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NOTE: At the time of publication, author Saswati Sarkar was affiliated with the Indian Institute of Science. Currently (February 2005), she is a faculty member in the Department of Electrical and Systems Engineering at the University of Pennsylvania.

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## Abstract

Cellular systems have hitherto been modeled mostly by graphs for the purpose of channel assignment. However, hypergraph modeling of cellular systems offers a significant advantage over graph modeling in terms of the total traffic carried by the system. For example, we show that a 37-cell system when modeled by a hypergraph carries around 30% more traffic than when modeled by a graph. We study the performance of channelized cellular systems modeled by hypergraphs in comparison with those modeled by graphs. For this purpose, we have evaluated the capacities of these cellular networks defined [3]. Evaluation of the capacity necessitates generation of maximal independent sets of hypergraphs. We describe some new algorithms that we have developed for this purpose.

## Comments

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# Hypergraph Models for Cellular Mobile Communication Systems

Saswati Sarkar and Kumar N. Sivarajan, *Member, IEEE*

**Abstract**—Cellular systems have hitherto been modeled mostly by graphs for the purpose of channel assignment. However, hypergraph modeling of cellular systems offers a significant advantage over graph modeling in terms of the total traffic carried by the system. For example, we shall show that a 37-cell system when modeled by a hypergraph carries around 30% more traffic than when modeled by a graph. We study the performance of channelized cellular systems modeled by hypergraphs in comparison with those modeled by graphs. For this purpose, we have evaluated the capacities of these cellular networks defined in [3]. Evaluation of the capacity necessitates generation of maximal independent sets of hypergraphs. We describe some new algorithms that we have developed for this purpose.

## I. INTRODUCTION

**T**HE TRAFFIC in cellular systems is usually too high to allow the use of a channel for one call at a time—radio channels must be used simultaneously for more than one call. This is known as *channel reuse*, and the cells using the same channel are termed *cochannel cells*. *Channel reuse* causes interference, which, in turn, degrades the transmission quality. However, if the cells reusing the same channel<sup>1</sup> simultaneously are far apart, then the interference produced is low. Consequently the deterioration in transmission quality becomes tolerable.

The usual approach is to determine the least distance  $D$  between cochannel cells<sup>2</sup> such that the transmission quality requirements such as the minimum signal-to-interference or carrier-to-interference ratio ( $S/I$  or  $C/I$ ) are met in all cells, even if all cells at a mutual distance of  $D$  or greater are using the same channel simultaneously. This distance  $D$  is known as the *reuse distance*. A cell can use a channel if no other cell within distance  $D$  is using the channel.

The situation can be represented by a *graph model*. In the *graph* representation of a cellular system, we have the following.

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<sup>1</sup>These cells are called cochannel cells.

<sup>2</sup>Determination of the distance between hexagonal cells using hexagonal coordinate system has been discussed in [5]. This distance depends on the cell radius  $R$ , which we shall assume to be  $1/\sqrt{3}$ , unless otherwise mentioned. This is equivalent to assuming that the distance between adjacent cells is unity.

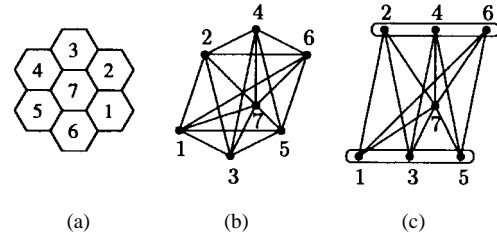


Fig. 1. (a) A 7-cell system. (b) Its graph representation with  $D = 2$ . The circles represent the vertices and the straight lines the edges of the graph. Note that the vertices corresponding to the cells separated by a distance two or more, e.g., two and five are not joined by edges. (c) Its hypergraph representation with the reuse conditions given in Example 1.2. The circles represent the vertices and the straight lines and the curved lines (ovals) the edges of the hypergraph.  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$ ,  $\{4, 5\}$ ,  $\{5, 6\}$ ,  $\{1, 6\}$ ,  $\{1, 7\}$ ,  $\{2, 7\}$ ,  $\{3, 7\}$ ,  $\{4, 7\}$ ,  $\{5, 7\}$ ,  $\{6, 7\}$ ,  $\{2, 4, 6\}$ , and  $\{1, 3, 5\}$  are the edges. Note that  $\{2, 4, 6\}$  is an edge, but not  $\{2, 4\}$ .

- 1) Each vertex represents a cell.
- 2) An edge exists between two vertices if and only if the distance between the corresponding cells is less than the reuse distance  $D$ .

### A. Example 1.1

Consider the seven-cell system in Fig. 1(a). With  $D = 2$ , the graph representation is given in Fig. 1(b).

Thus, a set of cells which can use a channel simultaneously forms an independent set<sup>3</sup> in the graph representing the cellular system. For example,  $\{2, 5\}$  forms an independent set of the graph shown in Fig. 1(b). These cells can reuse a channel simultaneously since the distance between them is  $\geq 2$ .

In this context, we recall a simple and commonly used fixed channel allocation algorithm, which is designed for regular hexagonal cellular systems. The required reuse distance given by  $\sqrt{\kappa} = \sqrt{i^2 + ij + j^2}$ , where  $i$  and  $j$  are some nonnegative integers, is determined based on the  $S/I$  ratio requirements.  $\kappa$  is called the *reuse ratio*. If  $n$  is the total number of channels,  $n/\kappa$  channels are allocated to each cell such that the cochannel cells are separated by a distance of at least  $\sqrt{3R\kappa}$ , where  $R$  is the cell radius. The graph model is a generalization of this scheme in as much as it allows any two cells to use the same channel only if they are separated by at least the reuse distance  $D$ , which is determined from the  $S/I$  ratio requirements. The quantity  $D^2/3R$  turns out to have the

<sup>3</sup>An *independent set* of a graph is a set of vertices such that no two vertices of the set are joined by an edge.

same value as  $\kappa$  for regular hexagonal systems. However, the application of the graph model is not restricted to the design of fixed channel allocation schemes. It may be used for designing many channel allocation algorithms, including dynamic channel allocation algorithms, as we shall discuss later.

The weakness of the graph model can be brought out by studying a regular hexagonal system. In the case of a regular hexagonal system, there can only be discrete distances  $D$ , such as  $1, \sqrt{3}, 2, \dots$ . The worst case transmission quality in the system depends on the reuse distance  $D$  only. Hence, there can only be discrete values of the worst case transmission quality possible, and these are generally quite far apart. So, if the required transmission quality falls between any two discrete values, say  $T_1$  and  $T_2$ , and say the corresponding reuse distances are  $D_1$  and  $D_2$ , then we have to settle for the transmission quality better than that which is required, and, hence, the greater of the two reuse distances. Thus, the full potential for channel reuse offered by the system is not realized.

We shall study only regular hexagonal systems in all subsequent examples, but all our observations apply to irregular systems as well—the graph model has the same weaknesses with respect to the hypergraph model, which we discuss next, for irregular systems as for regular systems.

Hypergraph modeling, introduced in [3], removes this weakness. A hypergraph  $H$  is formally defined as  $H = (V, E)$ , where  $V$  is the set of vertices and  $E$  is the set of edges, where each edge  $e$  is a nonempty subset of  $V$  such that  $\cup_{e \in E} e = V$  [1]. The main distinction between a graph and a hypergraph is that in a graph an edge can have no more than two vertices, but this restriction does not hold for a hypergraph. Hypergraph modeling of cellular systems is as follows.

