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Poisson Distributed Individuals Control Charts with Optimal Limits

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Abstract

The conventional method used in attribute control charts is the Shewhart three sigma limits. The implicit assumption of the Normal distribution in this approach is not appropriate for skewed distributions such as Poisson, Geometric and Negative Binomial. Normal approximations perform poorly in the tail area of the these distributions. In this research, a type of attribute control chart is introduced to monitor the processes that provide count data. The economic objective of this chart is to minimize the cost of its errors which is determined by the designer. This objective is a linear function of type I and II errors. The proposed control chart can be applied to Poisson, Geometric and Negative Binomial as the underlying distribution of count data. Control limits in this chart is calculated optimally since it is based on the probability distribution of the data and can detect a directional shift in the process rate. Some numerical results for the optimal design of the proposed control chart are provided. The expected cost of the control chart is compared to that of a one sided c chart. The effects of changing the available parameters on the cost, errors and the optimal limits of the proposed control chart are shown graphically.

Acknowledgments

I would like to express my gratitude and thanks to my advisor Dr. Chimka for the useful comments, remarks and engagement through completing this master thesis. I would also like to thank my committee members, Dr. Cassady and Dr. Rainwater for serving as my committee members and for their brilliant comments and suggestions.

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Chapter 1

Introduction

Statistical Process Control (SPC) as a subset of Statistical Quality Control (SQC) consists of tools and method to monitor, control and improve the products in manufacturing processes. Monitoring the process in manufacturing is required to ensure that it operates properly. One of the most important tools used in SPC is control chart. There are always variations in production due to common and special causes and control chart is the effective way used to inspect this variation and continually monitor the production process. One type of control chart is attribute chart which is used to monitor the processes that provide discrete or count data.

1.1 Research Background

The conventional method used in attribute control charts is c chart with three sigma limits which was proposed by Shewhart for the first time and has been widely applied for monitoring the manufacturing processes that provide count data. Since Poisson distribution is generally used to model the count data, c chart follows the assumption of the Poisson distribution for defect occurrence in a sample of constant size. Suppose that nonconformities or defects occur according to Poisson distribution in constant size samples; that is,

$$Pr(X = x) = \frac{e^{-\lambda} \times \lambda^x}{x!} \quad , \quad x = 0, 1, 2, ... \quad , \quad \lambda > 0 \tag{1.1}$$

Where λ is the parameter of the Poisson distribution and x is the number of defects in the sample. Since the mean and variance of the Poisson distribution are the same, the Upper Control Limit (UCL) and Lower Control Limit (LCL) with three sigma in the classical control chart are defined as follows,

$$UCL = \lambda + 3 \times \sqrt{\lambda} \tag{1.2}$$

$$CL = \lambda \tag{1.3}$$

$$LCL = \lambda - 3 \times \sqrt{\lambda} \tag{1.4}$$

When lower control limit is negative, set LCL $= 0$. In this control chart λ is a known parameter; otherwise, it may be estimated from a preliminary sample taken from the output of the process when it was in control, say $\bar{\lambda}$. On the other hand if the number of units in each sample is variable during the inspection, a U chart would be appropriate in which the average number of nonconformities per inspection unit would be compared to its control limits.

1.2 Research Motivation

The problem with this approach is that there is an implicit assumption of Normal distribution for data which is not appropriate to the skewed distributions such as Poisson. It means that the probability of defining an in control process as an out of control is not equally allocated in the tail area of the control chart. Normal approximation performs poorly in the tail area of the Poisson distribution and also when its parameter is small.

Another problem is that c chart is only intended to decrease the type I error to an specific amount and it does not consider a relative value to type II error in case of a change in the process.

Figure 1.1 shows an instance of Poisson probability distribution with $\lambda = 5$. This figure shows that Poisson distribution is not symmetric as it is in normal distribution, and any symmetric control chart such as c chart creates unequal tail area; therefore, the classical control chart cannot not be an efficient way to monitor this distribution.

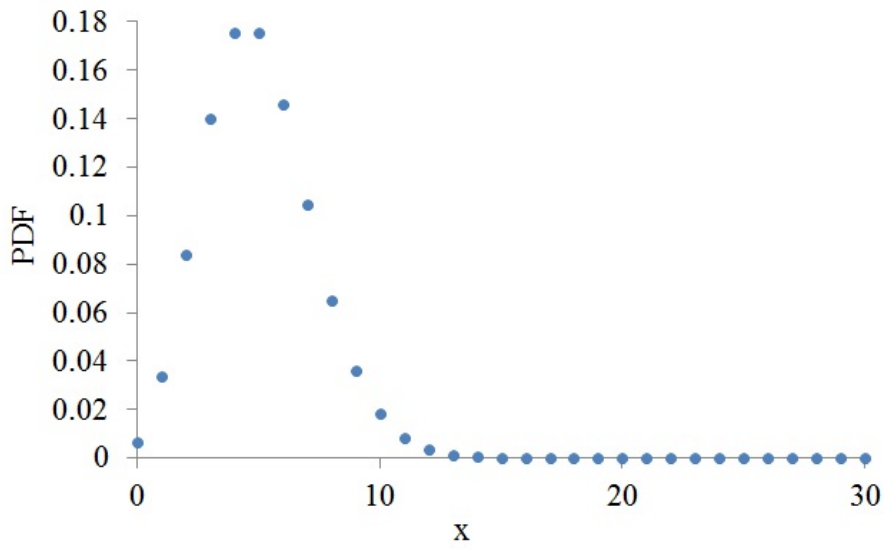

Figure 1.1: Poisson Probability Distribution

1.3 Research Objectives

The objective in this research is to create a attribute control chart that:

1. improves manufacturing processes that provide count data.
2. can be applied to Poisson count data.
3. causes the least cost from the view of the user.
4. has the capability to detect deterioration and improvement in the process quality.
5. can be applied to high quality processes where the nominal proportion of nonconformities is very small.

1.4 Research Contributions

The new approach proposed in this study is an optimal economic statistical alternative control chart to the conventional c chart for monitoring and improving the manufacturing processes, which

satisfies the economic requirement of the user in a way that the cost is assumed to be a linear function of type I and II errors. The control charts in this study are intended to minimize the cost of decision about the process whether it is in control or out of control. The firm will be able to use this approach to detect any shifts in its manufacturing process.

The proposed approach in this research will have the least possible cost that is charged to the users since this approach has optimal characteristic with its assumptions. On the other hand this approach has the capability to detect the downward shift in the processes that has nonconforming items to take action toward beneficial changes.

In contrast to normal approximation used in traditional c chart, research described here uses the exact probability of statistic distribution to design the control chart to monitor and improve the process.

Chapter 2

Literature Review

There have been numerous papers in the area of control charts. In this chapter several basic approaches will be discussed.

2.1 Shewhart Control Charts

This control chart was first proposed by Shewhart [26] [25]. Shewhart control charts are used to determine whether a process is in statistical control or not and to maintain the current status. The control chart uses 3-sigma limits as its limits that justify a fixed amount of type I error. This constant is on the assumption of that the data's distribution is normal and this causes a drawback for the control chart when the actual distribution is badly skewed since it may indicate lack of control when it actually is in control. So the normal approximation in 3-sigma cannot provide efficient control limits in this chart (Ryan [22], Woodall [32], Acosta-mejia [1]). C chart is one of them that is used to monitor the processes that provides count data and usually uses Poisson as the datas distribution which is a skewed distribution.

There have been many approaches developed by researchers to improve the attribute control chart by reducing false alarm rate and increase the power of attribute control chart especially in the case of the Poisson distribution. One approach has been introduced by Quesenberry [18] [17] that first standardizes the data and then plot them in -3 and 3 control limits. Another approach which is applied to data to acquire symmetry is the transforming approach. Ryan [22] presented different kinds of transformation in his book as well as tsia et al. [29]. Tsai et al. [29] proposed a square root transformation to improve the attribute control chart for both binomial and Poisson data in a way that the probability of a false alarm in this chart is close to nominal values. This approach converts the raw data to a form that approximately has normal distribution.

Other researchers suggested optimal control chart procedures for count data that produce a tail

area close to the nominal tail area. Ryan and Schwertman [21] proposed the optimal control limit that provides certain amount of type I error. They presented a regression model that can determine the limits. Kittlitz [9] presented an almost exact control limits that has nearly equal tail area for Poisson distribution.

Another problem with Shewhart attribute control charts is that it sometimes gives a negative lower control limits which cannot detect improvement in the process. Lucas et al. [13] developed a modified c chart that can signal an improvement based on the number of samples that has zero defects.

Some studies worked on the need for a general control chart for attribute data. Sellers [24] developed such control chart that can be applied to Poisson, Geometric and binomial distribution. Shore [27] introduced a general approach to attribute control chart that considers mean, standard deviation and the skewness of the underlying distribution. A method for detecting shifts in Poisson process considering variable sample size developed by Mei et al. [14]

2.2 Cumulative Sum

One of the most applicable forms of control charts is CUSUM or Cumulative Sum Control Charts. This control chart is first developed by Page [16]. The control chart uses all the information since the last action taken and has the ability to detect changes in the underlying parameter in two directions. Page argued that this control chart is more sensitive to small shifts than shewhart control chart. The main difference between CUSUM control chart and shewhart control chart is that it does not use only a single observation or a single sample. Both the c chart and the u chart are two procedures that use the information about the most recent sample to monitor the process to determine whether the process is out of control or not.

Many studies have been done in the area of CUSUM charts. Lucas [12] suggested a CUSUM procedure to monitor count data that can detect both upward and downward shift in the count level. The author also designed two modified CUSUM schemes that can detect the changes in count level faster than the simple CUSUM. Riaz et al. [19] proposed a CUSUM control chart with

runs rules and compared their control chart with simple CUSUM, WCUSUM, EWMA and Fast initial response CUSUM and showed that it has better performance in detecting small shift while maintaining its efficiency for detecting larger shifts. Yashchin [35] proposed a weighted CUSUM procedure (a generalized CUSUM) and asserted that it has better run length than classical CUSUM. Jiang et al. [7] developed a weighted CUSUM chart to monitor inhomogeneous Poisson process and detect different range of shifts when the sample size is not constant over time. This control chart can be applied to different distributions like the study done by Chen et al. [4] which used Geometric Poisson as the underlying distribution.

