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A Contact Stress Model for Determining Forces in an Equilibrium Grasp

Abstract

Most available methods that predict the forces necessary to grasp an arbitrary object treat the object and the fingers as rigid bodies and the finger/object interface as a point contact with Coulomb friction. For statically indeterminate grasps, therefore, while it is possible to find grasps that are stable, there is no unique determination of the actual forces at the contact points and equilibrium grasps are determined as many-parameter families of solutions. Also, these models sometimes lead to phenomenologically incorrect results which, while satisfactory from a purely mathematical viewpoint, are counterintuitive and not likely to be realized in practice. The model developed here utilizes a contact-stress analysis of an arbitrarily shaped object in a multi-fingered grasp. The fingers and the object are all treated as elastic bodies and the region of contact is modeled as a deformable surface patch. The relationship between the friction and normal forces is now nonlocal and nonlinear in nature and departs from the Coulomb approximation. The nature of the constraints arising out of conditions for compatibility and static equilibrium motivated the formulation of the model as a non-linear constrained minimization problem. The total potential energy of the system is minimized, subject to the nonlinear, equality and inequality constraints on the system, using the Schittkowski algorithm. The model is able to predict the magnitude of the inwardly directed normal forces, and both the magnitude and direction of the tangential (friction) forces at each finger/object interface for grasped objects in static equilibrium. Examples in two and three dimensions are presented along with application of the model to the grasp transfer maneuver.

Comments

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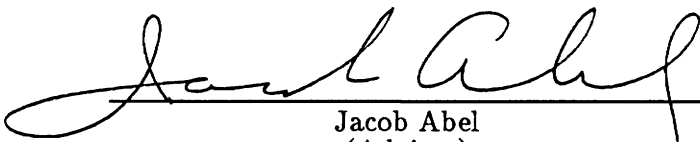
A CONTACT STRESS MODEL FOR
DETERMINING FORCES IN AN EQUILIBRIUM GRASP

Pramath Raj Sinha

Philadelphia, Pennsylvania

August 1989

A thesis presented to the Faculty of Engineering and Applied Science of the University of Pennsylvania in partial fulfillment of the requirements for the degree of Master of Science in Engineering for graduate work in Mechanical Engineering and Applied Mechanics.



Jacob Abel
(Advisor)



Noam Lior
(Graduate Group Chair)

*For
Amma and Babuji -
to whom I owe all
that I will ever be.*

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ABSTRACT

Most available methods that predict the forces necessary to grasp an arbitrary object treat the object and the fingers as rigid bodies and the finger/object interface as a point contact with Coulomb friction. For statically indeterminate grasps, therefore, while it is possible to find grasps that are stable, there is no unique determination of the actual forces at the contact points and equilibrium grasps are determined as many-parameter families of solutions. Also, these models sometimes lead to phenomenologically incorrect results which, while satisfactory from a purely mathematical viewpoint, are counterintuitive and not likely to be realized in practice. The model developed here utilizes a contact-stress analysis of an arbitrarily shaped object in a multi-fingered grasp. The fingers and the object are all treated as elastic bodies and the region of contact is modeled as a deformable surface patch. The relationship between the friction and normal forces is now nonlocal and nonlinear in nature and departs from the Coulomb approximation. The nature of the constraints arising out of conditions for compatibility and static equilibrium motivated the formulation of the model as a non-linear constrained minimization problem. The total potential energy of the system is minimized, subject to the nonlinear, equality and inequality constraints on the system, using the Schittkowski algorithm. The model is able to predict the magnitude of the inwardly directed normal forces, and both the magnitude and direction of the tangential(friction) forces at each finger/object interface for grasped objects in static equilibrium. Examples in two and three dimensions are presented along with application of the model to the grasp transfer maneuver.

Contents

| | | |
|----------|---|----------|
| 1 | Introduction | 1 |
| 1.1 | Previous Work | 1 |
| 1.1.1 | Grasping With a Mechanical Hand | 2 |
| 1.1.2 | Three Fingered Grasps, with Rigid Body Slip | 2 |
| 1.1.3 | Stiffness and Compliance of Fingers | 3 |
| 1.2 | Friction Models for Coordinated Grasping | 3 |
| 1.2.1 | Implications to Coordinated Grasping | 4 |
| 1.3 | Examples of the Failure of Available Models | 4 |
| 1.3.1 | Failure of the Coulomb Model | 4 |
| 1.3.2 | The Unidirectional Normal Force Constraint | 6 |
| 1.4 | Goals | 8 |
| 2 | Problem Formulation | 9 |
| 2.1 | The General Contact Problem | 9 |
| 2.2 | System Configuration | 10 |
| 2.2.1 | Size of the Contact Patch | 12 |
| 2.3 | Assumptions | 13 |
| 2.4 | Contact Problem Under a Normal Force | 14 |
| 2.4.1 | Condition of Equilibrium | 14 |
| 2.4.2 | The Concept of the Rigid Body Approach | 14 |
| 2.4.3 | Condition of Compatibility of Deformation | 15 |
| 2.4.4 | Finger Shape Definition | 16 |
| 2.4.5 | Criterion for Contact | 16 |
| 2.4.6 | General Model for Elastic Deformation | 17 |
| 2.4.7 | Potential Energy due to a Normal Force | 18 |
| 2.5 | Contact Problem Under a Tangential Force | 19 |
| 2.5.1 | Condition of Equilibrium | 21 |
| 2.5.2 | Condition of Compatibility of Deformation | 21 |
| 2.5.3 | Constraints on the Tangential Force | 22 |
| 2.5.4 | General Model for Elastic Deformation | 22 |
| 2.5.5 | Potential Energy due to a Tangential Force | 23 |
| 2.6 | Micromechanics of Friction | 24 |

| | | |
|----------|--|-----------|
| 2.6.1 | Empirical Determination of μ | 26 |
| 2.6.2 | A Nonlocal and Nonlinear Law | 26 |
| 2.6.3 | An Elastic Theory of Coulomb Friction | 26 |
| 2.6.4 | Determination of μ in our Examples | 27 |
| 2.7 | Models for Influence Matrix | 27 |
| 2.7.1 | The Elastic Half-Space | 27 |
| 2.7.2 | Deformations due to Concentrated Normal Force | 28 |
| 2.7.3 | Deformations due to Concentrated Tangential Force | 28 |
| 2.7.4 | Calculation of the Influence Coefficients | 29 |
| 2.8 | The Requirements for Static Equilibrium | 30 |
| 2.8.1 | Force Equilibrium | 30 |
| 2.8.2 | Moment Equilibrium | 31 |
| 2.9 | Directional Constraints on the Forces | 31 |
| 2.9.1 | Sense Constraint on the Direction of the Normal Forces | 31 |
| 2.9.2 | Constraint on the Tangential Forces | 32 |
| 2.10 | The Complete Model | 32 |
| 3 | Solution Technique | 34 |
| 3.1 | General Nonlinear Optimization | 35 |
| 3.1.1 | Convex Optimization | 35 |
| 3.1.2 | The Kuhn-Tucker Conditions | 35 |
| 3.1.3 | Quadratic Optimization | 36 |
| 3.2 | The Method of Wolfe | 37 |
| 3.3 | The Schittkowski Algorithm | 38 |
| 3.4 | Form of the Objective Function | 40 |
| 3.5 | An Example Program | 41 |
| 4 | Results | 47 |
| 4.1 | Examples in Two and Three Dimensions | 47 |
| 4.1.1 | Rectangular Plate in a Four-Finger Grasp | 47 |
| 4.1.2 | Disc in a Three-Finger Grasp | 48 |
| 4.1.3 | Pipe in a Three-Finger Grasp | 48 |
| 4.1.4 | Cube in a Three-Finger Grasp | 50 |
| 4.1.5 | Cube in a Three-Finger Grasp (Different Configuration) | 51 |
| 4.1.6 | Discussion | 53 |
| 4.2 | The Transfer Maneuver | 53 |
| 4.2.1 | Adaptation of the Algorithm | 53 |
| 4.2.2 | Example of Transfer of A Plate | 54 |
| 4.2.3 | Example of Transfer of A Disc | 54 |
| 4.2.4 | Discussion | 56 |

5 Conclusion **57**

5.1 Summary 57

5.2 Contributions 58

 5.2.1 Advantages 58

 5.2.2 Disadvantages 58

5.3 Future Research 59

List of Figures

| | | |
|----|---|----|
| 1 | Disc supported on two Fingers | 5 |
| 2 | Body Supported by Elastic Elements | 7 |
| 3 | System Configuration | 10 |
| 4 | Contact Point treated as a Patch of Rectangular Grid Elements | 11 |
| 5 | Discretized Contact Forces | 12 |
| 6 | An Undeformed Finger/Object Contact | 15 |
| 7 | A Deformed Contact Region | 19 |
| 8 | A Cube in a Three-Fingered Grasp | 41 |
| 9 | A Rectangular Plate Grasped by Four Fingers | 48 |
| 10 | A Disc Grasped by Three Fingers | 49 |
| 11 | A Pipe Grasped by Three Fingers | 49 |
| 12 | Three-finger Grasp for a Cube | 51 |
| 13 | Another Three-finger Grasp for a Cube | 52 |
| 14 | Transfer of a Plate | 54 |
| 15 | Changes in Forces during Transfer of a Plate | 55 |
| 16 | Transfer of a Disc | 55 |
| 17 | Changes in Forces and Moments during Transfer of a Disc | 56 |

Chapter 1

Introduction

A surprising and undesirable feature of modern robot manipulators is their relatively large size and small payload capacity. A PUMA 560 weighs approximately 300 lbs. yet can reliably pick up only about 25 lbs. The ability to use two manipulators to pick up the same object would increase the versatility of both manipulators. Two manipulators in contact with the same object form a closed loop mechanical system with two independently controlled sets of inputs. Each manipulator sees the other as an external disturbance transferred through the object. Therefore, an accurate model of the mechanics of interaction of each manipulator with the object itself is required so that each manipulator can identify the performance of the other. In the particular case of a quasi-static shift of weight from one manipulator to the other, a smooth partitioning of the load is required subject to the ability of each manipulator to support the object in a given configuration. This control requires an accurate model of the surface interaction at the contact between the manipulator and the object. In this thesis, we present an analytical investigation of the interaction between two or more fingers in contact with an arbitrary deformable object, with an intent to focus on the transfer of the object from one manipulator to another. The result of this effort will permit the design of control strategies for grasping with a mechanical hand as well as for the coordinated control of multiple robot systems.

1.1 Previous Work

The problem of how to position the fingers of a general purpose mechanical hand to grasp a general object has been the focus of much research [AHM85, MJ85, Sal83, TAP88]. In each case, the interaction between the object and the finger has been modeled using rigid body statics and, for rough objects, Coulomb friction. These problems are usually statically indeterminate, therefore, while it is possible to find stable grasps, there is no unique determination of the actual forces at the manipulator/object interface and equilibrium grasps are determined as many-parameter families of solutions.

1.1.1 Grasping With a Mechanical Hand

The analysis of the forces required to hold an object with a mechanical hand was introduced by Salisbury and Roth as a static equilibrium problem where the fingertip contacts were modeled as “point contacts with friction”. Clearly, if the number of fingers exceeds three then the system of forces is statically indeterminate. They consider the indeterminate force magnitudes as “internal forces” which may be manipulated to ensure a firm grasp. Jameson extends the work by including conditions which characterize the stability of a grasp. The essential idea is that the magnitude of the friction force at any point is to be less than the maximum sustainable between the finger and the object, as defined by the Coulomb coefficient of friction. Jameson recognizes that grasps relying on friction have additional degrees of freedom due to the undetermined direction of the friction force. He treats these unknowns in a manner similar to Salisbury, as internal degrees of freedom which, though unknown, can be used to guarantee a stable grasp. He extends his analysis to include the mechanical design of the gripping finger and the possible dependence of the applied joint torques, and uses an optimization technique to identify the “most stable” grasp configuration and joint torque specifications. This approach to grasp planning results in a “static” grasp by which the mechanical hand is certain to be able to support the object. However, because the problem is fundamentally statically indeterminate the actual friction and normal force at each fingertip cannot be predicted or planned. Any change in the configuration of the hand or load may require a completely different grasp, which may or may not be continuous with the first. In order to control the fingers through a range of configurations with varying load, we require an analytical model for the actual normal and friction forces at the contact between the fingers and the object.

1.1.2 Three Fingered Grasps, with Rigid Body Slip

An important phenomena in grasping is the potential for slip at the contact point. The coordinated control of a pair of manipulators or the transfer of an object between manipulators requires the ability to predict, and even control, the slip of the object. Abel, et al. [AHM85] and Holzmann and McCarthy [HM85] introduced the concept of incipient rigid body slip into the static equilibrium model, and showed how to compute the contact forces. The analysis of a three fingered grasp presented by Holzmann and McCarthy uses the fact that when a rigid object slips it must undergo a rigid body displacement. At the instant of slip the friction forces oppose the instantaneous virtual velocities of the contact points. The assumption that the friction forces are compatible with rigid motion at loadings where slip is only imminent permits the computation of friction forces in three fingered spatial grasps, and two fingered planar grasps. However, it turns out that not only are there grasps that slip and do not slip, but there are “paradoxical” grasps [HM85]. The paradoxical grasps, in order to satisfy equilibrium, have friction forces at some points which act in the direction of the impending motion contrary to the definition of friction. This model brings us close to the ability to determine the exact state of the forces at each fingertip, including the phenomena of slip, however, it is clear that a more general approach is required.

1.1.3 Stiffness and Compliance of Fingers

Cutkosky [Cut85] developed a procedure for discovering the properties of a grasp by moving the object slightly, observing the resulting finger motions and determining the changes in the forces on the object. The grasp properties of stiffness (related to the material properties of the finger), resistance to slipping and infinitesimal stability were introduced and it was shown that such properties could be used to compare grasps. For a given task, one could then choose the grip that would be the stiffest with respect to rotations or the grip that would resist the largest vertical force before slipping occurred.

While the limitation of the point-contact assumption is exposed in this analysis, the object is still modeled as being rigid and only the finger is treated as compliant. An interesting analysis of the interactions between different kinds of fingertips and the object is presented in this work. The characteristics of pointed, curved, and soft fingers are compared. The soft fingertip is modeled as a short elastic beam that is clamped between the object and the finger substrate. The fingertip is also softer than the object or the finger substrate which are both treated as rigid. Shear, bending and torsion of the fingertip is taken into account. However, the whole analysis in this work once again involves small motions about an initial position to calculate the grasping forces. Therefore, the mobility of the object is determined using the number of degrees of freedom associated with each contact as well as adopting the convention of twists and wrenches as used in the models mentioned in the two previous sections. In order to solve the statically indeterminate case, where no motion is possible for the given configuration, "virtual joints" are added to the finger to provide enough equations for the number of unknowns.

It is also established in Cutkosky's work that the hard curved fingertip and the very soft fingertip (two of the cases that are analyzed) represent extremes between which real, deformable robot fingertips may be expected to lie. However, the analysis for the individual cases is not combined to give a more complete model since it becomes very involved. This work is a step towards a more phenomenologically realistic model and the need for a better representation of the finger/object interaction is clearly established.

1.2 Friction Models for Coordinated Grasping

As indicated above, the classical Coulomb model for friction which, given its simplicity, yields remarkably good results in a wide variety of static and dynamic problems is not satisfactory in some grasping problems which involve general objects in multi-fingered grasps. When the problem is enlarged to the case in which there are more fingers than there are dimensions to the object, the Coulomb model may fail totally. The difficulty is bound up with the requirement to define an incipient or virtual motion for the object being grasped if the Coulomb model is to be used. When the number of contact points exceeds the dimensionality of the body, there may be no incipient motion which satisfies the rigid body assumption. An example in two dimensions is given below which illustrates this difficulty. In these situations, where intuition, considerations of geometry, or real world experience indicate that equilibrium grasps of rough objects are attainable but do not conform to the

Coulomb model, it is necessary to introduce a model which retains the features of the forces in the Coulomb model but which do not require the assumption of an incipient rigid body displacement for their definition.

1.2.1 Implications to Coordinated Grasping

When coordinated grasping of a single object by several manipulators is envisioned, the overconstrained state will be the norm rather than the exception, particularly if the object transfer is the task. In this case, there will always be an interval during the maneuver when at least four and probably six fingers will be in contact with the object. In this instance, for a body of arbitrary shape which is grasped at points that are chosen to meet criteria which are derived from considerations of grasp feasibility, there is little likelihood that a kinematically admissible virtual displacement will exist. Thus the determination of friction force magnitudes and directions from the Coulomb relations will be impossible. Secondly, the number of unknown force components will exceed the number of scalar equilibrium equations that can be written for the grasped object. Traditionally [Fun65], problems of this type are classified as statically indeterminate and must be solved by recourse to appropriate minimum principles which employ energy expressions or potentials in their formulations. In this setting, the equations of equilibrium become constraint relations among the unknown forces and may be appended to the energy function to be minimized using the Lagrange multiplier method. When this procedure is employed, an additional difficulty may be encountered wherein normal force values are determined which represent grasp forces which are outwardly directed from the object. This condition, of course, is impermissible and solutions of this character must be rejected.

