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# Review of the Literature on Time-Optimal Control of Robotic Manipulators

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# Review of the Literature on Time-Optimal Control of Robotic Manipulators

## **Abstract**

A task that robotic manipulators most frequently perform is motion between specified points in the working space. It is therefore important that these motions are efficient. The presence of the obstacles and other requirements of the task often require that the path is specified in advance. Robot actuators cannot generate unlimited forces/torques so it is reasonable to ask how to traverse the prespecified path in minimum time so that the limits on the actuator torques are not violated.

It can be shown that the motion which requires least time to traverse a path requires at least one actuator to operate on the boundary (maximum or minimum). Furthermore, if the path is parameterized, the equations describing the robot dynamics can be rewritten as functions of the path parameter and its first and second derivatives. In general, the actuator bounds will be transformed into the bounds on the acceleration along the path. These bounds will be functions of the velocity and position. It is possible to demonstrate that the optimal motion will be almost always bang-bang in acceleration. The task of finding the optimal torques thus reduces to finding the instants at which the acceleration will switch between the boundaries.

An algorithm for finding the time optimal motion along prespecified paths that explores this idea will be presented. It will be shown that so called singular arcs exist on which the algorithm will fail. Modification of the algorithm for such situations will be presented. Also, some properties of the solutions of the more general problem when the path is not known will be discussed. Lie-algebraic techniques will be shown to be a convenient tool for the study of such problems.

## **Comments**

University of Pennsylvania Institute for Research in Cognitive Science Technical Report No. IRCS-94-34.

# The Institute For Research In Cognitive Science

## Review of the Literature on Time-Optimal Control of Robotic Manipulators

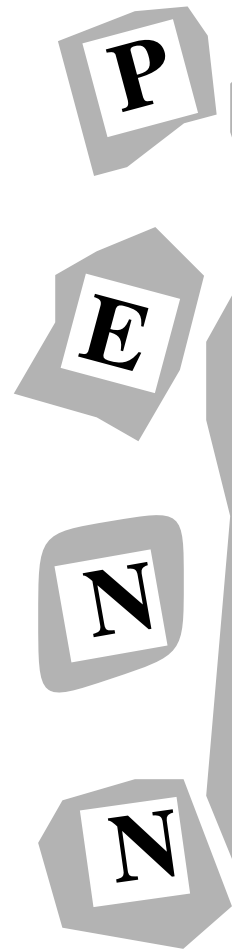
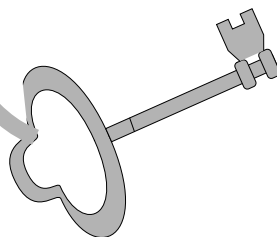
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December 1994

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# **Review of the Literature on Time-Optimal Control of Robotic Manipulators**

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April 1994

This report was submitted as a part of the qualifying requirements for the Ph.D. program of the Department of Computer and Information Science at the University of Pennsylvania. As such it does not represent the survey of all existing literature on the subject but just a selection of four papers chosen by the author.

## Abstract

A task that robotic manipulators most frequently perform is motion between specified points in the working space. It is therefore important that these motions are efficient. The presence of the obstacles and other requirements of the task often require that the path is specified in advance. Robot actuators cannot generate unlimited forces/torques so it is reasonable to ask how to traverse the prespecified path in minimum time so that the limits on the actuator torques are not violated.

It can be shown that the motion which requires least time to traverse a path requires at least one actuator to operate on the boundary (maximum or minimum). Furthermore, if the path is parameterized, the equations describing the robot dynamics can be rewritten as functions of the path parameter and its first and second derivatives. In general, the actuator bounds will be transformed into the bounds on the acceleration along the path. These bounds will be functions of the velocity and position. It is possible to demonstrate that the optimal motion will be almost always bang-bang in acceleration. The task of finding the optimal torques thus reduces to finding the instants at which the acceleration will switch between the boundaries.

An algorithm for finding the time-optimal motion along prespecified paths that explores this idea will be presented. It will be shown that so called singular arcs exist on which the algorithm will fail. Modification of the algorithm for such situations will be presented. Also, some properties of the solutions of the more general problem when the path is not known will be discussed. Lie-algebraic techniques will be shown to be a convenient tool for the study of such problems.

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# Chapter 1

## Introduction

### 1.1 Optimal control

When finding a control that would bring a mechanical system from an initial state to the desired goal configuration there are often many (usually at least continuum) possible solutions. In such cases it is desirable to find the solution which optimizes a certain criterion. Formally, the problem of finding the optimal control can be described as [13]:

**Problem 1** Consider an  $n$ -dimensional dynamical system

$$\dot{q} = f(q(t), u(t)) \quad (1.1)$$

and a functional

$$J = \psi(q(t_f)) + \int_{t_0}^{t_f} L(q(t), u(t)) dt \quad (1.2)$$

Functions  $f_i$  are defined on the direct product  $\mathbb{R}^n \times U$ , where  $U$  defines a set of admissible controls. Among all admissible controls  $u = u(t)$  which transfer the state of the system from  $q_0$  to  $q_f$ , find one for which the functional 1.2 reaches the least possible value. Time  $t_f$  need not be specified in advance.

Functions  $f$ ,  $L$  and  $\psi$  are assumed to be continuously differentiable. An autonomous system (without explicit dependence on time) is assumed in the formulation of the problem although the formulation could be easily generalized. Also, additional constraints that  $q$  and  $u$  must satisfy could be specified.

The optimal solution must satisfy necessary conditions given by the Pontryagin minimum principle [13]:

**Proposition 1 (Pontryagin Minimum Principle)** Define a Hamiltonian

$$H(q, u, \lambda) = L(q, u, t) + \lambda^T f(q, u, t) \quad (1.3)$$

where  $\lambda(t)$  is an  $n \times 1$  vector of adjoint variables. If the control  $u^*(t)$  is optimal and generates the trajectory  $q^*(t)$ , then there exist a nonzero solution  $\lambda^*(t)$  of the adjoint equations

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial q}\right)^T \lambda \quad (1.4)$$



such that for every  $t \in [t_0, t_f]$  and for every  $u \in U$

$$H(q^*, u^*, \lambda^*) \leq H(q^*, u, \lambda^*). \quad (1.5)$$

Furthermore, for every  $t \in [t_0, t_f]$

$$H(q^*, u^*, \lambda^*) = 0. \quad (1.6)$$

With the help of the Hamiltonian, the system equations and the adjoint equations can be rewritten in the following canonical form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial \lambda} \\ \dot{\lambda}^T &= -\frac{\partial H}{\partial q} \end{aligned} \quad (1.7)$$

## 1.2 Time-optimal control

If  $\psi = 0$  and  $L(q, u, t) = 1$  in the statement of Problem 1 the value of function  $J$  becomes exactly the time that the system takes to get from the initial state  $q_0$  to the desired state  $q_f$ . The input vector  $u$  that achieves this state transition is called time-optimal control. Pontryagin minimum principle describes how to compute the possible optimal solutions. Functions  $(q, u, \lambda)$  that satisfy the conditions of the minimum principle are called *extremals*. To calculate the extremals a boundary value problem must be solved for the set of equations 1.7. If additional constraints are imposed on  $q$  and  $u$ , a set of algebraic equations must be solved simultaneously with the boundary value problem. Finding the numerical solution of such a system is computationally very intensive. Furthermore, the Pontryagin minimum principle only states the necessary conditions; when the equations are nonlinear the extremals are not unique and additional tests are necessary to establish the optimality (Legendre-Clebsch test is an example of higher order conditions for optimality [13, 5]).

There are numerous engineering applications where time-optimal control is desired. Point-to-point motion of robotic manipulators is a typical example. In practice the size of the robot actuators is limited. More powerful motors are heavier and require more massive links. These in turn require higher torques to move. The size of the motors can thus not be increased over certain limit without decreasing the performance. To avoid the saturation of the motors the robot manufacturers usually impose quite conservative limits on the accelerations and velocities in their software. This means that point-to-point motions, which represent a large part of robotic operations, are far from being time-optimal.

## 1.3 Background

Kahn and Roth were the first to address time-optimal control of robotic manipulators [10]. A three-link serial mechanism with constant limits on the torques was studied. The path was not specified. The authors were able to show that at least one of the actuators will operate on the boundary. An approximate scheme based on the linearization of the robot dynamics was proposed to compute the optimal trajectories.

