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A paradigm of scientific discovery is defined within a first-order logical framework. It is shown that within this paradigm there exists a formal scientist that is Turing computable and universal in the sense that it solves every problem that any scientist can solve. It is also shown that universal scientists exist for no regular logics that extend first order logic and satisfy the Lowenheim-Skolem condition.

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A Universal Inductive Inference Machine*

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Abstract

A paradigm of scientific discovery is defined within a first-order logical framework. It is shown that within this paradigm there exists a formal scientist that is Turing computable and universal in the sense that it solves every problem that any scientist can solve. It is also shown that universal scientists exist for no regular logics that extend first order logic and satisfy the Lowenheim-Skolem condition.

1 Introduction

By a paradigm of inductive inference let us understand any specification of the concepts "scientist" and "inductive inference problem" along with a criterion that determines the conditions under which a given scientist is credited with solving a given problem. Hundreds of paradigms have been defined and investigated over the last twenty years. An excellent point of entry to this literature is Haussler & Pitt (1988) and Rivest & Haussler (1989).

Building on seminal papers by Shapiro (1981) and Glymour (1984), several recent paradigms have been defined within a first-order logical framework (see Osherson & Weinstein, 1989, and references cited there). The present paper discusses a paradigm of this character and shows it to contain a scientist that is simulable by Turing machine and universal in the following sense: Every problem that can be solved by some scientist in our paradigm (whether the scientist is machine simulable or not) can be solved by the universal one. The paradigm is defined in Section 2 and its universal scientist is exhibited in Section 3. An extension of our paradigm is introduced in Section 4. In Section 5 we characterize first-order logic by considering universal scientists for extensions of the predicate calculus. The remainder of this introduction attempts to motivate the ensuing definitions.

Let a countable, first-order language $\mathcal{L}$ be fixed, suitable for expressing scientific theories and data in some field of empirical inquiry. Prior research in the field is conceived as verifying

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a set T of $\mathcal{L}$-sentences, which constitute the axioms of a theory already known to be true. Each model of T thus represents a possible world consistent with background knowledge. Nature has chosen one of these models – say, structure $\mathcal{S}$ – to be actual; her choice is unknown to us. In the first paradigm that we define it is assumed that Nature's choice is limited to countable models of T. In the extended paradigm this assumption is lifted. For now, we suppose that $\mathcal{S}$ is countable.

Scientists are conceived as attempting to divine the truth-value in $\mathcal{S}$ of specific sentences not decided by T. Suppose that scientist Ψ wishes to determine the truth-value of θ in $\mathcal{S}$. At the start of inquiry, Ψ knows no more about $\mathcal{S}$ than what is implied by T. As inquiry proceeds, more and more information about $\mathcal{S}$ becomes available. This information has the following character. We conceive of Ψ as able to determine, for each atomic formula $\varphi(\bar{x})$ of $\mathcal{L}$ and any given tuple $\bar{a} \in |\mathcal{S}|$ whether or not $\bar{a}$ satisfies $\varphi(\bar{x})$ in $\mathcal{S}$. Ψ receives all of $|\mathcal{S}|$ in piecemeal fashion and bases its conjecture at a given moment on the finite subset of $|\mathcal{S}|$ examined by that time. In reponse to each new datum, Ψ emits a fresh conjecture about the truth of θ in $\mathcal{S}$, announcing either "true" or "false." To be counted as successful, Ψ's successive conjectures must stabilize to the correct one.

We turn now to the definitions that formalize the foregoing conception of scientific inquiry.

2 Paradigm

Let a countable first-order language $\mathcal{L}$ with identity be fixed. Until Section 4, all structures that arise in the discussion to follow are to be understood as countable structures interpreting $\mathcal{L}$. The sentences and formulas of $\mathcal{L}$ are denoted SEN and FORM respectively. $\varphi \in$ FORM is called "basic" just in case φ is an atomic formula or the negation of an atomic formula. The basic subset of FORM is denoted BAS. In the context of an assignment of variables to the unknown structure $\mathcal{S}$, members of BAS may be conceived as encoding facts of the form $\bar{a} \in P^{\mathcal{S}}$ or $\bar{a} \notin P^{\mathcal{S}}$, for $P \in \mathcal{L}$ as suggested in the last section. The set of all finite sequences over BAS is denoted SEQ. Given $\sigma \in$ SEQ, the set of formulas appearing in σ is denoted by *range*(σ). Members of SEQ of length n may be conceived as potential "evidential positions" of a scientist at moment n of his inquiry.

We rely in what follows on the standard account of consequence for open formulas, namely: Given sets Γ, Σ of formulas, we write $\Gamma \models \Sigma$ just in case for all structures $\mathcal{S}$ and assignments g to $\mathcal{S}$, if $\mathcal{S} \models \Gamma[g]$ then $\mathcal{S} \models \Sigma[g]$. By a *complete assignment* to a structure $\mathcal{S}$ is meant a mapping of the variables of $\mathcal{L}$ onto $|\mathcal{S}|$. Given $T \in$ SEN, the class of all (countable) models of T is denoted $\mathrm{MOD}_C(T)$. N is the set of positive integers.

We now consider the information available to a scientist working in an unknown structure. An *environment* is any ω-sequence over BAS. Given an environment e, the set of formulas appearing in e is denoted by *range*(e) and the initial finite sequence of length n in e is denoted $\bar{e}_n$. The following definition specifies the sense in which an environment can provide information about a structure.

(1) DEFINITION: Let environment e, structure $\mathcal{S}$, and assignment g to $\mathcal{S}$ be given. e is *for* $\mathcal{S}$ *via* g just in case $\mathit{range}(e) = \{\beta \in \text{BAS} \mid \mathcal{S} \models \beta[g]\}$.

Thus, when g is complete, an environment e for $\mathcal{S}$ via g provides basic information about every element of $|\mathcal{S}|$, using variables as codes for elements. It is easy to see that structures sharing an environment are isomorphic.

We take a scientist to be any function (partial or total, computable or uncomputable) from SEN $\times$ SEQ to $\{t, f\}$. Thus, a scientist Ψ may be conceived as a system that converts arbitrary $\theta \in$ SEN into a function $\lambda\sigma.\Psi(\theta, \sigma)$ that conjectures a truth-value for θ in whatever structure $\mathcal{S}$ has given rise to the data σ. To be successful on θ in $\mathcal{S}$, Ψ must "detect" the truth-value of θ in $\mathcal{S}$, as specified by the following definition.