2.3 Exponentially Weighted Moving Average

Exponentially Weighted Moving Average or EWMA is one type of control chart to monitor attribute data using all the information from the past. Many reserachers have studied this control chart in the area of quality control. Borror et al. [3] introduced an EWMA chart that considered a weighted average of past informationto the case of Poisson data. They evaluated their proposed control chart by using Markov chain approximation and argued that it has better ARL's than c chart. It also can detect downward shift since the lower limit is positive. Dong et al. [5] studied the EWMA control chart to detect an increase in incident rate and evaluated it by finding the probability of successful detection and ARL's. Weiß [30] proposed a one sided s EWMA to the dependent Poisson data and compared its ARL's with that of c chart and one sided CUSUM. Zhou et al. [37] developed a EWMA that accounts varying sample size and asserted that this control chart is more efficient than the classical EWMA. Shu et al. [28] applied the EWMA control chart to Poisson data to detect an increase in rate in their and used Markov chain to analyzed its performance. Khoo [8] suggested a Poisson moving average control chart to monitor count data.The author compared its performance to the c chart by evaluating its average run length and showed that it has smaller out of control ARL.

Other researchers studied the comparison between these control charts based on factors such as ARL. Ryan and Woodall [20] evaluated CUSUM and EWMA based on their Average Run Length

(ARL) with a simulation study, when the sample size varies. Han et al. [6] studied and compared the performance of CUSUM and EWMA for count data in their paper. White et al. [31] also presented a comparison of the c chart and CUSUM procedure based on ARL. They showed that CUSUM chart provided faster detection of a shift while maintaining a larger in-control ARL.

2.4 Geometric and Negative Binomial

There are other control charting procedures that are based on the number of conforming items found between nonconformities. Liu et al. [11] used the term ”Time Between Event” for this type of control chart. Cumulative Count of Conforming (CCC-r) control chart is another term used for this control chart which represents the number of units until rth nonconformities is found. These methods are usually applied to high quality processes where the proportion of nonconforming items is very small. Geometric or in the general case Negative Binomial are the two underlying distributions used. The traditional approach in this case is to use p chart which is based on the number of nonconformities in a unit sample which is not appropriate to small proportions. There have been many studies done in this area. Albers [2] analyzed this type of control chart and argued that as the proportion p increases Negative Binomial chart has better performance than Geometric one. Schwertman [23] expanded the work done by Xie et al. [33], which provided the control limits for different r, and also evaluated the ARL's to detect a change in process. Yang et al. [34] investigated the effect of sample size in a Geometric control chart when the proportion p is estimated. Lai and Govindaraju [10] showed how to reduce the variability in false and correct alarm rate in the design of control chart for high quality processes high quality. Ohta et al. [15] presented an optimal method to design a CCC-r chart from an economic view that can detect a positive shift in p and maximize the profit in each cycle. Yeh et al. [36] Proposed a EWMA control chart based a non transformed geometric count to monitor high quality processes.

Chapter 3

Methodology

In this chapter, the methodology used to design the proposed control chart will be explained in detail for three distributions: Poisson, Geometric and Negative Binomial.

3.1 Upward Shifts

Suppose the cost of the control chart for nonconformities, following a Poisson distribution, is a linear function of two types of errors; that is

$$Cost = z \times Type \text{ I } error + Type \text{ II } error \qquad , \qquad z > 0 \tag{3.1}$$

z denotes the relative cost of type I error to type II error. This function is true for every combination of Poisson mean and size of anticipated shift. This paper assumes that there is no LCL and UCL for upward shift and downward shift, respectively. The null hypothesis cannot be rejected because of small and large numbers of defects in an upward shift and downward shift, respectively. The process is considered out of control when the number of defects is greater than UCL for upward shift and less than LCL for downward shift.

Cost in Equation 3.1 is formulated as follows, where there is an upward shift assuming no lower control limit.

$$Cost(UCL = u) = z \times \left(1 - \sum_{x=0}^{u} \frac{e^{-\lambda_1} \times \lambda_1^x}{x!}\right) + \left(\sum_{x=0}^{u} \frac{e^{-\lambda_2} \times \lambda_2^x}{x!}\right) \tag{3.2}$$

Where λ_1 is the predefined mean of the Poisson distribution, and λ_2 is the actual mean considering an upward shift. Figure 3.1 shows an instance of cost versus upper control limit for an upward shift ($\lambda_1 = 5$, $\lambda_2 = 10$, $z = 1$).

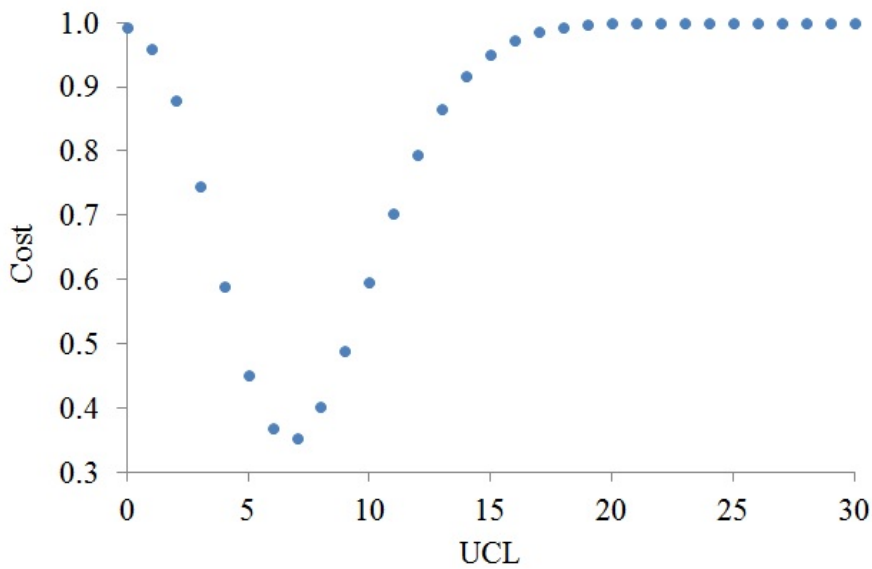

Figure 3.1: Cost versus Upper Control Limit

It can be observed from Figure 3.1 that cost of the control has decreasing trend to a point (say UCL*) before it starts to increase; therefore, it can be concluded that the difference between the costs of UCL and UCL-1 in which UCL $\leq$ UCL* are negative, and the difference between the costs of UCL and UCL-1 in which UCL$^* \leq$ UCL-1 are positive. Considering the objective, UCL* should be found to minimize cost of the control chart. The cost difference between any UCL and UCL-1 is represented in Equation 3.3.

$$CostDifference(X = x) = Cost(x) - Cost(x-1) = -z \times \left(\frac{e^{-\lambda_1} \times \lambda_1^x}{x!} \right) + \left(\frac{e^{-\lambda_2} \times \lambda_2^x}{x!} \right) \tag{3.3}$$

Figure 3.2 illustrates function CostDifference versus x for the instance of Figure 3.1 ($\lambda_1 = 5$, $\lambda_2 = 10$, $z = 1$).

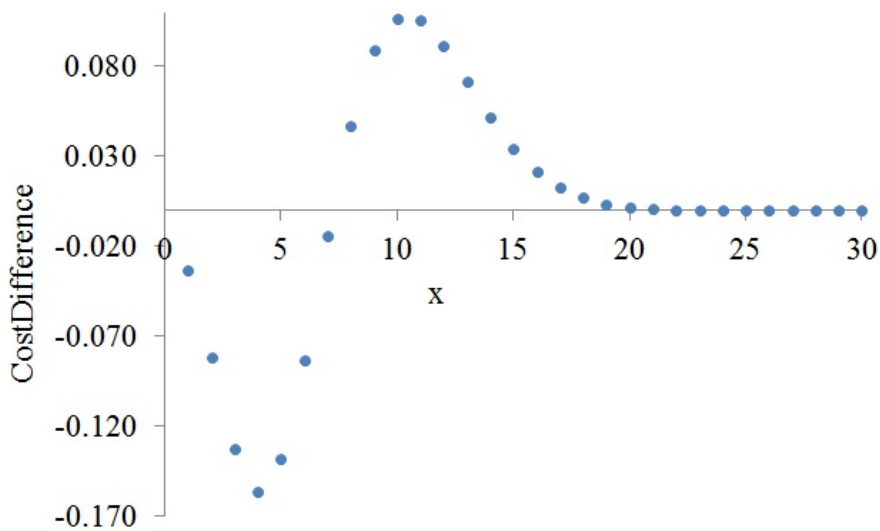

Figure 3.2: CostDifference versus x

As shown in Figure 3.2, for every x $\leq$ UCL* the value of CostDifference is negative and for all other x the value of CostDifference is positive. The main objective requires the greatest x that has a negative value in the function CostDifference. In order to do that, the following equation should be solved.

$$CostDifference(x) = 0 \tag{3.4}$$

The details to solve the above equation are shown in the Appendix 1. The solution to the Equation 3.4 is

$$X = \frac{ln(z) + \lambda_2 - \lambda_1}{ln(\lambda_2) - ln(\lambda_1)} \tag{3.5}$$

Since all the components of the above fraction are unique numbers, we can be sure that there is only one number which makes the function CostDifference equal to zero and there is no possibility of having another number that can satisfies Equation 3.4. This result demonstrates that the behavior of the graph in Figure 3.1 is true for all combinations of λ_1 and λ_2 ; there is a decreasing trend on

one side and an increasing trend on the other side of the curve separated by UCL*.

X in the above equation is not necessarily an integer, so it should be converted to the largest integer which has a negative value in function CostDifference. The final and the optimal UCL is

$$UCL^* = Floor\left(\frac{ln(z)+\lambda_2-\lambda_1}{ln(\lambda_2)-ln(\lambda_1)}\right) \tag{3.6}$$

And whenever the value for X is not nonnegative, it means there is no Upper Control Limit for this instance, and any number of defects is ;therefore, the process is considered out of control.

3.2 Downward Sifts

The general approch to find the limits of teh control chart to detect a downward shift is the same as upward shift. Note that there is no upper control limit in downward control chart. The equivalent cost for downward shift is

$$Cost(LCL=l) = z\times\left(\sum_{x=0}^{l-1}\frac{e^{-\lambda_1}\times\lambda_1^x}{x!}\right)+\left(1-\sum_{x=0}^{l-1}\frac{e^{-\lambda_2}\times\lambda_2^x}{x!}\right) \tag{3.7}$$

Figure 3.3 shows an instance of cost versus lower control limit for a downward shift ($\lambda_1 = 10$, $\lambda_2 = 5$, $z = 1$) .