A number of robotic tasks requires separate path planning. In such cases finding the time-optimal trajectory consists of two phases: a) path planning and b) optimization of the movement along the chosen path. Extensive amount of work has been done on the path planning. An important problem that is usually addressed during path planning is obstacle avoidance. The

early methods for path planning were influenced by the lack of methods for the control of the motion along the path. Constant velocity and acceleration bounds were assumed and the path was usually composed of circular and straight-line segments [4].

In the last ten years the control of the manipulators along prespecified paths became better understood and efficient algorithms were subsequently developed. Bobrow et al. [4] and Shin & McKay [17] independently developed similar methods to compute the optimal control for serial manipulators moving along a given path. Dynamic equations of the manipulator were reduced to a set of second order differential equations in the path parameter. The bounds on the actuator torques were transformed to the bounds on the acceleration along the path. So called velocity limit curve was obtained from these limits which defines the boundary of the feasible set in the phase-plane. By assuming that the control is bang-bang in the acceleration the authors were able to propose a scheme to obtain the switching points. Furthermore, Shin & McKay established that the optimal control will require only finitely many switching points on the portion of the path where the acceleration is saturated.

This work was followed by Pfeiffer and Johanni [12]. They noticed some additional properties of the velocity limit curve which allowed them to further simplify the computation of the switching points. Huang and McClamroch [9] used the method for contour following. Slotine & Yang [18] were able to add additional limitations to the velocity limit curve so that the original algorithm became more efficient.

McCarthy & Bobrow [11] formulated the equations for manipulators with arbitrary kinematic configuration (serial chain, parallel chain or the combination of the two) and showed that the limits on the internal forces can be handled in the same way as the limits on the actuator torques. They demonstrated that the linear programming can be used to calculate the acceleration bounds. This enabled them to compute the number of actuators that must be saturated.

Chen & Desrochers [7] tried to formally prove that the time-optimal motion along the path will be bang-bang in the accelerations. They followed the approach from [4] and [17] to reduce dynamic equations to a set of differential equations in the path parameter and then used a generalization of the Pontryagin minimum principle to show that the control must be bang-bang.

Shiller & Lu and Shiller [14, 16] realized that under some circumstances the method proposed in [4] and [17] fails to give the correct answer. They showed that paths exist along which the time-optimal control will not be bang-bang in the acceleration. They characterized the points where the acceleration will not be on the limit and called them singular points (if isolated) or singular arcs (if connected). Furthermore, they devised the optimal control at such points. Their findings require a revision of some of the works that use the aforementioned reduction. In [11] the singular arcs are excluded from the derivations and the claims are limited to the paths without singular arcs. The proof in [7] fails on singular arcs and the theorems should be reevaluated. The work by Shiller & Lu also implies that at least one of the actuators will operate at the limit, although they did not explicitly state this fact.

A parallel line of research was conducted for the time-optimal control of manipulators in the case when the path is not known. Such problems are much more complex since the equations cannot be projected into two dimensions. Some of the path-planning algorithms attempt at obtaining the time-optimal path. A review of this literature has not been done, though.

An approach to numerical approximation of the solution is presented in Shiller and Dubowsky [15]. They discretized the task space and represented all possible paths in a graph. The graph was pruned by estimating the cost of the curves so that the number of candidates was reduced to a reasonable number. The remaining curves were approximated with B-splines and the algorithm from [14] used to compute the time-optimal trajectory with local optimization of the control

points of the B-splines.

The method proposed in [15] does not give much insight into the structure of the optimal control. The first theoretical work that studied the problem for mechanical systems was done by Ailon & Langholz [1]. They have shown that if there is an admissible control for the mechanical system, there is also an optimal control which transfers the system from the initial to the desired state in the minimum time. Furthermore, they demonstrated that for a two-link manipulator the optimal control will be such that for any time-instant one of the actuator torques will reach its minimum or its maximum. The authors built their proof on the theory of ordinary differential equations, some topological properties of the set of admissible controls and Pontryagin minimum principle.

More indepth investigation of the properties of the optimal control was done by Sontag & Sussmann [20, 19]. They studied trajectories that satisfy the conditions of Pontryagin minimum principle on which the so called switching functions are equal to 0. It was shown that this cannot be true for all the actuators, at least one will have bang-bang control. Lie-algebraic properties of mechanical systems were derived and some additional results were proved for the systems where all except one of the switching functions are 0. The results were applied to a 2R two-degree-of-freedom planar manipulator.

Fourquet [8] extended the work of Sontag and Sussmann. In particular, he further classified the singular trajectories. This led him to simplification of the results in [20] for the two-degree-of-freedom manipulator.

## 1.4 Outline of the report

First the method developed by Bobrow et al. [4] for computing the time-optimal control of serial manipulators along a given path will be presented. The general form of the dynamic equations for the robotic manipulators will be reviewed. It will be shown how the equations can be reduced to a set of second order differential equations in the path parameter when the path of the end-effector is prescribed. Acceleration limits will be derived and the velocity limit curve will be introduced. With the use of the phase plane and the velocity limit curve it will be shown how the switching points can be obtained, provided that the control is bang-bang in the acceleration along the path.

A discussion of the article by McCarthy & Bobrow that extends the original algorithm for a general type of robotic systems will follow. It will be shown that the internal forces can be also limited. The acceleration limits will be shown to be the solution of a linear program. General theorems from the theory of the linear programming will allow to determine the number of the saturated actuators.

Following will be the article by Shiller and Lu [14] which shows that the algorithm by Bobrow et al. fails if the projected inertia vector contains zero components. Singular points and singular arcs will be defined and it will be shown how to alter the algorithm on such segments of the path.

Finally, the paper by Sontag & Sussmann [19] that presents some general facts about the time-optimal control when the path is not known will be discussed. Lie-algebraic properties of the mechanical systems will be explained. Switching functions and singular extremals will be defined. Theorems that describe the structure of the time-optimal control when all but one switching functions are identically equal to 0 will be presented.

## Chapter 2

# Time-Optimal Control of Robotic Manipulators Along Specified Paths (Bobrow et al.)

Numerous tasks require the robot to follow a prescribed path. This might be required for example to avoid obstacles in the working space or to avoid collisions with other robots. The path only determines the geometric location of the points in space, the velocity profile of the motion along the path is left unspecified. In such cases it is often desirable to traverse the path in the least possible time. If the actuators would not have any torque limits the traversal time could be brought arbitrary close to zero. In all practical cases the power and therefore the torques that the actuators can deliver are limited. The motion along the path is governed by the dynamic equations which are nonlinear. Geometric properties of the path (curvature) will be reflected in different terms in dynamic equations (inertial, centrifugal and Coriolis forces) during the motion. Any algorithm that attempts at finding the time-optimal solution must therefore consider manipulator dynamics.

### 2.1 Robot dynamics along a specified path

Let's consider an  $n$ -degree-of-freedom serial manipulator. Configuration of the manipulator is given by an  $n \times 1$  vector of joint coordinates  $q$ . The equations of motion for the manipulator can be obtained using Lagrange's equations and have the form:

$$M(q)\ddot{q} + \dot{q}^T C(q)\dot{q} + G(q) = \tau, \quad (2.1)$$

where  $M(q)$  is an  $n \times n$  symmetric, positive definite inertia matrix,  $C(q)$  is an  $n \times n \times n$  tensor of centrifugal and Coriolis coefficients,  $G(q)$  is an  $n \times 1$  vector of gravity terms and  $\tau$  is an  $n \times 1$  vector of actuator torques. We have assumed frictionless joints.

Let  $r(s)$  be an  $n \times 1$  vector function which prescribes a path in the task space. A scalar variable  $s$  parameterizes the path. The range of the actuator torques is given by the following inequalities:

$$T_{min}^i \leq \tau_i \leq T_{max}^i, \quad (2.2)$$

where  $T_{min}^i$  and  $T_{max}^i$  are given constants.

The problem of finding the time-optimal motion along the given path can be stated as:

**Problem 2** Find a set of actuator torques that satisfy the set of inequalities 2.2 such that the system governed by Eq. 2.1 traverses the given path in minimum time.

The path  $r(s)$  prescribes the position of the end effector in the task coordinates. But the relation between the task coordinates and the joint coordinates is also given by direct kinematics  $\Psi(q)$ :

$$r(s) = \Psi(q) \quad (2.3)$$

When the direct kinematics is one-to-one, it is possible to express the vector of the joint coordinates as a function of  $s$ :

$$q = f(s) = \Psi^{-1}(r(s)) \quad (2.4)$$

Function  $f$  is an  $n \times 1$  vector function. From Eq. 2.4 we can obtain joint velocities and accelerations as functions of the parameter  $s$  and its derivatives:

$$\begin{aligned} \dot{q} &= f'(s)\dot{s} \\ \ddot{q} &= f''(s)\dot{s}^2 + f'(s)\ddot{s} \end{aligned} \quad (2.5)$$

Prime denotes derivatives to  $s$  and dot denotes time derivatives.