(2) DEFINITION: Let $\theta \in$ SEN, structure $\mathcal{S}$ and scientist Ψ be given. Ψ *detects* θ *in* $\mathcal{S}$ just in case for every complete assignment g to $\mathcal{S}$, and every environment e for $\mathcal{S}$ via g, if $\mathcal{S} \models \theta$ then $\Psi(\theta, \bar{e}_n) = t$ for cofinitely many $n \in N$, and if $\mathcal{S} \models \neg\theta$ then $\Psi(\theta, \bar{e}_n) = f$ for cofinitely many $n \in N$.

Thus, we credit Ψ with detecting θ in $\mathcal{S}$ just in case Ψ's successive conjectures about the truth-value of θ in $\mathcal{S}$ eventually stabilize to the correct one in response to increasingly complete information about $\mathcal{S}$.

(3) DEFINITION: Let class $\mathcal{K}$ of structures, $\theta \in$ SEN and scientist Ψ be given. Ψ *detects* θ *in* $\mathcal{K}$ just in case for all $\mathcal{S} \in \mathcal{K}$, Ψ detects θ in $\mathcal{S}$. In this case θ is *detectable in* $\mathcal{K}$.

Pursuant to the conception of scientific inquiry described in Section 1, we shall be particularly concerned with detectability in elementary classes of structures. Given $\theta \in$ SEN and $T \subseteq$ SEN we ask whether θ is detectable in $\text{MOD}_C(T)$.

(4) EXAMPLE: For any $T \subseteq$ SEN and any existential $\theta \in$ SEN, θ is detectable in $\text{MOD}_C(T)$. Indeed, let scientist Ψ be such that for all existential $\theta \in$ SEN and all $\sigma \in$ SEQ, $\Psi(\theta, \sigma) = t$ if $\mathit{range}(\sigma) \models \theta$; $= f$ otherwise. Then, for any existential θ, Ψ detects θ in $\text{MOD}_C(\emptyset)$. A parallel argument shows that any universal $\theta \in$ SEN is detectable in $\text{MOD}_C(\emptyset)$. It can similarly be shown that any sentence translating "there are exactly n things," "there are at least n things," or "there are no more than n things" is detectable in $\text{MOD}_C(\emptyset)$.

(5) EXAMPLE: Suppose that the vocabulary of $\mathcal{L}$ is limited to the sole binary relation symbol $<$. Let T_1 be the axioms of the theory of strict linear orderings with a first but no last element. Let T_2 be the axioms of the theory of strict linear orderings with a last but no first element. Let $\theta = \exists x \forall y (x \neq y \rightarrow x < y)$. Then, θ is detectable in $\text{MOD}_C(T_1) \cup \text{MOD}_C(T_2)$ as witnessed by the following scientist Ψ. Let $v_1, v_2, \ldots$ list the variables of $\mathcal{L}$. Given $\sigma \in$ SEQ, let $\ell(\sigma)$ denote the least i such that for all j, $v_j < v_i \notin \mathit{range}(\sigma)$. Let $g(\sigma)$ denote the least i such that for all j, $v_i < v_j \notin \mathit{range}(\sigma)$. Define Ψ such that for all $\sigma \in$ SEQ:

$$\Psi(\theta,\sigma)=\begin{cases} t & \text{if } \ell(\sigma)<g(\sigma)\\ f & \text{otherwise.}\end{cases}$$

It is easy to verify that Ψ's conjectures stabilize to the truth-value of θ in any $\mathcal{S} \in \mathrm{MOD}_C(T_1) \cup \mathrm{MOD}_C(T_2)$.

(6) EXAMPLE: Suppose that $T \subseteq \mathrm{SEN}$ is model-complete. Then for all $\theta \in \mathrm{SEN}$, θ is detectable in $\mathrm{MOD}_C(T)$. This is an easy consequence of the fact that the model-completeness of T implies that for all $\theta \in \mathrm{SEN}$, there is existential $\varphi \in \mathrm{SEN}$ such that $T \vdash \varphi \leftrightarrow \theta$.

(7) EXAMPLE: Let T axiomatize the theory of strict linear orderings in a language with one binary relation symbol $<$. Then $\theta = \forall xy(x < y \rightarrow \exists z(x < z \wedge z < y))$ is not detectable in $\mathrm{MOD}_C(T)$. Indeed, it is shown in Osherson & Weinstein (1989, Sec. 3.2) that θ is not detectable in $\mathcal{K} = \{(Z, <^Z), (Q, <^Q)\}$, where Z are the integers and Q are the rationals. Observe that $\mathcal{K} \subseteq \mathrm{MOD}_C(T)$.

3 A universal scientist

By an "oracle machine" is meant a Turing machine with oracle in the sense of Rogers (1967, Sec. 9.2). Given oracle machine M and $T \subseteq \mathrm{SEN}$ we use M^T to denote the scientist computed by M equipped with an oracle for T.

(8) THEOREM: There is an oracle machine M such that for all $\theta \in \mathrm{SEN}$ and $T \subseteq \mathrm{SEN}$, if θ is detectable in $\mathrm{MOD}_C(T)$ then M^T detects θ in $\mathrm{MOD}_C(T)$.

To prove the theorem it is convenient to introduce a weaker notion of scientific success.

(9) DEFINITION: Let $\theta \in \mathrm{SEN}$, structure $\mathcal{S}$ and scientist Ψ be given. Ψ *t-detects* θ *in* $\mathcal{S}$ just in case for every complete assignment g to $\mathcal{S}$, and every environment e for $\mathcal{S}$ via g, $\mathcal{S} \models \theta$ if and only if $\Psi(\theta, \bar{e}_n) = t$ for cofinitely many $n \in N$. Let class $\mathcal{K}$ of structures, $\theta \in \mathrm{SEN}$ and scientist Ψ be given. Ψ *t-detects* θ *in* $\mathcal{K}$ just in case for all $\mathcal{S} \in \mathcal{K}$, Ψ t-detects θ in $\mathcal{S}$. In this case θ is *t-detectable in* $\mathcal{K}$.

Thus, for scientist Ψ to t-detect θ in $\mathcal{U}$ with $\mathcal{U} \models \neg\theta$, Ψ's conjectures must not stabilize to t on any environment for $\mathcal{U}$ via g (where g is a complete assignment to $\mathcal{U}$). However, Ψ need not stabilize to f on such environments. Theorem (8) follows from:

(10) THEOREM: There is an oracle machine N such that for all $\theta \in \mathrm{SEN}$ and $T \subseteq \mathrm{SEN}$:

(a) $N^T(\theta, \sigma)$ is defined for all $\sigma \in \mathrm{SEQ}$;

(b) if θ is t-detectable in $\mathrm{MOD}_C(T)$ then N^T t-detects θ in $\mathrm{MOD}_C(T)$.

