Transceiver Design and Interference Alignment in Wireless Networks: Complexity and Solvability

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Dedication

To my parents.

Abstract

This thesis aims to theoretically study a modern linear transceiver design strategy, namely interference alignment, in wireless networks. We consider an interference channel whereby each transmitter and receiver are equipped with multiple antennas. The basic problem is to design optimal linear transceivers (or beamformers) that can maximize the system throughput. The recent work [1] suggests that optimal beamformers should maximize the total degrees of freedom through the interference alignment equations. In this thesis, we first state the interference alignment equations and study the computational complexity of solving these equations. In particular, we prove that the problem of maximizing the total degrees of freedom for a given interference channel is NP-hard. Moreover, it is shown that even checking the achievability of a given tuple of degrees of freedom is NP-hard when each receiver is equipped with at least three antennas. Interestingly, the same problem becomes polynomial time solvable when each transmit/receive node is equipped with no more than two antennas.

The second part of this thesis answers an open theoretical question about interference alignment on generic channels: What degrees of freedom tuples $(d_1, d_2, ..., d_K)$ are achievable through linear interference alignment for generic channels? We partially answer this question by establishing a general condition that must be satisfied by any degrees of freedom tuple $(d_1, d_2, ..., d_K)$ achievable through linear interference alignment. For a symmetric system with $d_k = d$ for all k, this condition implies that the total achievable DoF cannot grow linearly with K (unlike the case in [1]), and is in fact no more than $K(M+N)/(K+1)$, where M and N are the number of transmit and receive antennas, respectively. We also show that this bound is tight when the number of antennas at each transceiver is divisible by the number of data streams.

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Chapter 1

Introduction

Consider a multiuser communication system in which a number of users must share common resources such as frequency, time, or space. The mathematical model for this communication scenario is the well-known *interference channel*, which consists of multiple transmitters simultaneously sending messages to their intended receivers while causing interference to each other. Interference channel is a generic model for multiuser communication and can be used in many practical applications such as Digital Subscriber Lines (DSL) [3], Cognitive Radio (CR) systems [4], ad-hoc wireless networks [5, 6] and cellular networks.

A central issue in the study of interfering multiuser systems is how to mitigate multiuser interference. In practice, there are several commonly used methods for dealing with interference. First, we can treat the interference as noise and just focus on extracting the desired signals (see [15], [21]). This approach is widely used in practice because of its simplicity and ease of implementation, but is known to be non-capacity achieving in general. An alternative technique is channel orthogonalization whereby transmitted signals are chosen to be nonoverlapping either in time, frequency or space, leading to Time Division Multiple Access, Frequency Division Multiple Access or Space Division Multiple Access respectively. While channel orthogonalization effectively eliminates multiuser interference, it can lead to inefficient use of communication resources and is also generally non-capacity achieving. Another interference management technique is to decode and remove interference. Specifically, when interference is strong relative to desired signals, a user can decode the interference first, then subtract it from the received signal, and finally decode its own message (see [8] and [11]). This method is less common in practice due to its complexity and security issues.

In a cellular system, multi-cell interference management is a major challenge. So far various base station cooperation techniques have been proposed to mitigate inter-cell interferences, including coordinated multi-point (CoMP) transmission, or network MIMO transmission [32–34]. In coordinated multipoint strategy, data to a single user is simultaneously transmitted from different base stations and the user jointly process the received signals from different base stations. Most of the CoMP proposed techniques in the literature require each base station to have full/partial channel state information (CSI) as well as the knowledge of actual independent data streams to all remote terminals. With the complete sharing of data streams and CSI, the multi-cell scenario is effectively reduced to a single cell interference management problem with either total [35] or per-group-of-antennas power constraints [36, 37]. While these techniques can offer significant improvement on data throughput, they also have several drawbacks including stringent requirement on base station coordination, the large demand on the communication bandwidth of backhaul links, and the heavy computational load associated with the increasing number of cells [38, 39].

Theoretically, what is the optimal interference management strategy? The answer is related to the characterization of capacity region of an interference channel, i.e., determining the set of rate tuples that can be achieved by the users simultaneously. For the noiseless case, the capacity region and the optimal precoding strategy of the two user interference channel is discussed in [8] and [7]. In spite of intensive research on this subject over the past three decades ([7] - [20]), the capacity region of interference channels is still unknown for general case (even for small number of users). The lack of progress to characterize the capacity region for a MIMO interference channel has motivated researchers to derive various approximations of the capacity region. For example, the maximum total degrees of freedom (DoF) corresponds to the first order approximation of sum-rate capacity of an interference channel at high SNR regime. Maximizing this approximation of sum-rate leads us to the interference alignment method [1].

Theoretically, what is the optimal transmit/receive strategy in a MIMO interference channel? The answer is related to the characterization of the capacity region of an interference channel, i.e., determining the set of rate tuples that can be achieved by the users simultaneously. In spite of intensive research on this subject over the past three decades, the capacity region of interference channels is still unknown (even for small number of users). The lack of progress to characterize the capacity region of the MIMO interference channel has motivated researchers to derive various approximations of the capacity region. For example, the maximum total degrees of freedom (DoF) corresponds to the first order approximation of sum-rate capacity at high SNR regime. Specifically, in a K-user interference channel, we define the degrees of freedom region as the following [1]:

$$\mathcal{D} = \Bigg\{(d_1, d_2, \ldots, d_K) \in \mathbb{R}_+^K \mid \forall (w_1, w_2, \ldots, w_K) \in \mathbb{R}_+^K, \\ \sum_{k=1}^{K} w_k d_k \leq \limsup_{\text{SNR}\to\infty} \left[\sup_{\mathbf{R}\in\mathcal{C}} \frac{1}{\log \text{SNR}} \sum_{k=1}^{K} w_k R_k \right] \Bigg\}, \tag{1.1}$$

where $\mathcal{C}$ is the capacity region and R_k is the rate of user k. We can further define the total DoF in the system as the following:

$$\eta = \max_{(d_1, d_2, \ldots, d_K)\in\mathcal{D}} d_1 + d_2 + \ldots + d_K.$$

Intuitively, the total DoF is the number of independent data streams that we can communicate interference-free in the channel.

It is well known that for a point-to-point MIMO channel with M antennas at the transmitter and N antennas at the receiver, the total DoF is $\eta = \min\{M, N\}$. Different approaches such as SVD precoder or V-BLAST can be used to achieve this DoF bound. For a 2-user MIMO fading interference channel with user k equipped with M_k transmit antennas and N_k receive antennas ($k = 1, 2$), Jafar and Fakhereddin [59] proved that the maximum total DoF is

$$\eta = \min\left\{M_1 + M_2, N_1 + N_2, \max\{M_1, N_2\}, \max\{M_2, N_1\}\right\}.$$

Moreover, this bound can be achieved using a *linear interference alignment*[1] scheme

[1] The concept of linear interference alignment was first introduced by [80]. There is a related, but different, notion of *signal level alignment* whereby the transmitted/received signals are aligned, not through linear beamformers, but through structured coding and/or phase arrangement. These schemes are nonlinear in the data symbols.

consisting of linear transmit and receive beamformers. This result shows that for the case of $M_1 = M_2 = N_1 = N_2$, the total DoF in the system is the same as the single user case. In other words, we do not gain more DoF by increasing the number of users from one to two. Interestingly, if statistically independent channel extensions are allowed either across time or frequency, Cadambe and Jafar [1] showed that the total DoF is $\eta = KM/2$ for a K-user MIMO interference channel, where M is the number of transmit/receive antennas per user. This result implies that each user can effectively utilize half of the total system resource in an interference-free manner. The principal assumption enabling this surprising result is that the channel extensions are i.i.d. and exponentially long in K, which can be impractical. However, if channel extensions are restricted to have a polynomial length or are not statistically independent, the total DoF for a MIMO interference channel is still largely unknown even for the Single-Input-Single-Output (SISO) interference channel. For the 3-user special case, reference [81] provided a characterization of the total achievable DoF as a function of the diversity. In the absence of channel extensions, various linear interference alignment algorithms have been proposed for the MIMO interference channel [2].

The main theoretical investigation pertaining to the current work is [45] by Yetis *et. al.* who studied the maximum achievable DoF for a MIMO interference channel without channel extension. In general, linear interference alignment can be described by a set of quadratic equations which correspond to the zero-forcing conditions at each receiver. For a K-user system, there are a total of $K(K-1)$ such coupled quadratic matrix equations whose unknowns are the transmit/receive beamforming matrices to be designed. Moreover, the achievability of a given tuple of DoF corresponds to these quadratic equations having a solution (in the form of beamforming matrices) whose individual matrix ranks are given by the DoFs. One can easily count the number of "independent unknowns" and the number of scalar equations in this quadratic system defining interference alignment. It is then tempting to conjecture, as was done in [45], that the interference alignment is feasible if and only if the number of equations is no more than the number of unknowns in each subsystem of the quadratic equations. When the latter is true, the authors of [45] called the corresponding system *proper*. However, except for some special cases involving a small number of users and antennas, the investigation of [45] was largely inconclusive.

In this thesis, we first consider the problem of designing the linear beamformers in a wireless interference channel. First, we study the complexity status of interference alignment problem in spatial domain and we show that the problem is NP-hard when the number of antennas at each node is at least three. Moreover, we show that the interference alignment problem is polynomial time solvable when the number of antennas at each node is at most two. In the second part of this thesis, we settle the conjecture of [45] completely in one direction, and partially in the other. In particular, we consider the case where no channel extension is allowed, and use results from the field theory to establish a general condition that must be satisfied by any DoF tuple achievable through linear interference alignment. This condition shows that the improperness property (in the sense of [45]) indeed implies the infeasibility of interference alignment. For the symmetric system with $M_k = M$ and $N_k = N$ for all k, this condition implies that the total achievable DoF cannot grow linearly with the number of users, and is in fact no more than $M + N - 1$. This is in sharp contrast to the case with independent channel extensions for which the total DoF can grow linearly with the number of users. For the converse direction, we show that if all users have the same DoF d and the number of antennas M_k, N_k are divisible by d for each k, then the properness of the quadratic system implies the feasibility of interference alignment for all generically generated MIMO interference channels. If in addition, $M_k = M$ and $N_k = N$ for all k and M, N are divisible by d, then our results imply that interference alignment is achievable if and only if $(M + N) \geq d(K + 1)$. In the simulation section, we use these established DoF bounds to numerically benchmark the performance of several existing algorithms for interference alignment and sum-rate maximization.

Chapter 2

Existing Interference Management Approaches in Multi-user Systems

In this section, we try to give a more general view over the different interference management approaches (other than interference alignment) in multi-user wireless systems.

2.0.1 Dirty Paper Coding and Successive Interference Cancelation

Dirty Paper Coding (DPC) and Successive Interference Cancelation (SIC) are two methods to cancel the multiuser interference at the transmitter and receiver, respectively.
The DPC result [55] simply states that the known noise at the transmitter can be precanceled without any cost (neither extra power is needed nor rate decrement happens). For example, in a broadcast (one-to-many) channel, when the transmitter transmits the signal of user k, it knows the codeword of the users $1, 2, \ldots, k-1$. Therefore, it can precancel the effect of them at receiver k. Therefore, user k sees no interference from the signals of users $1, 2, \ldots, k-1$. DPC and TDMA techniques can achieve any point in the capacity region of the broadcast channel [52–54]. However, so far the high complexity of the DPC technique at the transmitter prevented us to use it in practice.
SIC is a technique to iteratively decode the interference and subtract the interference from the received signal. By subtracting the interference from the received signal, the

SINR value of the signal increases and therefore, higher rates can be provided. The combination of SIC and TDMA can achieve any tuple of rates in the multi-access channel [56–58]. However, the complexity and the security issues arise from decoding the interference, made this approach far from being practical.

2.0.2 Maximizing a Utility of the System Using Optimization Techniques

One practical way to mitigate the interference between the users is to control the amount of interference by using linear beamformers at the transmitters and receivers. To this end, people consider a utility function of the system and maximize the utility subject to the existing constraints. One typical optimization problem is to maximize the total system throughput [31, 41, 43, 44], i.e.,

$$\begin{aligned} \max \quad & \sum_{k=1}^{K} R_k \\ \text{s.t.} \quad & \text{constraints} \end{aligned}$$

where R_k is the rate of user k and we can consider different practical constraint such as power budget, quality of service, etc. Unfortunately, in most of the cases, this problem becomes non-convex and computationally intractable. Furthermore, in lots of scenarios, this objective function leads to unfair resource allocation among different users. Hence, people also consider different objective functions in the above optimization problem. Some commonly used objective functions are as follows

- Sum rate utility function: $\mathcal{U}(R_1, \ldots, R_K) = \sum_{k=1}^{K} R_k$
- Harmonic mean utility function: $\mathcal{U}(R_1, \ldots, R_K) = \left(\sum_{k=1}^{K} R_k^{-1}\right)^{-1}$
- Geometric mean utility function: $\mathcal{U}(R_1, \ldots, R_K) = \left(\prod_{k=1}^{K} R_k\right)^{1/K}$
- Min rate utility function: $\mathcal{U}(R_1, \ldots, R_K) = \min_k R_k$

In addition to the above optimization problem, many people considered the power minimization problem.

When the channels are diagonal and no correlated signaling is allowed across different antennas, we are basically led to the dynamic spectrum management problem. The dynamic spectrum management problem, which is a key core in the performance of DSL systems, has recently been a topic of intensive research in the signal processing community. Different distributed and centralized algorithms has been proposed to the dynamic spectrum management problem [22–26, 28, 49–51]. The authors in [28] have studied this problem for different well-known utility functions and characterized the efficient solvability of the problems in different cases.

2.0.3 Coordinated Multi-Point (CoMP)

Coordinated multi-point transmission is a new strategy where base stations cooperate to transmit and receive the signals in order to mitigate the inter-user interference, effectively treating the whole multi-cell system as a giant MIMO broadcast/MAC channel. This cooperative strategy can improve the performance of cell-edge users in cellular networks [63–67]. Although in theory the raw system throughput can increase dramatically using CoMP strategy (linear in the number of users or total number of antennas), the system overhead caused by the data sharing and synchronization may prevent it from achieving the theoretical performance in practice. A major difficulty with CoMP approach is its requirement that the signals from different base stations for the same OFDM symbol should arrive at a given receiver at the same time. Due to large distances between different base stations, this same arrival time requirement is not possible unless appropriate timing offset is pre-compensated at the transmitting base stations. The latter would place a significant constraint on the length of cyclic prefix in a OFDM symbol. In fact, for a typical 1-2 mile spacing between base stations, the length of OFDM symbol would have to be longer than the data block length, rendering it rather inefficient.