- 1) Each cell corresponds to a vertex.
- 2) A *forbidden set* is a group of cells all of which cannot use a channel simultaneously. A *minimal forbidden set* is a forbidden set which is minimal with respect to this property, i.e., no proper subset of a *minimal forbidden set* is *forbidden*. An edge is a *minimal forbidden set*.
- 3) A set which is not *forbidden* is *independent*. A group of cells using the same channel cannot be forbidden. Hence, any group of cells which may use the same channel simultaneously forms an *independent set* of the underlying hypergraph. A *maximal independent set* is an independent set which is maximal with respect to the property of independence.

#### A. Advantage of Hypergraph Modeling

Hypergraph modeling of a cellular system offers much greater reuse of channels, while maintaining the required transmission quality. This can be explained through an example.

1) *Example 1.2:* Let us consider the seven-cell system shown in Fig. 1(a). The assumed model of interference is as follows.

- 1) Interference produced in cell  $u$  due to the use of the same channel in  $v$  is  $d(u, v)^{-4}$ , where  $d(u, v)$  is the center-to-center distance between cells  $u$  and  $v$ .
- 2) Total interference produced in cell  $u$  = interference produced by all other cells using the same channel =  $\sum_{v \in C} d(u, v)^{-4}$ , where  $C$  is the set of cells using the same channel as  $u$ , barring  $u$ . An *additive* model of interference is thus assumed.
- 3) The cell radius is assumed to be  $1/\sqrt{3}$ . Hence, the distance between adjacent cells is one.
- 4) Let the required transmission quality be that the maximum interference must be  $\leq 1/5$ .

2) *Graph Model:*  $D = \sqrt{3}$  produces a maximum interference of  $2/9$ , for example, if cells 2, 4, and 6 use the same channel simultaneously.

$D = 2$  produces a maximum interference of  $1/16$ , for example, if cells 1 and 4 use the same channel simultaneously.

Since the maximum interference cannot exceed  $1/5$ ,  $D = 2$  must be selected as the reuse distance. [Refer to Fig. 1(b).]

So, cells numbered 2 and 4 can *never* use the same channel simultaneously.

3) *Hypergraph Model:* A set of cells forms a forbidden set if they cannot use the same channel simultaneously, i.e., if the use of the same channel in all the cells violates the interference constraint. Here, the interference produced in cell 4 due to the use of the same channel in 2 is  $1/9$  and similarly for cell 2. This is below the given interference threshold. So, cells 2 and 4 do not form a *forbidden set*, and, hence, they form an *independent set*. [Refer to Fig. 1(c).] Thus, 2 and 4 can use the same channel at least *sometimes*. The word *sometimes* has been used because if 6 is using a channel, then both 2 and 4 cannot use the same channel, as that would produce an interference of  $2/9$  in 4, and, thus,  $\{2, 4, 6\}$  is a forbidden set. However, as stressed before, cells 2 and 4 can *never* use the same channel together if the system is modeled by a graph.

It was proved under certain assumptions in [3] that if the offered traffic intensity<sup>4</sup>  $r$  is less than or equal to a certain quantity  $r_0$ , which depends on the cellular system and the traffic pattern, there exists a channel assignment algorithm which achieves arbitrarily low-blocking probabilities if the number of available channels is sufficiently large. For  $r \geq r_0$ , no channel assignment algorithm can produce zero blocking for any number of channels.  $r_0$  has been termed the *capacity* of the system.  $r_0^{-1}$  is given by the following linear program:

$$\min \left\{ \sum_{j=1}^M X_j : X_j \geq 0, \sum_{j=1}^M X_j a_{ij} \geq p_i, i = 1, \dots, N \right\}. \quad (\text{LP1})$$

<sup>4</sup>Offered traffic intensity in the system is defined as follows. If  $A_i$  denotes the expected number of calls that would be in progress in cell  $i$ , if all call requests in that cell could be honored, and  $n$  denotes the number of channels available to the system, then the intensity of the offered traffic in cell  $i$  is  $r = A_i/n$  Erlangs per channel.

Here

$N$  number of cells in the system;  
 $p_i$  fraction of total traffic offered in cell  $i$ ;  
 $M$  number of maximal independent sets of the hypergraph representing the system

$$a_{ij} = \begin{cases} 1, & \text{if the } i\text{th cell is in the } j\text{th} \\ & \text{maximal independent set} \\ 0, & \text{otherwise} \end{cases}$$

$N_j$  size of the  $j$ th maximal independent set;  
 $X_j$  the  $j$ th variable of the linear program.

Thus,  $r_0$  is a measure of the performance of a cellular system. The assumptions in deriving this result are as follows.

- 1) A call is either accepted or blocked instantaneously. There is no queuing.
- 2) The underlying model of offered traffic is independent from cell to cell, and, thus, for example, handoffs are not allowed.
- 3) The underlying model of offered traffic satisfies the ‘‘asymptotic traffic property’’ (ATP), which states that

$$\lim_{n \rightarrow \infty} C(k, n)/n = \min(r, 1), \quad \text{where } \lim_{n \rightarrow \infty} k/n = r.$$

$C(k, n)$  is the carried traffic in a one-cell system when the offered traffic is  $k$  and the number of available channels is  $n$ .

This paper is organized as follows. Section II gives an algorithm for the generation of maximal independent sets of a hypergraph. Section III shows that hypergraph modeling gives better performance than graph modeling for a cellular system. The interference model we use in this section will consider the effect of shadow fading, unlike the simple model we used in the seven-cell example above. Section IV gives some approximations for the capacity  $r_0$  of the system.

It is worth mentioning at this point that, like the graph model, the application of the hypergraph model is not restricted to the design of any particular channel allocation scheme. As we shall point out later, hypergraph models can be used to generate very efficient fixed channel allocation and dynamic channel allocation schemes (which outperform those designed using the graph model).

## II. GENERATION OF MAXIMAL INDEPENDENT SETS OF A HYPERGRAPH

The evaluation of  $r_0$  requires generation of maximal independent sets of a hypergraph for systems modeled by hypergraphs. No algorithm for this purpose could be found in the literature. This section describes an algorithm for this problem. First, we shall discuss a simple algorithm which readily comes to mind. We shall call this the *straightforward algorithm*.

The straightforward algorithm starts from the set of all vertices and generates the subsets in a recursive manner. Whenever a subset is generated, it checks whether it is an independent set or not. If so, none of its subsets are generated, and it is to be stored. Before any new storage, all the independent sets stored until that point are checked to determine if any of the subsets of the independent set newly generated are present. If any such is found, it is eliminated. If the subset generated is not found to be an independent set, its subsets are generated and checked in the same manner.

Checking for independence can be done by testing whether any edge is a subset of the set being tested, if all the edges are known, or by testing for any other criterium for independence.

Any subset of an independent set is clearly an independent set, but never a maximal independent set. Since we are interested in the generation of maximal independent sets, once a subset has been found to be independent, its subsets need not be generated. However, if a subset is found to be an independent set, we cannot be sure that it is a maximal independent set because some of its supersets to be generated later may be found to be independent. Hence, storage and checking at each point is necessary.

The performance of the straightforward algorithm can be summarized as follows.

- 1) Memory requirement grows exponentially with the number of nodes because at least all maximal independent sets have to be stored and the number of such maximal independent sets grows exponentially with the number of vertices.
- 2) Speed is very slow. The time required to generate all the maximal independent sets grows exponentially with the number of vertices.