We can now introduce expressions 2.4 and 2.5 into equations of motion 2.1:

$$M(f)\{f''\dot{s}^2 + f'\ddot{s}\} + \{f'\dot{s}\}^T C(f)\{f'\dot{s}\} + G(f) = \tau \quad (2.6)$$

For brevity the explicit dependence of  $f$  on  $s$  has been omitted. Equation 2.6 can be rewritten as

$$m(s)\ddot{s} + c(s)\dot{s}^2 + g(s) = \tau \quad (2.7)$$

where  $m$ ,  $c$  and  $g$  are  $n \times 1$  vectors given by

$$\begin{aligned} m(s) &= M(f)f' \\ c(s) &= f'^T C(f)f' + M(f)f'' \\ g(s) &= G(f) \end{aligned} \quad (2.8)$$

We will refer to the vector  $m(s)$  as *the projected inertia vector*. Torque constraints (Eq. 2.2) can now be rewritten as

$$T_{min}^i \leq m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq T_{max}^i, \quad (2.9)$$

Matrix  $M$  was positive definite and we shall assume that the prescribed path  $r(s)$  is regular (the tangent vector  $f'(s)$  is nonzero). The projected inertia vector  $m(s)$  is therefore nonzero. If the  $i^{th}$  component of  $m$  is nonzero then the corresponding inequality can be rewritten as

$$L_i(s, \dot{s}) \leq \ddot{s} \leq U_i(s, \dot{s}) \quad (2.10)$$

where

$$L_i(s, \dot{s}) = \begin{cases} (T_{min}^i - c_i\dot{s}^2 - g_i)/m_i, & \text{if } m_i > 0, \\ (T_{max}^i - c_i\dot{s}^2 - g_i)/m_i, & \text{if } m_i < 0, \end{cases} \quad (2.11)$$

and

$$U_i(s, \dot{s}) = \begin{cases} (T_{max}^i - c_i\dot{s}^2 - g_i)/m_i, & \text{if } m_i > 0, \\ (T_{min}^i - c_i\dot{s}^2 - g_i)/m_i, & \text{if } m_i < 0. \end{cases} \quad (2.12)$$

If  $m_i = 0$ , the authors simply omit the corresponding inequality since it does not impose any constraints on the acceleration. However, as it will be seen in Ch. 4 such cases may lead to the so called singular arcs where the algorithm will fail.

Equations 2.10 define the range of the acceleration (as function of the position and the velocity along the path) for which the manipulator can be held on the path without violating any of the torque constraints 2.2. Each equation defines an interval of admissible accelerations. If the intervals do not have common intersection the manipulator will leave the path instantaneously. However, if the intersection is nonempty, it defines a set of admissible accelerations. The set is defined by:

$$L(s, \dot{s}) \leq \ddot{s} \leq U(s, \dot{s}) \quad (2.13)$$

where

$$L(s, \dot{s}) = \max_i L_i(s, \dot{s}), \quad (2.14)$$

and

$$U(s, \dot{s}) = \min_i U_i(s, \dot{s}). \quad (2.15)$$

The optimal control problem can now be restated as:

**Problem 3** *Given a system*

$$\ddot{s} = u(t) \quad (2.16)$$

*find a control  $u(t)$  which belongs to the admissible set*

$$L(s, \dot{s}) \leq u \leq U(s, \dot{s}) \quad (2.17)$$

*that transfers the system from a given initial position  $s_0$  to a desired final position  $s_f$  in a minimum time.*

Note that the control input is the acceleration along the path and that it uniquely determines the actuator torques through Eq. 2.7.

It is not difficult to see that the solution for the problem above must be bang-bang in the input variable  $u$  [5]. If we can prove that there are only finitely many switching points for control  $u$ , the problem of finding the optimal control reduces to finding the switching points.

## 2.2 Algorithm for finding the switching points

The cost function was originally defined as:

$$J = \int_{t_0}^{t_f} 1 dt \quad (2.18)$$

It is useful to reformulate the cost function by observing that

$$\dot{s} = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{\dot{s}} \quad (2.19)$$

Eq. 2.18 can be then rewritten as

$$J = \int_{t_0}^{t_f} \frac{ds}{\dot{s}} \quad (2.20)$$

The trajectory of the system 2.16 can be represented in the  $s$ - $\dot{s}$  phase plane. If the optimal control is bang-bang the trajectory must be at each point tangential to one of the two directions

defined by  $L(s, \dot{s})$  and  $U(s, \dot{s})$ . However, according to Eq. 2.17 the trajectory must lie in the admissible set where  $L(s, \dot{s}) \leq U(s, \dot{s})$  (unshaded area on Fig. 2.1). The boundary of this set is called *velocity limit curve* and is defined by equation

$$L(s, \dot{s}) = U(s, \dot{s}). \quad (2.21)$$

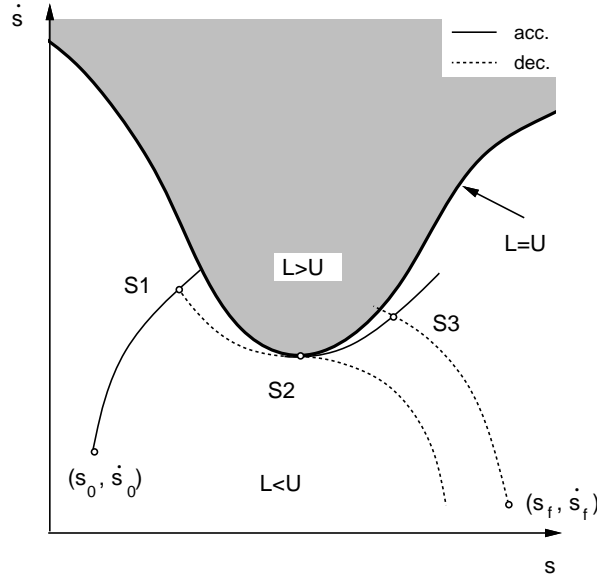


Figure 2.1: Velocity limit curve bounds the admissible region in the phase plane. Switching points occur (points  $S1$ ,  $S2$ , and  $S3$ ) when the trajectory switches from the acceleration to the deceleration.

According to Eq. 2.20 we should find a trajectory that lies entirely within the admissible set, satisfies the tangency constraint at each point and has the property that the velocity at each point is greater than the velocity on any other admissible trajectory. The switching points will occur where the trajectory switches between the acceleration and deceleration (points  $S1$ ,  $S2$  and  $S3$  on Fig. 2.1).

Consider first the case when there is only one switching point (Fig. 2.2). We know that the trajectory starts with the maximum acceleration and then switches to maximum deceleration. This suggests that we should integrate the equation

$$\ddot{s} = U(s, \dot{s}) \quad (2.22)$$

forward in time starting with the initial point  $(s_0, \dot{s}_0)$  (the initial and final velocity need not be necessarily 0), and the equation

$$\ddot{s} = L(s, \dot{s}) \quad (2.23)$$

backward in time starting with the final state  $(s_f, \dot{s}_f)$ . The switching point is given by the intersection of the two trajectories.

The algorithm that gives the switching points in the general case also uses the idea of integrating forward and backward in time with the maximum acceleration or the maximum deceleration. Velocity limit curve determines when to switch. As already said, the velocity

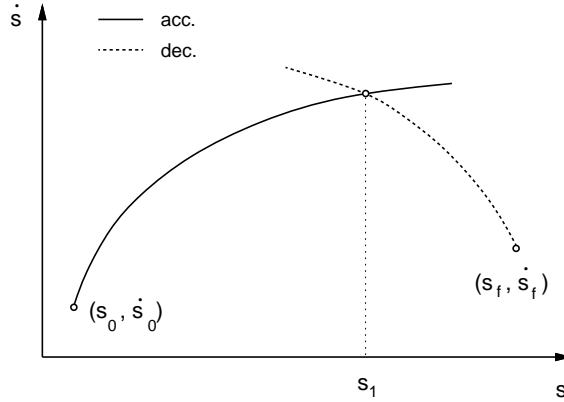


Figure 2.2: Obtaining the switching point when there is only one.

limit curve bounds the subset of the phase plane to which the trajectory must belong if the manipulator is to follow the path without violating actuator constraints. The trajectory is only allowed to be tangent to velocity limit curve, it cannot intersect it. Therefore the algorithm can be formulated as:

*Step 1* Integrate Eq. 2.22 forward starting at the initial point  $(s_0, \dot{s}_0)$  and Eq. 2.23 backward starting at  $(s_f, \dot{s}_f)$  to obtain segments  $F$  and  $B$ , respectively (Fig. 2.3). If the two intersect before intersecting the velocity limit curve there is only one switching point and we found it. Otherwise proceed to Step 2.

