution on the sphere. After that we will present the main result of this paper in Sect. 3. Section 4 is devoted to one example application, namely smoothing of 3D surfaces.

## 2 Mathematical Preliminaries

In this section we summarize some facts about spherical harmonics functions which we will use in the rest of this article. As an overall reference on this subject we refer to [5, 4]. In the following we parameterize the unit sphere  $S^2$  embedded in  $\mathbb{R}^3$  using standard spherical coordinates. Thus, an element of  $\eta \in S^2$  will be written as

$$\eta := (\cos(\varphi) \sin(\vartheta), \sin(\varphi) \sin(\vartheta), \cos(\vartheta)), \quad (1)$$

with  $\varphi \in [0, 2\pi)$  and  $\vartheta \in [0, \pi]$ .

The spherical harmonic functions  $Y_{lm} : S^2 \rightarrow \mathbb{C}$  are defined as the everywhere regular eigenfunctions of the spherical Laplace operator. These functions constitute a complete orthonormal system of the space of square integrable functions on the sphere  $L^2(S^2)$ .

$$Y_{lm}(\eta) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos(\vartheta)) e^{im\varphi}, \quad (2)$$

With  $l \in \mathbb{N}$  and  $|m| \leq l$ . Here  $P_l^m$  denote the associated Legendre functions

$$P_l^m(x) = \frac{(-1)^m (1-x^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l. \quad (3)$$

Any function on the sphere can be expanded into spherical harmonics:

$$f = \sum_{l \in \mathbb{N}} \sum_{|m| \leq l} f_{lm} Y_{lm}. \quad (4)$$

The coefficients  $f_{lm}$  can be extracted from a given function  $f$  as follows

$$f_{lm} = \int_{\eta \in S^2} f(\eta) \bar{Y}_{lm}(\eta) d\eta. \quad (5)$$

For the surface element on the sphere we use the shorthand notation  $d\eta := \sin(\vartheta) d\vartheta d\varphi$ . If  $f_{lm} = 0$  for all  $l > L$ ,  $f$  is called band-limited with band-width  $L$ . The set of coefficients  $f_{lm}$  is called the spherical Fourier transform of  $f$ .

Rotations in  $\mathbb{R}^3$  will be parameterized by Euler angles such that any  $g \in SO(3)$  will be written as

$$g(\gamma, \beta, \alpha) = R_z(\gamma) R_y(\beta) R_z(\alpha), \quad (6)$$

where  $R_y$  and  $R_z$  denote rotation about the y-, and z-axis, respectively. In matrix notation  $R_y$  and  $R_z$  take the form

$$R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix}, \quad R_z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (7)$$

Rotating a function  $f \in L^2(S^2)$  will be performed by the operator  $\Lambda(g)$  which is defined by

$$\Lambda(g)f(\eta) := f(g^{-1}\eta). \quad (8)$$

We will need the convolution operation in later in this paper. Spherical convolution can be defined as<sup>1</sup>

$$(f * h)(\eta) = \int_{g \in SO(3)} f(g\eta)h(g^{-1}\eta) dg, \quad \eta \in S^2. \quad (9)$$

Driscoll and Healy [7] proof a convolution theorem for this kind of spherical convolution

**Theorem 1.** *For functions  $f, h \in L^2(S^2)$  the spectrum of the convolution is a pointwise product of the spectra of  $f$  and  $h$*

$$(f * h)_{lm} = 2\pi \sqrt{\frac{4\pi}{2l+1}} f_{lm} h_{l0}. \quad (10)$$

### 3 Spherical Gaussians

#### 3.1 Mapping Planar Functions to the Sphere

Recently, the following definition of dilation on the sphere has been proposed [1]: Dilating a function on the sphere about the north-pole is performed by subsequently (1) projecting the function stereographically from the south-pole to the plane tangent to  $S^2$  at the north-pole, (2) dilate the mapped function within the tangent plane, and (3) map the dilated function back to the sphere by inverse stereographic projection. Along the same line the construction of filters on the sphere can be performed [6]. A filter is defined in  $\mathbb{R}^2$  and mapped by inverse stereographic projection to the sphere.