PROOF OF THEOREM (8) FROM THEOREM (10): Let oracle machine N witness Theorem (10). We construct an oracle machine M to witness Theorem (8). Define $start(T, \theta, \sigma)$ to be the least $i \leq length(\sigma)$ such that $N^T(\theta, \tau) = t$ for all $\tau \subseteq \sigma$ with $i \leq length(\tau)$. Define oracle machine M as follows. For all $T \subseteq \mathrm{SEN}, \theta \in \mathrm{SEN}$, and $\sigma \in \mathrm{SEQ}$:

$$M^T(\theta, \sigma) = \begin{cases} t & \text{if } start(T, \theta, \sigma) < start(T, \neg\theta, \sigma) \\ f & \text{otherwise.} \end{cases}$$

By clause (a) of Theorem (10), *start* can be computed by an oracle machine, so M is well defined. Now suppose that $T \subseteq \mathrm{SEN}$ and $\theta \in \mathrm{SEN}$ are such that θ is detectable in $\mathrm{MOD}_C(T)$. Then, it is trivial to verify that θ is t-detectable in $\mathrm{MOD}_C(T)$ and $\neg\theta$ is t-detectable in $\mathrm{MOD}_C(T)$. Hence, N^T t-detects θ in $\mathrm{MOD}_C(T)$ and N^T t-detects $\neg\theta$ in $\mathrm{MOD}_C(T)$. It is then easy to verify that M works as desired. ■

The remainder of this section is devoted to proving Theorem (10). The theorem is an immediate consequence of the next two lemmas, formulated with the help of the following definition.

(11) DEFINITION: Let $\theta \in \mathrm{SEN}$ and $T \subseteq \mathrm{SEN}$ be given. θ is *confirmable in T* just in case for all $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\theta\})$ there is existential-universal $\varphi \in \mathrm{SEN}$ such that:

(a) $\mathcal{S} \models \varphi$; and

(b) $T \cup \{\varphi\} \models \theta$.

(12) LEMMA: There is an oracle machine N such that for all $\theta \in \mathrm{SEN}$ and $T \subseteq \mathrm{SEN}$:

(a) $N^T(\theta, \sigma)$ is defined for all $\sigma \in \mathrm{SEQ}$;

(b) if θ is confirmable in T then N^T t-detects θ in $\mathrm{MOD}_C(T)$.

(13) LEMMA: For all $\theta \in \mathrm{SEN}$ and $T \subseteq \mathrm{SEN}$, if θ is t-detectable in $\mathrm{MOD}_C(T)$ then θ is confirmable in T.

The proof of Lemma (12) will be facilitated by the following.

(14) LEMMA: Let $\theta \in \mathrm{SEN}$ and $T \subseteq \mathrm{SEN}$ be given. If θ is confirmable in T, then for all $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\theta\})$ and all complete assignments g to $\mathcal{S}$ there is universal $\pi \in \mathrm{FORM}$ such that:

(a) $\mathcal{S} \models \pi[g]$; and

(b) $T \cup \{\pi\} \models \theta$.

PROOF OF LEMMA (14): Let θ and T be as specified by the lemma, and let $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\theta\})$ and complete assignment g to $\mathcal{S}$ be given. We must show that there is universal $\pi \in \mathrm{FORM}$ satisfying (14)a,b. Since θ is confirmable in T, there is existential-universal sentence $\varphi = \exists x_1 \ldots x_m \forall y_1 \ldots y_n \chi$, where χ is quantifier-free such that

(15) (a) $\mathcal{S} \models \varphi$; and

(b) $T \cup \{\varphi\} \models \theta$.

By (15)a there is a finite assignment p to $\mathcal{S}$ such that:

(16) (a) $domain(p) = \{x_1 \ldots x_m\}$, and

(b) $\mathcal{S} \models \forall y_1 \ldots y_n \chi[p]$.

Since g is complete, there are variables $w_1 \ldots w_m$ such that:

(17) $p(x_i) = g(w_i)$ for $1 \leq i \leq m$.

Let $v_1 \ldots v_n$ be distinct variables that are disjoint from the x_i's, y_i's, and w_i's. Let χ' be the result of simultaneously substituting the w_i's for the x_i's and the v_i's for the y_i's in χ. Let $\pi = \forall v_1 \ldots v_n \chi'$. By (16) and (17), $\mathcal{S} \models \pi[g]$, verifying (14)a. By (15)b, $T \cup \{\pi\} \models \theta$, which verfies (14)b. ■

PROOF OF LEMMA (12): The desired machine N is equipped with a device that progressively enumerates all consequences of an input, finite set of formulas. For $\varphi \in$ FORM and finite $\Sigma \subseteq$ FORM we write $\Sigma \models_j \varphi$ just in case φ appears in the enumeration of Σ's consequences by the jth step of computation. N is similarly equipped with a device that progressively queries its oracle about each sentence in turn. T_j denotes the finite set of sentences affirmed by the oracle T to be axioms by the jth step of this process. N relies as well on an internal enumeration of all universal formulas. Let π_i be the ith formula in this enumeration. Given $\sigma \in$ SEQ, we denote by $\sigma-$ the finite sequence that results from removing σ's last member; if $length(\sigma) = 0$ then $\sigma- = \sigma$.

Now let finite $\Sigma \subseteq$ SEN, $\theta \in$ SEN and $\sigma \in$ SEQ be given. We define $f(\Sigma, \theta, \sigma)$ to be the least $i < length(\sigma)$ such that:

(a) $range(\sigma) \not\models_{length(\sigma)} \neg\pi_i$;

(b) $\Sigma \cup \{\pi_i\} \models_{length(\sigma)} \theta$;

$f(\Sigma, \theta, \sigma) = 0$ if no such i exists. Obviously, f is a computable function. Finally, given $T \subseteq$ SEN, $\theta \in$ SEN, and $\sigma \in$ SEQ, $N^T(\theta, \sigma) = t$ if $f(T_{length(\sigma)}, \theta, \sigma) = f(T_{length(\sigma-)}, \theta, \sigma-) \neq 0$; $= f$ otherwise. It is clear that N satisfies (a) of the lemma.

To grasp the idea behind N's definition, imagine that N is examining environment e for $\mathcal{S}$ via g, where g is a complete assignment for $\mathcal{S}$. Then, N may be conceived as searching for the first universal formula in its internal enumeration that appears to witness (a) and (b) of Lemma (14). If no such candidate witness is found, then N responds with f; likewise, N responds with f each time it is forced to change candidates (either because the current candidate is shown to be inconsistent with $range(e)$ or because an earlier candidate is found). In contrast, N responds with t whenever its current candidate survives a subsequent test of primacy and consistency with $range(e)$.

