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ON THE RELIABILITY OF MULTISTATE SYSTEMS WITH IMPRECISE PROBABILITIES

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Розглядається обчислення надійності в складних системах за наявності випадкового набору оцінок працездатності елементів. Виявлено, що підхід Демпстер-Шефера є відповідним математичним інструментом, який відповідає поставленим задачам. Для випадку, коли взаємозалежності елементів невідомі, наведено також оцінки ефективності системи переконань і правдоподібність функції.
Ключові слова: складні системи, надійність, структурна функція, підхід Демпстер-Шефера

Рассматривается вычисления надежности в сложных системах при наличии случайного набора оценок работоспособности элементов. Выявлено, что подход Демпстер-Шефера является соответствующим математическим инструментом, который соответствует поставленным задачам. Для случая, когда взаимозависимости элементов неизвестны, приведены также оценки эффективности системы убеждений и правдоподобность функции.
Ключевые слова: сложные системы, надежность, структурная функция, подход Демпстер-Шефера

We consider the computation of multistate systems reliabilities in the presence of random set estimations for the elements' working abilities. It turns out that the Dempster-Shafer approach is a suitable mathematical tool. For the case that the interdependence of the elements is unknown, bounds for the system's performance belief and plausibility functions are given as well.
Keywords: multistate systems, reliability, structure function, Dempster-Shafer approach

1 INTRODUCTION

Consider a system **Σ** with *n* components $E_1,...,E_n$ (e.g., parallel, serial, etc...). The performance of each component is described by $x_i \in L_i$ for $i = 1,...,n$ with L_i being a complete lattice. Moreover, let $\boldsymbol{x} = (x_1,...,x_n)$. These are the basics for a rather general mathematical model of multistate systems where performance often means "working ability".

In applications, the L_i are usually finite sets (e.g. nonnegative integers) or real numbers from [0,1]. The system's performance is computed via the structure function $\Phi(\boldsymbol{x})$ (see. Def. 2.1). Concerning the elements performance, it is assumed that $p(x_i)$, i.e. the probability (density) for x_i taking values from L_i is known. (Thus the performance of E_i can be interpreted as a random variable on the states of E_i with range L_i.) This, however, may be unrealistic, because the available information for E_i often concerns regions of performances rather than single values.

Take for example L_i = [0,1]. Then the performance of E_i might be characterised by the statement "the probability of high performance is medium", "mean performance is likely", "low performance is not very probable". These linguistic statements are vague and one could try to grasp notions like "high", "medium", etc. by fuzzy sets on L_i (for the performance) and on [0,1] (for the probabilities). For the sake of lucidity we will, however, assume the performance regions to be crisp subsets of L_i and the probabilities to be crisp numbers. Thus we are led to classical the Dempster-Shafer Theory (DST).

Another problem concerns the correlation of the elements with respect to their performance. The assumption often made is that the elements behave independently, what is not always the case. Here, estimations for dependent elements are necessary.

2 MATHEMATICAL PREREQUISITES

Suppose to be given a system **Σ** with the above properties. Then the Cartesian product $P = L_1 \times ... \times L_n$ is a complete lattice as well, and we obviously have $\boldsymbol{x} \in P$. Further, let L be another complete lattice. We suppose all lattices to be bounded, i.e. for any of them there exist largest and smallest elements which we uniformly denote by 0 and 1. For the different partial orders within the lattices we always use "≤". The following definitions are well-known [3].

Definition 2.1. Let $\Phi: P \to L$ be an isotonic (non-decreasing) function (with respect to the partial order in P) with $\Phi(0,..., 0) = 0$, $\Phi(1,..., 1) = 1$. We call Φ the structure function of **Σ**.

Now we shortly present the basics of the Dempster-Shafer Theory [4].