1.3 Examples of the Failure of Available Models

1.3.1 Failure of the Coulomb Model

In Figure 1(a), we illustrate an elementary paradoxical grasp in two dimensions [Tri87]. A rigid uniform circular disc in the gravitational field is to be supported by two symmetrically placed rigid fingers with point contact. The resulting force diagram at equilibrium must be symmetric in order to satisfy moment equilibrium. From statics we know that :

$$\begin{aligned} \mathbf{f}_{1,x} + \mathbf{f}_{2,x} &= 0 & (1) \\ -w + \mathbf{f}_{1,y} + \mathbf{f}_{2,y} &= 0 & (2) \end{aligned}$$

Summing the moments of the forces about any point on the y axis except the center of the circle reveals that :

$$\mathbf{f}_{1,y} = \mathbf{f}_{2,y} = \frac{w}{2} \quad (3)$$

At the same time, it may be observed that if the normal forces are imposed by a device that permits no motion in the normal direction, the disc will have a center of rotation at

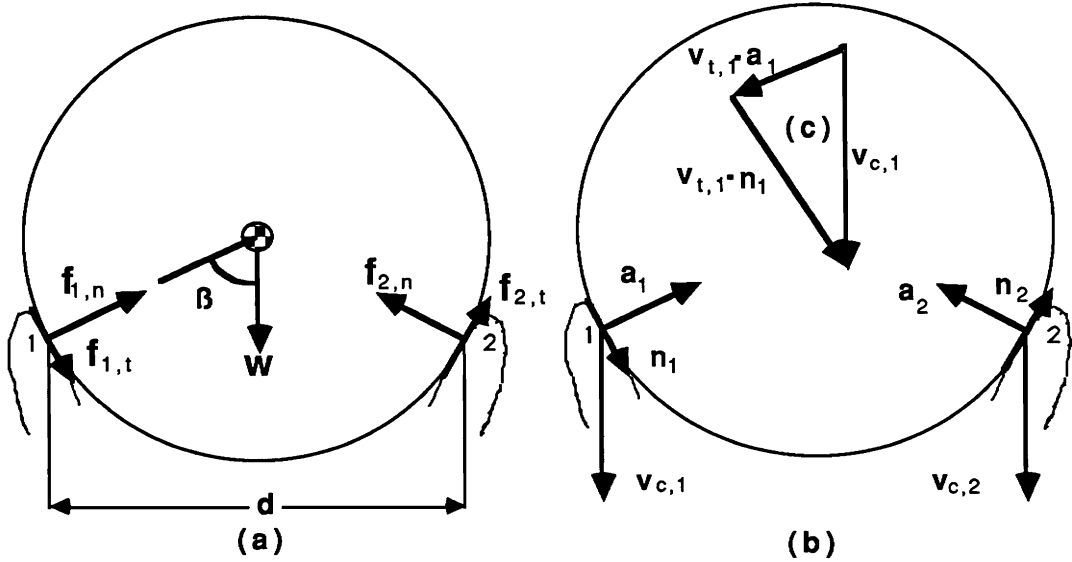


Figure 1: Disc supported on two Fingers

its geometric center. As a result the only kinematically admissible virtual displacement is a rotation about the center. In that case, according to the Coulomb theory, both friction forces must point in a direction that opposes the incipient motion at the point of contact. In other words, both must point in either the clockwise or counter-clockwise direction. If there is no impending rotation, then the friction forces must both be zero. Note that the friction forces must either produce moments of the same sense about the axis of the disc or no moment at all. Since the problem is symmetric, the solution must also be symmetric. Therefore, the only solution is that friction forces must be zero and the normal forces must support the weight of the disc :

$$\mathbf{f}_{1,t} = \mathbf{f}_{2,t} = 0 \quad (4)$$

$$\mathbf{f}_{1,n} = \mathbf{f}_{2,n} = \frac{w}{2\cos\beta} \quad (5)$$

Any asymmetric solution will cause the disc to rotate.

If we remove the rigid body assumption from the support, we realize that the above solution is incorrect. Assume that the fingers are made of steel. Next consider the following chain of interactions which occur as the weight of the disc is transferred to the fingers. As the normal forces grow, the supporting edges of the fingers recede in the direction of the contact normal, increasing the distance, d . Symmetry implies that the disc translate downward. The velocity of the contact points on the disc, $\mathbf{v}_{c,1}$ and $\mathbf{v}_{c,2}$ act in the negative y

direction. The velocity of the left edge of the finger, $\mathbf{v}_{t,1}$, can be resolved into the directions, \mathbf{a}_1 and \mathbf{n}_1 see Figure 1(b).

$$\mathbf{v}_{t,1} = -\alpha_1 \mathbf{a}_1 + \beta_1 \mathbf{n}_1 \quad \alpha_1 > 0 \quad (6)$$

To maintain contact without slipping $\mathbf{v}_{c,1}$ must be identical to $\mathbf{v}_{t,1}$ see Figure 1(c) (drawn in Figure 1(b)). Therefore, β_1 must be positive. This causes the support to bend toward the line of symmetry of the problem, giving rise to a friction force at the left contact applied to the disc :

$$\mathbf{f}_{1,t} = -\beta_1 \mathbf{n}_1 \quad \beta_1 > 0 \quad (7)$$

The friction force acting on the right edge of the disc is :

$$\mathbf{f}_{2,t} = \beta_1 \mathbf{n}_2 \quad (8)$$

If sliding occurs, the fingers' edges will tend toward their undeformed configurations with friction forces acting in the same directions as for the nonslipping case.

Whether sliding occurs or not, the friction forces act with opposite sense and equal magnitude about the axis of the disc. This result is contrary to the prediction using rigid body models for both objects since both normal and friction forces perform work.

The rigid body model can be easily discredited. Assume that the weight of the disc is initially zero, but in time increases to the weight, w (this is to simulate loading the disc onto the fingers). As the load on the fingers increases, the normal and friction forces have components in the positive y direction. Modeling all bodies as rigid, demands that the friction forces be zero and the normal forces support the entire load. Clearly, a real system will deform and the friction forces must participate in opposing the load. Using only rigid models fails to reflect accurately the contact forces, but the solution that it gives is correct for the gross motion of the system.

The above considerations lead us to use the rigid body models to approximate the gross motion of robotic manipulation systems, and to incorporate a contact model which possesses compliance to compute the friction forces.

1.3.2 The Unidirectional Normal Force Constraint

Here we show the effect on minimum energy solutions of the constraint that the normal force be directed toward the object surface. Figure 2 shows a simple one dimensional grasping problem which involves an object being supported in the gravitational field through elastic (spring) elements. The equilibrium equation is :

$$N_1 - N_2 = W \quad (9)$$

The potential energy of the system is :

$$V = \frac{1}{2} \left(\frac{N_1^2}{K} + \frac{N_2^2}{K} \right) + W \left(\frac{N_1}{K} - \frac{N_2}{K} \right) \quad (10)$$

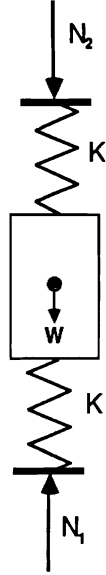


Figure 2: Body Supported by Elastic Elements

By introducing normalized variables, equations (9) and (10) become :

$$n_1 - n_2 = \frac{N_1}{W} - \frac{N_2}{W} = 1 \quad (11)$$

$$V^* = 2K \frac{V}{W} = n_1^2 + n_2^2 + 2(n_1 - n_2) \quad (12)$$

This problem is statically indeterminate. The equilibrium equation (11) reveals that there is a one parameter family of solutions in which either one of the forces may be chosen arbitrarily with the second being determined by the equation. The minimum of potential energy is found at the point $n_1 = \frac{1}{2}$, $n_2 = -\frac{1}{2}$. The value of the normalized potential energy is $V^* = 2.5$. This solution, while mathematically correct, violates the requirement that both normal forces point inward on the body, i.e. N_1 and N_2 must be positive. This requirement is an inequality constraint on the variables, N_1 and N_2 and cannot be treated by the Lagrange Multiplier Method. The lowest energy solution which also satisfies the normal force constraint is found at $n_1 = 1$ and $n_2 = 0$, i.e. the object is supported from below. The normalized potential energy is then : $V^* = 3.0$. This example illustrates the point made earlier that the straightforward application of the theorem of minimum potential energy may lead to solutions that violate the constraint imposed on the normal forces.

1.4 Goals

The subject of the preceding sections was intended to establish the need for an accurate model of the mechanics of surface interaction at the contact between the object and the manipulator. In this formulation we aim to develop a new quasi-elastic model of the manipulator/object contact which resolves the indeterminacy of the rigid body model, yet maintains its important phenomenological properties, the most important of which is the existence of a threshold at which rigid body slip occurs. The features of the "friction forces" that must be retained are : 1) tangential direction, 2) nonconservative nature, 3) existence of limiting values dependent on the normal force. In addition, we require that the "friction forces" derived from the model obey symmetry conditions which may be established from *a priori* considerations. It would also be desirable to have a model which will be simple enough to permit its introduction into practical control algorithms without excessive computational burden. To accomplish these objectives we will invoke some classical concepts from the field of contact stress and tribology which is in recognition of the fact that macroscopic descriptions of friction are always a reflection of many microscale phenomena which are too numerous or complex to analyze. This approach will also permit treatment of different material combinations for the manipulator and object. A model which possesses these features will retain the macroscale features of the Coulomb model yet permit the relaxation of the kinematic constraints which lead to the paradoxical equilibrium states. Also, this model will allow the prediction of contact forces during a shift of load from one manipulator to another, which may occur without overall configuration change.

In the next chapter, the problem is analyzed and the model developed in detail. In Chapter 3, the solution technique is presented along with reasons for why some other techniques did not work given the nature of the problem. Finally, in Chapter 4 the results of some sample problems are shown and the model is applied to the transfer maneuver problem mentioned earlier.

Chapter 2

Problem Formulation

Motivated by the need to treat the manipulator and object as deformable bodies and not as rigid bodies, concepts from solid mechanics are used to describe the phenomena of grasping. These concepts permit replacement of the classical Coulomb friction model with more phenomenologically realistic models.

The manipulator finger tips and the object are the only parts with which we will concern ourselves with in this thesis. They are both treated as linear elastic objects and the interaction at each object/finger contact is modeled as the classical case of two elastic bodies in contact.

2.1 The General Contact Problem

Hertz [Lov44] established his theory for elastic bodies in contact under a normal force. In his theory, he analyzes the contact area, the normal stress distribution and introduces the concept of the rigid body approach. However, the Hertz theory is restricted to frictionless surfaces and perfectly elastic solids.

Probably the most important extension of the Hertz contact theory consists of problems involving additional force systems superimposed upon the Hertz normal force [Joh85]. The case particularly relevant to our problem is that of two solids which have first been pressed together with a force along their common normal and which are subject to tangential forces which tend to slide one body relative to the other. Here, too, we need to distinguish between conforming and nonconforming contacts. Bodies which have dissimilar profiles are said to be nonconforming, and this is the case that we are interested in. When brought into contact without deformation they will first touch at a point (hereafter called the *point of contact* or *contact point*), and, even under a force, the dimensions of the contact patch are generally small compared to the bodies themselves. In these circumstances, the contact stresses and deformations consist of a “local stress distribution” and a “local deformation”.

First, we examine the finger/object interaction as a contact problem under a distributed normal force. Then the problem is analyzed with respect to the tangential forces that are consequent to the normal forces on the surface of contact. After that, an analysis of

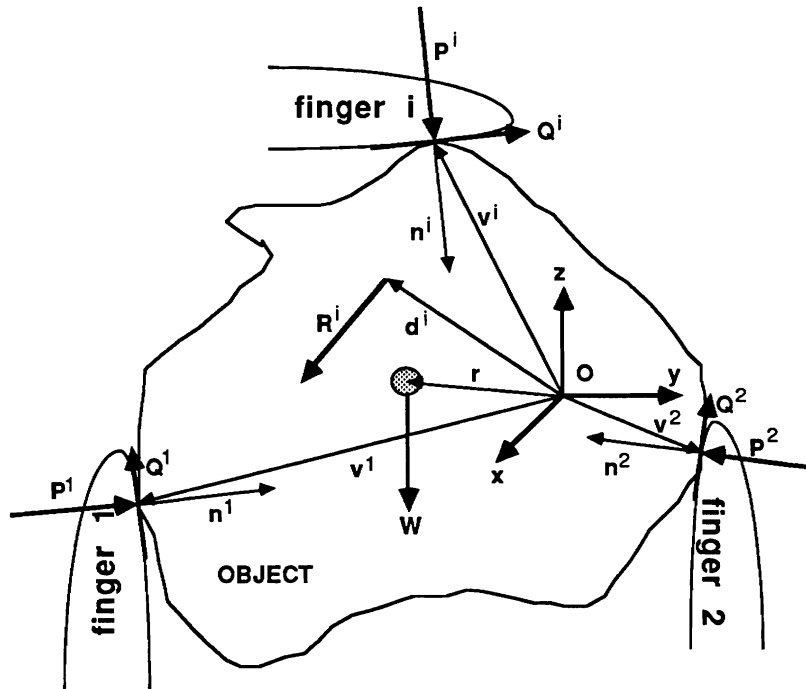


Figure 3: System Configuration

the micromechanics of friction is presented, allowing us to formulate a proper relationship between normal and tangential forces. Once the model for the normal and tangential forces at each contact has been formulated, it is linked with the conditions required for static equilibrium. The total potential energy of the system is then calculated, the reason for which will become clearer in the next chapter when solution techniques are discussed.

2.2 System Configuration

An arbitrarily shaped elastic object is grasped using many fingers.

Figure 3 shows the geometry and the nomenclature that we will use for our analysis. The figure is self-explanatory and the labels are as follows :

x, y, z represent a set orthogonal axes that form the reference coordinate system with its origin at an arbitrary reference point O within the object. For purposes of our analysis, the reference point is known.

r is the vector locating the center of mass of the object with respect to the reference point O . This is a known quantity.

W is the weight of the object acting in the negative z direction. This is a known quantity.

R^i , ($i = 1, \dots, K$) are the external forces other than the forces exerted by the

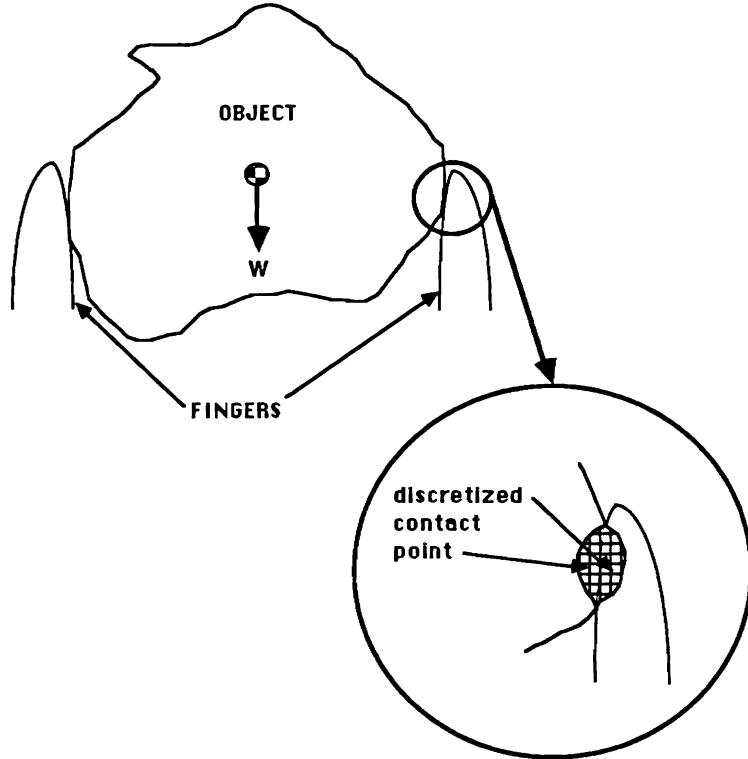


Figure 4: Contact Point treated as a Patch of Rectangular Grid Elements

fingers. All such forces are specified.

\mathbf{d}^i , ($i = 1, \dots, K$) are the position vectors of the points of application of the forces \mathbf{R}^i , with respect to the origin of the chosen reference frame \mathbf{O} . All such vectors are specified.

At each finger/contact i for $i = 1, \dots, M$, we have :

\mathbf{n}^i , the inwardly directed surface normal vector at the point of contact i . This vector is known.

\mathbf{v}^i , the position vector of the point on the surface through which the resultant normal force at the i^{th} fingertip acts with respect to \mathbf{O} . This vector is also known.

\mathbf{P}^i , is the resultant normal force at the the i^{th} fingertip. The magnitude of this force is unknown, however, it is constrained to act in the same direction as \mathbf{n}^i .

\mathbf{Q}^i , is the resultant tangential force at the the i^{th} fingertip. The magnitude of this force is unknown, however, it is constrained to act in the plane that is normal to \mathbf{n}^i or tangent to the surface at \mathbf{v}^i .

The material properties of the object and fingers are given. Given this information, we would like to predict the normal forces (and the resulting tangential forces) that each finger must exert to be able to grasp the object and support it in static equilibrium.

The resultant force transmitted from the fingertip to the object surface is resolved into a *normal force* (\mathbf{P}) acting along the common normal, which is generally compressive, and

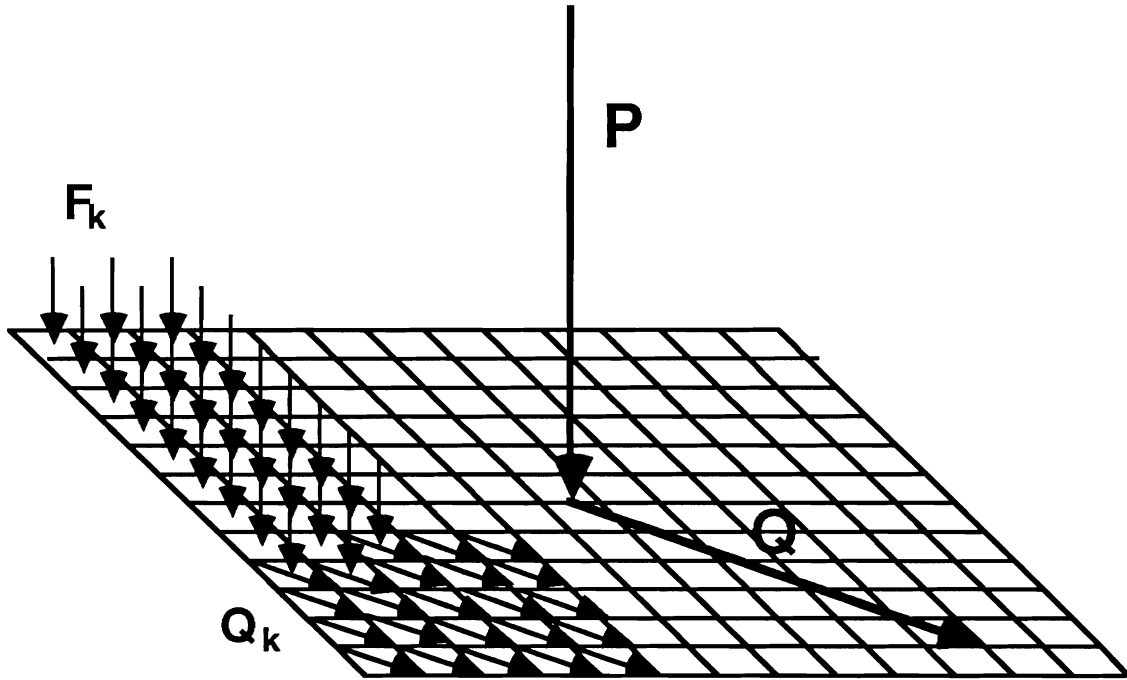


Figure 5: Discretized Contact Forces

a *tangential force* (\mathbf{Q}) attributed to friction. In contrast with the treatment in previously proposed models, the finger/object contact is not treated as a point but as a patch of rectangular grid elements with a finite area (see figure 4).

