Acta Math Vietnam (2016) 41:633–660
DOI 10.1007/s40306-016-0170-3

Color Partition Identities Arising from Ramanujan's Theta-Functions

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Received: 29 January 2015 / Revised: 28 June 2015 / Accepted: 6 July 2015 /
Published online: 18 February 2016

Abstract We establish several partition identities with distinct colors that arise from Ramanujan's theta-function identities and formulas for multipliers in the theory of modular equations. Also, we deduce few partition congruences as a corollary of some partition identities.

Keywords Color partition identities · Theta–functions · Partition congruences · Modular equations

Mathematics Subject Classification (2010) 11P83 · 05A17

1 Introduction

In [12], H. M. Farkas and I. Kra observed that certain theta constant identities can be interpreted into partition identities. The following theorem is the most elegant of their three partition theorems.

Theorem 1.1 *Let S denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 7. Then for each positive*

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integer k, the number of partitions of $2k$ *into even elements of S is equal to the number of partitions of* $2k+1$ *into odd elements of S.*

The generating function identity of Theorem 1.1 is

$$(-q;q^2)_\infty(-q^7;q^{14})_\infty - (q;q^2)_\infty(q^7;q^{14})_\infty = 2q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty, \quad (1.1)$$

where $(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$.

In [13], M. D. Hirschhorn gave a simple proof of Theorem 1.1. The referee of [13] observed that (1.1) is equivalent to a modular equation of degree 7 in Ramanujan Notebooks. B.C. Berndt [9] showed that modular equations of degree 5, 11, and 23 due to H. Schröter [19], Russell [16], and Ramanujan [15] also yield elegant partition identity similar to (1.1).

N. D. Baruah and Berndt [2, 3] found that there are many further modular equations and theta function identities of Ramanujan and Schröter-type, which yield elegant partition identities.

S. O. Warnaar [20] established a generalization of Theorem 1.1. In [14], S. Kim gives a bijective proof of Warnaar's generalization, which naturally gives a bijective proof of partition theorems due to Farkas and Kra [12]. C. Sandon and F. Zanella [17] extended Kim's ideas and consequently found a bijective proof of partition identities arising from modular equations of degree 5 and 11. In [18], they further found several new and non-trivial colored partition identities and conjectured 29 more identities. Berndt and R. R. Zhuo [10] proved three of the Sandon and Zanella conjectures using Ramanujan formulas of multipliers and the same authors proved all the remaining conjectures in [11]. In [4], Baruah and B. Bourah have also established 17 of those conjectures and proved analogues of all the remaining 12 using the theory of Ramanujan's theta function.

Baruah and K. K. Ojah have studied the partition function $p_{[c^l d^m]}(n)$ in [5] which is defined by

$$\sum_{n=0}^{\infty} p_{[c^l d^m]}(n)q^n := \frac{1}{(q^c;q^c)_\infty^l (q^d;q^d)_\infty^m}. \quad (1.2)$$

They have proved some analogues of Ramanujan's partition identities and also deduced several partition congruences.

In this paper, we present several new partition-theoretical interpretation that arise from existing Ramanujan's theta function identities.

We end this section by defining a modular equation in brief.

The complete elliptic integral of the first kind is defined for $0 < k < 1$ by

$$\begin{aligned} K := K(k) &:= \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \\ &= \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right). \end{aligned}$$

The number k is called the modulus of K and $k' := \sqrt{1-k^2}$ is called the complementary modulus and ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ denotes the ordinary hypergeometric function. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and $l' := \sqrt{1-l^2}$, respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \quad (1.3)$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.3). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α.

If

$$q := \exp\left(-\pi \frac{K'}{K}\right), \tag{1.4}$$

then from the theory of elliptic function [[6]. p. 101, Entry 6], we have

$$\varphi^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) =: z. \tag{1.5}$$

If we set $z_n := \varphi^2(q^n)$, then the multiplier m of degree n is defined by

$$m := \frac{z_1}{z_n}. \tag{1.6}$$

2 Definitions and Preliminary Results

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$(a_1, a_2, \ldots, a_k; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \ldots (a_k; q)_\infty,$$

and define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty.$$

Following Ramanujan, we define

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{2.1}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{2.2}$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \tag{2.3}$$

which are special cases of Ramanujan's general theta function [6]

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{2.4}$$

Using the Jacobi's famous triple product identity, the identity (2.4) reduces to

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \tag{2.5}$$

After Ramanujan, we also define

$$\chi(q) := (-q; q^2)_\infty. \tag{2.6}$$

We frequently use Euler's famous identity [1]

$$1/(q; q^2)_\infty = (-q; q)_\infty \tag{2.7}$$

i.e., the number of partitions of a positive integer n into odd parts is identical to the number of partitions of n into distinct parts.

Lemma 2.1 ([6, Entry 29, p. 45]) *If $ab = cd$, we have*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc), \tag{2.8}$$
$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af(b/c,ac^2d)f(b/d,acd^2). \tag{2.9}$$

Lemma 2.2 ([7, Entry 14, p.149]) *We have*

$$f(a,b)f(a^3,b^3) - f(-a,-b)f(-a^3,-b^3) = 2af(b/a,a^2)\psi(a^3b^3), \tag{2.10}$$
$$f(a,b)f(a^2b,ab^2) - f(-a,-b)f(-a^2b,-ab^2) = 2af(b/a,a^4b^2)\psi(ab). \tag{2.11}$$

3 Partitions and Theta Constant Identities of Degree 3 or 9

Theorem 3.1 *Let S denote the set consisting of three copies of odd positive integers, each of which is colored by one of three colors, and let one of which is not a multiple of 3. Let $A(N)$ denote the number of partitions of N into elements of S, let $B(N)$ denote the number of partitions of $2N$ into parts congruent to $\pm 2, \pm 4$ or $6 \pmod{12}$ with parts congruent to ± 4 or $6 \pmod{12}$ having one additional color and let $C(N)$ denote the number of partitions of $2N-2$ into even positive integers that are not multiples of 36 with parts congruent to $\pm 2, \pm 10, \pm 14$ or $18 \pmod{36}$ having one additional color. Then,*

(i) *for $N \geq 1$, $A(2N) = 3\,C(N)$ and*
(ii) *for $N \geq 0$, $A(2N+1) = 3\,B(N)$.*

Proof From [7, p. 202, Entry 50(i)]

$$\frac{\chi^3(q)}{\chi(q^3)} = 1 + 3q\frac{\psi(-q^9)}{\psi(-q)}. \tag{3.1}$$

Replacing q by $-q$ in (3.1), we obtain

$$\frac{\chi^3(-q)}{\chi(-q^3)} = 1 - 3q\frac{\psi(q^9)}{\psi(q)}. \tag{3.2}$$

Adding (3.1) and (3.2), we find that

$$\frac{\chi^3(q)}{\chi(q^3)} + \frac{\chi^3(-q)}{\chi(-q^3)} = 2 + 3q\left(\frac{\psi(-q^9)}{\psi(-q)} - \frac{\psi(q^9)}{\psi(q)}\right). \tag{3.3}$$

Setting $a = q$ and $b = q^5$ in (2.10) and then transforming the resulting identity to q-products, we obtain

$$\frac{(-q;q^2)_\infty}{(-q^9;q^{18})_\infty} - \frac{(q;q^2)_\infty}{(q^9;q^{18})_\infty} = 2q\frac{(-q^2;q^2)_\infty}{(-q^6;q^6)_\infty}(-q^{18};q^{18})^2_\infty. \tag{3.4}$$

Employing (2.2), (2.6), (2.7), and (3.4) in (3.3), we obtain the equality

$$\frac{(-q;q^2)^3_\infty}{(-q^3;q^6)_\infty} + \frac{(q;q^2)^3_\infty}{(q^3;q^6)_\infty} = 2 + 6q^2\frac{(q^{36};q^{36})_\infty}{(q^2;q^2)_\infty(q^{2\pm},q^{10\pm},q^{14\pm},q^{18};q^{36})_\infty}. \tag{3.5}$$

Now, subtracting (3.2) from (3.1), we find that

$$\frac{\chi^3(q)}{\chi(q^3)} - \frac{\chi^3(-q)}{\chi(-q^3)} = 3q\left(\frac{\psi(-q^9)}{\psi(-q)} + \frac{\psi(q^9)}{\psi(q)}\right). \tag{3.6}$$

We recall from [6, p. 345, Entry 1] that

$$\frac{\chi^3(-q^9)}{\chi(-q^3)}+q=\frac{\psi(q)}{\psi(q^9)}. \tag{3.7}$$

Replacing q by $-q$ in (3.7), and then adding the resulting identity with (3.7), we find that

$$\frac{\chi^3(q^9)}{\chi(q^3)}+\frac{\chi^3(-q^9)}{\chi(-q^3)}=\frac{\psi(-q)}{\psi(-q^9)}+\frac{\psi(q)}{\psi(q^9)}. \tag{3.8}$$

Setting $a=q$, $b=q^5$, and $c=d=-q^3$ in (2.8) and then writing the resulting identity in q-products, we have

$$\frac{(-q;q^2)_\infty}{(-q^3;q^6)^3_\infty}+\frac{(q;q^2)_\infty}{(q^3;q^6)^3_\infty}=2\frac{(q^4;q^4)^2_\infty(-q^6;q^6)^2_\infty}{(q^6;q^6)^2_\infty}. \tag{3.9}$$

Using (3.9) in (3.8) and then substituting the resulting identity in (3.6), we obtain an identity equivalent to

$$\frac{(-q;q^2)^3_\infty}{(-q^3;q^6)_\infty}-\frac{(q;q^2)^3_\infty}{(q^3;q^6)_\infty}=6q\frac{1}{(q^{2\pm},q^{4\pm},q^{4\pm},q^6,q^6;q^{12})_\infty}. \tag{3.10}$$

From (3.5) and (3.10), we deduce the partition identity claimed in Theorem 3.1(i) and (ii), respectively. □