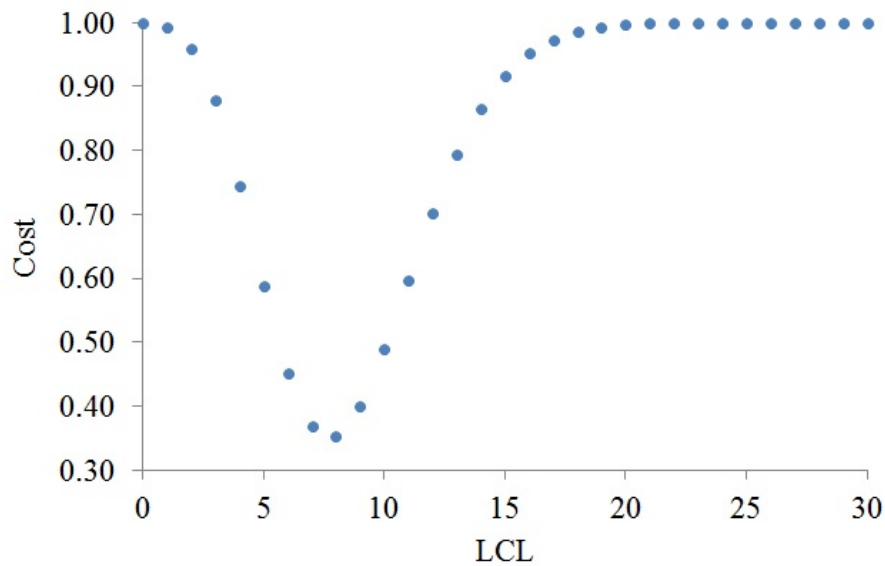

Figure 3.3: Cost versus Lower Control Limit

The cost difference between any LCL and LCL+1 is

$$CostDifference(X = x) = Cost(x) - Cost(x+1) = -z \times \left(\frac{e^{-\lambda_1} \times \lambda_1^x}{x!} \right) + \left(\frac{e^{-\lambda_2} \times \lambda_2^x}{x!} \right) \tag{3.8}$$

Figure 3.4 represents CostDifference versus x for the instance of Figure 3.3 ($\lambda_1 = 10$, $\lambda_2 = 5$, $z = 1$).

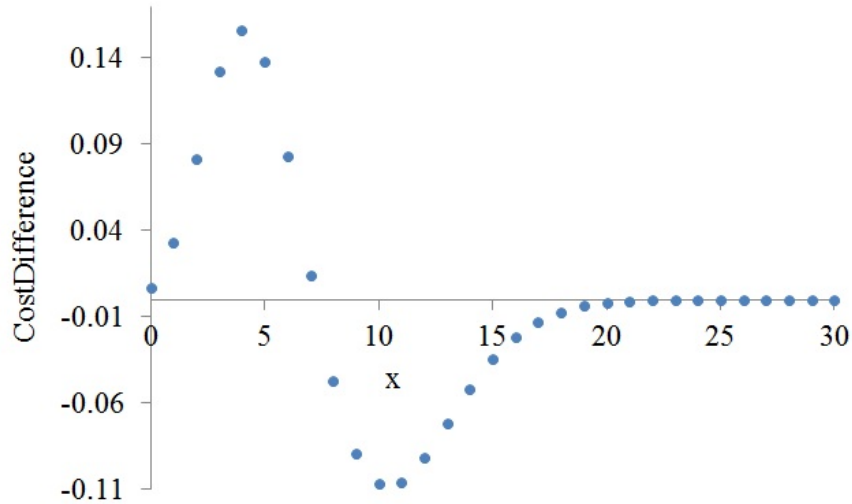

Figure 3.4: CostDifference versus x

It can be seen from the above figure that for every x$\geq$ UCL* the value of CostDifference is negative and for all other x's the value of CostDifference is positive. In order to minimize the cost, the smallest x should be found which has a negative value in the function CostDifference. To find this x, the following equation should be solved like in upward shift.

$$CostDifference(x) = 0 \tag{3.9}$$

The details to calculate the x which solves the above equation are shown in the Appendix 2. The result of this equation is

$$X = \frac{\lambda_1 - \lambda_2 - ln(z)}{ln(\lambda_1) - ln(\lambda_2)} \tag{3.10}$$

Again x in the above equation is not necessarily an integer number, so it should be converted to the smallest integer number which has a negative value in function CostDifference. The final and

the optimal LCL is

$$LCL^* = Ceiling\left(\frac{\lambda_1 - \lambda_2 - ln(z)}{ln(\lambda_1) - ln(\lambda_2)}\right) \tag{3.11}$$

Whenever the value for X is negative, it means that there is no Lower Control Limit for this instance, and any number of defects is acceptable; therefore, the process is considered in control. In the optimal bounds in Equations 3.6 and 3.10 that are shown below,

- $UCL^* = Floor\left(\frac{ln(z) + \lambda_2 - \lambda_1}{ln(\lambda_2) - ln(\lambda_1)}\right)$
- $LCL^* = Ceiling\left(\frac{\lambda_1 - \lambda_2 - ln(z)}{ln(\lambda_1) - ln(\lambda_2)}\right)$

Increasing z implicitly values type I error rather than type II error, so in order to minimize the cost, the optimal upper control limit should be increased in upward shift, and the optimal lower control limit should be decreased in downward shift, both of which can be obtained from their specific optimal bound. The optimal bounds have the appropriate behavior toward any changes in z value in upward and downward shifts, respectively.

Combining downward and upward shift in designing the control chart:

Suppose we are interested to design the proposed control chart that can detect an upward shift of size x_1 and a downward shift of size x_2 simultaneously with least cost based on the customized objective that the user defines. This objective is

$$Cost = z \times Type\ \text{I}\ error + Type\ \text{II}\ error \qquad , \qquad z > 0 \tag{3.12}$$

The second part of the objective, type II error, needs to know the alternative state of the process and it should be unique, so we cannot consider two shifts at the same time to design this control chart but We can assign probability to the alternative states to find the type II error and design it.

In this condition the objective would be

$$Cost = z \times Type\ \mathrm{I}\ error + \alpha \times Type\ \mathrm{II}\ error(\lambda + x_1) + (1-\alpha) \times TypeIIerror(\lambda - x_2) \tag{3.13}$$

in which z> 0 , $0 < \alpha < 1$, $x_1 > 0$ and $x_2 > 0$. Or in another way $Cost(UCL = u, LCL = l) =$

$$z \times \left(1 - \sum_{x=l}^{u} \frac{e^{-\lambda} \times \lambda^x}{x!}\right) + \alpha \times \left(\sum_{x=l}^{u} \frac{e^{-(\lambda+x_1)} \times (\lambda + x_1)^x}{x!}\right) + (1-\alpha) \times \left(\sum_{x=l}^{u} \frac{e^{(\lambda - x_2)} \times (\lambda - x_2)^x}{x!}\right) \tag{3.14}$$

Figure 3.5 shows how cost varies for different LCL and UCL for an instance with $\lambda = 10$, z = 1 , $x_1 = x_2 = 5$. You can see that there is one local minimum for this instance and the software *Mathematica* can find the exact value for the optimal UCL and LCL but I could not find a general expression to find the minimum point.

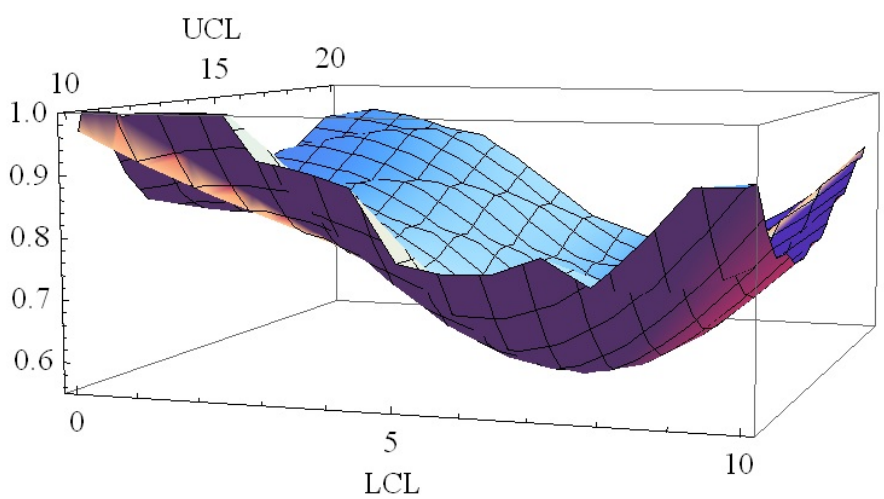

Figure 3.5: Cost versus UCL and LCL

3.3 Geometric

The previous sections were based on the assumption of Poisson distribution. But there are other cases where this assumption is violated and defects occur based on a Geometric distribution.

High quality processes is an example of this case where the nominal proportion of non-conforming units are quiet small. In such processes there is a large number of conforming items between two consecutive nonconformities and the decision is based on whether this number is small enough to say that the process is out of control or not. These processes should also have the capability of inspecting the units sequentially instead of inspecting them as a batch which most of the modern manufactures do. This section focuses on low failure rate which is common in health care monitoring as well as high quality processes.

Suppose that the probability of an item to be nonconforming is equal to p, then the probability of getting the nonconforming item after x conforming ones is

$$Pr(X = x) = p(1-p)^x \qquad , \qquad x = 0, 1, 2, 3, ... \tag{3.15}$$

The objective of the control chart is the same as the previous sections which is

$$Cost = z \times Type \text{ I } error + Type \text{ II } error \qquad z > 0 \tag{3.16}$$

Upward shift (deterioration) in this section means that the probability p increases and the number of conforming items between two consecutive nonconformities decreases. There is no UCL in the control chart and the objective is formulated as

$$Cost(LCL = l) = z \times \left(\sum_{x=0}^{l-1} p_1(1-p_1)^x - 1 \right) + \left(1 - \sum_{x=0}^{l-1} p_2(1-p_2)^x - 1 \right) \tag{3.17}$$

where p_1 is the predefined rate of nonconformities and p_2 is the actual rate considering an upward shift.

Figure 3.6 shows an instance of cost versus lower control limit for an upward shift ($p_1 = 0.1$, $p_2 = 0.15$, $z = 1$).

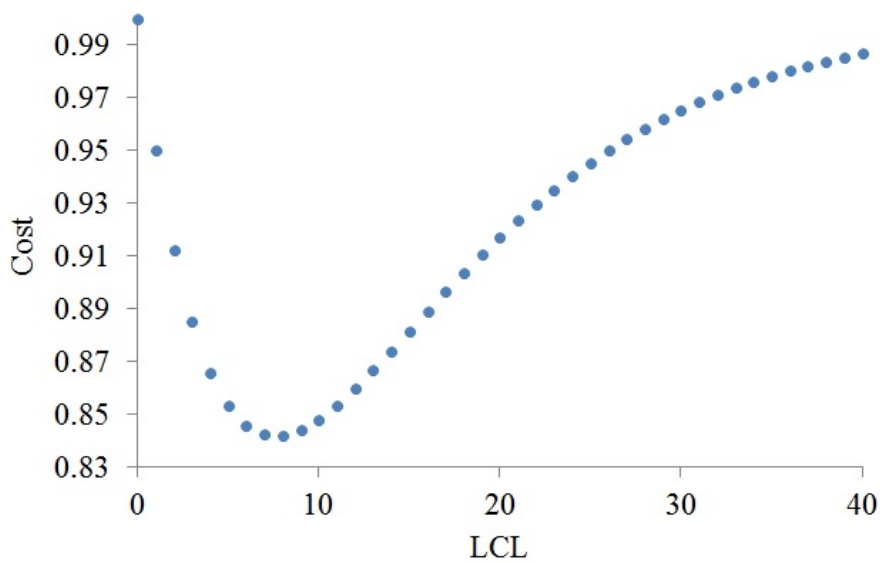

Figure 3.6: Cost versus Upper Control Limit

The behavior of the above figure is the same as the cost graph assuming Poisson distribution. In order to find the best bound in this case, we use the function CostDifference as the previous sections. Function CostDifference in this control chart would be

$$CostDifference(X = x) = -z \times p_1(1-p_1)^x + p_2(1-p_2)^x \tag{3.18}$$

Figure 3.7 shows how CostDifference varies as the lower control limit changes for the same instance in the previous figure ($p_1 = 0.1$, $p_2 = 0.15$, $z = 1$).

