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Geometry of locating sounds from differences in travel time: Isodiachrons

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Abstract
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Geometry of locating sounds from differences in travel time: Isodiachrons

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I. INTRODUCTION

Differences in the travel time of acoustic and electromagnetic waves are commonly used to locate objects. Applications include the global positioning system[1] (GPS) and the passive location of calling animals.[2–9] When the speed of the signal is constant, the difference in travel times can be converted to the difference in distances by multiplying the difference in travel times by the speed. Then the method of location is usually interpreted using hyperboloids[10,11] because the hyperboloid is the locus of points whose difference in distance from two points is constant. The hyperbola may have been discovered by the ancient Greek mathematician Menaechmus (circa 350 B.C.) (p. 280-281, Vol II, Ref. 12), but this is not certain as most original writings have been lost. Other geometrical interpretations of location have also been found.[11,13,14]

The single-speed approximation is good enough for some but not all applications. For greater accuracy, methods are used that allow the speed to differ from path to path, while still maintaining the picture of signals traversing straight line segments. Scientists have developed these approaches for application to acoustic navigation in the sea,[15–17] and to the passive location of calling animals in the sea and air.[9,18] Then the question arises as to whether there is a geometrical shape, which cannot be a hyperbola, that can be used to interpret such locations when a constant speed is not used.

The purpose of this paper is to show that such a shape exists and to use it to interpret these methods of location. The shape appears to have not been shown previously in the scientific or mathematical literature, except for a paper that gives its definition without showing its shape or describing its properties.[18] It is therefore necessary to use simple calculations to describe the properties of this shape, called an ''isodiachron,'' from the Greek words ''iso'' for same, ''dia'' for difference, and ''chron'' for time. It is the locus of points along which the difference in *travel time* is constant. It reduces to a hyperboloid when the average speed of the signal is the same on both paths. The isodiachron is the natural shape to intersect to find the location of an animal when the speed differs from path to path. In fact, the algorithm in Ref. 18 yields a probability density function for location using isodiachrons.

Unless noted otherwise, the phrase ''effective speed'' is defined to be the time for the acoustic or electromagnetic signal to propagate from the animal to the receiver divided by the Euclidean distance. Thus the effective speed includes all spatially and temporally varying effects including those due to refraction, diffraction, and, for acoustic signals, advection such as that due to winds or currents. Because of advection, the effective speed from the animal to the receiver can be different than from the receiver to the animal. The word ''speed'' will not include effects from advection.

Section II reviews some of the reasons and methods that have been used to infer location when the effective speed is spatially inhomogeneous. There must be other methods that have been used as well, and the list of examples here is not intended to be complete. What is important about this section is that it highlights some of the pitfalls in hyperbolic location methods in situations where the effective speed is not constant from path to path. Section III provides the calculations that define the isodiachron and its behavior. The paper ends with a short summary, and provides a speculation as to why the ancient Greek mathematicians did not apparently contemplate an isodiachron.

II. HYPERBOLIC LOCATION IS LESS ACCURATE AND SOMETIMES INCORRECT

A. Underwater navigation

Hydrophones are sometimes lowered from a ship to the seafloor to locate acoustic emissions from a variety of objects. There is considerable interest in determining the position of each hydrophone on the bottom so that locations of acoustic emissions can be made accurately.

Quite often, the positions of the hydrophones are estimated by towing an accurately located source from the ship while the hydrophones on the bottom pick up these calibration signals. Suppose the clock for the hydrophone time series has an unknown offset with respect to the clock governing the towed source. The unknown clock offset is removed from the problem by working with differences of signal travel time at the hydrophone. The travel times change primarily because the source transmits from different locations. Until recently, this was usually treated as a standard hyperbolic location problem.

In the sea, the effective speed of sound between the source and hydrophone varies with source location.[15–17] To explain why, assume for simplicity that the speed of sound varies with depth only. The actual time for sound to reach a hydrophone on the bottom from the surface depends on the speed of sound along a ray path, which is not straight because it bends due to refraction. If the ray path were straight, the effective speed of sound would be the same for all source locations at the surface. But the bending changes the effective speed of sound. For cases of interest, this effective speed can be pre-computed in a table of speed versus slant angle to and depth of the receiver.[15,17] This table can be accessed by a location algorithm.[15,16]

The equations relating location to differences in travel time are nonlinear.[15,16] When these equations are linearized about a good initial guess for hydrophone location, a least-squares problem for hydrophone location and clock offset between the source and hydrophone can be solved by iterating the linearized equations to minimize the residuals in a determined or overdetermined problem.[15,16] The method, called the ''inhomogeneous algorithm'' here, allows one to assume that different paths have the same or different effective speeds. The inhomogeneous algorithm looks up the effective speed along each path as it iterates for the location of the hydrophone on the bottom.[15,16] An estimate of the error obtained from hyperbolic location has been investigated as follows.

Using a realistic profile of sound speed in the Atlantic, a simulated hydrophone at 1600 m depth is estimated to have a depth error of about 3 m when located using hyperbolic methods (Table I, case 5, Ref. 16). When the effective speed of sound is allowed to vary from path to path from a precomputed database, the inhomogeneous algorithm yields the correct depth for the hydrophone (Table I, Case 6, Ref. 16). For real data with a similar geometry, the hyperbolic location algorithm yields a 3 m error compared with the inhomogeneous location algorithm (Cases 2 and 4 in Table 2.3 of Ref. 15). There are situations where 3 m errors in hydrophone locations are significant. Errors in the locations of receivers often translate to much larger errors in source location.[11,19]

B. Solutions for animal location in air

Naturalists, biologists, acousticians, and others estimate locations of sounds from differences of travel time on widely separated microphones in air.[20–25] Problems with hyperbolic location are highlighted by considering a geometry where an animal is located at Cartesian coordinate (20,100,7) m and its signals are monitored at five microphones at (0,0,0), (25,0,3), (50,3,5), (30,40,9), and (1,30,4) m respectively. For definiteness, assume the animal's call has a rms. bandwidth of 1000 Hz and, following the cross correlation of the signal between each pair of microphones, the peak signal-to-noise ratio is 20 dB. The lag of this peak has a standard deviation of 16 μs (Ref. 26), where the lag is the difference in the travel time of sound between the animal and two receivers. Such accuracy can be achieved in practice.

