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Abstract

This report characterizes the powerdomain constructions which have been used in the semantics of programming languages in terms of formulas of first order logic under a preordering of provable implication. This provides an intuitive representation which suggests a new form of powerdomain—called the *mixed* powerdomain—which expresses data in a different way from the well-known constructions from programming semantics. It can be shown that the mixed powerdomain has many of the properties associated with the convex powerdomain such as the possibility of solving recursive equations and a simple algebraic characterization.

1 Introduction.

A *powerdomain* is a "computable" analogue of the powerset operator. They were introduced in the 1970's as a tool for providing semantics for programming languages with non-determinism. However, it has been appreciated for some time that the elements of powerdomains have a strong logical intuition. This intuition has been captured rigorously through the use of modal logic [Win85], certain kinds of predicates [Hec90b, Hec90a] and Stone duality [Abr87a, Abr87b, Rob87, Abr88, Vic89]. In this report I want to point out that there is also a simple way of understanding (finite or compact) powerdomain elements as monadic second order formulas. A secondary objective is to convey the intuition behind the powerdomains to those who have found them difficult to appreciate under other formulations. This work can be viewed as a kind of tutorial supplement for [Gun91] (to which I refer the reader for a more complete and advanced discussion) or as a companion to some of the work on powerdomains in a database context [BDW88, BO86, BJO91].

The report is divided into four sections. The powerdomains are defined in the second section and an extended example using sets of records is discussed. In the third section the intuitions about information discussed in the second section are characterized using first order logic by proving a correspondence between powerdomain elements and certain first order formulas which represent second order predicates. Theorems establishing a precise relationship for the upper and lower powerdomains are proved. In the fourth section, the convex powerdomain is also characterized in these terms and a new powerdomain,

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the *mixed* powerdomain, is defined. The mixed powerdomain elements are then characterized with first order formulas as well.

2 Sets of data.

This section begins by providing precise definitions for the upper, lower and convex powerdomains. As a guide to intuition, we will then look at several examples of sets from the powerdomains of a simple datatype of records. Viewing things in such a concrete fashion aids one in seeing powerdomains as diverse theories of partially described sets and not just as a theories of the outcomes of non-deterministic computations.

Rather than follow the usual treatment which one can find in many places in the literature (see, for example, [Smy78] or [GS90]), I will reduce the domain-theoretic pre-requisites by working only with the action of the powerdomain operator on the *bases* of domains.[1] In this way, we may restrict our attention to the following simple class of directed graphs:

Definition: A *preorder* is a set A together with a binary relation $\gtrsim$ which is reflexive and transitive. We may write $y \lesssim x$ rather than $x \gtrsim y$. We write $x \eqsim y$ if $x \gtrsim y$ and $x \lesssim y$. ∎

A preorder is like a poset (partial order) except the anti-symmetry axiom need not hold. Intuitively, the elements of a preorder A may be thought of as propositions (of first order logic, say) under the preordering of provable implication. If we have propositions ϕ and ψ in A, then we may have $\phi \to \psi$ and $\psi \to \phi$ without it being the case that ϕ and ψ are the same proposition (although their truth values must be the same). For this reason and another (more important) reason mentioned below, it is more convenient to work with preorders than posets.

Let $\langle A, \gtrsim \rangle$ be a preorder and suppose $\mathcal{P}_f^* A$ is the collection of non-empty finite subsets of A. We define three preorderings on $\mathcal{P}_f^* A$ as follows. Suppose $u, v \in \mathcal{P}_f^* A$, then

- $u \gtrsim^\sharp v$ iff for every $x \in u$ there is a $y \in v$ such that $x \gtrsim y$,
- $u \gtrsim^\flat v$ iff for every $y \in v$ there is a $x \in u$ such that $x \gtrsim y$,
- $u \gtrsim^\natural v$ iff $u \gtrsim^\sharp v$ and $u \gtrsim^\flat v$

It is easy to check that each of these relations is, in fact, a preordering. The preorder $\langle \mathcal{P}_f^* A, \gtrsim^\sharp \rangle$ is called the *upper powerdomain* of $\langle A, \gtrsim \rangle$ and it is denoted $\langle A^\sharp, \gtrsim^\sharp \rangle$ (or just $A^\sharp$ when the preordering is clear). The preorder $\langle \mathcal{P}_f^* A, \gtrsim^\flat \rangle$ is called the *lower powerdomain* of $\langle A, \gtrsim \rangle$ and it is denoted $\langle A^\flat, \gtrsim^\flat \rangle$. Finally, the preorder $\langle \mathcal{P}_f^* A, \gtrsim^\natural \rangle$ is called the *convex powerdomain* of $\langle A, \gtrsim \rangle$ and it is denoted $\langle A^\natural, \gtrsim^\natural \rangle$.