Another major issue with CoMP is that the capacity of the backhaul links also limits the performance of the system. Since considering global knowledge and infinite capacity for the backhaul links is impractical, people have proposed more practical scenarios such as [68]:

- Local connectivity: in this scenario, two base stations are connected only if they are adjacent [69–75].

- Restricted connectivity to a central processor: only a subset of base stations is connected to each other (or a central processor) [76].
- Global but finite capacity backhaul links: all processors are connected to a central node via finite capacity links [77–79].

Compared to the networked MIMO CoMP strategy, the Coordinated Beamforming (CBF) strategy requires no data sharing between the base stations. In fact, different base stations encode the signals independently in the interference (interfering broadcast) channel. Hence, far less signaling is needed for synchronization and the system overhead is significantly reduced. Similarly, in the interference alignment (IA) strategy, only the channel state information is exchanged between the base stations, no sharing of data streams is required. Both CBF and IA allow more independent operations across base stations as compared to the centralized CoMP approach, while still achieving a high system throughput, at least theoretically, that is linear in the number of users or the number of antennas in the system.

2.0.4 Degrees of Freedom Characterization

The lack of progress to characterize the capacity region of the MIMO interference channel has motivated researchers to derive various approximations of the capacity region. One of the examples of this kind of approximations is the maximum total degrees of freedom (DoF) which corresponds to the first order approximation of sum-rate capacity of an interference channel at high SNR regime. In this approach, for a K user interference channel, we define the degrees of freedom region as the following [1]:

$$\mathcal{D} = \left\{ (d_1, d_2, \ldots, d_K) \in R_+^K | \forall (w_1, w_2, \ldots, w_K) \in R_+^K, \sum_{k=1}^{K} w_k d_k \leq \limsup_{\mathrm{SNR} \to \infty} \left[\sup_{\mathbf{R} \in \mathcal{C}} \frac{1}{\log \mathrm{SNR}} \sum_{k=1}^{K} w_k R_k \right] \right\},$$

where $\mathcal{C}$ is the capacity region and R_k is the rate of user k. In the approaches dealing with the DoF, the objective is to maximize the DoF in the system. In the consequent chapters, we provide more details on this approach, its computational complexity and the solution for this approach in the generic channel case.

Chapter 3

System Model, Assumptions, and Problem Statement

Consider a MIMO interference network consisting of K transmitter - receiver pairs, with transmitter k sending d_k independent data streams to receiver k. Let $\mathbf{H}_{kj}$ be an $M_j \times N_k$ matrix that represents the channel gain matrix from transmitter j to receiver k where M_j and N_k denote the number of antennas at transmitter j and receiver k, respectively. The received signal at receiver k is given by

$$\mathbf{y}_k = \sum_{j=1}^{K} \mathbf{H}_{kj}\mathbf{x}_j + \mathbf{n}_k$$

where $\mathbf{x}_j$ is an $M_j \times 1$ random vector that represents the transmitted signal of user j and $\mathbf{n}_k \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$ is a zero mean additive white Gaussian noise.

Throughout this thesis, we focus on *linear* transmit and receive strategies that can maximize system throughput. In this case, transmitter k uses a beamforming matrix $\mathbf{V}_k$ in order to send a signal vector $\mathbf{s}_k$ to its intended receiver k. On the other side, receiver k estimates the transmitted data vector $\mathbf{s}_k$ by using a linear beamforming matrix $\mathbf{U}_k$, i.e.,

$$\mathbf{x}_k = \mathbf{V}_k\, \mathbf{s}_k, \qquad \hat{\mathbf{s}}_k = \mathbf{U}_k^H \mathbf{y}_k$$

where the power of the data vector $\mathbf{s}_k \in \mathbb{R}^{d_k \times 1}$ is normalized such that $E[\mathbf{s}_k\mathbf{s}_k^H] = \mathbf{I}$, and $\hat{\mathbf{s}}_k$ is the estimate of $\mathbf{s}_k$ at the k-th receiver. The matrices $\mathbf{V}_k \in \mathbb{C}^{M_k \times d_k}$ and $\mathbf{U}_k \in \mathbb{C}^{N_k \times d_k}$ are the beamforming matrices at the k-th transmitter and receiver respectively. Without channel extension, the linear interference alignment conditions can be described by the following zero-forcing conditions [2, 45, 83]

$$\mathbf{U}_k^H \mathbf{H}_{kj} \mathbf{V}_j = \mathbf{0}, \quad \forall\ j \neq k, \tag{3.1}$$

$$\text{rank}\left(\mathbf{U}_k^H \mathbf{H}_{kk} \mathbf{V}_k\right) = d_k, \quad \forall\ k. \tag{3.2}$$

The first equation guarantees that all the interfering signals at receiver k lie in the subspace orthogonal to $\mathbf{U}_k$, while the second one assures that the signal subspace $\mathbf{H}_{kk}\mathbf{V}_k$ has dimension d_k and is linearly independent of the interference subspace. Intuitively, as the number of users K increases, the number of constraints on the beamformers $\{\mathbf{U}_k, \mathbf{V}_k\}$ increases quadratically in K, while the number of design variables in $\{\mathbf{U}_k, \mathbf{V}_k\}$ only increases linearly. This suggests the above interference alignment can not have a solution unless K or d_k is small.

The interference alignment conditions (3.1) and (3.2) imply that each transmitter k can use a linear transmit/receive strategy to communicate d_k interference free independent data streams to receiver k (per channel use). In this case, it can be checked that d_k represents the DoF achieved by the k-th transmitter/receiver pair in the information theoretic sense of (1.1). In other words, the vector $(d_1, d_2, ..., d_K)$ in (3.1) and (3.2) represents the tuple of DoF achieved by linear interference alignment. Intuitively, the larger the values of $d_1, d_2, ..., d_K$, the more difficult it is to satisfy the interference alignment conditions (3.1) and (3.2).

Chapter 4

Complexity Analysis of Interference Alignment

In this section, we show that for a given channel, not only the problem of finding the maximum DoF is NP-hard, but also the problem of checking the achievability of a given tuple of DoF, $(d_1, ..., d_K)$, is NP-hard when there are at least 3 antennas at each node. Then, we show that the same problem is polynomial solvable when the number of antennas at each transceiver is less than 3.

Notice that the interference alignment conditions in the k-th receiver are

$$\mathbf{U}_k^T \mathbf{H}_{kj} \mathbf{V}_j = 0, \qquad \forall j \neq k, \tag{4.1}$$

$$\operatorname{rank}\left(\mathbf{U}_k^T \mathbf{H}_{kk} \mathbf{V}_k\right) = d_k. \tag{4.2}$$

The first equation guarantees that all the interference is in the subspace orthogonal to $\mathbf{U}_k$ while the second one assures that the signal subspace $\mathbf{H}_{kk}\mathbf{V}_k$ has dimension d_k and is linearly independent of the interference subspace.

In the sequel, we examine the solvability of above interference alignment problem (4.1) - (4.2) in two different cases.

Theorem 1 *For a K user MIMO interference channel, maximizing the total DoF,* *namely,*

$$\begin{aligned} \max_{\{\mathbf{U}_k,\mathbf{V}_k\}_{k=1}^K} \quad & \sum_{k=1}^{K} d_k \\ \text{s.t.} \quad & \mathbf{U}_k^T \mathbf{H}_{kj} \mathbf{V}_j = 0, \qquad k = 1,..,K, \;\; j \neq k \\ & \operatorname{rank}\left(\mathbf{U}_k^T \mathbf{H}_{kk} \mathbf{V}_k\right) = d_k, \qquad k = 1,..,K \end{aligned}$$

is NP-hard. Moreover, if each node is equipped with at least 3 antennas, then the problem of checking the achievability of a given tuple of DoF, $(d_1, d_2, \ldots, d_K)$*, is also NP-hard.*

Proof The proof of the first part is based on a polynomial time reduction from the maximum independent set problem which is known to be NP-complete. For a given arbitrary graph $G = (V, E)$, where $|V| = K$, consider a K user interference channel that each receiver and transmitter has a single antenna. Moreover, the channel coefficients are given by:

$$h_{jk} = \begin{cases} 1, & \text{if } j = k \text{ or } (k,j) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

It can be checked that the receiver nodes can only achieve a DoF of either 0 or 1, and those receiver nodes achieving a DoF of 1 form an independent set in G. Thus, the problem of maximizing the total DoF for the above interference channel is equivalent to the problem of finding the maximum independent set of vertices in the graph G.
In order to prove the second part we use a polynomial reduction from the 3-colorability problem. The latter problem is to determine whether the nodes of a graph can be assigned one of the three possible colors so that no two adjacent nodes are colored the same. The 3-colorability problem is known to be NP-Complete. There are two main steps in the construction. In the first step, some dummy nodes are added to the channel in order to force a discrete structure such that each non-dummy node may only have one of the three possible cases. The second step is to define the direct channels in order to make a polynomial reduction from the 3-colorability of an arbitrary graph to this problem.

For an arbitrary graph G with N nodes, we will construct a special MIMO interference channel for which the achievability of one degree of freedom at each user is equivalent to the 3-colarability of G. In our construction, the MIMO interference channel will have two types of users: N main users, each equipped with 3 antennas at their transmitters and receivers and $11N$ dummy users which will be defined later. Hence the total number of users is $12N$. In the rest of the proof we suppose that each user (either the dummy user or the main user) wants to send one data stream. In other words we want to check if the tuple of all ones is achievable by the constructed interference channel or not.

We divide the dummy users into two groups. The number of dummy users in the first group is $2N$ and the number of dummy users in the second one is $9N$. Each dummy user in the first group has 3 antennas at its receiver and transmitter, while each dummy user in the second group has two antennas at its transmitter and receiver. Let us further arrange the $2N$ dummy users in the first group into N subsets each containing two users. We denote these subsets as $A_i, i = 1, ..., N, |A_i| = 2$. We also denote the users in the set A_i as $a_{i,1}$ and $a_{i,2}$, and associate them to the i-th main user. For notational consistency, we denote main user i as $a_{i,0}$. We will also use $a_{i,k,j}$ to denote the j-th transmit antenna of user $a_{i,k}$, where $1 \leq i \leq N$, $k = 0, 1, 2$ and $j = 1, 2, 3$. Similarly, we partition the set of $9N$ dummy users in the second group into N subsets $B_i, i = 1, ..., N$, each containing exactly 9 dummy users denoted by $b_{i,\ell}$, with $\ell = 1, .., 9$. Each of these 9 dummy users will have two receiving antennas which we denote as $b_{i,\ell,m}$, with $m = 1, 2$.

Now for any fixed i and j, we consider any size-2 subset of $\{a_{i,k,j} : k = 0, 1, 2\}$, e.g., $\{a_{i,0,j}, a_{i,1,j}\}$. For each fixed i and j, there are exactly 3 of these cardinality-2 subsets. Since there are 3 different choices of j, we have a total of 9 subsets of this kind for any fixed i. Let us index these 9 subsets by ℓ, $\ell = 1, .., 9$, and assign the ℓ-th subset to user $b_{i,\ell}$ in B_i. Now we define the links in the channel for the users in A_i and B_i. First, the channel matrices of all the direct links for any of the dummy users are $\mathbf{I}$ (where $\mathbf{I}$ is the identity matrix of the appropriate size). In addition, none of the dummy users in B_i $(i = 1, 2, ..., N)$ cause interference to the other users (which means that the channel gains between their transmit antennas and the other users' receive antennas are all zero). Now for the aforementioned ℓ-th subset which we denote as $S_{i,\ell} = \{a_{i,k_{\ell_1},j_{\ell_1}}, a_{i,k_{\ell_2},j_{\ell_2}}\}$, we connect $a_{i,k_{\ell_1},j_{\ell_1}}$ and $a_{i,k_{\ell_2},j_{\ell_2}}$ to $b_{i,\ell,1}$ and to $b_{i,\ell,2}$, respectively. Here by connecting a transmit antenna to a receive antenna we mean that the channel coefficient between these two antennas is 1. This situation is shown in the figure 4.1 for the case $S_{i,1} = \{a_{i,0,1}, a_{i,1,1}\}$. Furthermore, we assume that dummy users $a_{i,k},\ k = 1, 2$ do not suffer from any interference.

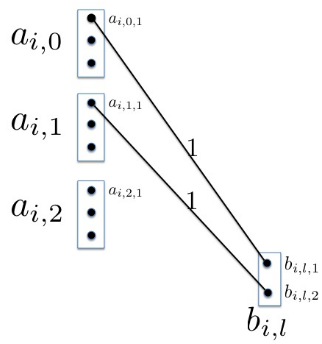

Figure 4.1: Channels to the dummy receiver $b_{i,\ell}$

Suppose that user $a_{i,k}$ $(k = 0, 1, 2)$ uses the transmit beamforming vector $(v_{i,k,1}, v_{i,k,2}, v_{i,k,3})$. Then the interference received at the dummy receiver of $b_{i,\ell}$ will be:

$$\mathbf{I}_{b_{i,\ell}} = (v_{i,k_{\ell_1},j_{\ell_1}} s_{i,k_{\ell_1}}, v_{i,k_{\ell_1},j_{\ell_2}} s_{i,k_{\ell_2}}) \tag{4.3}$$

where $s_{i,k}$ is the signal user $a_{i,k}$ intends to send. Notice that the signals which two different users want to transmit are statistically independent. As a consequence, if we want to have interference alignment at the receiver of $b_{i,\ell}$, so that this user can send its own data stream, it is necessary and sufficient to have $v_{i,k_{\ell_1},j_{\ell_1}} v_{i,k_{\ell_2},j_{\ell_2}} = 0$. Hence, having the interference alignment at $b_{i,\ell}$ for all $\ell = 1, .., 9$ is equivalent to the fact that users $a_{i,k}, k = 0, 1, 2$ cannot send their messages through the antennas with the same index, simultaneously. For example, if $v_{i,0,1} \neq 0$ then $v_{i,1,1}$ and $v_{i,2,1}$ have to be zero. On the other hand, considering the fact that each user needs to send one data stream, it follows that none of the users $a_{i,k},\ k = 0, 1, 2$, can send their message on two of their antennas simultaneously, because otherwise if for example $a_{i,0}$ sends its message on two antennas, then it would result in insufficient spatial dimension for either $a_{i,1}$ or $a_{i,2}$.