A sequential nonlinear Monte Carlo technique is used to estimate the probability density function for location from simulated lags.[18] The technique can accommodate spatially homogeneous or inhomogeneous effective speeds, and allows one to account for errors in the locations of the microphones. Realistic *prior* distributions of errors are permitted for all variables. Distributions of location can be compared with the same statistical assumptions except for the fact that in one case the effective speed is spatially homogeneous, and in the other, spatially inhomogeneous. Other algorithms may also be suitable for generating realistic location distributions, but it does not seem prudent to summarize or compare such techniques because the main point of this paper is not centered on a review of techniques.

Simulated lags are computed without noise for a speed of sound of 330 m/s and for a horizontal wind blowing at 10 m/s toward the positive y Cartesian axis. *A priori* distributions of the remaining variables are taken to be Gaussian but truncated at two standard deviations (Table I). The accurate locations of the receivers are typical for those surveyed optically. It is necessary to accommodate the effects of wind for hyperbolic location without allowing the effective speed to vary from path to path. This can be done in two ways, neither of which is satisfying.

The first accommodation is to let the necessarily spatially homogeneous effective speed vary by an amount equal to the variations from path to path, i.e., a standard deviation of 10 m/s (Table I, Hyperbolic Location 1). The second accommodation is to artificially increase the measured error in the difference in travel times from 16 μs to that which would be due to the change in lag due to path speed variations of $\delta c = \pm 10$ m/s over distances of the acoustic paths. An order-of-magnitude estimate of this effect can be obtained by using equal path lengths, L, given by the length scale of the array. The effect is

$$\sigma_\tau \approx \sqrt{(L\,\delta c/c^2)^2 + (L\,\delta c/c^2)^2}, \tag{1}$$

where to first order, the variation in travel time for one path is $L\,\delta c/c^2$ and σ_τ denotes the standard deviation of the difference in acoustic travel time between the animal and two

TABLE I. *A priori* distributions of independent variables involved in determining the location of a calling animal in air via hyperbolic and isodiachronic methods. Both methods yield distributions for animal location from the differences in simulated arrival times at five microphones. Hyperbolic location requires the effective speed to be the same for each acoustic path whereas isodiachronic location does not. Distributions (probability density functions) are Gaussian with indicated means and standard deviations except all are truncated at two standard deviations. "True" indicates a variable's mean is error-less. The y component of the wind is modeled for isodiachronic location but cannot be modeled in hyperbolic location. Instead, this wind is accounted for by including a variation of 10 m/s for the standard deviation of effective sound speed in hyperbolic location (method 1) or by artificially increasing the measured error in the lag from 16 μs to 0.006 49 s (method 2) via Eq. (1). *A priori* errors are zero for receiver one, the y and z coordinates of receiver two, and the z coordinate for receiver three. These coordinates merely define the origin and orientation of the coordinate system.

		A Priori distributions					
		Hyperbolic location 1		Hyperbolic location 2		Isodiachronic location	
Variable	Symbol	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Cartesian	$r_i(x)$	True	0.02 m	True	0.02 m	True	0.02 m
Receiver	$r_i(y)$	"	"	"	"	"	"
Coordinate	$r_i(z)$	"	"	"	"	"	"
Cartesian	$u(x)$	0	0	0	0	0	0
Wind	$u(y)$	0	0	0	0	0	10
Component	$u(z)$	0	0	0	0	0	0
Sound speed	c	330	10	330	0	330	0
Lag	τ_{ij}	True	16 μs	True	0.006 49 s	True	16 μs

receivers. For $L=50$ m, $\delta c=10$ m/s, and $c=330$ m/s, we get $\sigma_\tau=0.00649$ s. This artificial increase is about two orders of magnitude greater than the 16 μs accuracy that can be obtained for the signal limited by noise (Table I, Hyperbolic Location 2).

There is no difficulty accommodating inhomogeneous effective speeds with isodiachronic location. In this case, the speed of sound has zero variation about the mean of 330 m/s, and the y component of the wind is given a standard deviation of 10 m/s about a mean of 0 m/s (Table I, Isodiachronic Location).

Following application of the sequential nonlinear Monte Carlo algorithm, incorrect animal locations are obtained when the effective speed is assumed to be spatially homogeneous (Table I, Hyperbolic Location 1). Indeed, the animal's 100% confidence limits for y are 102.0 to 108.1 m, but its actual y coordinate is 100 m. The 100% limits do not extend to infinity because the *a priori* distributions of error are truncated at two standard deviations. So given *a priori* distributions of receiver locations, travel time differences, and environmental variations, this hyperbolic location method always yields incorrect answers for the location of the animal.

If the second hyperbolic location model is used with large errors in the lags (Table I, Hyperbolic Location 2), the 100% confidence limits are: x between -312 and $+207$ m, y between 55 and 12 000 m, and z between -1200 and $+410$ m. These bounds contain the correct location of the animal, but they are so large as to be useless. The 95% confidence limits are: x between 13 and 33 m, y between 57 and 540 m, and z between -66 and $+15$ m. These bounds are about the same scale as the array itself, and still quite large and probably not useful.

With isodiachronic location, 95% confidence limits for the animal are x:16.6–20.5 m, y:98.8–101.7 m, z:3.5–40.5 m. These are statistically consistent with the correct location at (20,100,7) m. The large variation in z stems from the fact that the animal and receivers are nearly coplanar. Other confidence limits could be given but they are not shown because the point is that isodiachronic location yields a correct answer at a stringent confidence of 95%, and therefore at 100% confidence as well. This demonstrates that hyperbolic methods yield incorrect or useless locations whereas the isodiachronic method yields useful and statistically correct locations.

C. Global positioning system

The GPS is used to locate receivers from differences of travel time from synchronized emissions of electromagnetic waves from satellites. One of the largest sources of location error is the variation of the group speed of electromagnetic waves for different paths through the ionosphere.[1] If one uses a GPS receiver that monitors only the single L1 frequency, the typical residual after correcting for the GPS-broadcasted ionospheric correction is 4 m, but could be many times that amount.[27] These errors translate to location errors of about 30 m both horizontally and vertically. If one assumes locations are obtained using a hyperbolic technique, no accommodation can be made for the differences in effective speed from path to path, and these errors would be difficult to suppress without further information. In this situation, one could use an algorithm for location that accommodates variations in the effective speed on a path-by-path basis. Such algorithms would yield more accurate estimates of location than hyperbolic algorithms.