The stereographic projection  $\Pi : S^2 \setminus \{(0, 0, -1)\} \rightarrow \mathbb{R}^2, (\varphi, \vartheta) \mapsto (x(\varphi, \vartheta), y(\varphi, \vartheta))$  is given by

$$x(\varphi, \vartheta) = \frac{2 \sin(\vartheta) \cos(\varphi)}{1 + \cos(\vartheta)}, \quad y(\varphi, \vartheta) = \frac{2 \sin(\vartheta) \sin(\varphi)}{1 + \cos(\vartheta)} \quad (11)$$

Lets consider the Gauss function in  $\mathbb{R}^2$ :

$$G(\mathbf{x}, t) := \frac{1}{4\pi kt} \exp\left(-\left[\frac{\mathbf{x}^2}{4kt}\right]\right). \quad (12)$$

---

<sup>1</sup> We say "can" since this is not the only possible definition of spherical convolution. It is as well possible to take out the integration over  $S^2$  only and have the result be a function on  $SO(3)$ . This is especially useful if we deal with non-isotropic, directional filters as in [3]

The inverse stereographic projection is actually not defined on the south-pole of the sphere. However, since the Gauss function goes to zero for  $|\mathbf{x}| \rightarrow \infty$  we will assign the value 0 to the south-pole.

It is easy to see that convolving two thus defined spherical Gauss functions the result will be greater than zero everywhere, including the south-pole. Thus, two-fold application of a smoothing procedure with this filter will lead to a result not obtainable by a single application of the Gaussian, whatever value for  $kt$  is chosen. We will thus abandon this approach. In the following we will propose the definition of a Gaussian filter on the sphere as the Green's function of the diffusion equation on the sphere.

### 3.2 Spherical Diffusion

In this section we derive the Green's function of the spherical diffusion equation. This Green's function can then be considered as an extension of the Gauss function to the sphere.

In image processing a scale space can be constructed by convolving the image with the Gaussian kernel

$$g(\mathbf{x}, t) = \frac{1}{4\pi kt} \exp\left(\frac{-\mathbf{x}^2}{4kt}\right) \quad (13)$$

This is equivalent to letting the image evolve under the diffusion equation

$$\Delta u(\mathbf{x}, t) = \frac{1}{k} \frac{\partial}{\partial t} u(\mathbf{x}, t) \quad (14)$$

The spherical diffusion equation is given by

$$\Delta_{S^2} u(\varphi, \vartheta, t) = \frac{1}{k} \frac{\partial}{\partial t} u(\varphi, \vartheta, t), \quad (15)$$

where  $\Delta_{S^2}$  is the spherical Laplace operator given by

$$\Delta_{S^2} = \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left( \sin(\vartheta) \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2(\vartheta)} \frac{\partial^2}{\partial \varphi^2} \quad (16)$$

For solving (15) we work with a product ansatz for  $u(\varphi, \vartheta, t)$

$$u(\varphi, \vartheta, t) = \Phi(\varphi)\Theta(\vartheta)T(t). \quad (17)$$

We will make use of the fact that spherical harmonics are eigenfunctions of the spherical Laplace operator:

$$\Delta_{S^2} Y_{lm} = -l(l+1)Y_{lm}. \quad (18)$$

Using this fact it is easily verified that

$$u_{lm}(\varphi, \vartheta, t) = Y_{lm}(\varphi, \vartheta) \exp(-l(l+1)kt) \quad (19)$$

solves (15). In order to obtain a Green's function we will impose the initial condition

$$u(\varphi, \vartheta, 0) = \delta_{S^2}(\varphi, \vartheta), \quad (20)$$

where the spherical Dirac function is defined by

$$f(n) = \int_{\eta \in S^2} f(\eta) \delta_{S^2}(\varphi, \vartheta) d\eta, \quad (21)$$

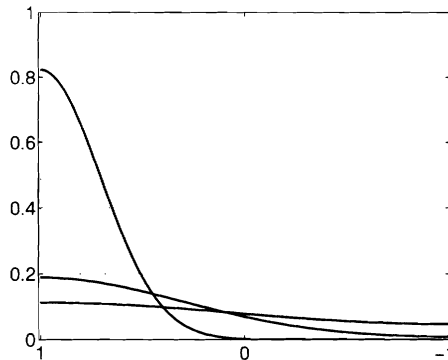
where  $n = (0, 0, 1)$  is the north-pole. Expanding  $\delta_{S^2}$  into spherical harmonics yields

$$\delta_{S^2} = \sum_{l \in \mathbb{N}} \sqrt{\frac{2l+1}{4\pi}} Y_{l0}. \quad (22)$$

Thus we obtain for the Green's function  $G$  the final result

$$G(\varphi, \vartheta, t) = \sum_{l \in \mathbb{N}} \sqrt{\frac{2l+1}{4\pi}} u_{l0}(\varphi, \vartheta, t) \quad (23)$$

