# THE RELATIVE INTENSITY OF HARMONICS OF A LECHER SYSTEM. (THEORETICAL).

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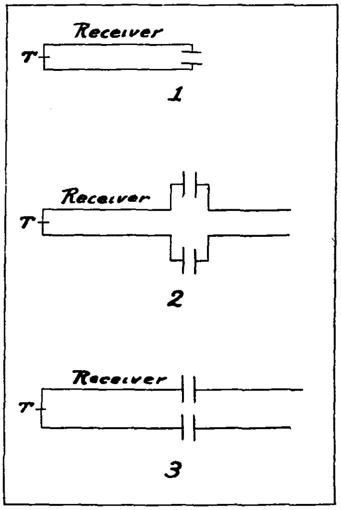
In the previous paper Blake and Jackson have shown the dependence of tone-intensity upon the edge-on distance between plates. They have found that the optimum condition for each tone is such that, for a given in-put of energy at the oscillator, the energy is distributed among the various tones as follows, the fundamental being taken as 100%: third, 61.63%, fifth 21.92%, seventh 6.57%, ninth 2.21%, eleventh 1.05%. In this paper a theory is given to account for these results.




Figures 1, 2 and 3

The theory makes use of the Kirchhoff-Abraham generalization of Kelvin's formula for the discharge of a condenser. As given in Abraham's "Theorie der Elektrizität," Vol. II, 2nd edition, we consider electric waves coming from the negative X direction (Figure 1) along two parallel wires that end in a condenser. Our experimental arrangement was necessarily different from that, but if we fix our attention upon the receiver alone and remember that the optimum oscillator-length is different for each tone it is obvious that, so far as the total energy on the receiver is concerned, our arrangement (Figure 2) is identical with Figure 1.

Later in this paper we shall discuss the conditions under which the practical case, Figure 2, may be said to conform to the ideal case, Figure 3. For the moment we shall call them identical.

Abraham's expression for the potential difference, $\phi$ per unit length of the wires (Figure 1) is

$$\phi = \frac{2A}{\kappa} \cos\left(\frac{2\pi x}{\lambda} - \gamma\right) \cos 2\pi\, nt$$

where A is a constant, $\kappa$ the capacity per unit length, $\lambda$ the wave-length, $\gamma$ the phase change due to the end-capacity and $n$ the frequency. We consider the total energy on the receiver that surges through the thermo-couple as made up of two parts, that on the wires themselves and that on the condenser. Suppose at a certain moment the condenser plates are charged to a maximum value. Due to the distributed capacity of the wires there is also at that moment a charge on them. A moment later these charges discharge through the thermocouple thus recharging the plates and wires with electricity of the opposite sign. During the half period of the galvanometer, viz., 1.4 sec., millions of vibrations surge through the thermocouple. The galvanometer needle moves off until the loss of energy by heat conduction and radiation from the thermocouple equals the input of energy. The scale usually moves off in a very vigorous fashion showing that the losses of energy are not appreciable till near the end of the half period of the galvanometer. We shall assume the rate of loss of heat energy by radiation from the junction to be independent of the frequency of the tone surging through it, and that Newton's law of cooling holds with respect to the surroundings.

In order to calculate the energy that surges through the thermocouple we must get the root-mean-square value of the potential as it is distributed both as to space and time. If we substitute $\frac{x}{n\lambda}$ for $t$ in the above expression we can say that the total energy on the wires is given by the expression

$$\frac{1}{2}CV^2 = \frac{1}{2}\kappa\left(\frac{s\lambda}{4} - \frac{\gamma\lambda}{2\pi}\right) \int_{-\frac{s\lambda}{4}+\frac{\gamma\lambda}{2\pi}}^{0} \left[\frac{2A}{\kappa}\cos\left(\frac{2\pi x}{\lambda} - \gamma\right)\cos\frac{2\pi x}{\lambda}\right]^2 dx$$

The distance from the thermocouple to the back of the receiver plates is $\frac{s\lambda}{4}-\frac{\gamma\lambda}{2\pi}$.

Hence the total capacity of the wires is $\kappa\left(\frac{s\lambda}{4}-\frac{\gamma\lambda}{2\pi}\right)$

The energy on the plates is represented by

$$\frac{1}{2}CV^2=\frac{1}{2}\kappa_0\left[\frac{2A}{\kappa}\ \frac{\gamma\lambda}{2\pi}\cos\gamma\right]^2\frac{s}{2}.$$

The condenser is at the point $x=o$. In the above expression for $\phi$ put $x=o$ and we have $\phi=\frac{2A}{\kappa}\cos\gamma$ per unit length. The equivalent wire-length of the condenser is $\frac{\gamma\lambda}{2\pi}$.