III. GEOMETRY OF ISODIACHRONS

Since there is a need for locating signals with high accuracy in spatially inhomogeneous fields of effective speed, it would be desirable to develop a geometrical interpretation of the problem as has been done when the effective speed is spatially homogeneous.[10,11,13,14]

A hyperboloid is the locus of points $\mathbf{s}$ whose difference in distance, d_{ij}, from two points is constant. These points satisfy

$$\|\mathbf{r}_i-\mathbf{s}\|-\|\mathbf{r}_j-\mathbf{s}\|=d_{ij}, \tag{2}$$

where the coordinates of the points (receivers here) are $\mathbf{r}_i$ and $\mathbf{r}_j$. Let t_i and t_j denote the time for sound to travel between the source and each receiver, respectively, and de-

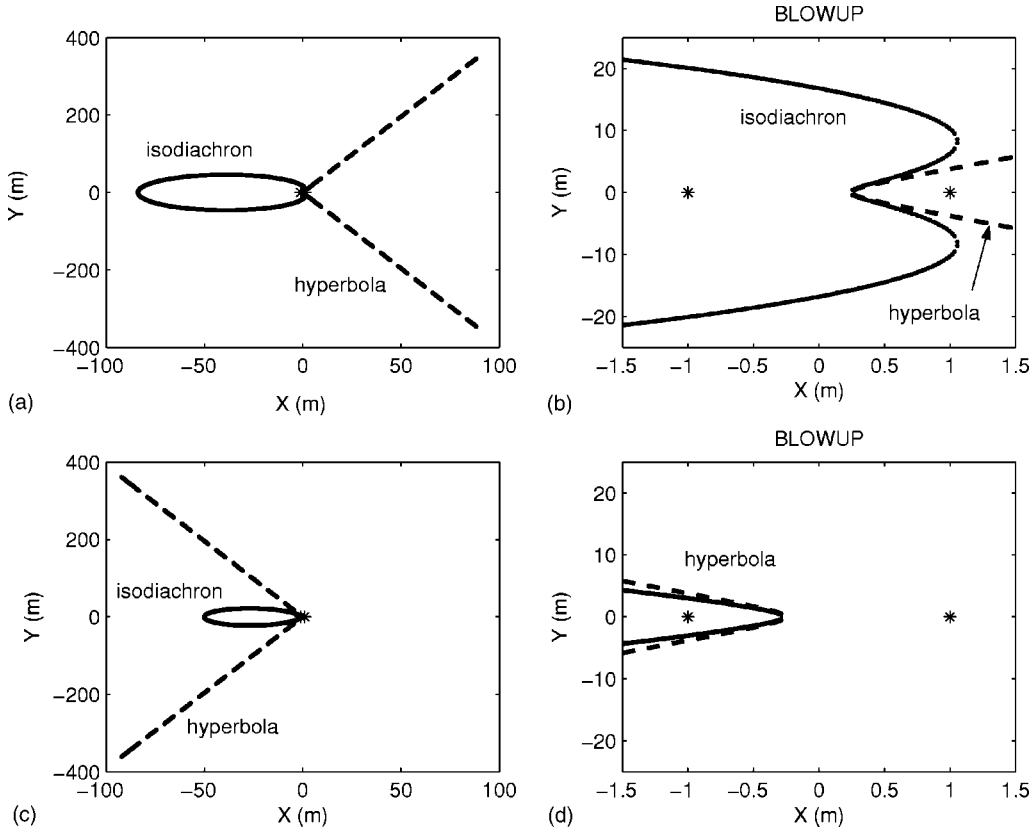

FIG. 1. A, B: Two-dimensional hyperbola (dashed) compared with two-dimensional isodiachron (solid). Locations of the two receivers (asterisks) are at Cartesian coordinates $(-1,0)$ and $(1,0)$. The effective speeds of sound between the calling animal and receivers one and two are 330 and 340 m s^{-1}, respectively. The lag, τ_{12}, is $+0.0015$ s. The hyperbola is computed for an effective speed of 330 m s^{-1}. C, D: Same except the lag is -0.0015 s.

fine the lag as $\tau_{ij} \equiv t_i - t_j$. When the effective speed is c, Eq. (2) is the same as

$$\|\mathbf{r}_i - \mathbf{s}\| - \|\mathbf{r}_j - \mathbf{s}\| = c\,\tau_{ij}, \tag{3}$$

which is described in Cartesian coordinates with a polynomial of degree two.

An isodiachron is defined to be the locus of points satisfying

$$t_i - t_j = \tau_{ij}, \tag{4}$$

where the effective speed depends on the spatial coordinates of the paths. Then Eq. (4) is

$$\frac{\|\mathbf{r}_i - \mathbf{s}\|}{c_i(\mathbf{s})} - \frac{\|\mathbf{r}_j - \mathbf{s}\|}{c_j(\mathbf{s})} = \tau_{ij}, \tag{5}$$

where $c_i(\mathbf{s})$ and $c_j(\mathbf{s})$ denote the effective speeds to receivers i and j, respectively, as a function of $\mathbf{s}$.

In some of what follows, it is useful to consider isodiachrons where $c_i(\mathbf{s})$ and $c_j(\mathbf{s})$ do not depend on location $\mathbf{s}$. This is called a "class one isodiachron." Isodiachrons of other classes are those given by Eq. (5) for which $c_i(\mathbf{s})$ and $c_j(\mathbf{s})$ depend on $\mathbf{s}$. Paradoxically, class one isodiachrons are useful for estimating probability density functions for the location of an animal in all realistic situations where $c_i(\mathbf{s})$ and $c_j(\mathbf{s})$ *do* depend on $\mathbf{s}$ because of another algorithm that uses class one isodiachrons in a particular way.[18] This paradox is resolved in Sec. III A.

The two-dimensional isodiachron approximates a hyperbola in a limited region (Fig. 1). Unlike a hyperbola, isodiachrons contain no points at infinity when $c_i(\mathbf{s})$ is unequal to $c_j(\mathbf{s})$ and when the receivers are separated by a finite distance. Instead, the assumption that the difference in propagation time be constant and that the effective speeds differ, constrains such isodiachrons to finite regions of space (Fig. 1). The proof which follows is true for isodiachrons of *all* classes and for all realistic effective speeds as long as $c_i(\mathbf{s})$ and $c_j(\mathbf{s})$ differ.