*Step 2* Suppose that the segment  $F$  intersects the velocity limit curve at  $(s_1, \dot{s}_1)$ . After that point the Eq. 2.17 is violated so the switching point is somewhere on the interval  $[s_0, s_1]$ . Let's pick  $s \in [s_0, s_1]$  and integrate Eq. 2.23 forward in time starting at the point on  $F$  which corresponds to  $s$ . One of the following can happen (Fig. 2.3):

- The trajectory will intersect the velocity limit curve at  $s = s_i$  (curve 1 on Fig. 2.3). This means that  $s$  was too big, since any admissible acceleration at  $s_i$  will force the manipulator to leave the path.
- The trajectory intersects the horizontal line  $\dot{s} = \dot{s}_f$  (curve 2 on Fig. 2.3). This means that  $s$  was too small: we could increase  $s$  for some  $\epsilon > 0$  so that the trajectory starting on  $F$  at  $s + \epsilon$  would still not intersect the velocity limit curve and which would obviously give shorter time according to Eq. 2.20.
- The trajectory is tangent to the velocity limit curve at some point  $(s_2, \dot{s}_2)$  and reaches the horizontal line  $\dot{s} = \dot{s}_f$  afterwards (curve 3 on Fig. 2.3). Then  $s$  is a switching point and  $s_2$  will be a new switching point.

We must therefore find  $s$  for which the third case will occur. Then the point of tangency  $(s_2, \dot{s}_2)$  can be taken as a new initial point and the algorithm repeated at Step 1.

An useful simplification that reduces the computation time is to rewrite Eq. 2.22 and 2.23 as

$$d\dot{s}/ds = U(s, \dot{s})/\dot{s} \quad (2.24)$$

$$d\dot{s}/ds = L(s, \dot{s})/\dot{s} \quad (2.25)$$



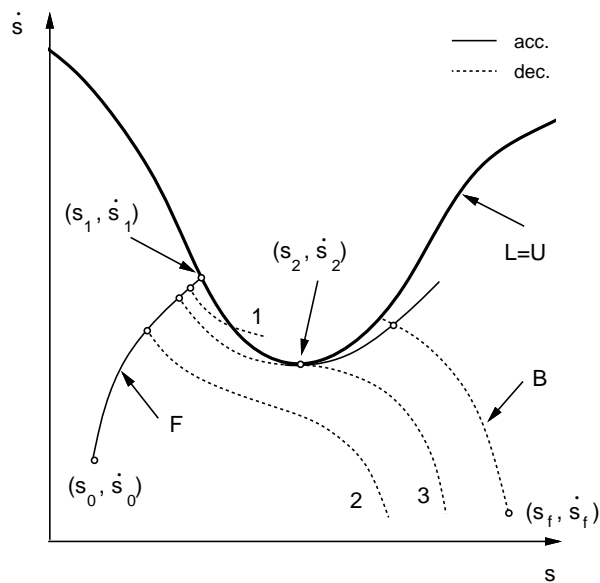


Figure 2.3: Finding the switching point.

It is worth noting that once the optimal trajectory for the interval  $[s_0, s_f]$  is known the optimal trajectory for any subinterval can be easily found: one just has to integrate Eq. 2.22 forward in time and Eq. 2.23 backward in time until the segments intersect the optimal trajectory. The optimal trajectory for the subinterval is obtained by taking the new segments together with the part of the old optimal trajectory between the intersections. Because of this property the optimal trajectory is also called *the switching curve*.

## 2.3 Critique

The authors were able to greatly simplify the original problem by observing that the equations of motion of the manipulator can be reduced to a set of second order differential equations in the path parameter. This, together with the transformation of the torque limits to the acceleration limits, transforms the problem of finding time-optimal control to much simpler problem in two dimensions. The acceleration limits suggest the definition of the velocity limit curve and lead to the idea of constructing the optimal trajectory in the  $s$ - $\dot{s}$  phase plane, which is the main contribution of the paper. The algorithm is quite intuitive and easy to understand, although it is not computationally as efficient and general as the algorithm described in [17].

The authors tried to formally prove the optimality of the resulting trajectory. They had to resort to some further assumptions about the optimal control in order to accomplish this. This assumptions reveal some weaknesses of the method. The assumptions are:

- The acceleration at any point along the path will be either equal to  $L(s, \dot{s})$  or to  $U(s, \dot{s})$ .
- The acceleration will switch only finitely many times between the above two values.
- The velocity limit curve is unique.

The first assumption is based on the fact that the optimal control of the system described by Eq. 2.16 and constraint equations 2.17 has to be bang-bang to satisfy Pontryagin minimum

principle. However, the authors did not correctly interpret the case when some of the components of the projected inertia vector  $m$  (Eq. 2.8) are equal to zero. In such case they simply disregarded the corresponding constraint equation. The constraint equations were derived from the corresponding actuator limits. Therefore, when actuator  $i$  is the limiting actuator and the corresponding component  $m_i$  is zero, the correct limiting equation is (compare with Eq. 2.9):

$$T_{min}^i \leq c_i(s)\dot{s}^2 + g_i(s) \leq T_{max}^i. \quad (2.26)$$

This equation does not occur in the statement of the Problem 3 which means that the problem is not equivalent to the original Problem 2. As a consequence it could happen that some of the actuator limits become violated. It will be explained in Ch. 4 how the correct formulation of Problem 3 admits optimal trajectories that are not bang-bang in the acceleration  $\ddot{s}$ . However, in most practical cases such anomalies do not occur and the original algorithm will perform well.

The authors did not try to substantiate the second assumption although the proof of optimality could not be completed without it. The proof itself can be carried out in a different way (as e.g. in [17]) but if the assumption is not true the algorithm will never stop. It is therefore necessary to establish whether there are cases when the acceleration would switch infinitely many times between the two boundaries during the motion (such phenomenon is called chattering). Shin and McKay [17] were able to prove that there will be only finitely many switching points on the optimal trajectory provided that the torque limits are analytic functions of  $s$  and  $\dot{s}$ . Finding the switching points can then be shown to be equivalent to finding zeroes of an analytic function. Analytic functions only have finitely many (isolated) zeroes so there will be only finitely many switching points. Of course, the optimal trajectory is unique so the properties of the trajectory do not depend on the particular algorithm.

The third assumption is not explicitly stated. However, the algorithm will fail if the admissible region in the phase-plane is not simply connected. The authors claim that their algorithm allows the torque limits to be arbitrary functions of the joint positions and velocities. This is not true, since the algorithm in its present form fails if the velocity limit curve is not unique. The algorithm by Shin & McKay is more general in this respect since it works for the cases when there are inadmissible islands in the admissible set.

## Chapter 3

# The Number of Saturated Actuators and Constraint Forces During Time-Optimal Movement of a General Robotic System (McCarthy & Bobrow)

The algorithm presented in the previous chapter was developed for serial mechanisms and unconstrained motion. A question arises whether it can be extended for more general structures and situations when additional constraints limit the motion of the manipulator. This is becoming an increasingly important issue if the applicability of serial manipulators is to be enhanced. Contacts with the environment are an integral part of any robot application and dynamic equations must be properly modified to describe such interactions. In such cases it is often necessary to impose limits on the internal forces. Examples include multiple arms holding an object. The forces on the object are usually limited and additional constraints are necessary to ensure that the object is firmly grasped.

### 3.1 Constrained dynamic systems

When the manipulator interacts with the environment the equations of motion must include the constraint forces. Formally, the constraints can be adjoined to the Lagrange equations using the method of Lagrange multipliers. It will be shown that the dynamic equations can still be reduced to a two dimensional space and that the algorithm from Ch. 2 can be used to obtain the optimal path.

