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THE RELATIVE GROWTH OF INFORMATION IN TWO-DIMENSIONAL PARTITIONS

Abstract

Let $\overline{x} \in [0,1)^2$. In this paper we find the rate at which knowledge about the partition elements $\overline{x}$ lies in for one sequence of partitions determines the partition elements it lies in for another sequence of partitions. This rate depends on the entropy of these partitions and the geometry of their shapes, and gives a two-dimensional version of Lochs' theorem.

1 Introduction.

Let $x \in [0,1)$ and suppose we are interested in two different number-theoretic expansions of x. Given n digits in one of the expansions, how many digits are determined in the other expansion?

In 1964, Lochs explored this question when comparing the decimal and continued fraction expansions of x. Let $x \in [0,1)$ be irrational with decimal expansion

$$x = .d_1 d_2 d_3 \ldots$$

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and continued fraction expansion

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [0; a_1, a_2, a_3, \ldots].$$

If only the first n decimals of x are known, then x lies in the decimal cylinder $[y, z]$ where $y = .d_1 d_2 \ldots d_n$ and $z = .d_1 d_2 \ldots d_n + 10^{-n}$. In order to find the number of digits in the continued fraction expansion thus determined, let $y = [0; b_1, b_2, \ldots, b_l]$ and $z = [0; c_1, c_2, \ldots, c_k]$ be their continued fraction expansion. Then

$$m(n, x) = \max \{i \leq \min(l, k) : b_j = c_j \text{ for all } j \leq i\}$$

is the number of digits determined. In other words, $m(n, x)$ is the largest integer such that $b_n(x) \subset c_{m(n,x)}(x)$, where $b_n(x)$ is the decimal cylinder of order n containing x, denoted $[y, z]$ above, and $c_{m(n,x)}(x)$ is the continued fraction cylinder of order $m(n, x)$ containing x. Lochs [Lo] proved the following theorem.

Theorem 1.1. *Let λ denote Lebesgue measure on $[0, 1)$. Then for λ-a.e. $x \in [0, 1)$,*

$$\lim_{n \to \infty} \frac{m(n, x)}{n} = \frac{6 \ln 2 \ln 10}{\pi^2}.$$

In 1999, Bosma, Dajani, and Kraaikamp [BDK] noticed that this problem could be rephrased in terms of dynamical systems. Define the maps $S : [0, 1) \to [0, 1)$ and $T : [0, 1) \to [0, 1)$ by

$$Sx = 10x (\text{mod } 1) \text{ and } Tx = \begin{cases} \frac{1}{x} (\text{mod } 1) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then $([0, 1), \mathcal{B}, \lambda, S)$ and $([0, 1), \mathcal{B}, \mu, T)$ are dynamical systems, where $\mathcal{B}$ is the Borel σ-algebra on $[0, 1)$ and μ is the Gauss measure on $[0, 1)$. Let the partitions P and Q be given by

$$P = \left(\left[0, \frac{1}{10}\right), \left[\frac{1}{10}, \frac{2}{10}\right), \ldots, \left[\frac{9}{10}, 1\right)\right), \; Q = \left(\ldots, \left(\frac{1}{4}, \frac{1}{3}\right], \left(\frac{1}{3}, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right)\right).$$

If we label P by $(p^0, p^1, \ldots, p^9)$ and Q by $(\ldots, q^3, q^2, q^1)$, then the decimal expansion of x is achieved by iterating x by S and letting $d_i = k$ iff $S^{i-1}x \in p^k$. Similarly, the continued fraction expansion of x is found by iterating x by T and setting $c_i = k$ iff $T^{i-1}x \in q^k$. Thus the expansions are actually the itineraries of x for a certain partition in a certain dynamical system, and the intervals determined by the first k terms of the expansion are the cylinder sets in the induced partitions $\bigvee_{i=0}^{k-1} S^{-i}P$, $\bigvee_{i=0}^{k-1} T^{-i}Q$.