We write the definition of an isodiachron

$$\tau_{ij} \equiv t_i - t_j = \frac{d_i}{c_i(\mathbf{s})} - \frac{d_j}{c_j(\mathbf{s})}, \tag{6}$$

where d_i and d_j are the distances between the animal and receivers i and j, respectively. By assumption, $c_i(\mathbf{s})$ is unequal to $c_j(\mathbf{s})$. For all points in space we have

$$d_i = d_j + \Delta, \tag{7}$$

where Δ must be less than or equal to the distance between the receivers, so

$$\Delta \leqslant |\mathbf{r}_i - \mathbf{r}_j| < \infty, \tag{8}$$

since the receiver separation, $|\mathbf{r}_i - \mathbf{r}_j|$, is finite. Substitute Eq. (7) into Eq. (6) and simplify to get

$$\tau_{ij} = d_j\left(\frac{1}{c_i(\mathbf{s})} - \frac{1}{c_j(\mathbf{s})}\right) + \frac{\Delta}{c_i(\mathbf{s})}. \tag{9}$$

All measured lags, τ_{ij}, are finite and the only way to obtain these finite values for $c_i(\mathbf{s}) \neq c_j(\mathbf{s})$ is to demand that d_j be

finite because Δ is finite [Eq. (8)]. Therefore, all points on an isodiachron are a finite distance from the receivers when $c_i(\mathbf{s}) \neq c_j(\mathbf{s})$.

Several facts concerning class one isodiachrons are useful. We derive an expression for $y(x)$, the locations of their x intercepts, and the bounds for their lags. Some of the more complicated algebraic expressions were obtained from a symbolic mathematical software program.

When the effective speeds, c_i and c_j, do not depend on $\mathbf{s}$, the isodiachron can be expressed in Cartesian coordinates with a polynomial of degree four.[18] The polynomial coefficients of degree three and four go to zero when c_i and c_j approach the same value c, leaving a second degree polynomial describing a hyperbola. For class one isodiachrons in two spatial dimensions, Eq. (5) reduces to

$$a_1 y^4 + a_2 y^2 + a_3 = 0, \tag{10}$$

where

$$a_1 \equiv \frac{c_j^4 - 2c_i^2 c_j^2 + c_i^4}{c_i^4 c_j^4}, \tag{11}$$

$$\begin{aligned} a_2 \equiv & 2(c_i^4 c_j^4)^{-1}(c_j^4 x^2 + c_j^4 x_0^2 + c_i^4 x^2 + c_i^4 x_0^2 - 2c_j^2 x^2 c_i^2 \\ & + 2c_j^4 x x_0 - 2c_j^2 c_i^2 x_0^2 - c_j^4 \tau_{ij}^2 c_i^2 - \tau_{ij}^2 c_i^4 c_j^2 - 2c_i^4 x x_0), \end{aligned} \tag{12}$$

$$\begin{aligned} a_3 \equiv & x^4(c_j^4 - 2c_i^2 c_j^2 + c_i^4)/(c_i^4 c_j^4) + (c_i^4 c_j^4)^{-1}(c_j^4 x_0^4 \\ & + 4c_j^4 x^3 x_0 + 6c_j^4 x^2 x_0^2 + 4c_j^4 x x_0^3 - 2c_j^2 x_0^4 c_i^2 + \tau_{ij}^4 c_i^4 c_j^4 \\ & - 4c_i^4 x^3 x_0 + 6c_i^4 x^2 x_0^2 - 4c_i^4 x x_0^3 - 2c_j^4 x^2 \tau_{ij}^2 c_i^2 \\ & + 4c_j^2 x^2 c_i^2 x_0^2 - 4c_j^4 x x_0 \tau_{ij}^2 c_i^2 - 2c_j^4 x_0^2 \tau_{ij}^2 c_i^2 + c_i^4 x_0^4 \\ & - 2\tau_{ij}^2 c_i^4 c_j^2 x^2 + 4\tau_{ij}^2 c_i^4 c_j^2 x x_0 - 2\tau_{ij}^2 c_i^4 c_j^2 x_0^2), \end{aligned} \tag{13}$$

where the receivers are at Cartesian coordinates $(-x_0,0)$ and $(x_0,0)$ and $x_0 > 0$. The solution for y in terms of x can be simplified by substituting $z = y^2$ in Eq. (10) which yields a quadratic equation in z. If z is real valued and non-negative, solutions for y are given by $\pm\sqrt{z}$ for a given value of x. Equation (10) describes a hyperbola when $c_i = c_j$.

The intersection of a class one isodiachron with the x axis occurs when y is zero in Eq. (5) which yields

$$c_j|x + x_0| - c_i|x - x_0| = c_i c_j \tau_{ij}, \tag{14}$$

where x is the x coordinate of $\mathbf{s}$. The solutions are

$$x = \frac{c_i c_j \tau_{ij} + (c_i + c_j)x_0}{c_i - c_j}; \quad x < -x_0, \tag{15}$$

$$x = \frac{c_i c_j \tau_{ij} + (c_i - c_j)x_0}{c_i + c_j}; \quad -x_0 \leq x \leq x_0, \tag{16}$$

$$x = \frac{c_i c_j \tau_{ij} - (c_i + c_j)x_0}{c_j - c_i}; \quad x_0 < x. \tag{17}$$

We now investigate the possible values for τ_{ij} given x_0, c_i and c_j. These are derived from Eqs. (15)–(17)

$$\tau_{ij} = \frac{-c_j(x + x_0) + c_i(x - x_0)}{c_i c_j}; \quad x < -x_0, \tag{18}$$

$$\tau_{ij} = \frac{c_j(x + x_0) + c_i(x - x_0)}{c_i c_j}; \quad -x_0 \leq x \leq x_0, \tag{19}$$

$$\tau_{ij} = \frac{c_j(x + x_0) - c_i(x - x_0)}{c_i c_j}; \quad x_0 < x. \tag{20}$$

For the first case ($x < -x_0$), τ_{ij} equals $-2x_0/c_j$ as $x \to -x_0$. For $x < -x_0$

$$\frac{\partial \tau_{ij}}{\partial x} = \frac{c_i - c_j}{c_i c_j}; \quad x < -x_0, \tag{21}$$

so when $c_i < c_j$

$$-2x_0/c_j \leq \tau_{ij} < \infty; \quad x < -x_0; \quad c_i < c_j. \tag{22}$$

Similarly when $c_i > c_j$

$$-\infty < \tau_{ij} \leq -2x_0/c_j; \quad x < -x_0; \quad c_i > c_j. \tag{23}$$

