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The Decomposition Theorem For Two-Dimensional Shifts Of Finite Type

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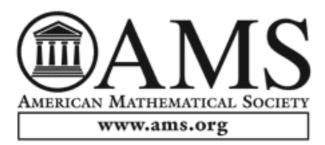K. M. Madden

THE DECOMPOSITION THEOREM
FOR TWO-DIMENSIONAL SHIFTS OF FINITE TYPE

AIMEE S. A. JOHNSON AND KATHLEEN M. MADDEN

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ABSTRACT. A one-dimensional shift of finite type can be described as the
collection of bi-infinite "walks" along an edge graph. The Decomposition The-
orem states that every conjugacy between two shifts of finite type can be
broken down into a finite sequence of splittings and amalgamations of their
edge graphs. When dealing with two-dimensional shifts of finite type, the ap-
propriate edge graph description is not as clear; we turn to Nasu's notion of
a "textile system" for such a description and show that all two-dimensional
shifts of finite type can be so described. We then define textile splittings and
amalgamations and prove that every conjugacy between two-dimensional shifts
of finite type can be broken down into a finite sequence of textile splittings,
textile amalgamations, and a third operation called an inversion.

1. INTRODUCTION

Let $\mathcal{A}$ be a finite alphabet. Then the full two-dimensional shift space on $\mathcal{A}$ is $\mathcal{A}^{Z^2}$, with shift map σ_a defined, for each $a \in Z^2$, by $[\sigma_a(y)]_b = y_{b+a}$ for all $b \in Z^2$. We let ρ denote the usual metric on $\mathcal{A}^{Z^2}$: $\rho(x,y) = 2^{-k}$ where k is the largest integer such that $x_b = y_b$ for all $b \in [-k,k]^2 \subset Z^2$. A two-dimensional shift space Y is a closed subset of $\mathcal{A}^{Z^2}$ which is invariant under all shift maps σ_a, $a \in Z^2$.

A two-dimensional shift of finite type (SFT) X is a shift space defined using two transition matrices A_1 and A_2 indexed by elements in $\mathcal{A}$. Thus

$$X = X(A_1, A_2) = \{x \in \mathcal{A}^{Z^2} : A_i(x_a, x_{a+e_i}) = 1 \text{ for all } a \in Z^2, i = 1, 2\}.$$

SFTs are more usually defined using allowable blocks, which can be shown to be equivalent to the above definition (see Proposition 2.3.9 in [LM95] for the one-dimensional result; the proof for higher dimensions is similar).

Difficulties arise in the two-dimensional case which do not occur in the traditional one-dimensional case. For example, given a single transition matrix, it is relatively easy to determine whether the corresponding one-dimensional SFT is nonempty. In two dimensions, the question of whether there are any two-dimensional arrays of symbols satisfying the transition requirements of A_1 and A_2 is referred to as the *nonemptiness problem* and is undecidable. The *extension problem*, which asks

whether a given admissible block will actually occur in a two-dimensional array in $X(A_1, A_2)$, is also undecidable.

In one dimension, a SFT is nonempty if and only if it contains periodic points. This is false in higher dimensions as the existence of aperiodic tiling dynamical systems illustrates. The undecidability results described above are closely related to this fact [B66], [R71].

There are settings in which these difficulties can be avoided. N. Markley and M. Paul consider matrix subshifts satisfying an overlapping condition [MP81], [MP281] and B. Kitchens and K. Schmidt restrict their attention to higher dimensional shifts for which $\mathcal{A}$ has a group structure [KS88], [KS92]. For these classes of two-dimensional SFTs , the nonemptiness and extension problems can be answered and progress has been made towards understanding other dynamical properties. (See again [MP81], [MP281], [KS88] and [KS92].) In general, however, little is known about higher dimensional SFTs. (For a complete list of references and a thorough overview of higher dimensional shifts see [LM95], Chapter 13.)

A useful tool in studying one-dimensional SFTs is the edge graph. Bi-infinite walks along an edge graph yield a SFT and conversely, using a higher block presentation if necessary, every one-dimensional SFT can be obtained in this way. The appropriate edge graph representation of a two-dimensional shift is not immediately obvious. In [N95], Nasu proposes the *textile system*, involving two edge graphs and two graph homomorphisms as a way of constructing two-dimensional SFTs. In fact, moving to a higher block presentation if necessary, all two-dimensional shifts can be represented in this way. In Section 2 we discuss these facts and propose the textile system as the appropriate two-dimensional "edge graph".

In Section 3 we define textile splittings and amalgamations. These definitions involve splitting (or amalgamating) one of the edge graphs in the textile system and extending to the other graph appropriately. The main theorem of this section is a two-dimensional decomposition theorem which states that any conjugacy between two-dimensional SFTs can be broken down into a finite sequence of textile splittings, amalgamations and a third operation which interchanges the horizontal and vertical directions. The formal definition of this operation, called a shift inversion, is given below.