To get a few examples, let us look at the powerdomains of a simple preorder of records. Our records will have between zero and four fields. the available fields are **name**, **age**, **socsec** and **married?**. The **age** and **socsec** fields may be filled with integers and the **married?** field may be filled with a boolean. The name field is a record with two fields: **first** and **second**. Each of these fields may be filled with a string. The type can be named by the following expression:

```
{ name : { first : string,
           last : string },
  age : int,
  socsec : int,
  married? : bool }
```

Here is a sample record r_1:

[1] This way of doing things has been discussed in numerous references. The information systems of Scott [Sco82] are a popular tool; preorders and domains are discussed in some detail in [Gun87]. The technique has been carried much further in [Abr88].

```
{ name = { first = "John",
           last = "Smith" },
  age = 28,
  socsec = 439048302,
  married? = true }
```

We will assume that records may have missing fields as in the following record r_2:

```
{ name = { first = "John" },
  age = 28 }
```

The record r_1 is *more informative* than r_2 because it provides more facts about the described individual "John". This concept of one record being more informative than another is basic to the discussion which follows. Records may have other relationships as well. In particular, there is an *inconsistency* between r_1, r_2 and the following record r_3:

```
{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2,
  married? = false }
```

We may model this collection of records and its associated information ordering as follows. First, we assume that we are given the types *string, int* and *bool* as *flat domains.* For example, the type of integers should contain the ordinary integers 1, -2, 0 and so on, together with a special *bottom* element $\perp$ which is intended to represent "no information". The ordering on these elements is given by taking $m \gtrsim n$ if and only if $n = \perp$ or $m = n$. For example, we *do not* have $28 \gtrsim 2$. This is what one would expect, after all; a record about a two year old John Smith is not less informative than a record about a 28 year old John Smith, these records are simply incompatible. The interpretation of strings is similar. The booleans are also a flat domain, but there are only three elements *true, false* and $\perp$. Now, the space of records is the product space

$$(string \times string) \times int \times int \times bool.$$

Of course, a record is interpreted in this space without regard to the order of its fields according to some convention perhaps (*e.g.* the first two strings are for the first and last names respectively; the first integer is the age and the second is the social security number). Missing record fields are interpreted as $\perp$. Records are ordered coordinate-wise. A pair of records r, r' is consistent if there is a record r'' such that $r'' \gtrsim r$ and $r'' \gtrsim r'$. Otherwise r and r' are inconsistent. Many of the sets in the powerdomain of our space of records will contain pairs of inconsistent records.

Our family of records is the raw material out of which we can build collections of data about some set of "real world entities". Some of our records probably make no real sense under any circumstances. For example:

```
{ name = { first = "John",
           last = "Smith" },
  age = 2,
  married? = true }
```

will probably not find its way into any useful database of records. There will also be pairs of records which are unlikely to be found together in the same database:

```
{ socsec = 229068403,
  age = 2 }

{ socsec = 229068403,
  age = 28 }
```

Moreover, most data items will be only partial descriptions (as is the case with most of the examples above). The question we need to answer is the following: how does a *set* of records provide a partial description of a set of real world entities?

Consider the following set s of records

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John" },
  age = 2 }
```

which might be the database [2] for a small nursery. When should we say of another set of records that it is *more informative* than the set of records above? Here is a first possibility s_1:

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  age = 2 }

{ name = { first = "Beth" }
  age = 3 }
```

This set seems more informative because it lists more of the children in the nursery and provides slightly more information about those who are enrolled (since we now have John's last name). In the lower powerdomain (pre)-ordering, $\succsim^\flat$, the set s_1 is greater (more informative) than s. But consider the following set s_2 of records:

```
{ name = { first = "Mary" },
  socsec = 439234970,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 429238406,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2 }
```

This seems more informative than s because it provides more information about the children in the class and eliminates the name of a child (Todd) who will not actually be attending. In the upper powerdomain ordering, $\succsim^\sharp$, the set s_2 is greater than s. However, it is *not* greater than s in the lower powerdomain ordering. Conversely s_2 is not greater than s in the upper powerdomain ordering.