As an immediate consequence of these two facts we have just mentioned, we can conclude that the transmit beamforming vector at each user $a_{i,k}, k = 0, 1, 2$, must be proportional to one of the vectors $[1, 0, 0]^T$, $[0, 1, 0]^T$ or $[0, 0, 1]^T$. This is true specially for the main user i. As we are not concerned about the constant factors, we have successfully imposed a discrete structure on the problem solution so far. Notice that each dummy user $b_{i,\ell}$ has a total of 2 dimensions in its receiver. Since we have aligned the interference at each dummy user $b_{i,\ell}$, these users can communicate their data streams easily along the remaining dimension left for them in their receivers and remove interference which lies in the other dimension. Moreover, since in our construction the dummy users $a_{i,k},\ k = 1, 2$ do not experience any interference from other users and their direct channel is $\mathbf{I}$, so these users can easily achieve one degree of freedom. Thus, we only need to take care of the main users.

For each of the N main users, we must pick one of the three transmit beamforming vectors $[1, 0, 0]^T$, $[0, 1, 0]^T$ or $[0, 0, 1]^T$ in order to achieve interference alignment at all the main receivers. We suppose all the direct channels for the main users, $\mathbf{H}_{ii}$, are $\mathbf{I}$. For the cross channels, we use the structure of graph $G = (V, E)$. For each edge (i, j) in G, we set $\mathbf{H}_{ij} = \mathbf{H}_{ji} = \mathbf{I}$. Otherwise we set $\mathbf{H}_{ij} = \mathbf{H}_{ji} = \mathbf{0}$ (zero matrix of appropriate size). Consequently, the main users i and j interfere with each other if and only if they are connected to each other in graph G. We claim that achieving interference alignment in the above MIMO interference channel is equivalent to 3-colorability of graph G. This is because each user can choose 3 possible beamforming vectors, each corresponding to a different color. If main user i chooses one of the three possible beamforming vectors (or one of the three colors), then this beamforming vector cannot be chosen by any other main users adjacent to the main user i in the graph G, otherwise the interference would appear in the desired signal space at the receiver of main user i. This establishes the equivalence between the 3-colorability of G and the achievability of one degree of freedom for each user in the constructed MIMO interference channel. Since 3-colorability problem is NP-hard, it follows that the problem of checking the feasibility of interference alignment is also NP-hard.

Theorem 1 shows that the problem of checking the achievability of a given tuple of DoF is NP-hard if all users (or at least a constant fraction of them) are equipped with at least three antennas. Our next result shows that when each user is equipped with no more than two antennas, the same problem can be solved in polynomial time. To this end, we need to define some notations and make some observations. First of all, the interference alignment problem is equivalent to finding the signal subspaces at the transmitters and the interference subspaces at the receivers such that the interference alignment conditions are satisfied, i.e.,

$$\begin{aligned} &d_k = \dim(\mathcal{S}_k) \\ &\mathbf{H}_{kk}\mathcal{S}_k \perp \mathcal{I}_k \\ &\mathbf{H}_{kj}\mathcal{S}_j \ \subseteq \ \mathcal{I}_k \quad \forall j \neq k, \end{aligned}$$

where $\mathcal{S}_k$ and $\mathcal{I}_k$ denote the signal subspace at the transmitter k and the interference subspace at receiver k, respectively. The operator $\perp$ represents the linear independence of two subspaces. The first condition implies that the signal space has dimension d_k while the second condition says that the interference subspace and the received signal subspace must be linearly independent. Finally, the third condition assures that the interference from other users lies in the interference subspace (which is linearly independent of the signal subspace).

Notice that in the 2-antenna case, if $d_j = d_k = 1$ and $\mathrm{rank}(\mathbf{H}_{kj}) = 2$, and the interference subspace $\mathcal{I}_k$ is known, then $\mathcal{S}_j$ can be uniquely determined by $\mathcal{S}_j = \mathbf{H}_{kj}^{-1}\mathcal{I}_k$, for any $j \neq k$. Conversely, if $\mathcal{S}_j$ is known, we can uniquely find the interference subspace of user k, i.e., $\mathcal{I}_k = \mathbf{H}_{kj}\mathcal{S}_j$. Thus, by starting from a node with a known subspace and traversing the interference links with full rank channel matrices, we can uniquely determine the signal subspaces in the transmitter sides and the interference subspaces at the receiver sides as long as they all have one DoF. Furthermore, if we find a loop of full rank interfering links, the signal subspaces at these nodes must be the eigenvector of the composite channel matrix of the corresponding loop. To make this point clear, consider a 4-user interference channel. If all interfering links are full rank, by starting from transmitter 1 and use the loop Tx1 $\rightarrow$ Rx2 $\rightarrow$ Tx3 $\rightarrow$ Rx4 $\rightarrow$ Tx1, we have the following relations

$$\mathcal{I}_2 = \mathbf{H}_{21}\mathcal{S}_1, \quad \mathcal{S}_3 = \mathbf{H}_{23}^{-1}\mathcal{I}_2, \quad \mathcal{I}_4 = \mathbf{H}_{43}\mathcal{S}_3, \quad \mathcal{S}_1 = \mathbf{H}_{41}^{-1}\mathcal{I}_4.$$

Thus, $\mathcal{S}_1$ must be the eigenvector of the loop channel matrix $\mathbf{H}_{41}^{-1}\mathbf{H}_{43}\mathbf{H}_{23}^{-1}\mathbf{H}_{21}$. Using this observation and the idea of traversing the full rank interfering channel links, we can establish the polynomial solvability of the problem of checking the achievability of a given tuple of DoF in Theorem 2.

Theorem 2 *For a K-user MIMO interference channel where each transmit/receive node is equipped with at most two antennas, the problem of checking the achievability of a given tuple of DoF is polynomial time solvable.*

Proof By assigning zero channel weight if necessary, we can assume without loss of generality that all transmitters/receivers are equipped with exactly two antennas, i.e., $M_k = N_k = 2$, for all $k = 1, 2, \cdots, K$. Furthermore, notice that if a user has zero DoF ($d_k = 0$), then we can assign the zero beamforming vector to this user and remove it (both its transmitter and receiver) from the system. Thus, we can assume $1 \leq d_k \leq 2$ for all $k = 1, 2, \cdots, K$. We further assume that all the direct channel matrices $\mathbf{H}_{kk}, k = 1, 2, \ldots, K$, are nonzero. Now the problem is to determine whether the given tuple of DoF $(d_1, d_2, \cdots, d_K)$ is achievable or not. To this end, we need to define two bipartite graphs over the nodes of the interference channel (one side of the graph consists of transmit nodes and the other consists of the receive nodes). In particular, we construct a bipartite graph G by connecting the transmit node of user i to the receive node of user j if and only if the channel between them is nonzero, i.e., $\mathbf{H}_{ji} \neq 0$. Furthermore, we construct a bipartite subgraph $G' = (V', E')$ of G by considering only the full rank links of G, i.e., connecting transmit node i to the receive node $j \neq i$ if and only if $\text{rank}(\mathbf{H}_{ji}) = 2$. Notice that the link between transmit node i and receive node i is not included in G' even if $\text{rank}(\mathbf{H}_{ii}) = 2$.

In what follows, we first consider a simple case which gives us the idea of how a loop of rank 2 interfering channels forces a discrete structure on the choice of signaling subspaces at the transmitters. Then, using this idea, we provide the proof for the general case.

Consider a connected component H of G where all the interfering links are full rank and connected, i.e., the induced subgraph of H over G' is connected and contains all the interfering links of H. We first argue that H can not contain the receive node of any user k with $d_k = 2$. Suppose the contrary. Then the direct channel matrix, $\mathbf{H}_{kk}$, must be full rank. [If $\mathbf{H}_{kk}$ is rank deficient, then the received signal subspace at receiver k has dimension at most 1, which would make it impossible to achieve $d_k = 2$.] We further claim that $\mathbf{H}$ cannot contain any other nodes. Since the direct link between the transmit and receive nodes of user k is not contained in H, it follows that the receive node of user k must be connected to another transmit node a in $\mathbf{H}$. Let this node a be associated with a user j ($j \neq k$). Notice that user j achieves a DoF at least 1 (since all zero DoF users have been removed from G). By definition, node a must be connected to the receive node of user k via a full rank cross talk channel matrix $\mathbf{H}_{kj}$. Thus, user j will cause a nonzero interference subspace to user k, contradicting $d_k = 2$. Since all users with DoF =0 has been removed from graph G, we must have $d_k = 1$ for all receive nodes in H. For the other case where node a is a receive node of user j, then a is linked to the transmit node of user k via a full rank channel matrix. In this case, user k will cause a 2-dimensional interference subspace to user j, making it impossible to have $d_j \geq 1$.

We now assume that all receive nodes in $\mathbf{H}$ have one DoF. We can start from an arbitrary initial node of $\mathbf{H}$ and use Breadth First Search (BFS) to find a spanning tree. Since each user has one DoF, the signal and interference spaces of all receive nodes in $\mathbf{H}$ are uniquely determined by the signal (or interference) space of the initial node. Since the initial node is arbitrary, this shows that the signal/interference spaces for all nodes in $\mathbf{H}$ are linearly related to each other (via some constant composite channel matrices, see the discussion before Theorem 2). Fixing any one uniquely determines the rest. For the remaining edges (or links) not in the spanning tree, they each create a unique loop in the tree. We can compute the composite channel matrices for these loops (see the discussion before Theorem 2). Notice that each loop matrix (size 2×2) has either one, two or infinitely many eigenvectors (when the composite channel matrix is a constant multiple of identity matrix). Suppose a loop matrix (starting from a given transmit node, say b, in the loop) has one or two unique eigenvectors, then the signal space of node b must be generated by one of these eigenvectors. In fact, since the beamforming vectors of nodes in $\mathbf{H}$ are linearly related, each loop in $\mathbf{H}$ places a restriction on the choice of beamforming vector of node b. Thus, for any fixed transmit node b in $\mathbf{H}$, there are multiple restriction sets, each corresponding to a loop in $\mathbf{H}$ caused by adding an edge to the minimum spanning tree and each containing one/two one-dimensional subspaces from which node b's signal space can be chosen. The receive nodes in $\mathbf{H}$ can achieve interference alignment if and only if these restricted sets of one-dimensional signal subspaces for node b share a common one-dimensional subspace. Moreover, to ensure each user in $\mathbf{H}$ achieves one DoF, we need to additionally make sure that the resulting interference subspaces at all receive nodes in H are linearly independent from the corresponding respective signal subspaces. Since the total number of restriction sets is at most linear in the number of edges in H and each restriction set contains at most two one-dimensional subspaces, checking if these restrictions have any common one-dimensional subspace can be carried out in $O(K^2)$ time. Moreover, for each common one-dimensional subspace, checking if the linear independence between the resulting signal subspace and interference subspace (already aligned) at each receive node can also be performed in time that is linear in the number of nodes in H, or in $O(K)$ time.

Now we are ready to look into the general case in which the rank 1 links are considered as well as the full rank links. Since there is no interfering link between different connected components of G, we can assign the signal subspace for each connected component separately. Notice that the number of connected components of G is at most K, we only need to assign transmit subspaces for every connected component of G in polynomial time.

Let $\mathbf{H}$ be a connected component of G. Let $\mathbf{H}' \subseteq G'$ be a subgraph of $\mathbf{H}$ which contains only links with full rank channel matrices. $\mathbf{H}'$ can be decomposed into various connected components of G'. By the argument above for such components, the signal/interference spaces for the nodes in these connected components (consisting of at least two nodes) can be assigned in one of the two ways:

(B1) The connected component contains a cycle with a channel matrix that is not equal to a constant multiple of the identity matrix. In the case, the beamforming vectors of all nodes can be determined from the eigenvector(s) of a certain loop channel matrix. In this case, there are at most two possible choices of signal/interference space for each node.

(B2) The connected component has no loops (i.e., forms a tree) or if every loop has a composite channel matrix that is a constant multiple of the identity matrix. In this case, the signal/interference spaces of all nodes are linearly related to one another. The signal/interference space of one node can be fixed at an arbitrary one-dimensional subspace. Once this is fixed, the signal/interference spaces of other nodes can be derived uniquely.

Consider a rank-1 interfering link in $\mathbf{H}$ with channel matrix $\mathbf{H}_{ij}$ $(i \neq j)$. If user j transmits in the null of $\mathbf{H}_{ij}$, then the signalling subspace of user j is known, i.e., $\mathcal{S}_j = \mathit{Null}(\mathbf{H}_{ij})$. Otherwise, the interference subspace at user i is known, i.e., $\mathcal{I}_i = \mathit{Range}(\mathbf{H}_{ij})$. This is because $d_i \geq 1$, so we have $\dim \mathcal{I}_i \leq 1$. This plus the fact that $\mathit{Range}(\mathbf{H}_{ij}) \subseteq \mathcal{I}_i$ implies $\mathcal{I}_i = \mathit{Range}(\mathbf{H}_{ij})$. Therefore, we can assign a Boolean variable x_{ij} to each rank-1 channel $\mathbf{H}_{ij}$, with "$x_{ij} = 1$" representing $\mathcal{S}_j = \mathit{Null}(\mathbf{H}_{ij})$ and "$x_{ij} = 0$" signifying $\mathcal{I}_i = \mathit{Range}(\mathbf{H}_{ij})$. In this way, we associate a Boolean variable x_{ij} for each rank-1 crosstalk channel matrix $\mathbf{H}_{ij}$ in H.

Next we represent the interference alignment condition at each receive node of H using the Boolean variables $\{x_{ij}\}$ (plus some auxiliary Boolean variables $\{y_i, z_{ij}, z_i\}$ defined below). Suppose user i's receive node is in $\mathbf{H}$. We consider the cases $d_i = 2$ and $d_i = 1$ separately.