Thus, it has exponential complexity both in time and space.

Another disadvantage of the straightforward algorithm is that all the maximal independent sets are generated at the end. Thus, even if we want only a few maximal independent sets, we have to wait for the algorithm to terminate.

We will now describe an algorithm which is much faster and whose memory requirement grows polynomially with the number of nodes. This algorithm has been developed by extending an algorithm for the generation of all maximal cliques of a graph given in [2].

This algorithm generates the maximal independent sets of a hypergraph. It consists of three sets:

- 1) `compsub`;
- 2) `candidates`;
- 3) `not`.

The set `compsub` is a set of vertices all of which form an independent set. The set `candidates` is the set of all vertices that are eligible to extend `compsub`, i.e., each of which forms an independent set with `compsub`.

The set `not` is the set of all vertices which at an earlier stage already served as an extension of the present configuration of `compsub` and are now explicitly excluded.

A recursively defined **extension** operator generates all extensions of the given configuration of `compsub` that it can make with the given set of `candidates` and that do not

contain any vertex in *not*. All extensions of *compsub* containing any vertex in *not* have already been generated. The basic mechanism now consists of the following five steps.

- 1) Selection of the first vertex in *candidates*.
- 2) Adding the selected candidate to *compsub*.
- 3) Creating a new set *candidates* from the old set by removing each vertex which does not form an independent set with the selected candidate and *compsub* and forming a new set *not* in a similar manner from the old set *not*.
- 4) If both *not* and *candidates* sets are empty, no further extension of the present configuration of *compsub* is possible, nor is there a larger independent set including the present configuration of *compsub* in the hypergraph since *not* is empty. Hence, *compsub* contains a maximal independent set, which is generated. If only *candidates* is empty, no further extension of the present configuration of *compsub* is possible and there exists a larger independent set including the present configuration of *compsub*. This independent set has been generated before. Thus, the algorithm backtracks. If *candidates* is nonempty (irrespective of whether *not* is nonempty), the **extension** operator is called to operate on the sets just formed.
- 5) Upon return, removal of the selected candidate from *compsub* and its addition to the old set *not*.

#### A. Performance of the New Algorithm

- 1) The time required to list the maximal independent sets was 0.3 s for the 19-cell system shown in Fig. 2, which is ten times faster than the straightforward algorithm. The required transmission quality is that the interference threshold should not exceed 0.375. The model of interference is the same as that of Example 1.2. For the 37-cell case (see Fig. 3) with the same interference threshold, 330 s were required, whereas the straightforward algorithm required more than two days. In both cases, an IBM RS/6000 model 340 was used. This is a vast improvement in speed. The improvement in speed results mainly from the absence of any comparisons with previously generated independent sets. The maintenance of the *not* set ensures that the independent set on output is a maximal independent set and removes the necessity for storage and comparisons to eliminate independent sets which are not maximal. However, the worst case time complexity of this algorithm is also exponential in the number of vertices since the number of maximal independent sets can grow exponentially with the number of vertices.
- 2) The memory requirement is obviously less than that required by the straight forward algorithm since storage of independent sets is not necessary. The maximum memory requirement is  $P + NP$ , where  $P$  is the maximum size of a maximal independent set and  $N$  is the number of nodes ( $P \leq N$ ). Hence, the memory requirement increases polynomially with the number of vertices.

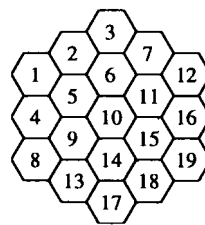


Fig. 2. 19-cell example.

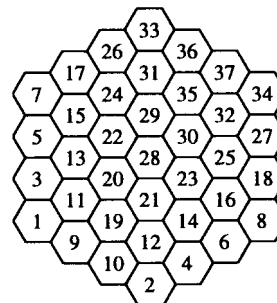


Fig. 3. 37-cell example.

- 3) Another advantage of this algorithm is that it generates the maximal independent sets progressively. This is unlike the straightforward algorithm, which can generate the maximal independent sets only at the end. It need not wait until the end to generate at least the first few maximal independent sets. So, if a certain number of maximal independent sets are required for some application, the procedure can be terminated after that number has been obtained, and all the maximal independent sets need not be unnecessarily generated.

In conclusion, this new algorithm gives a significant improvement in performance.

### III. COMPARISON OF THE PERFORMANCE OF CELLULAR SYSTEMS MODELED BY HYPERGRAPHS AND GRAPHS

In this section, we shall compare the performance of the same cellular system with the same transmission quality ( $S/I$ ) requirements, modeled by hypergraphs and graphs. Our performance measure will be the capacity of cellular systems. We shall point out that hypergraph modeling of cellular systems is better than graph modeling in general because hypergraph modeling enables the user to exploit the reuse conditions better. The required transmission quality is that the worst case  $S/I$  ratio at the base station should be greater than a given threshold, with a given probability. Reuse of channels causes interference at the base station as well as at the mobiles. Interference at the base station is produced by mobiles using the same channel in the interfering cells. We assume that the mean interference power is proportional to the fourth power of the distance between the interfering mobile in one cell and the base station of another cell. The *worst case*  $S/I$  ratio occurs in a cell when the signal at the base station is the least, i.e., the mobile in the same cell is farthest from the base station, and the interference is the greatest, i.e., when all the other cells which are allowed to use the same channel do so simultaneously and

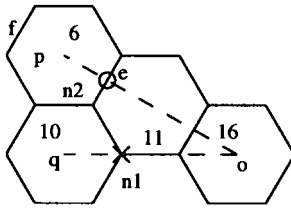


Fig. 4. Nearest points of cells 10 (center  $q$ ) and 6 (center  $p$ ) from the center of cell 16 (center  $o$ ) are  $n1$  and  $n2$ , respectively.  $n1$  is the nearest corner, and  $n2$  is not. This is because the segment  $op$  is perpendicular to edges  $e$  and  $f$  of cell 6. These cells belong to the 19-cell system shown in Fig. 2.

the mobile in each is at the closest point from the base station of the cell being considered. The total interference produced by mobiles in all the interfering cells is assumed to be the sum of that produced by each. Shadow fading is assumed to be present and will be modeled by a lognormal distribution. The base station is assumed to be at the center of each cell and antennas are assumed to be omnidirectional.

#### A. Remarks

It can be shown using simple geometry that if  $u$  and  $v$  are two cells and if  $o$  is the center of  $u$  and  $n$  is the nearest point of  $v$  from  $o$ , then  $n$  is that corner of  $v$  which is nearest from  $o$ <sup>5</sup> in all cases except one.<sup>6</sup> That case is the following. When the straight line joining  $o$  and the center of  $v$  is perpendicular to an edge  $e$  of  $v$ , then  $n$  is the point of intersection of the straight line joining  $o$  and the center of  $v$ , with the edge  $e$ .<sup>7</sup>

Under the above assumptions, the S/I ratio at the base station of a cell can be computed. We first introduce some notations.