These two alternative extensions should point out how the ordering of partial information suggests the intuitive significance of the set of records s. In the first case, under the lower ordering, s might be a list of children who have been enrolled in the nursery; more may enroll later. In the second case (under the upper ordering) s might be the list of all children who are on a waiting list; some children may drop off of the list but no new ones may enter (since the deadline for such entries has passed). In either case, a further refinement of the individual records through the addition of new fields results in a more informative set of records.

It is important to note that powerdomains are only preorderings and not posets (*i.e.* partial orderings). If the record

```
{ name = { last = "Smith" }
  age = 2 }
```

is added to s_1, there is *no change* in the intended meaning of the set of records with respect to the lower preordering. In other words, if s_1' is the larger

[2] I hope the reader will pardon my loose use of this term. It is not my intent to expound a serious theory of databases. The examples are meant to suggest the propositional consequences of the powerdomain orderings.

set, then $s_1 \gtrsim^\flat s_1'$ and also $s_1' \gtrsim^\flat s_1$. This is not true of the upper preordering. In that preordering, $s_1 \gtrsim^\sharp s_1'$, but $s_1' \not\gtrsim^\sharp s_1$. The following set of records

```
{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2 }

{ name = { first = "John" }
  age = 2 }
```

would not change, under either powerdomain ordering, if the following record were added:

```
{ name = { first = "John",
           last = "Smith" }
  age = 2 }
```

It may seem odd that we would allow in s_2 the possibility that a single record might split into two records as the record for John did. This seems more reasonable in other cases, however. For example, the singleton set of records containing only the record `age = 2` would indicate under the upper ordering that we are talking about a nursery of two year olds (whose names we do not yet know). In the lower ordering, this database would indicate only that there will be *some* two year old in the nursery (but there may also be some children of other ages). It is also possible for two data items to *merge* to form a new data item. For example, the following set of records:

```
{ name = { first = "Mary" } }

{ name = { first = "John" },
  age = 2 }

{ name = { last = "Smith" }
  socsec = 229068403 }

{ socsec = 429238406 }
```

is less descriptive (in either lower or upper ordering) than the set of records s_2 above.

We will look at some more examples of this kind when we get to the discussion of the convex ordering in a later section.

3 Powerdomains and logic.

Let us now try to relate the intuitions and preorderings discussed in the previous section to formulas of an appropriate logic. For this discussion first order predicate logic will be used because it is simple, well-known and seems to be sufficient for the job at hand. After some motivation, the upper and lower powerdomain operators on preorders will be precisely related to certain operations on collections of first order formulas.

In the examples provided in the previous section, we thought of sets of records as partial descriptions of sets of real world entities. However, one may dually think of a set of records as describing a *set* of "situations" compatible with the set of records. Each record can be treated as a predicate over a collection of *individuals.* For example, the record `name = first = "John"` is satisfied by individuals whose first name is "John". More concretely, we might think of individuals as *total records* (*i.e.* records with all fields filled in) for the example of the previous section.[3] If we view things this way, can we think of sets of records as predicates too? First of all, we must ask what is being predicated by a set of records. The answer seems clear: sets of individuals. Hence, a set of records should be considered a predicate over sets of individuals or, put succinctly, a second order predicate.

This seems to justify a leap into second order logic for a description of powerdomains. We expect to find that the different powerdomain orderings give

[3] It will not always be intuitively reasonable to view things in this way, although it works well for the example at hand.

rise to different second order predicates. However, a *first* order formula may be considered a second order predicate if it contains a unary predicate symbol. Suppose we are given a distinguished unary predicate symbol W and a collection of predicate symbols U. In a given model, a formula like $U(x)$ might be asserting that x is a two year old. With this interpretation, a first order formula such as

$$\phi \equiv \forall x.\ W(x) \rightarrow U(x)$$

asserts that everyone in the interpretation of W is a two year old. Hence ϕ itself becomes a predicate of W. Of course, there will be many predicates defined by first order formulas in this way, but which of them (if any) correspond to the elements of the powerdomains?