For the second case ($-x_0 \leq x \leq x_0$), τ_{ij} equals $-2x_0/c_j$ and $+2x_0/c_i$ at x equal to $-x_0$ and x_0, respectively. In between, we have

$$\frac{\partial \tau_{ij}}{\partial x} = 1/c_i + 1/c_j; \quad -x_0 \leq x \leq x_0, \tag{24}$$

which is positive. Thus the values of τ_{ij} increase linearly in this line segment and

$$-2x_0/c_j \leq \tau_{ij} \leq 2x_0/c_i; \quad -x_0 \leq x \leq x_0. \tag{25}$$

For the third case ($x_0 < x$), τ_{ij} equals $2x_0/c_i$ at $x = x_0$. For $x_0 < x$

$$\frac{\partial \tau_{ij}}{\partial x} = \frac{c_j - c_i}{c_i c_j}; \quad x_0 < x. \tag{26}$$

So when $c_i < c_j$,

$$2x_0/c_i < \tau_{ij} < \infty; \quad x_0 < x; \quad c_i < c_j \tag{27}$$

and when $c_i > c_j$

$$-\infty < \tau_{ij} < 2x_0/c_i; \quad x_0 < x; \quad c_i > c_j. \tag{28}$$

We see that τ_{ij} has a minimum value of $-2x_0/c_j$ when $c_i < c_j$ and that the x axis is crossed twice except when τ_{ij} is the minimum value in which case the class one isodiachron touches the x axis once at $-x_0$ (Fig. 2). Similarly, τ_{ij} has a maximum value of $2x_0/c_i$ when $c_i > c_j$, and the isodiachron crosses the x axis twice except when τ_{ij} is maximum in which case the isodiachron touches the x axis once at x_0 (Fig. 3).

We now prove that the lag bounds on the x axis ($\tau_{ij} \geq -2x_0/c_j$ for $c_i < c_j$ and $\tau_{ij} \leq 2x_0/c_i$ for $c_i > c_j$) are the bounds for all points on a class one isodiachron. This can be proved by showing that all such isodiachrons intersect the x axis because then the lag is constant everywhere on an isodiachron. We know that all class one isodiachrons are symmetric about the x axis because all class one isodiachrons have values of y given by $\pm\sqrt{z}$ [see sentences following Eq. (13)]. The isodiachron is a continuous function in $y(x)$ because it is a polynomial. For two points on an isodiachron given by $\pm y(x)$, there must then either be a curve joining them through infinity (which is impossible as shown above), or the curve must join them through finite values and thus

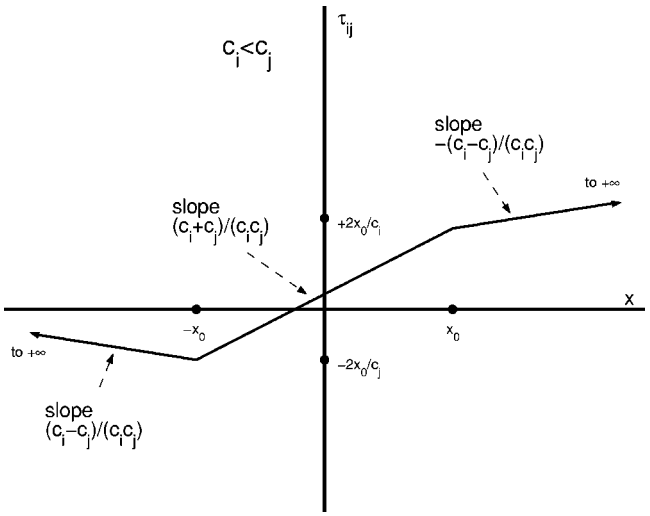

FIG. 2. The lags, τ_{ij}, for isodiachrons as a function of their x intercepts. Receivers i and j are at Cartesian (x,y) coordinates $(-x_0,0)$ and $(x_0,0)$, respectively. The effective speeds of the signal between receivers i and j and the isodiachron are c_i and c_j, respectively, and $c_i<c_j$. $\tau_{ij}=t_i-t_j$ where the times for the signal to travel between the isodiachron and receivers i and j are t_i and t_j respectively. Note that the lags must occur in the interval $-2x_0/c_j \leqslant \tau_{ij} < \infty$ [Eqs. (22), (25), and (27)].

cross the x axis at a value given by Eqs. (15–17). Thus the lag bounds given for class one isodiachrons on the x axis are valid for all class one isodiachrons because all class one isodiachrons touch the x axis.

A three-dimensional isodiachron can be formed by rotating the two-dimensional isodiachron around the x axis (Fig. 1). The closed form solution for isodiachronic location can yield four solutions from four receivers. This can be understood geometrically as follows. The first pair of receivers constrains the source to a class one isodiachron. A third receiver introduces a second isodiachron which can intersect the first one along two different closed curves (e.g., in the region between x between 0.5 and 1 in panel B of Fig. 1). A fourth receiver introduces a third isodiachron which can intersect the two closed curves at at most four points. In this case, a fifth receiver is needed to determine which of the four points is correct.

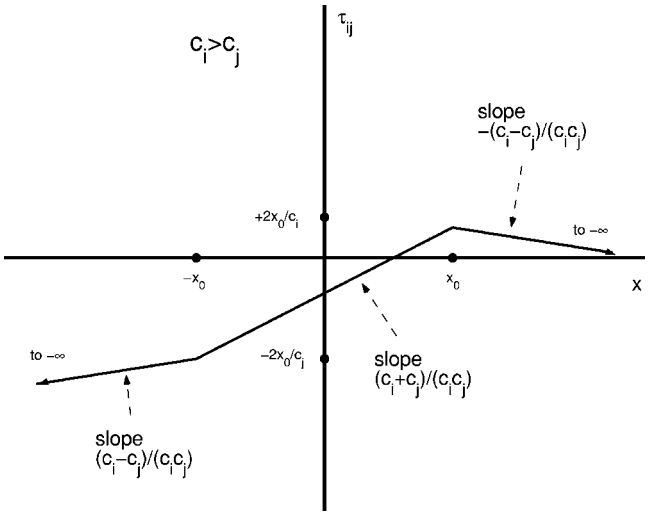

FIG. 3. Same as Fig. 2 except $c_i>c_j$. In this case all isodiachrons have lags $\tau_{ij} \leqslant 2x_0/c_i$ [Eqs. (23), (25), and (28)].