To verify that N satisfies (b) of the lemma, let $T \subseteq$ SEN and $\theta \in$ SEN be given, and suppose that θ is confirmable in T. It must be shown that for all $\mathcal{S} \in \mathrm{MOD}_C(T)$, all complete assigments g to $\mathcal{S}$, and all environments e for $\mathcal{S}$ via g, $\mathcal{S} \models \theta$ iff $N^T(\theta, \bar{e}_j) = t$ for cofinitely many $j \in N$. We first show that:

(18) If $\pi \in$ FORM is universal then $\mathcal{S} \models \pi[g]$ iff $range(e) \not\models \neg\pi$.

To prove (18) suppose first that π is universal and $\mathcal{S} \models \pi[g]$. Since the assignment g to $\mathcal{S}$ satisfies $range(e) \cup \{\pi\}$, it follows immediately that $range(e) \not\models \neg\pi$. For the other direction of (18) suppose that π is universal and $\mathcal{S} \models \neg\pi[g]$. Let $\varphi(\bar{x}, \bar{y})$ be quantifier-free and such that $\pi = \forall\bar{x}\varphi(\bar{x}, \bar{y})$. Then:

(19) $\mathcal{S} \models \exists\bar{x}\neg\varphi(\bar{x}, \bar{y})[g]$

Since g is complete, (19) implies:

(20) $range(e) \models \neg\varphi(\bar{z}, \bar{y})$

for some choice of variables $\bar{z}$. But (20) shows that $range(e) \models \neg\pi$.

To complete the proof, we consider two cases corresponding to $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\theta\})$ and $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\neg\theta\}$.

Case 1: $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\theta\})$. Since θ is confirmable in T, By Lemma (14) let i be least such that π_i satisfies (14)a,b. By the compactness and monotonicity of $\models$ and by (18), for all $k < i$ we have:

(21) (a) $range(\bar{e}_j) \models_j \neg\pi_k$ for cofinitely many $j \in N$, or

(b) $T_j \cup \{\pi_k\} \not\models_j \theta$ for all $j \in N$.

(21) implies that for all $k < i$, $f(T_j, \theta, \bar{e}_j) = k$ for only finitely many j. On the other hand, since π_i satisfies (14)a,b, we have:

(22) (a) $range(e) \not\models_j \neg\pi_i$ for all $j \in N$, and

(b) $T_j \cup \{\pi_i\} \models_j \theta$ for cofinitely many $j \in N$.

Hence, (21) and (22) imply that $f(T_j, \theta, \bar{e}_j) = i$ for cofinitely many $j \in N$. It follows that $\lambda\sigma.N^T(\theta, \bar{e}_j) = t$ for cofinitely many $j \in N$.

Case 2: $\mathcal{S} \in \mathrm{MOD}_C(T \cup \{\neg\theta\})$. By Lemma (14), it suffices to show:

(23) For all universal $\pi \in$ FORM, π does not satisfy (14)a,b.

For, (23) implies (via (18) and the monotonicity and compactness of $\models$) that for all $k \in N$ (21) holds. By the definition of f, this implies that for all $k \in N$, $f(T_j, \theta, \bar{e}_j) = k$ for only finitely many $j \in N$. It follows that $N^T(\theta, \bar{e}_j) = f$ for infinitely many $j \in N$, as desired.

To prove (23) suppose that universal π satisfies (14)b. Then $T \cup \{\neg\theta\} \models \neg\pi$ and thus $\mathcal{S} \models \neg\pi[g]$. Thus π does not satisfy (14)a. ■

The proof of Lemma (13) relies on two additional lemmas. These are now stated and proved. The following notation is used. Given $\sigma \in$ SEQ, the set of variables occurring in σ is denoted by $var(\sigma)$. Also, the conjunction of $range(\sigma)$ is denoted by $\bigwedge\sigma$.

(24) LEMMA: Let scientist Ψ, $\theta \in$ SEN and structure $\mathcal{S}$ be given with $\mathcal{S} \models \theta$. Suppose that Ψ t-detects θ in $\mathcal{S}$. Then there is $\sigma \in$ SEQ and $p : var(\sigma) \to |\mathcal{S}|$ such that:

(a) $\mathcal{S} \models \bigwedge \sigma[p]$;

(b) for all $\gamma \in$ SEQ, if

i. $\sigma \subseteq \gamma$ and

ii. $\mathcal{S} \models \exists x_1 \ldots x_k \bigwedge \gamma[p]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$

then $\Psi(\theta, \gamma) = t$.

PROOF: Let Ψ, θ, and $\mathcal{S}$ be as specified by the hypothesis of the lemma. Suppose for a contradiction that the consequent is false. Then:

(25) For all $\sigma \in$ SEQ and $p : var(\sigma) \to |\mathcal{S}|$, if $\mathcal{S} \models \bigwedge \sigma[p]$ then there is $\gamma \in$ SEQ such that:

(a) $\sigma \subseteq \gamma$;

(b) $\mathcal{S} \models \exists x_1 \ldots x_k \bigwedge \gamma[p]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$; and

(c) $\Psi(\theta, \gamma) \neq t$.

Using (25) we construct a complete assignment g to $\mathcal{S}$ and an environment e for $\mathcal{S}$ via g such that $\Psi(\theta, \bar{e}_i) \neq t$ for infinitely many $i \in N$. This contradicts the assumption that Ψ t-detects θ in $\mathcal{S}$. For the construction, let $\{a_i \mid i \in N\} = |\mathcal{S}|$, let $\{\beta_i \mid i \in N\} =$ BAS, and let $\{v_i \mid i \in N\}$ be the variables of $\mathcal{L}$.

We construct g and e in stages. At the mth stage we construct $e^m \in$ SEQ and $g^m : var(e^m) \to |\mathcal{S}|$. It will be the case that $e^0 \subseteq e^1 \subseteq \cdots$ and $g^0 \subseteq g^1 \subseteq \cdots$. We take $e = \bigcup_{m \in N} e^m$ and $g = \bigcup_{m \in N} g^m$. The construction ensures that for every $m \geq 0$ the following conditions hold:

(26) (a) $\{a_1 \ldots a_m\} \subseteq range(g^m) \subseteq |\mathcal{S}|$;

(b) $\{v_1 \ldots v_m\} \subseteq domain(g^m) = var(e^m)$;

(c) for all $i \leq m$, if $var(\beta_i) \subseteq domain(g^m)$ and $\mathcal{S} \models \beta_i[g^m]$, then $\beta_i \in range(e^m)$;

(d) $\mathcal{S} \models \bigwedge e^m[g^m]$;

(e) $\Psi(\theta, \tau) \neq t$ for at least m many $\tau \subseteq e^m$.