Definition 2.2. Let Ω be a sample space, P be a probability measure defined on a suitable σ-algebra over Ω (e.g. the set of all subsets of Ω). Further, let be given a system of sets ε (σ-algebra) and a set-valued function (random set) $X: \Omega \to$ ε. Then we define for any set Avε the function m_X : ε → [0,1]

by

$$m_X(A) = P(\omega : X(\omega) = A) \qquad (1)$$

where problems of measurability are left outside for simplicity. The lower index "X" will be omitted if misinterpretation is impossible. The system $\{A_1,\dots,A_N\}$ with A_i vε is called focal (w. r. to X) if all A_i are nonempty, the mass assignments $m(A_i)$ are positive for all i and the normalisation condition $\sum_i m(A_i)=1$ is fulfilled. Hence, the random set X can be given by $\{(A_1;m(A_1)),\dots,(A_N;m(A_N))\}$. Now we define the functions *bel*, *pl* (belief, plausibility) : ε → [0,1] by

$$bel(A)=\sum_{A_i\subseteq A} m(A_i),$$

$$pl(A)=\sum_{A_i\cap A\neq\varnothing} m(A_i). \tag{2}$$

Obviously, $bel(A) \le pl(A)$. We emphasise that the elements of ε may intersect. This is typical for situations with incomplete information. Presentations (1) and (2) are generalisations of the classical random variable which is recovered for atomic A_i (i.e., they are pairwise disjoint and $A_i\cap A\neq\varnothing$ implies $A_i\subseteq A$).

Next we need the following generalisation of DST to functions of random variables.

Definition 2.3. Suppose to be given M random sets X_i with ranges $rg(X_i)\in$ ε$_i$ characterised by focal elements $\{A^i_{k_i}\}$ and corresponding mass assignments $\{m^i_{k_i}\}$; i = 1,...,M. Here, $m^i_{k_i}=m(A^i_{k_i})$. Further, let be given a function f : $\mathop{\mathbf{X}}_{i=1}^{M} rg(X_i)\to$ ε, where ε is a suitable σ-algebra and **X** means the Cartesian product. Then we get the induced random set $Y=f(X_1,\dots,X_M)$ with focal elements $B_{k_1\dots k_M}=f(A^1_{k_1},\dots,A^M_{k_M})$ and given mass assignments

$$m_{k_1\dots k_M}=m(B_{k_1\dots k_M})=P(X_1=A^1_{k_1},\dots,X_M=A^M_{k_M}).$$

Notice that the entity $\{m_{k_1\dots k_M}\}$ is not necessarily normalised, because some of the $B_{k_1\dots k_M}$ may happen to be empty thus being excluded from further consideration. Hence, a normalisation should be performed in those cases and we may assume the above entity to be normal.

Now, for any $B\in$ε we get in analogy to (2)

$$bel(B)=\sum_{\substack{k_1,\dots,k_M\\ B_{k_1,\dots,k_M}\subseteq B}} m_{k_1,\dots,k_M},$$

$$pl(B)=\sum_{\substack{k_1,\dots,k_M\\ B_{k_1,\dots,k_M}\cap B\neq\varnothing}} m_{k_1,\dots,k_M}. \tag{3}$$

The assumption that the $m_{k_1\dots k_M}$ are known is rather restricting and may be unrealistic (as in statistics). If the random sets X_i are independent then one can set $m_{k_1\dots k_M}=m^1_{k_1}\cdot\ldots\cdot m^M_{k_M}$.

The case that information on X_i originates from several experts leads to Dempster's rule of combination and is considered, e.g. in [5].

In the case that the correlation between $X_1,\dots,X_M$ is unknown one can derive estimations as solutions of the following optimisation tasks (omitting non-negativity conditions)

$$\sum_{\substack{k_1,\dots,k_M\\ B_{k_1,\dots,k_M}\subseteq B}} m_{k_1,\dots,k_M} \xrightarrow[\{m_{k_1,\dots,k_M}\}]{} \min$$

$$\sum_{\substack{k_1,\dots,k_M\\ B_{k_1,\dots,k_M}\cap B\neq\varnothing}} m_{k_1,\dots,k_M} \xrightarrow[\{m_{k_1,\dots,k_M}\}]{} \max \tag{4}$$

$$\sum_{k_1,\dots,k_M}{}' \, m_{k_1,\dots,k_M}=m^i_{k_i};\ i=1,\dots,M$$

(here, prime means that the ith summand is omitted).