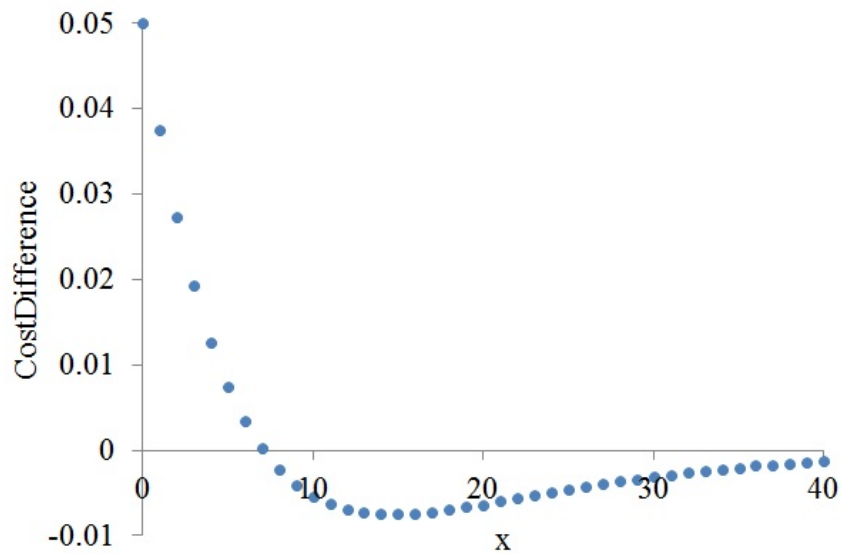

Figure 3.7: CostDifference versus Lower Control Limit

Again like in the previous section we should solve the following equation to find the optimal bound.

$$CostDifference(X = x) = 0 \tag{3.19}$$

The details how to solve this equation is shown in the Appendix 3. The result of the above equation is

$$X = \frac{ln(p_2) - ln(p_1) - ln(z)}{ln(1 - p_1) - ln(1 - p_2)} \tag{3.20}$$

So the optimal lower control limit would be

$$LCL^* = Ceiling\left(\frac{ln(p_2) - ln(p_1) - ln(z)}{ln(1 - p_1) - ln(1 - p_2)}\right) \tag{3.21}$$

The same approach for downward shift (improvement) from p_1 to p_2 where $p_1 > p_2$ gives

optimal upper control limit equal to

$$UCL^* = Floor\left(\frac{ln(z)+ln(p_1)-ln(p_2)}{ln(1-p_2)-ln(1-p_1)}\right) \tag{3.22}$$

3.4 Negative Binomial

In the previous section, the decision about whether the process is in control or out of control is based on the number of conforming items between two consecutive defects. Monitoring the number of items inspected before r nonconforming items requires Negative Binomial as the underlying distribution.

Suppose that the probability of an item to be nonconforming is equal to p, then the probability of reaching the rth nonconforming items after x inspections is

$$Pr(X=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad , \qquad x = r, r+1, r+2, ... \tag{3.23}$$

The objective of the control chart would be the same as the previous sections which is

$$Cost = z \times Type\ \text{I}\ error + Type\ \text{II}\ error \qquad z > 0 \tag{3.24}$$

Upward shift (deterioration) in this section means that the probability p increases and the number of conforming items between two consecutive nonconformities decreases. There is no UCL in the control chart and the objective is formulated as

$$Cost(LCL=l) = z \times \left(\sum_{x=0}^{l-1} \binom{x-1}{r-1} p_1^r (1-p_1)^{x-r}\right) + \left(1 - \sum_{x=0}^{u} \binom{x-1}{r-1} p_2^r (1-p_2)^{x-r}\right) \tag{3.25}$$

where p_1 is the predefined rate of nonconformities and p_2 is the actual rate considering a upward shift. Figure 3.8 shows an instance of cost versus lower control limit for an upward shift ($p_1 = 0.1$, $p_2 = 0.15$, r = 3, z = 1).

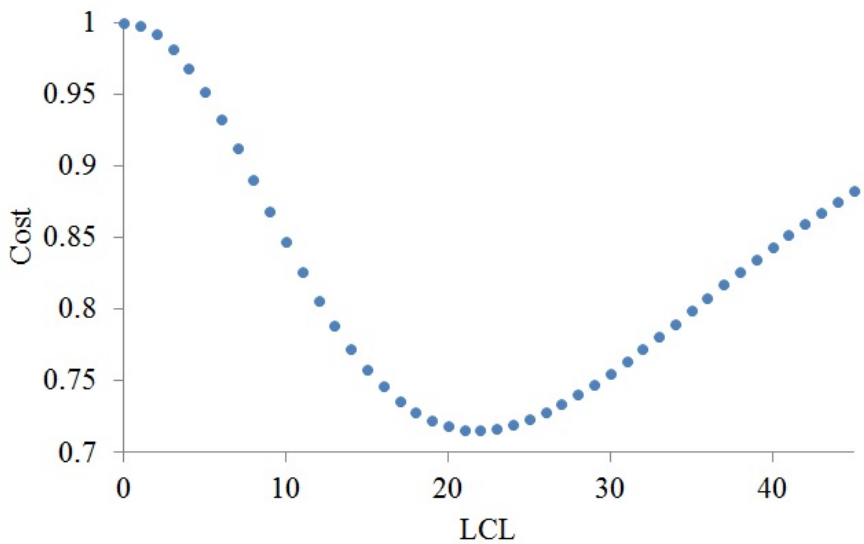

Figure 3.8: Cost versus Lower Control Limit

In the above figure, there is an increasing trend in one side and a decreasing trend on the other side of the graph, so we can use again the function CostDifference to help us to find the optimal bound. Function CostDifference would be

$$CostDifference(X = x) = -z \times \binom{x-1}{r-1} p_1^r (1-p_1)^{x-r} + \binom{x-1}{r-1} p_2^r (1-p_2)^{x-r} \tag{3.26}$$

Figure 3.9 shows how CostDifference varies as the lower control limit changes for the same instance in the previous figure ($p_1 = 0.1$, $p_2 = 0.15$, r = 3, z = 1).

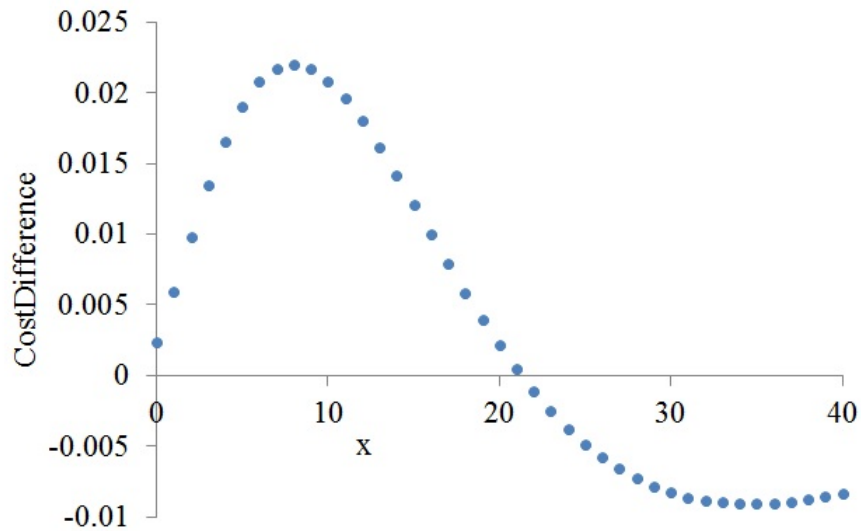

Figure 3.9: CostDifference versus Lower Control Limit

Again like in the previous section we should solve the following equation to find the optimal bound.

$$CostDifference(X = x) = 0 \tag{3.27}$$

The details to solve this equation are shown in the Appendix 4. The result of the above equation is

$$X = \frac{r \times (ln(p_2) - ln(p_1) + ln(1-p_1) - ln(1-p_2)) - ln(z)}{ln(1-p_1) - ln(1-p_2)} \tag{3.28}$$

So the optimal lower control limit would be

$$LCL^* = Ceiling\left(\frac{r \times (ln(p_2) - ln(p_1) + ln(1-p_1) - ln(1-p_2)) - ln(z)}{ln(1-p_1) - ln(1-p_2)}\right) \tag{3.29}$$

The same approach to detect a downward shift (improvement) from p_1 to p_2 where $p_1 > p_2$

gives upper control limit equal to

$$UCL^* = Floor\left(\frac{ln(z) + r \times (ln(p_1) - ln(p_2) + ln(1-p_2) - ln(1-p_1))}{ln(1-p_2) - ln(1-p_1)}\right) \tag{3.30}$$

Chapter 4

Discussion

In this chapter some numerical examples to design the proposed control chart are presented assuming that the distribution of the data is either Poisson, Geometric or Negative Binomial. Also the effects of the factors z and shift size on the design of the control chart and cost are examined graphically for some sample instances.

4.1 Upward Shift

In this section, upward shift in processes with Poisson as the underlying distribution of the data will be examined.

4.1.1 Results

Table 4.1 represents the control limits for several combinations of upward shift from λ_1 to λ_2 as the process parameters with different z values. As you can see from the table, the upper control limit has a nondecreasing pattern as the z value increases in each row which corresponds to an specific shift.

		z = 0.5		z = 1		z = 2	
λ_1	λ_2	LCL*	UCL*	LCL*	UCL*	LCL*	UCL*
1	2	None	0	None	1	None	2
1	3	None	1	None	1	None	2
1	4	None	1	None	2	None	2
1	5	None	2	None	2	None	2
2	3	None	0	None	2	None	4
2	4	None	1	None	2	None	3
2	5	None	2	None	3	None	4
2	6	None	3	None	3	None	4
5	6	None	1	None	5	None	9
5	7	None	3	None	5	None	8
5	8	None	4	None	6	None	7
5	9	None	5	None	6	None	7
5	10	None	6	None	7	None	8
10	14	None	9	None	11	None	13
10	16	None	11	None	12	None	14
10	18	None	12	None	13	None	14
10	20	None	13	None	14	None	15
20	40	None	27	None	28	None	29
40	70	None	52	None	53	None	54

Table 4.1: Upward Sample Results (Poisson)

4.1.2 Sensitivity Analysis

In this subsection we examine the behavior of the optimal bounds, cost and errors when the other parameters such as size of shift and z value change with the assumption that the data follow Poisson distribution.