We conclude this section with a review of some vocabulary and notation involving shift conjugacies and with the definition of a shift inversion. We also remark that similar results involving bipartite codes have been obtained by Hirashi Aso [A97]. Bipartite codes are used by Nasu [N86] to prove a variant of the Decomposition Theorem which states that any conjugacy between one-dimensional subshifts is the composition of a finite number of bipartite codes. Aso generalizes this result to two dimensions.

In two dimensions, a continuous, shift commuting map Φ between shifts $X \subseteq \mathcal{A}^{Z^2}$ and $\tilde{X} \subseteq \tilde{\mathcal{A}}^{Z^2}$ is a sliding block code. An $((m,n),(r,s))$−block code $\Phi : X \to \tilde{X}$ is defined via

$$\Phi(x)_{(i,j)} = \phi \begin{pmatrix} x_{(i-m,j+s)} & \cdots & x_{(i,j+s)} & \cdots & x_{(i+n,j+s)} \\ \vdots & & \vdots & & \vdots \\ x_{(i-m,j)} & \cdots & x_{(i,j)} & \cdots & x_{(i+n,j)} \\ \vdots & & \vdots & & \vdots \\ x_{(i-m,j-r)} & \cdots & x_{(i,j-r)} & \cdots & x_{(i+n,j-r)} \end{pmatrix}$$

where ϕ is a map from $((m,n),(r,s))$−blocks (as depicted above) of symbols from $\mathcal{A}$ to symbols in $\tilde{\mathcal{A}}$. Note that a $((0,0),(0,0))$−block map is defined using a map ϕ which sends symbols to symbols. We call m (r) the horizontal (vertical) memory and n (s) the horizontal (vertical) anticipation. If Φ is invertible, we say that X and $\tilde{X}$ are *conjugate*, denoted $(X, Z^2) \cong (\tilde{X}, Z^2)$.

In two dimensions, the choice of horizontal versus vertical direction is somewhat arbitrary and so we define an *inversion* which interchanges these directions. Let θ map a two-dimensional shift X to the two-dimensional shift $\theta(X)$ via $\theta(x)_{(i,j)} = x_{(j,i)}$ and let $\sigma_i = \sigma_{e_i}$, $i = 1, 2$. It is clear that $\sigma_1 \circ \theta = \theta \circ \sigma_2$ and $\sigma_2 \circ \theta = \theta \circ \sigma_1$ and thus $(X, \sigma_1, \sigma_2) \cong (\theta(X), \sigma_2, \sigma_1)$. Furthermore, $\theta^{-1} = \theta$. Note that if $\Phi : X \to \tilde{X}$ is an $((m,n),(r,s))$−block map, then $\theta \circ \Phi \circ \theta : \theta(X) \to \theta(\tilde{X})$ is an $((r,s),(m,n))$−block map.

2. Textile systems and SFTs

We begin this section by reviewing edge graphs before using them to define a textile system. For a more complete review, see [LM95].

Let G be an *edge graph* with vertices V_G and arcs E_G . For each $\alpha \in E_G$, let $i_G(\alpha)$ $(t_G(\alpha))$ be the initial (terminal) vertex of α. We say that G is essential if both i_G and t_G are onto.

Let X_G denote the collection of bi-infinite walks on G. That is,

$$x = \cdots x_{-1} x_0 x_1 x_2 \cdots$$

is in X_G if and only if $x_i \in E_G$ for all $i \in Z$ and $i_G(x_{i+1}) = t_G(x_i)$.

The *transition matrix* associated with an edge graph G is the $E_G \times E_G$ 0-1 matrix $A = [A_{\alpha\beta}]$ where $A_{\alpha\beta} = 1$ if and only if $t_G(\alpha) = i_G(\beta)$.

A *graph homomorphism* between graphs Γ and G is a pair of mappings $\phi_E : E_\Gamma \to E_G$ and $\phi_V : V_\Gamma \to V_G$ such that

$$i_G \circ \phi_E = \phi_V \circ i_\Gamma,$$
$$t_G \circ \phi_E = \phi_V \circ t_\Gamma.$$

A graph homomorphism $\phi : \Gamma \to G$ induces a one-block factor map $\Phi : X_\Gamma \to X_G$ in the obvious way.

Definition 2.1. A **textile system** consists of an essential edge graph Γ, a single vertex edge graph G, and two graph homomorphisms, $p, q : \Gamma \to G$, such that the quadruple $(i_\Gamma(\alpha), t_\Gamma(\alpha), p(\alpha), q(\alpha))$ uniquely determines $\alpha \in E_\Gamma$.

Definition 2.2. A **textile weaved by** T is a two-dimensional array

$$(\alpha_{(i,j)})_{i,j \in Z} \in E_\Gamma^{Z^2}$$

where $(\alpha_{(i,j)})_{i \in Z} \in X_\Gamma$ for all $j \in Z$ and $q(\alpha_{(i,j-1)}) = p(\alpha_{(i,j)})$ for all $i, j \in Z$.