Example 1 Let $N=2$. Then $A(4)=6$ and $C(2)=2$, we have the representation

$$3_r+1_r=3_r+1_o=3_r+1_b=3_o+1_r=3_o+1_o=3_o+1_b, 2_R=2_O.$$

Also, $A(5)=9$ and $B(2)=3$, and

$$\begin{aligned}5_o&=5_r=5_b=3_r+1_r+1_o=3_r+1_b+1_o=3_r+1_b+1_r=3_o+1_r+1_o\\&=3_o+1_b+1_o=3_o+1_b+1_r,\\2_R+2_R&=4_R=4_B.\end{aligned}$$

Corollary 3.1.1 *Let $p_{[1^33^{-1}]}(n)$ be the number of three colored partitions of n with one of the colors is not multiples of 3. We have*

$$\sum_{n=0}^{\infty}p_{[1^33^{-1}]}(2n)q^n=\frac{(q^2;q^2)^3_\infty(q^3;q^3)^2_\infty}{(q;q)^6_\infty(q^6;q^6)_\infty}+3q\frac{(q^2;q^2)^4_\infty(q^3;q^3)^3_\infty(q^{18};q^{18})^2_\infty}{(q;q)^8_\infty(q^6;q^6)^2_\infty(q^9;q^9)_\infty}. \tag{3.11}$$

Proof We have

$$\sum_{n=0}^{\infty}p_{[1^33^{-1}]}(n)q^n=\frac{(q^3;q^3)_\infty}{(q;q)^3_\infty}. \tag{3.12}$$

Thus

$$2\sum_{n=0}^{\infty}p_{[1^33^{-1}]}(2n)q^{2n}=\frac{(q^4;q^4)^3_\infty(q^6;q^6)^2_\infty}{(q^2;q^2)^6_\infty(q^{12};q^{12})_\infty}\left\{\frac{(-q;q^2)^3_\infty}{(-q^3;q^6)_\infty}+\frac{(q;q^2)^3_\infty}{(q^3;q^6)_\infty}\right\}. \tag{3.13}$$

Using (3.5) in (3.13) and then replacing q^2 with q, we obtain (3.11). □

Corollary 3.1.2 *For $r \geq 1$, we have*

$$p_{[1^33^{-1}]}(6n+4) \equiv 0 \pmod{6}, \tag{3.14}$$
$$p_{[1^33^{-1}]}(6n+6) \equiv 0 \pmod{6}, \tag{3.15}$$
$$p_{[1^33^{-1}]}(6\cdot 4^r n + 14\cdot 4^{r-1}) \equiv 0 \pmod{6}, \tag{3.16}$$
$$p_{[1^33^{-1}]}(12\cdot 4^r n + 26\cdot 4^{r-1}) \equiv 0 \pmod{6}, \tag{3.17}$$
$$p_{[1^33^{-1}]}(6\cdot 4^r n + 20\cdot 4^{r-1}) \equiv 0 \pmod{6}. \tag{3.18}$$

Proof For any positive integer k and prime number t, it is easy to see that

$$(q^k;q^k)_\infty^t \equiv (q^{tk};q^{tk})_\infty \pmod{t}. \tag{3.19}$$

Employing (3.19), we deduce that

$$\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2}{(q;q)_\infty^6(q^6;q^6)_\infty} + 3q\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^3(q^{18};q^{18})_\infty^2}{(q;q)_\infty^8(q^6;q^6)_\infty^2(q^9;q^9)_\infty} \equiv 1 + 3q\frac{(q^9;q^9)^3}{(q^3;q^3)} \pmod{6}. \tag{3.20}$$

Using (3.20) in (3.11), we obtain

$$\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(2n+2)q^n \equiv 3\frac{(q^9;q^9)^3}{(q^3;q^3)} \pmod{6}. \tag{3.21}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} in (3.21), we obtain (3.14) and (3.15), respectively. Now equating coefficients of q^{3n} on both sides of (3.21), we have

$$\sum_{n=0}^{\infty} p_{[1^33^{-1}]}(6n+2)q^n \equiv 3\frac{(q^3;q^3)^3}{(q;q)} \pmod{6}. \tag{3.22}$$

It follows from (1.2) and (3.22) that

$$p_{[1^33^{-1}]}(6n+2) \equiv 3\ p_{[1^13^{-3}]}(n) \pmod{6}. \tag{3.23}$$

In [21], E. X. W. Xia and O. X. M. Yao have proved that, for $r \geq 1$

$$p_{[1^13^{-3}]}\left(4^r n + \frac{7\cdot 4^{r-1}-1}{3}\right) \equiv 0 \pmod{2}, \tag{3.24}$$

$$p_{[1^13^{-3}]}\left(2\cdot 4^r n + \frac{13\cdot 4^{r-1}-1}{3}\right) \equiv 0 \pmod{2}, \tag{3.25}$$

$$p_{[1^13^{-3}]}\left(4^r n + \frac{10\cdot 4^{r-1}-1}{3}\right) = 0. \tag{3.26}$$

From (3.23) to (3.26), we obtain (3.16) to (3.18). □

Corollary 3.1.3 *We have*

$$p_{[1^33^{-1}]}(12n+10) \equiv 0 \pmod{18}, \tag{3.27}$$
$$p_{[1^33^{-1}]}(12n+6) \equiv 0 \pmod{18}, \tag{3.28}$$
$$p_{[1^33^{-1}]}(24n+14) \equiv 0 \pmod{18}, \tag{3.29}$$
$$p_{[1^33^{-1}]}(48n+26) \equiv 0 \pmod{18}. \tag{3.30}$$

Proof From [5, Corollary 4.18, p. 405], we have

$$p_{[1^3 3^{-1}]}(4n+2) \equiv 0 \pmod{9}. \tag{3.31}$$

Corollary 3.1.3 follows from the Corollary 3.1.2 and (3.31). □

Theorem 3.2 *Let S denote the set consisting of one copy of the positive integers that are not congruent to* $\pm 3 \pmod 9$ *and one additional copy of those positive integers that are multiples of 9. Let T denote the set consisting of one copy of the positive integers that are not multiples of 9. Let $A(N)$ and $B(N)$ be the number of partitions of $2N$ into, respectively, odd parts in S or into even positive integers that are not multiples of 9 and even parts in S or into even positive integers that are not multiples of 9. Let $C(N)$ be the number of partitions of N into odd parts in S and let $D(N)$ and $E(N)$ be the number of partitions of $2N$ into, respectively, odd parts in T and even number of even parts in T. Then,*

(i) *for $N \geq 1$, $A(N) = B(N-1)$,*
(ii) *for $N \geq 0$, $C(2N+1) = D(N)$ and $C(2N) = E(N)$.*

Proof Multiplying $\chi(-q)/\chi(-q^9)$ on both sides of the identity (3.7) and using the fact $\chi(q)\psi(-q) = f(-q^2)$, we find that

$$\frac{\chi(-q)\chi^2(-q^9)}{\chi(-q^3)} + q\frac{\chi(-q)}{\chi(-q^9)} = \frac{f(-q^2)}{f(-q^{18})}. \tag{3.32}$$

Replacing q by $-q$ in (3.32), we obtain

$$\frac{\chi(q)\chi^2(q^9)}{\chi(q^3)} - q\frac{\chi(q)}{\chi(q^9)} = \frac{f(-q^2)}{f(-q^{18})}. \tag{3.33}$$

Subtracting (3.32) from (3.33), we have

$$\frac{\chi(q)\chi^2(q^9)}{\chi(q^3)} - \frac{\chi(-q)\chi^2(-q^9)}{\chi(-q^3)} = q\left(\frac{\chi(q)}{\chi(q^9)} + \frac{\chi(-q)}{\chi(-q^9)}\right). \tag{3.34}$$

Adding (3.32) and (3.33), we deduce that

$$\frac{\chi(q)\chi^2(q^9)}{\chi(q^3)} + \frac{\chi(-q)\chi^2(-q^9)}{\chi(-q^3)} = q\left(\frac{\chi(q)}{\chi(q^9)} - \frac{\chi(-q)}{\chi(-q^9)}\right) + 2\frac{f(-q^2)}{f(-q^{18})}. \tag{3.35}$$

Using (3.4) in (3.35), we arrive at

$$\left(\frac{\chi(q)\chi^2(q^9)}{\chi(q^3)} + \frac{\chi(-q)\chi^2(-q^9)}{\chi(-q^3)}\right)\frac{f(-q^{18})}{f(-q^2)} = 2q^2\frac{\chi(-q^6)}{\chi(-q^2)\chi^2(-q^{18})}\frac{f(-q^{18})}{f(-q^2)} + 2. \tag{3.36}$$

Recall the formula for the multipliers for degree 9 [6, p. 352, Entry 3(x)],

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}, \tag{3.37}$$

where m is defined as in (1.6). The above modular equation can be transformed into (see [10])

$$\frac{(-q^2;q^2)_\infty}{(-q^{18};q^{18})_\infty} - \frac{(q^2;q^2)_\infty}{(q^{18};q^{18})_\infty} = q\left\{\frac{(-q;q^2)_\infty}{(-q^9;q^{18})_\infty} - \frac{(q;q^2)_\infty}{(q^9;q^{18})_\infty}\right\}. \tag{3.38}$$

Employing (3.38) in (3.35), we obtain

$$\frac{(-q;q^2)_\infty(-q^9;q^{18})^2}{(-q^3;q^6)}+\frac{(q;q^2)_\infty(q^9;q^{18})^2}{(q^3;q^6)}=\frac{(-q^2;q^2)_\infty}{(-q^{18};q^{18})_\infty}+\frac{(q^2;q^2)_\infty}{(q^{18};q^{18})_\infty}. \quad (3.39)$$

It is now clear that (3.36) and (3.34), (3.39) have the partition-theoretic interpretation as claimed in (i) and (ii) of Theorem 3.2, respectively. □

Example 2 Let $N=4$. Then $A(4)=B(3)=7$, we have the representation

$$7_r+1_r=5_r+1_r+\bar{2}=\bar{8}=\bar{6}+\bar{2}=\bar{4}+\bar{4}=\bar{4}+\bar{2}+\bar{2}=\bar{2}+\bar{2}+\bar{2}+\bar{2},$$
$$4_r+2_r=4_r+\bar{2}=\bar{4}+2_r=\bar{6}=\bar{4}+\bar{2}=\bar{2}+\bar{2}+2_r=\bar{2}+\bar{2}+\bar{2}.$$