If the construction succeeds, then (26)a,b ensure that g is a complete assignment to $\mathcal{S}$, (26)c,d ensure that e is for $\mathcal{S}$ via g, and (26)e ensures that $\Psi(\theta, \bar{e}_i) \neq t$ for infinitely many $i \in N$.

Stage 0: Set $e^0 = g^0 = \emptyset$.

Stage m+1: Suppose that e^m and g^m have been defined and satisfy (26)a-e. By (25) and (26)b,d choose $\gamma \in$ SEQ such that:

(27) (a) $e^m \subseteq \gamma$;

(b) $\mathcal{S} \models \exists x_1 \ldots x_k \bigwedge \gamma[g^m]$, where $var(\gamma) - var(e^m) = \{x_1 \ldots x_k\}$; and

(c) $\Psi(\theta, \gamma) \neq t$.

Observe that $\{x_1 \ldots x_k\} \cap domain(g^m) = \emptyset$. By (27)b, choose $b_1 \ldots b_k \in |\mathcal{S}|$ such that $\mathcal{S} \models \bigwedge \gamma[g^m; x_1|b_1 \ldots x_k|b_k]$. Let $q \in N$ be least such that $v_q \notin domain(g^m) \cup \{x_1 \ldots x_k\}$. Let r be least such that $a_r \notin range(g^m) \cup \{b_1 \ldots b_k\}$ (if $range(g^m) \cup \{b_1 \ldots b_k\} = |\mathcal{S}|$, then choose $a_r \in |\mathcal{S}|$ arbitrarily). Let $g^{m+1} = g^m \cup \{(x_i, b_i) \mid i \leq k\} \cup \{(v_q, a_r)\}$. Let B be the set of all $\beta_i \in$ BAS such that:

(a) $i \leq m+1$;

(b) $var(\beta_i) \subseteq domain(g^{m+1})$; and

(c) $\mathcal{S} \models \beta_i[g^{m+1}]$.

Let $\tau \in$ SEQ be such that $range(\tau) = B \cup \{v_q = v_q\}$. Let e^{m+1} be the result of concatenating τ to the end of γ. It may be seen that, with $m+1$ in place of m, g^{m+1} and e^{m+1} satisfy (26)a-e. ■

To formulate Lemma (28) we rely on the following definition. Given finite assignment p to structure $\mathcal{S}$, $\forall$-*type(p,$\mathcal{S}$)* $= \{\pi \in$ FORM $\mid \pi$ is universal, $var(\pi) \subseteq domain(p)$, and $\mathcal{S} \models \pi[p]\}$.

(28) LEMMA: Let $\theta \in$ SEN and $T \subseteq$ SEN be given. If θ is not confirmable in T then there is $\mathcal{S} \in$ MOD$_C(T \cup \{\theta\})$ such that for every finite assignment p to $\mathcal{S}$ there is $\mathcal{U} \in$ MOD$_C(T \cup \{\neg\theta\})$ and assignment q to $\mathcal{U}$ with $domain(q) = domain(p)$ and $\forall$-*type(p,$\mathcal{S}$)* $\subseteq$ $\forall$-*type(q,$\mathcal{U}$)*.

PROOF: Suppose that θ is not confirmable in T. Then by Definition (11) there is $\mathcal{S} \in$ MOD$_C(T \cup \{\theta\})$ such that:

(29) For every existential-universal $\varphi \in$ SEN, if $\mathcal{S} \models \varphi$ then $T \cup \{\neg\theta, \varphi\}$ is satisfiable.

Let p be a finite assignment to $\mathcal{S}$, and let $domain(p) = \{x_1 \ldots x_n\}$. By (29), for every finite $\{\pi_1 \ldots \pi_m\} \subseteq$ $\forall$-*type(p,$\mathcal{S}$)*, $T \cup \{\neg\theta, \exists x_1 \ldots x_n(\pi_1 \& \ldots \& \pi_m)\}$ is satisfiable. Hence, by the compactness and Löwenheim-Skolem theorems, there is $\mathcal{U} \in$ MOD$_C(T \cup \{\neg\theta\})$ and finite assignment q to $\mathcal{U}$ with $domain(q) = domain(p)$ such that $\mathcal{U} \models$ $\forall$-*type(p,$\mathcal{S}$)*$[q]$. ■

PROOF OF LEMMA (13): Let $\theta \in$ SEN and $T \subseteq$ SEN be such that:

(30) θ is t-detectable in MOD$_C(T)$.

We deduce a contradiction from the reductio assumption that:

(31) θ is not confirmable in T.

By (31) and Lemma (28) choose structure $\mathcal{S}$ such that:

(32) (a) $\mathcal{S} \in \text{MOD}_C(T \cup \{\theta\})$, and

(b) for every finite assignment p to $\mathcal{S}$ there is $\mathcal{U} \in \text{MOD}_C(T \cup \{\neg\theta\})$ and assignment q to $\mathcal{U}$ with $domain(q) = domain(p)$ and $\forall$-*type(p,$\mathcal{S}$)* $\subseteq$ $\forall$-*type(q,$\mathcal{U}$)*.

By (30) and Lemma (24) choose scientist Ψ, $\sigma \in \text{SEQ}$, and $p : var(\sigma) \rightarrow |\mathcal{S}|$ such that:

(33) (a) Ψ t-detects θ in $\text{MOD}_C(T)$;

(b) $\mathcal{S} \models \bigwedge \sigma[p]$;

(c) for all $\gamma \in \text{SEQ}$ if

i. $\sigma \subseteq \gamma$,

ii. $\mathcal{S} \models \exists x_1 \ldots x_k \bigwedge \gamma[p]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$

then $\Psi(\theta, \gamma) = t$.

By (32)b there is structure $\mathcal{U}$ and finite assignment q to $\mathcal{U}$ with $domain(q) = var(\sigma)$ such that:

(34) (a) $\mathcal{U} \in \text{MOD}_C(T \cup \{\neg\theta\})$, and

(b) $\forall$-*type(p,$\mathcal{S}$)* $\subseteq$ $\forall$-*type(q,$\mathcal{U}$)*.

Let f be a complete assignment to $\mathcal{U}$ that extends q, and let environment e be such that:

(35) (a) $\sigma \subseteq e$;

(b) e is for $\mathcal{U}$ via f.

Such an e may be chosen by (33)b and (34)b, since $\bigwedge \sigma$ is universal. We shall show:

(36) $\Psi(\theta, \bar{e}_m) = t$ for cofinitely many $m \in N$.

(36), (34)a, and (33)a yield the desired contradiction, completing the proof.