We can view a textile system as a means of associating the edges in Γ with a collection of Wang tiles: the edge α is associated with the tile depicted below.

If we let X_T be the set of all textiles weaved by T then X_T is a closed invariant subset of $E_\Gamma^{Z^2}$, and the textile dynamical system and the geometric tiling dynamical system are isomorphic.

We remark that in Nasu's original definition, Γ is not required to be essential and G is not required to be a single vertex graph. However, these are not fundamental restrictions. If Γ is not essential, then there exists an essential graph Γ' with $X_\Gamma = X_{\Gamma'}$. If G is not a single vertex graph, then we can construct G', consisting of a single vertex and the edges from G, and $T' = (p', q' : \Gamma \to G')$ with $p'(\alpha) = p(\alpha)$, $q'(\alpha) = q(\alpha)$ for $\alpha \in E_\Gamma$. It is clear that $X_T = X_{T'}$.

The inversion of a textile dynamical system X_T is also given by a textile system, known as the *dual* T^* of T. We define the dual Γ^T of Γ by

$$E_{\Gamma^T} = E_\Gamma \quad \text{and}$$
$$V_{\Gamma^T} = E_G,$$

and for $\alpha \in V_{\Gamma^T} = E_\Gamma$

$$i_{\Gamma^T}(\alpha) = p(\alpha) \quad \text{and}$$
$$t_{\Gamma^T}(\alpha) = q(\alpha).$$

The single vertex graph G^T has edges indexed by V_Γ. The inversion of the textile dynamical system X_T is then given by the textile system $T^* = (p^T, q^T : \Gamma^T \to G^T)$ where

$$p^T(\alpha) = i_\Gamma(\alpha),$$
$$q^T(\alpha) = t_\Gamma(\alpha).$$

It follows from the definition that $(T^*)^* = T$ and if $(\alpha_{(i,j)})_{i,j \in Z} \in X_T$, then $(\alpha_{(j,i)})_{i,j \in Z} \in X_{T^*}$.

A textile dynamical system is isomorphic to a geometric tiling dynamical system and thus it is clear that textile dynamical systems are two-dimensional SFTs. The converse is also true, as the following proposition demonstrates.

Proposition 2.3. *Let (X, σ_1, σ_2) be a SFT given by transition matrices A_1 and A_2. Then, moving to a higher block presentation of X if necessary, there exists a textile system $T = (p, q : \Gamma \to G)$ such that $X = X_T$.*

Proof. Consider the higher block presentation $\tilde{X}$ of X given by the $((0,1), (0,1))-$ block map $\Phi : X \to \tilde{X}$ where

$$\Phi(x)_{(i,j)} = \begin{matrix} cd \\ ab \end{matrix} \quad \text{when} \quad \begin{matrix} x_{(i,j+1)}\, x_{(i+1,j+1)} \\ x_{(i,j)}\, x_{(i+1,j)} \end{matrix} = \begin{matrix} cd \\ ab \end{matrix}.$$

Then $\tilde{X}$ is given by transition matrices $\tilde{A}_1$ and $\tilde{A}_2$ defined in the obvious way.

We show that $(\tilde{X}, Z^2)$ is given by a textile system $T = (p, q : \Gamma \to G)$. Let

$$\mathcal{S} = \{2 \times 2\ A_1, A_2 \text{ admissible blocks}\}.$$

Construct an edge graph Γ by

$$V_\Gamma = \{\text{columns of } 2 \times 2 \text{ blocks in } \mathcal{S}\},$$
$$E_\Gamma = \mathcal{S},$$

with

$$i_\Gamma(B) = \begin{matrix} c \\ a \end{matrix}, \qquad i_\Gamma(B) = \begin{matrix} d \\ b \end{matrix}$$

for all $B = \begin{smallmatrix} cd \\ ab \end{smallmatrix} \in E_\Gamma$. Then construct a single vertex edge graph G with edges labeled by

$$E_G = \{\text{rows of } 2 \times 2 \text{ blocks in } \mathcal{S}\}.$$

Define graph homomorphisms $p, q : \Gamma \to G$ via

$$p(B) = ab, \qquad q(B) = cd$$

for all $B = \begin{smallmatrix} cd \\ ab \end{smallmatrix} \in E_\Gamma$. Then $T = (p, q : \Gamma \to G)$ is a textile system and $X_T = \tilde{X}$. This follows from the fact that if $\alpha, \beta \in \mathcal{S}$ then

$$\tilde{A}_1(\alpha, \beta) = 1 \quad \text{if and only if}$$
$$i_\Gamma(\alpha) = t_\Gamma(\beta)$$

and

$$\tilde{A}_2(\alpha, \beta) = 1 \quad \text{if and only if}$$
$$q(\alpha) = p(\beta).$$

$\square$

Proposition 2.3 is not surprising since every two-dimensional SFT is isomorphic to a tiling dynamical system. (See for instance [M89].)