Case $d_i = 2$: In this case $\mathcal{I}_i = 0$, so we must have $x_{ij} = 1$. We rewrite this condition in the form of two 2-SAT clauses

$$x_{ij} \vee y_i, \quad x_{ij} \vee \bar{y}_i, \qquad \text{for all } j \neq i \text{ and } \mathrm{rank}(\mathbf{H}_{ij}) = 1, \tag{4.4}$$

where y_i is an auxiliary Boolean variable. In this case, the satisfaction of (5.7) and the condition that the receive node of user i is not connected to other users' transmit nodes via rank-2 links is equivalent to achieving one DoF for user i.

Case $d_i = 1$ *and* $\mathrm{rank}(\mathbf{H}_{ii}) = 1$: In this case, then the received signal subspace is $\mathbf{H}_{ii}\mathcal{S}_i = \mathit{Range}(\mathbf{H}_{ii})$ and $\dim \mathcal{I}_i = 1$, so that all the interference at the receive node of user i must be aligned in an one-dimensional subspace that is linearly independent of $\mathit{Range}(\mathbf{H}_{ii})$. We need to further consider several subcases, depending on if the receive node of user i is connected to other transmit nodes via rank-1 or rank-2 links. In particular, if the transmit nodes of users j and k are connected to receive node i via rank-1 links, then the interference alignment condition requires the satisfaction of the following 2-SAT clauses

$$
\begin{aligned}
x_{ij} \vee x_{ik}, &\quad \text{for all } j \neq k \neq i \text{ such that } \mathrm{rank}(\mathbf{H}_{ij}) = \mathrm{rank}(\mathbf{H}_{ik}) = 1 \text{ and } \mathit{Range}(\mathbf{H}_{ij}) \neq \mathit{Range}(\mathbf{H}_{ik}), \\
x_{ij} \vee z_{ij},\ x_{ij} \vee \bar{z}_{ij}, &\quad \text{for all } j \neq i \text{ such that } \mathrm{rank}(\mathbf{H}_{ij}) = 1 \text{ and } \mathit{Range}(\mathbf{H}_{ij}) = \mathit{Range}(\mathbf{H}_{ii}),
\end{aligned}
\tag{4.5}
$$

where z_{ij} is a dummy Boolean variable, and the last condition corresponds to the linear independence requirement of the signal/interference subspaces. Moreover, if there is a rank-2 link connecting the receive node of user i to the transmit node of user ℓ, $\ell \neq i$, i.e., $\mathbf{H}_{i\ell}$ is full rank, then the receive node of user i is in $\mathbf{H}'$. Consequently, the transmit strategy of user ℓ has only two possibilities B1 and B2 as outlined above. For the Case B1 where the transmit node of user ℓ can pick one of the two possible beamforming vectors $\mathbf{v}_\ell^0$, $\mathbf{v}_\ell^1$, we define a Boolean variable z_ℓ with "$z_\ell = 0$" representing $\mathbf{v}_\ell^0$ is chosen, while "$z_\ell = 1$" signifying $\mathbf{v}_\ell^1$ is chosen. Now the interference alignment for user i requires the satisfaction of following 2-SAT clauses

$$
\begin{aligned}
z_\ell \vee x_{ij}, &\quad \text{for all } j \neq \ell \neq i \text{ such that } \mathrm{rank}(\mathbf{H}_{ij}) = 1,\ \mathrm{rank}(\mathbf{H}_{i\ell}) = 2 \text{ and } \mathbf{H}_{i\ell}\mathbf{v}_\ell^0 \notin \mathit{Range}(\mathbf{H}_{ij}), \\
\bar{z}_\ell \vee x_{ij}, &\quad \text{for all } j \neq \ell \neq i \text{ such that } \mathrm{rank}(\mathbf{H}_{ij}) = 1,\ \mathrm{rank}(\mathbf{H}_{i\ell}) = 2 \text{ and } \mathbf{H}_{i\ell}\mathbf{v}_\ell^1 \notin \mathit{Range}(\mathbf{H}_{ij}).
\end{aligned}
\tag{4.6}
$$

If in Case B1 the transmit node of user ℓ must pick a unique vector $\mathbf{v}_\ell^0$, then we must have $z_\ell = 0$ and $x_{ij} = 1$ if $\mathbf{H}_{i\ell}\mathbf{v}_\ell^0 \notin \mathit{Range}(\mathbf{H}_{ij})$, and $z_\ell = 0$ if $\mathbf{H}_{i\ell}\mathbf{v}_\ell^0 \in \mathit{Range}(\mathbf{H}_{ij})$. The latter conditions are equivalent to the satisfaction of the following 2-SAT clauses:

$$
\begin{aligned}
\bar{z}_\ell \vee x_{ij},\ \bar{z}_\ell \vee \bar{x}_{ij},\ z_\ell \vee x_{ij}, &\quad \text{for all } j \neq \ell \neq i \text{ s.t. } \mathrm{rank}(\mathbf{H}_{ij}) = 1,\ \mathrm{rank}(\mathbf{H}_{i\ell}) = 2 \text{ and } \mathbf{H}_{i\ell}\mathbf{v}_\ell^0 \notin \mathit{Range}(\mathbf{H}_{ij}), \\
\bar{z}_\ell \vee x_{ij},\ \bar{z}_\ell \vee \bar{x}_{ij}, &\quad \text{for all } j \neq \ell \neq i \text{ s.t. } \mathrm{rank}(\mathbf{H}_{ij}) = 1,\ \mathrm{rank}(\mathbf{H}_{i\ell}) = 2 \text{ and } \mathbf{H}_{i\ell}\mathbf{v}_\ell^0 \in \mathit{Range}(\mathbf{H}_{ij}).
\end{aligned}
\tag{4.7}
$$

To ensure linear independence of the signal and interference subspaces for user i, we must make sure the satisfaction of the following 2-SAT clauses

$$
\begin{aligned}
\bar{z}_\ell \vee y_i,\ \bar{z}_\ell \vee \bar{y}_i, &\quad \text{for all } \ell \neq i \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = 2 \text{ and } \mathbf{H}_{i\ell}\mathbf{v}_\ell^1 \in \mathit{Range}(\mathbf{H}_{ii}), \\
z_\ell \vee y_i,\ z_\ell \vee \bar{y}_i, &\quad \text{for all } \ell \neq i \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = 2 \text{ and } \mathbf{H}_{i\ell}\mathbf{v}_\ell^0 \in \mathit{Range}(\mathbf{H}_{ii}),
\end{aligned}
\tag{4.8}
$$

where y_i is a dummy Boolean variable. Now we consider Case B2. Suppose the receive node of user i lies in a connected component $\mathbf{H}''$ of $\mathbf{H}'$. Then, for each pair of receive node of users i and ℓ in $\mathbf{H}''$ ($i \neq \ell$), there exists a (efficiently computable) nonsingular matrix $G_{i\ell}$ such that

$$
\mathcal{I}_i = G_{i\ell}\mathcal{I}_\ell.
$$

To ensure this condition, the following 2-SAT clauses must be satisfied for all transmit nodes j and k in H'':

$$x_{ij} \vee x_{\ell k}, \quad \text{for all } j \neq i,\ k \neq \ell \text{ s.t. } \mathrm{rank}(\mathbf{H}_{\ell k}) = \mathrm{rank}(\mathbf{H}_{ij}) = 1, \text{ and } G_{i\ell}\mathit{Range}(\mathbf{H}_{\ell k}) \neq \mathit{Range}(\mathbf{H}_{ij}). \tag{4.9}$$

Furthermore, to make sure that the signal and interference subspaces are linearly independent at the receive node of user i, we must have for all transmit node j in H'' that the following 2-SAT clauses are satisfied

$$x_{ij} \vee z_{ij},\ x_{ij} \vee \bar{z}_{ij}, \quad \text{for all } j \neq i \text{ s.t. } \mathit{Range}(\mathbf{H}_{ij}) = \mathit{Range}(\mathbf{H}_{ii}). \tag{4.10}$$

Finally, we notice that the Boolean variables $\{x_{i\ell}, z_{i\ell}\}$ all represent the signaling strategies of user ℓ. We must ensure that these signaling strategies are compatible. In other words, we can not simultaneously have both $\mathcal{S}_\ell = \mathit{Null}(\mathbf{H}_{i\ell})$ and $\mathcal{S}_\ell = \mathit{Null}(\mathbf{H}_{j\ell})$ $(j \neq i)$, unless of course the two null spaces are equal. This implies that we should have

$$\bar{x}_{i\ell} \vee \bar{x}_{j\ell}, \quad \text{for all } i \neq j \neq \ell \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = \mathrm{rank}(\mathbf{H}_{j\ell}) = 1,\ \mathit{Null}(\mathbf{H}_{i\ell}) \neq \mathit{Null}(\mathbf{H}_{j\ell}). \tag{4.11}$$

Moreover, if the transmit node of user ℓ is also in H'' and its transmit beamforming vector must be chosen from the set $\{\mathbf{v}_\ell^0, \mathbf{v}_\ell^1\}$ (Case B1). Then, by a similar argument, we must also ensure the following compatibility conditions:

$$\begin{aligned} &\bar{x}_{i\ell} \vee z_\ell, \quad \text{for all } i \neq j \neq \ell \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = 1,\ \mathrm{rank}(\mathbf{H}_{j\ell}) = 2,\ \mathbf{v}_\ell^0 \notin \mathit{Null}(\mathbf{H}_{i\ell}),\\ &\bar{x}_{i\ell} \vee \bar{z}_\ell, \quad \text{for all } i \neq j \neq \ell \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = 1,\ \mathrm{rank}(\mathbf{H}_{j\ell}) = 2,\ \mathbf{v}_\ell^1 \notin \mathit{Null}(\mathbf{H}_{i\ell}). \end{aligned} \tag{4.12}$$

In case of B2 (i.e., H'' is a tree or all loop matrices are constant multiples of identity matrix), then the transmit subspace of user ℓ (which lies in H'') can be chosen continuously (rather than from a discrete set $\{\mathbf{v}_\ell^0,\ \mathbf{v}_\ell^1\}$). In this case, the compatibility condition (4.11) is sufficient; there is no additional compatibility condition needed.

Case $d_i = 1$ *and* $\mathrm{rank}(\mathbf{H}_{ii}) = 2$: In this case, if the transmit node of user i is connected to a receive node of user j via a rank-1 link, then $x_{ji} = 1$ signifies the use of transmit beamforming subspace of $\mathit{Null}(\mathbf{H}_{ji})$ for user i; else if transmitter i is in H'' so that its transmit beamforming direction must be chosen from $\mathbf{v}_i^0$, $\mathbf{v}_i^1$, corresponding to $z_i = 0$ and 1 respectively (Case B1). [Case B2 corresponds to the continuous selection of beamforming vector for user i; no 2-SAT clause is needed in that case.] In the first case, the signal subspace at receive node of user i becomes $\mathbf{H}_{ii}Null(\mathbf{H}_{ji})$, while in the second case, the signal subspace is $\mathbf{H}_{ii}\mathbf{v}_i^0$, or $\mathbf{H}_{ii}\mathbf{v}_i^1$. We must make sure the signal subspace is linearly independent from the interference subspace of user i. This implies that the following 2-SAT clauses must be satisfied:

$$\begin{aligned}
\bar{x}_{ji} \vee x_{i\ell}, &\quad \text{for all } i \neq j \neq \ell \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = \mathrm{rank}(\mathbf{H}_{ji}) = 1,\ Range(\mathbf{H}_{i\ell}) = \mathbf{H}_{ii}Null(\mathbf{H}_{ji}), \\
x_{i\ell} \vee z_i, &\quad \text{for all } i \neq j \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = 1,\ i \in H'',\ \mathbf{H}_{ii}\mathbf{v}_i^0 \in Range(\mathbf{H}_{i\ell}), \\
x_{i\ell} \vee \bar{z}_i, &\quad \text{for all } i \neq j \neq \ell \text{ s.t. } \mathrm{rank}(\mathbf{H}_{i\ell}) = 1,\ i \in H'',\ \mathbf{H}_{ii}\mathbf{v}_i^1 \in Range(\mathbf{H}_{i\ell}).
\end{aligned} \tag{4.13}$$

It can be checked that the DoF tuple $(d_1, d_2, \ldots, d_K)$ is achievable if and only if conditions (5.7)-(4.13) are satisfied for some binary realizations of Boolean variables $\{x_{ij}, y_i, z_i, z_{ij}\}$. Moreover, the number of such 2-SAT clauses is polynomial in K (in fact $O(K^4)$). Hence, we have transformed the DoF feasibility problem in polynomial time to an instance of 2-satisfiability problem. The latter problem is known to be solvable in polynomial time.

Chapter 5

Solvability of Interference Alignment Equations for Generic Channels

5.1 Bounding the Total DoF Achievable via Linear Interference Alignment

Our goal is to study the solvability of the interference alignment problem (3.1)-(3.2) and derive a general condition that must be satisfied by any DoF tuple $(d_1, d_2, ..., d_K)$ achievable through linear interference alignment for generic choice of channel matrices. We will also provide some conditions under which this upper bound is achievable.

Let us denote the polynomial equations in (3.2) by the index set

$$\mathcal{J} \triangleq \{(k, j) \mid 1 \leq k \neq j \leq K\}.$$

The following theorem provides an upper bound on the total achievable DoF when no channel extension is allowed.

Theorem 3 *Consider a K-user flat fading MIMO interference channel where the channel matrices $\{\mathbf{H}_{ij}\}_{i,j=1}^{K}$ are generic (e.g., drawn from a continuous probability distribution). Assume no channel extension is allowed. Then any tuple of degrees of freedom*

$(d_1, d_2, ..., d_K)$ *that is achievable through linear interference alignment* (3.1) *and* (3.2) *must satisfy the following inequalities*

$$\min\{M_k, N_k\} \geq d_k, \quad \forall\, k, \tag{5.1}$$

$$\max\{M_k, N_j\} \geq d_k + d_j, \quad \forall\, k, j, k \neq j, \tag{5.2}$$

$$\sum_{k:(k,j)\in\mathcal{I}} (M_k - d_k)d_k + \sum_{j:(k,j)\in\mathcal{I}} (N_j - d_j)d_j \geq \sum_{(k,j)\in\mathcal{I}} d_k d_j, \quad \forall\, \mathcal{I} \subseteq \mathcal{J}. \tag{5.3}$$

Condition (5.3) in Theorem 3 can be used to bound the total DoF achievable in a MIMO interference channel. The following corollary is immediate.

