DOUBLE INTEGRAL VARIATION PROBLEMS WITH PRESCRIBED TRANSVERSALITY COEFFICIENTS*

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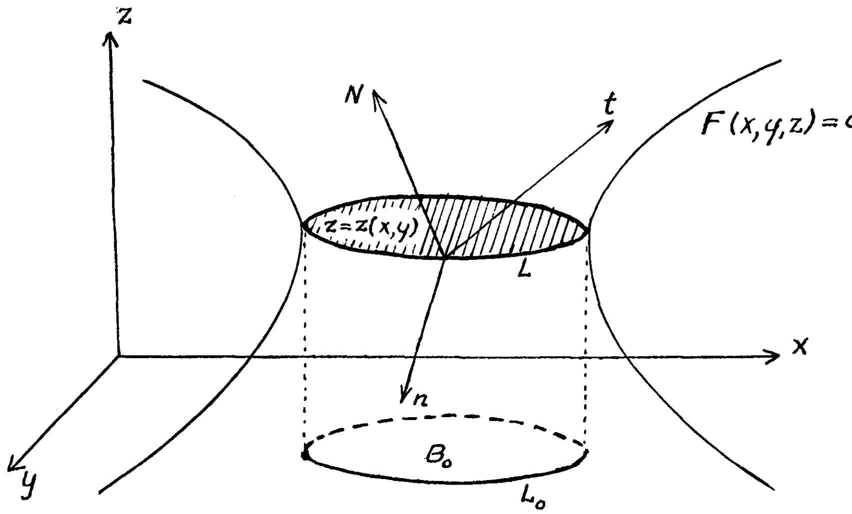
H. A. Simmons has recently published an interesting derivation of the transversality relationship for the variable limit problem of the calculus of variations for n-tuple integrals.[1] It is the purpose of this note to formulate and solve an inverse problem suggested by this transversality relationship. For the sake of perspicuity, attention is confined to the double integral case; but the argument made and the conclusions drawn are easily extended to the general multiple integral considered by Simmons.



Properties of the transversality coefficients of a regular double integral problem. Consider a double integral

$$(1)\qquad J=\iint_{B_0} f(x,y,z,p,q)\,dx\,dy,\quad z=z(x,y),\quad p=z_x,\quad q=z_y,$$

where f is of class C'' in a region R of sets of (x, y, z, p, q)—values and where the double integration is effected over the closed region B_0 (Fig. 1) in the xy-plane bounded by the xy-projection, L_0, of the curve of intersection, L, of the surface $z=z(x, y)$ with a prescribed surface $F(x, y, z)=O$, satisfying certain well known conditions[2] but otherwise arbitrarily chosen. A variable limit problem of the calculus of variations is then to find among surfaces $z=z(x, y)$ of class C'' lying in

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[1] H. A. Simmons, *Transactions, American Mathematical Society*, Vol. 36 (1934), pp. 29–43.

[2] In this connection, compare Bolza, *Vorlesungen über Variationsrechnung*, p. 653. Surfaces F = O satisfying the conditions enumerated by Bolza will be referred to as admissible.

the interior of the xyz-projection of R and bounded by their curves of intersection with the surface $F(x, y, z)=0$, one minimizing the integral J.

If $E: z=z(x, y)$ is a surface which furnishes a solution of the problem just formulated, then along L, the curve of intersection of E and $F=0$, a so-called transversality condition of the form

$$f_pF_x+f_qF_y-(f-pf_p-qf_q)F_z=0 \tag{2}$$

must necessarily be satisfied.[3] In particular, if the problem (1) is *regular* in the sense of Hilbert, i. e., if

$$d=\begin{vmatrix} f_{pp} & f_{pq} \\ f_{qp} & f_{qq} \end{vmatrix} \tag{3}$$

is different from zero in R, then the quantity $D=f-pf_p-qf_q$ does not vanish identically; for, if it did, the relation $D\equiv0$ would lead under differentiation with respect to p and q to the identities $pf_{pp}+qf_{pq}=0$, $pf_{qp}+qf_{qq}=0$ and, as $(p, q)\not\equiv(0, 0)$ the determinant d would, therefore, necessarily vanish in contradiction to our assumption that the problem is regular.

In what follows, we shall assume that the problem (1) is regular and restrict attention to that portion of the fundamental region R in which both f and D are different from zero. In this subregion of R the ratios

$$P'(x, y, z, p, q)=f_p/D; \quad Q'(x, y, z, p, q)=f_q/D, \tag{4}$$

are well defined and of class C'. These ratios will be referred to as the *transversality coefficients* of the variation problem (1). For a regular problem we therefore have defined in every point of L not only the normal $N: (p:q:-1)$ to the minimizing surface and the normal $n: (F_x:F_y:F_z)$ to the transversal surface $F=0$ but also a third direction $t: (P':Q':-1)$ given in terms of the transversality coefficients of the problem and, in view of (2), always orthogonal to n. The existence of such a direction t of course implies a relationship between the directions n and N; for example, if $f=(1+p^2+q^2)^{1/2}$ then it is found that t is N and hence $t\perp n$ implies $N\perp n$, i. e., for the problem of minimizing the area integral, the surfaces E and $F=0$ must necessarily intersect under a right angle.

By inspection, $1+pP'+qQ'=f/D$. Hence, in the subregion of R under consideration, the transversality coefficients necessarily satisfy the inequality

$$1+pP'+qQ'\neq0. \tag{5}$$

Geometrically, this inequation means that the normal N to the extremal surface through the point (x, y, z) does not lie in the plane with normal t passing through this point. Furthermore, it is easy to verify that

$$|\partial(P', Q')/\partial(p, q)|=fd/D^3. \tag{6}$$

Consequently, in the specified subregion of R, the Jacobian of the transversality coefficients with respect to p and q is necessarily different from zero.