By using the Shannon-McMillan-Breiman Theorem [B] and the refinement of the partitions under application of their associated maps, Bosma, Dajani, and Kraaikamp generalized Theorem 1.1 to a wider class of transformations. In particular, it follows from their generalization that

$$\lim_{n\to\infty} \frac{m(n,x)}{n} = \frac{h_\lambda(S)}{h_\mu(T)},$$

where $h_\lambda(S)$ and $h_\mu(T)$ denote the entropy of the dynamical systems $([0,1), \mathcal{B}, \lambda, S)$ and $([0,1), \mathcal{B}, \mu, T)$. Their proof assumes a certain regularity in the induced partitions, an assumption that was then dropped in the work of Dajani and Fieldsteel [DF], where it is proved that Lochs' theorem is true for any two sequences of interval partitions on $[0,1)$ satisfying the conclusion of the Shannon-McMillan-Breiman Theorem. This result of Dajani and Fieldsteel can be immediately generalized to higher dimensional actions on $[0,1)$, using a theorem of Lindenstrauss [Li] to yield a Shannon-McMillan-Breiman Theorem in this setting and then noting that the arguments in [DF] do not rely on the one-dimensionality of the action. Moreover, it is easy to see that the result of [DF] can be generalized to sequences of higher dimensional product partitions for which the projections of these partitions on each coordinate consist of intervals (see Final Remarks 2).

In this paper we consider the unit square $[0,1)^2$. We prove a version of Lochs' theorem for any two sequences of partitions of $[0,1)^2$ satisfying certain geometric conditions and the conclusion of the Shannon-McMillan-Breiman Theorem. In the next section we will discuss our assumptions on the shapes of the partitions. In one dimension, all partition elements were intervals. In two dimensions, the variety of partition shapes seen can be much greater and the geometry of these shapes will play a role in the result. In Section 3 we state and prove a two-dimensional version of Lochs' theorem. We conclude with some final remarks about general partitions and about some special partitions in higher dimensions.

2 Partitions.

We are interested in pairs of sequences of partitions, $\mathcal{P} = \{P_n\}$, $\mathcal{Q} = \{Q_n\}$, of $[0,1)^2$. We denote the elements of partition P_n by p_n^i and Q_n by q_n^i. We denote by λ Lebesgue measure on $[0,1)^2$. There are certain criteria that we will assume about these partitions.

Assumptions A

*A*1. For every n, P_n consists of squares.

*A*2. For every n, Q_n consists of convex polygons.

*A*3. There exist constants R, $S > 0$ and $\beta \geq 2$ such that for every n and every i,

$$R\,\lambda(q_n^i) \leq \text{ (diameter of } q_n^i\ )^{\beta} \leq S\,\lambda(q_n^i).$$

Assumption $A3$ restricts the type of convex polygons that can be seen in Q_n. Since a convex polygon q_n^i is contained in a square of side length equal to the diameter of q_n^i, we have that $\lambda(q_n^i) \leq \text{ (diameter of } q_n^i)^2$, so the form seen in $A3$ is natural. In order to get both sides of the inequality, it is not usually possible to use $\beta = 2$. Under a very mild condition the following two lemmas tell us why we must have $\beta \geq 2$ and why, if β exists, it is unique.

Lemma 2.1. *Let $\{A_n\}$ be a sequence of partitions of $[0,1)^2$. Let $\alpha < 2$. Suppose that for every $\epsilon > 0$, there exist n and i with $\lambda(a_n^i) < \epsilon$. Then there is no $S > 0$ such that for every n and i, (diameter of $a_n^i)^\alpha \leq S\,\lambda(a_n^i)$.*

Proof. It is always true that $\lambda(a_n^i) \leq \text{ (diameter of } a_n^i)^2$. Now suppose that for every n and i, (diameter of $a_n^i)^\alpha \leq S\lambda(a_n^i)$. We then have

$$\lambda(a_n^i) \leq \left((\text{diameter of } a_n^i)^{\alpha}\right)^{\frac{2}{\alpha}} \leq S^{\frac{2}{\alpha}}\lambda(a_n^i)^{\frac{2}{\alpha}}.$$