Let $N=9$. Then $C(19)=D(9)=4$, and

$$19_r=13_r+5_r+1_r=11_r+7_r+1_r=9_r+9_b+1_r,$$
$$17+1=15+3=13+5=11+7.$$

Also, $C(18)=E(9)=4$, and

$$17_r+1_r=13_r+5_r=11_r+7_r=9_r+9_b,$$
$$16+2=14+4=12+6=10+8.$$

Theorem 3.3 *Let S denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are not congruent to* $\pm 3 \pmod 9$. *Let T denote the set consisting of the positive integers that are not multiples of 9. Let $A(N)$ be the number of partitions of N into odd elements of S, let $B(N)$ be the number of partitions of $2N$ into even elements of S, let $C(N)$ and $D(N)$ be the number of partitions of $2N$ into, respectively, odd elements of T and even elements of T and let $E(N)$ be the number of partitions of $2N$ into even number of even parts in T. Then, for $N\ge 0$,*

$$A(2N)=C(N) \text{ and } A(2N+1)=3\,D(N)-B(N)=D(N)+E(N).$$

Proof Multiplying (3.1) by $\chi(q^9)/\chi(q)$ on both sides and using the fact $\chi(q)\psi(-q)=f(-q^2)$, we have

$$\frac{\chi^2(q)\chi(q^9)}{\chi(q^3)}=\frac{\chi(q^9)}{\chi(q)}+3q\frac{f(-q^{18})}{f(-q^2)}. \quad (3.40)$$

Replacing q by $-q$ in (3.40), we obtain

$$\frac{\chi^2(-q)\chi(-q^9)}{\chi(-q^3)}=\frac{\chi(-q^9)}{\chi(-q)}-3q\frac{f(-q^{18})}{f(-q^2)}. \quad (3.41)$$

Adding (3.40) and (3.41), we find that

$$\frac{\chi^2(q)\chi(q^9)}{\chi(q^3)}+\frac{\chi^2(-q)\chi(-q^9)}{\chi(-q^3)}=\frac{\chi(-q^9)}{\chi(-q)}+\frac{\chi(q^9)}{\chi(q)}. \quad (3.42)$$

Subtracting (3.41) from (3.40), we obtain

$$\frac{\chi^2(q)\chi(q^9)}{\chi(q^3)}-\frac{\chi^2(-q)\chi(-q^9)}{\chi(-q^3)}=6q\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty}+\left(\frac{\chi(q^9)}{\chi(q)}-\frac{\chi(-q^9)}{\chi(-q)}\right). \quad (3.43)$$

Using the identity (3.4) in (3.43), we deduce that

$$\frac{\chi^2(q)\chi(q^9)}{\chi(q^3)} - \frac{\chi^2(-q)\chi(-q^9)}{\chi(-q^3)} = 6q\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty} - 2q\frac{(-q^2;q^2)^2_\infty(-q^{18};q^{18})_\infty}{(-q^6;q^6)_\infty}. \tag{3.44}$$

Consider the modular equation of degree 9 [6, p. 352, Entry 3(xi)],

$$\frac{3}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} \tag{3.45}$$

where m is defined by (1.6). The identity (3.45) can be converted to q-product identity

$$\frac{(q^9;q^{18})_\infty}{(q;q^2)_\infty} - \frac{(-q^9;q^{18})_\infty}{(-q;q^2)_\infty} = q\left\{3\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty} - \frac{(-q^{18};q^{18})_\infty}{(-q^2;q^2)_\infty}\right\}. \tag{3.46}$$

Applying (3.46) in (3.43), we have

$$\frac{\chi^2(q)\chi(q^9)}{\chi(q^3)} - \frac{\chi^2(-q)\chi(-q^9)}{\chi(-q^3)} = 2q\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty} + q\left\{\frac{(q^{18};q^{18})_\infty}{(q^2;q^2)_\infty} + \frac{(-q^{18};q^{18})_\infty}{(-q^2;q^2)_\infty}\right\}. \tag{3.47}$$

From (3.42) and (3.44), (3.47), we deduce the partition identity as claimed in the theorem. □

Example 3 Let $N = 3$. Then $A(6) = C(3) = 4$, and

$$5_r + 1_r = 5_r + 1_b = 5_b + 1_r = 5_b + 1_b,$$
$$5 + 1 = 3 + 3 = 3 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1.$$

Also, $A(7) = 4$, $D(3) = 3$, $B(3) = 5$ and $E(3) = 1$, and

$$7_r = 7_b = 5_r + 1_r + 1_b = 5_b + 1_r + 1_b,$$
$$6 = 4 + 2 = 2 + 2 + 2,$$
$$6_r = 4_r + 2_r = 4_r + 2_b = 4_b + 2_b = 4_b + 2_r,$$
$$4 + 2.$$

Theorem 3.4 *Let S denote the set consisting of one copy of the positive integers and five additional copies of those positive integers that are multiples of 3. Let T denote the set consisting of four copies of the positive integers that are not multiples of 3. Let $A(N)$ and $B(N)$ be the number of partitions of $2N$ into, respectively, odd parts in S or into four different colors of positive integers that are not multiples of 3 and even parts in S or into four different colors of positive integers that are not multiples of 3. Let $C(N)$ be the number of partitions of N into odd parts in S and let $D(N)$ and $E(N)$ be the number of partitions of $2N$ into, respectively, odd parts in T and even number of even parts in T. Then,*

(i) *for $N \geq 1$, $A(N) = 4B(N-1)$,*
(ii) *for $N \geq 0$, $C(2N+1) = D(N)$ and $C(2N) = E(N)$.*

Proof From [6, p. 345, Entry 1(i)]

$$\frac{\chi^9(-q^3)}{q\chi^3(-q)} + 1 = \frac{\psi^4(q)}{q\psi^4(q^3)}. \tag{3.48}$$

Multiplying (3.48) by $q\chi^4(-q)/\chi^4(-q^3)$ on both sides and using the fact $\psi(q)\chi(-q) = f(-q^2)$, we arrive at

$$\chi(-q)\chi^5(-q^3) + q\frac{\chi^4(-q)}{\chi^4(-q^3)} = \frac{f^4(-q^2)}{f^4(-q^6)}. \tag{3.49}$$

Replacing q by $-q$ in (3.49), we obtain

$$\chi(q)\chi^5(q^3) - q\frac{\chi^4(q)}{\chi^4(q^3)} = \frac{f^4(-q^2)}{f^4(-q^6)}. \tag{3.50}$$

Adding (3.49) and (3.50), we arrive at

$$\chi(q)\chi^5(q^3) + \chi(-q)\chi^5(-q^3) = q\left(\frac{\chi^4(q)}{\chi^4(q^3)} - \frac{\chi^4(-q)}{\chi^4(-q^3)}\right) + 2\frac{f^4(-q^2)}{f^4(-q^6)}. \tag{3.51}$$

Setting $a = c = q$ and $b = d = q^5$ in (2.2) and (2.3), then multiplying both resulting identities, we obtain an identity equivalent to

$$\frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} - \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4} = 8q(-q^2;q^2)_\infty(-q^6;q^6)_\infty^5. \tag{3.52}$$

Using (3.52) in (3.51), we find an identity equivalent to

$$\begin{aligned}&\{(-q;q^2)_\infty(-q^3;q^6)_\infty^5 + (q;q^2)_\infty(q^3;q^6)_\infty^5\}\frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}\\&= 8q^2(-q^2;q^2)_\infty(-q^6;q^6)_\infty^5\frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4} + 2.\end{aligned} \tag{3.53}$$

Consider the multiplier of degree 3 [6, p. 230, Entry 5(vii)],

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}, \tag{3.54}$$

where m is defined by (1.6). The above modular equation can be transcribed into (see [10])

$$\frac{(-q^2;q^2)_\infty^4}{(-q^6;q^6)_\infty^4} - \frac{(q^2;q^2)_\infty^4}{(q^6;q^6)_\infty^4} = q\left\{\frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} - \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4}\right\}. \tag{3.55}$$

Employing (3.55) in (3.51), we obtain

$$(-q;q^2)_\infty(-q^3;q^6)_\infty^5 + (q;q^2)_\infty(q^3;q^6)_\infty^5 = \frac{(-q^2;q^2)_\infty^4}{(-q^6;q^6)_\infty^4} + \frac{(q^2;q^2)_\infty^4}{(q^6;q^6)_\infty^4}. \tag{3.56}$$

Subtracting (3.49) from (3.50), we deduce that

$$(-q;q^2)_\infty(-q^3;q^6)_\infty^5 - (q;q^2)_\infty(q^3;q^6)_\infty^5 = q\left\{\frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} + \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4}\right\}. \tag{3.57}$$

It is now easy to see that (3.53) and (3.56), (3.57), respectively, have partition-theoretic interpretation claimed in Theorem 3.4 (i) and (ii). □

Example 4 Let $N = 2$. Then $A(2) = 20$ and $B(1) = 5$, and

$4_R = 4_B = 4_G = 4_P = 3_r + 1_r = 3_b + 1_r = 3_g + 1_r = 3_o + 1_r = 3_p + 1_r = 3_y + 1_r$
$= 2_R + 2_R$, 9 further partitions of the form $2 + 2$,
$2_r = 2_R = 2_B = 2_G = 2_P$.
Also, $C(4) = E(2) = 6$, *and*
$3_r + 1_r$, 5 further partitions of the form $3 + 1$,
$2_{r_1} + 2_{b_1}$, 5 further partitions of the form $2 + 2$.