[3]See, for example, H. A. Simmons, *Transactions, American Mathematical Society*, Vol. 28 (1926), pp. 235–251.

Variation Problems with prescribed transversality coefficients.

Suppose now that a pair of functions $P(x, y, z, p, q)$, $Q(x, y, z, p, q)$ of class C' and satisfying the inequalities $1+pP+qQ\neq 0$, $|\partial(P, Q)/\partial(p, q)|\neq 0$ in a region S of (x, y, z, p, q)—values is given. It is natural to inquire if there always exists a regular problem in S of which the given functions P, Q are the transversality coefficients. To answer this question we return to the transversality relation (2) and observe that for a definite but arbitrarily chosen admissible surface $F=0$, this condition established a certain relationship between the elements (x, y, z, p, q) and (x, y, z, F_x, F_y, F_z), that is to say, between the surface element (x, y, z, p, q) and the plane of directions $dx : dy : dz$ with normal n. Hence, if the given functions P, Q are to be the transversality coefficients of a problem (1) the equations

(7) $$PF_x+QF_y-F_z=0$$

and (2) must establish the same relation between surface elements (x, y, z, p, q) and planes with normal n for every admissible choice of $F=0$. This can only be true if P and Q are such that every admissible choice of $F=0$ for which the left member of (2) vanishes, also makes the left member of (7) vanish and conversely. But then corresponding coefficients in these two relations must be proportional. Consequently, a pair of functions, P, Q as described can be the transversality coefficients of a regular problem (1) if, and only if, there exists a function $f(x, y, z, p, q)$ satisfying the relations

(8) $$f-pf_p-qf_q=h, \quad f_p=hP, \quad f_q=hQ,$$

where $h\neq 0$ is a function of x, y, z, p, q. Elimination of h in (8) leads to the following system of non-homogeneous partial differential equations which must be satisfied by f regarded as a function of the independent variables p, q and the parameters x, y, z,

(9) $$\begin{aligned} Pf-(1+pP)f_p-qPf_q &=0, \\ Qf-pQf_p-(1+qQ)f_q &=0. \end{aligned}$$

To ascertain the integrability conditions for the system (9), we transform it into a system of equations linear and homogeneous in the first partial derivatives of a function $g(x, y, z, p, q, f)$ with $g_f\neq 0$, so that when g is known, f is determined by means of the equation $g=$constant. The resulting system is

(10) $$\begin{aligned} U_1g &= (1+pP)g_p+qPg_q+fPg_f=0, \\ U_2g &= pQg_p+(1+qQ)g_q+fQg_f=0. \end{aligned}$$

The equations (10) are obviously independent since the matrix of coefficients contains a second order determinant with the value $1+pP+qQ$ and, therefore, by hypothesis different from zero. Hence if the commutator $(U_1, U_2)g$ of this pair of equations is not a linear combination of U_1g and U_2g, the only solution for g is a constant and, therefore, a function f as required in (8) does not exist. It is easily shown that a necessary and sufficient condition for $(U_1, U_2)g$ to be a linear combination of the left members of the original equations is that P and Q satisfy the relation $C=0$, where

(11) $$C=p(PQ_p-QP_p)+q(PQ_q-QP_q)+(Q_p-P_q).$$

The results so far obtained may be summarized in the following theorem:

THEOREM 1. *Necessary and sufficient conditions for a pair of functions $P(x, y, z, p, q)$, $Q(x, y, z, p, q)$ to be the transversality coefficients of a regular problem* (1) *in a region S of (x, y, z, p, q)—values are that in S the functions P, Q be of class C' and satisfy the inequalities* $1+pP+qQ\neq 0$, $|\partial(P, Q)/\partial(p, q)|\neq 0$ and the relation $C=0$.

We shall refer to a pair of functions P, Q satisfying the conditions enumerated in Theorem 1 as an admissible pair of functions. Suppose now that an admissible pair of functions is prescribed. Our next problem is to determine the most general integrand function $f(x, y, z, p, q)$ of a regular problem (1) having this pair as its transversality coefficients. To solve this problem we introduce the line integral

(12) $$K=\int_{0,0}^{p,q}(P\,dp+Q\,dq)/(1+pP+qQ).$$

A necessary condition for this line integral to be independent of the path of integration is that $c=0$, where

(13) $$c=(\partial/\partial q)[P/(1+pP+qQ)]-(\partial/\partial p)[Q/(1+pP+qQ)].$$

This condition is also a sufficient condition for independence of path in a space of suitably simple connectivity properties. Since it is easy to show that $c=C$, it follows that when $C=0$, as it does for the pair P, Q under consideration, the line integral K is independent of the path in such a space.

A particular solution of the homogeneous system (10) is given by the function

(14) $$g(x, y, z, p, q, f)=f.e^{-K(x, y, z, p, q)},$$

and, therefore, since the general integral of a complete system of two equations in three independent variables is an arbitrary function of a single particular integral of the system, the most general solution of (10) is given by

(15) $$g=A(f.e^{-K}, x, y, z)$$

where A is the symbol of an arbitrary function. The most general non-singular solution f of (9) is now obtained by solving for f the relation $A=0$. Whence

THEOREM 2. *If a pair of functions* P(x, y, x, p, q), Q(x, y, z, p, q) *satisfy the conditions of Theorem 1, the most general regular problem* (1) *for which the functions* P, Q *are the transversality coefficients has an integrand function*

(16) $$f(x, y, z, p, q)=a(x, y, z).e^{K}$$

where a *is different from zero and of class* C'' *but is otherwise an arbitrary function of its arguments and where* K *has the value* (12).