Figure 1 shows  $G$  for different values of  $kt$ . We show only the dependence on  $\cos(\vartheta)$  since  $G$  does not depend on  $\varphi$ . We can now perform



**Fig. 1.** The Green's function  $G$  from (23). The horizontal axis shows  $\cos(\vartheta)$ . The vertical axis shows  $G$

spherical diffusion with any given function  $f \in L^2(S^2)$  as initial condition of the diffusion process by convolving  $f$  with the Green's function (23).

$$f_t(\varphi, \vartheta) := (f * G(\cdot, t))(\varphi, \vartheta) \quad (24)$$



Since  $G$  is a superposition of zonal spherical harmonics, i.e. spherical harmonics with  $m = 0$ ,  $G$  is rotationally symmetric. We can use this by observing that the convolution by integration over  $SO(3)$  as given in (9) contains an integration about the rotational degree of freedom of  $G$  about its center. This integration does merely contribute a factor  $2\pi$ . We thus replace (9) by

$$(f \star h)(\eta) = \int_{g \in S^2} f(g\eta)h(g^{-1}\eta) dg, \quad \eta \in S^2. \quad (25)$$

For isotropic filters  $h$  we find

$$(f \star h)(\eta) = \frac{1}{2\pi}(f * h)(\eta). \quad (26)$$

We make use of this fact and perform the diffusion process on the spectra of  $f$  and  $G$ .

$$(ft)_{lm} = \sqrt{\frac{4\pi}{2l+1}} f_{lm} G(\cdot, t)_{l0} \quad (27)$$

$$= f_{lm} \exp(-l(l+1)kt). \quad (28)$$

This is the main result of this paper. It is now easy to proof that spherical diffusion fulfills the half-group property. Convolving  $f$  with  $G(\cdot, 0)$  yields

$$(f_0)_{lm} = f_{lm} \exp(-l(l+1)k \cdot 0) = f_{lm}. \quad (29)$$

Thus for  $t = 0$  spherical diffusion has no effect on  $f$ , as expected. Furthermore, applying diffusion to an already diffused image turns out to have the same effect than diffusing the image once, where the time-parameter is the some of the time-parameters of the concatenated diffusions.

$$((fs)_t)_{lm} = f_{lm} \exp(-l(l+1)ks) \exp(-l(l+1)kt) \quad (30)$$

$$= f_{lm} \exp(-l(l+1)k(s+t)) = (f_{s+t})_{lm} \quad (31)$$

After briefly describing our implementation in the following section we will present results in Sect. 4

## 4 Surface Smoothing

We show one example for the smoothing filter proposed in this article, i.e. convolution with the newly defined spherical Gaussian.

The surface representation we use is based on the assumption that we deal with a star-shaped object. That means that there exists a point within the object, such that each ray originating from this point intersects the object's surface exactly once. This allows us to assign to each ray (representing an orientation in space and thus an element of the sphere  $S^2$ ) a real number, namely the distance of the ray-surface-intersection to the origin of the ray. This results in the definition of a surface as a real function on the sphere. One smoothing example is shown in Fig. 2.

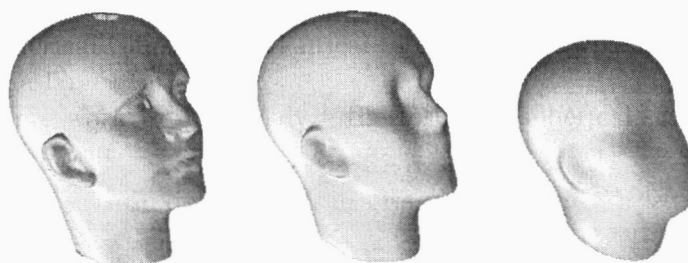


Fig. 2. From left to right: Original data. Smoothed with  $kt = 0.01$ . Smoothed with  $kt = 0.001$ .

## 5 Conclusion

In this paper we proposed a new smoothing operation for surface data defined as a scalar function on the sphere. The smoothing is performed as a diffusion on the sphere. This leads to a natural definition of a "Gaussian function" on the sphere.

This Gaussian function fulfills the half-group property, i.e. there exist a neutral element (the spherical delta function) and twofold convolution with a spherical Gaussian is identical to convolution with a single Gaussian with a larger variance.

In future work we will analyze the resulting spherical scale space in more detail.

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