To prove (36), let $m \geq length(\sigma)$ be given. Let $\gamma = \bar{e}_m$, so by (35)a $\sigma \subseteq \gamma$. By (33)c it suffices to show:

(37) $\mathcal{S} \models \exists x_1 \ldots x_k \bigwedge \gamma[p]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$

We demonstrate (37) by contradiction. The falsity of (37) implies:

(38) $\mathcal{S} \models \forall x_1 \ldots x_k \neg \bigwedge \gamma[p]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$

Hence by (34)b:

(39) $\mathcal{U} \models \forall x_1 \ldots x_k \neg \bigwedge \gamma[q]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$

However, since $\gamma \subseteq e$ and $q \subseteq f$, (35)b implies:

(40) $\mathcal{U} \models \exists x_1 \ldots x_k \bigwedge \gamma[q]$, where $var(\gamma) - var(\sigma) = \{x_1 \ldots x_k\}$

which contradicts (39). ■

This completes the proof of Theorem (10), hence of Theorem (8). It is worth noting that the witness provided to Theorem (8) computes a total function in both its arguments for any oracle.

Lemmas (12) and (13) yield the following characterization of detectability in elementary classes of structures.

(41) COROLLARY: Let $\theta \in$ SEN and $T \subseteq$ SEN be given. Then θ is detectable in $\text{MOD}_C(T)$ iff both θ and $\neg\theta$ are confirmable in T.

4 Detection in uncountable structures

Within the paradigm defined in Section 2, scientists are conceived as examining every element of the structure giving rise to their environment. This conception must be modified in order to extend our paradigm to uncountable structures. In the present section we use the term "structure" without cardinality restrictions. Generalizing from the countable case, our intention is to show the scientist a representative sample of an unknown structure $\mathcal{S}$. Samples of this kind are provided by $\mathcal{S}$'s elementary substructures, defined as follows.

(42) DEFINITION: (Robinson, 1956) Let structure $\mathcal{S}$ be given. Substructure $\mathcal{T} \subseteq \mathcal{S}$ is *elementary* just in case for all $\varphi(\bar{x}) \in$ FORM and $\bar{a} \in |\mathcal{T}|$, $\mathcal{T} \models \varphi(\bar{x})[\bar{a}]$ iff $\mathcal{S} \models \varphi(\bar{x})[\bar{a}]$. We write $\mathcal{T} \preceq \mathcal{S}$ in case $\mathcal{T}$ is an elementary substructure of $\mathcal{S}$.

Similarly to the countable case, samples from a domain will be coded as variables. Since there are only countably many variables, the following version of the Löwenheim-Skolem theorem is central to our paradigm (recall that $\mathcal{L}$ has been assumed to be countable).

(43) LEMMA: Let structure $\mathcal{S}$ be given. Then there is countable $\mathcal{T} \preceq \mathcal{S}$.

PROOF: Chang & Keisler (1973, Theorem 3.1.6). ■

An assignment g to a structure $\mathcal{S}$ will be called "elementary" just in case $range(g)$ induces an elementary substructure of $\mathcal{S}$. By the lemma, elementary assignments exist for every structure. Our extended paradigm may now be defined as follows.

(44) DEFINITION: Let $\theta \in$ SEN, structure $\mathcal{S}$, and scientist Ψ be given. Ψ *strongly detects θ in $\mathcal{S}$* just in case for every elementary assignment g to $\mathcal{S}$, and every environment e for $\mathcal{S}$ via g, if $\mathcal{S} \models \theta$ then $\Psi(\theta, \bar{e}_n) = t$ for cofinitely many $n \in N$, and if $\mathcal{S} \models \neg\theta$ then $\Psi(\theta, \bar{e}_n) = f$ for cofinitely many $n \in N$. Let collection $\mathcal{K}$ of structures be given. Ψ *strongly detects θ in $\mathcal{K}$* just in case for all $\mathcal{S} \in \mathcal{K}$, Ψ strongly detects θ in $\mathcal{S}$. In this case, θ is *strongly detectable in $\mathcal{K}$*.

Old and new paradigms are related by the following proposition, which follows immediately from Definition (44).

(45) PROPOSITION: Let $\theta \in$ SEN, collection $\mathcal{K}$ of structures, and scientist Ψ be given. Then Ψ strongly detects θ in $\mathcal{K}$ iff for every $\mathcal{S} \in \mathcal{K}$ and every countable $\mathcal{T} \preceq \mathcal{S}$, Ψ detects θ in $\mathcal{T}$.

The new paradigm offers a stronger criterion of success than the original paradigm, even with respect to countable structures. This is the content of the following proposition.

(46) PROPOSITION: Suppose that $\mathcal{L}$ contains the one-place predicate P, the unary function symbol S, and the two-place relation symbol $<$. Then there is $\theta \in$ SEN and collection $\mathcal{K}$ of countable structures such that θ is detectable in $\mathcal{K}$ but θ is not strongly detectable in $\mathcal{K}$.

PROOF: Let Z_1, Z_2, Z_3 be three copies of the integers. Let Q be the rationals. Let $\mathcal{S} = (Z_1 + Z_3, P^{\mathcal{S}}, S^{\mathcal{S}}, <^{\mathcal{S}})$ where $P^{\mathcal{S}} = Z_1$, $S^{\mathcal{S}}$ is successor, and $<^{\mathcal{S}}$ is the natural order on $Z_1 + Z_3$. Let $\mathcal{U} = (Z_1 + Z_2 + Q, P^{\mathcal{U}}, S^{\mathcal{U}}, <^{\mathcal{U}})$ where $P^{\mathcal{U}} = Z_1 \cup Z_2$, $S^{\mathcal{U}}$ is successor on $Z_1 + Z_2$ and identity on Q, and $<^{\mathcal{U}}$ is the natural order on $Z_1 + Z_2 + Q$. Let $\theta = \forall xy(\neg Px \wedge \neg Py \wedge x < y \rightarrow \exists z(x < z \wedge z < y))$. We claim that θ, $\mathcal{K} = \{\mathcal{S}, \mathcal{U}\}$ witness the proposition.