The general form of the equations for the constrained dynamic system is

$$M(q)\ddot{q} + \dot{q}^T C(q)\dot{q} + G(q) = B(q)\tau + J(q)^T \lambda \quad (3.1)$$

Symbols  $M$ ,  $C$  and  $G$  are as in Eq. 2.1,  $\tau$  is now a  $p \times 1$  vector of actuator torques,  $B(q)$  is an  $n \times p$  matrix describing how the torques act on the configuration coordinates  $q$ ,  $\lambda$  is an  $m \times 1$  vector of Lagrange multipliers and  $J(q)$  is an  $m \times n$  matrix defining the velocity constraints

$$J(q)\dot{q} = 0. \quad (3.2)$$

Vector  $\lambda$  denotes the forces that are required to maintain the constraints.

In addition to the limits on the actuator torques

$$T_{min}^i \leq \tau_i \leq T_{max}^i, \quad (3.3)$$

one can also limit the constraint forces:

$$\Lambda_{min}^j \leq \lambda_j \leq \Lambda_{max}^j. \quad (3.4)$$

By using the procedure from section 2.1, equations 3.1 can be reduced to a system of  $n$  equations in  $s$

$$m(s)\ddot{s} + c(s)\dot{s}^2 + g(s) = b(s)\tau + j(s)^T\lambda \quad (3.5)$$

Quantities  $m$ ,  $c$  and  $g$  are defined in Eq. 2.8. In addition

$$\begin{aligned} b(s) &= B(f), \\ j(s) &= J(f), \end{aligned} \quad (3.6)$$

where  $f$  is the given path (see Eq. 2.4).

Equations 3.5 are linear in  $\ddot{s}$ ,  $\tau$  and  $\lambda$ . Furthermore, the constraints (Eq. 3.3 and 3.4) are also linear in these variables. The maximum allowable acceleration  $U(s, \dot{s})$  can therefore be obtained by finding

$$\max \ddot{s} \quad (3.7)$$

subject to equations 3.5 and inequality constraints 3.3 and 3.4. The minimum allowable acceleration  $L(s, \dot{s})$  can be obtained in the same way by replacing  $\max$  with  $\min$  in Eq. 3.7.

## 3.2 Linear program

The presented problem is a typical linear programming problem. Introduce

$$\begin{aligned} x_i &= \tau_i - T_{min}^i &> 0, & \quad i = 1, \dots, p \\ x_{i+p} &= T_{max}^i - \tau_i &> 0, & \quad i = 1, \dots, p \\ x_{j+2p} &= \lambda_j - \Lambda_{min}^j &> 0, & \quad j = 1, \dots, m \\ x_{j+2p+m} &= \Lambda_{max}^j - \lambda_j &> 0, & \quad j = 1, \dots, m \\ x_{2(p+m)+1} &\geq 0, \\ x_{2(p+m)+2} &\geq 0. \end{aligned} \quad (3.8)$$

with additional relations

$$\begin{aligned} x_i + x_{i+p} &= T_{max}^i - T_{min}^i, & \quad i = 1, \dots, p \\ x_{j+2p} + x_{j+2p+m} &= \Lambda_{max}^j - \Lambda_{min}^j, & \quad j = 1, \dots, m \\ x_{2(p+m)+1} - x_{2(p+m)+2} &= \ddot{s} \end{aligned} \quad (3.9)$$

Expressing  $\tau$ ,  $\lambda$  and  $\ddot{s}$  with  $x$  and substituting into Eq. 3.5 we obtain a standard form of the linear program with  $2(p+m+1)$  variables and  $p+m+1+n$  equations.

The theory of linear programming says that at least  $2(p+m+1) - (p+m+1+n) = p+m+1-n$  variables will be equal to 0 in the optimal solution (which means that they will lie on the boundary). Since an arbitrary constant can be added to  $x_{2(p+m)+1}$  and  $x_{2(p+m)+2}$  without changing the maximum (minimum) value of  $\ddot{s}$ , the variables which are equal to 0 must be among  $x_1, \dots, x_{2(p+m)}$ . This is equivalent to saying that at least  $p+m+1-n$  torques or internal forces will lie on the boundary.

For the special case of a serial link manipulator with the same number of links and actuators,  $p = n$  and  $m = 0$ . The result thus implies that at least one of the actuators has to be saturated.

### 3.3 Critique

The article extends and formalizes the work from [4]. It is shown that the method is applicable for robotic mechanisms with arbitrary structure. Also, constraints can be adjoined to the equations of motion with Lagrange multipliers. This allows more convenient derivation of the dynamic equations and also accommodates constraints on the internal forces in addition to the limits on the actuator torques. It is shown that when the path of the system is prescribed the equations of motion can be reduced to the system of second order differential equations in the path parameter  $s$ . Therefore, the dimension of the system is reduced from  $2n$  to  $2$  ( $n$  is the number of generalized coordinates). The resulting system is linear in the acceleration  $\ddot{s}$ , torques  $\tau$  and internal forces  $\lambda$ . The constraints are also linear. Therefore, finding the maximum and minimum allowable acceleration as a function of  $s$  and  $\dot{s}$  becomes equivalent to solving a linear program. From the theory of linear programming it follows that  $p + m + 1 - n$  internal forces and torques will be on the boundary, where  $n$  is the number of generalized coordinates,  $p$  the number of actuators and  $m$  the number of (holonomic or nonholonomic) constraints. This implies that for the case of non-redundant  $n$ -link serial manipulator ( $p = n$  and  $m = 0$ ) at least one of the actuators will be saturated.

The assumptions of the article follow those in [4]. Therefore, the results are not valid on singular arcs (see Ch. 4) where the dynamic relation between the velocity and the acceleration must be used to determine the acceleration. Linear programming cannot accommodate such relations.

The presentation generalizes the method of reducing equations of motion to equations in path parameter. However, the algorithms presented in [4] or [17] are based on the assumption that expressions for the acceleration limits are analytic. This allows efficient computation of the velocity limit curve. The limit curve obtained with the presented method will be given numerically so the applicability of the method in these algorithms is limited.

## Chapter 4

# Computation of Path Constrained Time Optimal Motions With Dynamic Singularities (Shiller & Lu)

In some instances the methods presented thus far fail to give the correct result. The transformation of the Problem 2 to Problem 3 (Sec. 2.1) is consistent only when all the actuator limits can be converted to appropriate acceleration bounds. When the components of the projected inertia vector (Eq. 2.8) become zero the corresponding actuator will not directly limit the acceleration but rather the velocity. The acceleration will be indirectly limited through the dynamics of the system.

### 4.1 Critical and singular points and arcs

Let's state the problem of time-optimal control again:

**Problem 4** *Given a trajectory  $q = f(s)$  minimize the time*

$$J = \int_{t_0}^{t_f} 1 dt, \quad (4.1)$$

*that the system described by equations*

$$M(q)\ddot{q} + \dot{q}^T C(q)\dot{q} + G(q) = \tau \quad (4.2)$$

*takes to transfer from the initial configuration  $q_0$  to the desired configuration  $q_f$  so that the following constraints on the actuator torques are satisfied:*

$$T_{min}^i \leq \tau \leq T_{max}^i. \quad (4.3)$$

It was shown in chapters 2 and 3 that the problem above can be reduced to the following form:

**Problem 5** *Minimize the time that the system*

$$\ddot{s} = u \quad (4.4)$$

*needs to reach the final position  $s_f$  from the initial position  $s_0$  subject to the constraints*

$$T_{min}^i \leq m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq T_{max}^i. \quad (4.5)$$

For the definitions of symbols see section 2.1.

Note that the constraints 4.5 are linear in  $\dot{s}^2$  and  $\ddot{s}$  and therefore define a polygon in the  $\dot{s}^2$ - $\ddot{s}$  phase plane (Fig. 4.1). The acceleration limits  $L(q, \dot{q})$  and  $U(q, \dot{q})$  were defined in Eq. 2.14 and 2.15. On the Fig. 4.1 they correspond to the lower and upper edge of the polygon, respectively. By solving  $L(q, \dot{q}) = U(q, \dot{q})$  the velocity limit curve  $\dot{s}_{max}(s)$  is obtained. This velocity corresponds to the rightmost vertex of the polygon (point  $\dot{s}_a^2$  on the abscissa). The figure shows a slice of the allowable space for  $s = const$ .