$R$	The cell radius, which is also the maximum distance traversed by the signal received at the base station.
$\gamma$	The path-loss exponent, which we assume to be four.
$P_s/R^\gamma$	The signal power, which is received at the base station. $P_s$ is assumed to have a lognormal distribution on account of shadow fading and $R^\gamma$ the attenuation because of propagation.
$S_0$	The signal power transmitted from the mobile.
$\sigma^2$	The variance associated with the lognormal fading of the signal.
$I$	The interference at the base station.
$r_i$	The distance between the center of the cell concerned (at which we are computing the S/I ratio) and the closest point of the $i$ th interfering cell, i.e., the distance traversed by the interfering signal from interfering cell $i$ .
$P_i/r_i^\gamma$	The interference power from interfering cell $i$ received at the base station. $P_i$ is assumed to have a lognormal distribution on account of shadow fading. The factor $r_i^\gamma$ appears on account of attenuation due to propagation.

<sup>5</sup>See Fig. 4 for an illustration in the case of the 19-cell system shown in Fig. 2.

<sup>6</sup>See Fig. 4.

<sup>7</sup>This straight line will also be perpendicular to  $f$ , the edge parallel to  $e$ , but we will consider the nearest of the two parallel edges as  $e$ .

$S_i$	The signal power transmitted from the interfering mobile in cell $i$ : we assume $S_i = S_0$ for all $i$ .
$\sigma_i^2$	The variance associated with lognormal fading of the interference produced by the $i$ th interfering cell: $\sigma_i = \sigma$ for all $i$ .

In this notation, the received signal power at the base station can be written as

$$S = \frac{P_s}{R^\gamma}$$

and the received interference power at the base station can be written as

$$I = \sum_i P_i/r_i^\gamma$$

where the summation is over all the interfering cells. (All summations in this example will be over all interfering cells.) The S/I ratio can thus be written as

$$S/I = \frac{P_s}{R^\gamma} / \sum_i P_i/r_i^\gamma.$$

Since  $P_s$  and  $P_i$  are lognormal, in the above notation

$$\ln P_s \sim N(S_0, \sigma^2)$$

and

$$\ln P_i \sim N(S_i, \sigma_i^2), \quad \text{for all } i$$

where  $N(m, \sigma^2)$  denotes the normal probability distribution with mean  $m$  and variance  $\sigma^2$ . This model for S/I is inspired by [6] and [7].

Note that  $S$  is a lognormal random variable. Since  $\ln S = \ln P_s - 4 \ln R$  and  $\ln P_s \sim N(S_0, \sigma^2)$ ,  $\ln S \sim N(S_0 - 4 \ln R, \sigma^2)$ .

Let  $I_i$  be the  $i$ th term in the summation that yields  $I$ . Just as  $S$  is a lognormal random variable, so is  $I_i$ . In fact

$$\ln I_i \sim N(S_0 - 4 \ln r_i, \sigma^2), \quad (1)$$

We assume that the  $I_i$ 's are mutually independent. Since the sum of independent lognormal random variables can be approximated by a lognormal random variable (refer to [8]), we can approximate  $I$  by a lognormal random variable, i.e.,  $\ln I$  is distributed as  $N(m_I, \sigma_I^2)$ , where

$$\sigma_I^2 = \ln \left[ e^{\sigma^2 - 1} \frac{\sum_i r_i^{-8}}{\left( \sum_i r_i^{-4} \right)^2} + 1 \right]$$

and

$$m_I = S_0 + \sigma^2/2 + \ln \sum_i r_i^{-4} - \sigma_I^2/2.$$

(Refer to the Appendix for the determination of  $\sigma_I^2$  and  $m_I$ .)

Since the ratio of two independent lognormal random variables is a lognormal random variable and  $S$  and  $I$  are mutually independent lognormal random variables, S/I is a lognormal random variable.  $\ln(S/I) \sim N(S_0 - 4 \ln R - m_I, \sigma^2 + \sigma_I^2)$ .

Let  $S/I = e^Z$ . Then

$$Z \sim N(m_{SI}, \sigma_{SI}^2)$$

where

$$m_{SI} = S_0 - 4 \ln R - m_I$$

and

$$\sigma_{SI}^2 = \sigma^2 + \sigma_I^2.$$

Then, the condition that the S/I ratio be greater than some threshold  $SI_0$  with probability  $p$  can be written as

$$P(e^Z > SI_0) > p$$

$$P(e^Z \leq SI_0) \leq 1 - p$$

$$P(Z \leq \ln SI_0) \leq 1 - p$$

$$\begin{aligned} \frac{1}{\sqrt{2\pi\sigma_{SI}^2}} \int_{-\infty}^{\ln SI_0} e^{-(z-m_{SI})^2/2\sigma_{SI}^2} dz &\leq 1 - p \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\ln SI_0 - m_{SI})/\sigma_{SI}} e^{-z^2/2} dz &\leq 1 - p \\ \frac{1}{\sqrt{2\pi}} \int_{(m_{SI} - \ln SI_0)/\sigma_{SI}}^{\infty} e^{-z^2/2} dz &\leq 1 - p. \end{aligned} \quad (2)$$

The last step holds if  $\ln SI_0 \leq m_{SI}$ , which is usually the case since typically  $p > 0.5$ . The condition (2) can be written as

$$Q[(m_{SI} - \ln SI_0)/\sigma_{SI}] \leq 1 - p$$

or

$$m_{SI} - \ln SI_0 \leq \sigma_{SI} Q^{-1}(1 - p) \quad (3)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2} dy.$$

Inequality (3) gives the independence condition in the hypergraph model, i.e., a set is independent if and only if inequality (3) is satisfied for each cell in the set. For the graph model, the reuse distance  $D$  is determined from inequality (3).

Let us consider the 37-cell system shown in Fig. 3. The required transmission quality is that the minimum S/I ratio in each cell should exceed a given threshold with 90% probability. The variance associated with shadow fading is assumed to be 1.0. We consider two cases: uniform and nonuniform traffic.

#### A. Case 1. Uniform Traffic

In the first case, we assume the traffic is uniformly distributed amongst cells, i.e.,  $p_i = p_j$  for all  $i$  and  $j$ . The capacities ( $r_0$ ) of this cellular system when modeled by a graph and by a hypergraph are evaluated for various thresholds. The results are plotted in Fig. 5. The following observations can be made.

- 1) The capacities of the cellular system modeled by a graph increase in discrete steps at the S/I thresholds corresponding to discrete values of  $D$ . So, if the S/I threshold is close to, but to the right of one of the jumps in Fig. 5, the capacity of the system remains at the value

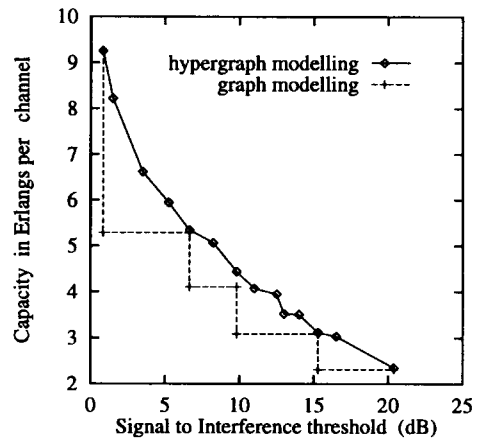


Fig. 5 Capacity versus S/I threshold for a 37-cell system modeled by a hypergraph and a graph. The necessary S/I threshold must be exceeded with 90% probability. Offered traffic is uniform.

corresponding to the immediately lower step. Since the difference in capacities obtained for S/I thresholds corresponding to consecutive  $D$  values is significant, this is a serious shortcoming of the graph model. The hypergraph model shows a more continuous change in capacity with the allowable S/I threshold. Thus, the system can be *tuned* to our needs, and this model is very effective in the intermediate region between two jumps in the graph model. This tuning aspect can be further illustrated by another set of data.