Denoting the extremal values of (4) by $\underline{bel}(B)$ and $\overline{pl}(B)$ one gets the obvious inclusion

$$\underline{bel}(B)\le bel(B)\le pl(B)\le \overline{pl}(B). \tag{5}$$

Remark 2.1. Solving (3) and (4) becomes rather time-consuming for higher dimensions. To keep ef-forts minimal, one should take sets B which are of special interest for the random set Y. In practise, often ε$_i$ and ε are set systems on the real axis. This may lead to interval computation for (3) and (4). For B one can take the set μ(z) = {xvP : $x \le z$} thus obtaining the plausibility and belief distribution functions

$\overline{F}$, $\underline{F}$ from

$$\overline{F}(z)=pl(\mu(z)),\quad \underline{F}(z)=bel(\mu(z)). \tag{6}$$

Example 2.1. Consider two independent random sets $X_1=\{([0,0.4];0.2),([0.3,0.8];0.67),([0.7,1];0.13)\}$ and $X_2=\{([0,0.6];0.67),([0.8,1];0.33)\}$ characterising the working ability of the two elements in a serial system. Hence, we take function f as *min* (acting on intervals by bounds). After simple computations we get $Y=\{([0,0.4];0.2),([0,0.6];0.54),([0.3,0.8];0.22),([0.7,1];0.04)\}$.

Assume we want to know *bel* and *pl* for an "acceptable" work ability of the system characterised by the interval B = [0.65,1]. From (3) we get $bel(B)$ = 0.04, $pl(B)$ = 0.22+0.04 = 0.26, what is not very high, because both systems mainly work at medium level.

Therefore, the question for "medium" working ability given by B = [0.3,0.6] will be answered by $bel(B)$ = 0.54+0.22 = 0.76, $pl(B)$ = 0.2+0.54+0.22 =0.96.

3 APPLICATION TO SYSTEM RELIABILITY

In principle, the above apparatus easily applies to reliability determination of multistage systems. The in-formation on the elements performance is given by the random sets X_i with focal elements $A^i_{k_i}\subseteq$ ε$_{L_i}$ (the latter being a suitable extension of L_i).

The role of the function f is now played by the structure function Φ that maps (in analogy to f) into ε$_L$, the latter being the corresponding extension of L. Often, the system is a connection of parallel-serial subsystems what may ease the computation of Φ (e.g. by paths or cuts). A popular choice for L_i and L is the unit interval [0,1]. Usually, one aims at computing the probability for a certain minimal level α of the system's performance, it is $\Phi(\boldsymbol{x}) \geq \alpha$. This leads to $B_{k_1 \dots k_M} = \Phi\left(A^1_{k_1}, \dots, A^n_{k_M}\right)$ whereby the focal elements of X_i may be taken as intervals in the continuous case, i.e. $A^i_{k_i} = \left[\underline{a}^i_{k_i}, \overline{a}^i_{k_i}\right]$. For B we take $[\alpha,1]$. Due to the isotonicity of Φ we get for (3)

$$bel(\alpha) = \sum_{\substack{k_1,\dots,k_n \\ \Phi\left(\underline{a}^1_{k_1},\dots,\underline{a}^n_{k_n}\right) \geq \alpha}} m_{k_1,\dots,k_M},$$

$$pl(\alpha) = \sum_{\substack{k_1,\dots,k_n \\ \Phi\left(\overline{a}^1_{k_1},\dots,\overline{a}^n_{k_n}\right) \geq \alpha}} m_{k_1,\dots,k_M} \qquad (7)$$

where we used $bel(\alpha)$, $pl(\alpha)$ for $bel(B)$, $pl(B)$.

Though (7) is computationally easier to handle than the general task (3), it may be of advantage to de-compose the system Σ into smaller parts what is typical for parallel-serial systems. The most elementary subsystems are those consisting of two elements. As a result we obtain random sets describing the behaviour of the subsystems and which can be combined to get the final estimation with respect to (7) or (4).

4 CONCLUSION

In the present paper we considered possibilities to compute reliabilities of multistate systems in the presence of random set estimations for the elements' working ability (performance). It turned out that the Dempster-Shafer approach is a suitable mathematical tool. For the case that the interdependence of the elements is unknown, bounds for the system's performance belief and plausibility functions are given as well.

From a practical point of view it may be useful to consider fuzzy focal elements and/or fuzzy sets B witch will be a topic for future research. We also refer to [1,2,6] where generalized implication operators are used to characterize the degree of inclusion of fuzzy sets.

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