At each contact patch the normal force (\mathbf{P}) and the tangential force (\mathbf{Q}) are discretized into smaller components F_k and G_k , respectively - each component acting over a grid element (see figure 5). F_k and G_k represent the distributed normal and tangential pressure on the grid element k , where element k is located by the vector \mathbf{l}_k^i relative to the point located by \mathbf{v}^i . The discretized forces are not necessarily uniform as shown in the figure.

2.2.1 Size of the Contact Patch

The size of the contact patch is determined the by material properties and the accuracy desired. To some extent, it will also be determined by the shape of the finger and how the finger deforms. In the algorithm presented here, there is no prescribed method to predict the size of the patch. This parameter will have to be determined by *a priori* information and the level of accuracy desired. The results of a particular computation will always reveal if the assumed contact patch was large enough since its boundary would be a locus of zero normal forces.

2.3 Assumptions

To carry out the analysis for the above configuration it is necessary to adopt a model that incorporates a force/deformation analysis for the fingers and the object. The model developed in the following sections incorporates the following simplifying assumptions :

- The fingers and the object obey the laws of linear elasticity.
- Surfaces are smooth on the microscale and have continuous first derivatives. The measure of microscale depends on the size of the contact patch and how finely we choose to discretize the contact area. This assumption is permits a varying measure for what we mean by microscale.
- The deformations are small and the pressure is distributed over an area which is small compared to the dimensions of the object. This assumption implies that the deformation at a particular grid element is a result of only the force acting on that element and the forces acting on the other grid elements of that particular contact patch. To be more specific, the local deformations at a particular fingertip are not affected by the forces at another fingertip.
- The forces acting on the object act through certain locations that can be well defined in the undeformed state. Typically, the force transmitted at a point of contact has the effect of compressing deformable solids so that the bodies make contact over an area of finite size. In this analysis, the moments of the forces are calculated such that the locations of the forces are exactly as they *would be* in the undeformed state. Once the bodies start deforming, it is assumed, however, that the deformations are so small with respect to the dimensions of the object and the finger, that the locations of these forces do not change with respect to the chosen reference point, **O** in Figure 3, and resultant changes in the moments are not taken into consideration.
- The analysis is static. There is no consideration of dynamic terms and no explicit treatment of the slipping motion. One needs to distinguish between the gross relative sliding motion between the object and the finger and the microscale slip that is considered in this analysis. However, the model can predict when a finger will start to slip along the outer surface of the object.
- The analysis does not attempt to solve for the optimum grasp for a given task but provides a mechanism for evaluating the forces at the finger/object interface given a particular set of contact locations.
- The analysis is not concerned with geometric constraints, such as whether the manipulator is actually able to achieve a given grasp, or whether it is possible to place fingers underneath an object lying on a flat surface. It is assumed that the grasp under consideration already satisfies such criteria.

- In the development of the model, only the interaction between the fingertips and the object is considered. The compliance of the finger itself, the hand or the arm is not considered. To develop the overall characteristics, the compliance of the hand and the arm could be added to this analysis by incorporating effective linear elastic properties that reflect hand and arm compliance.

2.4 Contact Problem Under a Normal Force

In this section we will analyze the contact problem under the influence of normal forces only [CS71]. The superscript i for the i^{th} finger has been omitted in this section, but it is understood that this analysis has to be done for each finger.

2.4.1 Condition of Equilibrium

The sum of all forces F_k acting at the discrete points ($k = 1, \dots, N$ where N is the number of candidate points for contact) must balance the applied force (\mathbf{P}) normal to the surface. The equilibrium condition can therefore be written as :

$$\sum_{k=1}^N F_k = |\mathbf{P}| \quad (13)$$

2.4.2 The Concept of the Rigid Body Approach

Before we proceed any further it is important that the concept of the *rigid body approach* be made clear since it plays an important role in the formulation of the contact problem under both the normal and the tangential forces.

Figure 6 shows two elastic bodies (the finger and the object) being pressed together by a force \mathbf{P} , the line of action of which is perpendicular to the common tangent plane π and passes through the contact point (given here by the intersections of Z_1 , and Z_2 with the tangent plane π). The bodies deform under the action of the force \mathbf{P} in the region adjacent to the point of contact and move closer to each other. Let $-\partial_1$ and $-\partial_2$ denote the projections of the translatory displacements of the first and second bodies along the Z_1 , and Z_2 axes, respectively (which as can be seen, are directed into the respective bodies). One can also define ∂_1 and ∂_2 as the displacements of the points of the first and second bodies, *sufficiently far away* from the contact area, in the direction of the force \mathbf{P} . The sum $\alpha = \partial_1 + \partial_2$ will be called the *rigid body movement or approach*. A similar analysis can be done for the contact under the influence of the tangential forces.

To make the point clearer, let us focus on some body coordinate reference frames for the finger (T_1) and the object (T_2) whose origins are located far away from the proposed area of contact. Initially, the finger and the object are at a distance from each other. If the finger now approaches the object, its body coordinate system will move by a finite distance, say d , before the fingertip comes into contact with the object surface. The finger tip and the object surface now start deforming under the effect of the forces (\mathbf{P}) pressing the finger

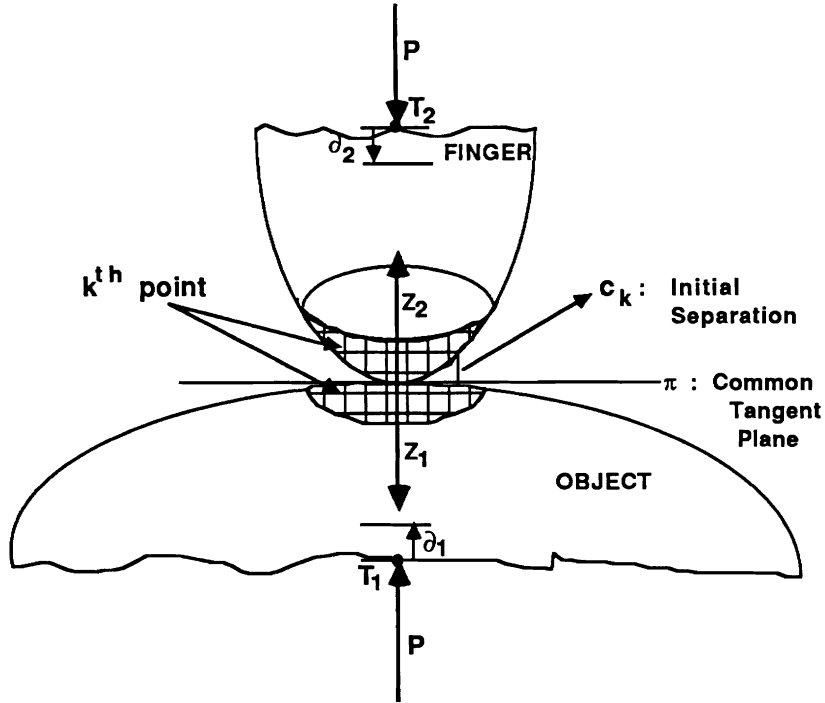


Figure 6: An Undeformed Finger/Object Contact

and the object together. While a small region around the area of contact deforms, the rest of the finger and the object remains undeformed. Consequently, as the contacting surfaces deform, the undeformed regions move towards each other (with translatory displacements ∂_1 and ∂_2 , respectively) to maintain the compatibility for contact. This relative approach of the two body coordinate systems ($\alpha = d + \partial_1 + \partial_2$) is what we once again choose to call *rigid body movement or approach*.

In this analysis, d is always taken to be zero since we assume the finger to be just touching the object in the undeformed state. α^i will represent the rigid body movement in the normal direction at the i^{th} fingertip and β^i will represent the rigid body movement in the tangential direction at the i^{th} fingertip. Alternatively, we can view α^i as the projection of the resultant rigid body movement along the surface normal vector n^i and β^i as the projection of the resultant rigid body movement on the tangent plane π .

2.4.3 Condition of Compatibility of Deformation

Before deformation, the separation between two corresponding surface points (shown as the k_{th} point in figure 6) is given by c_k . The c_k 's define the shape of the contacting surfaces. During the compression, distant points in the two bodies, T_1 and T_2 , move towards the contact point (given here by the intersections of Z_1 and Z_2 with the tangent plane π), parallel to the axes Z_1 and Z_2 , by displacements ∂_1 and ∂_2 , respectively. The resultant

displacement is given by α . Due to the contact pressure, the point k on the surface of the finger is displaced parallel to Z_2 by an amount $w_{k(2)}$ (measured positive into the finger) relative to the distant point T_2 . Similarly, the point k on the surface of the object is displaced parallel to Z_1 by an amount $w_{k(1)}$ (measured positive into the object) relative to the distant point T_1 . If, after deformation, the k_{th} points on the two surfaces are coincident within the contact surface then we obtain a compatibility constraint :

$$w_{k(1)} + w_{k(2)} + c_k - \alpha = 0$$

If k_{th} points on the two surfaces are outside the contact area so that they do not touch, it follows that

$$w_{k(1)} + w_{k(2)} + c_k - \alpha > 0$$

Therefore, at any point k in the proposed region of contact, the sum of the elastic deformations and any initial separations must be greater than or equal to the rigid-body approach. This condition is represented as :

$$w_{k(1)} + w_{k(2)} + c_k - \alpha \geq 0 \quad (14)$$

where,

- c_k is the initial separation at point k ,
- $w_{k(1)}, w_{k(2)}$ are elastic deformations at point k , in the direction of the normal force at point k in the object and finger, respectively,
- and α is the rigid body movement in the direction of the normal force, as described in the previous section.

2.4.4 Finger Shape Definition

From the previous section it is clear that the shape of the finger is an important feature of the problem. The c_k 's need to be determined for all the grid elements of the contact patch in order that the condition for compatibility and the criterion for contact can be evaluated. Since we start out by assuming an initial point of contact (located by \mathbf{v}^i), the value of c_k at that point will be zero. If the finger tip conforms with the object surface then all the c_k 's will be zero. Otherwise the variable gap between the surface and object is defined discretely by c_k .

2.4.5 Criterion for Contact

At any point k , the left hand side of the inequality constraint (14) may be strictly positive or identically zero. Defining a new variable Y_k , the inequality (14) can be rewritten :

$$w_{k(1)} + w_{k(2)} + c_k - \alpha - Y_k = 0 \quad (15)$$

where, Y_k , a "slack" variable, satisfies:

$$Y_k \geq 0.$$

When the surfaces touch, $Y_k = 0$ and there is a finite normal force being transmitted across the contact. The criterion for contact is, therefore :

$$\begin{aligned} \text{If } Y_k = 0, \quad \text{then } F_k &\geq 0 \\ \text{If } Y_k > 0, \quad \text{then } F_k &= 0 \end{aligned} \tag{16}$$

and the solution for the discrete contact problem is the set of forces $F_k (k = 1, \dots, N)$ which satisfies equations (14)-(16).

2.4.6 General Model for Elastic Deformation

The continuous pressure distribution is approximated by a set of forces, acting at discrete points. Since both bodies obey the laws of linear elasticity, the elastic deformation in the normal direction at a point k is a linear summation of the influence of all the forces F_j and G_j (discretized tangential force (\mathbf{Q})) acting on the interface. Accordingly,

$$w_k = \sum_{j=1}^N (a_{kj}^n F_j + a_{kj}^t G_j) \tag{17}$$

where a_{kj}^n is the normal deformation at point k due to a unit normal force (signified by the superscript n) at point j and a_{kj}^t is the normal deformation at point k due to a unit tangential force (signified by the superscript t) at point j .

The Contact Problem under a Normal Force may now be formally stated as follows :

Find a solution (F, α, Y) which satisfies the following constraints :

$$S^n F + S^t G - \alpha e - IY = -c \tag{18}$$

$$e^T F = |\mathbf{P}| \tag{19}$$

Either $F_k = 0$ and $G_k = 0$, or $Y_k = 0$

$$F_k \geq 0, \quad Y_k \geq 0, \quad \alpha \geq 0$$

where,

$$S_{kj}^n = a_{kj(1)}^n + a_{kj(2)}^n$$

$$S_{kj}^t = a_{kj(1)}^t + a_{kj(2)}^t$$

and $a_{kj(1)}, a_{kj(2)}$ are the influence coefficients for the deformation of the object and the finger in the normal direction, respectively. Also,

$S^n = N \times N$ matrix of influence coefficients for normal forces

$S^t = N \times N$ matrix of influence coefficients for tangential forces

$F = N \times 1$ vector of normal forces

$G = N \times 1$ vector of tangential forces
 $Y = N \times 1$ vector of slack variables
 $e = N \times 1$ vector of ones
 $c = N \times 1$ vector of initial separations
 $I =$ the $N \times N$ Identity matrix
 $\alpha =$ rigid-body approach in the normal direction
 $P =$ applied normal force

2.4.7 Potential Energy due to a Normal Force

For N discrete forces, the strain energy for the two bodies can be written as :

$$SE = \frac{1}{2} \sum_{j=1}^N F_j (w_{j(1)} + w_{j(2)}) \quad (20)$$

Substituting (17) into (20), gives

$$\begin{aligned}
 SE &= \frac{1}{2} \sum_{k=1}^N F_k \left[\sum_{j=1}^N (a_{kj(1)}^n + a_{kj(2)}^n) F_j + \sum_{j=1}^N (a_{kj(1)}^t + a_{kj(2)}^t) G_j \right] \\
 &= \frac{1}{2} \sum_{k=1}^N \sum_{j=1}^N (S_{kj}^n F_k F_j + S_{kj}^t F_k G_j)
 \end{aligned}$$

or in matrix notation :

$$SE = \frac{1}{2} (F^T S^n F + F^T S^t G) \quad (21)$$

where, the superscript T implies the transpose of the vector.

The loss in potential energy of the forces acting on both bodies is :

$$\text{loss in PE} = - \sum_{j=1}^N F_j (w_{j(1)} + w_{j(2)}) \quad (22)$$

The potential energy for the system of the two bodies in contact is therefore :

$$PE = - \frac{1}{2} \sum_{j=1}^N F_j (w_{j(1)} + w_{j(2)}) \quad (23)$$

This can also be proved independently from the classical concepts in elasticity. The theorem concerning the potential energy of deformation [Lov44] states that:

The potential energy of deformation of a body, which is in equilibrium under given load, is equal to half the work done by the external forces, acting through the displacements from the unstressed state to the state of equilibrium.

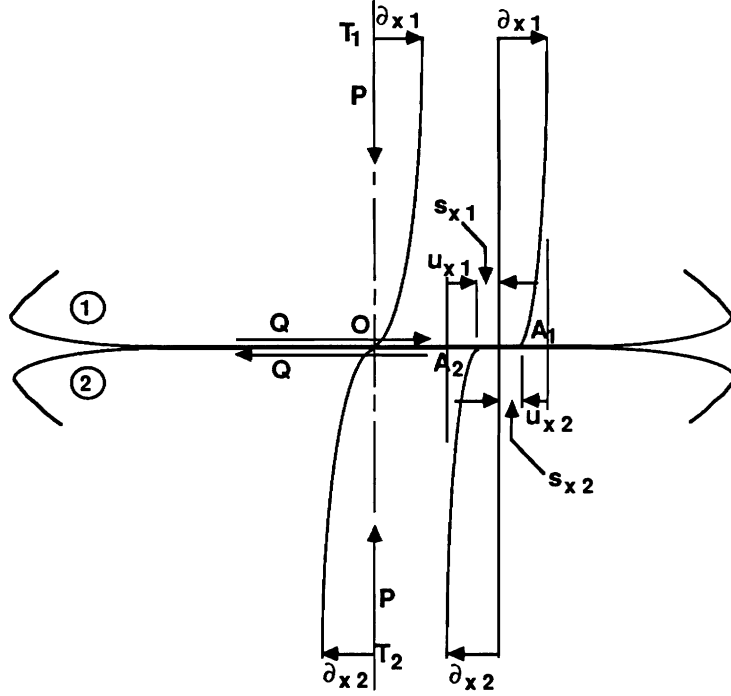


Figure 7: A Deformed Contact Region

Substituting (14)-(16) into (23) we obtain :

$$\begin{aligned}
 PE &= -\frac{1}{2} \sum_{j=1}^N F_j (\alpha - c_j) \\
 &= -\frac{1}{2} |P| \alpha + \frac{1}{2} \sum_{j=1}^N F_j c_j
 \end{aligned}
 \tag{24}$$

Equation (21) shows the strain energy represented by a quadratic form. Since strain energy is always positive for all forces and is zero only when all the forces are zero, the quadratic form is positive definite. Thus the matrices S^t and S^n are positive definite.

2.5 Contact Problem Under a Tangential Force

A tangential force whose magnitude is less than the force of limiting friction, when applied to two bodies pressed into contact, will not give rise to a sliding motion but, nevertheless, will induce frictional tractions at the contact interface. In this section we shall examine the tangential surface tractions which arise from a combination of normal and tangential forces which do not cause the bodies to slide relative to each other.

The problem is illustrated in Figure 7 [Joh85]. The normal force (\mathbf{P}) gives rise to a contact area and pressure distribution which we assume to be given by the theory developed in the previous section. The effect of the tangential force (\mathbf{Q}) is to cause the bodies to deform in shear, as indicated by the distorted center line in Figure 7. Points on the contact surface will undergo tangential displacements u_x and u_y relative to distant points T_1 and T_2 in the undeformed region of each body. Clearly, if there is no sliding motion between the two bodies as a whole, there must be at least one point at the interface where the surfaces deform without relative motion; but it does not follow that there is no slip anywhere within the contact area. In fact, it will be shown that the effect of a tangential force equal to the limiting friction force ($|\mathbf{Q}| = \mu(\mathbf{P})|\mathbf{P}|$ where $\mu(\mathbf{P})$ is the effective(nonlinear) coefficient of friction) is to cause a small relative motion, referred to as “slip” or “microslip”, over part of the interface. The remainder of the interface deforms without relative motion and in such regions the surfaces are said to adhere or there is “no slip”.