- 2) *Tuning*: The minimum S/I ratio attained versus the S/I threshold specified has been plotted in Fig. 6 for the 37-cell system and in Fig. 7 for the 19-cell system given in Fig. 2 for both the hypergraph and graph models. The curve for graph modeling has discrete steps or jumps. This is fairly intuitive as the graph model cannot achieve a continuous range of values of the S/I ratio. For hypergraph modeling, the curve for the 19-cell system has steps, but these are much smaller than for graph modeling and the jumps are also much closer to each other. For hypergraph modeling of the 37-cell system, the curve is almost continuous. This throws light on the fact that even the hypergraph model cannot achieve all values of the S/I ratio. But the values of S/I ratio it can achieve are closer to each other, and this proximity increases as the number of cells increases. Hence, the hypergraph model has a curve with smaller jumps. However, the curve is almost continuous for the 37-cell system and an actual system will have a large number of cells. So, almost any S/I ratio value can be obtained for an actual system. Thus, the hypergraph model can be *tuned* to our needs.
- 3) For S/I thresholds corresponding to the discrete values of  $D$  in the graph model, the performance of both the graph and hypergraph models is more or less the same.

#### B. Case 2. Nonuniform Traffic

In the second case, we assume the traffic is higher in central cells and less as we move toward the outskirts. The precise



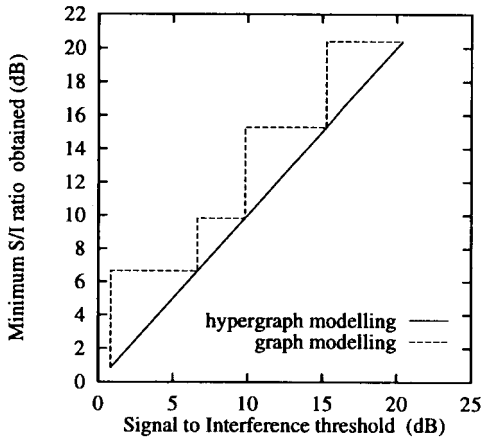


Fig. 6. Minimum S/I ratio in the system with 90% probability versus S/I threshold for the 37-cell system.

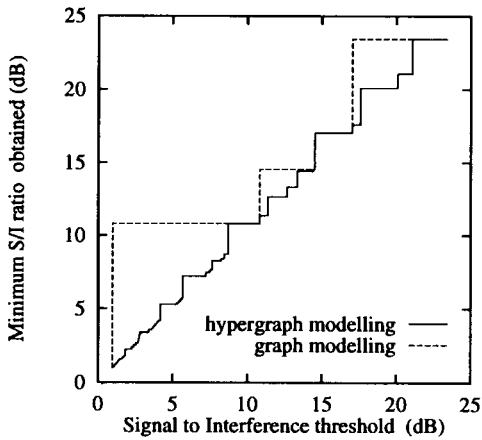


Fig. 7. Minimum S/I ratio in the system with 90% probability versus S/I threshold for the 19-cell system.

model of traffic<sup>8</sup> is

$$p_i = \alpha/q_i, \quad i = 1, 2, \dots, N.$$

$\alpha \rightarrow$  normalizing constant so that  $\sum_{i=1}^N p_i = 1$ ,  $q_i \rightarrow$  ring number of cell, e.g.,  $q_i = 1$  for the cell at the center, i.e., cell 28 in the 37-cell system,  $q_i = 2$  for cells in the next ring, i.e., cells 22, 29, 30, etc.

$$p_i = \begin{cases} 1/50, & i \in \{1, 2, \dots, 10, 17, 18, \\ & 26, 27, 33, 34, 36, 37\} \\ 2/75, & i \in \{11, 12, \dots, 16, 19, 24, \\ & 25, 31, 32, 35\} \\ 1/25, & i \in \{20, 21, 22, 23, 29, 30\} \\ 2/25, & i = 28. \end{cases}$$

The capacities of the 37-cell cellular system for both the graph and hypergraph models have been plotted in Fig. 8. Clearly, the hypergraph model outperforms the graph model in all cases.

<sup>8</sup>Unless otherwise mentioned, nonuniform traffic will refer to this distribution of traffic in the rest of the paper.

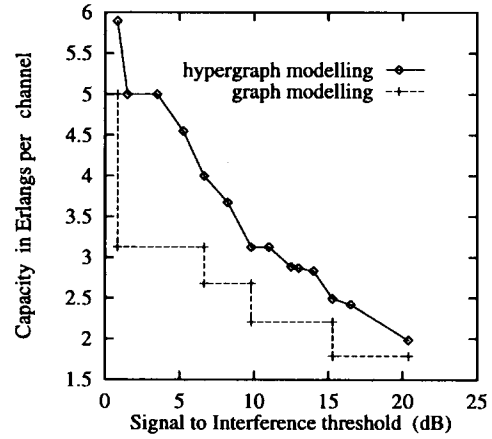


Fig. 8. Capacity versus S/I threshold for hypergraph and graph modeling of the 37-cell system. Offered traffic is nonuniform. The necessary S/I threshold must be exceeded with 90% probability.

### C. Discussion

The difference in the observations for the two cases discussed above can be explained as follows.

Refer to the linear program LP1. Each variable  $X_j$  corresponds to a maximal independent set, and each cell corresponds to a constraint. Thus, there can be only  $N$ -basis variables, and  $N$  is generally much less than  $M$ .<sup>9</sup> Thus, at most  $N$ , maximal independent set variables need to be given nonzero values. These  $N$  will be selected out of  $M$  in such a manner as to satisfy the constraints, assigning as low values to  $X_j$ 's as possible. Now, if the maximal independent sets of the hypergraph model corresponding to the variables that are assigned nonzero values are present in the graph model as well,<sup>10</sup> then the graph and hypergraph model will show no difference in performance. This is precisely the case for the uniform traffic model at S/I thresholds corresponding to the discrete  $D$  values in the graph model. In the 37-cell system of Fig. 3, at a S/I threshold of 0.831 345 dB ( $D = 2.0$  in the graph model), only four maximal independent sets ( $\{11, 12, 15, 16, 28, 31, 32\}$ ,  $\{2, 6, 9, 13, 17, 21, 25, 29, 33, 37\}$ ,  $\{3, 4, 7, 8, 19, 22, 23, 26, 27, 35\}$ , and  $\{1, 5, 10, 14, 18, 20, 24, 30, 34, 36\}$ ) are assigned nonzero values by the linear program. These maximal independent sets are those of the graph model as well because the mutual distance between any two cells in the same set is greater than or equal to 2.0.

In the uniform traffic case, the linear program in both the graph and hypergraph models tends to assign nonzero values to the variables corresponding to those maximal independent sets which contain the largest number of cells. The approach of the linear program will not be the same for nonuniform traffic. It will tend to assign nonzero values to maximal independent sets which include cells with large traffic values. The hypergraph model offers a wider choice in this respect. For a S/I threshold of 0.831 345 dB, the hypergraph model has maximal independent sets which include the cell at the center,

<sup>9</sup> $N$  and  $M$  have the same significance as before.

<sup>10</sup>Any maximal independent set of the graph model will always be an independent set of the hypergraph model for the same transmission quality (S/I) threshold.

with some at distance  $\sqrt{3}$  and some at distance 2.0 from it. These are cells with more traffic than cells at the periphery. The graph model for  $D = 2.0$  has no maximal independent set that includes the central cell and another at distance  $\sqrt{3}$  from it. Hence, the hypergraph model does significantly better than the graph model, even for the interference thresholds corresponding to the discrete  $D$  values of the graph model.