Corollary 3.4.1 *Let $p_{[1^1 3^5]}(n)$ be the number of six colored partitions of n with five of the six colors appearing only in parts that are multiples of 3. Then*

$$\sum_{n=0}^{\infty} p_{[1^1 3^5]}(2n)q^n = 4q\frac{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^{10}}{(q;q)_\infty^3(q^3;q^3)_\infty^{15}} + \frac{(q;q)_\infty^2(q^2;q^2)_\infty(q^6;q^6)_\infty^5}{(q^3;q^3)_\infty^{14}}. \tag{3.58}$$

Proof We have

$$\sum_{n=0}^{\infty} p_{[1^1 3^5]}(n)q^n = \frac{1}{(q;q)_\infty(q^3;q^3)_\infty^5}. \tag{3.59}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} p_{[1^1 3^5]}(n)q^n + \sum_{n=0}^{\infty} p_{[1^1 3^5]}(n)(-q)^n \\ = \frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty^5}{(q^2;q^2)_\infty^2(q^6;q^6)_\infty^{10}}\left\{(-q;q^2)_\infty(-q^3;q^6)_\infty^5 + (q;q^2)_\infty(q^3;q^6)_\infty^5\right\}.\end{aligned} \tag{3.60}$$

Employing (3.53) in (3.60), then comparing even terms from both sides of the resulting identity, we obtain (3.58). □

Corollary 3.4.2 *We have*

$$p_{[1^1 3^5]}(4n + 2) \equiv 0 \pmod{2}, \tag{3.61}$$

$$p_{[1^1 3^5]}(8n + 4) \equiv 0 \pmod{2}. \tag{3.62}$$

Proof Using (3.19), we deduce that

$$\frac{(q;q)_\infty^2(q^2;q^2)_\infty(q^6;q^6)_\infty^5}{(q^3;q^3)_\infty^{14}} \equiv \frac{(q^4;q^4)_\infty}{(q^{12};q^{12})_\infty} \pmod{2}. \tag{3.63}$$

From (3.58) and (3.63), we obtain (3.61) and (3.62). □

Theorem 3.5 *Let S denote the set consisting of five copies of the positive integers and one additional copy of those positive integers that are multiples of 3. Let T denote the set consisting of four different colors of the positive integers that are not multiples of 3. Let $A(N)$ be the number of partitions of N into odd elements of S, let $B(N)$ be the number of partitions of $2N$ into even elements of S, let $C(N)$ and $D(N)$ be the number of partitions of $2N$ into, respectively, odd elements of T and even elements of T and let $E(N)$ be the number of partitions of $2N$ into even number of even parts in T. Then, for $N \geq 0$,*

$$A(2N) = C(N) \text{ and } A(2N + 1) = 9\,D(N) - 4\,B(N) = 4\,D(N) + E(N).$$

Proof From [6, p. 345, Entry 1(ii)]

$$1+3q\frac{\psi(-q^9)}{\psi(-q)}=\left(1+9q\frac{\psi^4(-q^3)}{\psi^4(-q)}\right)^{1/3}. \tag{3.64}$$

Invoking (3.1) in (3.64), we find that

$$\frac{\chi^9(q)}{\chi^3(q^3)}=1+9q\frac{\psi^4(-q^3)}{\psi^4(-q)}. \tag{3.65}$$

Multiplying (3.65) by $\chi^4(q^3)/\chi^4(q)$, we obtain

$$\chi^5(q)\chi(q^3)=\frac{\chi^4(q^3)}{\chi^4(q)}+9q\frac{f^4(-q^6)}{f^4(-q^2)}. \tag{3.66}$$

Replacing q by $-q$, we get

$$\chi^5(-q)\chi(-q^3)=\frac{\chi^4(-q^3)}{\chi^4(-q)}-9q\frac{f^4(-q^6)}{f^4(-q^2)}. \tag{3.67}$$

Adding (3.66) and (3.67), we arrive at

$$(-q;q^2)_\infty^5(-q^3;q^6)_\infty+(q;q^2)_\infty^5(q^3;q^6)_\infty=\frac{(q^3;q^6)_\infty^4}{(q;q^2)_\infty^4}+\frac{(-q^3;q^6)_\infty^4}{(-q;q^2)_\infty^4}. \tag{3.68}$$

Subtracting (3.67) from (3.66), we obtain

$$\chi^5(q)\chi(q^3)-\chi^5(-q)\chi(-q^3)=\frac{\chi^4(q^3)}{\chi^4(q)}-\frac{\chi^4(-q^3)}{\chi^4(-q)}+18q\frac{f^4(-q^6)}{f^4(-q^2)}. \tag{3.69}$$

Using (3.52) in (3.69), we deduce an identity equivalent to

$$(-q;q^2)_\infty^5(-q^3;q^6)_\infty-(q;q^2)_\infty^5(q^3;q^6)_\infty=18q\frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}-8q(-q^2;q^2)_\infty^5(-q^6;q^6)_\infty. \tag{3.70}$$

Consider the multiplier of degree 3 [6, p. 230, Entry 5(vii)],

$$\frac{9}{m^2}=\left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}, \tag{3.71}$$

where m is defined as in (1.6). The above modular equation can be transformed into the q-products

$$\frac{(q^3;q^6)_\infty^4}{(q;q^2)_\infty^4}-\frac{(-q^3;q^6)_\infty^4}{(-q;q^2)_\infty^4}=q\left\{9\frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}-\frac{(-q^6;q^6)_\infty^4}{(-q^2;q^2)_\infty^4}\right\}. \tag{3.72}$$

Employing (3.72) in (3.69), we arrive at

$$\begin{aligned}(-q\ ;q^2)_\infty^5&(-q^3;q^6)_\infty-(q;q^2)_\infty^5(q^3;q^6)_\infty\\&=8q\frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}+q\left\{\frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty^4}+\frac{(-q^6;q^6)_\infty^4}{(-q^2;q^2)_\infty^4}\right\}.\end{aligned} \tag{3.73}$$

It is now readily seen that (3.68) and (3.70), (3.73), have the partition-theoretic interpretation as claimed in Theorem 3.5. □

Example 5 Let $N = 2$. Then $A(4) = C(2) = 35$, and

$$\begin{aligned}3_r + 1_r &= 3_b + 1_r = 3_g + 1_r = 3_o + 1_r = 3_p + 1_r = 3_y + 1_r, \text{ 24 additional} \\ &\quad \text{representations of the form } 3+1 \\ &= 1_r + 1_b + 1_g + 1_o, \text{ 4 further representations of the form } 1+1+1+1, \\ 1_R + 1_R &+ 1_R + 1_R, \text{ 34 further partitions of the form } 1+1+1+1.\end{aligned}$$

Also, $A(5) = 66$, $B(2) = 15$, $D(2) = 14$ and $E(2) = 10$, and

$$\begin{aligned}5_r &= 5_b = 5_g = 5_o = 5_p = 3_r + 1_r + 1_r, \text{ 59 further partitions of the form } 3+1+1 \\ &= 1_r + 1_b + 1_g + 1_o + 1_p, \\ 4_r &= 4_b = 4_g = 4_o = 4_p = 2_r + 2_b, \text{ 9 further partitions of the form } 2+2, \\ 4_R &= 4_B = 4_G = 4_O = 2_R + 2_R = 2_R + 2_B = 2_R + 2_G = 2_R + 2_O, \text{ 6 further} \\ &\quad \text{representations of the form } 2+2.\end{aligned}$$

Corollary 3.5.1 *Let $p_{[1^5 3^1]}(n)$ be the number of six colored partitions of n with one of the six colors appearing only in parts that are multiples of 3. Then*

$$\sum_{n=0}^{\infty} p_{[1^5 3^1]}(2n+1) q^n = 9\frac{(q^2;q^2)_\infty^5 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty}{(q;q)_\infty^{14}} - 4\frac{(q^2;q^2)_\infty^{10}(q^6;q^6)_\infty^2}{(q;q)_\infty^{15}(q^3;q^3)_\infty^3}. \quad (3.74)$$

Proof We have

$$\sum_{n=0}^{\infty} p_{[1^5 3^1]}(n) q^n = \frac{1}{(q;q)_\infty^5 (q^3;q^3)_\infty}. \quad (3.75)$$

Therefore,

$$\begin{aligned}&2\textstyle\sum_{n=0}^{\infty} p_{[1^5 3^1]}(2n+1) q^{2n+1} \\ &= \frac{(q^4;q^4)_\infty^5 (q^{12};q^{12})_\infty}{(q^2;q^2)_\infty^{10}(q^6;q^6)_\infty^2}\left\{(-q;q^2)_\infty^5(-q^3;q^6)_\infty - (q;q^2)_\infty^5(q^3;q^6)_\infty\right\}.\end{aligned} \quad (3.76)$$

Employing (3.70) in (3.76), then comparing odd terms from both sides of the resulting identity, we obtain (3.74). □

Corollary 3.5.2 *We have*

$$p_{[1^5 3^1]}(4n+3) \equiv 0 \pmod{2}, \quad (3.77)$$

$$p_{[1^5 3^1]}(8n+5) \equiv 0 \pmod{2}. \quad (3.78)$$

Proof Using (3.19), we deduce that

$$9\frac{(q^2;q^2)_\infty^5 (q^3;q^3)_\infty^2 (q^6;q^6)_\infty}{(q;q)_\infty^{14}} \equiv \frac{(q^{12};q^{12})_\infty}{(q^4;q^4)_\infty} \pmod{2}. \quad (3.79)$$

From (3.74) and (3.79), we obtain (3.77) and (3.78). □

Theorem 3.6 *Let S denote the set consisting of six copies of the positive integers and three additional copies of those positive integers that are not multiples of 3. Let $A(N)$ be the number of partitions of N into odd parts in S, let $B(N)$ be the number of partitions of $2N$ into even parts in S. Let $C(N)$ be the number of partitions of $2N-2$ into parts congruent* *to* $\pm 2, \pm 4$ *or* $6 \pmod{12}$ *having four different colors, with parts congruent to* $6 \pmod{12}$ *having two additional color and with parts congruent to* $\pm 2 \pmod{12}$ *having one additional color and let* $D(N)$ *be the number of partitions of* $2N$ *into parts congruent* $\pm 1, \pm 4$ *or* $\pm 5 \pmod{12}$ *having four distinct colors. Then,*

(i) *for* $N \geq 1$, $A(2N) = 36\, C(N) = 4\, B(N)$ *and*
(ii) *for* $N \geq 0$, $A(2N+1) = 9\, D(N)$.