Let us attempt to work out an example similar to those in the previous section. Recall the set s of records:

```
{ name = { first = "Mary" },
  age = 2 }

{ name = { first = "Todd" },
  age = 2 }

{ name = { first = "John" },
  age = 2 }
```

Let M, T and J be unary predicate symbols for having first name "Mary", "Todd" and "John" respectively. Under the lower powerdomain ordering, what is this collection of records telling us about the set of children in our hypothetical nursery? The first record of s seems to assert that *there is a child named "Mary" in the nursery.* If W is a predicate symbol which we are interpreting as the children in the nursery, this can be represented by the formula

$$\exists x.\ W(x) \wedge M(x)$$

which we may express more succinctly as $W \cap M \neq \emptyset$. Actually, the first record expresses a bit more than this. Let O be a predicate which is being interpreted as the set of all two year olds. Then the first record says: $W \cap M \cap O \neq \emptyset$. In summary, s corresponds to the following proposition:

$$\begin{array}{l} W \cap (M \cap O) \neq \emptyset \wedge \\ W \cap (T \cap O) \neq \emptyset \wedge \\ W \cap (J \cap O) \neq \emptyset \end{array}$$

As an exercise, the reader may express s_1 in this way and show that the resulting proposition implies the one above.

Now, what about the upper powerdomain ordering? Under this ordering, each record expresses a range of possibilities. The three records together assert that the children of the nursery (or those on its waiting list if that is preferred interpretation) are all named "Mary", "Todd" or "John". More specifically, a child on the waiting list must be a two year old "Mary", a two year old "Todd" or a two year old "John". However, this does not preclude the possibility that there is no "Todd" who is actually waiting for entry. If W is a new unary predicate symbol to be interpreted as the individuals in the nursery, then this assertion may be summarized as

$$\forall x.\ W(x) \rightarrow \theta \qquad (1)$$

where θ is the disjunction

$$(M(x) \wedge O(x)) \vee (T(x) \wedge O(x)) \vee (J(x) \wedge O(x)).$$

The formula (1) may also be expressed with set-theoretic notation:

$$W \subseteq (M \cap O) \cup (T \cap O) \cup (J \cap O).$$

Again, the reader may find it instructive to express s_2 in this way and check that the resulting proposition implies this one.

It is tempting, at this point, to "think semantically" and try to view the powerdomains in terms of sets of individuals. This can be misleading, however. Given a predicate symbol U, let $[\![U]\!]$ be the interpretation of U in a fixed model. In particular, for the

upper ordering, we may have

$$[\![U_1]\!] \cup \cdots \cup [\![U_m]\!] = [\![V_1]\!] \cup \cdots \cup [\![V_n]\!]$$

without it being the case that the $[\![U_i]\!] \subseteq [\![V_j]\!]$ or $[\![V_j]\!] \subseteq [\![U_i]\!]$ for *any* pair of predicate symbols U_i and V_j. It seems, therefore, that although the formulas

$$\phi \equiv W \subseteq U_1 \cup \cdots \cup U_n$$

and

$$\psi \equiv W \subseteq V_1 \cup \cdots \cup V_m$$

define the same family of predicates, this does not follow from the ordering under inclusion of the sets $[\![U]\!]$ for unary predicate symbols U of the language. For a *fixed* model, the interpretations of the predicates $\phi(W)$ and $\psi(W)$ may have more relationships than one can "obtain" from the ordering of the sets $[\![U]\!]$. One may place some *ad hoc* assumptions on the model to make things work out better. However, the treatment which I provide below uses *non-standard models* to hide this problem.

To crystalize this discussion by proving some theorems, it is necessary to be somewhat more formal about the ground rules. Some notation is helpful. Fix a first order language $\mathcal{L}$ of unary predicate symbols and a set T of formulas of the form $U \subseteq V$ where U and V are unary predicates in the language. Given a set of formulas Φ, the theory T *induces* a preordering on the formulas of Φ by provable implication. In other words, the induced preorder has, as its elements, formulas $\phi \in \Phi$ and it is preordered by taking $\phi \gtrsim \phi'$ iff $T \vdash \phi \rightarrow \phi'$. For the remainder of this paper, fix the theory T and assume that W is a new unary predicate symbol not in the language of T. It will simplify matters to assume that $U \subseteq V$ is in T whenever $T \vdash U \subseteq V$. Let A be the preorder which T induces on formulas of the form $U(x)$ where U is a unary predicate symbol of $\mathcal{L}$. Then we have the following:

Theorem 1 *The preorder which T induces on formulas of the form*

$$W \subseteq U_1 \cup \cdots \cup U_n$$

is exactly the upper powerdomain of A.