To proceed with an analysis we must consider the conditions governing “no slip” and “slip”. In Figure 7, A_1 and A_2 denote two points on the interface which are coincident before the application of the tangential force. Under the action of the force, points in the body such as T_1 and T_2 , distant from the interface, move through effectively rigid displacements (as described before) $\partial_{x1}, \partial_{y1}$ and $\partial_{x2}, \partial_{y2}$ (for simplicity the y dimension is not shown in the figure) while A_1 and A_2 experience tangential elastic displacements u_{x1}, u_{y1} and u_{x2}, u_{y2} relative to T_1 and T_2 . If the absolute displacements of A_1 and A_2 may be written,

$$\begin{aligned} s_x &= s_{x1} - s_{x2} \\ &= (u_{x1} - \partial_{x1}) - (u_{x2} - \partial_{x2}) \\ &= (u_{x1} - u_{x2}) - (\partial_{x1} - \partial_{x2}) \end{aligned}$$

A similar relation governs the tangential displacements in the y direction. If the points A_1 and A_2 are located in a “no slip” region the slip s_x and s_y will be zero so that

$$\begin{aligned} u_{x1} - u_{x2} &= (\partial_{x1} - \partial_{x2}) = \partial_x \\ u_{y1} - u_{y2} &= (\partial_{y1} - \partial_{y2}) = \partial_y \end{aligned}$$

We note that the right hand sides of the above equations denote relative tangential displacements between two bodies as a whole under the action of the tangential force. Thus, ∂_x and ∂_y are constant, independent of the position of A_1 and A_2 within the “no slip” region. Further, if the two bodies have the same elastic moduli, since they are subjected to mutually equal and opposite surface tractions, we can say that $u_{x2} = -u_{x1}$ and $u_{y2} = -u_{y1}$. The condition of no slip can then be stated as : *all surface points within a “no slip” region undergo the same tangential displacement.* The statement is also true when the elastic constants are different, but the overall relative displacements ∂_x and ∂_y are then divided unequally between the two bodies.

At points within the “no slip” region the resultant tangential forces cannot exceed their limiting values. This restriction can be stated as :

$$|\mathbf{Q}| \leq \mu(\mathbf{P})|\mathbf{P}|$$

where, $\mu(\mathbf{P})$ is the effective(nonlinear) coefficient of friction. In regions where there is slip between the surfaces, the conditions of the compatibility equations stated earlier are violated, but the tangential and normal forces are related by :

$$|\mathbf{Q}| = \mu(\mathbf{P})|\mathbf{P}|$$

We will use these results to formulate the tangential deformations and forces in our grasping problem. For the purposes of keeping the analysis uncomplicated, the subsequent formulation was done assuming that the x axis points in the direction of the tangential force (\mathbf{Q}). However, this does not make the formulation any less general and in fact, we will need to employ the results in two dimensions when we start to solve the problem.

2.5.1 Condition of Equilibrium

In this case, the discrete forces G_k represent the discretized shear or tangential forces over the individual grid elements [Cho86]. The sum of all forces G_k acting at the discrete points ($k = 1, \dots, N$ where N is the number of candidate points for contact) must balance the tangential force (\mathbf{Q}) due to the normal force (\mathbf{P}). The equilibrium condition can therefore be written as :

$$\sum_{k=1}^N G_k = |\mathbf{Q}| \quad (25)$$

It is also important to remember that the force G_k really acts in two dimensions (in the plane π given in Figure 6). If we define a two-dimensional coordinate system on the plane of contact with local axes x and y then G_k will have the components G_{kx} and G_{ky} and G_k will be given by :

$$G_k = \sqrt{(G_{kx})^2 + (G_{ky})^2}$$

2.5.2 Condition of Compatibility of Deformation

At any point k in the proposed region of contact, the sum of the elastic deformations must be less than or equal to the rigid-body approach. This condition is represented as :

$$\begin{aligned} u_{k(1)} + u_{k(2)} - \beta &= 0 && \text{for no slip} \\ u_{k(1)} + u_{k(2)} - \beta &< 0 && \text{for slip} \end{aligned} \quad (26)$$

where,

$u_{k(1)}, u_{k(2)}$ are discretized elastic deformations at point k in the direction of the tangential force at point k in the object and finger, respectively, and β is the relative rigid body movement in the direction of the tangential force, as explained in the earlier sections (analogous to $\partial_x = (\partial_{x1} - \partial_{x2})$ in Figure 7).

2.5.3 Constraints on the Tangential Force

The tangential forces are related to the normal forces and cannot exceed a certain threshold. We can state this constraint in a familiar manner :

$$\begin{aligned} G_k &< \mu(F_k)F_k && \text{for no slip} \\ G_k &= \mu(F_k)F_k && \text{for slip} \end{aligned} \tag{27}$$

where,

$\mu(F_k)$ is the effective(nonlinear) coefficient of friction.

2.5.4 General Model for Elastic Deformation

The continuous tangential force distribution is approximated by a set of forces, acting at discrete points. Since both bodies obey the laws of linear elasticity, the elastic deformation in the tangential direction at a point k is a linear summation of the influence of all the forces F_j and G_j acting on the interface. Accordingly,

$$u_k = \sum_{j=1}^N (b_{kj}^t G_j + b_{kj}^n F_j) \tag{28}$$

where b_{kj}^n is the tangential deformation at point k due to the unit normal force at point j and where b_{kj}^t is the tangential deformation at point k due to the unit tangential force at point j .

Introducing a set of non-negative slack variables Z_{1k} , equation (26) can be rewritten as follows :

$$u_{k(1)} + u_{k(2)} + Z_{1k} - \beta = 0 \tag{29}$$

where,

$$Z_{1k} = 0 \quad \text{in the no slip region}$$

and,

$$Z_{1k} > 0 \quad \text{in the slip region.}$$

Introducing a set of non-negative slack variables Z_{2k} , equation (27) can be rewritten as follows :

$$G_k + Z_{2k} = \mu(F_k)F_k \tag{30}$$

where,

$$Z_{2k} > 0 \quad \text{in the no slip region}$$

and,

$$Z_{2k} = 0 \quad \text{in the slip region.}$$

Since point k must be either in a slip or no slip region, we can say :

$$\text{either } Z_{1k} = 0 \quad \text{or} \quad Z_{2k} = 0 \quad (31)$$

The Contact Problem under a Tangential Force may now be formally stated as follows :

Find a solution (G, β, Z_1, Z_2) which satisfies the following constraints :

$$T^t G + T^n F - \beta e + I Z_1 = 0 \quad (32)$$

$$G + Z_2 = \mu(F) F \quad (33)$$

$$e^T G = |\mathbf{Q}| \quad (34)$$

Either $Z_{1k} = 0$, or $Z_{2k} = 0$ and,

$$F \geq 0, \quad Z_{1k} \geq 0, \quad Z_{2k} \geq 0, \quad \beta \geq 0$$

where,

$$T_{kj}^t = b_{kj(1)}^t + b_{kj(2)}^t$$

$$T_{kj}^n = b_{kj(1)}^n + b_{kj(2)}^n$$

and $b_{kj(1)}, b_{kj(2)}$ are the influence coefficients for the deformation of the object and the finger, respectively. Also,

$T^t = N \times N$ matrix of influence coefficients for tangential forces

$T^n = N \times N$ matrix of influence coefficients for normal forces

$G = N \times 1$ vector of forces

$F = N \times 1$ vector of forces

$Z_1 = N \times 1$ vector of slack variables

$Z_2 = N \times 1$ vector of slack variables

$e = N \times 1$ vector of ones

$I =$ the $N \times N$ Identity matrix

$\beta =$ relative rigid-body approach in the tangential direction

$\mathbf{Q} =$ tangential force associated with the normal force \mathbf{P}

2.5.5 Potential Energy due to a Tangential Force

For N discrete forces, the strain energy for the two bodies can be written as :

$$SE = \frac{1}{2} \sum_{j=1}^N G_j (u_{j(1)} + u_{j(2)}) \quad (35)$$

Substituting (28) into (35), gives

$$\begin{aligned} \text{SE} &= \frac{1}{2} \sum_{k=1}^N G_k \left[\sum_{j=1}^N (b_{kj(1)}^t + b_{kj(2)}^t) G_j + \sum_{j=1}^N (b_{kj(1)}^n + b_{kj(2)}^n) F_j \right] \\ &= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N (T_{kj}^t G_k G_j + T_{kj}^n G_k F_j) \end{aligned}$$

or in matrix notation :

$$\text{SE} = \frac{1}{2} (G^T T^t G + G^T T^n F) \quad (36)$$

The loss in potential energy of the forces acting on both bodies is :

$$\text{loss in PE} = - \sum_{j=1}^N G_j (u_{j(1)} + u_{j(2)}) \quad (37)$$

The potential energy for the system of the two bodies in contact is therefore :

$$\text{PE} = - \frac{1}{2} \sum_{j=1}^N G_j (u_{j(1)} + u_{j(2)}) \quad (38)$$

Substituting (26) and (29) into (38) we obtain :

$$\begin{aligned} \text{PE} &= - \frac{1}{2} \sum_{j=1}^N G_j (\beta - Z_{1j}) \\ &= - \frac{1}{2} |\mathbf{Q}| \beta + \frac{1}{2} \sum_{j=1}^N G_j Z_{1j} \end{aligned} \quad (39)$$

Equation (33) shows once again that the matrices T^t and T^n are positive definite.

2.6 Micromechanics of Friction

A very concise treatment of the micromechanics of friction is presented by Oden and Pires in their paper on nonlocal and nonlinear friction laws [OP83]. Historically, it was in 1781 that the French engineer C.A. Coulomb published his "Théorie des Machines Simples" in which he presented his celebrated law of friction. This work earned him a double prize from the Royal Academy of Sciences in 1785. The classical Coulomb law of static dry friction, of course, asserts that

relative sliding between two bodies in contact along plane surfaces will occur when the net shear force parallel to the plane reaches a critical value proportional to the net normal force pressing the two bodies together. The constant of proportionality is called the coefficient of friction.

Oden and Pires argue that as a basis for contact problems in the theory of elasticity, Coulomb's law is not acceptable from either a physical or mathematical point of view. Physically, it can be said that Coulomb's law is capable of describing only the friction effects between effectively rigid bodies and the gross sliding of one body relative to another. Indeed, it seems that Coulomb himself never intended that his law be applied pointwise in boundary-value problems in elasticity; the foundations of continuum mechanics were only fully developed many decades after Coulomb proposed his law, and the first successful formulation of a contact problem in elasticity came over a full century after Coulomb's work.

The paper emphasizes the point that there are several aspects of actual friction phenomena between metallic bodies that suggest alternative friction laws which represent a marked departure from the classical formulations. The first is the obvious nonlocal character of the mechanism by which normal forces are distributed on contact surfaces. These stresses are transmitted over junctions formed by deformed asperities and are not concentrated at isolated points on the contact surface. Secondly, on application of loads, experiments show that there always exists a small tangential displacement of points on the contact surfaces due to the elastic and elastoplastic deformation of these junctions; sliding occurs when these junctions are actually fractured. Since these junctions can be recovered upon a quasi-static reversal of loads, the actual "adhesion-sliding" friction mechanism is highly non-linear and depends on the properties of the contact surfaces. *Therefore, to be able to represent the surface interaction, what we require is a nonlinear, nonlocal friction law* [OP83].

From the analysis of the Sections on normal and tangential forces presented above, we can assert that our model does incorporate a nonlocal friction law. Basically, a nonlocal friction law proposes that *impending motion at a point of contact between two deformable bodies will occur when the shear stress at that point reaches a value proportional to a weighted measure of the normal stresses in a neighborhood of the point*. This in fact holds for the analysis developed above. In Section 2.4 on normal forces, we first determine the character of the effective neighborhood when a finger exerts a normal force. And we saw from Section 2.5 that the manner in which the neighborhood stresses contribute to the slipping condition depends on the influence coefficients which in turn depend on the material properties of the materials, as we shall see in the next section.

To incorporate the nonlinear behavior, the effects of the tangential elastic-plastic deformations of the contact junctions mentioned earlier, need to be incorporated. Once again, we have accommodated such effects which allow for small but nonzero, elastic tangential displacements at the contact surface for tangential forces below a certain critical level. For shear forces at or near this critical level, substantially larger motions can occur which effectively represent large tangential motions such as sliding. This critical value may be proportional to a weighted measure of the normal forces in a neighborhood of the point on the contact surface.

To be consistent with the nonlocal and nonlinear nature of the analysis that has been developed, it is also important to develop relationships between the normal and tangential forces such that we can accurately calculate $\mu(F_k)$ that we have left undetermined so far.

2.6.1 Empirical Determination of μ

The most reliable method to obtain μ would be to actually measure the variation of the maximum attainable friction forces with respect to varying normal loads. From this an analytical representation for μ could be found and incorporated into the model. This would save the trouble of representation that other models mentioned below present and would also allow for the measurement of μ for a widely varying choice of materials.

2.6.2 A Nonlocal and Nonlinear Law

Oden and Pires [OP83] presented in 1983, a model that results in a nonconventional friction law which is given in terms of three positive material parameters: μ , ρ and ϵ . The parameter μ is the coefficient of friction, although its actual interpretation is somewhat more complex than that made in classical mechanics. The parameter ρ quantifies the nonlocal character of the response; for $\rho = 0$ a fully local law is obtained. Finally, ϵ is a measure of the tangential stiffness of the elastic-plastic junctions on the contact surface; the case $\epsilon = 0$ corresponds to a fully rigid response - full adhesion or full sliding of contact surfaces. Thus, by allowing ρ and ϵ to approach zero, we can recover the classical, local, pointwise formulation of contact problems based on Coulomb's law.

2.6.3 An Elastic Theory of Coulomb Friction

In 1973, Piero Villagio [Vil79] analyzed the contact problem in plane elasticity and presented a result where the dependence of the friction force on the normal force is nonlinear, so that Coulomb's law can be accepted only as a first approximation to describe friction. Another unexpected property of the elastic theory is that the nonconstant ratio between friction force and the normal force is not smaller than unity and tends to infinity as the normal load increases. The friction coefficient is a function of the bulk moduli, the Poisson's ratios of the two materials in contact and the principal curvatures of the surfaces. In this model the coefficient of friction is given by :

$$\begin{aligned}\mu(\mathbf{P}) &= \frac{8.42 a(\mathbf{P}) K_2}{\pi^3 R_1 K} & (40) \\ \text{where, } a(\mathbf{P}) &= \sqrt{\frac{2|\mathbf{P}|}{\pi K}} \\ K &= \frac{R_1^{-1} - R_2^{-1}}{(1 - \nu_1)E_1^{-1} + (1 - \nu_2)E_2^{-1}} \\ K_2 &= \frac{R_2^{-1}}{2(1 - \nu_2)G_2^{-1}}\end{aligned}$$

where,

ν_1, ν_2 are the Poisson ratios of the finger and object, respectively,
 E_1, E_2 are the elastic moduli of the finger and object, respectively,

R_1, R_2 are the radii of curvature at the contact of the finger and object, respectively ($R_1 < R_2$).

In general, $\mu(\mathbf{P})$ is a monotonically increasing function on \mathbf{P} . Values of μ smaller than 1, customarily assumed in using Coulomb's law, occur only for relatively small normal loads.

2.6.4 Determination of μ in our Examples

Most of the above models make it very difficult to calculate μ unless the surface geometry of the bodies in contact is known or some assumptions are made about them. In our examples, we have chosen a constant value of μ . This, however, does not mean that we are using the Coulomb friction model. As explained in Section 2.6, the analysis takes into account both the nonlocal and nonlinear nature of the tangential forces. There is a threshold that is proportional to the applied normal force which depends on the material properties of the bodies in contact. Most importantly, there is now a strain energy of deformation associated with the tangential (friction) forces which is a marked contrast to rigid body Coulomb models. What we have here is a "quasi-coulombic" or an "elastic slip" model that is significantly different and phenomenologically more accurate than the rigid body Coulomb model.

2.7 Models for Influence Matrix

The model allows the use of any valid set of influence coefficients. It could be chosen from experimental measurements or by using finite element analysis. We have used the results for deformation in an elastic half-space.

2.7.1 The Elastic Half-Space

Nonconforming elastic bodies in contact, whose deformation is sufficiently small for the linear strain theory of elasticity to be applicable, inevitably make contact over an area whose dimensions are small compared with the radii of curvature of the undeformed surfaces. The contact stresses are highly concentrated close to the contact region and decrease rapidly in intensity with distance from the point of contact, so that the region of practical interest lies close to the contact interface. Thus, provided the dimensions of the bodies themselves are large compared with the dimensions of the contact area, the stresses in this region are not critically dependent upon the shape of the bodies distant from the contact area, nor upon the precise way in which they are supported. The stresses may be calculated to a good approximation by considering each body as a semi-infinite elastic solid bounded by a plane surface: i.e. an elastic half space. This idealization, in which bodies of arbitrary surface profile are regarded as semi-infinite in extent and having a plane surface, is made almost universally in elasticity theory. It simplifies the boundary conditions and makes available the large body of theory which has been developed for the elastic half-space [Joh85].

Motivated by the discussion above, it was chosen to model the finger and the object as an elastic half-space, and the forces and deformations were related using results for point

loading of an elastic half space. These results for a concentrated force can be superposed to find the deformations produced by normal and tangential forces distributed over an area of the surface, as in our case.

In our problem, the deformations are produced in an elastic half space, bounded by the plane surface $z = 0$ (plane π in Figure 6), under the action of normal and tangential forces applied to a closed area S of the surface in the neighborhood of the contact point. The loading at each fingertip is two dimensional: the normal force given by \mathbf{P} and the tangential force \mathbf{Q} , which lies in the contact plane.

The classical approach to finding the stresses and displacements in an elastic half-space due to surface tractions is due to Boussinesq (1885) and Cerruti (1882) who made use of the theory of potential. This approach is presented by [Lov44, Joh85]: only relevant results will be presented here. To interpret these results physically, a Cartesian reference frame with its origin at the contact point needs to be defined. The z axis is normal to the surface, and points into the solid, parallel to the normal force (\mathbf{P}). The x axis is oriented such that it points in the direction of the tangential force (\mathbf{Q}) and the y axis lies in the contact plane, perpendicular to both the x and the z axes, completing a right handed system.