Proof Replacing q by $-q$ in (3.65), we have

$$\frac{\chi^9(-q)}{\chi^3(-q^3)} = 1 - 9q\frac{\psi^4(q^3)}{\psi^4(q)}. \tag{3.80}$$

Add (3.65) with (3.80). Then invoking (3.52) and Euler's identity (2.7), we find that

$$\frac{(-q;q^2)_\infty^9}{(-q^3;q^6)_\infty^3} + \frac{(q;q^2)_\infty^9}{(q^3;q^6)_\infty^3} = 2 + 72\frac{q^2}{(q^{2\pm};q^{12})_\infty^5 (q^6;q^{12})_\infty^6 (q^{4\pm};q^{12})_\infty^4}. \tag{3.81}$$

Subtracting (3.80) from (3.65) and then employing (2.7), we have

$$\frac{(-q;q^2)_\infty^9}{(-q^3;q^6)_\infty^3} - \frac{(q;q^2)_\infty^9}{(q^3;q^6)_\infty^3} = 9q\left\{\frac{(q^3;q^6)_\infty^4}{(q;q^2)_\infty^4} + \frac{(-q^3;q^6)_\infty^4}{(-q;q^2)_\infty^4}\right\}\frac{(q^{12};q^{12})_\infty^4}{(q^4;q^4)_\infty^4}. \tag{3.82}$$

From [7, p. 376, Entry 37(vii)],

$$\frac{\psi^3(q)}{\psi(q^3)} + \frac{\psi^3(-q)}{\psi(-q^3)} = 2\frac{\psi^3(q^2)}{\psi(q^6)}. \tag{3.83}$$

Cube both sides of the identity (3.83). Then rewriting in q-products, we have

$$\frac{(-q;q^2)_\infty^9}{(-q^3;q^6)_\infty^3} + \frac{(q;q^2)_\infty^9}{(q^3;q^6)_\infty^3} = 8\frac{(-q^2;q^2)_\infty^9}{(-q^6;q^6)_\infty^3} - 6. \tag{3.84}$$

Now, Theorem 3.6 (i) follows from the identities (3.81) and (3.84) and (ii) follows from (3.82). □

Example 6 Let $N = 2$. Then $A(4) = 180$, $C(2) = 5$ and $B(2) = 45$ and

$$\begin{aligned} 3_r + 1_r &= 3_r + 1_b = 3_r + 1_g = 3_r + 1_o, \text{ 50 further representations of the form } 3+1 \\ &= 1_r + 1_b + 1_g + 1_o, \text{ 125 further partitions of the form } 1+1+1+1, \\ &\quad 2_R = 2_B = 2_G = 2_O = 2_V, \\ 4_r &= 4_b = 4_g = 4_o = 4_y = 4_v = 4_p = 4_i = 4_m = 2_r + 2_b = 2_r + 2_g, \text{ 34 further} \\ &\quad \text{partitions of the form } 2+2. \end{aligned}$$

Also, $A(5) = 351$ and $D(2) = 39$

$$\begin{aligned} 5_r &= 5_b = 5_g = 5_y = 5_o = 5_p = 5_i = 5_m = 5_v = 3_r + 1_r + 1_b, \text{ 215 further partitions} \\ &\quad \text{of the form } 3+1+1 \\ &= 1_r + 1_b + 1_g + 1_o + 1_y, \text{ 125 further partitions of the form } 1+1+1+1+1, \\ 4_R &= 4_B = 4_G = 4_O = 1_R + 1_R + 1_R + 1_R, \text{ 34 further representations of the form} \\ &\quad 1+1+1+1. \end{aligned}$$

Corollary 3.6.1 *We have* $A(2N) \equiv 0 \pmod{36}$ *and* $A(2N+1), B(N) \equiv 0 \pmod 9$.

Theorem 3.7 *Let S denote the set consisting of two copies of the positive integers that are not multiples of 4 and one additional copy of those positive integers that are not multiples of 3 or 4. Let T denote the set consisting of 2 copies of the positive integers that are not congruent to* $2 \pmod 4$ *and one additional copy of those positive integers that are not multiples of 3 or not congruent to* $2 \pmod 4$. *Let $A(N)$ and $B(N)$ be the number of partitions of $2N$ into, respectively, parts in S and parts in T. Then, for $N \geq 1$,*

$$A(N) = 2\,B(N).$$

Proof Replacing q by q^2 in (3.83), multiplying by $\psi^3(q)/\psi(q)$ on both sides of the resulting equation, we deduce that

$$\frac{\psi^3(q^2)\psi^3(q)}{\psi(q^6)\psi(q^2)} + \frac{\psi^3(-q^2)\psi^3(q)}{\psi(-q^6)\psi(q^2)} = 2\frac{\psi^3(q^4)\psi^3(q)}{\psi(q^{12})\psi(q^2)}. \tag{3.85}$$

Replacing q by $-q$ in (3.85), then add the resulting equation with (3.85), we have the equivalent q-product identity

$$\begin{aligned} &\frac{(-q;q)^3_\infty(-q^{12};q^{12})_\infty}{(-q^4;q^4)^3_\infty(-q^3;q^3)_\infty} + \frac{(q;-q)^3_\infty(-q^{12};q^{12})_\infty}{(-q^4;q^4)^3_\infty(q^3;-q^3)_\infty} + 2 \\ &= 2\left\{\frac{(-q;q)^3_\infty(-q^6;q^{12})_\infty}{(-q^2;q^4)^3_\infty(-q^3;q^3)_\infty} + \frac{(q;-q)^3_\infty(-q^6;q^{12})_\infty}{(-q^2;q^4)^3_\infty(q^3;-q^3)_\infty}\right\}. \end{aligned} \tag{3.86}$$

Equating the coefficients of q^{2N} on both sides of the equation, we complete the proof. □

Example 7 Let $N = 2$. Then $A(2) = 18$ and $B(2) = 9$, and

$$\begin{aligned} 3_r+1_r &= 3_r+1_b = 3_r+1_g = 3_b+1_r = 3_b+1_b = 3_b + 1_g = 2_r + 2_b = 2_r + 2_g = 2_b + 2_g \\ &= 2_r + 1_r + 1_b = 2_r + 1_r + 1_g = 2_r + 1_g + 1_b \text{ 6 further partitions} \\ &\quad \text{of the form } 2+1+1, \\ 4_r &= 4_b = 4_g = 3_r + 1_r = 3_r + 1_b = 3_r + 1_g = 3_b + 1_r = 3_b + 1_b = 3_b + 1_g. \end{aligned}$$

Theorem 3.8 *Let S denote the set consisting of one copy of the positive integers that are not congruent to* $6 \pmod{12}$ *and another copy of the positive integers that are either congruent to* $\pm 5 \pmod{12}$ *or are multiples of 4. Let $A(N)$ be the number of partitions of $2N + 1$ into odd parts in S and Let $B(N)$ be the number of partitions of $2N$ into odd parts in S. Then $A(N) = B(N)$.*

Proof Setting $a = q$, $b = q^3$ in (2.11), we obtain the equality

$$f(q,q^3)f(q^5,q^7) - f(-q,-q^3)f(-q^5,-q^7) = 2qf(q^2,q^{10})\psi(q^4). \tag{3.87}$$

Invoking (2.5), (2.6), and (2.7) in (3.87), we have

$$(-q;q^2)_\infty(-q^5,-q^7;q^{12})_\infty - (q;q^2)_\infty(q^5,q^7;q^{12})_\infty = 2q\frac{(-q^2;q^2)_\infty}{(-q^6;q^{12})_\infty}(-q^4;q^4)_\infty. \tag{3.88}$$

Equating the coefficients of q^{2N+1} on both sides of the equation, we complete the proof. □

Example 8 Let $N = 7$. Then $A(7) = B(7) = 9$, and

$$\begin{aligned}
15_r &= 11_r + 3_r + 1_r = 9_r + 5_r + 1_r = 9_r + 5_b + 1_r = 7_r + 7_b + 1_r = 7_r + 5_r + 3_r \\
&= 7_b + 5_r + 3_r = 7_r + 5_b + 3_r = 7_b + 5_b + 3_r, \\
14_r &= 12_r + 2_r = 12_b + 2_r = 10_r + 4_r = 10_r + 4_b = 8_r + 4_r + 2_r = 8_r + 4_b + 2_r \\
&= 8_b + 4_r + 2_r = 8_b + 4_b + 2_r.
\end{aligned}$$

4 Partitions and Theta Constant Identities of Degree 5

Theorem 4.1 *Let $A(N)$ denote the number of partitions of N into five distinct colors, with one color, appearing at most once and only in odd parts that are not multiples of 5 and remaining colors also appearing at most once and only in odd parts, let $B(N)$ denote the number of partitions of $2N-2$ into two distinct colors of even positive integers that are not multiples of 20, with parts congruent to ± 2 or $\pm 6 \pmod{20}$ having one additional color and with parts congruent to $10 \pmod{20}$ having additional 2 colors and let $C(N)$ be the number of partitions of $2N$ into 2 distinct colors of positive integers that are not multiples of 5, wherein odd parts are distinct. Then,*

(i) *for $N \geq 1$, $A(2N) = 10\,B(N)$ and*
(ii) *for $N \geq 0$, $A(2N+1) = 5\,C(N)$.*

Proof From [7 p. 202, Entry 50(ii)]

$$\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)}. \tag{4.1}$$

Replacing q by $-q$ in (4.1), we obtain

$$\frac{\chi^5(-q)}{\chi(-q^5)} = 1 - 5q\frac{\psi^2(q^5)}{\psi^2(q)}. \tag{4.2}$$

Adding (4.1) and (4.2), we find that

$$\frac{\chi^5(q)}{\chi(q^5)} + \frac{\chi^5(-q)}{\chi(-q^5)} = 2 + 5q\left(\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\psi^2(q^5)}{\psi^2(q)}\right). \tag{4.3}$$

Set $a = q$, $b = q^9$, $c = q^3$, and $d = q^7$ in (2.8) and (2.9). Then multiplying the resulting identities, we obtain the following equivalent q-product identity

$$\frac{(-q;q^2)^2_\infty}{(-q^5;q^{10})^2_\infty} - \frac{(q;q^2)^2_\infty}{(q^5;q^{10})^2_\infty} = 4q(-q^2;q^2)_\infty(-q^{10};q^{10})^3_\infty. \tag{4.4}$$