There are two possible behavior expected for the optimal upper control limit when the size of shift increases:

1. If type II error costs more than type I error ($z \leq 1$) then as the size of shift increases, the optimal control limit increases.

2. If type I error costs more than type II error ($z > 1$) then the optimal upper control limit has convex behavior toward the changes in size of shift.

Here is an example for the these cases. Figure 4.1 shows how the optimal upper control limit varies with different z values when the size of the shift increases for a process with $\lambda_1 = 10$. In this figure, the instance with $z = 0.5$ and $z = 1$ has increasing trend but for $z = 2$ it has convex behavior. The decreasing part in the optimal bound for $z = 2$ is because the z value is large and the peak of the two distributions are very close so it is better to have a larger in control region to decrease the type I error since the type II error will be large. This case can be helpful in practice when the user is not sure about determining the λ_2. In this case there is a minimum optimal upper control limit that the user can not violate in order to decrease the cost of the control chart. This minimum optimal bound can be found by plugging the solution to the derivative of the Equation 3.5 in Equation 3.6.

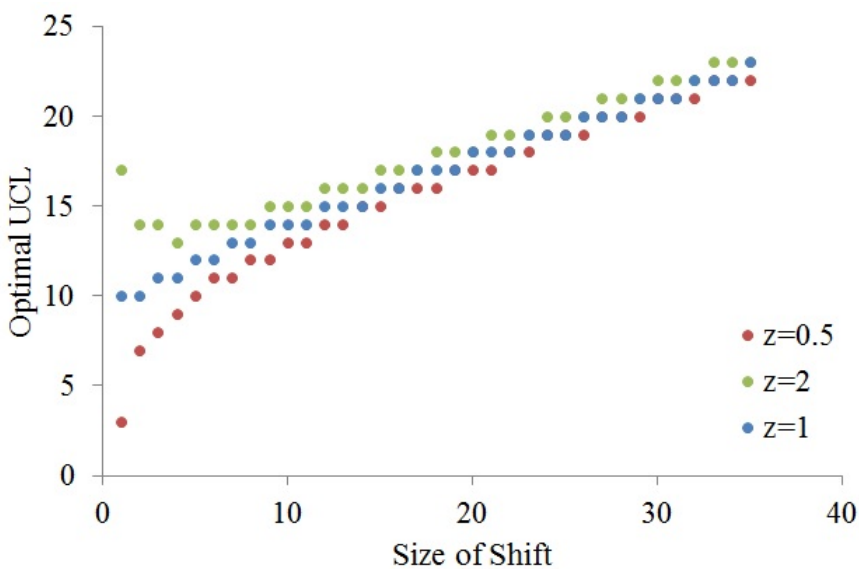

Figure 4.1: Optimal Upper Control Limit versus Size of Shift

Figure 4.2 shows how type I error varies as the size of shift increases for $\lambda_1 = 10$ and different z values. Type I error associated with z= 0.5 and z= 1 has decreasing trend because the optimal region has increasing trend but for z= 2 it has concave behavior since the in control region has convex behavior.

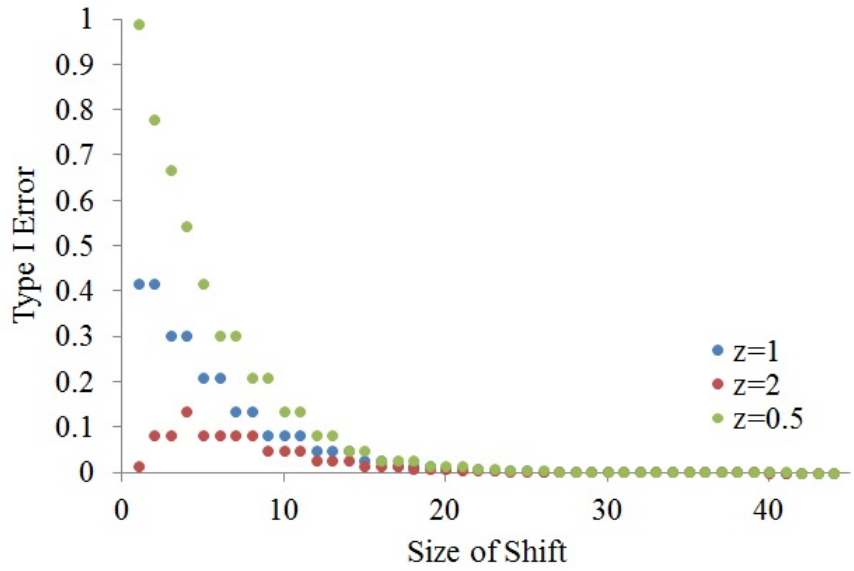

Figure 4.2: Type I Error versus Size of Shift

Figure 4.3 shows how type II error varies as the size of shift increases for $\lambda_1 = 10$ and different z values. For z= 2, the optimal bound first decreases and that is why type II decreases but when the optimal bound starts to increase, size of shift is large and that is why type II keeps decreasing.

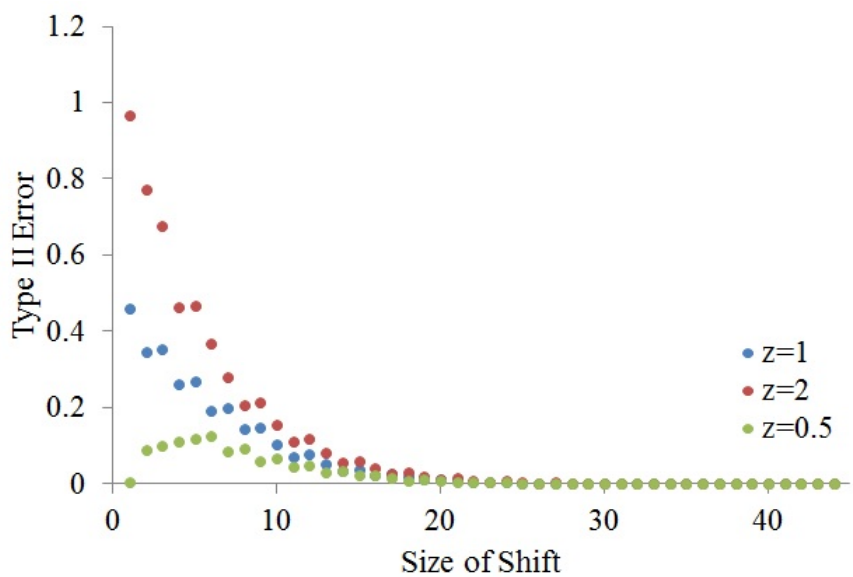

Figure 4.3: Type II Error versus Size of Shift

Figure 4.4 shows how the difference in type I and II errors varies as the size of shift increases for a process with $\lambda_1 = 10$ and z=1. The inconsistency of the difference between two types of errors for the small shift is due to two things. First is the discrete property of the Poisson and the way the optimal limit is rounded, but as the size of shift increases it will lose its effect. The second reason is that when the size of shift is relatively small, it means the peaks of the probability density function for both distributions (λ_1 and λ_2) are very close. That is why any control limit between (λ_1 and λ_2) is going to result in relatively large type I and type II errors.

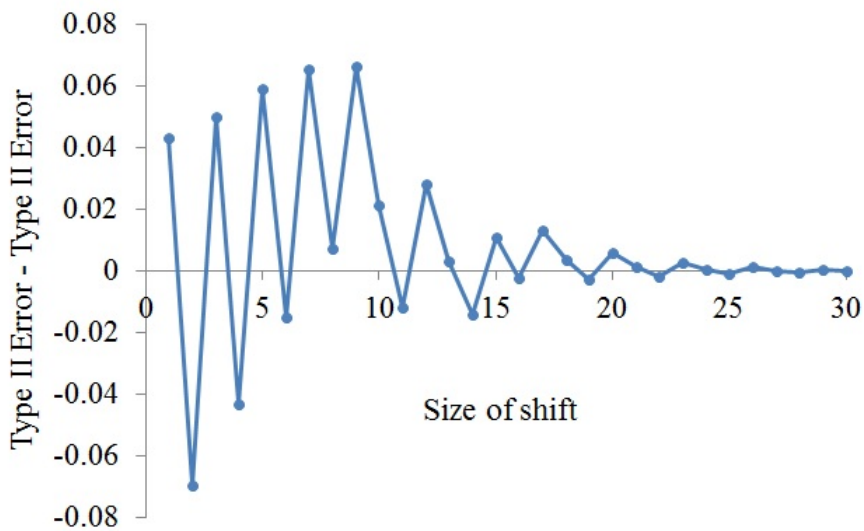

Figure 4.4: Difference between Two Types of Errors versus Size of Shift

Figure 4.5 Shows how cost changes as the size of shift increases for $\lambda_1 = 10$ and different z values. The decreasing trend for all three z's in this figure, is because of the fact that detecting a larger shift is easier than a small shift with less errors. As the size of the anticipated shift increases, the peak of the two distributions will not be closer, so any control limit in between will have a smaller error rates and cost.

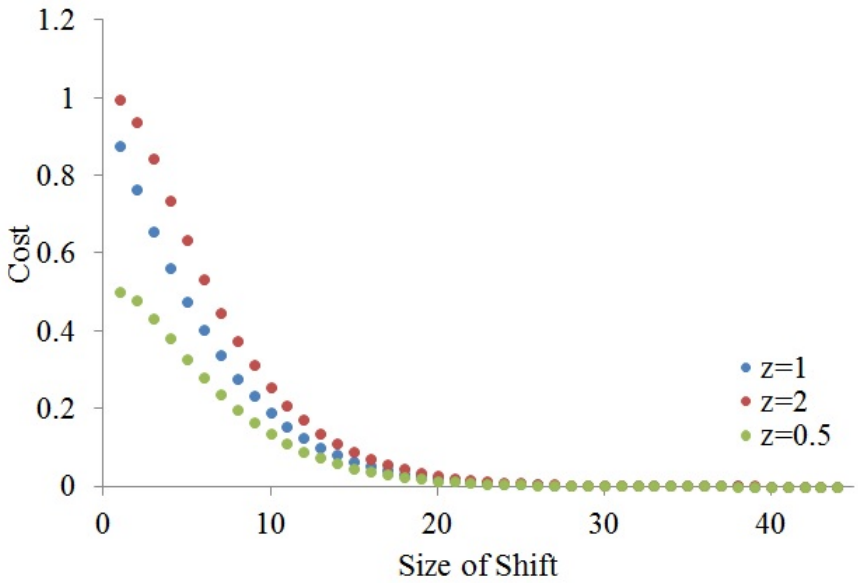

Figure 4.5: Cost versus Size of Shift

Figure 4.6 shows how optimal Upper Control limit varies when z increases for an upward shift from $\lambda_1 = 5$ to $\lambda_2 = 20$. In this instance, increasing z implicitly values type error rather than type II error, so in order to minimize the cost which includes type I error with increasing coefficient, the optimal upper control limit should be increased to expand the in control region.