Hence $S^{-\frac{2}{\alpha}} \leq \lambda(a_n^i)^{\frac{2}{\alpha}-1}$, and thus $S^{-\frac{2}{\alpha}} \leq \epsilon^{\frac{2}{\alpha}-1}$ for every $\epsilon > 0$, which yields a contradiction. □

Lemma 2.2. *Let $\{A_n\}$ be a sequence of partitions of $[0,1)^2$ and suppose that for every $\epsilon > 0$, there exist n and i with $\lambda(a_n^i) < \epsilon$. Then there is at most one $\beta \geq 2$ for which there exist constants $R, S > 0$ such that for every n and i,*

$$R\,\lambda(a_n^i) \leq (\textit{diameter of } a_n^i)^{\beta} \leq S\,\lambda(a_n^i).$$

Proof. Suppose we have, for every n and i,

$$R\,\lambda(a_n^i) \leq (\text{diameter of } a_n^i)^{\beta} \leq S\,\lambda(a_n^i). \tag{2.1}$$

Suppose it is also the case that there exist positive constants E and ς such that, for every n and i,

$$E\,\lambda(a_n^i) \leq (\text{diameter of } a_n^i)^{\beta+\varsigma}. \tag{2.2}$$

Then from the right hand side of (2.1) we have

$$(\text{diameter of } a_n^i)^{\beta+\varsigma} \leq S^{\frac{\beta+\varsigma}{\beta}} \lambda(a_n^i)^{\frac{\beta+\varsigma}{\beta}},$$

and hence from (2.2), $E \leq S^{\frac{\beta+\varsigma}{\beta}} \lambda(a_n^i)^{\frac{\varsigma}{\beta}}$. By assumption, we then get for arbitrary $\epsilon > 0$, $E \leq S^{\frac{\beta+\varsigma}{\beta}} \epsilon^{\frac{\varsigma}{\beta}}$ and we have a contradiction.

Similarly, suppose it is also the case that there exists a constant F such that, for every n and i,

$$(\text{diameter of } a_n^i)^{\beta-\varsigma} \leq F\lambda(a_n^i), \tag{2.3}$$

where $\varsigma > 0$ is such that $\beta - \varsigma > 0$. Then from the left hand side of (2.1) we have

$$R^{\frac{\beta-\varsigma}{\beta}} \lambda(a_n^i)^{\frac{\beta-\varsigma}{\beta}} \leq (\text{diameter of } a_n^i)^{\beta-\varsigma},$$

and hence from (2.3), $R^{\frac{\beta-\varsigma}{\beta}} \leq F\lambda(a_n^i)^{\frac{\varsigma}{\beta}}$. As before, this implies $R^{\frac{\beta-\varsigma}{\beta}} \leq F\epsilon^{\frac{\varsigma}{\beta}}$ for arbitrary $\epsilon > 0$, yielding a contradiction. □

Recall that the partitions Q_n consist of convex polygons q_n^i. We will be interested in the set of points in the polygons lying close to the boundary, defined as follows.

Definition 2.3. Let q be a convex polygon in $[0,1)^2$. The **frame of q of width δ** is the set

$$\mathcal{F}(q,\delta) = \{\overline{x} : \overline{x} \in q \text{ and } d(\overline{x}, \partial q) \leq \delta\},$$

where ∂q is the boundary of q and d indicates Euclidean distance on the plane.

The proportion of q taken up by its frame is small when δ is small. The next lemma provides a bound that will be useful in the next section.

Lemma 2.4. *Let q be a convex polygon in $[0,1)^2$, such that*

$$(\textit{diameter of } q)^{\beta} \leq S\,\lambda(q)$$

for some constants S and β. Then the proportion of q taken up by its frame of width δ is bounded above by $\dfrac{4S^{\frac{1}{\beta}}\delta}{\lambda(q)^{\frac{\beta-1}{\beta}}}$.