It was shown in [3] that an  $X$ -fixed algorithm<sup>11</sup> maximizes the total traffic carried in any system and, hence, minimizes the average blocking probability, when the number of channels ( $n$ ) is very large ( $n \rightarrow \infty$ ) and the assumptions stated above hold. The average blocking probability obtained in the 37-cell system when modeled by a hypergraph and using the  $X$ -fixed algorithm for an arbitrarily large number of channels and also for finite numbers of channels ( $n = 1200$  and  $3000$ ) is plotted in Figs. 9 and 10. The average blocking probability obtained in the same system for the same numbers of channels when the system is modeled by a graph has also been plotted. These figures clearly bring out the superiority of the hypergraph model over the graph model. Let us consider some performance figures when the number of channels available is 1200. If traffic is uniform, the system yields a 5% blocking probability when the offered traffic is 4503 Erlangs when modeled by a hypergraph and 3474 Erlangs when modeled by a graph. In this case, hypergraph modeling gives almost a 30% increase in carried traffic over graph modeling for the same blocking probability. For nonuniform traffic, the figures are 3286 and 2486 Erlangs, respectively, for hypergraph and graph modeling. It turns out that hypergraph modeling yields a 32% improvement over graph modeling in this case.

The system always performs worse for a finite number of channels compared to the infinite channel case both for graph and hypergraph modeling.

The  $X$ -fixed algorithm performs better when the system is modeled by a hypergraph than when it is modeled by a graph because the hypergraph model offers a much greater choice of cochannel cells<sup>12</sup> for the same S/I ratio requirements. For example, the set of cells  $\{1, 2, 24, 25\}$  is a cochannel set in the 37-cell system of Fig. 3 when the required transmission quality is that the minimum S/I ratio must exceed 12.5 dB with a probability of 0.9 when the system is modeled by a hypergraph, but is not so in the graph model. The  $X$ -fixed algorithm at  $r = 3.89$  Erlangs/channel allocates a certain number of channels to each cell in this cochannel set when the system is modeled by hypergraph, but it cannot do so for the graph model. This feature of hypergraph models can be utilized

<sup>11</sup>The  $X$ -fixed algorithm allocates  $[x_i n]$  channels to the  $i$ th cell, where  $n$  is the total number of channels available and  $x_i = \sum_{j=1}^M X_j a_{ij}$ , where the  $X_j$ 's constitute an optimal solution to the linear program  $\{\max \sum_{i=1}^N (\sum_{j=1}^M X_j a_{ij} - z_i) : X_j \geq 0, j = 1, 2, \dots, M, \sum_{j=1}^M X_j = 1, z_i \geq 0, \sum_{j=1}^M X_j a_{ij} - z_i \leq p_i r, i = 1, 2, \dots, N\}$ . The  $X$ -fixed algorithm is a fixed channel allocation algorithm for fixed  $r$ , but the allocation of channels is load dependent.

<sup>12</sup>The maximal independent sets of the hypergraph/graph models of the system are the cochannel cells, in as much as all cells of a maximal independent set can use the same channel simultaneously. Also, all cells of no proper superset of a maximal independent set can use a channel simultaneously.

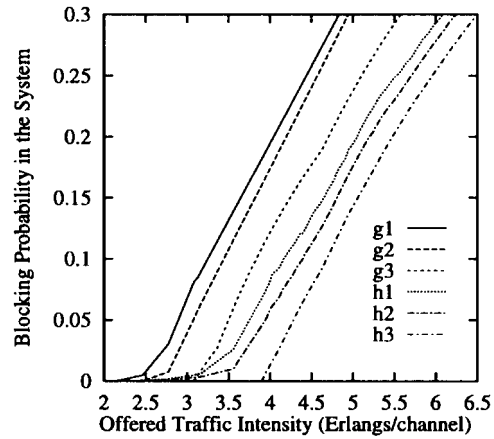


Fig. 9. Blocking probability obtained with the  $X$ -fixed algorithm in the 37-cell system with the required transmission quality being that the minimum S/I ratio must exceed 12.5 dB with a probability of 0.9. Offered traffic intensity is the same in all the cells (uniform distribution of traffic). g1, g2, and g3 are the blocking probability curves for the system modeled by a graph. g1 is the curve for 1200 channels, g2 for 3000 channels, and g3 for an arbitrarily large number of channels ( $n \rightarrow \infty$ ). h1, h2, and h3 are the blocking probability curves for the system modeled by a hypergraph. h1 is the curve for 1200 channels, h2 for 3000 channels, and h3 for an arbitrarily large number of channels ( $n \rightarrow \infty$ ).

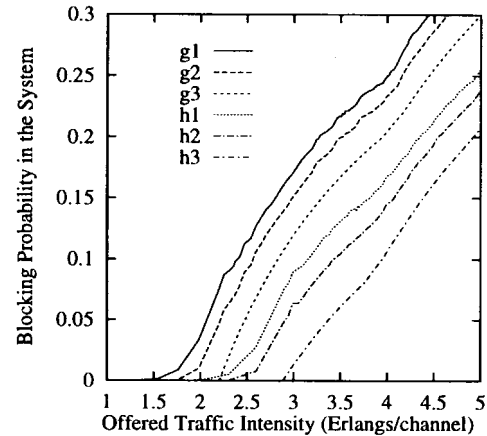


Fig. 10. Blocking probability obtained with the  $X$ -fixed algorithm in the 37-cell system with required transmission quality being that the minimum S/I ratio must exceed 12.5 dB with a probability of 0.9. Offered traffic intensity is nonuniformly distributed amongst the cells. g1, g2, g3, h1, h2, and h3 have the same interpretation as in Fig. 9.

to design some dynamic channel allocation (DCA) schemes, which yield considerably lower blocking probabilities than when the graph model is used.

If the DCA takes the decision whether or not to assign a channel to a new call based on the current measured value of the S/I ratio, then in effect it allows a set of cells to reuse the same channel only if it is an independent set of the hypergraph modeling the system. However, many DCA's, for reasons of implementation complexity or inability to make real-time measurements, take this decision based on the knowledge of cochannel sets computed *a priori*. In this situation, the hypergraph model is at an advantage over the graph model simply because it allows a wider choice of cochannel cells. The simulation results bear testimony to this fact. We have presented the simulation results for the 19-cell system of Fig. 2

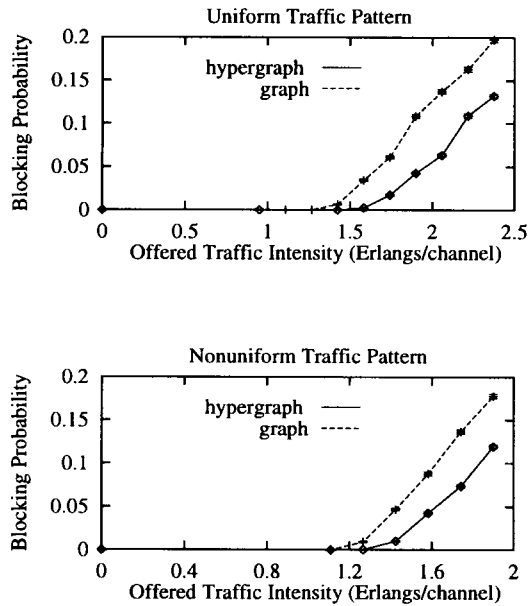


Fig. 11. DCA simulation results for a 19-cell system. The minimum  $S/I$  ratio must exceed 12.5 dB with a probability of 0.9. 120 channels are available. In the case of nonuniform traffic, the pattern is similar to that for the 37-cell system. Maximum traffic is offered in the central cell (cell 10), half of this traffic is offered in each cell in the next ring (cells 5, 6, 9, 11, 14, and 15), and the offered traffic in any other cell is one third of the traffic offered in the central cell. The probability that the true blocking probabilities lie within the error bars around the estimated blocking probabilities is 0.9.

in Fig. 11. We have simulated a DCA which accepts a call whenever it is possible to do so without any rearrangements, satisfying the reuse constraints. It takes decisions based on its knowledge of cochannel cells computed *a priori*.