2.7.2 Deformations due to Concentrated Normal Force

The deformation u_{kj}^{xp} at point k in the x direction, due to a unit normal force at point j , is given by :

$$u_{kj}^{xp} = -\frac{(1-2\nu)(1+\nu)}{2\pi E} \frac{x_{kj}}{r_{kj}^2}$$

The deformation u_{kj}^{yp} at point k in the y direction, due to a unit normal force at point j , is given by :

$$u_{kj}^{yp} = -\frac{(1-2\nu)(1+\nu)}{2\pi E} \frac{y_{kj}}{r_{kj}^2}$$

The deformation w_{kj}^p at point k in the normal direction, due to a unit normal force at point j , is given by :

$$w_{kj}^p = \frac{(1-\nu)(1+\nu)}{\pi E} \frac{1}{r_{kj}}$$

where r_{kj} is the distance of the point k from the point j , and x_{kj} and y_{kj} are the projections of r_{kj} along the x and y axes, respectively. E is the Young's modulus and ν is the Poisson's ratio for the elastic solid.

2.7.3 Deformations due to Concentrated Tangential Force

The deformation u_{kj}^{xq} at point k in the x direction, due to a unit tangential force at point j , is given by :

$$u_{kj}^{xq} = \frac{1+\nu}{\pi E r_{kj}} \left[1 - \nu - \frac{\nu x_{kj}^2}{r_{kj}^2} \right]$$

The deformation u_{kj}^{yq} at point k in the y direction, due to a unit tangential force at point j , is given by :

$$u_{kj}^{yq} = \frac{1 + \nu}{2\pi E} \left[\frac{2\nu(x_{kj})(y_{kj})}{r_{kj}^2} \right]$$

The deformation w_{kj}^q at point k in the normal direction, due to a unit tangential force at point j , is given by :

$$w_{kj}^q = \frac{(1 - 2\nu)(1 + \nu)}{2\pi E} \frac{x_{kj}}{r_{kj}^2}$$

where r_{kj} is the distance of the point k from the point j and x_{kj} and y_{kj} are the projections of r_{kj} along the x and y axes, respectively.

2.7.4 Calculation of the Influence Coefficients

Now we have the normal force F_j and the tangential force G_j acting at each point j . For the deformation in the normal direction w_{kj} , we obtain the expression :

$$\begin{aligned} w_{kj} &= w_{kj}^p + w_{kj}^q \\ &= \frac{(1 + \nu)}{2\pi E r_{kj}^2} [2F_j(1 - \nu)r_{kj} + Q_j(1 - 2\nu)x_{kj}] \end{aligned}$$

For the deformation in the x direction u_{kj}^x , we obtain the expression :

$$\begin{aligned} u_{kj}^x &= u_{kj}^{xq} + u_{kj}^{xp} \\ &= \frac{(1 + \nu)}{2\pi E r_{kj}^2} \left[2G_j \left(1 - \nu - \frac{\nu x_{kj}^2}{r_{kj}^2} \right) r_{kj} - F_j(1 - 2\nu)x_{kj} \right] \end{aligned}$$

For the deformation in the y direction u_{kj}^y , we obtain the expression :

$$\begin{aligned} u_{kj}^y &= u_{kj}^{yq} + u_{kj}^{yp} \\ &= \frac{(1 + \nu)}{2\pi E r_{kj}^2} [G_j 2\nu(x_{kj})(y_{kj}) - F_j(1 - 2\nu)x_{kj}] \end{aligned}$$

For our analysis, we are interested in only the first two equations, since we are not considering stresses or deformations in the y direction on the contact plane. This is the direction perpendicular to the direction in which the tangential force is acting. However, the expression for u_{kj}^y is useful nevertheless, as we will see in the next chapter when we attempt to solve the problem.

Therefore,

$$a_{kj}^n = \frac{(1 - \nu^2)}{\pi E} \frac{1}{r_{kj}} \quad k \neq j \quad (41)$$

$$a_{kk}^n = \frac{0.95(1 - \nu^2)}{E\sqrt{A_k}}$$

$$a_{kj}^t = \frac{(1 - 2\nu)(1 + \nu) x_{kj}}{2\pi E r_{kj}^2} \quad (42)$$

$$b_{kj}^t = \frac{1 + \nu}{\pi E r_{kj}} \left[1 - \nu - \frac{\nu x_{kj}^2}{r_{kj}^2} \right] \quad k \neq j \quad (43)$$

$$b_{kk}^t = \frac{1}{\sqrt{\pi A_k}} \frac{(1 + \nu)(2 - \nu)}{E}$$

$$b_{kj}^n = -\frac{(1 - 2\nu)(1 + \nu) x_{kj}}{2\pi E r_{kj}^2} \quad (44)$$

where,

A_k is the area of the k_{th} grid element.

Looking at Equations (41) and (43) it would seem that the expressions for a_{kk}^n and b_{kk}^t would turn out to be singular. However, the expressions for a_{kk}^n and b_{kk}^t can be derived by considering F_k and G_k to represent the distributed normal and tangential pressure, respectively, over the the grid element k [Lov44, TG70]. Indeed, this is how they are defined in Section 2.2.

2.8 The Requirements for Static Equilibrium

Now that we have developed a model for the description of the surface interaction and the forces at each contact between the finger and object, we need to integrate the model with the requirements for static equilibrium. Let us assume that our object is grasped by M fingers. Refer to Figure 3 for explanation of labels. The normal force that each finger exerts on the object is then given by the vector \mathbf{P}^i ($i = 1, \dots, M$) and the tangential force as a result of \mathbf{P}^i will be \mathbf{Q}^i ($i = 1, \dots, M$).

2.8.1 Force Equilibrium

We require that the sum of forces in the \mathbf{x} , \mathbf{y} and \mathbf{z} directions be equal to zero. If we denote all the external forces (other than gravity) by \mathbf{R}^i ($i = 1, \dots, K$) and the weight of the object as W we obtain the following :

$$\sum_{i=1}^M \sum_{k=1}^N (F_k^i + G_k^i)_x + \sum_{i=1}^K R_x^i = 0 \quad (45)$$

$$\sum_{i=1}^M \sum_{k=1}^N (F_k^i + G_k^i)_y + \sum_{i=1}^K R_y^i = 0 \quad (46)$$

$$\sum_{i=1}^M \sum_{k=1}^N (F_k^i + G_k^i)_z + \sum_{i=1}^K R_z^i - W = 0 \quad (47)$$

for each direction x , y and z , respectively, assuming that the object weight W acts in the negative z direction. We could also express this requirement more compactly in the vector notation as :

$$\sum_{i=1}^M (\mathbf{P}^i + \mathbf{Q}^i) + \sum_{i=1}^K \mathbf{R}^i - W\mathbf{k} = 0 \quad (48)$$

2.8.2 Moment Equilibrium

We require that the sum of the moments due to all forces about some conveniently chosen reference point \mathbf{O} within the body be equal to zero. If the vector locations of each finger/object contact with respect to the reference point is specified by \mathbf{v}^i ($i = 1, \dots, M$), then the expression for moment equilibrium in vector notation is :

$$\sum_{i=1}^M (\mathbf{v}^i \times \mathbf{P}^i + \mathbf{v}^i \times \mathbf{Q}^i) + \sum_{i=1}^K (\mathbf{d}^i \times \mathbf{R}^i) + \mathbf{r} \times (-W)\mathbf{k} = 0 \quad (49)$$

where,

\mathbf{d}^i is the vector location of the force \mathbf{R}^i with respect to the chosen reference point and,

\mathbf{r} is the vector location of the center of mass with respect to the reference point.

Equation (49) expresses the moment equilibrium condition in a very compact manner. However, it is not quite an accurate description of the way moments are accounted for in this model. In the actual implementation, the moments are calculated for each of the discretized forces F_k^i and G_k^i . All these forces *do not* act through the points located by \mathbf{v}^i , but through points that are located by \mathbf{l}_k^i relative to \mathbf{v}^i . \mathbf{l}_k^i is a vector of magnitude proportional to the the number of grid elements that the point k is away from the point located by \mathbf{v}^i (in a sense, it is the distance of point k from the point located by \mathbf{v}^i). It is important to make the point here that, contrary to what Equation (49) seems to show, the discretized forces are not all assumed to act through the point specified by \mathbf{v}^i , and their relative locations due to the discretization are indeed taken into account.

2.9 Directional Constraints on the Forces

An important constraint to consider is that normal forces and the tangential forces can only act in certain directions at each area of contact. The obvious constraint is that the tangential forces are tangential and the normal forces are normal to the surface. In addition, the sense of the normal forces is constrained.

2.9.1 Sense Constraint on the Direction of the Normal Forces

Each finger is capable of exerting normal forces that are directed *into* the the object. Vectorially, the normal force at each finger is constrained to act in the direction of the inwardly

directed surface normal vector (\mathbf{n}^i). This constraint can be expressed as follows :

$$\mathbf{P}^i \cdot \mathbf{n}^i > 0 \quad (i = 1, \dots, M) \quad (50)$$

2.9.2 Constraint on the Tangential Forces

The tangential forces, by definition lie in the tangent plane (given by π in Figure 6) at the point of contact but their direction in the tangent plane is not constrained. We can express this condition as follows :

$$\mathbf{Q}^i \times \mathbf{n}^i = 0 \quad (i = 1, \dots, M) \quad (51)$$

2.10 The Complete Model

Our system is described as an arbitrarily shaped object of mass W being held in static equilibrium by M fingers. The normal and tangential forces at each finger/object contact must satisfy the following constraints :

From force equilibrium,

$$\sum_{i=1}^M \sum_{k=1}^N (F_k^i + G_k^i)_x + \sum_{i=1}^K R_x^i = 0 \quad (52)$$

$$\sum_{i=1}^M \sum_{k=1}^N (F_k^i + G_k^i)_y + \sum_{i=1}^K R_y^i = 0 \quad (53)$$

$$\sum_{i=1}^M \sum_{k=1}^N (F_k^i + G_k^i)_z + \sum_{i=1}^K R_z^i - W = 0 \quad (54)$$

From moment equilibrium,

$$\sum_{i=1}^M (\mathbf{v}^i \times \mathbf{P}^i + \mathbf{v}^i \times \mathbf{Q}^i) + \sum_{i=1}^K (\mathbf{d}^i \times \mathbf{R}^i) + \mathbf{r} \times (-W)\mathbf{k} = 0 \quad (55)$$

And for $i = 1, \dots, M$,

from the directional constraints on the forces - equations (50) and (51),

$$\mathbf{P}^i \cdot \mathbf{n}^i > 0 \quad (i = 1, \dots, M) \quad (56)$$

$$\mathbf{Q}^i \times \mathbf{n}^i = 0 \quad (i = 1, \dots, M) \quad (57)$$

from the discretization of the normal and tangential forces over the contact patch - equations (13) and (25), respectively,

$$\sum_{k=1}^N F_k^i = |\mathbf{P}^i| \quad (58)$$

$$\sum_{k=1}^N G_k^i = |\mathbf{Q}^i| \quad (59)$$

$$(60)$$

remembering that

$$G_k = \sqrt{(G_{kx})^2 + (G_{ky})^2}$$

where, the subscripts x and y now correspond to the local coordinate frame attached to the contact plane at each fingertip.

From the compatibility of deformation and criterion of contact in the normal direction (equations (15)-(17)),

$$\sum_{j=1}^N (S_{kj}^{ni} F_j^i + S_{kj}^{ti} G_j^i) + c_k^i - \alpha^i - Y_k^i = 0 \quad (61)$$

coupled with,

$$\text{If } Y_k^i = 0, \quad \text{then } F_k^i \geq 0 \quad (62)$$

$$\text{If } Y_k^i > 0, \quad \text{then } F_k^i = 0$$

from the compatibility of deformation in the tangential direction (equations (28) and (29)),

$$\sum_{j=1}^N (T_{kj}^{ti} G_j^i + T_{kj}^{ni} F_j^i) + Z_{1k}^i - \beta^i = 0 \quad (63)$$

and from the constraint on the tangential forces (equation (30)),

$$G_k^i + Z_{2k}^i - \mu(F_k^i)F_k^i = 0 \quad (64)$$

with the final restriction that,

$$\text{either } Z_{1k}^i = 0 \quad \text{or} \quad Z_{2k}^i = 0 \quad (65)$$

In the above, $c_k^i, W, \mathbf{R}^i, \mathbf{v}^i, \mathbf{d}^i, \mathbf{r}$ are given and we can obtain S_{kj}^{ti}, S_{kj}^{ni} and T_{kj}^{ti}, T_{kj}^{ni} from the discussion in Section 2.7 and μ as a function of F_k^i as discussed in Section 2.6.

Our objective is to solve this set of equations to get F_k^i and G_k^i for $(k = 1, \dots, N)$ and $(i = 1, \dots, M)$. We discuss the solution technique in the next chapter.

Chapter 3

Solution Technique

Having completed the analysis of the interaction between the fingers and the object, it is necessary to find an efficient solution technique for the problem that has been posed. A general solution procedure is required which will be capable of solving a system of equations where the number of unknowns exceeds the number of prescribed constraints. An optimization technique using the Lagrange Multiplier method would be ideal in this case except for the fact that some of the constraint equations involve “either/or” conditions or complementarity conditions, for example, the set of equations (61) and (62) and equations (63), (64), and (65). Thus, we are forced to look for a mathematical programming technique that will help us find the optimum solution. Even in the field of mathematical programming, however, the algorithms are not built to incorporate complementarity conditions and a modified technique had to be found.

In order to use a mathematical programming technique, an objective function must be formulated such that the optimization can be carried out subject to the given constraints. Given the nature of the model developed, it was chosen to minimize the potential energy of the finger/object system. Since the interactions between the finger and the object are modeled as an elastic phenomenon, the stresses and deformations act in such a way that the total potential energy of the system is minimized. This can be stated more formally as the Theorem of minimum Potential Energy [Lov44]:

The displacement which satisfies the differential equations of equilibrium, as well as the conditions at the bounding surface, yields a smaller value for the potential energy of deformation than any other displacement, which satisfies the same conditions at the bounding surface.

This, then, is the reason for the formulation of the total potential energy expressions of the previous chapter.

Getting back to the choice of a solution technique, a linear programming technique can be ruled out since the objective function is nonlinear (actually, quadratic) in addition to the constraint equations being nonlinear - primarily due to the dependence of the friction coefficient on the normal forces. In the search for a suitable method for nonlinear optimization, let us briefly examine some of the principal ideas.

3.1 General Nonlinear Optimization

Within the framework of this thesis, it is not possible to treat the theory of nonlinear optimization in detail. To look at the relevant part of the theory [KTZ71], let us first define a convex function. A function $F(\mathbf{x})$, ($\mathbf{x}^T = (x_1, \dots, x_n)$) on R^n is called convex on the convex domain M if for any two points \mathbf{x}^1 and \mathbf{x}^2 of M

$$F(\lambda\mathbf{x}^1 + (1 - \lambda)\mathbf{x}^2) \leq \lambda F(\mathbf{x}^1) + (1 - \lambda)F(\mathbf{x}^2)$$

holds for $0 < \lambda < 1$.

3.1.1 Convex Optimization

Let $F(\mathbf{x})$ and $f_j(\mathbf{x})$, $j = 1, \dots, m$ be convex functions of the n variables x_1, \dots, x_n . Convex optimization becomes the problem of minimizing the function $F(\mathbf{x})$ subject to the constraints

$$\begin{aligned} f_j(\mathbf{x}) &\leq 0 & (j = 1, \dots, m) \\ \mathbf{x} &\geq 0. \end{aligned} \tag{66}$$

The function $F(\mathbf{x})$ is called the objective function and a point \mathbf{x} satisfying the constraints $f_j(\mathbf{x}) \leq 0$ in (66) is called a feasible point.

3.1.2 The Kuhn-Tucker Conditions

The Kuhn-Tucker theorem is the central theorem for nonlinear optimization. It represents a generalization of the classical method of Lagrange multipliers for the determination of extrema under constraints, to include the case when these constraints not only contain equations but inequalities as well. More precisely it provides necessary and sufficient conditions for a certain $\hat{\mathbf{x}}$ to be a solution of the problem (66).

These conditions utilize a so-called generalized Lagrangian function Φ . If m new variables u_1, \dots, u_m , the Lagrange multipliers, are introduced and combined to form a vector \mathbf{u} , the Φ is a function of $m + n$ variables (\mathbf{x}, \mathbf{u}) given by

$$\Phi(\mathbf{x}, \mathbf{u}) = F(\mathbf{x}) + \sum_{j=1}^m u_j f_j(\mathbf{x}).$$

The Kuhn-Tucker theorem now states :

A vector $\hat{\mathbf{x}}$ is a solution of problem (66) if and only if a vector $\hat{\mathbf{u}}$ exists such that

$$\hat{\mathbf{x}} \geq 0, \quad \hat{\mathbf{u}} \geq 0$$

and

$$\Phi(\hat{\mathbf{x}}, \mathbf{u}) \leq \Phi(\mathbf{x}, \mathbf{u}) \leq \Phi(\mathbf{x}, \hat{\mathbf{u}})$$

for all

$$\mathbf{x} \geq 0, \quad \mathbf{u} \geq 0.$$

A detailed implication of this theorem can be found in any standard book on Nonlinear Programming. We shall just examine the application of this theorem to the quadratic programming problem.