Proof. We are interested in $\frac{\lambda(\mathcal{F}(q,\delta))}{\lambda(q)}$. Note that

$$\lambda(\mathcal{F}(q,\delta)) \leq (\text{perimeter of } q) \cdot \delta.$$

As already observed before, the polygon q is easily seen to be contained in a square of side length equal to the diameter of q and thus

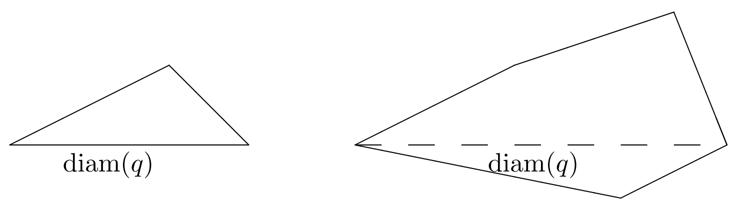
$$\text{(perimeter of } q) \leq 4 \cdot \text{(diameter of } q),$$

which gives us

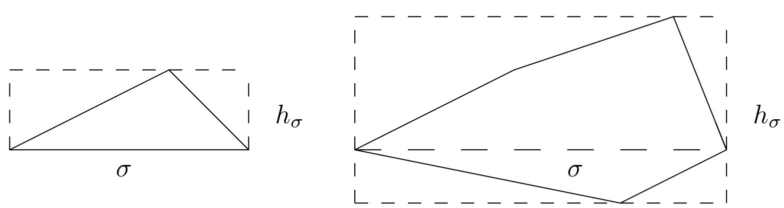
$$\lambda(\mathcal{F}(q,\delta)) \leq 4 \cdot \text{(diameter of } q) \cdot \delta \leq \; 4\delta S^{\frac{1}{\beta}} \lambda(q)^{\frac{1}{\beta}}$$

which can be plugged into the above to yield the result. $\square$

Note that if q is a convex polygon, then the diameter is given by the largest length of the line segments connecting the vertices of q. This may be the length of a side of q or the length of a line segment in the interior of q (see figure below).



Note that a line segment in q whose length equals the diameter of q is in general not unique. Let σ be the diameter of q and let r_σ be the minimal rectangle containing q with sides parallel (or perpendicular) to a line segment in q of length σ. Finally, let h_σ be the height of r_σ (see figure below).



Then, $h_\sigma \cdot \sigma = \lambda(r_\sigma) \leq 2\lambda(q)$. We refer to r_σ as a "diameter rectangle of q" of height h_σ.

The following obvious observation will be used in the next section.

Lemma 2.5. *Let p be a square of side length x and r be a rectangle of height h. If $x > h$, then p is* **not** *contained in r.*

We end this section with additional criteria for the partition sequences $\mathcal{P} = \{P_n\}$ and $\mathcal{Q} = \{Q_n\}$. In the sequel, $\log x$ ($x > 0$) always means the logarithm of x with respect to base 2. Denote by $p_n(\overline{x})$ the element of partition P_n which contains $\overline{x}$, and by $q_m(\overline{x})$, the element of partition Q_m which contains $\overline{x}$. The following definition is inspired by the Shannon-McMillan-Breiman Theorem.

Definition 2.6. Let $\mathcal{P} = \{P_n\}$ be a sequence of partitions. Let $c \geq 0$. We say that $\mathcal{P}$ has entropy c a.e. with respect to λ if

$$-\frac{\log \lambda(p_n(\overline{x}))}{n} \to c \quad \text{for } \lambda\text{-a.e. } \overline{x}.$$

Assumptions B

Let $\mathcal{P} = \{P_n\}$ and $\mathcal{Q} = \{Q_n\}$ be sequences of partitions of $[0,1)^2$ such that

$B1$. For some constant $c > 0$, $\mathcal{P}$ has entropy c a.e. with respect to λ, and

$B2$. For some constant $d > 0$, $\mathcal{Q}$ has entropy d a.e. with respect to λ.