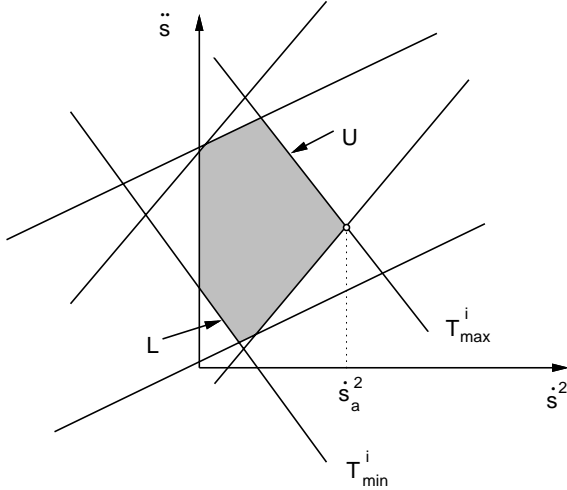


Figure 4.1: The admissible region in the  $\dot{s}^2$ - $\ddot{s}$  plane.

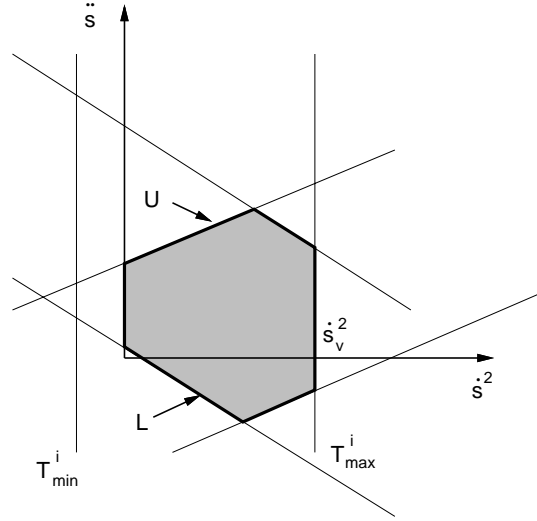


Figure 4.2: Admissible polygon at a critical point.

In some cases the edges of the polygon become vertical (Fig. 4.2). This happens at the degenerate points where  $m_i = 0$  for some  $i$ . The corresponding constraint equation becomes:

$$T_{min}^i \leq c_i(s)\dot{s}^2 + g_i(s) \leq T_{max}^i. \quad (4.6)$$

The inequality defines a limit on the velocity

$$\dot{s} \leq \dot{s}_v^i \quad (4.7)$$

where

$$\dot{s}_v^i(s) = \begin{cases} (T_{max}^i - g_i)/c_i, & \text{if } c_i > 0, \\ (T_{min}^i - g_i)/c_i, & \text{if } c_i < 0, \end{cases} \quad (4.8)$$

If the velocity limit 4.7 is not observed the manipulator will either leave the path or violate the torque limit. The methods from the previous two chapters incorrectly handle such cases since they disregard the velocity limits resulting from the zero components of the projected inertia vector. The problem can be easily alleviated by redefining the velocity limit curve:

$$\dot{s}_{max}(s) = \min\{\dot{s}_a(s), \dot{s}_v^i(s)\} \quad (4.9)$$

When  $\dot{s}_{max}(s) = \dot{s}_v^i(s)$  the  $i^{th}$  actuator becomes saturated and we have an additional bound on the maximum acceleration:

$$\ddot{s} \leq \ddot{s}_{max} \quad (4.10)$$

where

$$\ddot{s}_{max} = \frac{d\dot{s}_{max}}{dt} = \frac{\partial \dot{s}_{max}}{\partial s} \dot{s}_{max} \quad (4.11)$$

If such a point is isolated it is called a *critical point*, otherwise we are on a *critical arc*. When the acceleration is actually limited by this value,  $\ddot{s}_{max} < U(q, \dot{q})$ , the maximum acceleration  $U(q, \dot{q})$  cannot be used without violating the constraints. Such point is called a *singular point* and if we have a connected subset of singular points they form a *singular arc*. At isolated singular points the derivative in 4.11 need not be defined and the right limit must be used. In any case the trajectory will slide along the velocity limit curve until a point is reached which is not singular.

In general the velocity limit curve will consist of three types of points (Fig. 4.3):

**Regular points:** At the regular points the velocity limit is given by equation  $L(q, \dot{q}) = U(q, \dot{q})$  and only a single value of the acceleration is admissible. Point *A* on Fig. 4.3 is a regular point.

**Critical points:** The velocity limit at critical points is given by Eq. 4.6. The values of  $L(q, \dot{q})$  and  $U(q, \dot{q})$  are not equal. However, the acceleration given by the velocity limit (Eq. 4.11) is greater than  $U(q, \dot{q})$  so  $U(q, \dot{q})$  can be used as maximum acceleration. On Fig. 4.3 *B* is a critical point.

**Singular points:** The velocity limit at singular points is also given by Eq. 4.6 and the values of  $L(q, \dot{q})$  and  $U(q, \dot{q})$  are not equal. But in this case the acceleration bound given by Eq. 4.11 is smaller than  $U(q, \dot{q})$  and it must be used as maximum acceleration. Point *C* on Fig. 4.3 is a singular point.

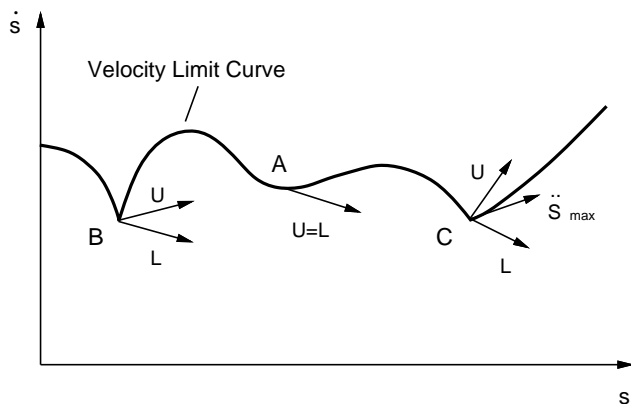


Figure 4.3: Regular point (*A*), critical point (*B*), and singular point (*C*).

## 4.2 Modified algorithm

Geometric properties of the velocity limit curve suggest an algorithm which is computationally more efficient than the algorithm proposed by Bobrow et al. in [4]. At every point in the phase plane the acceleration is limited by  $L(q, \dot{q})$  and  $U(q, \dot{q})$ . These two tangential directions define a cone of feasible accelerations (Fig. 4.4). At a regular point on the velocity limit curve the two acceleration bounds are equal and the cone degenerates to a single vector. At a singular point the maximum feasible acceleration is defined by the tangent to the velocity limit curve and it



lies inside the cone. The optimal trajectory will therefore follow the singular arcs. At a critical point the cone is defined but the maximum feasible acceleration is given by  $U(q, \dot{q})$ . Above the velocity limit curve no feasible acceleration exists.

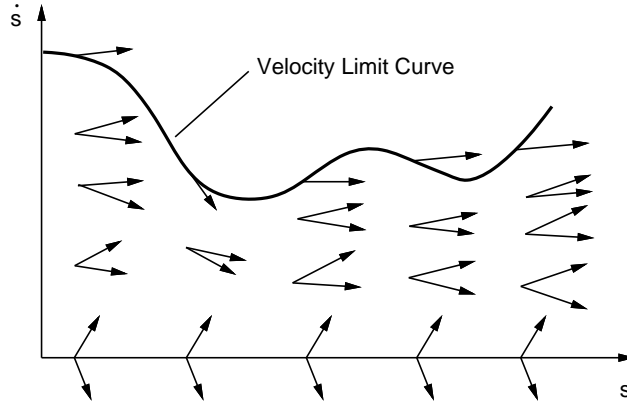


Figure 4.4: Cone defining the feasible accelerations.

Parts of the velocity limit curve that are regular will determine the switching points. At such points it is useful to know the difference between the feasible acceleration and the acceleration which would force the trajectory to be tangent to the velocity limit curve

$$\phi(s) = U(s, \dot{s}) - \ddot{s}_{max} = L(s, \dot{s}) - \ddot{s}_{max} \quad (4.12)$$

The sign of  $\phi(s)$  defines where we can move from the velocity limit curve. If  $\phi(s) > 0$ , the admissible acceleration will drive the manipulator off the trajectory. Such points are called *sinks*. When  $\phi(s) < 0$ , the permissible acceleration will force us to leave the velocity limit curve, therefore such points are called *sources*. Where  $\phi(s)$  switches sign, more precisely, switches from sink to source, we have a switching point.

We can now formulate the algorithm for finding the optimal trajectory:

*Step 1* From the initial point  $(s_0, \dot{s}_0)$  integrate forward with the maximum feasible acceleration.

If the starting point is not on the velocity limit curve or if it is a regular or a critical point on the velocity limit curve, this will be  $U(s, \dot{s})$ . If the point is singular the acceleration is given by Eq. 4.11.