Corollary 1 *Assume the setting of Theorem 3. Then the following upper bounds hold true.*

(a) In the case of $d_k = d$ for all k, interference alignment is impossible unless

$$d \leq \frac{1}{K(K+1)} \sum_{k=1}^{K} (M_k + N_k).$$

(b) In the case of $M_k + N_k = M + N$, interference alignment requires

$$\left(\sum_{k=1}^{K} d_k\right)^2 + \sum_{k=1}^{K} d_k^2 \leq (M+N) \sum_{k=1}^{K} d_k$$

which further implies

$$\sum_{k=1}^{K} d_k < (M+N).$$

Part (b) of Corollary 1 shows that the total achievable DoF in a MIMO interference channel is bounded by a constant $M + N - 1$, regardless of how many users are present in the system. While this bound is an improvement over the single user case which has a maximum DoF of $\min\{M, N\}$, it is significantly weaker than the maximum achievable total DoF for a diagonal frequency selective (or time varying) interference channel. The latter grows linearly with the number of users in the system [1].

The rest of this section is devoted to the proof of Theorem 3 and its converse.

Since we will use several concepts and results from the field theory [82] and algebraic geometry [85, 86], we first provide a brief review of the necessary algebraic background.

5.1.1 Algebraic Preliminaries

Let $\mathcal{K}, \mathcal{F}$ be two fields such that $\mathcal{K} \subseteq \mathcal{F}$. In this case, we say $\mathcal{F}$ is an extension of $\mathcal{K}$, denoted by $\mathcal{F}/\mathcal{K}$. Let us use $\mathcal{K}[z_1, z_2, \ldots, z_n]$ to denote the ring of polynomials with coefficients drawn from $\mathcal{K}$. We say $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{F}$ are *algebraically dependent* over $\mathcal{K}$ if there exists a nonzero polynomial $f(z_1, z_2, \ldots, z_n) \in \mathcal{K}[z_1, z_2, \ldots, z_n]$ such that

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0. \tag{5.4}$$

Otherwise, we say that they are *algebraically independent* over $\mathcal{K}$. The largest cardinality of an algebraically independent set is called the transcendence degree of $\mathcal{F}$ over $\mathcal{K}$. An element $\alpha \in \mathcal{F}$ is said to be algebraic over $\mathcal{K}$ if there exists a nonzero polynomial $f \in \mathcal{K}[z]$ such that $f(\alpha) = 0$; else, we say α is transcendental over $\mathcal{K}$.

Example 1. Let $\mathcal{K} = \mathbb{C}$ be the field of complex numbers and $\mathcal{F} = \mathbb{C}(x_1, x_2)$ be the field of rational functions in variables x_1, x_2. Then, the polynomials

$$g_1 = x_1^2 x_2, \quad g_2 = x_2^2, \quad g_3 = x_1 x_2$$

are algebraically dependent over $\mathbb{C}$ because $f(g_1, g_2, g_3) = 0$ identically for all (x_1, x_2), where $f(z_1, z_2, z_3) = z_1^2 z_2 - z_3^4$.

Example 2. The two complex numbers $a = \sqrt{\pi},\ b = 3\pi + 2$ are algebraically dependent over the field of rational numbers because by defining $f(z_1, z_2) = 3z_1^2 - z_2 + 2$, we have $f(a, b) = 0$.

Notice that the definition of algebraic independence is in many ways similar to the standard notion of linear independence from linear algebra. In fact, if the function f in (5.4) is required to be linear, then algebraic independence reduces to the usual concept of linear independence. Similar to linear algebra, we can define a basis for the field $\mathcal{F}$

using the notion of algebraic independence. In particular, given any algebraically independent set S over the field $\mathcal{K}$, let $\mathcal{K}(S)$ denote the field of rational functions in S with coefficients taken from the field $\mathcal{K}$. For any field extension $\mathcal{F}/\mathcal{K}$, it is always possible to find a set S in $\mathcal{F}$, algebraically independent over $\mathcal{K}$, such that $\mathcal{F}$ is an algebraic extension of $\mathcal{K}(S)$. Such a set S is called a transcendence basis of $\mathcal{F}$ over $\mathcal{K}$. All transcendence bases have the same cardinality, equal to the transcendence degree of the extension $\mathcal{F}/\mathcal{K}$. If every element in $\mathcal{F}$ is algebraic over $\mathcal{K}$, then we say $\mathcal{F}/\mathcal{K}$ is an algebraic extension. In this case, the transcendence degree of $\mathcal{F}$ over $\mathcal{K}$ is zero.

Example 3. The two polynomials g_1 and g_2 in Example 1 are algebraically independent over $\mathbb{C}$. Together, they constitute a transcendental basis for $\mathbb{C}(x_1, x_2)$ over $\mathbb{C}$.

The following table shows similar concepts between linear algebra and transcendental field extension (see [82, 86] for more details).

Linear algebra	**Transcendental field extension**
linear independence	algebraic independence
$A \subseteq \text{span}(B)$	A algebraically dependent on B
linear basis	transcendence basis
dimension	transcendence degree

In linear algebra, it is well known that any $(n+1)$ vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n+1}$ in an n-dimensional vector space must be linearly dependent. In other words, there exists a nonzero linear function $f(z_1, z_2, ..., z_{n+1})$ such that $f(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n+1}) = 0$. A similar result holds for algebraic independence. For example, any $(n+1)$ polynomials g_1, g_2,..., g_{n+1} defined on n variables $(x_1, x_2, ..., x_n)$ must be algebraically dependent. Consequently, there exists a nonzero polynomial $f(z_1, z_2, ..., z_{n+1})$ such that

$$f(g_1, g_2, ..., g_{n+1}) = 0, \quad \forall\ (x_1, x_2, ..., x_n).$$

Example 1 is an instance of this property with $n = 2$. The following example states this property, to be used in the proof of Theorem 3, in a more formal setting.

Example 4. Let $\mathbb{C}(z_1, z_2, \ldots, z_n)$ denote the field of rational functions in n variables with coefficients in $\mathbb{C}$. The set $\{z_1, z_2, \ldots, z_n\}$ is a maximal algebraically independent set in $\mathbb{C}(z_1, z_2, \ldots, z_n)$. Hence the transcendence degree of the field extension $\mathbb{C}(z_1, z_2, \ldots, z_n)/\mathbb{C}$ is n. Furthermore, for any m polynomials

$$g_1(z_1, z_2, \ldots, z_n),\ g_2(z_1, z_2, \ldots, z_n),\ \ldots,\ g_m(z_1, z_2, \ldots, z_n),$$

where $m > n$, there exists a nonzero polynomial $f(\cdot)$ such that $f(g_1, g_2, \ldots, g_m) = 0,\ \forall\, z_1, z_2, \ldots, z_n$.

Next we describe a useful local expansion of a multivariate polynomial function. Recall that for any univariate polynomial f and any $\bar{x} \in \mathbb{C}$, there holds

$$f(x) = f(\bar{x}) + (x - \bar{x})g(x),\ \text{for all } x \in \mathbb{C},$$

where g is some polynomial dependent on $\bar{x}$ and the coefficients of f only. Similarly, for a n-variate polynomial f defined on the variables $\mathbf{x} = (x_1, x_2, ..., x_n)$ and any $\bar{\mathbf{x}} \in \mathbb{C}^n$, we have

$$f(\mathbf{x}) = f(\bar{\mathbf{x}}) + \sum_{i=1}^{n} (x_i - \bar{x}_i) g_i(\mathbf{x}) = f(\bar{\mathbf{x}}) + (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{g}(\mathbf{x}),\ \forall\, \mathbf{x} \in \mathbb{C}^n,$$

where each g_i is some polynomial dependent on $\bar{\mathbf{x}}$ and the coefficients of f only. If we replace the scalar variable x_i by a matrix variable $\mathbf{X}_i$, then we can write

$$f(\mathbf{X}) = f(\bar{\mathbf{X}}) + \sum_{i=1}^{n} \mathrm{Tr}\left((\mathbf{X}_i - \bar{\mathbf{X}}_i) G_i(\mathbf{X})\right),\ \forall\, \mathbf{X}, \tag{5.5}$$

where each G_i is a matrix whose entries are polynomials dependent on the entries of $\bar{\mathbf{X}}$ and the coefficients of f only. The local expansion (5.5) will be used in the proof of Theorem 3.

To prove the converse of Theorem 3, we will use the concepts of Zariski topology and a Zariski constructible set. We briefly review these concepts next (see [85] for more details). Consider $\mathbb{C}^n$, the n-dimensional vector space over the field of complex numbers $\mathbb{C}$. [One can replace $\mathbb{C}$ by any algebraically closed field.] The Zariski topology for $\mathbb{C}^n$ is defined by specifying its closed sets, and these are taken simply to be all the algebraic sets in $\mathbb{C}^n$. That is, the closed sets under Zariski topology are those of the form

$$S = \{\mathbf{x} \in \mathbb{C}^n \mid f_i(\mathbf{x}) = 0,\ i = 1, 2, ..., m\}$$

where $\{f_i\}_{i=1}^m$ is any set if polynomials with coefficients taken from $\mathbb{C}$. For example, the entire space $\mathbb{C}^n$ is Zariski closed (Take $m = 1$ and f_1 to be the zero function, i.e., $f_1(x) = 0,\ \forall\ x$). All other Zariski closed sets have zero measure. A nonempty Zariski open set (the complement of a Zariski closed set) always has dimension n. If a property holds over a Zariski open set, we say the property holds *generically.*

In topology, a set is locally closed if it is the intersection of an open set with a closed set. A *constructible set* is defined as a finite union of locally closed sets. Thus, a Zariski constructible set is simply a finite collection of sets, each defined by the feasible set of finitely many polynomial equations and polynomial inequalities. Clearly, if a Zariski constructible set has dimension n, then it must contain a Zariski open subset.

Let $\phi_1, \phi_2, \ldots, \phi_n$ be polynomials in $x_1, x_2, \ldots, x_n$ with coefficients from $\mathbb{C}$. They define a map $\Phi : \mathbb{C}^n \mapsto \mathbb{C}^n$ as follows: $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \ldots, \phi_n(\mathbf{x})) \in \mathbb{C}^n$. Chevalley's Theorem says that the image of this map is a constructible set (see [86] for more details).

Example 5. Let $\Phi : \mathbb{C}^2 \mapsto \mathbb{C}^2$ be defined by $\Phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi_2(\mathbf{x}))$ where $\phi_1(x) = x_1$ and $\phi_2(x) = x_1x_2$. Let $\mathcal{L}$ be the line $\{\mathbf{x} \in \mathbb{C}^2 : x_1 = 0\}$. The image of Φ is the union of two locally closed sets, $\mathbb{C}^2 \backslash \mathcal{L}$ (which is in fact open) and the point $(0, 0)$ (which is indeed closed).

Let the image of Φ be the union of locally closed subsets $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_p$ where $\mathcal{W}_i = \mathcal{U}_i \bigcap \mathcal{V}_i$ and $\mathcal{V}_i$ is closed and $\mathcal{U}_i$ is open. Assume the Jacobian of $\phi_1, \phi_2, \ldots, \phi_n$ is nonsingular at some point $\mathbf{x} \in \mathbb{C}^n$. The Implicit Function Theorem says that the image of Φ contains a small open disc around $\Phi(\mathbf{x})$, hence the measure of the image is nonzero. This implies that for some i, $\mathcal{V}_i = \mathbb{C}^n$ and $\mathcal{W}_i = \mathcal{U}_i$, i.e., the image of the map $\Phi(\cdot)$ contains a Zariski open set. Thus, if a certain property is shown to hold over the image of a polynomial map $\Phi : \mathbb{C}^n \mapsto \mathbb{C}^n$ whose Jacobian is nonsingular at some point, then this property must hold generically. We will use this approach to establish the generic feasibility of interference alignment for certain MIMO interference channels (Theorem 4).

5.1.2 Proof of Theorem 3

We now use the transcendental field extension theory to establish Theorem 3.

Proof The inequality (5.1) is obvious due to (3.2). To prove (5.2), assume $M_j \leq N_k$. Since $\mathbf{H}_{kj}$ is generic, $\text{rank}(\mathbf{H}_{kj}\mathbf{V}_j) = d_j$. Furthermore, due to (3.2), the beamformer $\mathbf{U}_k$ must be full rank and hence $d_k + d_j$ must be no more than the total dimension N_k. Similar argument shows that $d_k + d_j \leq M_j$ when $M_j \geq N_k$. Thus, $d_k + d_j \leq \max\{M_j, N_k\}$.