Proof: Suppose we are given formulas

$$\phi \equiv W \subseteq U_1 \cup \cdots \cup U_n$$
$$\psi \equiv W \subseteq V_1 \cup \cdots \cup V_m$$

It is not at all difficult to see that if, for each predicate U_i, there is predicate V_j such that $U_i \subseteq V_j$ is in the theory T, then

$$T \vdash \phi \rightarrow \psi.$$

What is less obvious is the fact that this is *the only way* such an implication can be proved. Suppose we know that $T \vdash \phi \rightarrow \psi$. By the Soundness Theorem for First Order Logic, we know that

$$T \models \phi \rightarrow \psi \tag{2}$$

Suppose that (2) holds, but there is a predicate U_i such that $U_i \subseteq V_j$ is not in T for any V_j. We demonstrate a contradiction. Define a model $\mathcal{A}$ of $T \cup \{\phi\}$ as follows. The universe of $\mathcal{A}$ is the set of predicate symbols of $\mathcal{L}$ (this does not include W). If U is a predicate symbol of $\mathcal{L}$, it is interpreted in $\mathcal{A}$ as the set of predicate symbols $V \in \mathcal{L}$ such that $U \subseteq V$ is in T. The predicate symbol W is interpreted as the set $\{U_1, \ldots, U_n\}$. Let $[\![U]\!]$ be our notation for the interpretation of a predicate symbol U. I claim that $\mathcal{A} \models T \cup \{\phi\}$. If $U \subseteq V$ is in T and $U' \in [\![U]\!]$, then $U' \subseteq U$ is in T so $U' \subseteq V$ is in T. Thus $U' \in [\![V]\!]$ and it follows that $[\![U]\!] \subseteq [\![V]\!]$ as desired. That $\mathcal{A} \models \phi$ follows immediately from the interpretation of W. On the other hand, I also claim that $\mathcal{A} \not\models \psi$. Since there is no V_j such that $U_i \subseteq V_j$ is in T, the element U_i is not in $[\![V_1]\!] \cup \cdots \cup [\![V_m]\!]$ and therefore $W \not\subseteq [\![V_1]\!] \cup \cdots \cup [\![V_m]\!]$. ∎

Theorem 2 *The preorder which T induces on formulas of the form*

$$(W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)$$

is exactly the lower powerdomain of A.

Proof: Define formulas

$$\phi' \equiv (W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)$$
$$\psi' \equiv (W \cap V_1 \neq \emptyset) \wedge \cdots \wedge (W \cap V_m \neq \emptyset)$$

If, for each V_j there is a predicate U_i such that $U_i \subseteq V_j$ is in T, then it is easy to show that

$$T \vdash \phi' \rightarrow \psi'$$

Conversely, if this holds then we also have

$$T \models \phi' \rightarrow \psi' \qquad (3)$$

Suppose that (3) holds, but there is a predicate V_j such that $U_i \subseteq V_j$ is not in T for any U_i. I will demonstrate a contradiction. Let $\mathcal{A}$ be the model of T given in the proof of Theorem 1. Obviously $\mathcal{A} \models \phi'$. However, $[\![V_j]\!] \cap [\![W]\!]$ is the emptyset since there is no U_i in $[\![V_j]\!]$. ∎

4 Other powerdomains?

In this section I will look at a few more second order predicates such as the ones which were used to characterize the upper and lower powerdomains in the previous section. I begin by discussing the convex ordering and its information-theoretic significance using sets of records. A logical characterization of the convex powerdomain is then provided and a correspondence theorem similar to Theorems 1 and 2 will be given. I will then define a close relative of the *sandwich powerdomain* of Buneman, Davidson, Ohori and Watters [BDW88, BO86, BJO91] which has been used used for the semantics of databases.

Under the convex ordering, *none* of the three sets of records s, s_1, s_2 given earlier are related. The following set s_3 satisfies $s_3 \gtrsim^\natural s$:

```
{ name = { first = "Mary" },
  socsec = 4392349703,
  age = 2 }

{ name = { first = "Todd",
           last = "Smith" },
  socsec = 923799210,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 429238406,
  age = 2 }

{ name = { first = "John",
           last = "Smith" },
  socsec = 229068403,
  age = 2 }
```

Note that no new names were added in s_3 as we added the name "Beth" in s_1 (although the two John Smith's were disambiguated), and no names were removed from s as we removed "Todd" in s_2. On the other hand, the records of s_3 are considerably more specific than those in s. For example, if we assume that now two children have the same social security number, then no further refinement of s_3 will have more or less than four children. (However, sets with multiple names associated with the same social security number are permitted in the convex powerdomain.) As with the other powerdomains, it is easy to produce examples which show that the convex powerdomain of a poset may not satisfy anti-symmetry. The following can be proved by combining the proofs of Theorems 1 and 2:

Theorem 3 *The preorder which T induces on formulas of the form*

$$(W \subseteq U_1 \cup \cdots \cup U_n) \wedge$$
$$(W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)$$

is exactly the convex powerdomain of A. ∎

The convex powerdomain is generally considered to be more "natural" than the upper and lower powerdomains; this view is supported, for example, by the categorical characterizations of the three powerdomains [HP79, GS90] as well as considerations

from the semantics of concurrency. However, when one views the three powerdomains from the standpoint of this paper, the convex powerdomain seems to entail a peculiar assumption. Each of the records in a database under the convex ordering must convey *both* upper and lower information; or, to put it another way, the upper and lower information conveyed by the database must be conveyed by the *same* set of predicates. We are permitted to use formulas of the form

$$\begin{array}{l}(W \subseteq U_1 \cup \cdots \cup U_n) \wedge \\ (W \cap U_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U_n \neq \emptyset)\end{array} \tag{4}$$

but not formulas of the the more general form

$$\begin{array}{l}(W \subseteq U_1 \cup \cdots \cup U_m) \wedge \\ (W \cap U'_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U'_n \neq \emptyset)\end{array} \tag{5}$$

While it makes perfectly good sense to make a restriction to formulas as in (4), it also seems reasonable, in some circumstances, *not* to make this restriction. The use of formulas such as those in (5) in the theory of databases has been discussed in several publications [BDW88, BO86, BJO91] using an operator known as the *sandwiches powerdomain*. Although questions about the categorical and topological significance of sandwiches are only beginning to be investigated, their information-theoretic significance and potential applications suggest interesting lines of investigation. I now define an operator which has a strong kinship to the sandwiches domain and demonstrate a logical characterization for it.

Definition: Let $\langle A, \succsim \rangle$ be a preorder. A *mix* (on A) is a pair $(u, v) \in \mathcal{P}^*_f A \times \mathcal{P}^*_f A$ such that $v \succsim^\sharp u$. We define the *mixed powerdomain* $A^{(\sharp,\flat)}$ to be the set of mixes on A under the preorder given by taking $(u, v) \precsim (u', v')$ iff $u \precsim^\sharp u'$ and $v \precsim^\flat v'$. As with other preorders, we write $x \succsim y$ if $y \precsim x$. We also write $x \approx y$ if $x \succsim y$ and $x \precsim y$. ∎

As aside on uniformity of notation, we might have written $\precsim^{(\sharp,\flat)}$ rather than $\precsim$, but this becomes a rather cumbersome notation in calculations.

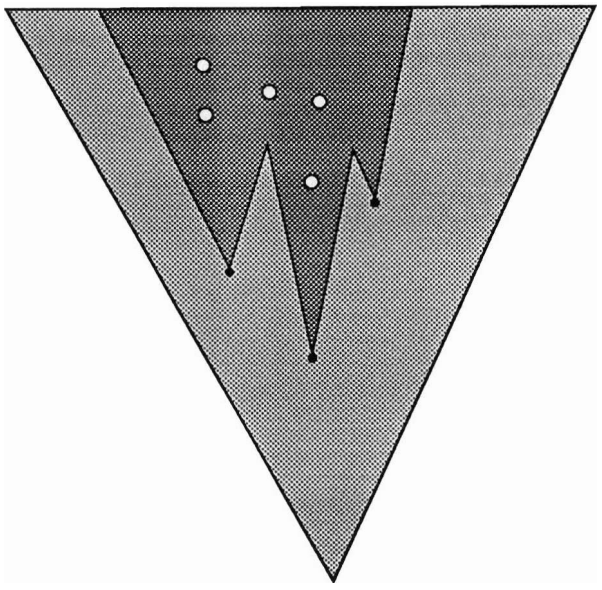

Figure 1: *A mixed powerdomain element (u, v) is illustrated above. The elements of the set u are indicated as closed circles (dots). They determine a shaded upper set within which the elements of v must lie. The elements of v are represented as open circles.*

The choice of preordering on the pairs $(u,v) \in A^{(\sharp,\flat)}$ is unsurprising. It is slightly less clear why only pairs (u,v) with $v \gtrsim^\sharp u$ are used. To understand this restriction and get a feeling for the mixed powerdomain, it is best to look at some examples. Rather than representing elements of the mixed powerdomain with a pair of sets of records it is convenient to write a set of records which are *tagged* to indicate whether they belong in the first or second coordinate of the pair. I will use a tag # for the records in the first coordinate (since this looks like the $\sharp$ sign) and a tag b for records in the second coordinate (since this looks like a $\flat$ sign). Forget, for the moment, about the condition that $v \gtrsim^\sharp u$ and consider the following set of (tagged) records t:

```
b{ name = { first = "Mary" } }

b{ name = { first = "Todd" } }

b{ name = { first = "John" } }

#{ age = 2 }
```