Invoking (2.2), (2.6), (2.7), and (4.4) in (4.3), we obtain the equality

$$\frac{(-q;q^2)^5_\infty}{(-q^5;q^{10})_\infty} + \frac{(q;q^2)^5_\infty}{(q^5;q^{10})_\infty} = 2 + 20q^2\frac{(q^{20};q^{20})^2_\infty}{(q^2;q^2)^2_\infty(q^{2\pm},q^{6\pm},q^{10},q^{10};q^{20})_\infty}. \tag{4.5}$$

Now, subtracting (4.2) from (4.1) and using (2.7), we find that

$$\frac{(-q;q^2)^5_\infty}{(-q^5;q^{10})_\infty} - \frac{(q;q^2)^5_\infty}{(q^5;q^{10})_\infty} = 5q\left\{\frac{(-q;q^2)^2_\infty}{(-q^5;q^{10})^2_\infty} + \frac{(q;q^2)^2_\infty}{(q^5;q^{10})^2_\infty}\right\}\frac{(q^{10};q^{10})^2_\infty}{(q^2;q^2)^2_\infty}. \tag{4.6}$$

Equations (4.5) and (4.6) give the partition-theoretic interpretation given in Theorem 4.1. □

Example 9 Let $N = 3$. Then $A(6) = 80$ and $B(3) = 8$, and we record the interpretation

$$\begin{aligned}5_r + 1_r &= 5_r + 1_b, \text{ 18 further representations of the form } 5+1\\ &= 3_r + 3_b = 3_r + 3_o, \text{ 8 further representations of the form } 3+3\\ &= 3_r + 1_r + 1_b + 1_o, \text{ 49 further partitions of the form } 3+1+1+1,\\ 4_{r_1} = 4_{b_1} &= 2_{r_1} + 2_{r_1} = 2_{r_1} + 2_{b_1} = 2_{b_1} + 2_{b_1} = 2_{r_1} + 2_{g_1} = 2_{b_1} + 2_{g_1} = 2_{g_1} + 2_{g_1}.\end{aligned}$$

Now, Let $N = 2$. Then $A(5) = 55$ and $C(2) = 11$, and

$$\begin{aligned}5_r &= 5_b = 5_o = 5_g = 3_r + 1_r + 1_b, \text{ 49 more partitions of the form } 3+1+1\\ &= 1_r + 1_b + 1_o + 1_g + 1_y,\\ 4_R &= 4_B = 2_R + 2_R = 2_R + 2_B = 2_B + 2_B = 3_R + 1_R = 3_R + 1_B = 3_B + 1_R = 3_B + 1_B\\ &= 2_R + 1_R + 1_B = 2_B + 1_R + 1_B.\end{aligned}$$

Corollary 4.1.1 *Let $p_{[1^5 5^{-1}]}(n)$ be the number of 5 colored partitions of n with one of the colors appearing only in parts that are not multiples of 5. Then*

$$\sum_{n=0}^{\infty} p_{[1^5 5^{-1}]}(2n)q^n = \frac{(q^5;q^5)_\infty^2 (q^2;q^2)_\infty^5}{(q^{10};q^{10})_\infty (q;q)_\infty^{10}} + 10q\frac{(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3}{(q;q)_\infty^3 (q^5;q^5)_\infty}. \tag{4.7}$$

Proof We have

$$\sum_{n=0}^{\infty} p_{[1^5 5^{-1}]}(n)q^n = \frac{(q^5;q^5)_\infty}{(q;q)_\infty^5} = \frac{(q^5;q^{10})_\infty (q^{10};q^{10})_\infty}{(q;q^2)_\infty^5 (q^2;q^2)_\infty^5}.$$

Therefore

$$2\sum_{n=0}^{\infty} p_{[1^5 5^{-1}]}(2n)q^{2n} = \frac{(q^{10};q^{10})_\infty^2 (q^4;q^4)_\infty^5}{(q^{20};q^{20})_\infty (q^2;q^2)_\infty^{10}} \left\{ \frac{(-q;q^2)_\infty^5}{(-q^5;q^{10})_\infty} + \frac{(q;q^2)_\infty^5}{(q^5;q^{10})_\infty} \right\}. \tag{4.8}$$

Using (4.5) in (4.8) and then replacing q^2 by q, we obtain (4.7). □

Corollary 4.1.2

$$p_{[1^5 5^{-1}]}(2n+2) \equiv 0 \pmod{10}. \tag{4.9}$$

Proof Employing (3.19), we deduce that

$$\frac{(q^5;q^5)_\infty^2 (q^2;q^2)_\infty^5}{(q^{10};q^{10})_\infty (q;q)_\infty^{10}} \equiv 1 \pmod{10}. \tag{4.10}$$

Congruence (4.9) follows from (4.7) and (4.10). □

Theorem 4.2 *Let S denote the set consisting of one copy of the positive integers and three additional copies of those positive integers that are multiples of 5. Let T denote the set consisting of two copies of the positive integers that are not multiples of 5. Let $A(N)$ be the number of partitions of $2N$ into odd parts in S or into two colors of even parts that are not multiples of 5 and let $B(N)$ be the number of partitions of $2N-2$ into even parts in S or into two colors of even parts that are not multiples of 5. Let $C(N)$ be the number of partitions of N into odd parts in S and let $D(N)$ and $E(N)$ be the number of partitions of $2N$ into, respectively, odd parts in T and even number of even parts in T. Then,*

(i) *for $N \geq 1$, $A(N) = 2B(N)$ and*

(ii) *for* $N \geq 0$, $C(2N+1) = D(N)$ *and* $C(2N) = E(N)$.

Proof From [8, p. 365, Entry 18]

$$\psi^2(-q) + q\psi^2(-q^5) = \frac{f(q^5)\phi(q^5)}{\chi(q)}. \tag{4.11}$$

Multiplying (4.11) by $\chi^2(q)/f^2(-q^{10})$ on both sides and then invoking (2.2)–(2.3) and (2.6), we find that

$$(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 = \frac{(q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty^2} + q\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2}. \tag{4.12}$$

Replacing q by $-q$ in (4.12), we find that

$$(q;q^2)_\infty(q^5;q^{10})_\infty^3 = \frac{(q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty^2} - q\frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}. \tag{4.13}$$

Subtracting (4.13) from (4.12), we obtain the equality

$$(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 - (q;q^2)_\infty(q^5;q^{10})_\infty^3 = q\left\{\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} + \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}\right\}. \tag{4.14}$$

On the other hand adding (4.12) with (4.13), we obtain

$$\begin{aligned}&(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 + (q;q^2)_\infty(q^5;q^{10})_\infty^3\\ &= 2\frac{(q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty^2} + q\left\{\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}\right\}.\end{aligned} \tag{4.15}$$

Employing (4.4) in (4.15), we have

$$\begin{aligned}&\left\{(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 + (q;q^2)_\infty(q^5;q^{10})_\infty^3\right\}\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2}\\ &= 4q^2(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty^3\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2} + 2.\end{aligned} \tag{4.16}$$

Consider the multiplier of degree 5 [6, p. 281, Entry 13(xii)],

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}, \tag{4.17}$$

where m is defined as in (1.6). The above modular equation can be transcribed into (see [10])

$$\frac{(-q^2;q^2)_\infty^2}{(-q^{10};q^{10})_\infty^2} - \frac{(q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty^2} = q\left\{\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}\right\}. \tag{4.18}$$

Using (4.18) in (4.15), we deduce that

$$(-q;q^2)_\infty(-q^5;q^{10})_\infty^3 + (q;q^2)_\infty(q^5;q^{10})_\infty^3 = \frac{(-q^2;q^2)_\infty^2}{(-q^{10};q^{10})_\infty^2} + \frac{(q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty^2}. \tag{4.19}$$

Now, Theorem 4.2 follows from (4.16), (4.14), and (4.19). □

Example 10 Let $N = 3$. Then $A(3) = 16$ and $B(3) = 8$, and

$$\begin{aligned}6 &= \bar{6} = 5_r + 1_r = 5_b + 1_r = 5_g + 1_r = 5_o + 1_r = 4 + 2 = \bar{4} + 2 = 4 + \bar{2} = \bar{4} + \bar{2}\\ &= 3_r + 2 + 1_r = 3_r + \bar{2} + 1_r = 2 + 2 + 2 = 2 + \bar{2} + \bar{2} = 2 + 2 + \bar{2} = \bar{2} + \bar{2} + \bar{2},\\ 4_r &= 4 = \bar{4} = 2_r + 2 = 2_r + \bar{2} = 2 + 2 = 2 + \bar{2} = \bar{2} + \bar{2}.\end{aligned}$$

Now, let $N = 7$. Then $C(14) = E(7) = 12$, and

$$13_r + 1_r = 11_r + 3_r = 9_r + 5_r = 9_r + 5_b = 9_r + 5_g = 9_r + 5_o = 5_r + 5_b + 3_r + 1_r,$$

5 further representations of the form $5 + 5 + 3 + 1$,

$$12 + 2 = 12 + \bar{2} = \overline{12} + 2 = \overline{12} + \bar{2} = 8 + 6 = 8 + \bar{6} = \bar{8} + 6 = \bar{8} + \bar{6} = 6 + 4 + 2 + \bar{2}$$
$$= 6 + \bar{4} + 2 + \bar{2} = \bar{6} + 4 + 2 + \bar{2} = \bar{6} + \bar{4} + 2 + \bar{2}.$$

Corollary 4.2.1 *Let $p_{[1^1 5^3]}(n)$ be the number of four colored partitions of n with three of the four colors appearing only in parts that are multiples of 5. We have*

$$\sum_{n=0}^{\infty} p_{[1^1 5^3]}(2n) q^n = 2q \frac{(q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^6}{(q;q)_\infty^3 (q^5;q^5)_\infty^9} + \frac{(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3}{(q^5;q^5)_\infty^8}. \tag{4.20}$$