3. A TWO-DIMENSIONAL DECOMPOSITION THEOREM

In one dimension, the decomposition theorem states that any conjugacy between SFTs can be decomposed into a finite sequence of splittings and amalgamations. We will prove an analogous theorem for two dimensions. The crucial lemmas will show that horizontal and vertical memory and anticipation of a block map can be reduced using a notion of textile splittings and amalgamations. A textile splitting (or amalgamation) of $T = (p, q : \Gamma \to G)$ will involve a splitting (or amalgamation) of Γ, extended appropriately. Γ determines the horizontal transition rules and thus, such operations will be able to decrease horizontal memory and anticipation. To decrease vertical memory and anticipation, we will first interchange the horizontal and vertical direction with an inversion and secondly apply textile splittings and amalgamations.

Let $T = (p, q : \Gamma \to G)$ be a textile system with $V_\Gamma = \{v_1, \ldots, v_n\}$, $E_\Gamma = \{\alpha_1, \ldots, \alpha_r\}$, and $E_G = \{\beta_1, \ldots, \beta_t\}$. We next define $\tilde{T} = (\tilde{p}, \tilde{q} : \tilde{\Gamma} \to G)$, the *textile out-splitting* of T. First, we out-split the graph Γ by refining the natural partition of E_Γ; for each $v \in V_\Gamma$, we partition $\mathcal{E}_v = \{\alpha \in E_\Gamma : i_\Gamma(\alpha) = v\}$ into subsets $\mathcal{E}_v^1, \ldots, \mathcal{E}_v^{m(v)}$. Then set

$$V_{\tilde{\Gamma}} = \left\{ v_1^1, \ldots, v_1^{m(v_1)}, v_2^1, \ldots, v_2^{m(v_2)}, \ldots, v_n^1, \ldots v_n^{m(v_n)} \right\}$$

and

$$E_{\tilde{\Gamma}} = \left\{ \alpha_1^1, \ldots, \alpha_1^{m(t_\Gamma(\alpha_1))}, \ldots, \alpha_r^1, \ldots, \alpha_r^{m(t_\Gamma(\alpha_r))} \right\}$$

with $i_{\tilde{\Gamma}}(\alpha_i^j) = i_\Gamma(\alpha_i)^l$ where $\alpha_i \in \mathcal{E}_{i_\Gamma(\alpha_i)}^l$ and $t_{\tilde{\Gamma}}(\alpha_i^j) = t_\Gamma(\alpha_i)^j$. This new graph is denoted $\tilde{\Gamma}$. Next, extend p and q by setting $\tilde{p}(\alpha_i^j) = p(\alpha_i)$ and $\tilde{q}(\alpha_i^j) = q(\alpha_i)$ for all $\alpha_i^j \in E_{\tilde{\Gamma}}$.

In a similar way we can construct the *textile in-splitting* T' of T by refining the partition of incoming edges.

In the following two lemmas we verify that $\tilde{T}$ is indeed a textile system and that $X_{\tilde{T}} \cong X_T$.

Lemma 3.1. $\tilde{T} = (\tilde{p}, \tilde{q} : \tilde{\Gamma} \to G)$ *is a textile system.*

Proof. Clearly $\tilde{\Gamma}$ is essential if Γ is, and $\tilde{p}$, $\tilde{q}$ are graph homomorphisms. It remains only to check that the quadruple $\left(i_{\tilde{\Gamma}}(\alpha_i^j), t_{\tilde{\Gamma}}(\alpha_i^j), \tilde{p}(\alpha_i^j), \tilde{q}(\alpha_i^j) \right)$ uniquely determines $\alpha_i^j \in E_{\tilde{\Gamma}}$. Suppose that $\alpha_i^j \neq \alpha_k^l$ and consider their respective quadruples.

Case 1. Suppose $i = k$. Thus $j \neq l$ and $t_{\tilde{\Gamma}}(\alpha_i^j) = t_\Gamma(\alpha_i)^j \neq t_\Gamma(\alpha_i)^l = t_{\tilde{\Gamma}}(\alpha_i^l)$ and the second entry will distinguish the two quadruples.

Case 2. Suppose $i \neq k$ and $\alpha_i \in \mathcal{E}_{i_\Gamma(\alpha_i)}^x$, $\alpha_k \in \mathcal{E}_{i_\Gamma(\alpha_k)}^z$. If $i_\Gamma(\alpha_i) \neq i_\Gamma(\alpha_k)$ or if $x \neq z$, then $i_{\tilde{\Gamma}}(\alpha_i^j) = i_\Gamma(\alpha_i)^x \neq i_\Gamma(\alpha_k)^z = i_{\tilde{\Gamma}}(\alpha_k^l)$ and the quadruples are distinguished by their first element. Similarly, if $t_\Gamma(\alpha_i) \neq t_\Gamma(\alpha_k)$ or $j \neq l$, the quadruples are distinguished by their second element. Finally, if $i_\Gamma(\alpha_i) = i_\Gamma(\alpha_j)$, $x = z$, $t_\Gamma(\alpha_i) = t_\Gamma(\alpha_k)$, and $j = l$, then because T is a textile system, we know $(i_\Gamma(\alpha_i), t_\Gamma(\alpha_i), p(\alpha_i), q(\alpha_i)) \neq (i_\Gamma(\alpha_k), t_\Gamma(\alpha_k), p(\alpha_k), q(\alpha_k))$. So either