This is very similar in information content to the set of records s which were considered earlier. It describes a group of two year olds which must include a "Mary", a "Todd" and a "John". Here is another set of records t_1 similar to s_1:

```
b{ name = { first = "Mary" },
   age = 2 }

b{ name = { first = "Todd" },
   age = 2 }

b{ name = { first = "John",
            last = "Smith" },
   age = 2 }

b{ name = { first = "Beth" }
   age = 3 }

#{ age = 2 }

#{ age = 3 }
```

which allows that the nursery is now enrolling three year olds as well as two year olds. However, the following set of records is *nonsense:*

```
b{ name = { first = "Mary" },
   age = 2 }

b{ name = { first = "Todd" },
   age = 2 }

b{ name = { first = "John",
            last = "Smith" },
   age = 2 }

b{ name = { first = "Beth" }
   age = 3 }

#{ age = 2 }
```

because Beth is incorrectly recorded as a three year old or the new admissions policy has not be properly entered. In order for a set of mixed records such as these to make sense, it is essential that, for each $\flat$-record, there is a $\sharp$-record which applies to it. Otherwise, the set of mixed records is "inconsistent." As another example, a dating service may have a database d:

```
b{ name = { first = "Sharon" },
   age = 26,
   married? = false}

b{ name = { first = "David" },
   age = 28,
   married? = false}

b{ name = { first = "Mabel" },
   age = 58,
   married? = false}
```

```
b{ name = { first = "Lee" },
   age = 55,
   married? = false}

#{ married? = false }
```

but trouble may arise from adding a record such as

```
b{ name = { first = "John" }
   age = 30,
   married? = true }
```

The sandwiches powerdomain is defined to include records like t above; t is not in the mixed powerdomain because the $\flat$-records are missing their age fields.

Definition: A *sandwich* is a pair

$$(u, v) \in \mathcal{P}_f^* A \times \mathcal{P}_f^* A$$

such that there is a set $w \in \mathcal{P}_f^* A$ such that $w \gtrsim^\sharp u$ and $w \gtrsim^\flat v$. The *sandwich powerdomain* of A is the set of sandwiches under the ordering $(u, v) \precsim (u', v')$ iff $u \gtrsim^\sharp u'$ and $v \gtrsim^\flat v'$

Obviously, any mix is a sandwich. Unfortunately, the logical interpretation of the sandwich powerdomain in the sense of this paper does not seem to be straight-forward.

To characterize the mixed powerdomain logically, it is necessary to generalize from formulas such as (4) to a set of formulas such as (5). It is easiest to do this for a subset of the mixes which is isomorphic to the mixed powerdomain. Suppose $(u, v) \in A^{(\sharp,\flat)}$. Since $u \gtrsim^\sharp v$, we have $u \cup v \cong u$. In particular, $(u \cup v, v)$ is a mix which is equivalent to (u, v). Since we are only really interested in the equivalence classes of mixes, we might therefore have included this condition our earlier definition of the mixed powerdomain. However, this would have complicated the examples slightly, since some elements would need to be listed twice.

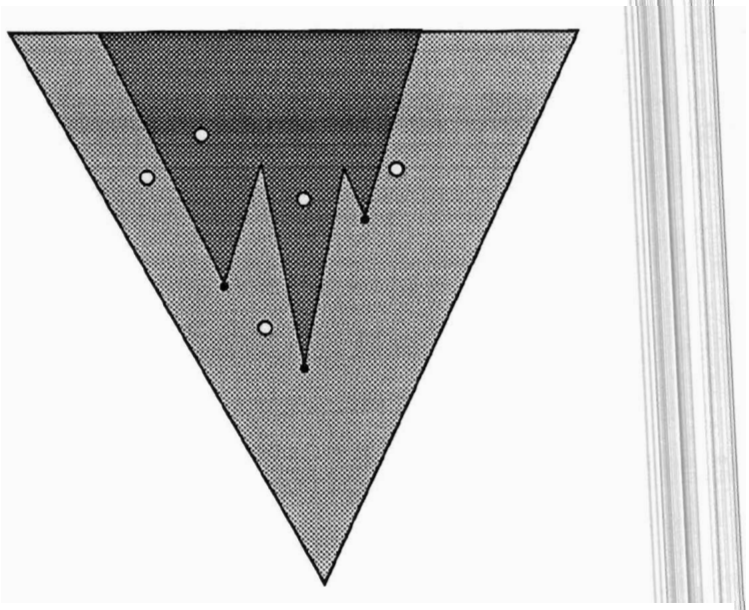

Figure 2: *A sandwich (u, v) is illustrated above. The elements of the set u are indicated as closed circles (dots). They determine a shaded upper set. The elements of v are represented as open circles; each element of v is required to have an upper bound in the shaded region.*