For simplicity of notations, we prove (5.3) for the case $\mathcal{I} = \mathcal{J}$. When $\mathcal{I} \subset \mathcal{J}$, the proof is the same except that we need to focus on a subset of equations/variables. Now, we prove (5.3) for the case of $\mathcal{I} = \mathcal{J}$ by contradiction. Assume the contrary that

$$\sum_{k=1}^{K}(M_k - d_k)d_k + \sum_{j=1}^{K}(N_j - d_j)d_j < \sum_{k,j=1,k\neq j}^{K} d_k d_j, \tag{5.6}$$

and the interference alignment conditions in (3.1) and (3.2) are satisfied. The interference alignment condition (3.2) implies that $\mathbf{U}_k$ and $\mathbf{V}_k$ must have full column rank. By applying appropriate linear transformations to the rows of $\mathbf{U}_k$ and $\mathbf{V}_k$, we can write

$$\mathbf{U}_k = \mathbf{P}_k^u \begin{bmatrix} \mathbf{I} \\ \bar{\mathbf{U}}_k \end{bmatrix} \mathbf{Q}_k^u, \quad \mathbf{V}_k = \mathbf{P}_k^v \begin{bmatrix} \mathbf{I} \\ \bar{\mathbf{V}}_k \end{bmatrix} \mathbf{Q}_k^v, \quad \forall k, \tag{5.7}$$

where $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$ are some matrices of size $(N_k - d_k) \times d_k$ and $(M_k - d_k) \times d_k$ respectively. The matrices $\mathbf{P}_k^u$ and $\mathbf{P}_k^v$ are square permutation matrices of size $N_k \times N_k$ and $M_k \times M_k$ respectively, while $\mathbf{Q}_k^u, \mathbf{Q}_k^v$ are some invertible matrices of size $d_k \times d_k$. Define $\bar{\mathbf{H}}_{ij} = \mathbf{P}_i^{u\,-1}\mathbf{H}_{ij}\mathbf{P}_j^{v\,-1}$ to be the permuted version of $\mathbf{H}_{kj}$. We can partition the matrix $\bar{\mathbf{H}}_{kj}$ as

$$\bar{\mathbf{H}}_{kj} = \begin{bmatrix} \bar{\mathbf{H}}_{kj}^{(1)} & \bar{\mathbf{H}}_{kj}^{(2)} \\ \bar{\mathbf{H}}_{kj}^{(3)} & \bar{\mathbf{H}}_{kj}^{(4)} \end{bmatrix}$$

where $\bar{\mathbf{H}}_{kj}^{(1)}$ is of size $d_k \times d_j$. Since the channel matrices $\{\mathbf{H}_{kj}\}_{k\neq j}$ are drawn from a continuous probability distribution, the transformed channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ remain generic. Rewriting the linear interference alignment condition (3.1) in terms of $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$, we obtain

$$\begin{bmatrix} \mathbf{I} & \bar{\mathbf{U}}_k^H \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_{kj}^{(1)} & \bar{\mathbf{H}}_{kj}^{(2)} \\ \bar{\mathbf{H}}_{kj}^{(3)} & \bar{\mathbf{H}}_{kj}^{(4)} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \bar{\mathbf{V}}_j \end{bmatrix} = \mathbf{0} \tag{5.8}$$

or equivalently

$$\bar{\mathbf{H}}_{kj}^{(1)} + \bar{\mathbf{U}}_k^H \bar{\mathbf{H}}_{kj}^{(3)} + \bar{\mathbf{H}}_{kj}^{(2)} \bar{\mathbf{V}}_j + \bar{\mathbf{U}}_k^H \mathbf{H}_{kj}^{(4)} \bar{\mathbf{V}}_j = \mathbf{0}, \quad \forall\, j \neq k. \tag{5.9}$$

The above system of quadratic equations, first derived in [45], is equivalent to the interference alignment condition (3.1). The number of scalar equations in (5.9) is

$$\sum_{j,k=1,j\neq k}^{K} d_k d_j,$$

while the total number of scalar variables (i.e., the scalar entries of the unknown matrices $\{\bar{\mathbf{U}}_k\}$'s and $\{\bar{\mathbf{V}}_k\}$'s) is

$$\sum_{k=1}^{K}(M_k - d_k)d_k + \sum_{k=1}^{K}(N_k - d_k)d_k = \sum_{k=1}^{K}(M_k + N_k - 2d_k)d_k.$$

So if

$$\sum_{k=1}^{K}(M_k + N_k - 2d_k)d_k < \sum_{j,k=1,j\neq k}^{K} d_k d_j, \tag{5.10}$$

then we would have more constraints than unknowns in the interference alignment condition (5.9), which we will argue cannot hold.

Let us consider the field $\mathcal{F}$ defined over the field of complex numbers $\mathbb{C}$, consisting of all rational functions in the entries of the matrices $\{\bar{\mathbf{U}}_k\}_{k=1}^K$ and $\{\bar{\mathbf{V}}_k\}_{k=1}^K$. Note that the entries of the matrices $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$ form a transcendence basis for $\mathcal{F}$ over $\mathbb{C}$. Thus, the transcendence degree of $\mathcal{F}$ is $\sum_{k=1}^K (M_k + N_k - 2d_k)d_k$, which is equal to the number of entries in the matrices $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$.

Now, let us consider the matrices $\mathbf{H}_{kj}^{(2)}, \mathbf{H}_{kj}^{(3)}, \mathbf{H}_{kj}^{(4)}$ for all $k, j, k \neq j$ and define the matrix $\mathbf{F}_{kj}$:

$$\mathbf{F}_{kj}(\bar{\mathbf{U}},\bar{\mathbf{V}}) \triangleq -\left(\bar{\mathbf{U}}_k^H\bar{\mathbf{H}}_{kj}^{(3)} + \bar{\mathbf{H}}_{kj}^{(2)}\bar{\mathbf{V}}_j + \bar{\mathbf{U}}_k^H\bar{\mathbf{H}}_{kj}^{(4)}\bar{\mathbf{V}}_j\right), \tag{5.11}$$

for all k, j with $k \neq j$. Note that $\mathbf{F}_{kj}$ is a $d_k \times d_j$ matrix, with each entry being a quadratic polynomial function of the entries in the matrices $\bar{\mathbf{U}}_k$ and $\bar{\mathbf{V}}_k$. As a result, the entries of $\mathbf{F}_{kj}$ belong to the field $\mathcal{F}$. Moreover, if (5.10) holds, then the number of quadratic polynomials given in the matrices $\{\mathbf{F}_{kj}\}_{k\neq j}$ is strictly larger than the transcendence degree of $\mathcal{F}$ over $\mathbb{C}$. Hence, as we discussed in the algebraic preliminaries (Section 5.1.1; see also [82, Chapter 8]), these quadratic polynomials in $\mathcal{F}$ must be algebraically dependent. This implies that there exists a nonzero polynomial p which vanishes at the quadratic polynomials corresponding to the entries of the matrices $\{\mathbf{F}_{kj}\}_{k\neq j}$, i.e.,

$$p\left(\mathbf{F}_{12}(\bar{\mathbf{U}},\bar{\mathbf{V}}), \mathbf{F}_{13}(\bar{\mathbf{U}},\bar{\mathbf{V}}), \ldots, \mathbf{F}_{K(K-1)}(\bar{\mathbf{U}},\bar{\mathbf{V}})\right) = 0,$$

for all $\{\bar{\mathbf{U}}_k,\ \bar{\mathbf{V}}_k\}_{k=1}^K$. Notice that the polynomial p is independent of the channel matrices $\left\{\bar{\mathbf{H}}_{kj}^{(1)}\right\}_{k\neq j}$, even though it does depend on the matrices $\left\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\right\}_{k\neq j}$. When viewed as a polynomial of the matrix variable $\mathbf{X} := \left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \ldots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right)$, $p(\cdot)$ can be expanded locally at $\bar{\mathbf{X}} := (\mathbf{F}_{12}(\bar{\mathbf{U}},\bar{\mathbf{V}}), \mathbf{F}_{13}(\bar{\mathbf{U}},\bar{\mathbf{V}}), \ldots, \mathbf{F}_{K(K-1)}(\bar{\mathbf{U}},\bar{\mathbf{V}}))$ using (5.5):

$$\begin{aligned} &p\left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \ldots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right) \\ &\quad = p\left(\mathbf{F}_{12}(\bar{\mathbf{U}},\bar{\mathbf{V}}), \mathbf{F}_{13}(\bar{\mathbf{U}},\bar{\mathbf{V}}), \ldots, \mathbf{F}_{K(K-1)}(\bar{\mathbf{U}},\bar{\mathbf{V}})\right) \\ &\qquad + \sum_{k\neq j} \mathrm{Tr}\left((\bar{\mathbf{H}}_{kj}^{(1)} - \mathbf{F}_{kj}(\bar{\mathbf{U}},\bar{\mathbf{V}}))\mathbf{Q}_{kj}(\bar{\mathbf{U}},\bar{\mathbf{V}})\right), \end{aligned}$$

for all $\{\bar{\mathbf{U}}_k,\ \bar{\mathbf{V}}_k\}_{k=1}^K$, where $\mathbf{Q}_{kj}$ is some polynomial matrix of size $d_j \times d_k$. Combining the above two identities yields

$$\begin{aligned} &p\left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \ldots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right) \\ &\quad = \sum_{k\neq j} \mathrm{Tr}\left((\bar{\mathbf{H}}_{kj}^{(1)} - \mathbf{F}_{kj}(\bar{\mathbf{U}},\bar{\mathbf{V}}))\mathbf{Q}_{kj}(\bar{\mathbf{U}},\bar{\mathbf{V}})\right). \end{aligned} \tag{5.12}$$

Notice that this equality holds for all choices of $\{\bar{\mathbf{U}}_k,\ \bar{\mathbf{V}}_k\}_{k=1}^K$. If the interference alignment condition (5.9) holds, then we have

$$\bar{\mathbf{H}}_{kj}^{(1)} - \mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = 0, \quad \text{for all } k,\ j \text{ with } k \neq j,$$

for some special choices of the matrices $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^{K}$. Substituting this condition into the right hand side of (5.12), we obtain

$$p\left(\bar{\mathbf{H}}_{12}^{(1)}, \bar{\mathbf{H}}_{13}^{(1)}, \ldots, \bar{\mathbf{H}}_{K(K-1)}^{(1)}\right) = 0. \tag{5.13}$$

Notice that the polynomial p is independent of the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$. Under our channel model, the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ are drawn from a continuous probability distribution. It follows that the condition (5.13) cannot hold unless p is identically zero, which contradicts the requirement $p \neq 0$.

Theorem 3 settles the conjecture of [45] in one direction, namely, the improperness of polynomial system (3.1) and (3.2) implies the infeasibility of interference alignment. From the proof of Theorem 3, it can be seen that the upper bound (5.3) holds for any choice of fixed channel matrices $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ as long as the channel matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ are generic.

Also, we remark that the proof technique for Theorem 3 can be used to bound the DoF for a single antenna parallel interference channel (e.g., the OFDM channel). In particular, consider a single input single output interference channel with M channel extensions, i.e., the channel matrices are diagonal and of the size $M \times M$. Assuming each user transmits one data stream ($d_k = 1$ for all k), we can check that the properness of the interference alignment condition (3.1)-(3.2) is equivalent to $K + 1 \leq 2M$ (see [45, Theorem 1]). Using a completely identical proof, we can show that the properness condition $K + 1 \leq 2M$ is a necessary condition for the feasibility of interference alignment. This implies that for the single beam case the total DoF per channel extension is upper bounded by 2, regardless of the number of channel extensions. This DoF bound has also been proposed recently in [88].

5.1.3 The Converse Direction

In the remainder of this section, we consider the converse of Theorem 3. In particular, we show that the upper bound in Theorem 3 is tight for a special case where all users have the same DoF d and number of antennas is divisible by d. In this case, we have $K(K-1)$ matrix equations in (5.9), each giving rise to d^2 scalar equations. For any subset of these matrix equations indexed by $\mathcal{I}$, with $\mathcal{I} \subseteq \mathcal{J}$, the number of corresponding scalar equations is equal to $d^2|\mathcal{I}|$, whereas the number of scalar variables involved in the equations indexed by $\mathcal{I}$ is

$$\left(\sum_{k:(k,j)\in\mathcal{I}} (M_k - d) + \sum_{j:(k,j)\in\mathcal{I}} (N_j - d) \right) d.$$

The next result shows that the bound in Theorem 3 is tight if the polynomial system (5.9) defining interference alignment is proper, i.e., for each $\mathcal{I} \subseteq \mathcal{J}$, the number of variables involved in each set of equations indexed by $\mathcal{I}$ is no less than $d^2|\mathcal{I}|$, the number of scalar equations. The proof of this result uses the Implicit Function Theorem which involves checking the Jacobian matrix of the polynomial map (5.11) is nonsingular at some channel realization $\{\bar{\mathbf{H}}_{kj}\}_{k\neq j}$. Notice that the feasibility of interference alignment condition (5.9) at a given channel realization $\{\bar{\mathbf{H}}_{kj}\}_{k\neq j}$ is equivalent to $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ being contained in the image of the polynomial map (5.11) which is defined by $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$. Fix a generic choice of $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ for which the Jacobian of the polynomial map (5.11) is nonsingular. The Implicit Function Theorem allows us to establish the existence of a locally invertible map from the space of channel submatrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$ to the space of beamforming matrices, and that the image of this polynomial map (5.11) is locally full-dimensional. Therefore, for all channel submatrices near the given channel realization $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$, the interference alignment condition (5.9) can be satisfied by some beamforming matrices. By Chevalley's Theorem from algebraic geometry [85] (see also the discussion at the end of Section 5.1.1), the "local full-dimensionality" of the image of (5.11) implies that this image, which is a constructible set, must contain a nonempty Zariski open set. As a result, the whole image of polynomial map (5.11) contains all generically generated channel sub-matrices

$\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k\neq j}$. Since the choice of channel submatrices $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k\neq j}$ is also generic, this then establishes the feasibility of interference alignment for all generically generated channel matrices $\{\bar{\mathbf{H}}_{kj}\}_{k\neq j}$.

Theorem 4 *Assume that all users have the same DoF $d_k = d$, where $1 \leq d \leq \min\{M_k, N_k\}$, $\forall k$. Furthermore, suppose that M_k and N_k are divisible by d for all k. Then interference alignment is achievable for generic channel coefficients if and only if for each subset $\mathcal{I}$ of equations in* (5.9)*, the number of variables involved in these equations is no less than the number of matrix equations times d^2, or equivalently,*

$$|\mathcal{I}|d \leq \sum_{k:(k,j)\in\mathcal{I}} (M_k - d) + \sum_{j:(k,j)\in\mathcal{I}} (N_j - d), \qquad \forall\, \mathcal{I} \text{ with } \mathcal{I} \subseteq \mathcal{J}. \tag{5.14}$$

Proof First of all, the "only if" direction is a direct consequence of Theorem 3. We now focus on the "if" direction. Consider the polynomial map that we get by concatenating all maps in (5.11) for all $(k, j) \in \mathcal{J}$, i.e.,

$$\begin{aligned}
\mathbf{F}_{12}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{12}^{(3)} + \bar{\mathbf{H}}_{12}^{(2)} \bar{\mathbf{V}}_2 + \bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{12}^{(4)} \bar{\mathbf{V}}_2\right), \\
\mathbf{F}_{13}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{13}^{(3)} + \bar{\mathbf{H}}_{13}^{(2)} \bar{\mathbf{V}}_3 + \bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{13}^{(4)} \bar{\mathbf{V}}_3\right), \\
&\vdots \\
\mathbf{F}_{K(K-1)}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_K^H \bar{\mathbf{H}}_{K(K-1)}^{(3)} + \bar{\mathbf{H}}_{K(K-1)}^{(2)} \bar{\mathbf{V}}_{K-1} + \bar{\mathbf{U}}_K^H \bar{\mathbf{H}}_{K(K-1)}^{(4)} \bar{\mathbf{V}}_{K-1}\right),
\end{aligned} \tag{5.15}$$

which maps the variables $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_k\}_{k=1}^K$ to the $\{\mathbf{F}_{k,j}\}_{k\neq j}$ space. We will first show that for a specific set of channel matrices, the rank of the Jacobian of this polynomial map is $K(K-1)d^2$, equal to the number of equations. Hence, if we restrict the equations to a subset of variables of size $K(K-1)d^2$, the determinant of the Jacobian matrix of the polynomial map (5.15) does not vanish identically. This step will establish the existence of a locally invertible map from the space of beamforming matrices to the space of channel matrices. By Chevalley's Theorem (see [85, Chapter 2, 6.E.]), this image is a constructible subset under Zariski topology. This, plus the fact that the image is locally full-dimensional, implies that the interference alignment condition (5.9) is feasible for all generically chosen channel matrices. This then will show the "if" direction of Theorem 4.