$$\tilde{p}(\alpha_i^j) = p(\alpha_i) \neq p(\alpha_k) = \tilde{p}(\alpha_k^l) \quad \text{or} \quad \tilde{q}(\alpha_i^j) = q(\alpha_i) \neq q(\alpha_k) = \tilde{q}(\alpha_k^l).$$

□

Lemma 3.2. *Let $\tilde{T}$ be a textile out-splitting of T. Then*
$$(X_{\tilde{T}}, \sigma_1, \sigma_2) \cong (X_T, \sigma_1, \sigma_2).$$

Proof. Define a $((0,0),(0,0))$–block map $\Phi : X_{\tilde{T}} \to X_T$ via $\phi(\alpha_i^j) = \alpha_i$ and a $((0,1),(0,0))$–block map $\Psi : X_T \to X_{\tilde{T}}$ via $\psi(\alpha_i\alpha_j) = \alpha_i^k$, where $\alpha_j \in \mathcal{E}_{i_\Gamma(\alpha_j)}^k$.

It is not difficult to check that $\Phi(X_{\tilde{T}}) \subseteq X_T$ and $\Psi(X_T) \subseteq X_{\tilde{T}}$. It is clear that $\Phi(\Psi(x))_{(i,j)} = x_{(i,j)}$ for all $x \in X_T$ since adding and removing superscripts has no effect. Thus we only need check that $\Psi(\Phi(x))_{(i,j)} = x_{(i,j)}$ for all $x \in X_{\tilde{T}}$. Let $x \in X_{\tilde{T}}$. We need to show that

$$\Psi(\Phi(x))_{(i,j)} = \psi(\phi(x_{(i,j)})\phi(x_{(i+1,j)})) = x_{(i,j)}.$$

Suppose $x_{(i,j)} = \alpha_m^k$, $x_{(i+1,j)} = \alpha_n^l \in E_{\tilde{\Gamma}}$. Because $(x_{(i,j)})_{i \in Z} \in X_{\tilde{\Gamma}}$, $t_{\tilde{\Gamma}}(\alpha_m^k) = t_\Gamma(\alpha_m)^k = i_{\tilde{\Gamma}}(\alpha_n)^s = i_{\tilde{\Gamma}}(\alpha_n^l)$, where $\alpha_n \in \mathcal{E}_{i_\Gamma(\alpha_n)}^s$. Thus $t_\Gamma(\alpha_m) = i_\Gamma(\alpha_n)$, $k = s$ and $\alpha_n \in \mathcal{E}_{i_\Gamma(\alpha_n)}^k$. So

$$\psi\left(\phi(x_{(i,j)})\phi(x_{(i+1,j)})\right) = \psi\left(\phi(\alpha_m^k)\phi(\alpha_n^l)\right) = \psi(\alpha_m\alpha_n) = \alpha_m^k = x_{(i,j)}.$$

□

When the partition $\mathcal{P}$ consists of singleton sets then $\tilde{T}$ is called the *complete out-splitting* of T and, in general, if $\tilde{T}$ is a textile out-splitting of T then we call $\Phi : X_{\tilde{T}} \to X_T$ the *out-amalgamation code* from $X_{\tilde{T}}$ to X_T and we call $\Psi : X_T \to X_{\tilde{T}}$ the *out-splitting code* from X_T to $X_{\tilde{T}}$.

We can define a *textile in-splitting* T' of T analogously by using an in-splitting Γ' of Γ. We will then have a $((1,0),(0,0))$–block in-splitting code and a $((0,0),(0,0))$–block in-amalgamation code, both giving $X_{T'} \cong X_T$.

In the remainder of this section, we give a proof of the two dimensional decomposition theorem. In Lemma 3.8 we show that a higher block presentation of a textile

system is the composition of a finite sequence of textile splittings, amalgamations and inversions. Then, by moving to a higher block presentation if necessary, we may assume that the conjugacy φ between two textile systems is a $((0,0),(0,0))-$ block map with an $((m,n),(r,s))-$block inverse. If $m = n = r = s = 0$, then this conjugacy is just a relabeling of the symbols and as such is a trivial splitting. So we would like a way to reduce the horizontal and vertical memory and anticipation of the inverse of φ. We will reduce the horizontal memory and anticipation in Lemmas 3.3 and 3.4 by using textile splittings and amalgamations. Then, in Corollary 3.6, we interchange the vertical and horizontal directions using an inversion and reduce the vertical memory and anticipation, again using textile splittings and amalgamations.