If the sequence of partitions $\mathcal{Q}$ satisfies assumption $B2$, it follows directly from Lemma 2.1 and Lemma 2.2 that there exists at most one $\beta \geq 2$ for which assumption $A3$ holds. For ease of notation we will call $p_n(\overline{x})$ (respectively $q_n(\overline{x})$) (n,η)-good if

$$2^{-n(c+\eta)} \leq \lambda(p_n(\overline{x})) \leq 2^{-n(c-\eta)}$$
$$\text{(respectively } 2^{-n(d+\eta)} \leq \lambda(q_n(\overline{x})) \leq 2^{-n(d-\eta)}).$$

3 Main Theorem.

As before, λ denotes the Lebesgue measure on $[0,1)^2$. Let $\mathcal{P} = \{P_n\}$ and $\mathcal{Q} = \{Q_n\}$ be sequences of partitions of $[0,1)^2$ satisfying assumptions A and B from Section 2. For each $n \in \mathbb{N}$ and $\overline{x} \in [0,1)^2$, let

$$m_{\mathcal{P},\mathcal{Q}}(n,\overline{x}) = \sup \{m : p_n(\overline{x}) \subset q_m(\overline{x})\}.$$

Theorem 3.1. *For λ-a.e. $\overline{x} \in [0,1)^2$,*

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \to \frac{\beta}{2(\beta-1)}\frac{c}{d},$$

where $\beta \geq 2$ is the constant from assumption $A3$ and c,d are the constants from the assumptions $B1$ and $B2$.

The proof is in two parts. We will first show that

$$\liminf_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \geq \frac{\beta}{2(\beta-1)}\frac{c}{d} \quad \text{for } \lambda\text{-a.e. } \overline{x}, \tag{3.1}$$

and then

$$\limsup_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \leq \frac{\beta}{2(\beta-1)}\frac{c}{d} \quad \text{for } \lambda\text{-a.e. } \overline{x}. \tag{3.2}$$

Together these give the result.

PROOF. To prove (3.1), let $0<\epsilon<1$. For each n, let

$$\tilde{m}(n) = \lfloor (1-\epsilon)\frac{\beta}{2(\beta-1)}\frac{c}{d}n \rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Choose $\eta>0$ so small that $\zeta := \epsilon\frac{c}{2} - \eta(\frac{c}{2d}(1-\epsilon)+\frac{1}{2}) > 0$. Consider the set of points

$$D_n(\eta) = \left\{ \overline{x} : \begin{array}{l} p_n(\overline{x}) \text{ is } (n,\eta)\text{-good}, \\ q_{\tilde{m}(n)}(\overline{x}) \text{ is } (\tilde{m}(n),\eta)\text{-good, and} \\ p_n(\overline{x}) \not\subset q_{\tilde{m}(n)}(\overline{x}) \end{array} \right\}.$$

If $\overline{x} \in D_n(\eta)$, then $\overline{x}$ lies in an element of P_n that intersects at least 2 elements of $Q_{\tilde{m}(n)}$. Thus $\overline{x}$ must lie in the frame of $q_{\tilde{m}(n)}(\overline{x})$ of width δ, where δ is the diameter of $p_n(\overline{x})$. Since $p_n(\overline{x})$ is a square, we know its diameter δ is $\sqrt{2}[\lambda(p_n(\overline{x})]^{\frac{1}{2}}$. Since $p_n(\overline{x})$ is (n,η)-good, $\lambda(p_n(\overline{x})) \leq 2^{-n(c-\eta)}$; thus we know $\overline{x}$ must lie in the frame of $q_{\tilde{m}(n)}(\overline{x})$ of width $\sqrt{2}\; 2^{-\frac{n}{2}(c-\eta)}$. We can thus bound the measure of $D_n(\eta)$ by the sum of all the frames of the $(\tilde{m}(n),\eta)$-good elements of $Q_{\tilde{m}(n)}$, of width $\sqrt{2}\; 2^{-\frac{n}{2}(c-\eta)}$.