#### IV. APPROXIMATIONS TO THE CAPACITY

Although, in general, hypergraph modeling outperforms graph modeling, the performance of the system when modeled by a graph closely approaches the performance of the hypergraph model at some values of the offered load (see Figs. 5 and 8). Thus, before the system designer decides whether to design his channel allocation scheme based on the hypergraph model or on the graph model, he may like to estimate the extent of the superiority of the more complex hypergraph model at the load at which he would like to operate. He can do so by comparing the capacities. However, he has to compute the maximal independent sets of the hypergraph/graph models of the system for this purpose. The number of maximal independent sets of the underlying graph/hypergraph model increases exponentially with the number of cells. Hence, the complexity of generation of these maximal independent sets also increases exponentially with the number of cells. In a real system, there will be a large number of cells. Hence, evaluation of the capacity will be difficult in a real system, unless very powerful computing resources are available. Thus, the need arises for finding good and easily computable approximations to the capacity. If easily computable upper and lower bounds on the capacity  $r_0$  can be found and these are close to each other, then our objective is achieved.

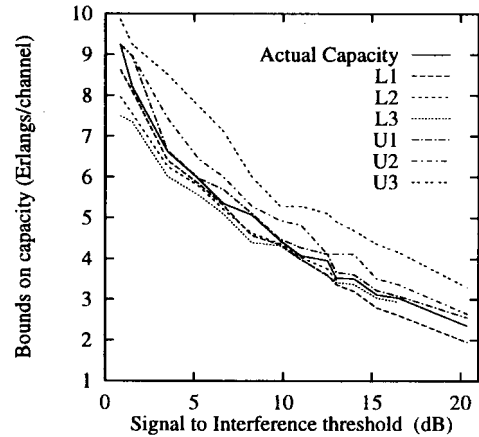


Fig. 12. Approximations to the capacity of the system. Both upper and lower bounds on the capacity have been shown. A 37-cell system with uniform traffic is studied. The required transmission quality is that a given  $S/I$  threshold must be exceeded with 90% probability. L1, L2, and L3 are lower bounds on the capacity. U1, U2, and U3 are upper bounds on the capacity.

##### A. Lower Bound on Capacity

If the linear program LP1 is run with some of the maximal independent sets instead of all the maximal independent sets, it is same as forcing some variables  $X_j$  to zero. So, the solution<sup>13</sup> will not be optimal, but greater than the optimal and the reciprocal of it will give a lower bound on  $r_0$ . Actually, as discussed before, only  $N$  out of  $M$  ( $N \ll M$ ) variables need to have nonzero values. So, if the linear program is run with only those  $N$  variables, the solution will be optimal, but since we do not know beforehand which  $N$  variables will be basis variables, we need to run the linear program with all  $M$  variables, to get the optimal solution.

The linear program was run with different numbers of maximal independent sets for various  $S/I$  thresholds. The results have been plotted in Fig. 12 for the 37-cell system modeled by a hypergraph, with uniform traffic distribution. The solid line gives the actual capacity. The curves below the actual line give the lower bounds.

- 1) L1 was obtained with 1/4th of the total number of maximal independent sets. It is very close to the actual curve. However, if generation of all the maximal independent sets has exponential complexity, generation of 1/4 of the total number of maximal independent sets also has exponential complexity. So, it is not a good idea to use this approximation.
- 2) The next lower bound L2 was obtained with around  $20N$  maximal independent sets, i.e., 1000 maximal independent sets for a 37-cell system. The lower bound thus obtained is fairly close to the actual curve. Generation of  $20N$  maximal independent sets has a complexity that increases polynomially with the number of cells  $N$ .
- 3) L3 was obtained with around  $10N$  maximal independent sets, i.e., precisely 400 maximal independent sets for a 37-cell system. As expected, this is not very close to the actual curve.

<sup>13</sup>A feasible solution may not exist. This will happen if the maximal independent sets are so chosen that one or more cells does not occur in any maximal independent set. This case has to be avoided.

The same curves were obtained for the same system with nonuniform traffic. The lower bound computed using 25% of the maximal independent sets (L1) almost coincides with the actual curve (the solid one). The lower bounds<sup>14</sup> obtained in the other cases are also fairly close to the actual curve at most places. But there are gaps at a few places. This shows that the procedure may not always be reliable. The proximity of the lower bounds to the actual capacity curve in most cases indicates that a channel assignment algorithm may do quite well, even if it uses only some of the cochannel sets generated by the hypergraph model.

### B. Upper Bound on Capacity

Let us consider the dual of the linear program LP1 for the computation of  $r_0^{-1}$

$$\left\{ \max \sum_{i=1}^N Y_i p_i : Y_i \geq 0, \sum_{i=1}^N Y_i a_{ij} \leq 1, j = 1, \dots, M \right\}. \quad (\text{LP2})$$

$a_{ij}$ ,  $M$ ,  $N$ , and  $p_i$  have the same significance as before.  $Y_i$ 's are the variables of the linear program.

Here, each cell corresponds to a variable  $Y_i$  and each maximal independent set to a constraint. The following strategy can be adopted.

If some cell variables are forced to zero, the linear program will yield a solution<sup>15</sup> less than the optimal, and the reciprocal of that will thus be greater than  $r_0$ , and, hence, an upper bound on  $r_0$ . If  $C'$  is the set of cell variables to be forced to zero and  $H'$  is the hypergraph underlying the system formed by removing the cells in  $C'$  from the original system (the fraction of traffic in the remaining cells is not increased, i.e.,  $p_i$ 's in these remain the same), then generating all the maximal independent sets of  $H'$  and running linear program LP2 on them will yield the same result as forcing some variables corresponding to the cells of the original system to zero. Since the number of maximal independent sets increases exponentially with the number of cells, removal of even a small number of cells from the original system will reduce the number of maximal independent sets considerably. For example, removal of only six cells from the outermost ring from the 37-cell system reduces the number of maximal independent sets by about a factor of 3/4 when the S/I threshold is such that the number of maximal independent sets is large.

This technique was tried with the system of 37 cells with uniform traffic. The results have been plotted in Fig. 12. The solid curve, as mentioned before, gives the actual capacity versus S/I thresholds. The curves above it are upper bounds.

- 1) The closest upper bound U1 was obtained by removing six cells at distance 3 from the central cell. This is very close to the actual curve.
- 2) The next higher curve U2 was obtained by removing six cells at distance 3 from the central cell and six others

at distance  $\sqrt{7}$  from the central cell, i.e., 25 cells only were considered. The number of maximal independent sets varies from 63 to 1486, whereas in the actual system the range is from 217 to 33 536. This curve is also fairly close to the actual curve.