But the plates are not charged all the while, hence $V^2$ in the expression for the energy on the plates must be taken

$$V^2=\frac{4A^2}{\kappa^2}\frac{\gamma^2\lambda^2}{4\pi^2}\cos^2\gamma\cdot\frac{1}{\tau}\int\cos^2\nu\, tdt.$$

$$\text{Now } \frac{1}{\tau}\int_0^\tau\cos^2\nu\, tdt=\frac{1}{\nu\tau}\int_0^{2\pi}\cos^2 x dx=\frac{\pi}{\nu\tau}.$$

But the plates are charged $n$ times a second, and since $\nu=2\pi n$ and $n\tau=1$, multiplication by $n$ gives $\frac{n}{2}=\frac{n_0 s}{2}$ where $n_0=$ frequency of the fundamental tone and s the frequency number.

Integrating the above expression for the energy on the wires we get as the expression for the total energy E,

$$E=\frac{n_0A^2\lambda^2}{4\pi^2\kappa}\left[s\gamma^2\cos^2\gamma\frac{\kappa_0}{\kappa}+\frac{1}{4}\left(\frac{s\pi}{2}-\gamma\right)\left\{\left(\frac{s\pi}{2}-\gamma\right)\left(2\cos^2\gamma+1\right)\right.\right.$$
$$\left.\left.+3\sin\gamma\cos^3\gamma\,(1+\sin^2\gamma)-5\sin^5\gamma\cos\gamma+4\sin\gamma\cos\gamma\right\}\right].$$

Now the constant A depends upon the manner of setting the receiver into vibration. Our case is similar to the acoustical case treated by Lord Rayleigh.* The disturbing force which varies as $\cos\frac{2\pi x}{\lambda}$ is not applied at a single point, but is distributed over the distance $\frac{2\gamma\lambda}{2\pi}$. The disturbing force over one-half of this distance concerns itself with the Lecher system.

* Theory of Sound. Vol. I, p. 189.

That over the other half has to do with the receiver. Accordingly we can take

$$A=\frac{A_0}{2}\int_0^{\frac{2\gamma\lambda}{2\pi}}\cos\frac{2\pi x}{\lambda}\,dx=\frac{A_0\lambda}{2\pi}\sin\gamma\cos\gamma \text{ and } A^2=\frac{A_0^2\,\lambda^2\sin^2\lambda\cos^2\lambda}{4\pi^2}.$$

Call $\frac{A_0^2\,n_0}{16\pi^4\,\kappa}=C.$ Then our expression for E becomes

$$E=C\,\lambda^4\sin^2\gamma\cos^2\gamma\left[s\gamma^2\cos^2\gamma\frac{\kappa_0}{\kappa}+\frac{1}{4}\left(\frac{s\pi}{2}-\gamma\right)\left\{\left(\frac{s\pi}{2}-\gamma\right)\left(2\cos^2\gamma+1\right)\right.\right.$$
$$\left.\left.+3\sin\gamma\cos^3\gamma\,(1+\sin^2\gamma)-5\sin^5\gamma\cos\gamma+4\sin\gamma\cos\gamma.\right\}\right].$$

Now Abraham's theory for the transmission of electric waves along a pair of parallel wires does not take into account the necessary bending of the wires leading up to the plates. For obvious reasons the ideal case shown in Figure 3 cannot be realized unless one has a rather large distance between the Lecher wires, which means a long bridge and larger phase-changes due to the bridge length. Blake and Sheard have shown that there are two factors, which they called $\theta\ (y)$ and $\phi(y)$ controlling the relation between the edge-on distance between the plates and the tone intensity. The former represents the electrostatic leakage to plates of the same circuit, the latter the phase-changes due to that portion of the wires at right angles to their main length. They have shown that these two factors act in opposite directions and the preceding paper confirms this. Abraham's theory does not consider either of these factors. We have made no attempt to determine the nature of either of these functions of $y$, (though the experimental data of Blake and Jackson are probably sufficient to determine both). They have determined, however, the optimum value of $y$ for each frequency, that is, the value of $y$ at which the tone intensity is a maximum. Obviously, this optimum value of $y$ is that value at which these two factors nullify each other.

If then we compare the tone-intensities for the various optimum values of $y$ they should agree with the simple theory of Abraham that neglects the two factors mentioned above. In making the theoretical calculations, however, some circumspection must be used. For instance, theory and experiment should not be expected to agree except at or near those values of $y$ where $\kappa_0/\kappa$ is perceptibly constant, at least so long as the forms of $\theta(y)$ and $\phi(y)$ remain unknown.

Now in the work of Blake and Jackson the per cents of error were least for $y=8$, as column 25 of their Table I shows as also does their Figure 7. We have carried through the calculation of $E$ for the following values of $y$: 7.15, 8.0, 9.0, 11.1 cm., and have collected the results into Table I.