From Lemma 2.4, we know the proportion of an element $q_{\tilde{m}(n)}$ of the partition $Q_{\tilde{m}(n)}$ taken up by its frame of width $\sqrt{2}\; 2^{-\frac{n}{2}(c-\eta)}$ is bounded above by

$$\frac{4\sqrt{2}\; S^{\frac{1}{\beta}}\, 2^{-\frac{n}{2}(c-\eta)}}{\lambda(q_{\tilde{m}(n)})^{\frac{\beta-1}{\beta}}},$$

where S is from assumption A3. Since $q_{\tilde{m}(n)}$ is $(\tilde{m}(n),\eta)$-good, we have that

$$\lambda(q_{\tilde{m}(n)})^{\frac{\beta-1}{\beta}} \geq 2^{-\tilde{m}(n)\frac{\beta-1}{\beta}(d+\eta)}.$$

Hence

$$\lambda(q_{\tilde{m}(n)})^{\frac{\beta-1}{\beta}} \geq 2^{-(1-\epsilon)\,\frac{c}{d}\,\frac{n}{2}(d+\eta)}.$$

Plugging this into the above, we see that the proportion of $q_{\tilde{m}(n)}$ taken up by its frame is bounded above by

$$\frac{4\sqrt{2}\ S^{\frac{1}{\beta}}\ 2^{-\frac{n}{2}(c-\eta)}}{2^{-(1-\epsilon)\ \frac{c}{d}\ \frac{n}{2}(d+\eta)}} = 4\sqrt{2}\ S^{\frac{1}{\beta}} 2^{[(1-\epsilon)\ \frac{c}{d}\ \frac{n}{2}(d+\eta)-\frac{n}{2}\ (c-\eta)]}$$
$$= 4\sqrt{2}\ S^{\frac{1}{\beta}} 2^{-n[\epsilon\frac{c}{2}-\eta((1-\epsilon)\frac{c}{2d}+\frac{1}{2})]} = 4\sqrt{2}\ S^{\frac{1}{\beta}} 2^{-n\zeta},$$

by the definition of ζ above. Thus the area of the frame of an $(\tilde{m}(n),\eta)$-good $q_{\tilde{m}(n)}$ is bounded above by $4\sqrt{2}\ S^{\frac{1}{\beta}} 2^{-n\zeta}\lambda(q_{\tilde{m}(n)})$, and thus

$$\lambda(D_n(\eta)) \leq 4\sqrt{2}\ S^{\frac{1}{\beta}} 2^{-n\zeta}.$$

Then $\sum_{n=1}^{\infty}\lambda(D_n(\eta)) < \infty$, which implies $\lambda(\{\overline{x} : \overline{x} \in D_n(\eta) \text{ infinitely often}\}) = 0$.

Since $\tilde{m}(n)$ goes to infinity as n does, it follows that for λ-a.e. $\overline{x} \in [0,1)^2$, there exists an $N = N(\overline{x}) \in \mathbb{N}$ such that for all $n \geq N$, $p_n(\overline{x})$ is (n,η)-good and $q_{\tilde{m}(n)}(\overline{x})$ is $(\tilde{m}(n),\eta)$-good and $\overline{x} \notin D_n(\eta)$. But knowing that $p_n(\overline{x}) \subset q_{\tilde{m}(n)}(\overline{x})$ means that $m_{\mathcal{P},\mathcal{Q}}(n,\overline{x}) \geq \tilde{m}(n)$. Thus for λ-a.e. $\overline{x} \in [0,1)^2$,

$$\liminf_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \geq \liminf_{n\to\infty} \frac{\tilde{m}(n)}{n} = (1-\epsilon)\,\frac{\beta}{2(\beta-1)}\,\frac{c}{d}.$$

Since ϵ was arbitrary, this gives the first part of our proof.

To prove (3.2), let $\epsilon > 0$. It is sufficient to show, for λ-a.e. $\overline{x}$, that

$$\limsup_{n\to\infty} \frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \leq \frac{\beta}{2(\beta-1)}\,\frac{c}{d}\,(1+\epsilon).$$

Choose $0 < \eta < d$ so small that $\zeta := \epsilon\frac{c}{2} - \eta(\frac{c}{2d}(1+\epsilon)+\frac{1}{2})\ > 0$. Take $\overline{x}$ from the set of full measure on which the assumptions B hold. Let

$$\hat{m}(n) = \lceil \frac{\beta}{2(\beta-1)}\,n\,\frac{c}{d}\,(1+\epsilon)\rceil,$$

where $\lceil x \rceil$ is the smallest integer larger than or equal to x.