Because class one isodiachronic and hyperbolic surfaces can deviate significantly from one another in the vicinity of the receivers, hyperbolic locations can yield incorrect answers while isodiachronic locations are correct, even when accounting for errors.

A. Class one isodiachrons are useful when the effective speed is spatially inhomogeneous

The paradox is that class one isodiachrons are useful for estimating the probability density function for an animal's location when the effective speed varies in any realistic manner, including effects from advection. Understanding the paradox comes from the way this isodiachron is used by a Monte Carlo algorithm.[18]

A constellation is the minimum number of receivers needed to yield unambiguous solutions for location. For three-dimensional locations without prior knowledge of the animal's location, the constellation consists of four or five receivers, depending on the location of the animal.[18]

An analytical solution for the animal's location is available for any constellation.[18] The solution requires (1) the effective speeds between the animal and each receiver, (2) locations of the receivers, and (3) the values of the lags. When the effective speed includes advective effects, the algorithm needs to do an extra step because the effective speed depends on the location of the animal, but one does not initially know the location of the animal without using the analytical solution. A solution to this problem is given later, but for now it is important to state that no first guess for the animal's location is made by any hyperbolic location technique. The relevant part of the Monte Carlo algorithm is explained next.[18]

When the effective speeds are unaffected by advection, the Monte Carlo algorithm adopts any realistic *prior* probability density functions for (1) the effective speed between the animal and each receiver, (2) the errors in the Cartesian coordinate of each receiver in the constellation, and (3) the measured lags. A sample is drawn from each of the distributions yielding a set called a ''configuration.'' A configuration that does not yield at least one real-valued solution for location is discarded because the samples could not have jointly occurred. A ''valid configuration'' is one where there is at least one real-valued analytical solution for location. Each real-valued analytical solution for location is a point at which class one isodiachrons from all possible receiver pairs intersect. The Monte Carlo algorithm has established one set of effective speeds between the animal and each receiver in the constellation from the valid configuration. The Monte Carlo algorithm continues to find a sufficient number of valid configurations such that convergence is obtained for the probability distribution of the animal's location. The collection of effective speeds from all Monte Carlo runs provides an estimate for all the effective speeds that are consistent with the data, the cloud of possible animal locations, and the clouds of possible receiver locations.

Class one isodiachrons are a mathematically and computationally convenient and efficient means for obtaining the

distribution of the animal's location because they accommodate an analytical solution for location when the effective speed differs from path to path and they accommodate models for sound speed and advection that are realistic when used with the Monte Carlo algorithm.[18] More specifically, the analytical solution for location is derived[18] by using the fact that $\|\mathbf{r}_i-\mathbf{s}\|^2=c_i^2t_i^2$, and then subtracting the equation for $i=1$ from the equations for $i>1$ where c_i is a specified effective speed that is independent of the animal's location. When the equation for $i=1$ is subtracted from any other equation for $i>1$, the resulting equation specifies that the animal resides somewhere on the locus of points for which the difference in travel time to receivers $i>1$ and $i=1$ is a constant. This is a class one isodiachron. With a constellation, one has enough difference equations to yield an analytical solution for location.[18]

Consider an effective speed that is affected by advection, such as wind. A configuration is drawn from prior distributions of the (1) wind, (2) speeds, (3) receiver coordinates, (4) lags, and (5) the location of the animal, **s**. This configuration contains draws from the *prior* wind and source distributions. These were not drawn when advection was unimportant. One can always form a prior distribution for the location of the animal because one can use a uniform distribution in space with boundaries that are so large as to encompass every possible location. For example, one could know that sounds from a cricket would originate within 200 m of a receiver. Next, the effective speed between the animal and each receiver is obtained from

$$c_i=C_i+\mathbf{U}\circ(\mathbf{s}-\mathbf{r}_i)/d_i, \tag{29}$$

where C_i is the draw from the speed distribution (the scalar field), **U** is the draw from the vector wind distribution and the open circle denotes dot product. Note that this draw for the effective speed depends on a random guess for the location of the animal. The analytical solution for location is now obtained as before, yielding location $\mathbf{s}_1$. The closest real-valued solution, $\mathbf{s}_1$, to the randomly chosen location for the animal, **s**, is accepted if

$$|\mathbf{s}_1-\mathbf{s}|<\epsilon, \tag{30}$$

where ϵ is some small value such as 0.1 m. If all analytical solutions, $\mathbf{s}_1$, are complex, they are discarded and a fresh draw is made for a configuration. Otherwise, the effective speed is updated from Eq. (29) using $\mathbf{s}_1$ in place of **s**. This procedure iterates a maximum number of times. On each iteration the analytical solution for location is accepted if the difference between the analytical solution for **s** and the most recent guess for **s** is less than ϵ. We then have a valid configuration. The configuration is discarded if the maximum allowed number of iterations is exceeded, and one starts with fresh draws for the five categories of variables until one has sufficient numbers of valid configurations to yield accurate estimates of the distribution of the animal's location. Because effects from advection are incorporated into the effective speed, the geometrical interpretation for the analytical solution for location is based on class one isodiachrons as before. We see that realistic variations of advection and sound speed are accounted for in the probability distribution of the animal's location.

B. Example in air

Consider a two-dimensional geometry where five receivers are located at Cartesian coordinates (0,0), (25,0), (50,3), (30,40), and (5,30) m (Fig. 4). An animal is located at (22,2) m. The speed of sound is assumed to be a typical 330 m/s, and a wind is blowing in the positive y direction at 10 m/s. With R receivers there are

$$N_\tau=R(R-1)/2, \tag{31}$$

possible lags $(\tau_{ij}, i=1,\ldots,R-1; j=i+1,\ldots,R)$ so for $R=5$, we get $N_\tau=10$. All ten lags are computed without error and, for each, the isodiachron and hyperbola are drawn. The hyperbolas are drawn for an effective speed of 330 m/s. Some of the hyperbolas look like the isodiachrons and others do not (Fig. 5). Isodiachrons all intersect the animal location exactly, but the hyperbolas do not, as it is impossible for them to accommodate variations in effective speed from path to path with the hyperbolic assumption. No attempt has been made to find a single effective speed that minimizes the residuals from a central intersection point, but this is not important to do in this context because the hyperbolas would not intersect at a point anyway, and some of their shapes are quite different than the isodiachrons (Fig. 5).