Proof We have

$$\sum_{n=0}^{\infty} p_{[1^1 5^3]}(n) q^n = \frac{1}{(q;q)_\infty (q^5;q^5)_\infty^3} = \frac{1}{(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3 (q;q^2)_\infty (q^5;q^{10})_\infty^3},$$

so that

$$2\sum_{n=0}^{\infty} p_{[1^1 5^3]}(2n) q^{2n}$$
$$= \frac{(q^4;q^4)_\infty (q^{20};q^{20})_\infty^3}{(q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^6} \left\{(-q;q^2)_\infty (-q^5;q^{10})_\infty^3 + (q;q^2)_\infty (q^5;q^{10})_\infty^3\right\}. \tag{4.21}$$

Using (4.16) in (4.21), we deduce (4.20). □

Corollary 4.2.2 *We have*

$$p_{[1^1 5^3]}(4n+2) \equiv 0 \pmod{2}. \tag{4.22}$$

Proof Employing (3.19) in (4.20), we deduce that

$$\sum_{n=0}^{\infty} p_{[1^1 5^3]}(2n) q^n \equiv \frac{(q^2;q^2)_\infty}{(q^{10};q^{10})_\infty} \pmod{2}, \tag{4.23}$$

which yields (4.22). □

Theorem 4.3 *Let S denote the set consisting of three copies of the positive integers and one additional copy of those positive integers that are multiples of 5. Let T denote the set consisting of two distinct colors of the positive integers that are not multiples of 5. Let $A(N)$ be the number of partitions of N into odd parts in S, let $B(N)$ be the number of partitions of $2N$ into even parts in S, let $C(N)$ and $D(N)$ be the number of partitions of $2N$ into, respectively, odd parts in T and even parts in T and let $E(N)$ be the number of partitions of $2N$ into even number of even parts in T. Then, for $N \geq 1$,*

$$A(2N) = C(N) \text{ and } A(2N+1) = 5\,D(N) - 2\,B(N) = 2\,D(N) + E(N).$$

Proof From [8, p. 366, Entry 19]

$$\psi^2(-q) + 5q\psi^2(-q^5) = \frac{\phi^2(q)}{\chi(q)\chi(q^5)}. \tag{4.24}$$

Equivalent q-product identity of (4.24) is

$$(-q;q^2)_\infty^3(-q^5;q^{10})_\infty = \frac{(-q^5;q^{10})_\infty^2}{(-q;q^2)_\infty^2} + 5q\frac{(-q^{10};q^{10})_\infty^2}{(-q^2;q^2)_\infty^2}. \tag{4.25}$$

Replacing q by $-q$, we obtain

$$(q;q^2)_\infty^3(q^5;q^{10})_\infty = \frac{(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2} - 5q\frac{(-q^{10};q^{10})_\infty^2}{(-q^2;q^2)_\infty^2}. \tag{4.26}$$

Adding (4.26) and (4.25), we find that

$$(-q;q^2)_\infty^3(-q^5;q^{10})_\infty + (q;q^2)_\infty^3(q^5;q^{10})_\infty = \frac{(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2} + \frac{(-q^5;q^{10})_\infty^2}{(-q;q^2)_\infty^2}. \tag{4.27}$$

Now, subtracting (4.26) from (4.25), we find that

$$\begin{aligned}&(-q;q^2)_\infty^3(-q^5;q^{10})_\infty - (q;q^2)_\infty^3(q^5;q^{10})_\infty \\ &= 10q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2} - \left\{\frac{(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2} - \frac{(-q^5;q^{10})_\infty^2}{(-q;q^2)_\infty^2}\right\}.\end{aligned} \tag{4.28}$$

Employing (4.4) in (4.28), we deduce that

$$\begin{aligned}&(-q;q^2)_\infty^3(-q^5;q^{10})_\infty - (q;q^2)_\infty^3(q^5;q^{10})_\infty \\ &= 10q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2} - 4q(-q^2;q^2)_\infty^3(-q^{10};q^{10})_\infty.\end{aligned} \tag{4.29}$$

Consider the multiplier of degree 5 [6, p. 281, Entry 3(xii)],

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}, \tag{4.30}$$

where m is defined by (1.6). The above modular equation can be transformed into the q-products

$$\frac{(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2} - \frac{(-q^5;q^{10})_\infty^2}{(-q;q^2)_\infty^2} = q\left\{5\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2} - \frac{(-q^{10};q^{10})_\infty^2}{(-q^2;q^2)_\infty^2}\right\}. \tag{4.31}$$

Employing (4.31) in (4.28), we have

$$\begin{aligned}&(-q;q^2)_\infty^3(-q^5;q^{10})_\infty - (q;q^2)_\infty^3(q^5;q^{10})_\infty \\ &= 4q\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2} + q\left\{\frac{(q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^2} + \frac{(-q^{10};q^{10})_\infty^2}{(-q^2;q^2)_\infty^2}\right\}.\end{aligned} \tag{4.32}$$

Theorem 4.3 readily follows from (4.27), (4.29), and (4.32). □

Example 11 Let $N = 3$. Then $A(6) = C(3) = 18$, and

$$\begin{aligned}5_r + 1_r &= 5_b + 1_r \text{ 10further partitions of the form} 5+1 \\ &= 3_r + 3_b = 3_r + 3_g = 3_b + 3_g = 3_r + 1_r + 1_b + 1_g = 3_r + 1_r + 1_b + 1_g \\ &= 3_r + 1_r + 1_b + 1_g,\end{aligned}$$

$$\begin{aligned}3+3 &= 3+\bar{3} = \bar{3}+\bar{3} = 3+1+1+1, \text{ 7 further representations of the form } 3+1+1+1 \\ &= 1+1+1+1+1+1, \text{ 6 further partitions of the form} 1+1+1+1+1+1.\end{aligned}$$

Also, $A(7) = 24$, $D(3) = 10$, $B(3) = 13$ and $E(3) = 4$, and

$$\begin{aligned}
7_r &= 7_b = 7_g = 5_r + 1_r + 1_b,\ \text{11 more representations of the form } 5+1+1\\
&= 3_r + 3_b + 1_r = 3_r + 3_g + 1_r\ \text{7 further partitions of the form } 3+3+1,\\
6 &= \bar{6} = 4+2 = 4+\bar{2} = \bar{4}+2 = \bar{4}+\bar{2} = 2+2+2 = 2+2+\bar{2} = 2+\bar{2}+\bar{2} = \bar{2}+\bar{2}+\bar{2},\\
6_r &= 6_b = 6_g = 4_r + 2_r,\ \text{8 further partitions of the form } 4+2\\
&= 2_r + 2_b + 2_r,\\
4 &+ 2 = 4+\bar{2} = \bar{4}+2 = \bar{4}+\bar{2}.
\end{aligned}$$

Corollary 4.3.1 *Let $p_{[1^3 5^1]}(n)$ be the number of four colored partitions of n with one of the colors appearing only in parts that are multiples of 5. We have*

$$\sum_{n=0}^{\infty} p_{[1^3 5^1]}(2n+1)q^n = 5\frac{(q^2;q^2)_\infty^3 (q^{10};q^{10})_\infty}{(q;q)_\infty^8} - 2\frac{(q^2;q^2)_\infty^6 (q^{10};q^{10})_\infty^2}{(q;q)_\infty^9 (q^5;q^5)_\infty^3}. \tag{4.33}$$

Proof The proof of (4.33) is similar to the proof of (4.20), except that in place of (4.16), (4.29) is used. □

Corollary 4.3.2 *We have*

$$p_{[1^3 5^1]}(4n+3) \equiv 0 \pmod 2. \tag{4.34}$$

Proof Employing (3.19) in (4.33), we deduce that

$$\sum_{n=0}^{\infty} p_{[1^3 5^1]}(2n+1)q^n \equiv \frac{(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty} \pmod 2. \tag{4.35}$$

which yields (4.34). □

5 Partitions and Theta Constant Identities of Degree 7

Theorem 5.1 *Let S denote the set consisting of one copy of the positive integers that are not multiples of 4 and one additional copy of the positive multiples of 7 that are not multiples of 4. Let T denote the set consisting of one copy of the positive integers that are not congruent to* 2 (mod 4) *and one additional copy of the positive multiples of 7 that are not congruent to* 2 (mod4). *Let A(N) be the number of partitions of* $2N+1$ *into parts in S and B(N) be the number of partitions of* $2N-1$ *into parts in T. Then, for* $N \geq 1$,

$$A(N) = 2\,B(N).$$

Proof Replacing q by q^2 in (1.1), multiplying by $(-q;q^2)_\infty(-q^7;q^{14})_\infty$ on both sides of the resulting equation and then using (2.7), we deduce that

$$\frac{(-q;q)_\infty(-q^7;q^7)_\infty}{(-q^4;q^4)_\infty(-q^{28};q^{28})_\infty} - \frac{(-q;q^2)_\infty(-q^7;q^{14})_\infty}{(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty} = 2q^2\frac{(-q;q)_\infty(-q^7;q^7)_\infty}{(-q^2;q^4)_\infty(-q^{14};q^{28})_\infty}. \tag{5.1}$$

Replacing q by $-q$ in (5.1), subtracting the resulting identity from (5.1) and then using (1.1), we have

$$\begin{aligned}&\frac{(-q;q)_\infty(-q^7;q^7)_\infty}{(-q^4;q^4)_\infty(-q^{28};q^{28})_\infty}-\frac{(q;-q)_\infty(q^7;-q^7)_\infty}{(-q^4;q^4)_\infty(-q^{28};q^{28})_\infty}-2q\\&=2q^2\left\{\frac{(-q;q)_\infty(-q^7;q^7)_\infty}{(-q^2;q^4)_\infty(-q^{14};q^{28})_\infty}-\frac{(q;-q)_\infty(q^7;-q^7)_\infty}{(-q^2;q^4)_\infty(-q^{14};q^{28})_\infty}\right\}.\end{aligned}\tag{5.2}$$

Equating the coefficients of q^{2N+1} on both sides of the equation, we complete the proof. □