Lemma 3.3. Let $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ and 2, be textile systems. Suppose $\varphi : X_{T_1} \to X_{T_2}$ is a $((0,0),(0,0))-$block conjugacy with an $((m,n),(r,s))-$block inverse. Then there are textile out-splittings $\tilde{T}_k$ of T_k such that the following diagram commutes:

$$
\begin{array}{ccc}
X_{T_1} & \xrightarrow{\;\varphi\;} & X_{T_2} \\
\Psi_1 \downarrow & & \downarrow \Psi_2 \\
X_{\tilde{T}_1} & \xrightarrow{\;\tilde{\varphi}\;} & X_{\tilde{T}_2}
\end{array}
$$

where Ψ_1, Ψ_2 are out-splittings and where $\tilde{\varphi}$ is a $((0,0),(0,0))-$block conjugacy with an $((m,n-1),(r,s))-$block inverse.

Proof. For $v \in V_{\Gamma_1}$, we partition $\mathcal{E}_v$ by the image of φ into $\bigcup_{a \in E_{\Gamma_1}} \{ b \in \mathcal{E}_v : \varphi(b) = \varphi(a) \}$. (In a slight abuse of notation, we are using φ to denote both the $((0,0),(0,0))-$block conjugacy between X_{T_1} and X_{T_2} and the map from symbols in E_{Γ_1} to symbols in E_{Γ_2} which determines this conjugacy.) Then we can denote

$$
V_{\tilde{\Gamma}_1} = \left\{ v^\alpha : v \in V_{\Gamma_1},\ \alpha \in E_{\Gamma_2} \text{ with } \varphi(a) = \alpha \text{ for some } a \in \mathcal{E}_v \right\},
$$

$$
E_{\tilde{\Gamma}_1} = \left\{ a^\beta : a \in E_{\Gamma_1} \text{ with } \varphi(b) = \beta \text{ for some } b \in E_{\Gamma_1} \text{ with } t_{\Gamma_1}(a) = i_{\Gamma_1}(b) \right\}
$$

and

$$
i_{\tilde{\Gamma}_1}\left(a^\beta\right) = i_{\Gamma_1}(a)^{\varphi(a)}, \qquad t_{\tilde{\Gamma}_1}\left(a^\beta\right) = t_{\Gamma_1}(a)^\beta.
$$

For $a^\beta \in E_{\tilde{\Gamma}_1}$, we define $\tilde{p}_1(a^\beta) = p_1(a)$. The homomorphism $\tilde{q}_1$ is defined similarly.

As shown in Lemmas 3.1 and 3.2, $\tilde{T}_1 = (\tilde{p}_1, \tilde{q}_1 : \tilde{\Gamma}_1 \to G_1)$ is a textile system and $\Psi_1 : X_{T_1} \to X_{\tilde{T}_1}$ is a conjugacy via $\Psi_1(x)_{(i,j)} = \psi_1\left(x_{(i,j)} x_{(i+1,j)}\right) = x_{(i,j)}^{\varphi(x_{(i+1,j)})}$.

Now let $\tilde{\Gamma}_2$ be a complete out-splitting of Γ_2. So

$$
V_{\tilde{\Gamma}_2} = \left\{ v^\alpha : v \in V_{\Gamma_2},\ \alpha \in A_{\Gamma_2} \text{ with } i_{\Gamma_2}(\alpha) = v \right\},
$$

$$
E_{\tilde{\Gamma}_2} = \left\{ \alpha^\beta : \alpha, \beta \in A_{\Gamma_2} \text{ with } t_{\Gamma_2}(\alpha) = i_{\Gamma_2}(\beta) \right\},
$$

and

$$
i_{\tilde{\Gamma}_2}\left(\alpha^\beta\right) = i_{\Gamma_2}(\alpha)^\alpha, \qquad t_{\tilde{\Gamma}_2}\left(\alpha^\beta\right) = t_{\Gamma_2}(\alpha)^\beta.
$$

For $\alpha^\beta \in E_{\tilde{\Gamma}_2}$, we define $\tilde{p}_2(\alpha^\beta) = p_2(\alpha)$. The homomorphism $\tilde{q}_2$ is defined similarly.
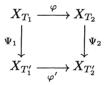
As shown in Lemmas 3.1 and 3.2 , $\tilde{T}_2 = (\tilde{p}_2, \tilde{q}_2 : \tilde{\Gamma}_2 \to G_2)$ is a textile system and $\Psi_2 : X_{T_2} \to X_{\tilde{T}_2}$ is a conjugacy via $\Psi_2(x)_{(i,j)} = \psi_2 \left(x_{(i,j)} x_{(i+1,j)} \right) = x_{(i,j)}^{\,x_{(i+1,j)}}$.