Take $N = N(\overline{x})$ so large that

- for all $n \geq N$, $\lambda(p_n(\overline{x})) \geq 2^{-n(c+\eta)}$,
- for all $m \geq \hat{m}(N)$, $\lambda(q_m(\overline{x})) \leq 2^{-m(d-\eta)}$, and
- $N > \frac{-\log\frac{1}{2}R^{\frac{1}{\beta}}}{\zeta}$.

We want to show that $p_n(\overline{x}) \not\subset q_{\hat{m}(n)+l}(\overline{x})$, for all $l \geq 0$ and $n \geq N$. Let $l \geq 0$ and $n \geq N$, and let h be the height of a diameter rectangle of $q_{\hat{m}(n)+l}$. According to Lemma 2.5, it is sufficient to show that the length of a side of the square $p_n(\overline{x})$ is larger than h. We know

$$\begin{aligned} h &\leq \frac{2\lambda(q_{\hat{m}(n)+l}(\overline{x}))}{\operatorname{diam}(q_{\hat{m}(n)+l}(\overline{x}))} \leq \frac{2}{R^{\frac{1}{\beta}}}\lambda(q_{\hat{m}(n)+l}(\overline{x}))^{\frac{\beta-1}{\beta}} \\ &\leq \frac{2}{R^{\frac{1}{\beta}}}2^{-(\hat{m}(n)+l)(d-\eta)\frac{\beta-1}{\beta}} \leq \frac{2}{R^{\frac{1}{\beta}}}2^{-\hat{m}(n)(d-\eta)\frac{\beta-1}{\beta}} \leq \frac{2}{R^{\frac{1}{\beta}}}2^{-\frac{n}{2}\frac{c}{d}(1+\epsilon)(d-\eta)}. \end{aligned}$$

On the other hand, the side length of $p_n(\overline{x})$ is $\lambda(p_n(\overline{x}))^{\frac{1}{2}}$, which is bounded below by $2^{-\frac{n}{2}(c+\eta)}$. We want to show that

$$\text{side of } p_n(\overline{x}) \; \geq 2^{-\frac{n}{2}(c+\eta)} > \frac{2}{R^{\frac{1}{\beta}}}2^{-\frac{n}{2}\frac{c}{d}(1+\epsilon)(d-\eta)},$$

which is an upper bound for h. But $2^{-\frac{n}{2}(c+\eta)} > \frac{2}{R^{\frac{1}{\beta}}}2^{-\frac{n}{2}\frac{c}{d}(1+\epsilon)(d-\eta)}$ can be rewritten as $\frac{1}{2}R^{\frac{1}{\beta}} > 2^{-n\zeta}$, and we see that N was chosen large enough so this would be true. Hence, $m_{\mathcal{P},\mathcal{Q}}(n,\overline{x}) < \hat{m}(n)$ for $n \geq N(\overline{x})$ and part (3.2) follows. $\square$

We end this paper with some remarks, concerning sequences of partitions $\mathcal{P} = \{P_n\}$ and $\mathcal{Q} = \{Q_n\}$ of $[0,1)^D$, where D is an arbitrary positive integer. For ease of notation, we denote by λ the Lebesgue measure on $[0,1)$ as well as the Lebesgue measure on $[0,1)^D$. Furthermore, we let $\overline{x} = (x_1,\ldots,x_D)$.

Final Remarks.

1. Suppose the sequences of partitions $\mathcal{P}$ and $\mathcal{Q}$ satisfy the assumptions $B1$ and $B2$. It follows from the proof of Theorem 4 in [DF] that for λ-a.e. $\overline{x}$, the inequality

$$\limsup_{n\to\infty}\frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \leq \frac{c}{d}$$

holds *without* imposing further conditions on the partition elements, where c and d refer to the values in assumptions $B1$ and $B2$.