TABLE I.

| $y$ Cm. | $\lambda_s$ Cm. | $\kappa_0/\kappa$ Cm. | $\gamma$ Degrees | $\gamma$ Min. | $E\times10^{-6}$ Arbitrary Units (C=1) | E Percent. |
|---|---|---|---|---|---|---|
| **7.15** | 442.90 | 14.60 | 11 | 42.1 | 3663.2 | 100.00 |
| | 145.81 | | 32 | 10.6 | 2143.6 | 58.52 |
| | 86.01 | | 46 | 50.8 | 692.4 | 18.90 |
| | 60.47 | | 56 | 36.5 | 209.0 | 5.70 |
| | 46.43 | | 63 | 9.3 | 71.47 | 1.95 |
| | 37.58 | | 67 | 43.3 | 28.77 | 0.78 |
| **7.15** | 442.90 | 14.60 | | | | 100.00 |
| | 145.84 | 14.64 | | | | 58.56 |
| | 86.00 | 14.58 | | | | 18.89 |
| | 60.44 | 14.42 | | | | 5.76 |
| | 46.34 | 13.71 | | | | 2.12 |
| | 37.40 | 11.61 | | | | 1.13 |
| **8.0** | 440.07 | 13.70 | 11 | 4.1 | 3107.3 | 100.00 |
| | 145.14 | | 30 | 40.3 | 1921.0 | 61.82 |
| | 85.78 | | 45 | 6.0 | 683.2 | 21.99 |
| | 60.41 | | 54 | 56.3 | 220.6 | 7.10 |
| | 46.41 | | 61 | 40.1 | 76.64 | 2.47 |
| | 37.60 | | 66 | 24.3 | 32.33 | 1.04 |
| **9.0** | 441.00 | 13.17 | 10 | 37.8 | 2838.8 | 100.00 |
| | 145.62 | | 29 | 36.5 | 1823.5 | 64.24 |
| | 86.19 | | 43 | 50.0 | 691.5 | 24.36 |
| | 60.74 | | 53 | 43.3 | 234.5 | 8.26 |
| | 46.71 | | 60 | 33.4 | 86.20 | 3.04 |
| | 37.86 | | 65 | 24.9 | 36.07 | 1.27 |
| **9.0** | 440.50 | 13.06 | | | | 100.00 |
| | 145.60 | 13.15 | | | | 65.56 |
| | 86.36 | 13.58 | | | | 25.32 |
| | 60.88 | 13.84 | | | | 8.23 |
| | 46.82 | 14.21 | | | | 2.24 |
| | 37.94 | 14.48 | | | | 1.12 |
| **11.1** | 440.28 | 11.92 | 9 | 39.2 | 2235.6 | 100.00 |
| | 145.68 | | 27 | 12.5 | 1551.5 | 69.41 |
| | 86.45 | | 40 | 54.3 | 677.8 | 30.32 |
| | 61.07 | | 50 | 48.4 | 257.5 | 11.52 |
| | 47.01 | | 57 | 53.1 | 98.38 | 4.40 |
| **11.1** | 439.28 | 11.63 | | | | 100.00 |
| | 145.86 | 12.08 | | | | 74.60 |
| | 87.00 | 13.12 | | | | 34.09 |
| | 61.62 | 14.54 | | | | 10.97 |
| | 47.40 | 15.33 | | | | 3.64 |

The agreement between theory and experiment is remarkably good for $y = 8.0$ cm., or for the mean values for $y = 7.15$, 8.0 and 9.0 cm.*, as shown in Table II. Expressed in per cent the largest error is nearly eleven per cent, but when, as here, relative intensities are compared, it seems far more reasonable to express the error in terms of the fundamental intensity. When this is done the maximum error is less than one-half per cent. In the experimental work, although we tried to read to fractions of a division for small galvanometer throws, nevertheless such things as the wandering of the zero during a reading, slight unsteadiness of the zero particularly on windy days, together with slight errors of experiment, mentioned above but uneliminated, served to make the experimental error as high as five per cent, at least for the tones beyond the fifth.

TABLE II.

| $y$ cm. | ENERGY IN PER CENT | | | | | |
|---|---|---|---|---|---|---|
| | S=1 | 3 | 5 | 7 | 9 | 11 |
| 7.15 | 100 | 58.52 | 18.90 | 5.70 | 1.95 | 0.78 |
| 8.0 | 100 | 61.82 | 21.99 | 7.10 | 2.47 | 1.04 |
| 9.0 | 100 | 64.24 | 24.36 | 8.26 | 3.04 | 1.27 |
| Mean | 100 | 61.53 | 21.75 | 7.02 | 2.49 | 1.03 |
| Experimental | 100 | 61.63 | 21.92 | 6.57 | 2.21 | 1.05 |

Expressed in per cent of the fundamental intensity the error is thus small and we can accordingly say that a satisfactory theory has been worked out to explain the observed curves of Blake and Jackson's Figure 6.

* The value of $\kappa_0/\kappa$ used in Table II is not exactly the value given in Table I of Blake and Jackson's paper. The calculation was made taking $\kappa_0/\kappa = 13.17$ cm., instead of 13.03 cm. This can affect our results but very little, for a change of 1 per cent in $\kappa_0/\kappa$ with its consequent change in $\lambda$ affects the relative intensity of the various tones from one to three per cent at most. The variation is not always in the same direction for the different tones, however.

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