3.1.3 Quadratic Optimization

From our point of view, the special case of quadratic optimization is of particular interest. the objective function is now assumed to have the form

$$\Phi(\mathbf{x}) = \mathbf{a}^T \mathbf{x} + \mathbf{x}^T \mathbf{C} \mathbf{x}$$

where \mathbf{C} is a symmetric and positive definite or semidefinite matrix. Assuming the constraints to be linear, and given by

$$\begin{aligned} \mathbf{A} \mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq 0. \end{aligned}$$

The Lagrange function for the problem now has the form

$$\Phi(\mathbf{x}, \mathbf{u}) = \mathbf{a}^T \mathbf{x} + \mathbf{x}^T \mathbf{C} \mathbf{x} + \mathbf{x}^T (\mathbf{A} \mathbf{x} - \mathbf{b}).$$

Letting,

$$\frac{\partial \Phi}{\partial \mathbf{x}} = \mathbf{v} \quad \text{and} \quad -\frac{\partial \Phi}{\partial \mathbf{u}} = \mathbf{u}$$

we have

$$\mathbf{v} = \mathbf{a} + 2\mathbf{C}\mathbf{x} + \mathbf{A}^T \mathbf{u} = \frac{\partial \Phi}{\partial \mathbf{x}}$$

and

$$\mathbf{y} = -\mathbf{A}\mathbf{x} + \mathbf{b} = -\frac{\partial \Phi}{\partial \mathbf{u}}.$$

With the substitutions the Kuhn-Tucker conditions for the quadratic problem with equality constraints are as follows :

$$\begin{aligned} \mathbf{A} \mathbf{x} &= \mathbf{b} \\ 2\mathbf{C}\mathbf{x} + \mathbf{v} + \mathbf{A}^T \mathbf{u} &= -\mathbf{a} \\ \mathbf{x} &\geq 0, \quad \mathbf{v} \geq 0 \\ \mathbf{x}^T \mathbf{v} &= 0 \end{aligned}$$

3.2 The Method of Wolfe

Wolfe's method published in 1959 [Wol59] is adapted to work well with the revised simplex method. Once again it would be futile to go into a rigorous analysis of the Wolfe method and a summary would quickly explain why the method seemed to be suitable for solving our problem and why it failed.

In the Wolfe method, additional variables are introduced into the system in such way that a feasible basis solution can be given immediately for which the conditions of the previous section can be satisfied. The revised simplex method of linear programming is used to make these additional variables disappear again. Care must be taken during the iterative process, however, that the additional condition $\mathbf{x}^T \mathbf{v} = 0$ is satisfied at each step. The way this is actually implemented in the algorithm is of particular interest to us because the model we want to solve has two very similar constraints. Refer to Equation (62) which can be stated as

$$F^T Y = 0$$

and Equation (65) which can be stated as

$$Z_1^T Z_2 = 0.$$

In the revised simplex method the operations are very much the same as the Gaussian Elimination operations for matrices. In this case, however, there is a tableau corresponding to the constraint equations and a set of independent (basis) variables and a set of dependent variables. The simplex method prescribes the selection of the pivot element at each iteration and the pivoting operation leads to the replacement of an independent (basis) variable by a dependent variable and vice versa. The way in which the complementarity constraints are implemented is that during the revised simplex method the variable entering the basis is allowed to enter the basis only if

- either the complementary variable is *not present* in the basis
- or if the complementary variable will be forced to *leave* the basis if the variable is to enter the basis.

For example, suppose the simplex method leads to the choice of F_k^i as the entering variable at a particular iteration step. A check must be made to see if the Y_k^i corresponding to F_k^i is in the basis. If Y_k^i is in the basis, it must correspond to the same row as the pivoting element, if F_k^i is to be allowed to enter the basis. If Y_k^i is in the basis and does not correspond to the row of the pivot element then F_k^i may not enter the basis and a new entering variable must be chosen.

While it was convenient to incorporate complementarity conditions using Wolfe's algorithm, the method presented two major drawbacks with respect to the solution of our problem. It was anticipated that it would be possible to linearize the constraints in some way, possibly by using the intermediate solution at a particular iteration. But since the

algorithm is not iterative, it was not capable of providing an interim solution at a particular iteration. In view of the fact that our model needed to have nonlinear constraints (precisely because that was the original goal of this thesis), it would be a big disadvantage to be unable to implement such constraints.

Also, since all the variables were restricted to be positive, the only way the variables for friction force could be represented, was as a difference of two variables. This would greatly increase the number of variables and make them very unmanageable when they had to be closely monitored during the optimization steps, in order to impose the complementarity conditions. Moreover, since this method only accepted equality constraints, additional slack variables had to be added and solved, for each of the inequality constraints. It should be pointed out that Wolfe's algorithm adds $2n + m$ variables (where n is the number of unknown variables and m is the number of constraints) even before it can start solving the optimization problem. It seems an unnecessarily large burden even for today's fast computers to handle.

The reason for abandoning this algorithm was also the availability of the new IMSL routines which were more efficient and considerably faster than Wolfe's algorithm. In particular, the routines NCONF and NCONG [FOR87], which implemented Schittkowski's algorithm were very well suited for our application.

Once again it is beyond the scope of this thesis to describe the theory behind Schittkowski's algorithm completely. What follows is a short summary of what the process does and the reader is directed to the appropriate references for details.

3.3 The Schittkowski Algorithm

An algorithm for nonlinearly constrained optimization was developed by Schittkowski in 1986 [Sch86]. It uses a successive quadratic programming method [Sto85] to solve the general nonlinear programming problem. The problem can be stated as :

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} f(x) \\ & \text{subject to } g_j(x) = 0, \quad \text{for } j = 1, \dots, m_e \\ & \quad \quad g_j(x) \geq 0, \quad \text{for } j = m_e + 1, \dots, m \\ & \quad \quad x_l \leq x \leq x_u, \end{aligned}$$

where all functions are assumed to be continuously differentiable. The method, based on the iterative formulation and solution of quadratic programming subproblems, obtains subproblems by using a quadratic approximation of the Lagrangian and by linearizing the constraints. That is,

$$\begin{aligned} & \min_{d \in \mathbb{R}^n} \frac{1}{2} d^T B_k d + \nabla f(x_k)^T d \\ & \text{subject to } \nabla g_j(x_k)^T d + g_j(x_k) = 0, \quad \text{for } j = 1, \dots, m_e \\ & \quad \quad \nabla g_j(x_k)^T d + g_j(x_k) \geq 0, \quad \text{for } j = m_e + 1, \dots, m \end{aligned}$$

$$x_l - x_k \leq d \leq x_u - x_k,$$

where B_k is a positive definite approximation of the Hessian, and x_k is the current iterate. Let d_k be the solution of the subproblem. A line search is used to find a new point x_{k+1} ,

$$x_{k+1} = x_k + \lambda d_k, \lambda \in (0, 1),$$

such that a “merit function” will have a lower function value at the new point. Here the augmented Lagrange function [Sch86] is used as a merit function.

When optimality is not achieved, B_k is updated according to the BFGS-quasi Newton method [Sch80]. This algorithm may generate infeasible points during the solution process, but in this application that does not pose a problem. For more theoretical and practical details see Stoer [Sto85], Schittkowski [Sch80, Sch86] and Gill et al. [PWMW85].

This algorithm admits nonlinear constraints. It actually linearizes the constraints by using the values of the variables from the previous iteration to linearize the constraints for the current iteration.

The form in which the algorithm is implemented in the routine NCONF is particularly conducive to the solution of our problem. The problem can be stated in very much the same way as it was stated at the end of the previous chapter. The complementarity condition can also be implemented in an elegant fashion.

For example, take the constraints given by equations (63), (64) and (65) :

$$\begin{aligned} \sum_{j=1}^N (T_{kj}^{ti} G_j^i + T_{kj}^{ni} F_j^i) + Z_{1k}^i - \beta^i &= 0 \\ G_k^i + Z_{2k}^i - \mu(F_k^i) F_k^i &= 0 \end{aligned}$$

with the restriction that,

$$\text{either } Z_{1k}^i = 0 \text{ or } Z_{2k}^i = 0$$

Since the algorithm handles inequality constraints, we do not need the slack variables at all. The above constraints can be represented as :

Either (for the nonslipping condition),

$$\beta^i - \sum_{j=1}^N (T_{kj}^{ti} G_j^i + T_{kj}^{ni} F_j^i) = 0 \quad (67)$$

$$\mu(F_k^i) F_k^i - G_k^i \geq 0 \quad (68)$$

or (for the slipping condition),

$$\beta^i - \sum_{j=1}^N (T_{kj}^{ti} G_j^i + T_{kj}^{ni} F_j^i) \geq 0 \quad (69)$$

$$\mu(F_k^i) F_k^i - G_k^i = 0 \quad (70)$$

Since this algorithm uses an iterative scheme, this “either/or” constraint can be embedded into the program. At the end of each iteration, for this particular constraint, the value of F_k^i and the corresponding G_k^i are examined. If equation (68) is currently satisfied, the next iteration uses equations (67) and (68) as constraints. Alternatively, equations (69) and (70) are used as constraints in the next iteration. Implementing this scheme was well within the bounds of the routine, NCONF and seemed to cause no problems even though a provision for such an implementation had not been made.

Results of some representative problems that were solved using this routine are given in the next chapter.

3.4 Form of the Objective Function

Before we go any further, it is important that we define the objective function for optimization. It was decided earlier to minimize the total potential energy of the system. From the expressions derived in the previous chapter, the potential energy due to the influence of the normal forces is given by,

$$\begin{aligned} \text{PE} &= \frac{1}{2}(F^T S^n F + F^T S^t G) \\ &= -\frac{1}{2} \sum_{j=1}^N F_j(\alpha - c_j) \end{aligned}$$

and under the influence of tangential forces,

$$\begin{aligned} \text{PE} &= \frac{1}{2}(G^T T^t G + G^T T^n F) \\ &= -\frac{1}{2} \sum_{j=1}^N G_j(\beta - Z_{1j}) \end{aligned}$$

Therefore, the total potential energy of the system is

$$\text{TPE} = \frac{1}{2}(F^T S^n F + F^T S^t G) + \frac{1}{2}(G^T T^t G + G^T T^n F) \quad (71)$$

$$= -\frac{1}{2} \sum_{j=1}^N [F_j(\alpha - c_j) - G_j(\beta - Z_{1j})] \quad (72)$$

To formulate a quadratic optimization problem that is well behaved we chose to minimize the difference of Equations (71) and (72), since that way all the variables that were in the problem would be included in the objective function. Therefore, the objective function for minimization is given by Equation (71) - Equation (72),

$$F^T(S^n F + S^t G - c) + G^T(T^t G + T^n F - Z_1) + e^T(F\alpha + G\beta) \quad (73)$$

where,

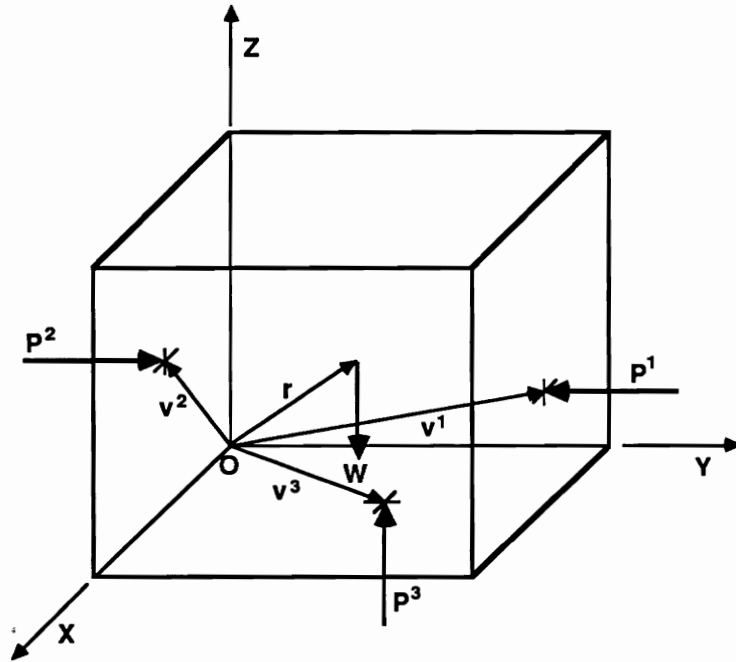


Figure 8: A Cube in a Three-Fingered Grasp

- $S^t = N \times N$ matrix of influence coefficients for tangential forces
- $S^n = N \times N$ matrix of influence coefficients for normal forces
- $T^t = N \times N$ matrix of influence coefficients for tangential forces
- $T^n = N \times N$ matrix of influence coefficients for normal forces
- $G = N \times 1$ vector of forces
- $F = N \times 1$ vector of forces
- $Z_1 = N \times 1$ vector of slack variables
- $e = N \times 1$ vector of ones
- $c = N \times 1$ vector of initial separations (shape information)
- $\beta =$ rigid-body approach in the tangential direction
- $\alpha =$ rigid-body approach in the normal direction

3.5 An Example Program

A cube is held at three points [HM85](see figure 8). The points of contact with reference to the origin O are :

$$\begin{aligned} \mathbf{v}^1 &= (0.5, 1.0, 0.3) \\ \mathbf{v}^2 &= (0.5, 0.0, 0.5) \\ \mathbf{v}^3 &= (0.6, 0.5, 0.0) \end{aligned}$$

and the associated surface normals are :

$$\begin{aligned}\mathbf{n}^1 &= (0.0, -1.0, 0.0) \\ \mathbf{n}^2 &= (0.0, 1.0, 0.0) \\ \mathbf{n}^3 &= (0.0, 0.0, 1.0)\end{aligned}$$

The object is assumed to weigh 1lb, that is we have an external force $W\mathbf{k} = -1.0\mathbf{k}$ acting on the object. Its center of mass is given by :

$$\mathbf{r} = (0.5, 0.5, 0.5)$$

The first step would be to choose the dimensions of our contact patch. Let us suppose we choose each patch to be a square grid of size 5×5 elements, and actual dimensions of 0.1×0.1 , so that each grid element is 0.02×0.02 .

Now the force, \mathbf{P}^1 can be discretized as

$$|\mathbf{P}^1| = \sum_{k=1}^{25} F_{ky}^1$$

Similarly, the forces, \mathbf{P}^2 and \mathbf{P}^3 can be written as

$$|\mathbf{P}^2| = \sum_{k=1}^{25} F_{ky}^2$$

$$|\mathbf{P}^3| = \sum_{k=1}^{25} F_{kz}^3$$

Notice that we have partially incorporated the unidirectional constraint on the normal forces, and to make that constraint complete, all we need to specify is that

$$F_{ky}^1 \geq 0 \quad \text{for all } k = 1, \dots, 25$$

$$F_{ky}^2 \geq 0 \quad \text{for all } k = 1, \dots, 25$$

$$F_{kz}^3 \geq 0 \quad \text{for all } k = 1, \dots, 25$$

This can be specified as a one line data statement in the program. The constraints also get implicitly enforced when we formulate the force and moment equilibrium equations.

Now the tangential forces at each grid element need to be discretized. At the contact point 1, the tangential force \mathbf{Q}^1 is constrained to lie parallel to the xz plane. However, we do not know the actual direction of this force in the yz plane. Therefore, we will represent the tangential forces in this manner,

$$|\mathbf{Q}^1| = \sum_{k=1}^{25} \sqrt{(G_{kx}^1)^2 + (G_{kz}^1)^2}$$

$$|\mathbf{Q}^2| = \sum_{k=1}^{25} \sqrt{(G_{kx}^2)^2 + (G_{kz}^2)^2}$$

$$|\mathbf{Q}^3| = \sum_{k=1}^{25} \sqrt{(G_{kx}^3)^2 + (G_{ky}^3)^2}$$

And in this case,

$$-\infty < G_{kx}^1, G_{kz}^1, G_{kx}^2, G_{kz}^2, G_{kx}^3, G_{ky}^3 < \infty \quad \text{for all } k = 1, \dots, 25$$

The limits could really be specified more accurately, but that is how the routine handles it when a variable is not restricted in sign. Once again this can be specified in a single data statement along with the sign constraint for the normal forces.

The objective function will be given by,

$$(F^i)^T(S^n F^i + S^t G^i - c^i) + (G^i)^T(T^t G^i + T^n F^i - Z_1^i) + e^T(F^i \alpha^i + G \beta^i)$$

where, for ($i = 1, 2, 3$) and the superscript T signifies the transpose of a matrix or vector,

$F^i = 25 \times 1$ vector of the discretized normal forces at the i^{th} fingertip

$G^i = 25 \times 1$ vector of the discretized tangential forces at the i^{th} fingertip, each element of G^i is of the form $\sqrt{(G_{km}^i)^2 + (G_{kn}^i)^2}$, where m and n are the directions in which the tangential force G^i is resolved at the i^{th} fingertip. For example, at the fingertip 1, $G_k^1 = \sqrt{(G_{kx}^1)^2 + (G_{kz}^1)^2}$

$c^i = 25 \times 1$ vector of the initial separations at each discretized grid element of the i^{th} fingertip

$\alpha^i =$ relative rigid body movement at the i^{th} fingertip in the direction of the normal force

$\beta^i =$ relative rigid body movement at the i^{th} fingertip in the direction of the tangential force

S^n, S^t, T^n, T^t are as specified in the previous section.

Enforcing conditions for force equilibrium in the x, y and z directions, we obtain the following three equations :

$$\begin{aligned} \sum_{k=1}^{25} (G_{kx}^1 + G_{kx}^2 + G_{kx}^3) &= 0 \\ \sum_{k=1}^{25} (F_{ky}^2 - F_{ky}^1 + G_{ky}^3) &= 0 \\ \sum_{k=1}^{25} (F_{kz}^3 + G_{kz}^1 + G_{kz}^2) - 1 &= 0 \end{aligned}$$

Enforcing conditions for moment equilibrium about the x, y and z axes, we obtain the following three equations :

$$\begin{aligned} \sum_{k=1}^{25} [(0.3 + \delta_{kz}^1) F_{ky}^1 - (0.5 + \delta_{kz}^2) F_{ky}^2 + (0.5 + \delta_{ky}^3) F_{kz}^3 + G_{kz}^1] - 0.5 &= 0 \\ \sum_{k=1}^{25} [0.3 G_{kx}^1 + 0.5 (G_{kx}^2 - G_{kz}^1 - G_{kz}^2) - (0.6 + \delta_{kx}^3) F_{kz}^3] + 0.5 &= 0 \\ \sum_{k=1}^{25} [(0.5 + \delta_{kx}^2) F_{ky}^2 - (0.5 + \delta_{kx}^1) F_{ky}^1 - 0.5 G_{kx}^3 - G_{kx}^1 + 0.6 G_{ky}^3] &= 0 \end{aligned}$$

where,

δ_{km}^i is the distance of the k^{th} contact point ($k = 1, \dots, 25$) at the i^{th} fingertip ($i = 1, 2, 3$) from the point located by \mathbf{v}^i , measured along the m^{th} axis ($m = x, y, z$).