The location of the animal in Fig. 4 coincides with the point of intersection of the isodiachrons because this example uses error-less values for the variables that determine location (lags, receiver locations, and effective speeds). The presence of errors dictates that there are an infinite number of possible animal locations consistent with measurements. With truncated prior distributions of error for the pertinent variables (lags, receiver locations, and effective speeds), the infinite number of possible locations can be confined to a finite region. The nonlinear Monte Carlo algorithm[18] draws from the prior distributions of these variables to find animal locations for which all ten isodiachrons intersect at one point. These locations form clouds of feasible locations of the animal. The feasible locations near the animal have sets of 10 isodiachrons that look like those in Fig. 4.

C. Example in ocean

Consider a two-dimensional geometry where five receivers are located at Cartesian coordinates (0,0), (4000,0), (1500,3000), $(-50,-2000)$, and $(2000,-3000)$ m (Fig. 6). Suppose these receivers are located near a zonal front where the speed of sound is 1500 m/s to the south and 1525 m/s to the north of the x axis (Fig. 6). Suppose a whale calls at $(3960,-20)$ m. Then the effective speed of sound is 1500 m/s for all receivers except the one at (1500,3000) m where it is 1524.8 m/s because the signal crosses the front to the north. Hyperbolic locations assume the effective speed is 1512.5 m/s. This is the average of the speeds on either side of the front. Isodiachrons and hyperbolas are drawn without data error.

Some of the hyperbolas look like isodiachrons, and others do not (Fig. 6). When isodiachronic location[18] is used to

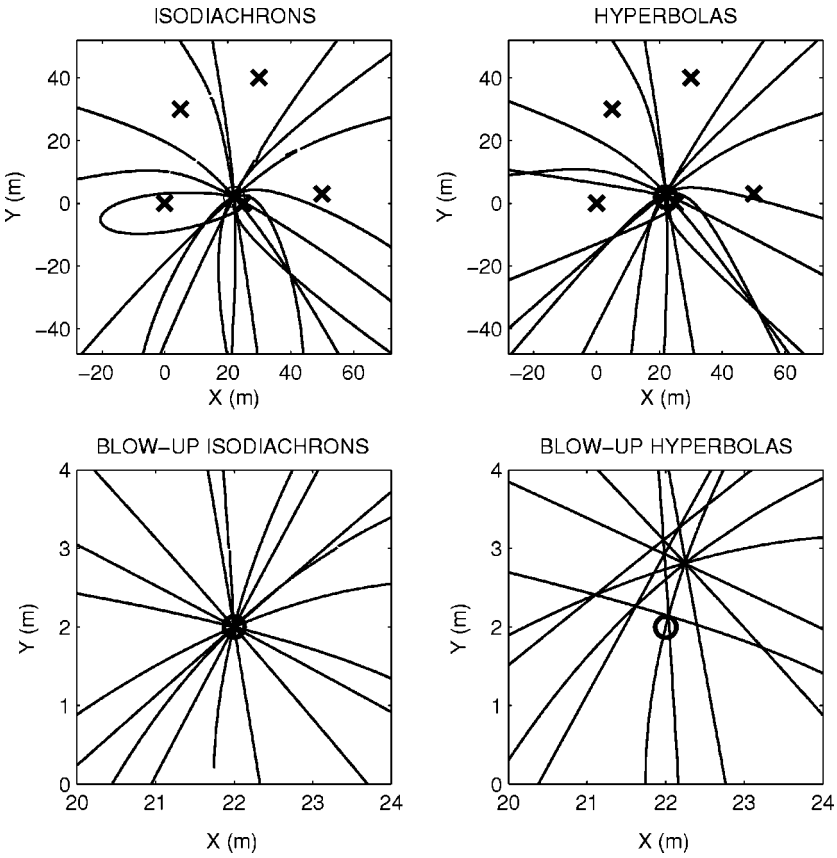

FIG. 4. *Left column*: The ten isodiachrons corresponding to the ten lags derived from an animal at Cartesian coordinate (22,2) m (circle) in air and recorded at five receivers (X's). One of the receivers at (25,0) m is close to the animal so its X is not resolved but is visible in Fig. 5. The bottom shows the isodiachrons within a few meters of the animal. *Right column*: Same except these are the ten hyperbolas. The speed of sound is 330 m/s and a spatially homogeneous wind is blowing in the positive *y* direction at 10 m/s. The hyperbolas are derived by assuming the effective speed of sound is 330 m/s. Note the hyperbolas do not intersect at the same point nor do they intersect the animal (bottom right). Some of the isodiachrons are similar to the hyperbolas, and some are quite different (Fig. 5).

locate the whale, the $N_\tau = 10$ isodiachrons always intersect at the same point. The hyperbolas can never intersect at the same point, and their mismatches indicate the inability of hyperbolic geometry to find a correct location when the effective speed differs from path to path. Even when one accounts for errors in the effective speeds, the lags, and the locations of the hydrophones, isodiachrons always intersect at the same point and hyperbolas do not.[18]

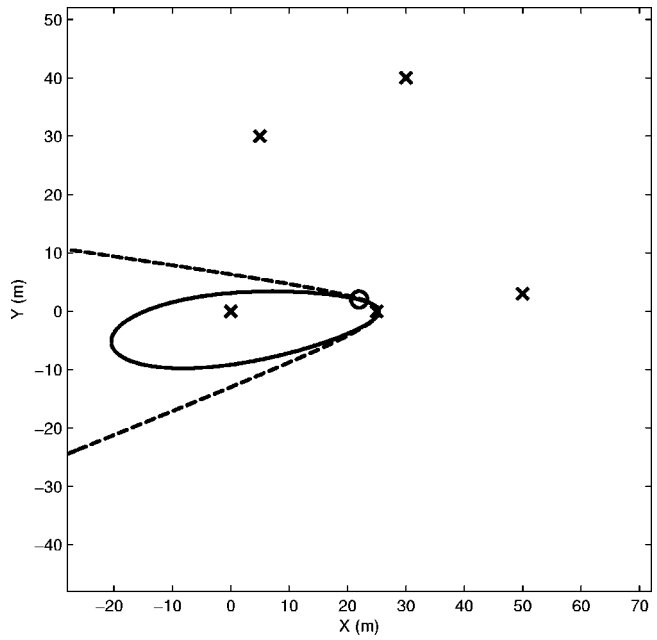

FIG. 5. Same as Fig. 4 except only the isodiachron (solid) and hyperbola (dashed) corresponding to one particular lag are shown.