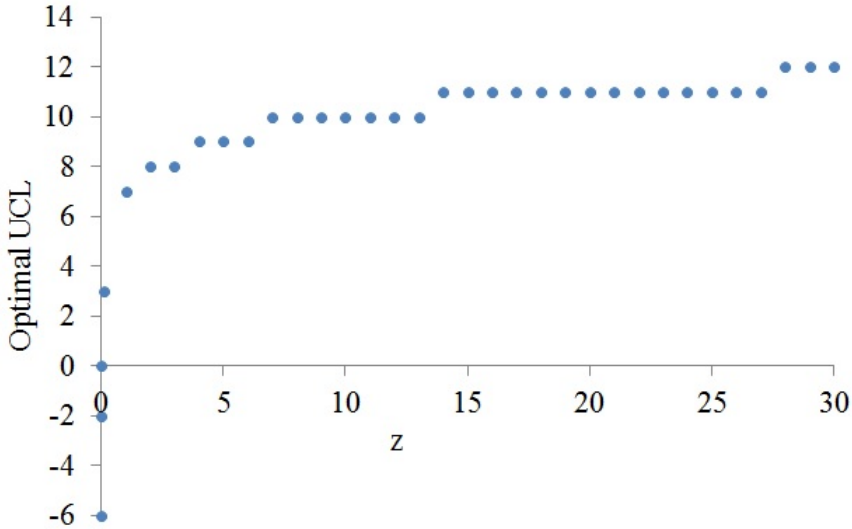

Figure 4.6: Optimal Upper Control Limit versus z

4.2 Downward Shift

In this section, downward shifts in processes with POisson as the underlying distribution of the data will be examined.

4.2.1 Results

Table 4.2 represents the control limits of the proposed control chart for several combinations of downward shift from λ_1 to λ_2 as the process parameters with different z values. In each row of the above table, the lower control limit has nonincreasing trend as the coefficient of the type I error (z) increases to lower the type I error.

		z = 0.5		z = 1		z = 2	
λ_1	λ_2	LCL*	UCL*	LCL*	UCL*	LCL*	UCL*
3	2	5	None	3	None	1	None
3	1	3	None	2	None	2	None
4	3	6	None	4	None	2	None
4	2	4	None	3	None	2	None
4	1	3	None	3	None	2	None
5	4	8	None	5	None	2	None
5	3	6	None	4	None	3	None
5	2	5	None	4	None	3	None
5	1	3	None	3	None	3	None
8	7	13	None	8	None	3	None
8	5	8	None	7	None	5	None
8	3	6	None	6	None	5	None
8	1	4	None	4	None	4	None
10	8	13	None	9	None	6	None
10	6	10	None	8	None	7	None
10	4	8	None	7	None	6	None
10	2	6	None	5	None	5	None
20	5	12	None	11	None	11	None
40	10	23	None	22	None	22	None

Table 4.2: Downward Sample Results (Poisson)

4.2.2 Sensitivity Analysis

The same sensitivity analysis can goes to the control chart that is intended to detect improvement in a process assuming that Poisson is the underlying distribution of the data.

Figure 4.8 shows how the optimal lower control limit varies with different z values, when size of the downward shift increases for a process with $\lambda_1 = 10$. This instance has the opposite behavior as its upward shift; that is, a decreasing trend with $z = 0.5$ and $z = 1$ and a concave behavior with $z = 2$ for in control region.

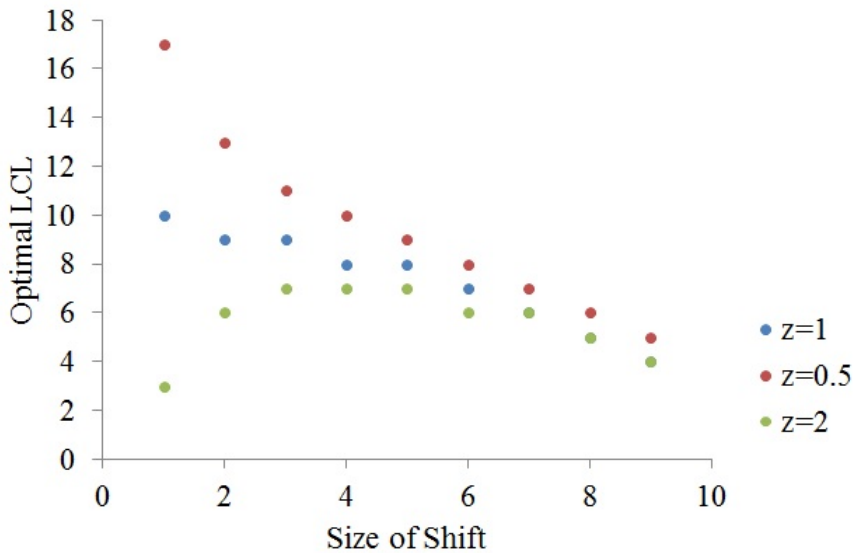

Figure 4.7: Optimal Upper Control Limit versus Size of Shift

Figure 4.9 shows how type I error varies as the size of shift increases for $\lambda_1 = 10$ and different z values. This figure is in accordance to the previous figure. It means as the in control region increases in the previous figure, the type I error decreases.

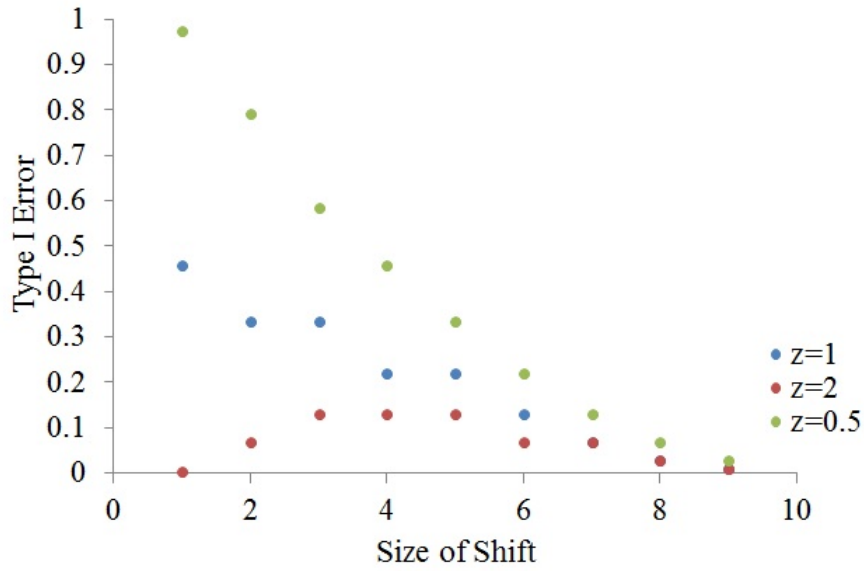

Figure 4.8: Type I Error versus Size of Shift

Figure 4.10 shows how type II error varies as the size of shift increases for $\lambda_1 = 10$ and different z values. This figure has the same behavior as the upward shift for this instance.

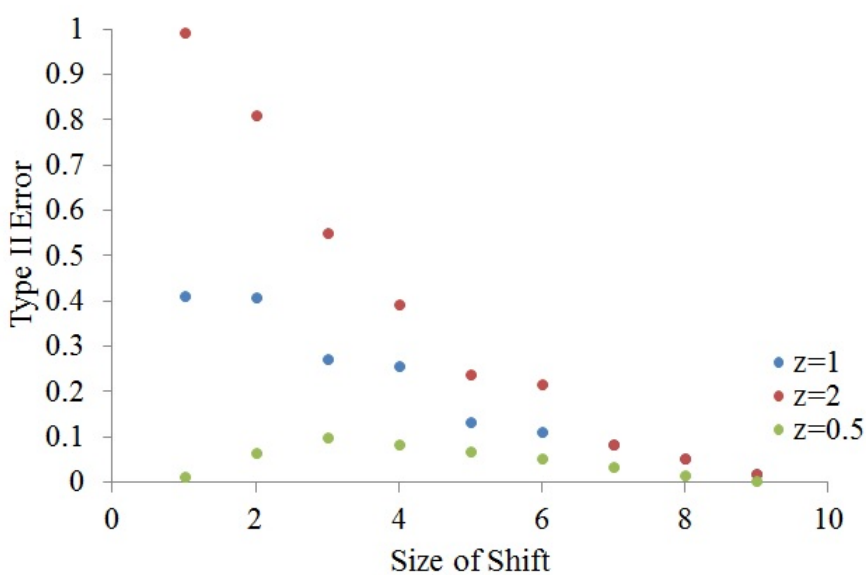

Figure 4.9: Type II Error versus Size of Shift

Figure 4.11 shows how the difference in type I and II errors varies as the size of shift varies for a

process with $\lambda_1 = 10$ and z=1. The inconsistency of the difference between two types of errors for the small shift is due to the same reasons mentioned for upward shift for this instance.

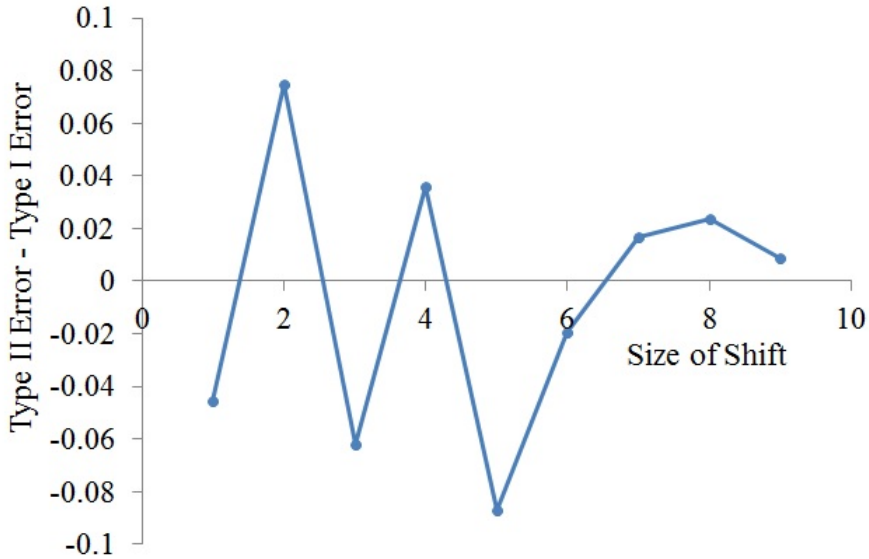

Figure 4.10: Difference between Two Types of Errors versus Size of Shift

Figure 4.12 Shows how cost changes as the size of shift increases for $\lambda_1 = 10$ and different z values. The decreasing trend of cost function in this figure is due to the fact that detecting a larger shift is easier with less errors.