From the compatibility of deformation in the normal direction, we get the following condition,

For i varying from 1 to M and k varying from 1 to 25 for each i , we calculate

$$Y_k^i = \sum_{j=1}^N (S_{kj}^{ni} F_j^i + S_{kj}^{ti} G_j) + c_k^i - \alpha^i$$

$$\text{If } Y_k^i = 0, \quad \text{then } F_k^i \geq 0$$

and, the constraint is,

$$\sum_{j=1}^N (S_{kj}^{ni} F_j^i + S_{kj}^{ti} G_j) + c_k^i - \alpha^i = 0$$

$$\text{If } Y_k^i > 0, \quad \text{then set } F_k^i = 0.$$

Similarly, from the compatibility of deformation in the tangential direction and the criterion for the friction limit, we get the following condition,

For i varying from 1 to M and k varying from 1 to 25 for each i , we check if

$$\mu(F_k^i) F_k^i - G_k^i = 0$$

then,

$$\begin{aligned} \mu(F_k^i) F_k^i - G_k^i &= 0 \quad \text{and} \\ \sum_{j=1}^N \beta^i - (T_{kj}^{ti} G_j^i + T_{kj}^{ni} F_j) &\geq 0 \end{aligned}$$

else,

$$\begin{aligned} \mu(F_k^i)F_k^i - G_k^i &\geq 0 \quad \text{and} \\ \sum_{j=1}^N \beta^i - (T_{kj}^{ti}G_j^i + T_{kj}^{ni}F_j^i) &= 0 \end{aligned}$$

Putting it all together we can say that

$$\text{Minimize} \quad (F^i)^T(S^n F^i + S^t G^i - c^i) + (G^i)^T(T^t G^i + T^n F^i - Z_1^i) + e^T(F^i \alpha^i + G \beta^i)$$

Such that,

$$\begin{aligned} \sum_{k=1}^{25} (G_{kx}^1 + G_{kx}^2 + G_{kx}^3) &= 0 \\ \sum_{k=1}^{25} (F_{ky}^2 - F_{ky}^1 + G_{ky}^3) &= 0 \\ \sum_{k=1}^{25} (F_{kz}^3 + G_{kz}^1 + G_{kz}^2) - 1 &= 0 \\ \sum_{k=1}^{25} [(0.3 + \delta_{kz}^1)F_{ky}^1 - (0.5 + \delta_{kz}^2)F_{ky}^2 + (0.5 + \delta_{ky}^3)F_{kz}^3 + G_{kz}^1] - 0.5 &= 0 \\ \sum_{k=1}^{25} [0.3G_{kx}^1 + 0.5(G_{kx}^2 - G_{kz}^1 - G_{kz}^2) - (0.6 + \delta_{kx}^3)F_{kz}^3] + 0.5 &= 0 \\ \sum_{k=1}^{25} [(0.5 + \delta_{kx}^2)F_{ky}^2 - (0.5 + \delta_{kx}^1)F_{ky}^1 - 0.5G_{kx}^3 - G_{kx}^1 + 0.6G_{ky}^3] &= 0 \end{aligned}$$

The logic for the satisfaction of the complementary constraints that effectively establish the extent of the contact patch can be expressed as follows :

For i varying from 1 to M and k varying from 1 to 25 for each i , if

$$\alpha^i = \sum_{j=1}^N (S_{kj}^{ni}F_j^i + S_{kj}^{ti}G_j^i) + c_k^i$$

then for the next iteration,

$$\sum_{j=1}^N (S_{kj}^{ni}F_j^i + S_{kj}^{ti}G_j^i) + c_k^i - \alpha^i = 0$$

else,

$$\text{set } F_k^i = 0.$$

And, if

$$\mu(F_k^i)F_k^i - G_k^i = 0$$

then for the next iteration,

$$\begin{aligned} \mu(F_k^i)F_k^i - G_k^i &= 0 \quad \text{and} \\ \sum_{j=1}^N \beta^i - (T_{kj}^{ti}G_j^i + T_{kj}^{ni}F_j) &\geq 0 \end{aligned}$$

else,

$$\begin{aligned} \mu(F_k^i)F_k^i - G_k^i &\geq 0 \quad \text{and} \\ \sum_{j=1}^N \beta^i - (T_{kj}^{ti}G_j^i + T_{kj}^{ni}F_j) &= 0 \end{aligned}$$

That completes the program. In the following chapter we will look at some of the results obtained from using this scheme on elementary grasping problems and the results of this particular example will also be discussed.

Chapter 4

Results

The IMSL routine implementing Schittkowski's algorithm was combined with the analytical model developed in Chapter 2, to calculate forces in some elementary grasps. The scheme followed in each of the examples is similar to the one developed in the last section of Chapter 3, and details of the formulation for individual examples will not be discussed here.

The routine worked very efficiently when implemented on a Sun 4/280 computer running Sun OS 4.0. This machine is equipped with a floating point coprocessor which greatly enhanced the speed of the computation and the time taken to achieve an optimal solution. A quantitative analysis of the computational aspects was not attempted here, but would be worth looking at if this model is to be implemented in real-time.

4.1 Examples in Two and Three Dimensions

In the examples that follow, the orientation of normal forces at each of the contact points (\mathbf{n}^i) and the location (\mathbf{v}^i) of the point of contact of the fingers relative to a chosen reference point (\mathbf{O}) was all the information the algorithm was provided, in addition, of course, to the body weight (\mathbf{W}), the location of the center of mass (\mathbf{r}) and the material properties of the object and the finger. The friction forces were restricted to be perpendicular to the direction of the normal force which, in turn, was constrained to act inwardly. The actual directions of the friction forces, however, were output as a solution as were the magnitudes of the normal and the friction forces. The figures represent the results qualitatively and the length of the arrows representing the forces are not proportional to the actual magnitudes obtained. Obviously, the actual values obtained satisfy the requirements for force and moment equilibrium. Here are three 2-dimensional examples.

4.1.1 Rectangular Plate in a Four-Finger Grasp

Our first example models a rectangular plate being grasped by four fingers (see figure 9). From our results, the plate seems to be fully supported by fingers 1, 2 and 3 with finger 4 exerting no forces ($\mathbf{P}^4 = 0$ and $\mathbf{Q}^4 = 0$). Fingers 1 and 3 exert vertical friction forces (\mathbf{Q}^1 and \mathbf{Q}^3 , respectively), which seems intuitively correct, with the friction forces actually

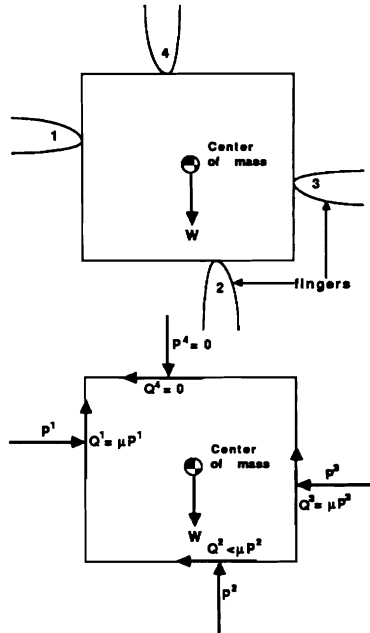


Figure 9: A Rectangular Plate Grasped by Four Fingers

reaching the maximum attainable values (equal to μP^1 and μP^3 , respectively). While finger 2 exerts a nonzero normal force, the frictional force is still less than the limiting value ($Q^2 < \mu P^2$) and acts towards the left to maintain equilibrium.

4.1.2 Disc in a Three-Finger Grasp

The next case of a disc (see figure 10) being supported by 3 fingers is an interesting example of a problem that yields phenomenologically incorrect results when solved using a rigid body model. In fact, we discussed this particular example in Section 1.3.1, (see Figure 1) where some of the friction forces that were obtained, act to rotate the disc, which violates the conditions for static equilibrium. In our case, however, we get nonzero friction forces at fingers 1 and 2 acting upward and outward, which makes more sense intuitively and is consistent with the deformation process that was discussed in Section 1.3.1. Finger 3 does not exert any force ($P^3 = 0$ and $Q^3 = 0$) when the disc is in equilibrium and the magnitudes of the friction forces at fingers 1 and 2 are at their attainable maximum limits ($Q^1 = \mu P^1$ and $Q^2 = \mu P^2$). In this particular example, the fingers were positioned symmetrically at a 45 degree angle with respect to the horizontal.

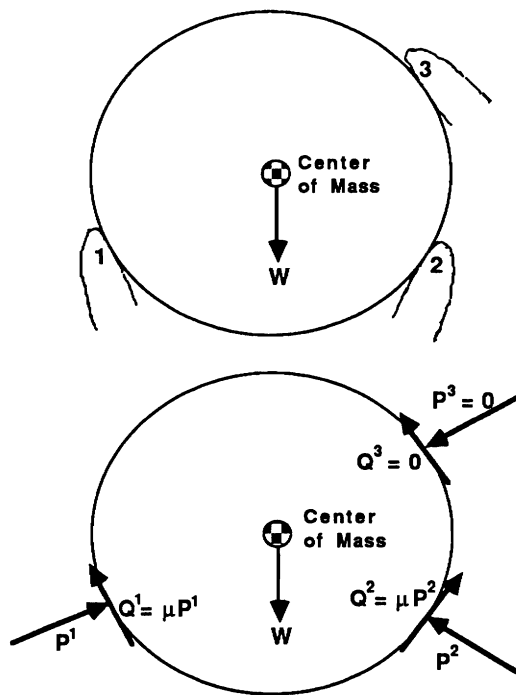


Figure 10: A Disc Grasped by Three Fingers

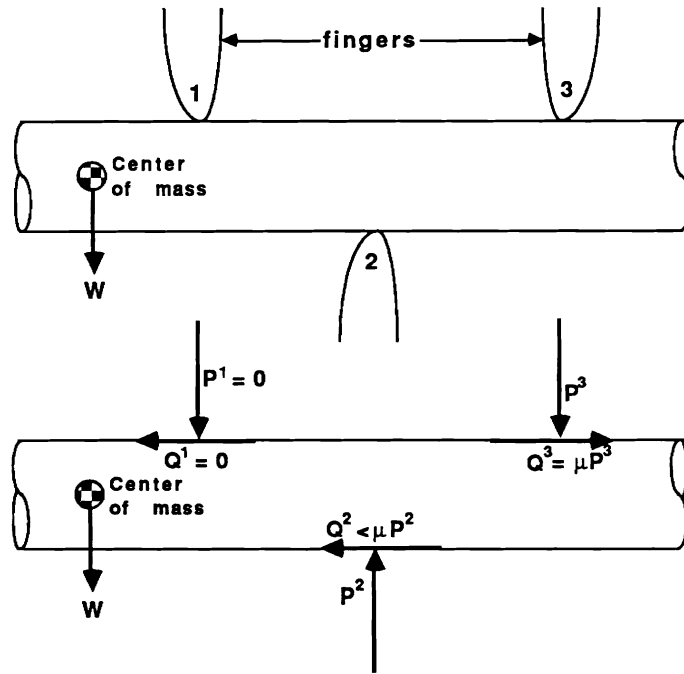


Figure 11: A Pipe Grasped by Three Fingers

4.1.3 Pipe in a Three-Finger Grasp

The other illustrative example is that of a long cylindrical object being grasped by three fingers (see figure 11), however, unlike the previous cases the center of mass lies beyond the region of grasping. Finger 1 does not seem to play any role in supporting the object in equilibrium ($P^1 = 0$ and $Q^1 = 0$) while fingers 2 and 3 seem to behave very much the same way as when we exert forces with our own fingers while holding a pen or pencil in this fashion. Finger 2 provides the most support but the friction forces arising out of it (acting towards the center of mass) are below their maximum limit ($Q^2 < \mu P^2$). Meanwhile, finger 3 pushes downward and away from the center of mass with a limiting friction force ($Q^3 = \mu P^3$). It should be mentioned here that in this example, the length of the cylinder does not matter. The relative locations of the finger and the center of mass are all that determines the solution. In fact, in this example, as in the other two mentioned above, the same solutions would be valid if the body was extended in the third dimension, as long as the relative positions of the fingers and the center of mass did not change. Of course, any change in body weight will alter the magnitudes of the forces obtained, but qualitatively the solutions will not change.

4.1.4 Cube in a Three-Finger Grasp

This example (see figure 12) is the same as the one we used in the previous chapter to show how a model program would be developed. The points of contact with reference to the origin \mathbf{O} are :

$$\mathbf{v}^1 = (0.5, 1.0, 0.3)$$

$$\mathbf{v}^2 = (0.5, 0.0, 0.5)$$

$$\mathbf{v}^3 = (0.6, 0.5, 0.0)$$

and the associated surface normals are :

$$\mathbf{n}^1 = (0.0, -1.0, 0.0)$$

$$\mathbf{n}^2 = (0.0, 1.0, 0.0)$$

$$\mathbf{n}^3 = (0.0, 0.0, 1.0)$$

The object is assumed to weigh 1lb, that is we have an external force $W\mathbf{k} = -1.0\mathbf{k}$ acting on the object. Its center of mass is located by :

$$\mathbf{r} = (0.5, 0.5, 0.5)$$

This example is identical to the one solved by Holzmann and McCarthy in 1985 [HM85], using a rigid body model and finding the friction forces such that they opposed the incipient twist in the cube. We tried to solve the problem using the same friction coefficients used by them and got results that are very similar to those obtained by them. Unlike their method, however, this method did not require the specification of the magnitudes of the normal forces, nor did it require the calculation of the instantaneous motion or twist direction.

In the rigid body example, the normal forces at the fingertips are set to the values

$$\mathbf{P}^1 = 0.5, \quad \mathbf{P}^2 = 0.4, \quad \mathbf{P}^3 = 0.8.$$

From our analysis, we get

$$\mathbf{P}^1 = 0.54, \quad \mathbf{P}^2 = 0.46, \quad \mathbf{P}^3 = 0.75.$$

In the rigid body example, the friction forces at the fingertips are found to be

$$\mathbf{Q}^1 = 0.2, \quad \mathbf{Q}^2 = 0.09, \quad \mathbf{Q}^3 = 0.22.$$

From our analysis, we get

$$\mathbf{Q}^1 = 0.22, \quad \mathbf{Q}^2 = 0.11, \quad \mathbf{Q}^3 = 0.21.$$

The directions of the friction forces are also nearly the same as those predicted by the Holzmann and McCarthy model.

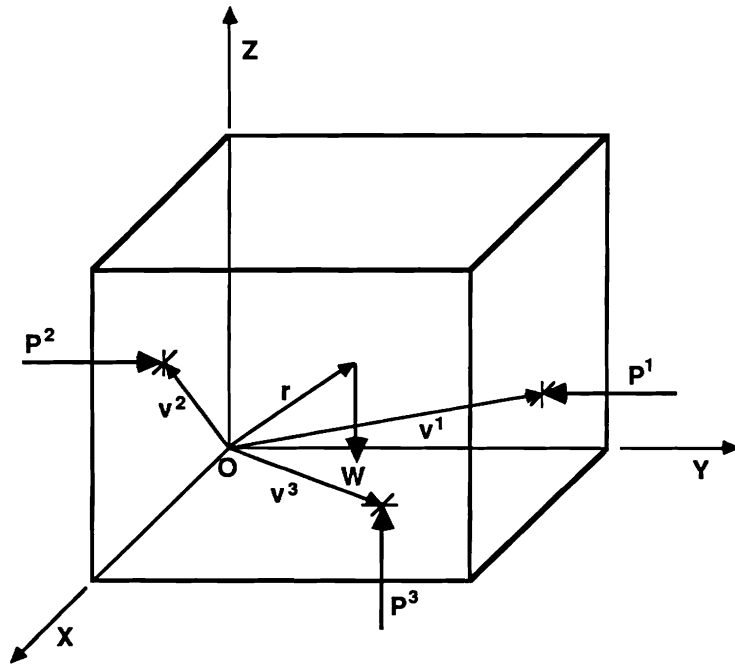


Figure 12: Three-finger Grasp for a Cube

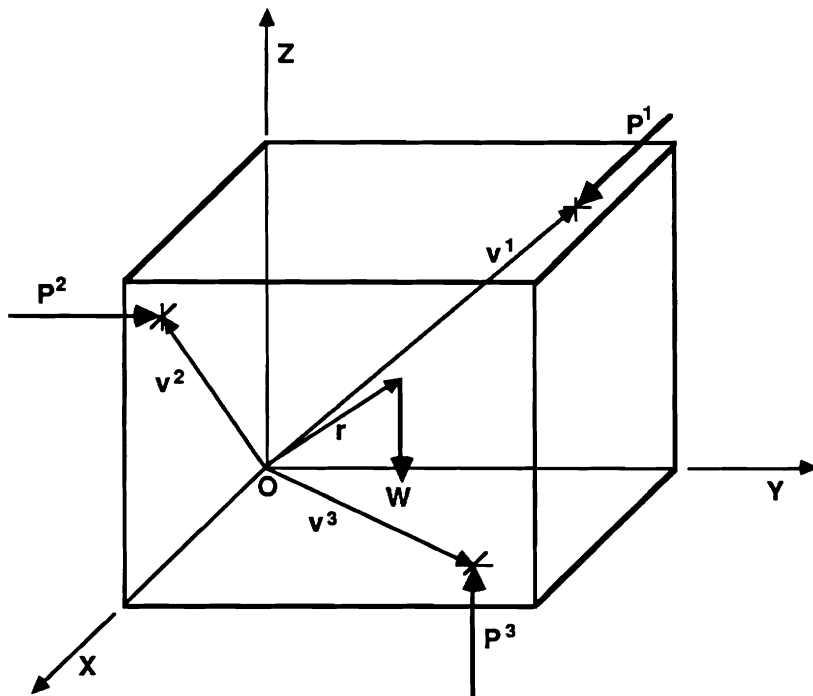


Figure 13: Another Three-finger Grasp for a Cube

4.1.5 Cube in a Three-Finger Grasp (Different Configuration)

This is another example (see figure 13) taken from the paper [HM85] by Holzmann and McCarthy. This case was the example of a paradoxical grasp where, the friction forces obtained did not oppose the instantaneous motion of the object. This was inconsistent with the assumptions of the rigid body and Coulomb Friction model that were the basis of the analysis, and therefore, the grasp was classified as being infeasible.