Now we define the $((0,0),(0,0))-$ block map $\tilde{\varphi} : X_{\tilde{T}_1} \to X_{\tilde{T}_2}$ via

$$\tilde{\varphi}(x)_{(i,j)} = \tilde{\varphi}\left(x_{(i,j)}^{\varphi(x_{(i+1,j)})} \right) = \varphi(x_{(i,j)})^{\varphi(x_{(i+1,j)})}.$$

Clearly the diagram commutes and thus $\tilde{\varphi}$ is one-to-one and onto. It remains only to check that $\tilde{\varphi}^{-1} = \psi_1 \circ \varphi^{-1} \circ \psi_2^{-1}$ is a $((m, n-1), (r, s))-$ block map. That is, we must show that for any $x \in X_{\tilde{T}_2}$, the coordinates in an $((m, n-1), (r, s))-$ block about $x_{(0,0)}$ determine $\tilde{\varphi}^{-1}(x)_{(0,0)}$. But this follows from the observation that a $((m, n-1), (r, s))-$ block about $x_{(0,0)}$ determines both $\varphi(x_{(1,0)})$ and a $((m, n), (r, s))-$ block about $\psi_2^{-1}(x)_{(0,0)}$. $\qquad\square$

The proof of this lemma is similar to the proof of an analogous one-dimensional result. See [LM95], Lemma 7.1.3.

Lemma 3.4. *Let $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ and 2, be textile systems. Suppose $\varphi : X_{T_1} \to X_{T_2}$ is a $((0,0),(0,0))-$block conjugacy with an $((m,n),(r,s))-$ block inverse. Then there are textile in-splittings T'_k of T_k such that the following diagram commutes:*

$$
\begin{array}{ccc}
X_{T_1} & \xrightarrow{\ \varphi\ } & X_{T_2} \\
{\scriptstyle \Psi_1}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_2} \\
X_{T'_1} & \xrightarrow[\ \varphi'\ ]{} & X_{T'_2}
\end{array}
$$

where Ψ_1, Ψ_2 are in-splittings and where φ' is a $((0,0),(0,0))-$ block conjugacy with an $((m-1,n),(r,s))-$ block inverse.

Proof. The proof of Lemma 3.4 is analogous to the proof of Lemma 3.3. $\qquad\square$

Corollary 3.5. *Let $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ and 2, be textile systems. Suppose $\varphi : X_{T_1} \to X_{T_2}$ is a $((0,0),(0,0))-$block conjugacy with an $((m,n),(r,s))-$ block inverse. Then there are textile systems $\tilde{T}_k$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
X_{T_1} & \xrightarrow{\ \varphi\ } & X_{T_2} \\
{\scriptstyle \tilde{\eta}_1}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\eta}_2} \\
X_{\tilde{T}_1} & \xrightarrow[\ \tilde{\varphi}\ ]{} & X_{\tilde{T}_2}
\end{array}
$$

where the $\tilde{\eta}_k$'s are the composition of a finite collection of in- and out-splittings and $\tilde{\varphi}$ is a $((0,0),(0,0))-$block conjugacy with an $((0,0),(r,s))-$block inverse.
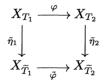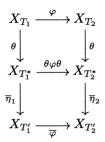
Corollary 3.6. *Let $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ and 2, be textile systems. Suppose $\varphi : X_{T_1} \to X_{T_2}$ is a $((0,0),(0,0))-$block conjugacy with an $((m,n),(r,s))-$ block inverse. Then there are textile systems T'_k such that the following diagram*

commutes:

$$
\begin{array}{ccc}
X_{T_1} & \xrightarrow{\ \varphi\ } & X_{T_2} \\
\Big\downarrow{\scriptstyle\theta} & & \Big\downarrow{\scriptstyle\theta} \\
X_{T_1^*} & \xrightarrow{\ \theta\varphi\theta\ } & X_{T_2^*} \\
\Big\downarrow{\scriptstyle\bar{\eta}_1} & & \Big\downarrow{\scriptstyle\bar{\eta}_2} \\
X_{T_1'} & \xrightarrow[\ \bar{\varphi}\ ]{} & X_{T_2'}
\end{array}
$$

where the $\bar{\eta}_k$'s are the composition of a finite collection of in- and out-splittings and $\bar{\varphi}$ is a $((0,0),(0,0))-$ block conjugacy with an $((0,0),(m,n))-$ block inverse.