Recall that T is a set of formulas of the form $U \subseteq V$ where U and V are unary predicates in a fixed first order language $\mathcal{L}$. A is the preorder which T induces on formulas of the form $U(x)$ where U is a unary predicate symbol of $\mathcal{L}$.

Theorem 4 *The preorder which T induces on formulas of the form*

$$(W \subseteq U_1 \cup \cdots \cup U_m \cup U'_1 \cup \cdots \cup U'_n) \wedge (W \cap U'_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U'_n \neq \emptyset)$$

is isomorphic to the mixed powerdomain of A.

Proof: Suppose we have formulas

$$\phi \equiv W \subseteq U_1 \cup \cdots \cup U_m \cup U'_1 \cup \cdots \cup U'_n$$
$$\phi' \equiv (W \cap U'_1 \neq \emptyset) \wedge \cdots \wedge (W \cap U'_n \neq \emptyset)$$
$$\psi \equiv W \subseteq V_1 \cup \cdots \cup V_p \cup V'_1 \cup \cdots \cup V'_q$$
$$\psi' \equiv (W \cap V'_1 \neq \emptyset) \wedge \cdots \wedge (W \cap V'_q \neq \emptyset)$$

and define

$$\mathcal{U} = \{U_1, \ldots, U_m\}$$
$$\mathcal{U}' = \{U'_1, \ldots, U'_n\}$$
$$\mathcal{V} = \{V_1, \ldots, V_p\}$$
$$\mathcal{V}' = \{V'_1, \ldots, V'_q\}$$

We must show that

$$T \vdash (\phi \wedge \phi') \rightarrow (\psi \wedge \psi') \qquad (6)$$

if and only if

1. for each predicate symbol $U \in \mathcal{U} \cup \mathcal{U}'$ there is a predicate symbol $V \in \mathcal{V} \cup \mathcal{V}'$ such that $U \subseteq V$ is in T, and
2. for each predicate symbol $V \in \mathcal{V}'$ there is a predicate symbol $U \in \mathcal{V}$ such that $U \subseteq V$ is in T, and

As with the earlier proofs of this kind, the harder part of the proof is showing that (6) implies items (1) and (2). As before, we utilize the Soundness Theorem to prove each of these items by contradiction. Define a model $\mathcal{A}$ of $T \cup \{\phi, \phi'\}$ as follows. The universe of $\mathcal{A}$ is the set of predicate symbols of $\mathcal{L}$ (this does not include the distinguished predicate symbol W). If U is a predicate symbol of $\mathcal{L}$, it is interpreted in $\mathcal{A}$ as the set of predicate symbols $V \in \mathcal{L}$ such that $U \subseteq V$ is in T. The predicate symbol W is interpreted as the set $\{U_1, \ldots, U_m, U'_1, \ldots, U'_n\}$. That $\mathcal{A}$ is a model of $T \cup \{\phi, \phi'\}$ follows immediately from the definitions.

Now, suppose that (1) *fails*. Then there is some U such that

$$U \notin [\![V_1]\!] \cup \ldots \cup [\![V_p]\!] \cup [\![V'_1]\!] \cup \ldots \cup [\![V'_q]\!]$$

Since $U \in [\![W]\!]$, it follows that $[\![W]\!] \not\subseteq [\![V_1]\!] \cup \ldots \cup [\![V_p]\!] \cup [\![V'_1]\!] \cup \ldots \cup [\![V'_q]\!]$ and therefore $\mathcal{A}$ does not satisfy ψ. Suppose that (2) *fails*. Then there is some V'_j such that $U'_i \notin [\![V'_j]\!]$ for each U'_i. To get the desired contradiction, we want to use a new model $\mathcal{A}'$ which is the same as $\mathcal{A}$ except $[\![W]\!] = U'_1, \ldots, U'_n$. Clearly, $\mathcal{A}' \models T \cup \{\phi, \phi'\}$. But $[\![V'_j]\!] \cap [\![W]\!] = \emptyset$ so $\mathcal{A}' \not\models \psi'$. ∎

5 Acknowledgements.

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