If the trajectory passes the final point  $s_f$ , go to Step 4. Otherwise the trajectory has hit the velocity limit curve at some point  $(s_1, \dot{s}_1)$ , go to Step 2.

*Step 2* If the point  $(s_1, \dot{s}_1)$  is regular then go to Step 3. Otherwise we are either at a critical or a singular point. Take the point  $(s_1, \dot{s}_1)$  as a new initial point and go to Step 1.

*Step 3* Along the velocity limit curve search for the point  $(s_2, \dot{s}_2)$  where  $\phi(s)$  changes sign.

This is a switching point. From the point  $(s_2, \dot{s}_2)$  integrate backward with the maximum feasible deceleration  $L(q, \dot{q})$  until the trajectory intersects previously obtained segments of the trajectory. The point of intersection is a switching point. Then take  $(s_2, \dot{s}_2)$  as a new initial point and go to Step 1.

*Step 4* From the final point  $(s_f, \dot{s}_f)$  integrate backward with  $L(q, \dot{q})$  until intersecting the previously obtained segments of the trajectory.

### 4.3 Geometric characterization of the singular arcs

Although the theoretical investigation predicts the singular arcs it is not clear whether they can really occur. Recall that the singular arcs occur when one of the components of the projected inertia vector  $m(s)$  becomes 0. Eq. 2.8 suggests

$$m(s) = M(f)f' = [M_1 f', \dots, M_n f'] \quad (4.13)$$

where  $M_i$  is the  $i^{\text{th}}$  row of the inertia matrix. The possible locations of the critical points are the curves where one of the components of  $m(s)$  becomes 0. Since vectors  $M_i$  are not zero, these are the curves for which the tangent vector is perpendicular to the corresponding row  $M_i$ . Such curves are called *zero inertia curves* and can be obtained by solving a differential equation

$$M_i \sigma'_i = 0. \quad (4.14)$$

for the function  $\sigma_i(s)$ . Points where the path is tangent to zero inertia curve  $\sigma_i$  are possible critical points. Critical arc can occur when a segment of the path matches the zero inertia curve  $\sigma_i$ . The necessary condition for occurrence of critical points and arcs is that the actuator  $i$  becomes the saturated actuator (which means that the vertical line is an edge of the polygon on the figure 4.1). Furthermore, if the maximum admissible acceleration  $U(s, \dot{s})$  is greater than the acceleration given by Eq. 4.11, critical points and arcs become singular.

### 4.4 Critique

The paper by Shiller & Lu is a very detailed exposition of the issues involved in the time-optimal control of the path-constrained motion of robotic manipulators. The authors reinterpret the findings in [4], [17] and [12] and generalize the methods and algorithms which were presented there. By geometrical reasoning about the polygon representing the permissible region in the  $\dot{s}^2$ - $\ddot{s}$  plane they explain what the maximal and the minimal accelerations are. They show that the point of the velocity limit curve corresponds to a vertex of the polygon. It is explained how the vertex can degenerate to an edge leading to the appearance of the critical points and arcs. By studying the constraints on the critical arcs the authors observe that the maximum acceleration could be limited by an expression that depends on the dynamics (Eq. 4.11) and not only on the state of the system. In some cases this acceleration limit leads to the singular points and arcs. On such segments the acceleration is not given by the bounds  $L(s, \dot{s})$  and  $U(s, \dot{s})$  as was wrongly assumed in the previous works. The authors propose a corrected version of the algorithm presented in [12] to account for singular points and arcs. The algorithm is therefore the most general solution for obtaining the time-optimal control along the prespecified path.

In the paper it is assumed that the actuator bounds are constant. The resulting admissible set in the  $\dot{s}^2$ - $\ddot{s}$  plane is a polygon and as a consequence the admissible region in the  $s$ - $\dot{s}$  plane is simply connected. More general torque constraints result in topologically more complicated regions, both in  $\dot{s}^2$ - $\ddot{s}$  and in  $s$ - $\dot{s}$  plane. In such cases the geometric interpretation of the critical and singular points is still valid. However, the proposed algorithm is not general enough for the regions which are not simply connected. In practice such regions will occur if the friction is included in the dynamic equations or if more realistic non-constant torque limits are used. It might be therefore more appropriate to modify the algorithm presented in [17] which accommodates admissible regions with more complex topology.

## Chapter 5

# Time-Optimal Control of Manipulators (Sontag & Sussmann)

In the previous chapters we have been studying time-optimal control along a given path. A path is usually given by an off-line obstacle-avoidance algorithm or some other path-planning method. We have seen that efficient algorithms exist which construct the time-optimal trajectory.

There is no equivalent procedure that would give the optimal solution in the general case when also the path has to be found as a part of the optimization task. The application of Pontryagin minimum principle transforms the problem to solving two-point boundary-value problem. In addition, the influence functions on the constraint surfaces must be found by some iterative method. Alternatively, dynamic programming can be used. In both cases the methods are computationally demanding. It is therefore desirable to understand the properties of the time-optimal solutions. This could potentially lead to the development of efficient algorithms.

### 5.1 Lie theoretic properties of mechanical systems

Once more, let's recall the equations of motion for a mechanical manipulator:

$$M(q)\ddot{q} + N(q, \dot{q}) = \tau \quad (5.1)$$

We collected the centrifugal terms, Coriolis terms and gravitational forces into the vector  $N(q, \dot{q})$ . The inertia matrix  $M(q)$  is positive definite, so the equations can be rewritten as an affine system

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^n g_i u_i. \quad (5.2)$$

with

$$\begin{aligned} u &= \tau, \\ x &= \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \\ f(x) &= \begin{bmatrix} \dot{q} \\ -L(q)N(q, \dot{q}) \end{bmatrix}, \\ G(x) &= \begin{bmatrix} 0 \\ L(q) \end{bmatrix}. \end{aligned} \quad (5.3)$$

Abbreviation  $L(q) = M(q)^{-1}$  is used. The matrix  $M(q)$  is symmetric, positive definite, which implies that  $L(q)$  is also symmetric and positive definite. Let's denote by  $\mathcal{G}$  the space generated by columns of  $G$ , that is  $\mathcal{G} = \text{span}\{g_1, \dots, g_n\}$ .

We shall define the notion of the Lie bracket  $[u, v]$  of vector fields  $u$  and  $v$ :

$$[u, v] = \frac{\partial v}{\partial x}u - \frac{\partial u}{\partial x}v \quad (5.4)$$

The Lie bracket of two vector fields is another vector field, so the operation can be iterated. We will shorten the notation by writing  $uv$  in place of  $[u, v]$  and more generally  $f_1 f_2 \dots f_n$  for  $[f_1, [f_2, \dots [f_{n-1}, f_n] \dots]]$ . It is easily verified that the form of vector fields  $f$  and  $g_i$  implies that the matrix  $\{f g_1, \dots, f g_n\}$  has the form

$$\{f g_1, \dots, f g_n\} = - \begin{bmatrix} L(q) \\ R(q) \end{bmatrix} \quad (5.5)$$

The form of  $R(q)$  is not important for our discussion. Equations 5.3, 5.5 and the positive-definiteness of  $L(q)$  imply that vectors  $\{g_1, \dots, g_n, f g_1, \dots, f g_n\}$  are linearly independent for each  $q$ . Some additional observations can be made. Let  $\mathcal{Z}_i$  denote the set of all functions on  $\mathbb{R}^{2n}$  which are polynomials of degree at most  $i$  as functions of  $\dot{q}_1, \dots, \dot{q}_n$  (by convention  $\mathcal{Z}_i = \{0\}$  if  $i < 0$ ). Now define a class  $\mathcal{S}_i$  of the vector fields with a property that the first  $n$  components belong to  $\mathcal{Z}_i$  and the last  $n$  components to  $\mathcal{Z}_{i+1}$ . It is then not too difficult to check that if  $u \in \mathcal{S}_i$  and  $v \in \mathcal{S}_j$  then  $[u, v] \in \mathcal{S}_{i+j}$ .