2. Suppose that for each n, P_n and Q_n are product partitions consisting of rectangles:

$$P_n = P_n^1 \times \cdots \times P_n^D \,,\, Q_n = Q_n^1 \times \cdots \times Q_n^D,$$

where P_n^i and Q_n^i, $1 \leq i \leq D$, are interval partitions. Assume that for $1 \leq i \leq D$, the sequences of partitions $\mathcal{P}^i = \{P_n^i\}$ and $\mathcal{Q}^i = \{Q_n^i\}$ have positive entropy c_i, respectively d_i, a.e. with respect to λ. Furthermore, assume that Q_{n+1} refines Q_n for each n. In general, the sequence of partitions $\mathcal{Q}$ does not satisfy assumption $A3$. However, Theorem 4 in [DF] states that for $1 \leq i \leq D$,

$$\frac{m_{\mathcal{P}^i,\mathcal{Q}^i}(n,x_i)}{n} \to \frac{c_i}{d_i} \text{ for } \lambda\text{-a.e. } x_i.$$

Since for each n, Q_{n+1} refines Q_n, we have that

$$m_{\mathcal{P},\mathcal{Q}}(n,\overline{x}) = \min_{1\leq i\leq D} m_{\mathcal{P}^i,\mathcal{Q}^i}(n,x_i).$$

Hence, for λ-a.e. $\overline{x}$,

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \to \min_{1\leq i\leq D}\{\frac{c_i}{d_i}\}.$$

The condition in the previous remark stating that Q_{n+1} refines Q_n for each n is not necessary as the following example shows. This example also illustrates the fact that for all $\beta \geq 2$ and $c, d > 0$, there exist sequences of partitions $\mathcal{P}$ and $\mathcal{Q}$ satisfying the assumptions A and B.

Example. Let $\beta \geq 2$ and $c, d > 0$ be given. Define the partition elements of Q_n by

$$q_n^{(i,j)} = \left[\frac{i}{\lfloor 2^{\frac{d(\beta-1)n}{\beta}}\rfloor}, \frac{i+1}{\lfloor 2^{\frac{d(\beta-1)n}{\beta}}\rfloor}\right) \times \left[\frac{j}{\lfloor 2^{\frac{dn}{\beta}}\rfloor}, \frac{j+1}{\lfloor 2^{\frac{dn}{\beta}}\rfloor}\right),$$

$0 \leq i \leq \lfloor 2^{\frac{d(\beta-1)n}{\beta}}\rfloor - 1, 0 \leq j \leq \lfloor 2^{\frac{dn}{\beta}}\rfloor - 1$, and define the partitions Q_n^1 and Q_n^2 by the relation $Q_n = Q_n^1 \times Q_n^2$. One easily verifies that $\{Q_n^1\}$ and $\{Q_n^2\}$ have entropy $d_1 = \frac{d(\beta-1)}{\beta}$ and $d_2 = \frac{d}{\beta}$, respectively, a.e. with respect to λ. Define the partition $P_n = P_n^1 \times P_n^2$ simply by partitioning $[0,1)^2$ into $\lfloor 2^{\frac{cn}{2}}\rfloor \times \lfloor 2^{\frac{cn}{2}}\rfloor$ squares of equal side length and note that $\{P_n^1\}$ and $\{P_n^2\}$ have entropy $c_1 = \frac{c}{2}$ and $c_2 = \frac{c}{2}$, respectively, a.e. with respect to λ. It is straightforward to show that the assumptions A and B are all satisfied. It follows from Theorem 3.1, that for λ-a.e. $\overline{x}$,

$$\frac{m_{\mathcal{P},\mathcal{Q}}(n,\overline{x})}{n} \to \frac{\beta}{2(\beta-1)}\frac{c}{d} = \min\{\frac{c_1}{d_1}, \frac{c_2}{d_2}\}.$$

However, if for instance d_1 or d_2 is small enough, Q_{n+1} does not refine Q_n for infinitely many n.

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