ON THE COMMUTATIVITY OF THE BOUNDARY AND INTERIOR OPERATORS IN A TOPOLOGICAL SPACE.—N. Levine [2] discovered that, in a topological space, the interior and closure operators will commute if and only if the set on which they operate is the symmetric difference of a set that is both open and closed and a set that is nowhere dense. The intent of this paper is to characterize those sets for which the interior and boundary operators will commute.

The following notation will be used:

cA—closure of A
CA—complement of A
Int A—interior of A
BA—boundary of A
$A \nabla B$—symmetric difference of A and B

Lemma 1: $\{X,T\}$ is a topological space and A and E subsets of X, then $\text{Int}(A \cap E) = \text{Int}A \cap \text{Int}E$.

Lemma 2: If $\{X,T\}$ is a topological space, A and E are subsets of X, and A is open and E is dense, then $c(A \cap E) = cA \cap cE = cA$. This is an exercise on p. 57 in Kelley's book (1955).

Lemma 3: If $\{X,T\}$ is a topological space and A is a subset of X, then $\text{IntBA} \cap \text{BIntA} = \emptyset$.

Proof: $\text{IntBA} \cap \text{BIntA} = \text{Int}(cA \cap cCA) \cap c\text{IntA} \cap cC\text{IntA} = \text{IntcA} \cap \text{IntcCA} \cap c\text{IntA} \cap cC\text{IntA} = \text{IntcA} \cap Cc\text{IntA} \cap c\text{IntA} \cap cC\text{IntA} = \emptyset$.

Lemma 4: If $\{X,T\}$ is a topological space and A is a subset of X, then $\text{IntBA} = \text{BIntA}$ if and only if $\text{IntcA} = c\text{IntA} = \text{IntA}$.

Proof: Necessity. Suppose IntBA = BIntA. Then these sets must both be empty in order to be equal since by lemma 2 they have nothing in common. Thus

(1) $\text{IntBA} = \text{IntcA} \cap Cc\text{IntA} = \emptyset$ and

(2) $\text{BIntA} = c\text{IntA} \cap cCA = \emptyset$

Equations 1 and 2 imply $\text{IntcA} \subset c\text{IntA}$ and $c\text{IntA} \subset CcCA = \text{IntA}$ respectively. Since $\text{IntA} \subset \text{IntcA}$, it follows that $\text{IntA} = \text{IntcA} = c\text{IntA}$.

Sufficiency. Suppose $\text{IntcA} = c\text{IntA} = \text{IntA}$. This implies that $\text{BIntA} = \emptyset$ and $\text{IntBA} = \emptyset$, and thus the two sets are equal.

Theorem: If $\{X,T\}$ is a topological space and A is a subset of X, then IntBA = BIntA if and only if $A = E \cup P$, where E is open and closed, P is nowhere dense, and $E \cap P = \emptyset$.

Proof: Using lemma 4, the proof reduces to showing that $\text{IntcA} = c\text{IntA} = \text{IntA}$ if and only if $A = E \cup P$, where E is open and closed, P is nowhere dense, and $E \cap P = \emptyset$.

Necessity. Suppose $\text{IntcA} = c\text{IntA} = \text{IntA}$. By Levine's theorem (Levine, 1961), it follows that if $\text{IntcA} = c\text{IntA}$, then $A = E \nabla P$ where E is open and closed and P is nowhere dense. Thus it is left to establish what further conditions the second equality places on E and P. In Levine's proof, $E = c\text{IntA}$. Thus $E = \text{IntA}$ (i.e. $E = \text{Int}(E \nabla P)$). $\text{Int}(E \nabla P) = CcC[\ (E \cap CP) \cup (CE \cap P)\ ] = Cc[\ (CE \cap CP) \cup (P \cap E)\ ] = C[c(CE \cap CP) \cup c(P \cap E)\ ]$. By lemma 2 and the fact that CP is dense it follows that $C[c(CE \cap CP) \cup c(P \cap E)\ ] = CcCE \cap Cc(P \cap E) = E \cap Cc(P \cap E)$. Therefore, $E = E \cap Cc(P \cap E)$ which implies $P \cap E = \emptyset$.

Sufficiency. Suppose $A = E \cup P$, E is open and closed, P is nowhere dense, and $E \cap P = \emptyset$. By Levine's theorem, $c\text{IntA} = \text{IntcA}$. $\text{IntA} = \text{Int}(E \cup P) = Cc(CE \cap CP) = CcCE = E$. Thus IntA is closed and it follows that $C\text{IntA} = \text{IntcA} = \text{IntA}$.—

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