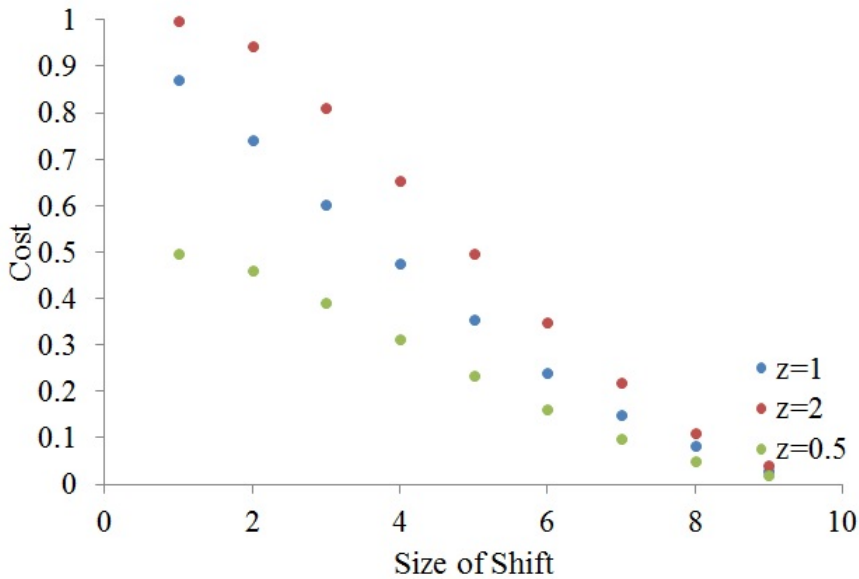

Figure 4.11: Cost versus Size of Shift

Figure 4.13 shows how optimal Lower Control limit varies when z increases for a downward shift from $\lambda_1 = 20$ to $\lambda_2 = 5$. In this instance, as z increases, the in control region expands to decrease type I error and consequently the cost of the control chart.

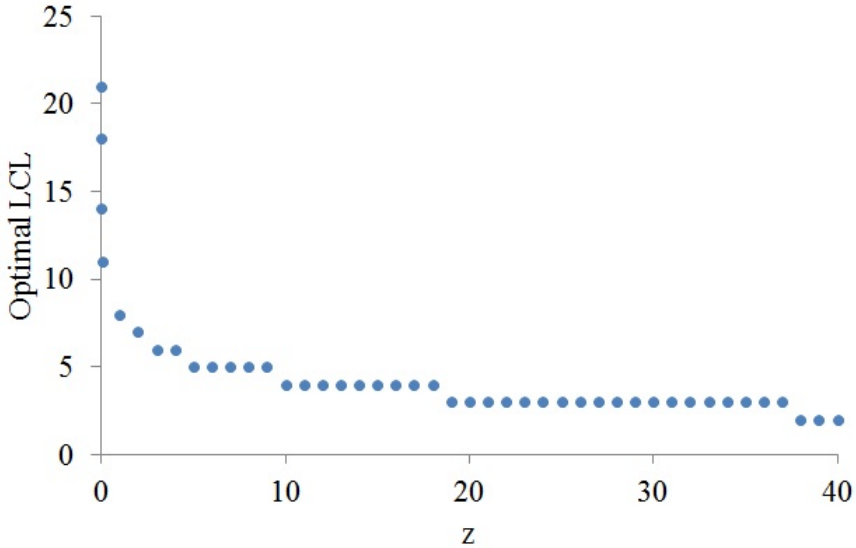

Figure 4.12: Optimal Lower Control Limit versus z

Figure 4.15 presents the cost associated with the proposed control chart and the conventional c

chart for a process with $\lambda_1 = 10$ and z=1. It is assumed that the c chart is one sided. As it can be seen in the figure, the proposed control chart returns lower cost compared to the one sided c chart when a directional change in the process is to be detected, especially when the size of the shift is small.

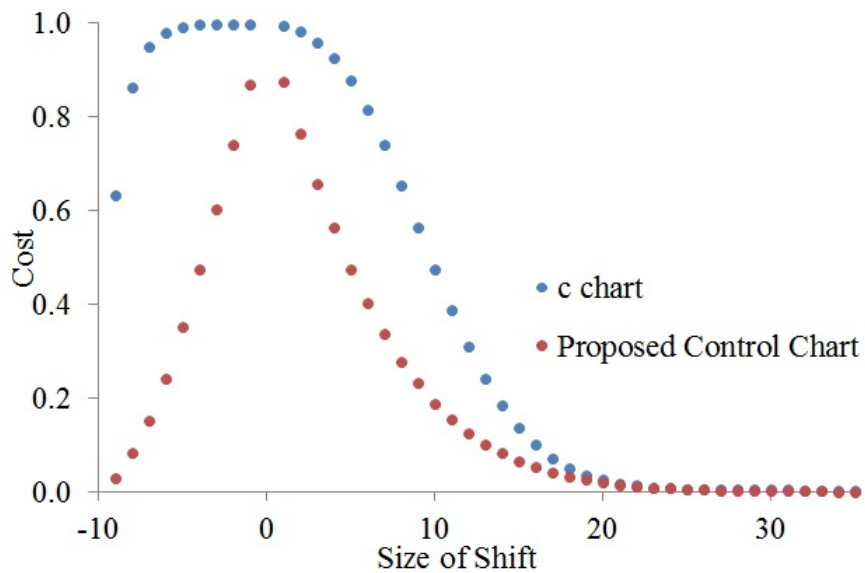

Figure 4.13: Cost versus Size of Shift

4.3 Geometric

4.3.1 Results

In this subsection, some numerical examples of the control limits for the proposed control chart are presented assuming that the nonconforming items occur based on a Geometric distribution. Table 4.3 represents the control limits for several combinations of upward shifts from p_1 as the parameter of the in control state to p_2 as the parameter for the alternative state with different z values. In this table, ”None” as the lower control limit means that the process is always considered in control. Any number of conforming items between two nonconformities is acceptable.

		z = 0.5		z = 1		z = 2	
p_1	p_2	LCL^*	UCL^*	LCL^*	UCL^*	LCL^*	UCL^*
0.005	0.0055	1569	None	190	None	None	None
0.005	0.006	871	None	182	None	None	None
0.005	0.0065	634	None	174	None	None	None
0.005	0.007	512	None	168	None	None	None
0.005	0.0075	437	None	162	None	None	None
0.01	0.011	781	None	95	None	None	None
0.01	0.012	433	None	91	None	None	None
0.01	0.013	315	None	87	None	None	None
0.01	0.014	255	None	84	None	None	None
0.01	0.015	217	None	81	None	None	None
0.015	0.0165	518	None	63	None	None	None
0.015	0.018	288	None	60	None	None	None
0.015	0.0195	209	None	58	None	None	None
0.015	0.02	169	None	56	None	None	None
0.015	0.02	144	None	54	None	None	None
0.1	0.11	71	None	9	None	None	None
0.1	0.12	39	None	9	None	None	None
0.1	0.13	29	None	8	None	None	None
0.1	0.14	23	None	8	None	None	None
0.1	0.15	20	None	8	None	None	None

Table 4.3: Upward Sample Results (Geometric)

Table 4.3 represents the control limits for several combinations of downward shifts from p_1 to p_2 with different z values. In this table, ”None” as the upper control limit means that the process is always considered out of control. Any number of conforming items between two consecutive nonconformities shows that the process is out of control.

		z = 0.5		z = 1		z = 2	
p_1	p_2	LCL^*	UCL^*	LCL^*	UCL^*	LCL^*	UCL^*
0.01	0.009	None	None	None	104	None	790
0.01	0.008	None	None	None	110	None	454
0.01	0.007	None	None	None	117	None	346
0.01	0.006	None	None	None	126	None	298
0.01	0.005	None	None	None	137	None	275
0.015	0.0135	None	None	None	69	None	524
0.015	0.012	None	None	None	73	None	301
0.015	0.0105	None	None	None	78	None	230
0.015	0.009	None	None	None	84	None	198
0.015	0.0075	None	None	None	91	None	182
0.1	0.09	None	None	None	9	None	72
0.1	0.08	None	None	None	10	None	41
0.1	0.07	None	None	None	10	None	32
0.1	0.06	None	None	None	11	None	27
0.1	0.05	None	None	None	12	None	25
0.15	0.135	None	None	None	6	None	45
0.15	0.12	None	None	None	6	None	26
0.15	0.105	None	None	None	6	None	20
0.15	0.09	None	None	None	7	None	17
0.15	0.075	None	None	None	8	None	16

Table 4.4: Downward Sample Results (Geometric)

4.4 Negative Binomial

4.4.1 Results

In this section, some numerical results are presented to design the proposed control chart for a process that follows Negative Binomial as the defect occurence distribution. Table 4.5 represents the control limits for several combinations of upward shifts from p_1 to p_2 with different z values and $r = 2$. p_1 is the parameter of the process when it is in control and p_2 is its parameter when it is out of control. In this table, ”None” as the lower control limit means that the process is always in the state of control.

		z = 0.5		z = 1		z = 2	
p_1	p_2	LCL^*	UCL^*	LCL^*	UCL^*	LCL^*	UCL^*
0.005	0.0055	1761	None	382	None	None	None
0.005	0.006	1054	None	365	None	None	None
0.005	0.0065	810	None	350	None	None	None
0.005	0.007	681	None	337	None	None	None
0.005	0.0075	600	None	325	None	49	None
0.01	0.011	877	None	191	None	None	None
0.01	0.012	526	None	183	None	None	None
0.01	0.013	404	None	175	None	None	None
0.01	0.014	340	None	169	None	None	None
0.01	0.015	300	None	163	None	26	None
0.015	0.0165	582	None	128	None	None	None
0.015	0.018	349	None	122	None	None	None
0.015	0.0195	268	None	117	None	None	None
0.015	0.021	226	None	113	None	None	None
0.015	0.0225	199	None	109	None	18	None
0.1	0.11	82	None	20	None	0	None
0.1	0.12	50	None	19	None	0	None
0.1	0.13	38	None	18	None	0	None
0.1	0.14	33	None	17	None	2	None
0.1	0.15	29	None	17	None	5	None

Table 4.5: Upward Sample Results (Negative Binomial)

Table 4.6 represents the control limits for several combinations of downward shifts from p_1 to p_2 with different z values and r = 3. In this table, the rows with ”None” as the upper control limit means that the process is alwasy out of control.