First we informally describe a scientist Ψ that detects θ in $\mathcal{K}$. By an "S-chain" from variable v_i to variable v_j is meant a set $\{Sv_i = v_{i_1}, Sv_{i_1} = v_{i_2}, \ldots, Sv_{i_k} = v_j\} \subseteq$ BAS. Suppose that Ψ is working in an environment e either for $\mathcal{S}$ via g or for $\mathcal{U}$ via f (where g and f are complete assignments to $\mathcal{S}$ and $\mathcal{U}$, respectively). Ψ begins by searching in e for variables v_i, v_j of lowest index with $\{Pv_i, Pv_j, v_i < v_j\} \subseteq range(e)$. For as long as no S-chain from v_i to v_j appears in e, Ψ conjectures "true" (since it appears that $v_i \in Z_1$ and $v_j \in Z_2$, and thus represent elements of $|\mathcal{U}|$). If Ψ finds an S-chain from v_i to v_j in e then Ψ searches for variables v_p, v_q of lowest index with $\{\neg Pv_p, \neg Pv_q, v_p < v_q\} \subseteq range(e)$. For as long as no v_r is found with $\{\neg Pv_r, v_p < v_r, v_r < v_q\} \subseteq range(e)$, Ψ conjectures "false" (since it appears that $v_p, v_q \in Z_3 = \overline{P^{\mathcal{S}}}$). If Ψ finds such a v_r in e, Ψ searches for new variables v_i, v_j of next lowest index with $\{Pv_i, Pv_j, v_i < v_j\} \subseteq range(e)$ and proceeds as before. It is easy to see that Ψ's conjectures stabilize to the correct truth-value for θ.

It remains to show that θ is not strongly detectable in $\mathcal{K}$. Define $\mathcal{T} = (Z_1 + Q, P^{\mathcal{T}}, S^{\mathcal{T}}, <^{\mathcal{T}})$ where $P^{\mathcal{T}} = Z_1$, $S^{\mathcal{T}}$ is successor on Z_1 and identity on Q, and $<^{\mathcal{T}}$ is the natural order on $Z_1 + Q$. Then $\mathcal{T} \preceq \mathcal{U}$. We shall show that θ is not detectable in $\{\mathcal{S}, \mathcal{T}\}$. By Proposition (45) this suffices to prove that θ is not strongly detectable in $\{\mathcal{S}, \mathcal{U}\}$, completing the proof. Let scientist Ψ detect θ in $\mathcal{T}$. Let $\sigma \in$ SEQ and $p : var(\sigma) \rightarrow |\mathcal{T}|$ be as specified in Lemma (24) (with $\mathcal{T}$ in place of $\mathcal{S}$). Then $\Psi(\theta, \gamma) = t$ for all $\gamma \supseteq \sigma$ such that $\mathcal{T} \models \wedge\gamma[h]$, where $h \supseteq p$. It is clear that $\mathcal{S} \models \wedge\sigma[g]$ for some complete assignment g to $\mathcal{S}$. So we may choose environment e for $\mathcal{S}$ via g such that $\sigma \subseteq e$. It is easy to verify that for all $i \geq length(\sigma)$, $\mathcal{T} \models \wedge\bar{e}_i[h]$ for some $h \supseteq p$. Thus, for all $i \geq length(\sigma)$, $\Psi(\theta, \bar{e}_i) = t$. Since $\mathcal{S} \models \neg\theta$, Ψ does not detect θ in $\mathcal{S}$. ■

In view of Proposition (45) it is easy to modify the proof of Theorem (8) to demonstrate the existence of a universal learner for the present paradigm. For $T \subseteq$ SEN, let MOD(T) be the class of structures (of arbitrary cardinality) that satisfy T. We have:

(47) THEOREM: There is an oracle machine M such that for all $\theta \in$ SEN and $T \subseteq$ SEN, if θ is strongly detectable in MOD(T) then M^T strongly detects θ in MOD(T).

5 A characterization of first-order logic

The present section is devoted to characterizing first-order logic within the framework established in the preceding discussion. To extend our paradigm of inductive inference beyond first-order logic it will be helpful to restrict attention to logics of a finitary character. We proceed as follows. Let HF be the set of hereditarily finite sets. We fix a countably infinite subset VAR of HF, called "variables." By a *vocabulary* is meant a countable set of constants, finitary relation symbols, and finitary function symbols, all drawn from HF and disjoint from VAR. Relative to a choice of vocabulary, the definition of an environment for a countable structure carries over from Section 2 virtually without modification. Specifically:

(48) DEFINITION: Let vocabulary V and countable structure $\mathcal{S}$ for V be given.

(a) BAS_V denotes the set of first-order, basic formulas over V and VAR.

(b) The set of finite sequences over BAS_V is denoted by SEQ_V.

(c) A *V-environment* is an ω-sequence over BAS_V.

(d) A *complete assignment to $\mathcal{S}$* is a mapping of VAR onto $|\mathcal{S}|$.

(e) Given complete assignment g to $\mathcal{S}$, and V-environment e, e is *for $\mathcal{S}$ via g* just in case $range(e) = \{\beta \in \mathrm{BAS}_V \mid \mathcal{S} \models \beta[g]\}$ (where $\models$ is first-order satisfaction).

As before, structures for V that share a V-environment are isomorphic.

By a *logic* we mean a pair $(L, \models_L)$ of mappings defined on the set of vocabularies and meeting the following conditions for each vocabulary V:

(a) $L(V) \subseteq$ HF;

(b) $\models_{L(V)}$ is a relation between the class of structures for V, and $L(V)$.

In studies of comparative logic discussion is typically limited to logics possessing certain properties familiar from the predicate calculus, for example, that isomorphic structures satisfy the same sentences. These properties are brought together in Definitions 1.1.1, 1.2.1 - 1.2.3 of Ebbinghaus (1985) under the term "regular." Henceforth we use the expression "regular logic" in the sense of Ebbinghaus. First- and second-order logic — denoted $\mathcal{L}^1 = (L^1, \models)$ and $\mathcal{L}^2 = (L^2, \models_{L^2})$, respectively — are regular.

We now generalize the definitions of Section 2 in order to define an inductive inference paradigm corresponding to an arbitrary logic. For simplicity, only countable structures will be considered.

(49) DEFINITION: Let logic $\mathcal{L} = (L, \models_L)$ and vocabulary V be given.

(a) A *scientist for $\mathcal{L}$ and V* is any function from $L(V) \times \text{SEQ}$ to $\{t, f\}$.

(b) Let $\theta \in L(V)$, countable structure $\mathcal{S}$ and scientist Ψ for $\mathcal{L}$ be given. Ψ *$\mathcal{L}$-detects θ in $\mathcal{S}$* just in case for every complete assignment g to $\mathcal{S}$, and every environment e for $\mathcal{S}$ via g, if $\mathcal{S} \models_{L(V)} \theta$ then $\Psi(\theta, \bar{e}_n) = t$ for cofinitely many $n \in N$, and if $\mathcal{S} \not\models_{L(V)} \theta$ then $\Psi(\theta, \bar{e}_n) = f$ for cofinitely many $n \in N$.

(c) Let class $\mathcal{K}$ of countable structures, $\theta \in L(V)$ and scientist Ψ for $\mathcal{L}$ be given. Ψ *$\mathcal{L}$-detects θ in $\mathcal{K}$* just in case for all $\mathcal{S} \in \mathcal{K}$, Ψ $\mathcal{L}$-detects θ in $\mathcal{S}$. In this case, θ is *$\mathcal{L}$-detectable in $\mathcal{K}$*.