Observing that  $f \in \mathcal{S}_1$  and  $g_i \in \mathcal{S}_{-1}$  we can conclude that for all  $i, j \leq n$

$$\begin{aligned} g_i g_j &= 0, \\ g_i f g_j &\in \mathcal{G}. \end{aligned} \quad (5.6)$$

## 5.2 Singular extremals

Suppose we have an affine system of the form

$$\dot{x} = f(x) + G(x)u \quad (5.7)$$

where  $f$  is an  $n \times 1$  vector function and  $G$  an  $n \times m$  matrix function, and we want to minimize the time that it takes to transfer the system from an initial state  $x_0$  to a final state  $x_f$  subject to the constraints on the control vector

$$L_i \leq u_i \leq U_i. \quad (5.8)$$

Note that the constraints can be rewritten in the form  $C(u) \leq 0$  where  $C$  is a  $2m \times 1$  vector function. The solution of the time-optimal control problem is given by the Pontryagin minimum principle. First, the Hamiltonian is defined

$$H(x, u, \lambda) = 1 + \lambda^T (f(x) + G(x)u) + \mu^T C(x, u) \quad (5.9)$$

where  $\lambda$  is an  $n \times 1$  vector of influence functions and  $\mu$  is a  $2m \times 1$  vector of multipliers. The solution must satisfy the equations

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda} \\ \dot{\lambda}^T &= -\frac{\partial H}{\partial x} \end{aligned} \quad (5.10)$$

and in addition

$$\frac{\partial H}{\partial u} = 0, \quad (5.11)$$

$$\mu_i(t) \begin{cases} = 0, & \text{if } C_i(x, u) < 0 \\ > 0, & \text{if } C_i(x, u) = 0 \end{cases} \quad (5.12)$$

As mentioned, the solutions  $(x, u, \lambda)$  that satisfy the above equations are called extremals. It can be shown that when the function

$$\psi_i = \lambda^T g_i, \quad (5.13)$$

is nonzero, the corresponding input variable  $u_i$  must take the value on one of the boundaries (if  $\psi_i > 0$  then  $u_i = L_i$  otherwise  $u_i = U_i$ ). If the function  $\psi_i$  changes sign, the input  $u_i$  switches from one boundary to the other. From this reason the function  $\psi_i$  is also called *switching function*. When the switching function is zero the corresponding control is said to be singular. If the switching function only has finite number of zeroes, the control  $u_i$  will be bang-bang. When the switching function  $\psi_i$  is identically equal to zero the corresponding extremal is said to be  $u_i$ -singular. The extremal is *singular* if it is  $u_i$ -singular for some  $i$ .

To implement the optimal control the calculated optimal trajectory is often used as the open-loop trajectory, the system is linearized along the trajectory and a linear controller is used to regulate the deviations from the trajectory. When the control is  $u_i$ -singular, input  $u_i$  has no effect on the Hamiltonian (see Eq. 5.9). Therefore, the linearized system will not be controllable. This motivates the study of singular extremals.

Now fix an extremal and take the switching function  $\psi_i$ . We can calculate the derivative:

$$\begin{aligned} \psi'_i &= \dot{\lambda}^T g_i + \lambda \dot{g}_i \\ &= - \left( \frac{\partial(f + Gu)^T}{\partial x} \lambda \right)^T g_i + \lambda^T \left( \frac{\partial g_i}{\partial x} (f + Gu) \right) \\ &= \lambda^T \left( \frac{\partial g_i}{\partial x} (f + Gu) - \frac{\partial(f + Gu)}{\partial x} g_i \right) \\ &= \lambda^T [f + Gu, g_i] \end{aligned} \quad (5.14)$$

Second equality follows from Eq. 5.10. The linearity of the Lie bracket allows us to rewrite Eq. 5.14 as

$$\psi'_i = \lambda^T [f, g_i] + \sum_j u_j \lambda^T [g_j, g_i] = \lambda^T [f, g_i] \quad (5.15)$$

The last equality follows from Eq. 5.6. Now, let  $N_i$  denote the set of limit points of zeroes of  $\phi_i$  and  $N'_i$  the set of limit points of zeroes of  $\phi'_i$ . The switching function is continuously differentiable, therefore  $N_i \subset N'_i$ . At points in  $N_i$  the following equations hold:

$$\begin{aligned} \lambda^T g_i &= 0, \\ \lambda^T [f, g_i] &= 0. \end{aligned} \quad (5.16)$$

Suppose that the above equations hold for all  $i$ . Vectors  $g_i$  and  $f g_i$  form a basis for  $\mathbb{R}^{2n}$ . Eq. 5.16 thus imply that  $\lambda = 0$  which contradicts the requirement of the Pontryagin minimum principle that  $\lambda$  should be nontrivial on the extremal. We can therefore formulate the following

**Proposition 2** *If the extremal  $(x, u, \lambda)$  is  $u_j$ -singular for all  $j \neq i$ , then  $u_i$  is bang-bang.*

By considering the second derivative of the switching functions and again using the argument that  $\lambda$  is nontrivial one can state an even stronger claim

**Proposition 3** *If  $(x, u, \lambda)$  is  $u_j$ -singular for all  $j \neq i$  and the trajectory  $x(t)$  remains in a set  $\mathcal{P}_i$  consisting of the points where the vectors*

$$\{g_j, j = 1, \dots, n\} \cup \{fg_j, j = 1, \dots, n, j \neq i\} \cup \{ffg_j, j = 1, \dots, n, j \neq i\}$$

*span the entire space, then  $u_i$  will be constant (equal to either  $L_i$  or  $U_i$ ).*

It can be shown that if the trajectory remains in a further restricted subset of  $\mathcal{P}_i$ , the controls  $u_j, j \neq i$  can be calculated from the value of  $u_i$ .

### 5.3 Critique

The paper by Sontag & Sussmann is the first that investigates the structure of the general time-optimal control for mechanical systems. By using the Lie-algebraic techniques the authors are able to show a number of special properties of such systems. They show that one of the controls must be bang-bang if all the others are singular. They also show how to calculate the control if the trajectory possesses some further properties.

The paper opens a vast new area for research and a lot of questions remain unanswered. The authors concentrate on the trajectories that are singular for all but one input variable. It is very unlikely that such case would appear in practice. For example, if we know what the time-optimal path is and apply the algorithms from the first chapters, we will get a control which saturates one actuator at a time, however the saturated actuator changes along the path.

It is not clear how to apply the theorems that were derived in the paper. They require characterizations of the set  $\mathcal{P}_i$ . The authors use their theory for a two-link manipulator and even in such simple case they used a symbolic package and an extensive search to find the expressions defining  $\mathcal{P}_i$  and further restricting it to a set where the other control can be calculated. The work by Fourquet [8] extends and simplifies some of the results for the 2-link manipulator however even for this case the structure of more general extremals remains unknown.

## Chapter 6

# Conclusion

In the last ten years substantial amount of work has been done on the time-optimal control of robotic manipulators. Two predominant directions of research are present:

- Optimization of motion along prespecified path.
- Optimization of motion when only the boundary conditions are specified.

The body of work that deals with the first problem is quite extensive. The first efficient algorithm for finding the optimal trajectory was proposed in 1985 independently by [4] and [17]. The algorithm has been later simplified [12] and modified to account for singular trajectories [14].

Through these works a number of facts have been established. It has been shown that the equations of motions with torque limits can be reduced to a second order differential equation with the limits on the acceleration and the velocity [4, 17]. The acceleration limits can be obtained by solving a linear program [11]. The time-optimal solution will require at least one actuator to be saturated [14]. When only the acceleration limits are active, the trajectory will require bang-bang control of acceleration and if the expressions for the joint variables in terms of the path parameter are piecewise analytic, there will be only finite number of switches [17]. On singular arcs the velocity limits are active and the acceleration is given by the time derivative of the velocity limit [14].

Presently, a major thrust of research is directed towards the extension of the algorithm for the multiple-arm configurations [3, 2, 6]. Recently, an extension of the algorithm that includes the actuator dynamics has also been proposed [21]. The problem which has not been addressed in the literature is how to obtain the optimal trajectory for kinematically redundant manipulators. In such a case the equations of motion cannot be reduced solely to differential equations in the path parameter and the proposed phase-plane based algorithm will not be adequate. It would be also interesting to study the time-optimal control for manipulators with kinematic and actuator redundancy where some additional cost function has to be minimized.

The investigation of the time-optimal control without path constraints has been very limited. The article by Sontag & Sussmann [19] opened many important questions which still remain to be solved. The Lie-algebraic techniques that were used for the investigation are undoubtedly a useful tool for the study of the optimal control of mechanical manipulators in general.

Sontag & Sussmann investigated only the case when all but one of the controls are singular. The study was motivated by the fact that linearization along such trajectories will yield uncontrollable system. However, the experiments with the time-optimal motions along a prespecified path show that different actuators become saturated as the motion progresses. This suggests

that the same will be true for the time-optimal solution in the general case. The practical value of the presented results thus seems to be very limited.



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