Proof. We see that because φ is a $((0,0),(0,0))-$ block conjugacy with an $((m,n),$ $(r,s))-$ block inverse, $\theta\circ\varphi\circ\theta$ is a $((0,0),(0,0))-$ block conjugacy and $(\theta\circ\varphi\circ\theta)^{-1} = \theta\circ\varphi^{-1}\circ\theta$ is an $((r,s),(m,n))-$ block map. Thus Corollary 3.6 follows from Corollary 3.5. $\qquad\square$

Proposition 3.7. *Let $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ and 2, be textile systems. Suppose $\varphi : X_{T_1} \to X_{T_2}$ is a $((0,0),(0,0))-block$ conjugacy with an $((m,n),(r,s))-$ block inverse. Then φ is the composition of a finite sequence of textile splittings, amalgamations and shift inversions.*

Proof. Use Corollaries 3.6 and 3.5 to obtain $\varphi = \tilde{\eta}_2^{-1}\circ\theta\circ\bar{\eta}_2^{-1}\circ\bar{\varphi}\circ\bar{\eta}_1\circ\theta\circ\tilde{\eta}_1$, where $\bar{\varphi}$ is a $((0,0),(0,0))-$block conjugacy with a $((0,0),(0,0))-$block inverse, that is, a relabeling. $\qquad\square$

   In general a conjugacy between shift spaces is a $((m,n),(r,s))-$block map but the following lemma shows that by moving to a higher block presentation we may assume it is a $((0,0),(0,0))-$block map.

Lemma 3.8. *Let $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ and 2, be textile systems. Let $\varphi : X_{T_1} \to X_{T_2}$ be a $((m,n),(r,s))-block$ conjugacy and let $\hat{X}_{T_1}$ be the $((m,n),(r,s))-$ higher block presentation of X_{T_1}. Then there exists a map $\eta : X_{T_1} \to \hat{X}_{T_1}$ which is a sequence of splittings and inversions, such that $\varphi\circ\eta^{-1}$ is a $((0,0),(0,0))-block$ conjugacy.*

Proof. Clearly, $\varphi\circ\eta^{-1}$ is a $((0,0),(0,0))-$block conjugacy. We need to show that η is a sequence of splittings and inversions.
   First note that $\hat{X}_{T_1}$ can be written as a textile system, using the technique of Theorem 2.1, with $((m,n),(r,s))-$blocks playing the role of the $((0,1),(0,1))-$blocks used there.
   A complete out-splitting of a textile system yields a higher block presentation by

$$
\Psi(x)_{(i,j)} = x_{(i,j)}^{x(i+1,j)}.
$$

Similarly, a complete in-splitting gives

$$
\Psi(x)_{(i,j)} = {}^{x(i-1,j)}x_{(i,j)}.
$$

Thus we can find Ψ_1, a sequence of splittings, such that $\eta\circ\Psi_1^{-1}$ is a $((0,0),(r,s))-$block conjugacy. Then $\theta\circ\eta\circ\Psi_1^{-1}\circ\theta$ is an $((r,s),(0,0))-$block conjugacy on the dual spaces and the process can be repeated to find Ψ_2, a sequence of splittings, so

that $\theta \circ \eta \circ \Psi_1^{-1} \circ \theta \circ \Psi_2^{-1}$ is a $((0,0),(0,0))$−block conjugacy, or simply a relabeling. Let us denote this by $\bar{\eta}$. Then $\eta = \theta \circ \bar{\eta} \circ \Psi_2 \circ \theta \circ \Psi_1$, and we have the necessary result. $\square$

We are now ready to state the main result of this section, a two-dimensional decomposition theorem:

Theorem 3.9. *Let* $T_k = (p_k, q_k : \Gamma_k \to G_k)$, $k = 1$ *and* 2, *be textile systems. Every conjugacy between* X_{T_1} *and* X_{T_2} *is the composition of a finite sequence of textile splittings, amalgamations and shift inversions.*

Proof. This theorem follows from Proposition 3.7 and Lemma 3.8. $\square$

By Proposition 2.3, we can view a two-dimensional SFT as a textile system and thus apply the last theorem to that scenario. We restate the result as follows:

Corollary 3.10. *Given two two-dimensional SFTs,* $X = X(A_1, A_2)$ *and* $Y = Y(B_1, B_2)$, *any conjugacy between them is the composition of a finite sequence of textile splittings, amalgamations and shift inversions.*

In one dimension, the Decomposition Theorem for edge shift conjugacies results in the notion of strong shift equivalence for matrices; a splitting or an amalgamation is described by a matrix condition and the Decomposition Theorem allows us to deduce that two edge shifts are equivalent if and only if their adjacency matrices are related by a sequence of these matrix conditions. It is possible to find matrix conditions for textile splittings and amalgamations. For example, suppose (A_1, A_2) $((A_1', A_2'))$ are the transition matrices for the edge graphs Γ and Γ^T in textile system T (T'). Then T' is an outsplitting of T if and only if there exist matrices R and S with $A_1 = RS$, $A_1' = SR$, and $R^T A_2 R = A_2'$ where R is an "enhanced identity" matrix, that is an identity matrix with some columns repeated. The matrix condition for out amalgamations has the roles of A_1 and A_1' reversed; this leads to difficulty in defining a symmetric matrix relation. Another difficulty lies in the fact that edge labelings play a more fundamental role in two dimensions thus requiring the use of 0-1 transition matrices to determine SFTs. These matrices are not closed under matrix operations. These difficulties remain to be surmounted and we are left with the open question: can Corollary 3.10 be used to classify conjugacies between two-dimensional SFTs using a matrix condition?

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