As before, we shall be particularly concerned with $\mathcal{L}$-detection within the elementary classes of structures determined by subsets of $L(V)$. Given logic $(L, \models_L)$ and $T \subseteq L(V)$, we denote by MOD($\mathcal{L}$,V,T) the class of all countable structures $\mathcal{S}$ for V such that $\mathcal{S} \models_{L(V)} T$.

In contrast to Theorem (8) there are regular logics for which no universal inference machine exists. For example, there is no such machine for $\mathcal{L}^2$. Indeed:

(50) PROPOSITION: There is vocabulary V and $T \subseteq L^2(V)$ such that for all oracle machines M there is $\theta \in \mathrm{L}^2(V)$ such that

(a) θ is detectable in MOD($\mathcal{L}^2$,V,T), but

(b) M^T does not detect θ in MOD($\mathcal{L}^2$,V,T).

PROOF: Let $\mathcal{N}$ be the standard model of arithmetic. Let V contain just the vocabulary of arithmetic, and let sentence $T \in L^2(V)$ characterize $\mathcal{N}$ up to isomorphism. Choose complete assignment g to $\mathcal{N}$ and environment e for $\mathcal{N}$ via g such that $\{\sigma \in \text{SEQ}_V \mid \sigma \subseteq e\}$ is recursive. It is easy to see that such g and e exist. For a contradiction, suppose that M is an oracle machine such that for all $\theta \in L^1(V) \subseteq L^2(V)$, M^T detects θ in MOD($\mathcal{L}^2$,V,T) $= \{\mathcal{S} \mid \mathcal{S} \cong \mathcal{N}\}$. Since T is a single sentence there is Turing machine M' without oracle that behaves like M^T. Hence, given $\theta \in L^1(V)$, $\mathcal{N} \models \theta$ iff $M'(\theta, \bar{e}_i) = t$ for cofinitely many $i \in N$. But this exhibits $\{\theta \in L^1(V) \mid \mathcal{N} \models \theta\}$ as arithmetic, which is impossible. ■

The reasoning used to establish Proposition (50) points to a characterization of $\mathcal{L}^1$. Toward this end we formulate a strengthened sense of universal scientist, applicable to an arbitrary logic. Given logic $\mathcal{L} = (L, \models_L)$, vocabulary V, and $\theta \in L(V)$, we denote by $V(\theta)$ the intersection of V and the transitive closure of θ (i.e., $V(\theta)$ is the vocabulary occurring in θ). Also, given $T \subseteq L(V)$, MOD$^\theta$($\mathcal{L}$,V,T) denotes $\{\mathcal{S}\lceil V(\theta) \mid \mathcal{S} \in \text{MOD}(\mathcal{L},V,T)\}$ (that is, $\mathcal{U} \in$ MOD$^\theta$($\mathcal{L}$,V,T) iff $\mathcal{U}$ is the reduct to $V(\theta)$ of some countable $\mathcal{S}$ with $\mathcal{S} \models_{L(V)} T$).

(51) DEFINITION: Let logic $\mathcal{L} = (L, \models_L)$, vocabulary V, and oracle machine M be given.

(a) M is *PC-universal for $\mathcal{L}$ and V* just in case for every $T \subseteq L(V)$ and every $\theta \in L(V)$ if θ is $\mathcal{L}$-detectable in MOD$^\theta$($\mathcal{L}$,V,T) then M^T $\mathcal{L}$-detects θ in MOD$^\theta$($\mathcal{L}$,V,T).

(b) $\mathcal{L}$ has the *PC universal property* just in case for every vocabulary V there is an oracle machine M that is PC-universal for $\mathcal{L}$ and V.

Minor modifications to the proof of Theorem (8) yield the following.

(52) THEOREM: $\mathcal{L}^1$ has the PC-universal property.

Our characterization of $\mathcal{L}^1$ also relies on the Löwenheim-Skolem property. A logic $(L, \models_L)$ has this property just in case for every vocabulary V and every $\theta \in L(V)$, if $\mathcal{S} \models_{L(V)} \theta$ for some structure $\mathcal{S}$, then $\mathcal{U} \models_{L(V)} \theta$ for some countable structure $\mathcal{U}$. Finally, given logics $\mathcal{L} = (L, \models_L)$ and $\mathcal{L}' = (L', \models_{L'})$, we say that $\mathcal{L}'$ *effectively extends* $\mathcal{L}$ just in case

(a) for all vocabularies V there is computable $h : L(V) \to L'(V)$ such that for all $\theta \in L(V)$, MOD($\mathcal{L}$,V,θ) = MOD($\mathcal{L}'$,V,$h(\theta)$); and

(b) there is a vocabulary V and $\theta' \in L'(V)$ such that for every $\theta \in L(V)$, MOD($\mathcal{L}$,V,θ) $\neq$ MOD($\mathcal{L}'$,V,θ').

(53) THEOREM: Let regular logic $\mathcal{L}$ be given. If $\mathcal{L}$ has the Löwenheim-Skolem property and $\mathcal{L}$ effectively extends $\mathcal{L}^1$ then $\mathcal{L}$ does not have the PC-universal property.

Thus, $\mathcal{L}^1$ is a maximal logic with the Löwenheim-Skolem and PC-universal properties.

PROOF: Let logic $\mathcal{L} = (L, \models_L)$ be given such that $\mathcal{L}$ is regular, $\mathcal{L}$ has the Löwenheim-Skolem property, and $\mathcal{L}$ effectively extends $\mathcal{L}^1$. By Flum (1985, Corollary 2.1.3), $(\omega, <)$ is PC in $\mathcal{L}$. By the regularity of $\mathcal{L}$ and the fact that $\mathcal{L}$ effectively extends $\mathcal{L}^1$, there is vocabulary $V \supseteq \{<, +, \cdot\}$ and $A \in L(V)$ such that $\mathcal{S} \in$ MOD($\mathcal{L}$,V,A) iff $\mathcal{S}\lceil\{<, +, \cdot\} \cong (\omega, <, +, \cdot)$. Hence, every $\theta \in L(\{<, +, \cdot\})$ is $\mathcal{L}$-detectable in MOD$^\theta$($\mathcal{L}$,V,A). On the other hand, minor modification to the proof of (50) shows that there is no oracle machine M such that M^A $\mathcal{L}$-detects θ in MOD$^\theta$($\mathcal{L}$,V,A), for every $\theta \in L(\{<, +, \cdot\})$. Hence no oracle machine is PC-universal for $\mathcal{L}$ and V. ■

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