D. Other examples

Isodiachrons could be used to locate animals when the effective speeds are greatly different from path to path. For example, a modeling study[28] indicates that low-frequency sounds from a fin whale could travel to hydrophones through different paths. Some receivers close to the whale could pick up only the first acoustic path through the sea, while other distant receivers could pick up only the acoustic path that propagates below the sea floor because the paths through the water could be blocked by seamounts. The effective speed through the water and solid Earth can differ by more than a factor of 2.[28]

There are other possibilities. Sounds created by some animals can reach receivers through both the air and the solid Earth.[29,30] Seals flap their flippers on the surface and the sound propagates in air and water to distant receivers.[31] All these animals could be located with isodiachrons.

IV. CONCLUSION

When one seeks accurate locations for a calling animal from measurements of the time differences of signals, consideration must be paid to the differences in effective speed along different paths if such differences exist. Differences in effective speed are significant enough in air and water to have led researchers to adopt models for location that allow the effective speeds to differ from path to path.[15–18] Path-to-path variations in effective speed are inconsistent with a geometrical interpretation based on a hyperbola. Instead, one can visualize a new geometrical shape, called an isodiachron, to interpret location in this situation. The isodiachron is the

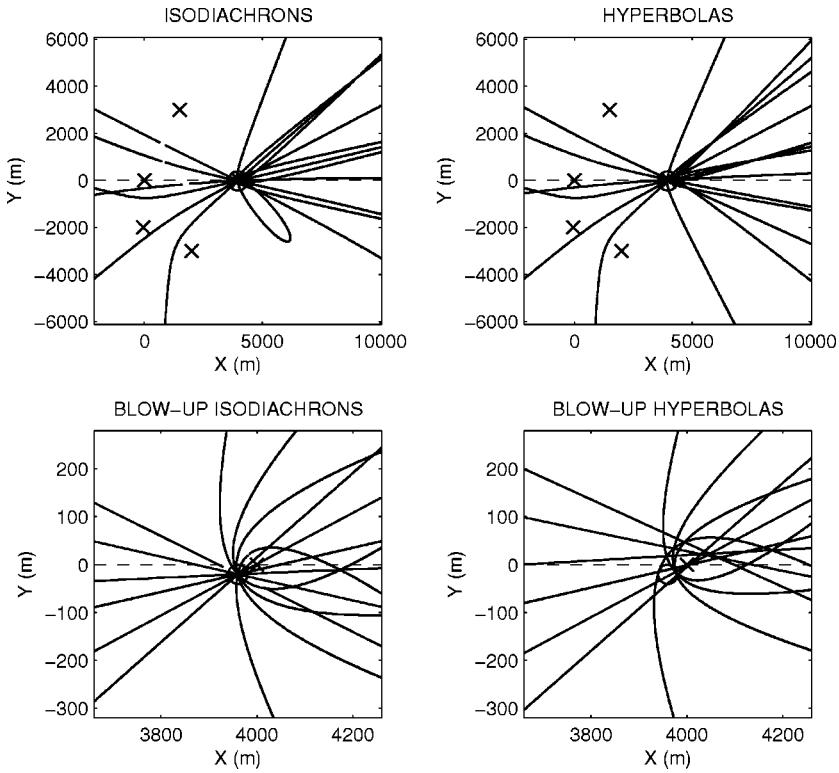

FIG. 6. *Left column*: The ten isodiachrons corresponding to the ten lags derived from a simulated call at Cartesian coordinate $(3960,-20)$ m (circle) in water and recorded at five hydrophones (X's). One of the hydrophones at (4000,0) m is close to the animal and can barely be made out on the bottom row where the isodiachrons are shown within a few hundreds of meters of the animal. *Right column*: Same except these are the ten hyperbolas. The speed of sound is 1500 and 1525 m/s on the lower and upper regions of the figures, respectively. The frontal region separating these speeds occurs at $y=0$ (dashed). The hyperbolas are derived by assuming the effective speed of sound is 1512.5 m/s, the average of the speeds on either side of the front. Note the hyperbolas do not intersect at the same point nor do they intersect the animal (bottom right). Some of the isodiachrons are similar to the hyperbolas, and some are quite different.

locus of points corresponding to a constant difference in travel time along straight paths between the animal and two receivers. Isodiachrons differ significantly in shape from hyperboloids for many problems of practical interest (Figs. 4–6).

It seems interesting to speculate why ancient Greek mathematicians did not think about isodiachrons. The hyperbola was discovered by them (pp. 280, 281, Vol II, Ref. 12). Why did they conceive of hyperbolas? Perhaps the answer to this question is not currently known due to the loss of ancient documents. But something is known about the motivation for geometry and in the use of a hyperbola during these times. The word ''geometry'' comes from the Greek words geo for Earth and metron which means to measure. The emphasis on measuring the Earth was a principal motivation for the Greek's development of this branch of mathematics.[12]

The earliest known written reference to a hyperbola mentions Menaechmus who used it to solve the problem of how long to make the side of a cube so as to double its volume, a problem that today is solved by knowing how to compute a cube root of a number (pp. 280, 281, Vol. II, Ref. 12). There are two ancient documents describing the origin of this problem. The first explains that God asked the Delians via an oracle to double the size of an altar to rid themselves of a plague (p. 257, Vol. I, Ref. 12). The workman did not know how to double a volume, and asked Plato for help, who himself did not know the method of solution. The second document explains that an ancient tragic poet portrayed Minos preparing a tomb for Glaucus in the shape of a cube whose sides were a hundred feet in length. There was a desire to double the volume as to be more suitable for royal burial (pp. 257–259, Vol. I, Ref. 12). Thus the hyperbola's first known reference occurs in the context of solving a practical problem.

The idea of measuring location from the propagation time of signals is conveniently done using electronic equipment developed in the modern age. It is plausible that ancient Greeks would not consider the isodiachron because it would perhaps have been too distant from the problems of their day, though they may have had the mathematical tools needed to derive the geometrical shape.

Besides allowing a general physical interpretation for location in spatially inhomogeneous media of effective speed, isodiachrons are the geometrical shapes that are intersected for an analytical solution for location[18] (Figs. 4, 6). Isodiachrons revert to hyperboloids when the effective speed is spatially homogeneous.

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