To show the nonsingularity of the Jacobian matrix, we need to remove some redundant variables in $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ (this occurs when there are more variables than equations), and then construct a specific set of channel matrices $\{\mathbf{H}_{kj}^{(2)}, \mathbf{H}_{kj}^{(3)}, \mathbf{H}_{kj}^{(4)}\}_{k\neq j}$ and a solution $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ at which the Jacobian matrix of (5.15) is nonsingular. Before providing a rigorous description for such a construction, we first consider a toy example with $K = 3$ users where $M_k = 3, N_k = 2, d_k = 1$, for $k = 1, 2, 3$. For this specific example, the assumption (5.14) is satisfied and the equations in (5.15) can be rewritten as

$$
\begin{aligned}
\mathbf{F}_{12}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{12}^{(3)} + \bar{\mathbf{H}}_{12}^{(2)} \bar{\mathbf{V}}_2 + \bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{12}^{(4)} \bar{\mathbf{V}}_2\right), \\
\mathbf{F}_{13}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{13}^{(3)} + \bar{\mathbf{H}}_{13}^{(2)} \bar{\mathbf{V}}_3 + \bar{\mathbf{U}}_1^H \bar{\mathbf{H}}_{13}^{(4)} \bar{\mathbf{V}}_3\right), \\
\mathbf{F}_{21}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_2^H \bar{\mathbf{H}}_{21}^{(3)} + \bar{\mathbf{H}}_{21}^{(2)} \bar{\mathbf{V}}_1 + \bar{\mathbf{U}}_2^H \bar{\mathbf{H}}_{21}^{(4)} \bar{\mathbf{V}}_1\right), \\
\mathbf{F}_{23}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_2^H \bar{\mathbf{H}}_{23}^{(3)} + \bar{\mathbf{H}}_{23}^{(2)} \bar{\mathbf{V}}_3 + \bar{\mathbf{U}}_2^H \bar{\mathbf{H}}_{23}^{(4)} \bar{\mathbf{V}}_3\right), \\
\mathbf{F}_{31}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_3^H \bar{\mathbf{H}}_{31}^{(3)} + \bar{\mathbf{H}}_{31}^{(2)} \bar{\mathbf{V}}_1 + \bar{\mathbf{U}}_3^H \bar{\mathbf{H}}_{31}^{(4)} \bar{\mathbf{V}}_1\right), \\
\mathbf{F}_{32}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) &= -\left(\bar{\mathbf{U}}_3^H \bar{\mathbf{H}}_{32}^{(3)} + \bar{\mathbf{H}}_{32}^{(2)} \bar{\mathbf{V}}_2 + \bar{\mathbf{U}}_3^H \bar{\mathbf{H}}_{32}^{(4)} \bar{\mathbf{V}}_2\right),
\end{aligned}
$$

where $\bar{\mathbf{V}}_k = [v_{k_1}\ v_{k_2}]^T \in \mathbb{C}^{2\times 1}$, $\bar{\mathbf{U}}_k = [u_k] \in \mathbb{C}$, for $k = 1, 2, 3$, and $\bar{\mathbf{H}}_{kj}^{(2)} = [\bar{h}_{kj}^{(2),1}\ \bar{h}_{kj}^{(2),2}]^T \in \mathbb{C}^{2\times 1}$, $\bar{\mathbf{H}}_{kj}^{(3)} = [\bar{h}_{kj}^{(3)}] \in \mathbb{C}$, for $k \neq j$. If we set $\bar{\mathbf{H}}_{kj}^{(4)} = 0$ for all channels, one can write the Jacobian of $[\mathbf{F}_{12}\ \mathbf{F}_{13}\ \mathbf{F}_{21}\ \mathbf{F}_{23}\ \mathbf{F}_{31}\ \mathbf{F}_{32}]$ with respect to the variables $[u_1\ u_2\ u_3\ v_{1_1}\ v_{1_2}\ v_{2_1}\ v_{2_2}\ v_{3_1}\ v_{3_2}]$ as

$$
\begin{bmatrix}
-\bar{h}_{12}^{(3)} & -\bar{h}_{13}^{(3)} & 0 & 0 & 0 & 0 \\
0 & 0 & -\bar{h}_{21}^{(3)} & -\bar{h}_{23}^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & -\bar{h}_{31}^{(3)} & -\bar{h}_{32}^{(3)} \\
0 & 0 & -\bar{h}_{21}^{(2),1} & 0 & -\bar{h}_{31}^{(2),1} & 0 \\
0 & 0 & -\bar{h}_{21}^{(2),2} & 0 & -\bar{h}_{31}^{(2),2} & 0 \\
-\bar{h}_{12}^{(2),1} & 0 & 0 & 0 & 0 & -\bar{h}_{32}^{(2),1} \\
-\bar{h}_{12}^{(2),2} & 0 & 0 & 0 & 0 & -\bar{h}_{32}^{(2),2} \\
0 & -\bar{h}_{13}^{(2),1} & 0 & -\bar{h}_{23}^{(2),1} & 0 & 0 \\
0 & -\bar{h}_{13}^{(2),2} & 0 & -\bar{h}_{23}^{(2),2} & 0 & 0
\end{bmatrix}.
$$

One can easily observe that by removing the variables $\{v_{1_1}, v_{2_1}, v_{3_2}\}$ and setting

$$\bar{h}_{12}^{(3)} = \bar{h}_{23}^{(3)} = \bar{h}_{31}^{(3)} = \bar{h}_{13}^{(2),1} = \bar{h}_{21}^{(2),2} = \bar{h}_{32}^{(2),2} = 1,$$
$$\bar{h}_{13}^{(3)} = \bar{h}_{21}^{(3)} = \bar{h}_{32}^{(3)} = \bar{h}_{12}^{(2),2} = \bar{h}_{31}^{(2),2} = \bar{h}_{23}^{(2),1} = 0,$$

the Jacobian of the mapping (5.15) with respect to the remaining variables becomes

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is clearly nonsingular since there exists exactly one nonzero element in each column/row.

Next we argue that the above construction procedure can be generalized to the case where M_k and N_k are divisible by d, provided that the assumption (5.14) is satisfied. The construction of these channel/beamforming matrices and the removal of redundant variables are outlined below. First, we set $\mathbf{H}_{kj}^{(4)} = \mathbf{0}$, for all $k \neq j$. Then we choose $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ arbitrarily. It remains to specify $\{\bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(2)}\}_{k\neq j}$. We should do so to ensure that the corresponding Jacobian matrix of (5.15) at $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ is nonsingular. Since M_k and N_k are divisible by d, we can partition our variables into blocks of size $d \times d$ and rewrite the mapping (5.15) as

$$\mathbf{F}_{kj}(\bar{\mathbf{U}}, \bar{\mathbf{V}}) = -\left[\bar{\mathbf{U}}_{k_1}^H \; \bar{\mathbf{U}}_{k_2}^H \dots \bar{\mathbf{U}}_{k_{s_k}}^H\right] \begin{bmatrix} \bar{\mathbf{H}}_{kj}^{(3),1} \\ \bar{\mathbf{H}}_{kj}^{(3),2} \\ \vdots \\ \bar{\mathbf{H}}_{kj}^{(3),s_k} \end{bmatrix} - \left[\bar{\mathbf{H}}_{kj}^{(2),1} \; \bar{\mathbf{H}}_{kj}^{(2),2} \dots \bar{\mathbf{H}}_{kj}^{(2),t_j}\right] \begin{bmatrix} \bar{\mathbf{V}}_{j_1} \\ \bar{\mathbf{V}}_{j_2} \\ \vdots \\ \bar{\mathbf{V}}_{j_{t_j}} \end{bmatrix}, \quad \forall\, k \neq j, \tag{5.16}$$

where $s_k = \frac{M_k}{d} - 1, t_j = \frac{N_j}{d} - 1$, and $\bar{\mathbf{U}}_{k_i}, \bar{\mathbf{V}}_{j_\ell}, \bar{\mathbf{H}}_{kj}^{(2),i}, \bar{\mathbf{H}}_{kj}^{(3),\ell} \in \mathbb{C}^{d\times d}$. Consider a bipartite graph G where the vertices are partitioned into two sets $\mathcal{X}$ and $\mathcal{Y}$. Each block of variables will correspond to a node in $\mathcal{X}$, while each matrix equation in (5.16) will correspond to a node in $\mathcal{Y}$. We draw an edge between a node $x \in \mathcal{X}$ and a node $y \in \mathcal{Y}$ if the block of variables corresponding to node x appears in the equation corresponding to node y. When viewed on the bipartite graph G, the assumption (5.14) simply says that for any given set of nodes $\mathcal{S} \subseteq \mathcal{Y}$, the cardinality of the neighbors of $\mathcal{S}$ in $\mathcal{X}$ is no smaller than the cardinality of $\mathcal{S}$. This condition is precisely what is required to ensure the existence of a complete matching in G covering all nodes in $\mathcal{Y}$ (Hall's theorem, see [87, Theorem 3.1.11]). Now consider a fixed complete matching in G. Let $A \subseteq \mathcal{X}$ be the set of vertices that are not matched to a node in $\mathcal{Y}$. Then, we can set to zero all the blocks of the variables corresponding to the vertices in A, i.e., we can remove them from our equations. Now we choose the rest of the channel matrices so that the determinant of the Jacobian with respect to the remaining variables is nonzero. To this end, we set $\bar{\mathbf{H}}_{kj}^{(3),p} = 0$ if the node for $\bar{\mathbf{U}}_{k_p}$ is not matched to the node in $\mathcal{Y}$ corresponding to the equation $\mathbf{F}_{kj}$. Similarly, we set $\bar{\mathbf{H}}_{kj}^{(2),q} = 0$ if $\bar{\mathbf{V}}_{j_q}$ is not matched to $\mathbf{F}_{kj}$. Moreover, we set all the remaining channel sub-matrices to the $d \times d$ identity matrix. Since this construction is based on a complete matching, it is not hard to see that the Jacobian for the whole system is a block permutation matrix, with nonzero blocks equal to the negative $d \times d$ identity matrix. Hence the determinant of the Jacobian matrix is equal to the product of the determinant of all nonzero blocks (up to sign), which is clearly nonzero in our case. This completes the description of the procedure to remove potential redundant variables, as well as the procedure to construct all the channel matrices $\{\mathbf{H}_{kj}^{(2)}, \mathbf{H}_{kj}^{(3)}, \mathbf{H}_{kj}^{(4)}\}_{k \neq j}$ and the beamforming solution $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$. The Jacobian matrix of (5.15) is nonsingular at this constructed channel realization and beamforming solution. Figure 5.1 illustrates the construction of graph G and a complete matching (in solid lines) for the aforementioned toy example.

To complete the proof, we fix a generic choice of $\{\bar{\mathbf{H}}_{kj}^{(2)}, \bar{\mathbf{H}}_{kj}^{(3)}, \bar{\mathbf{H}}_{kj}^{(4)}\}_{k \neq j}$ for which the Jacobian of (5.15) is nonsingular. Let n be the total number of remaining scalar variables in $\{\bar{\mathbf{U}}_k, \bar{\mathbf{V}}_j\}_{k,j}$ after removing the redundant variables. Notice that n is the same as the number of scalar equations, i.e., $n = d^2K(K-1)$. Let $R_1 = \mathbb{C}[h_1, h_2, \ldots, h_n]$ and $R_2 = \mathbb{C}[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_{n-m}]$ be two polynomial rings where $\bar{u}_i$'s and $\bar{v}_j$'s are the entries of the matrices $\{\bar{\mathbf{U}}_k\}_{k=1}^K$ and $\{\bar{\mathbf{V}}_k\}_{k=1}^K$ (after removing the redundant variables), and $h_1, h_2, \ldots, h_n$ are the entries of the matrices $\{\bar{\mathbf{H}}_{kj}^{(1)}\}_{k \neq j}$. Consider $\{f_i\}_{i=1}^n$ (the

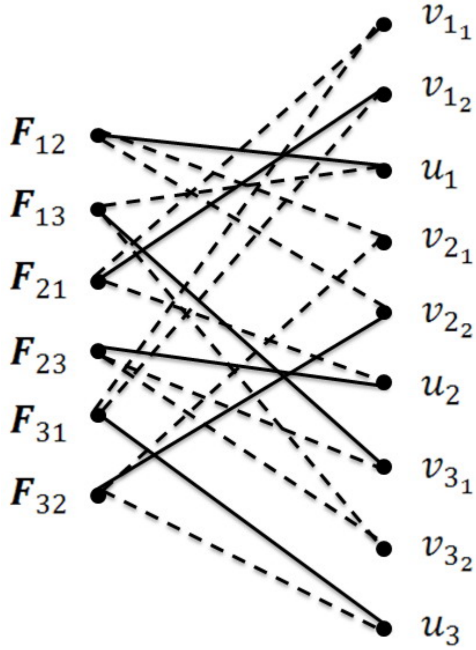

Figure 5.1: The bipartite graph G and a complete matching for the toy example

components of $\mathbf{F}_{kj}$'s in (5.15)) as the functions of $\bar{u}$'s and $\bar{v}$'s, i.e., f_i's are polynomials in R_2. These polynomials define a map a_ϕ which maps a point $c = (c_1, c_2, \dots, c_n)$ to $(f_1(c), f_2(c), \dots, f_n(c))$. According to the Chevalley Theorem (see [85, Chapter 2, 6.E.]), the image of this map is a Zariski constructible subset of $A^n_{\mathbb{C},1}$, where $A^n_{\mathbb{C},1}$ is the corresponding affine space of R_1. Since the Jacobian of the set $\{f_1, f_2, \dots, f_n\}$ with respect to the variables $\{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m, \bar{v}_1, \dots, \bar{v}_{n-m}\}$ is nonsingular generically for all channel realizations, it follows from the Implicit Function Theorem that the dimension of the image of a_ϕ is n. Note that the image of a_ϕ is a Zariski constructible subset of $A^n_{\mathbb{C},1}$ (see Chevalley Theorem [85], Ch. 2, 6.E.) and it has full dimension. Hence, the image contains a Zariski open subset of $A^n_{\mathbb{C},1}$ (see the discussion in section 5.1.1). Let $\mathcal{U}$ be that Zariski open subset of $A^n_{\mathbb{C},1}$ in the image. Since $\mathcal{U}$ is in the image of the map a_ϕ, there exists a solution for interference alignment equations for any choice of $\{\bar{\mathbf{H}}^{(1)}_{kj}\}_{k\neq j}$ in $\mathcal{U}$, which implies that interference alignment is feasible for generic choice of $\{\bar{\mathbf{H}}^{(1)}_{kj}\}_{k\neq j}$. Since the choice of channel matrices $\{\bar{\mathbf{H}}^{(2)}_{kj}, \bar{\mathbf{H}}^{(3)}_{kj}, \bar{\mathbf{H}}^{(4)}_{kj}\}_{k\neq j}$ is also generic, this completes the proof of the "if" direction.