The points of contact with reference to the origin \mathbf{O} are :

$$\begin{aligned}\mathbf{v}^1 &= (0.0, 0.75, 0.75) \\ \mathbf{v}^2 &= (0.75, 0.0, 0.75) \\ \mathbf{v}^3 &= (0.75, 0.75, 0.0)\end{aligned}$$

and the associated surface normals are :

$$\begin{aligned}\mathbf{n}^1 &= (1.0, 0.0, 0.0) \\ \mathbf{n}^2 &= (0.0, 1.0, 0.0) \\ \mathbf{n}^3 &= (0.0, 0.0, 1.0)\end{aligned}$$

The object is assumed to weigh 1lb, that is we have an external force $W\mathbf{k} = -1.0\mathbf{k}$ acting on the object. Its center of mass is located by :

$$\mathbf{r} = (0.5, 0.5, 0.5)$$

From our analysis, we get

$$\begin{aligned}\mathbf{P}^1 &= 0.29, \quad \mathbf{P}^2 = 0.77, \quad \mathbf{P}^3 = 0.29 \\ \mathbf{Q}^1 &= 0.15, \quad \mathbf{Q}^2 = 0.38, \quad \mathbf{Q}^3 = 0.15.\end{aligned}$$

Further, \mathbf{Q}^1 has components in the +ve y and -ve z directions. \mathbf{Q}^2 has components in the -ve x and -ve z directions and \mathbf{Q}^3 has components in the -ve x and +ve y directions.

While it is true that our model was unable to find feasible solutions for effective values of $\mu < 0.5$, it was able to find optimal solutions for “rougher” ($\mu \geq 0.5$) fingers, like the one obtained here. The directions for the friction forces are nearly the same as predicted by the rigid body model, however, since there is no assumption of the friction forces opposing any instantaneous twists, there is no reason to believe that this solution is inconsistent. A point to note here is that the forces at fingers 1 and 2 are symmetric, which is what would be expected since their locations are symmetric about the center of mass.

4.1.6 Discussion

The examples shown above illustrate the successful implementation of the models and techniques developed in the previous chapter. It is clear that the model is able to predict the forces that result at the points of interaction when an arbitrarily shaped body is held in a multi-fingered grasp. There are no inherent inconsistencies that prevent the acceptance of a certain solution. If the grasp is impossible, the iterative minimization scheme does not converge and this indeed happened when some infeasible examples were analyzed.

4.2 The Transfer Maneuver

We now attempt to use the analysis in a quasi-dynamic way to predict the changes in the forces at the finger/object contacts when an object is transferred from one set of fingers to another without change in its configuration. In both of the examples presented here, the object is transferred from one two-finger grasp to another two-finger grasp.

4.2.1 Adaptation of the Algorithm

We first calculate the forces required to support the body in static equilibrium using the two fingers, say, 1 and 2. Once these forces are calculated, they are treated as external forces acting on the body with the solution being aimed at identifying the forces on the other two fingers, say, 3 and 4 (which are now assumed to be in contact). However, while the normal forces at the fingers 1 and 2 are treated as known, the friction forces are still calculated by the algorithm since we have no control over them in a real transfer situation. The normal forces at the fingers 1 and 2 are now slowly decreased (in a predetermined fashion) and the new forces in the other two fingers 3 and 4 are recalculated. Thus, by continuously varying the normal forces in the fingers 1 and 2 we are able to obtain the normal forces and the friction forces that need to be exerted by the other two fingers to hold the body in equilibrium. This will constitute the response of, and taking on the load, by the two new fingers as the original fingers remove their support, and this can be carried out till the normal forces in the fingers 1 and 2 become zero, with the object being totally supported by the fingers 3 and 4.

The same result can be also achieved by increasing the normal forces on fingers 3 and 4 till the forces required of fingers 1 and 2 to support the object in static equilibrium go to zero.

4.2.2 Example of Transfer of A Plate

The first example considered here is that of a plate being transferred from the grasp of fingers 1 and 2 to the grasp of fingers 3 and 4. The fingers are placed such that finger 1 is opposite finger 2 and finger 3 is opposite finger 4 (see Figure 14). The results are shown in Figure 15.

Due to the symmetry of the grasps, it turned out that $\mathbf{P}^1 = \mathbf{P}^2$ and $\mathbf{Q}^1 = \mathbf{Q}^2$ as was expected. Similarly, $\mathbf{P}^3 = \mathbf{P}^4$ and $\mathbf{Q}^3 = \mathbf{Q}^4$. For the first half of the transfer (time < 10) the forces \mathbf{P}^1 and \mathbf{P}^2 were decreased along the path shown in the plot. All the other forces were predicted by the algorithm. For the second half of the transfer maneuver, the forces \mathbf{P}^3 and \mathbf{P}^4 were increased as shown in the plot (time > 10), and the rest of the forces were predicted by the algorithm. In this example, all the friction forces were at their maximum attainable limit at all stages of the transfer maneuver.

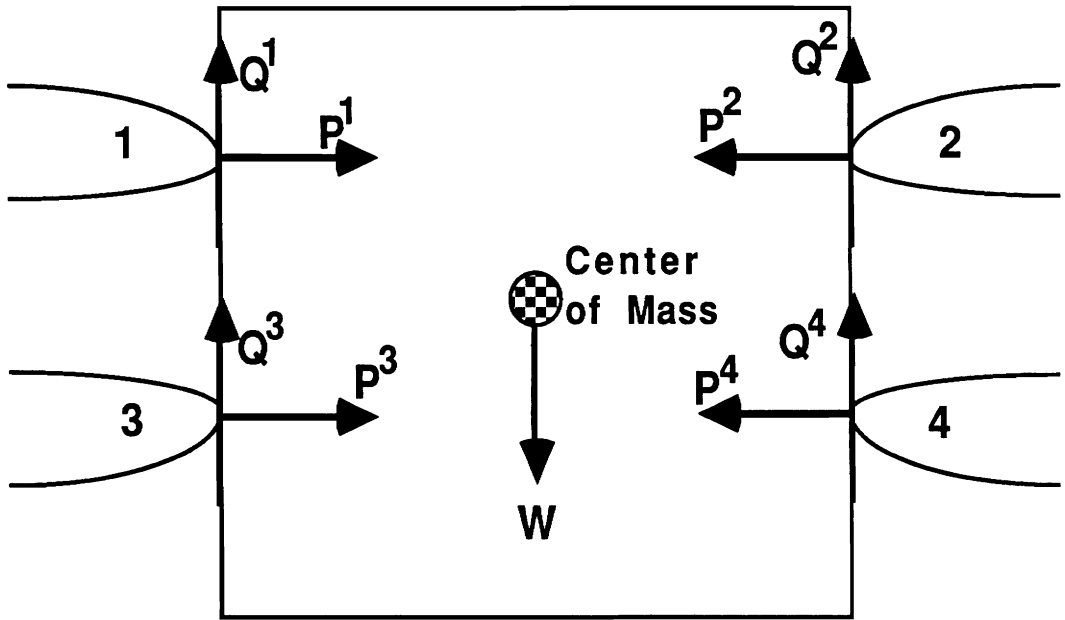


Figure 14: Transfer of a Plate

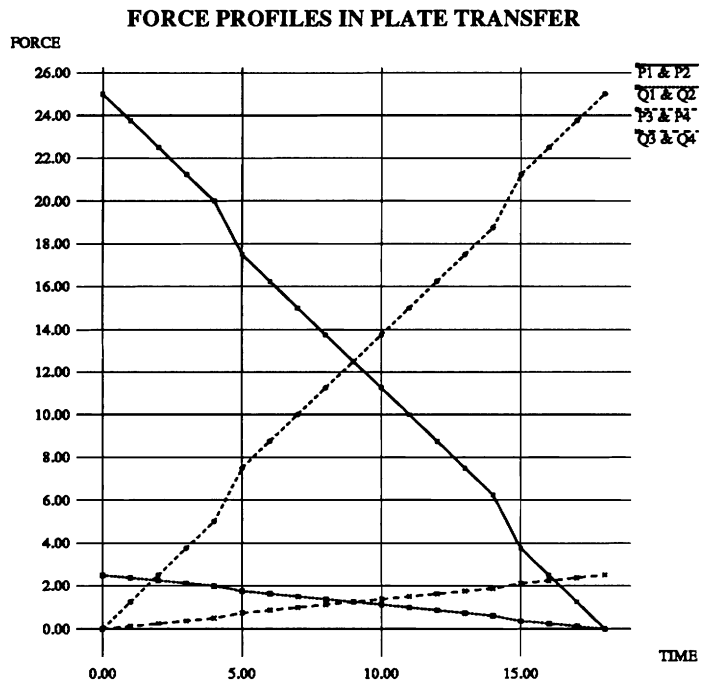


Figure 15: Changes in Forces during Transfer of a Plate

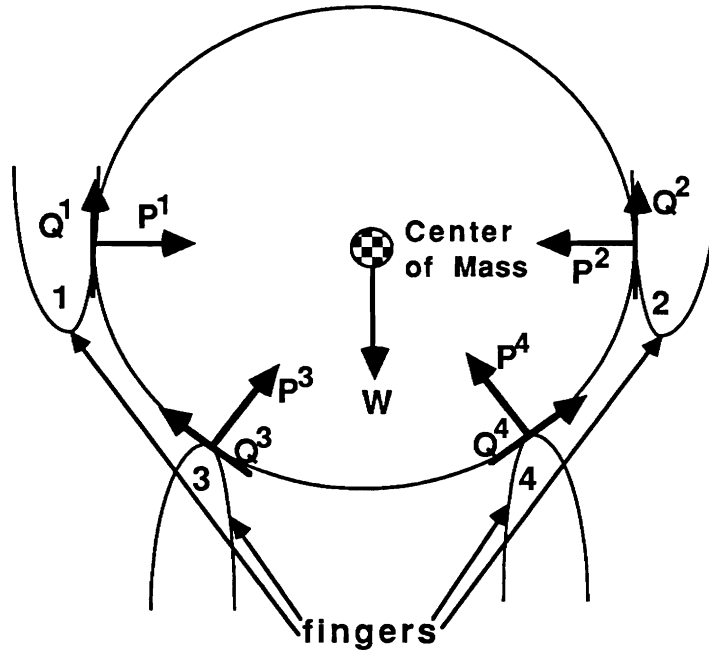


Figure 16: Transfer of a Disc

4.2.3 Example of Transfer of A Disc

The example considered here is that of a disc being initially grasped by two fingers, 3 and 4 (see figure 16), symmetrically positioned at a 45 degree angle as shown. The disc is now transferred to the grasp of fingers 1 and 2, that are positioned diametrically opposite. The results are shown in Figure 17. Moments were plotted for the tangential forces in this example because the tangential forces are a sum of discretized forces that do not all point in the same direction. This is due to the fact that they are tangent to the disc at each grid element. Therefore, it makes more sense to plot their moments.

In this example, the normal forces on finger 1 and 2 were increased linearly till the forces exerted by fingers 3 and 4 became zero. Once again, it turned out that $P^1 = P^2$ and $Q^1 = Q^2$ as was expected. Also, $P^3 = P^4$ and $Q^3 = Q^4$. The friction forces attained their maximum limiting values at each of the fingers.

4.2.4 Discussion

The examples of the previous two sections show that it is possible to use the contact stress model to calculate forces in an object transfer maneuver, where the problem is statically indeterminate and is characterized by continuously varying forces. In fact, the shift in forces from one system to another can be continuously predicted by the model and this would be of great use in controlling robot fingers when an object is handed from one hand to another.

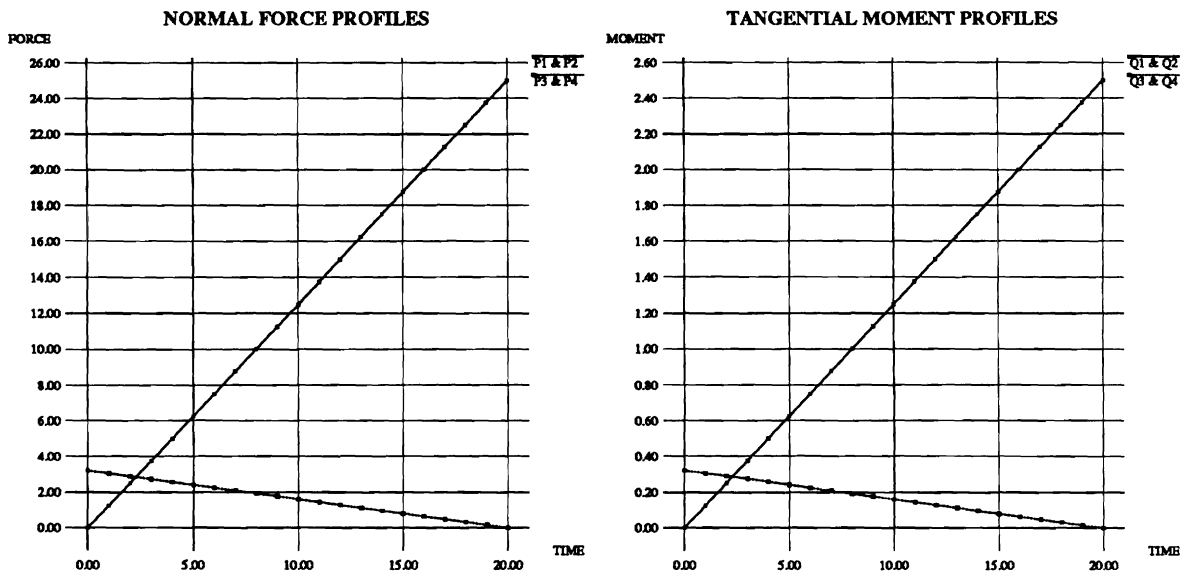


Figure 17: Changes in Forces and Moments during Transfer of a Disc

The changes in forces need not follow a particular profile, that is, they do not necessarily have to be increased or decreased in a linear fashion.

Chapter 5

Conclusion

In this chapter, we will summarize what has been presented in this thesis and point out some of the major contributions, along with the some of the advantages and disadvantages of using the proposed model.

5.1 Summary

In Chapter 1, a review of some of the available grasping models was presented. It was clear that these models have limitations that arise out of the inherent assumptions on which the models are based. The results of applying these models were shown to be sometimes phenomenologically or intuitively incorrect, and they did not seem to handle statically indeterminate problems very well. This provided the motivation for this thesis and the goal was to create a model that accurately modeled the surface interactions at the manipulator/object interface and eliminated some of the inconsistencies of the earlier models.

In Chapter 2, the model formulation was presented. The interactions at the manipulator/object interface were modeled using some concepts from classical elasticity theory, and were then coupled with the equations satisfying static equilibrium. Using the theorem of minimum potential energy, we ended up with a constrained minimization problem. The central idea was to minimize the total potential energy of the system subject to the constraint equations arising out of the compatibility and equilibrium conditions, to give the interaction forces at each finger tip.

In Chapter 3, the solution technique was presented. The problem to be solved is a nonlinear optimization problem subject to nonlinear, equality and inequality constraints. One of the few robust algorithms available to solve such a system the one by Schittkowski [Sch86] that has been implemented as subroutine in the IMSL library [FOR87]. That is the algorithm that was used and a sample formulation was done to show the complexity of the solution procedure.

In Chapter 4, some representative examples were solved in two and three dimensions. It was also demonstrated that the model can be used to solve for forces in the grasp transfer maneuver. In this case, the forces in the “releasing” grasp were specified and the forces in

the new grasp were solved for.

5.2 Contributions

At the beginning of this thesis the need was expressed to establish an accurate model of the mechanics of surface interaction at the contact between the grasped object and the manipulator. We have succeeded in developing a model based on elasticity theory which eliminates the indeterminacy of the rigid body model. There is still a limiting value on the frictional forces, except that now, by the inherent nature of the analysis, these forces are nonlocal and nonlinear in behavior and there is a strain energy of deformation associated with them. We now have a model that solves for the forces of interaction for statically determinate and indeterminate grasps which leads to results that are intuitively correct and satisfy symmetry conditions.

5.2.1 Advantages

- The object can be arbitrarily shaped.
- There is no limit on the number of fingers that grasp the object.
- There are no kinematic constraints on the system and the model does not require any assumptions related to the incipient motion of the object.
- The model can be used to predict grasping forces for materials of different properties. For example, it can be as easily applied to grasping a hard metal object as to grasping a highly deformable rubber object.
- The material properties of the fingers can be made different depending on which fingers we would not want to slip.
- This model could be very well used to find optimal grasps or predict the stability of grasps. This would give the advantage of having the same model for planning and executing the grasp as well as for making certain that the grasp can withstand external disturbances.
- The model can be adapted to solve for forces during grasp transfer and this has been shown in the previous chapter.

5.2.2 Disadvantages

- The method requires the knowledge of the material properties of both the manipulator and the object.
- Even though the object may be arbitrarily shaped, some knowledge of the local geometry around the points where the fingertips touch the object is required. For very smooth objects, the surface asperities will not need to be modeled accurately but for

rough objects the surface asperities would have to be modeled accurately, and this may be hard to do.

- Conjugate to the advantage of having an accurate model, the disadvantage is the requirement of heavy processing effort. There is a definite tradeoff between phenomenological accuracy and computational burden, and to determine the acceptable level of complexity one would need to evaluate this model by experimentation.

5.3 Future Research

This thesis provides a useful tool to examine forces generated in a typical grasping problem. However, the applicability of this method in transfer problems has not been fully shown. This model may be used to examine the very character of the transfer problem and inherent differences in the nature of the force changes and responses in the maneuvers, described as the “hand-off” or cooperative transfer and “taking away” or non-cooperative transfer.

This method could also be used to examine what kind of forces are generated in response to task-induced forces and displacements. For example, the model could be useful for determining forces when the grasped object runs into interference during assembly operations. In addition, this method would be very effective in fine motion or manipulation tasks, where an accurate model of the surface interactions is required.

While we have demonstrated how the model can predict the forces in a grasp that satisfies the conditions for static equilibrium, a possible extension of this analysis would be to apply it to grasp planning or to predict how compliant the fingertip should be to provide a stable grasp, or conversely how compliant the object should be, given fingers that have certain material properties. In other words, the theoretical model could be used to solve the reverse problem wherein given the forces, one would like to predict the material properties of the object.

Another possible area of research is that of applying this model to walking machines. The forces of interaction between the foot and the terrain could be analyzed using similar concepts. Such an analysis could be particularly useful since the terrain is almost always deformable and not a rigid body as most current models assume.

Finally, the greatest support to this thesis would be the experimental verification of the results - a favorable comparison between the results of the computer simulations presented here and the grasping forces generated in an actual grasp.

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