- 3) In both the cases above, cells were removed only from the outskirts of the system. Now, we compute the same curve after removing the central cell (cell 28) and five others from the next ring (i.e., at distance 1 from the central cell). The result is the curve farthest from the solid curve, i.e., U3 in Fig. 12. The approximation we get in this case is very poor even though only six cells have been removed. So, the choice of the cells to be removed is crucial in order to get a good upper bound. This is intuitive. Removal of a cell amounts to not allocating any channel to that cell, while keeping the traffic in the remaining cells unchanged. The total number of available channels is also unchanged. Hence, the remaining cells can use a greater number of channels. The calls attempted in the cells removed are not considered to be blocked either. Now, a channel allocated to a cell in the central part of the system prevents a greater number of cells in the system from using it as compared to that allocated to a cell in the outskirts. Hence, the removal of a cell in the central part gives a greater upper bound than that obtained by removing a cell in the outskirts.

The same strategy was tried with the 37-cell system with nonuniform traffic. The results are plotted in Fig. 13. The solid curve gives the actual curve. U1, U2, and U3 were obtained in the same manner as for the uniform traffic case. U1 and U2 are both quite close to the actual curve with U2 being slightly worse (as expected), but easier to compute. While U3 uses the same number of cells as U1, its performance is considerably worse. This is not only due to the reasons explained in the uniform traffic case (channels allocated to central cells are prevented from being reused in a greater number of cells), but also because the nonuniform traffic is such that cells closer to the center of the system need many more channels than those in the periphery.

## V. CONCLUSION

The numerical results presented clearly demonstrate the superiority of hypergraph modeling of cellular systems over graph modeling. Although we have mostly dealt with infinite channel systems, the numerical results obtained for the finite channel case (refer to Figs. 9 and 10) sufficiently illustrate the superiority of hypergraph modeling over graph modeling. Usually, the number of channels available is large, so that the actual carried traffic and blocking probability obtained with any algorithm are close to those obtained with the infinite channel assumption. This is also brought out by Figs. 9 and 10. Also, it was shown in [3] that the capacity of a system does not change when the assumption that the offered traffic is independent from cell to cell is relaxed. Thus, our results hold for mobile cellular networks with handovers and intercell calls. However, the computational complexity is high for hypergraph

<sup>14</sup>L2 with  $20N$  maximal independent sets and L3 with  $10N$  maximal independent sets.

<sup>15</sup>A feasible solution will always exist in this case.

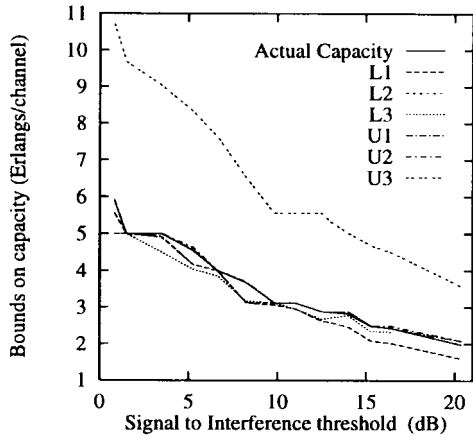


Fig. 13. Approximations to the capacity of the system. Both upper and lower bounds on the capacity have been shown. A 37-cell system with nonuniform traffic is studied. The required transmission quality is that a given S/I threshold must be exceeded with 90% probability. L1, L2, and L3 are lower bounds on the capacity. U1, U2, and U3 are upper bounds on the capacity.

modeling, but in most cases the bounds discussed above are very close to the actual results, and these are also easy to evaluate.

#### APPENDIX

##### ESTIMATION OF THE PARAMETERS OF $I$

Note that  $I$ , the total received interference power, is a random variable which is the sum of mutually independent lognormal random variables. First, we give expressions for the mean and variance of a lognormal random variable.

##### A. Mean of a Lognormal Random Variable

Let  $X$  be a lognormal random variable.  $\ln X \sim N(m, \sigma^2)$ . Let  $Y = \ln X$ . Hence,  $Y \sim N(m, \sigma^2)$ . Since  $X = e^Y$ ,  $E(X) = E(e^Y)$ , which is the characteristic function of a  $N(m, \sigma^2)$  random variable. Thus

$$E(X) = e^{m+\sigma^2/2}. \quad (4)$$

##### B. Variance of a Lognormal Random Variable

Let  $X$  be a lognormal random variable. As before, let  $Y = \ln X$ . Hence,  $2Y \sim N(2m, 4\sigma^2)$ . Since  $X = e^Y$ ,  $E(X^2) = E(e^{2Y})$ , which is the characteristic function of a  $N(2m, 4\sigma^2)$  random variable. Thus

$$E(X^2) = e^{2m+2\sigma^2}$$

and

$$\begin{aligned} \text{var}(X) &= E(X^2) - [E(X)]^2 \\ &= e^{2m+2\sigma^2} (e^{\sigma^2} - 1). \end{aligned} \quad (5)$$

Recall that  $I$  is the sum of lognormal random variables  $I_i$ ,  $i = 1, 2, \dots, N$ , where  $I_i$ 's have been assumed to be mutually independent.  $\ln I_i \sim N(S_0 - 4 \ln r_i, \sigma^2)$  for each  $i$ . Also, recall that the sum of independent lognormal random variables can be approximated by a lognormal random variable (refer to [8]). Thus,  $\ln I$  is distributed as  $N(m_I, \sigma_I^2)$ . The

parameters of  $I$ , i.e.,  $m_I$  and  $\sigma_I$ , can be determined in various ways. The method which gives the closest approximation depends on the expected range of values occupied by the random variable [8]. For simplicity, we select the method which determines the parameters such that  $I$  has the same first and second moments for all ranges. Let  $\ln I = N(m_I, \sigma_I^2)$ . From [8]

$$\sigma_I^2 = \ln \left( \frac{V^2}{\eta^2} + 1 \right) \quad (6)$$

$$m_I = \ln \eta - \sigma_I^2/2 \quad (7)$$

where

$$\begin{aligned} V^2 &= \sum_i \text{var}(I_i) \\ &= \sum_i e^{2(S_0 - 4 \ln r_i) + \sigma^2} (e^{\sigma^2} - 1) \end{aligned}$$

[using (1) and (5)]

$$= e^{2S_0} (e^{\sigma^2} - 1) \sum_i r_i^{-8}. \quad (8)$$

$$\begin{aligned} \eta &= \sum_i E(I_i) \\ &= \sum_i e^{S_0 - 4 \ln r_i + \sigma^2/2} \end{aligned} \quad (9)$$

[using (1) and (4)]

$$\begin{aligned} &= e^{S_0 + \sigma^2/2} \sum_i r_i^{-4} \\ \Rightarrow \eta^2 &= e^{2S_0 + \sigma^2} \left( \sum_i r_i^{-4} \right)^2. \end{aligned}$$

Hence

$$\sigma_I^2 = \ln \left[ e^{\sigma^2 - 1} \frac{\sum_i r_i^{-8}}{\left( \sum_i r_i^{-4} \right)^2} + 1 \right]$$

[substituting (8) and (10) into (6) and cancelling the common terms] and

$$m_I = S_0 + \sigma^2/2 \ln \sum_i r_i^{-4} - \sigma_I^2/2$$

[substituting (9) into (7)].

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