		z = 0.5		z = 1		z = 2	
p_1	p_2	LCL^*	UCL^*	LCL^*	UCL^*	LCL^*	UCL^*
0.01	0.009	None	None	None	210	None	897
0.01	0.008	None	None	None	223	None	566
0.01	0.007	None	8	None	237	None	466
0.01	0.006	None	83	None	255	None	427
0.01	0.005	None	139	None	277	None	414
0.015	0.0135	None	None	None	140	None	595
0.015	0.012	None	None	None	148	None	376
0.015	0.0105	None	6	None	158	None	310
0.015	0.009	None	56	None	170	None	284
0.015	0.0075	None	93	None	184	None	276
0.1	0.09	None	None	None	21	None	83
0.1	0.08	None	None	None	22	None	53
0.1	0.07	None	2	None	23	None	44
0.1	0.06	None	9	None	25	None	41
0.1	0.05	None	14	None	27	None	40
0.15	0.135	None	None	None	14	None	53
0.15	0.12	None	None	None	14	None	34
0.15	0.105	None	2	None	15	None	29
0.15	0.09	None	6	None	16	None	27
0.15	0.075	None	10	None	18	None	26

Table 4.6: Downward Sample Results (Negative Binomial)

Table 4.7 represents the cost, type I error and type II error associated with the optimal control chart for an upward shift from $p_1 = 0.01$ to $p_2 = 0.015$ with $z = 1$ and different r values. As r gets larger, the effect of probability error will decrease. It means that the number of conforming items until the rth nonconfromities will be a better representation of the process; therefore, the errors and cost will decrease. Number of conforming items between two consecutive defects follows a Geometric distribution, so the average of this r numbers to find the rth defects converge to Normal distribution $(\frac{1}{p}, 0)$ when r gets large. It means that there will be less errors with data from the process and less error with the decision of determining whether the process is out of control or not. It is better for the user to use the Negative Binomial control chart with larger r than Geometric control chart to monitor the process. Another factor that may affect the decision of the user is time. With choosing larger r, the time to make the first decision about the process will get much more than the smaller r, so the user can choose a r to trade of between cost and time to monitor the

process.

r	LCL*	Type I error	Type II error	Cost
1	82	0.5614	0.2896	0.8510
2	163	0.4889	0.2933	0.7822
3	244	0.4465	0.2850	0.7314
4	325	0.4156	0.2743	0.6898
5	406	0.3908	0.2634	0.6542
6	487	0.3699	0.2528	0.6227
7	568	0.3518	0.2426	0.5944
8	649	0.3357	0.2330	0.5687
9	730	0.3212	0.2238	0.5450
10	811	0.3080	0.2151	0.5231

Table 4.7: The Effect of r on Binomial Control Chart

Chapter 5

Conclusion and Future Works

5.1 Conclusions

In this research, a new control chart is described to monitor and improve the quality of the processes that provide count data. The control chart is designed to decrease the cost charged to the user in long term. The user has the ability to define the cost as a linear function of the control chart's error rates (type I and II errors). The control limits are optimaly chosen in this control chart since it is based on the exact probability distribution of the data. The proposed control chart can be applied to Poisson, Geometric and Negative Binomial as the distribution of the count data. In addition, it has the ability to detect both deteriorartion and improvement in the quality. The proposed control chart is a superior alternative to the classical Shwhart chart which is only intended to provide a fixed rate of type I error in a case that type II error cost the user as well.
Numerical samples of the control limits for different possible shifts are represented for Poisson, Geometric and Negative Binomial control chart. The effect of shift size in the parameter of interest and the relative cost of the errors on the design of the control chart , cost and error rates are shown graphically.

5.2 Future Works

1. In this research, the control chart is designed from an economic view of the user and it does not account for ARL performance. One may adjust the control limits in the proposed chart to give a desired performance such as the in control and out of control ARL. These can be added as a constriant for the defined objective.

2. For the Negative Binomial chart we know that as r gets larger the control chart can detect small shift easier in p with less cost. One may consider the cost of inspecting items to

obtain the rth nonconformities in the objective of the control chart. So the optimal limits are determined after accounting a trade off between costs of errors and inspection in each cycle.

3. The proposed control chart, like other kinds of individuals control charts, has large type I error rate when a small size of shift is desired to be detected. Future studies can include modification of this control chart to a CUSUM procedure which detects the shift based on a sequence of observations.

4. The control chart in this reserach is designed to detect only step shifts in the underlying distribution's parameters. One may consider designing a control chart with the same objective to detect other kinds of trends in the undelying distribution parameter such as a linear change.

5. In the proposed Poisson control chart, it is assumed that the rate of nonconformities is fixed in the statistical control. This rate is based on a fixed unit of sample. One may investigate the effect of varying sample sizes over time as they do in practice on designing the control limits .

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Appendix 1

Upward Shift

$$CostDifference(x) = 0$$

$$-z \times \left(\frac{e^{-\lambda_1} \times \lambda_1^x}{x!} \right) + \left(\frac{e^{-\lambda_2} \times \lambda_2^x}{x!} \right) = 0 \qquad \text{combining the two fractions} \rightarrow$$

$$\frac{-z \times e^{-\lambda_1} \times \lambda_1^x + e^{-\lambda_2} \times \lambda_2^x}{x!} = 0 \qquad \text{equaling the numerator to zero} \rightarrow$$

$$-z \times e^{-\lambda_1} \times \lambda_1^x + e^{-\lambda_2^x} \times \lambda_2^x = 0 \qquad \text{setting the base to ”e”} \rightarrow$$

$$-e^{ln(z)} \times e^{-\lambda_1} \times (e^{ln(\lambda_1)})^x + e^{-\lambda_2} \times (e^{ln(\lambda_2)})^x = 0$$

$$-e^{ln(z)} \times e^{-\lambda_1} \times e^{x \times ln(\lambda_1)} + e^{-\lambda_2} \times e^{x \times ln(\lambda_2)} = 0$$

$$-e^{ln(z) - \lambda_1 + x \times ln(\lambda_1)} + e^{-\lambda_2 + x \times ln(\lambda_2)} = 0$$

$$e^{-\lambda_2 + x \times ln(\lambda_2)} = e^{ln(z) - \lambda_1 + x \times ln(\lambda_1)} \qquad \text{taking the natural log of the two sides} \rightarrow$$

$$-\lambda_2 + x \times ln(\lambda_2) = ln(z) - \lambda_1 + x \times ln(\lambda_1)$$

$$X = \frac{ln(z) + \lambda_2 - \lambda_1}{ln(\lambda_2) - ln(\lambda_1)}$$

Appendix 2

Downward Shift

$$CostDifference(x) = 0$$

$$-z \times \left(\frac{e^{-\lambda_1} \times \lambda_1^x}{x!} \right) + \left(\frac{e^{-\lambda_2} \times \lambda_2^x}{x!} \right) = 0 \qquad \text{combining the two fractions} \rightarrow$$

$$\frac{-z \times e^{-\lambda_1} \times \lambda_1^x + e^{-\lambda_2} \times \lambda_2^x}{x!} = 0 \qquad \text{equaling the numerator to zero} \rightarrow$$

$$-z \times e^{-\lambda_1} \times \lambda_1^x + e^{-\lambda_2^x} \times \lambda_2^x = 0 \qquad \text{setting the base to ”e”} \rightarrow$$

$$-e^{ln(z)} \times e^{-\lambda_1} \times (e^{ln(\lambda_1)})^x + e^{-\lambda_2} \times (e^{ln(\lambda_2)})^x = 0$$

$$-e^{ln(z)} \times e^{-\lambda_1} \times e^{x \times ln(\lambda_1)} + e^{-\lambda_2} \times e^{x \times ln(\lambda_2)} = 0$$

$$-e^{ln(z) - \lambda_1 + x \times ln(\lambda_1)} + e^{-\lambda_2 + x \times ln(\lambda_2)} = 0$$

$$e^{-\lambda_2 + x \times ln(\lambda_2)} = e^{ln(z) - \lambda_1 + x \times ln(\lambda_1)} \qquad \text{taking the natural log of the two sides} \rightarrow$$

$$-\lambda_2 + x \times ln(\lambda_2) = ln(z) - \lambda_1 + x \times ln(\lambda_1)$$

$$X = \frac{\lambda_1 - \lambda_2 - ln(z)}{ln(\lambda_1) - ln(\lambda_2)}$$

Appendix 3

Geometric

$$CostDifference(x) = 0$$

$$-z \times p_1(1-p_1)^x + p_2(1-p_2)^x = 0$$

$z \times p_1(1-p_1)^x = p_2(1-p_2)^x$ taking the ”x” to one side of the equality $\rightarrow$

$\frac{z \times p_1}{p_2} = (\frac{1-p_2}{1-p_1})^x$ taking the natural log of the two sides $\rightarrow$

$$ln(\frac{z \times p_1}{p_2}) = x \times ln(\frac{1-p_2}{1-p_1})$$

$$x = \frac{ln(\frac{z \times p_1}{p_2})}{ln(\frac{1-p_2}{1-p_1})}$$

$$x = \frac{ln(z) + ln(p_1) - ln(p_2)}{ln(1-p_2) - ln(1-p_1)}$$

Appendix 4

Negative Binomial

$CostDifference(x) = 0$

$-z \times \binom{x-1}{r-1} p_1^r (1-p_1)^{x-r} + \binom{x-1}{r-1} p_2^r (1-p_2)^{x-r} = 0$

$z \times \binom{x-1}{r-1} p_1^r (1-p_1)^{x-r} = \binom{x-1}{r-1} p_2^r (1-p_2)^{x-r}$ canceling the combination and taking the "x" to one side of the equality $\rightarrow$

$z \times (\frac{p_1}{p_2})^r = (\frac{1-p_2}{1-p_1})^{x-r}$

$z \times (\frac{p_1}{p_2})^r \times (\frac{1-p_2}{1-p_1})^r = (\frac{1-p_2}{1-p_1})^x$ taking the natural log of the two sides $\rightarrow$

$ln(z \times (\frac{p_1}{p_2})^r \times (\frac{1-p_2}{1-p_1})^r) = x \times ln(\frac{1-p_2}{1-p_1})$

$$x = \frac{ln(z \times (\frac{p_1}{p_2})^r \times (\frac{1-p_2}{1-p_1})^r)}{ln(\frac{1-p_2}{1-p_1})}$$

$$x = \frac{ln\left(z \times \left(\frac{p_1(1-p_2)}{p_2(1-p_1)}\right)^r\right)}{ln(\frac{1-p_2}{1-p_1})}$$

$$x = \frac{ln(z) + r \times \big(ln(p_1) - ln(p_2) + ln(1-p_2) - ln(1-p_1)\big)}{ln(1-p_2) - ln(1-p_1)}$$