Notice that the condition (5.14) is equivalent to the properness of the polynomial system (5.9) defining interference alignment. For symmetric systems with $M_k = M$, $N_k = N$ for all k, this condition simplifies to $M + N \geq d(K+1)$ (see [45, Theorem 1]). Thus, each user can achieve d degrees of freedom as long as $M + N \geq d(K+1)$ and that d divides both M and N. In a concurrent work, the authors of [46] obtained a similar result under a different set of assumptions. More specifically, they considered the symmetric case with $M_k = N_k = M,\ d_k = d$ for all k, and proved that the feasibility of interference alignment in this case is equivalent to $2M \geq d(K+1)$. This result and Theorem 4 are complementary to each other. In particular, Theorem 4 is applicable to non-symmetric systems, but does require an extra condition about the divisibility of the number of antennas by the number of data streams. When K is odd and $(K+1)d = 2M$, then M must be divisible by d. This case is then covered by both Theorem 4 and the result in [46]. However, for the case where K is even and $(K+1)d \leq 2M$, Theorem 4 is no longer applicable, whereas [46] shows that the interference alignment is achievable.

A few other remarks are in order.

1. Reference [45] also considered the case $d_k = 1$ and used the Bernshtein's theorem to numerically compute the number of solutions, and therefore prove the feasibility, for the resulting polynomial system (3.1)-(3.2) when the number of antennas are small. In contrast, Theorem 4 shows the feasibility of single beam interference alignment for all values of M_k, N_k as long as the system is proper.

2. As shown in Theorem 3, the condition (5.2) is necessary. For example, the system $K = 2, M = N = 3, d = 2$ satisfies the inequality (5.3). However, the system of equations (3.1)-(3.2) is infeasible for generic choice of channel coefficients. This further shows that the *properness* property in [45] does not imply feasibility in general, a fact that was first pointed out in [45, example 17].

3. Theorem 4 does not contradict the NP-hardness result of [83]. Given a set of channel matrices, checking the feasibility of the interference alignment conditions (3.1)-(3.2) when $M_k \geq 3$ and $N_k \geq 3$, is NP-hard. It is true that, under the setting of Theorem 4, the interference alignment fails only for a measure zero set of channels. However, for systems not satisfying the conditions of Theorem 4, checking the feasibility of interference alignment can be hard. Moreover, the

results of [83] imply that, even if a given tuple of DoF is known to be achievable via interference alignment, finding the actual linear transmit/receive beamformers to achieve it is still a NP-hard problem when the number of users is large.

4. The condition (5.14) implies the condition (5.2) if the number of antennas at each transceiver is divisible by d. In fact, by choosing $\mathcal{I} = \{(k, j)\}$, condition (5.14) implies that $d \leq M_k + N_j - 2d$ and hence the condition (5.2) is satisfied.

5. Theorem 4 assumes that both M_k and N_k are divisible by d. This condition can be weakened for a symmetric system where $M_k = M,\ N_k = N,\ d_k = d$, for all k. In particular, assume that only M (not N) is divisible by d and $M, N \geq d$. If the properness condition $(K+1)d \leq M + N$ holds, then we can construct a reduced MIMO interference channel with $N' = M - d(K+1)$ receive antennas for each user, where $M + N' = d(K+1)$ and M, N' are divisible by d. By Theorem 4, the interference alignment condition for the reduced interference channel is feasible and therefore, so is the interference alignment condition for the original channel since the latter has more antennas. This shows that if M is divisible by d and $M, N \geq d$, then the interference alignment system (3.1)–(3.2) is feasible for generic choice of channel coefficients if and only if $(K+1)d \leq M + N$. By symmetry, the same conclusion holds for the case where N is divisible by d.

5.2 Simulation Results

In this section, we use the theoretical DoF upper bounds to benchmark some existing algorithms for interference alignment and sum-rate maximization. We generate MIMO interference channels using the standard Rayleigh fading model. The numerical experiments are averaged over 10 Monte Carlo runs.

In the first numerical experiment, we consider a MIMO interference channel with 3 users. We further consider two different cases of $M = N = 2, d = 1$ and $M = N = 4, d = 2$. In both cases, the bounds in Theorem 3 and Theorem 4 are satisfied, suggesting the interference alignment is achievable. We plot the sum-rate results of the Distributed Interference Alignment (DIA) algorithm [2] (see also [29]). The "Predicted Slope" line has the slope corresponding to the DoF in the system and it starts from the same point as the DIA method. The closeness of the two curves suggests that the expected interference alignment has been achieved.

In the second numerical experiment, we consider a MIMO interference channel where

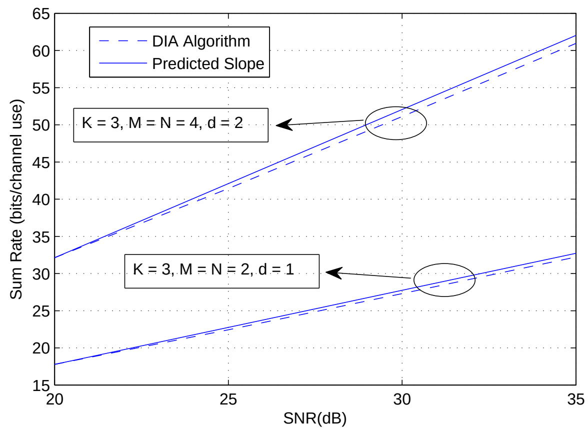

Figure 5.2: Sum-rate versus SNR(dB)

each transmitter/receiver is equipped with 3 antennas. For different number of users in the system, we maximize the sum-rate using the WMMSE algorithm [84] at increasingly high SNRs. We estimate the slope of the sum-rate versus SNR and use it to approximate the achievable total DoF. We then compare it with the value of theoretical upper bound given by the conditions in Theorem 3. The maximum gap of the two curves is one, but it is not clear if the gap is due to the weakness of the WMMSE algorithm or the upper bound.

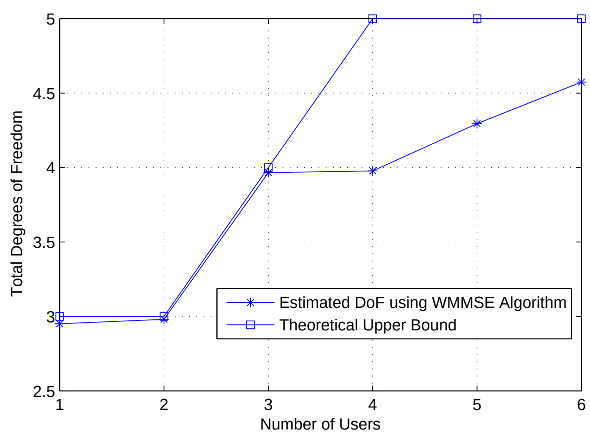

Figure 5.3: Achievable DoF and theoretical upper bound

Chapter 6

Remarks and Suggestions for Future Work

According to the recent work of Jafar [1], interference alignment (IA) is capacity achieving at high SNR when the interference channel exhibits sufficient diversity either in time or frequency (in the sense that the channel response is i.i.d. across sufficiently long time or sufficiently wide frequency band). Moreover, linear transmit and receive strategies suffice. This work strongly suggests that one should try to design optimal linear transceivers to achieve interference alignment.

The major findings of our investigation are as follows.

1. IA can offer a huge performance gain when compared to the traditional orthogonal transmit/receive strategies, especially when the number of interfering users are large. Unfortunately, realizing this potential gain in practice will be very challenging. This is because of the following main obstacles:

   - We have shown that computing optimal *spatial* interference alignment is computationally intractable for systems with large number of users.
   - The constructive results of [1] showed that the interference alignment is possible with exponentially many channel extensions or infinite diversity in the system. Obviously, in practice we need to align the interference in systems with significantly small channel extensions and limited diversity. These considerations make Jafar's construction impractical. In this thesis, we consider

no channel extension case and prove that unlike Jafar's work the total degrees of freedom will no longer grow linearly with the number of users in this case. We have also showed that our bound is tight for the single data stream/user case.

- Perfect CSI is required for the computation of an optimal IA scheme. However, acquiring perfect CSI is difficult when the number of interfering users in the system is large. For a K user system, there are $O(K^2)$ links to be estimated, which could require substantial communication resource. In fact, the overhead may be so high for large K that the resources available for actual data transmission could be significantly reduced.

Based on the above remarks, it is clear that, to make interference alignment practical, the following issues deserve further investigation. Answers to these questions, even if partial, may shed light on the effectiveness of IA in practice.

1. **Grouping of users:** Since IA for large number of users is impractical, it is important to develop methods for grouping users into groups of small size and perform IA within each group, while putting different groups over orthogonal channels. What is a good strategy to group users for a MIMO interference channel? Clearly, any grouping strategy must take into account the linear spaces spanned the channel matrices. Due to the use of MIMO beamforming strategy, the size or the magnitude of the channel matrices do not capture the compatibility of users in a given group. This suggests that simple location based grouping strategies are not likely to work.

2. **Interference alignment with limited channel extensions:** Channel extension for an interference channel corresponds to the number of different time slots for transmission or the number of subcarriers in the system. Although Jafar showed how to achieve perfect alignment with infinitely many channel extension, the maximum DoF that one can achieve with limited channel extensions (say, polynomial in the number of users) is still unknown. In this work, we only consider the case of no channel extension.

3. **Interference alignment in the systems with finite diversity:** The diversity

in the system shows how different subcarriers in the system are related to each other. The surprising result of [1] corresponds to alignment for systems with infinite diversity, i.e., channel in different subcarriers are completely independent. However, in real systems the channel gain for adjacent subcarriers can be correlated. Therefore, we need to study interference alignment for systems with limited diversity. Is it possible to achieve a high DoF when the channel response across different subcarriers are correlated? How does correlation impact the maximum achievable DoF?

4. **Interference alignment in interfering broadcast (multicast) channel:** Currently, most of the results on interference alignment is for interference channel. Although many real systems can be decomposed into interference channel model, we may be able to gain more in practice by considering more practical channel models like interfering broadcast channels or interfering multicast channel. In these models, one transmitter transmits to different receivers at the same time and the transmitted signal from other transmitters cause interference to some of the receivers. The study of these models, which are more practical, should be considered in the next step. Can we design, either analytically or computationally, IA schemes for these channels, especially when the transmitters and receivers are equipped with multiple antennas? A key practical requirement for such IA constructions is that the symbol/channel extensions must be limited to polynomial in the number of users.

5. **Interference alignment and KKT conditions of sum MSE (rate) minimization (maximization):** Intuitively, there should be a relationship between these two problems. This relation can be studied and based on the properties of KKT points. Understanding of this relationship may enable us to suggest more efficient practical algorithms for (partial) interference alignment.

6. **Robustness and low channel feedback:** The number of required bits for feedback in interference channel is studied in [89]. This study propose a beamforming and feedback scheme for interference alignment using imperfect CSI. In fact, it has been shown that if the number of feedback bits increases linearly with the number of users, then Jafar's interference alignment scheme still works in the channel.

However, this work is also impractical due to infinite number of channel extensions and assuming infinite diversity in the system. It would be interesting to study the possibility of interference alignment with limited feedback, channel extension, and channel diversity. Moreover, it is absolutely essential in practice that any devised linear transceiver strategies be robust against channel variations. We need to investigate robust design of linear transceivers to approximately achieve interference alignment.

Chapter 7

Conclusion

This thesis aims to study the interference alignment equations for linear transceiver design in wireless networks.

The recent work [1] shows that the optimal strategy, that can maximize the total degrees of freedom (DoF), is interference alignment approach. This result is exciting, but not practical because it requires exponentially many channel extensions and infinite channel diversity.

What is the achievable DoF under practical assumptions such as limited channel extension and limited diversity in the system? This thesis tries to answer this question to some extent.

First we analyze the computational complexity of solving the interference alignment equations with no channel extension. More precisely, we prove that maximizing the total degrees of freedom in the system is NP-hard. Moreover, it is shown that even checking the feasibility of interference alignment is NP-hard when there are at least three antennas at each node. The same problem is proved to be polynomial time solvable when we have at most two antennas at each node.

In the second part of this thesis, we again consider the interference alignment problem with no channel extension and showed that unlike [1], the total DoF in the system cannot grow linearly with the number of users. In particular, it is proved that the total DoF is bounded by $M + N$ for a symmetric system, where M is the number of antennas at the transmitter and N is the number of receive antennas. Furthermore, we demonstrate the tightness of this upper bound for a special case.

As a result, although interference alignment initially seems to be a promising and exciting strategy in dealing with interference, the gain is not that large when more practical considerations are come into picture. Furthermore, it is computationally difficult to achieve interference alignment in practice. These results show some major drawbacks of interference alignment which prevent it from being practical.

Another interesting question related to this thesis is about the total degrees of freedom when there are limited and more than one channel extensions/channel diversity. This is an important open problem which needs to be studied in future.

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