Example 12 Let $N=6$. Then $A(6)=12$ and $B(6)=6$, and

$$\begin{aligned}13 &= 11+2=10+3=10+2+1=9+5=9+3+2=7+6=\bar{7}+6\\&=7+5+1=\bar{7}+5+1=7+3+2+1=\bar{7}+3+2+1,\\11&=8+3=7+4=\bar{7}+4=7+3+1=\bar{7}+3+1.\end{aligned}$$

6 Partitions and Theta Constant Identities of Degree 15

Theorem 6.1 *Let $A(N)$ denote the number of partitions of $2N+1$ into parts congruent to* $\pm 12, \pm 24, \pm 20 \pmod{60}$ *or into four distinct colors, with two colors appearing at most once and only in odd parts and the remaining two colors, also appearing at most once and only in odd multiples of 15 and let $B(N)$ denote the number of partitions of $2N$ into even positive integers that are not multiples of 60, with parts congruent to* $30 \pmod{60}$ *having one additional color, or into two distinct colors with one color appearing at most once and only in odd multiples of three and the other color, also appearing at most once and only in odd multiples of 5. Then*

$$A(N)=2\,B(N).$$

Proof From [6, p. 377, Entry 9(ii)]

$$\varphi(-q^6)\varphi(-q^{10})+2q\psi(q^3)\psi(q^5)=\varphi(q)\varphi(q^{15}).\tag{6.1}$$

Replacing q by $-q$ in (6.1), then subtracting the resulting identity from (6.1), we find that

$$\varphi(q)\varphi(q^{15})-\varphi(-q)\varphi(-q^{15})=2q\left(\psi(q^3)\psi(q^5)+\psi(-q^3)\psi(-q^5)\right).\tag{6.2}$$

Rewriting (6.2) in q-products, we have

$$\begin{aligned}&\frac{(-q;q^2)_\infty^2(-q^{15};q^{30})_\infty^2-(q;q^2)_\infty^2(q^{15};q^{30})_\infty^2}{(q^{12\pm},q^{20\pm},q^{24\pm},q^{60};q^{60})_\infty}\\&=2q\left\{\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty+(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{60})_\infty}\right\}.\end{aligned}\tag{6.3}$$

It is now easily seen that (6.3) has partition-theoretical interpretation claimed in Theorem 6.1. □

Example 13 Let $N=4$. Then $A(4)=12$ and $B(4)=6$, we have the representations

$$\begin{aligned}9_r&=9_b=5_r+3_r+1_r=5_r+3_r+1_b=5_r+3_b+1_r=5_r+3_b+1_b=5_b+3_r+1_r\\&=5_b+3_r+1_b=5_b+3_b+1_r=5_b+3_b+1_b=7_r+1_r+1_b=7_b+1_r+1_b,\\8&=6+2=\bar{5}+\bar{3}=4+4=4+2+2=2+2+2+2.\end{aligned}$$

Theorem 6.2 *Let $A(N)$ denote the number of partitions of $2N+1$ into two distinct colors, with one color appearing at most once and only in odd parts and the remaining color, also appearing at most once and only in odd parts that are not multiples of 3 or 5, but both. Let $C(N)$ denote the number of partitions of $2N$ into even positive integers that are not multiples of 12 or 20, with parts congruent to $30 \pmod{60}$ having one additional color. Then $A(0)=2$, and for $N \geq 1$,*

$$A(N) = C(N).$$

Proof From [6, p. 377, Entry 9(v) and (vi)]

$$\varphi(q)\varphi(q^{15}) - \varphi(q^3)\varphi(q^5) = 2qf(-q^2)f(-q^{30})\chi(q^3)\chi(q^5), \tag{6.4}$$

$$\varphi(q)\varphi(q^{15}) + \varphi(q^3)\varphi(q^5) = 2f(-q^6)f(-q^{10})\chi(q)\chi(q^{15}). \tag{6.5}$$

Adding (6.4) and (6.5), we obtain an identity equivalent to

$$\frac{\varphi(q)\varphi(q^{15})}{\chi(q^3)\chi(q^5)} = qf(-q^2)f(-q^{30}) + f(-q^6)f(-q^{10})\frac{\chi(q)\chi(q^{15})}{\chi(q^3)\chi(q^5)}. \tag{6.6}$$

Replacing q by $-q$ in (6.6), then subtracting the resulting identity from (6.6), we obtain an identity equivalent to the following q-product

$$\begin{aligned}&\frac{(-q;q^2)_\infty^2(-q^{15};q^{30})_\infty^2}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty^2(q^{15};q^{30})_\infty^2}{(q^3;q^6)_\infty(q^5;q^{10})_\infty}\\ &= 2q + \frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}\left\{\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty}\right\}.\end{aligned} \tag{6.7}$$

From [2, p. 1042], we note that

$$\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty} = 2q(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty. \tag{6.8}$$

Employing (6.8) in (6.7), we arrive at the partition identity claimed in our Theorem 6.2. □

Example 14 Let $N=6$. Then $A(6)=C(6)=10$, we have the representations

$$\begin{aligned}13_r &= 13_b = 11_r + 1_r + 1_b = 11_b + 1_r + 1_b = 9_r + 3_r + 1_r = 9_r + 3_r + 1_b\\ &= 7_r + 5_r + 1_r = 7_r + 5_r + 1_b = 7_b + 5_r + 1_r = 7_b + 5_r + 1_b,\\ 10+2 &= 8+4 = 8+2+2 = 6+6 = 6+4+2 = 6+2+2+2 = 4+4+4\\ &= 4+4+2+2 = 4+2+2+2+2 = 2+2+2+2+2+2.\end{aligned}$$

Theorem 6.3 *Let S denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 15. Let T denote the set consisting of the positive integers that are not multiples of 3 or 5. Let $A(N)$ be the number of partitions of N into odd parts in S, let $B(N)$ be the number of partitions of $2N-2$ into even parts in S, let $C(N)$ and $D(N)$ be the number of partitions of $2N$ into, respectively, odd parts in T and even parts in T and let $E(N)$ be the number of partitions of $2N$ into even number of even parts in T. Then,*

(i) *for $N \geq 1$, $A(2N) = D(N) - B(N) = E(N)$ and*
(ii) *for $N \geq 0$, $A(2N+1) = C(N)$.*

Proof From [6, p. 377, Entry 9(vii)], we note that

$$\left(\psi(q^3)\psi(q^5) - q\psi(q)\psi(q^{15})\right)\varphi(-q^3)\varphi(-q^5) = f(-q)f(-q^3)f(-q^5)f(-q^{15}). \tag{6.9}$$

Transcribing (6.9) into q-products, we deduce that

$$\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty} - q\frac{(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(q;q^2)_\infty(q^{15};q^{30})_\infty} = (q;q^2)_\infty(q^{15};q^{30})_\infty. \tag{6.10}$$

Replacing q by $-q$ in (6.10), we obtain

$$\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty} + q\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty} = (-q;q^2)_\infty(-q^{15};q^{30})_\infty. \tag{6.11}$$

Subtracting (6.10) from (6.11), we find that

$$\begin{aligned}&(-q\ ;q^2)_\infty(-q^{15};q^{30})_\infty - (q;q^2)_\infty(q^{15};q^{30})_\infty\\ &\quad= q\left\{\frac{(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(q;q^2)_\infty(q^{15};q^{30})_\infty} + \frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}\right\}.\end{aligned} \tag{6.12}$$

On the other hand, adding (6.10) and (6.11), we find that

$$\begin{aligned}(-q;q^2)_\infty(-q^{15};\ q^{30})_\infty + (q;q^2)_\infty(q^{15};q^{30})_\infty &= 2\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}\\ &-q\left\{\frac{(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(q;q^2)_\infty(q^{15};q^{30})_\infty} - \frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}\right\}.\end{aligned} \tag{6.13}$$

Employing, (6.8) in (6.13), we obtain the equality

$$\begin{aligned}&(-q;\ q^2)_\infty(-q^{15};q^{30})_\infty + (q;q^2)_\infty(q^{15};q^{30})_\infty\\ &\quad= 2\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty} - 2q^2(-q^2;q^2)_\infty(-q^{30};q^{30})_\infty.\end{aligned} \tag{6.14}$$

Suppose that the moduli α, β, γ and δ are of degree 1, 3, 5, and 15, respectively. If $m = z_1/z_3$ and $m' = z_5/z_{15}$. Then [6, p. 384, Entry 11(viii)],

$$\sqrt{\frac{m'}{m}} = \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}. \tag{6.15}$$

The above modular equation can be transformed into the q-product

$$\begin{aligned}&\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty} - \frac{(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty}{(-q^2;q^2)_\infty(-q^{30};q^{30})_\infty}\\ &\quad= q\left\{\frac{(q^3;q^6)_\infty(q^5;q^{10})_\infty}{(q;q^2)_\infty(q^{15};q^{30})_\infty} - \frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}\right\}.\end{aligned} \tag{6.16}$$

Invoking (6.16) in (6.13), we have

$$\begin{aligned}&(-q\ ;q^2)_\infty(-q^{15};q^{30})_\infty + (q;q^2)_\infty(q^{15};q^{30})_\infty\\ &\quad= \frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty} + \frac{(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty}{(-q^2;q^2)_\infty(-q^{30};q^{30})_\infty}.\end{aligned} \tag{6.17}$$

It is now easy to see that (6.14), (6.17), and (6.12) are equivalent to the statements (ii) and (i), respectively, in Theorem 6.3. □

Example 15 Let $N = 10$. Then $A(20) = E(10) = 8$, $D(10) = 16$, and $B(10) = 8$, we have the representations

$$\begin{aligned}19_r + 1_r &= 17_r + 3_r = 15_r + 5_r = 15_b + 5_r = 13_r + 7_r = 11_r + 9_r\\ &= 11_r + 5_r + 3_r + 1_r = 9_r + 7_r + 3_r + 1_r,\\ 16 + 4 &= 14 + 2 + 2 + 2 = 8 + 8 + 2 + 2 = 8 + 4 + 4 + 4 = 8 + 4 + 2 + 2 + 2 + 2\\ &= 4 + 4 + 4 + 4 + 2 + 2 = 4 + 4 + 2 + 2 + 2 + 2 + 2 + 2\\ &= 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2,\\ 16 + 4 &= 16 + 2 + 2 = 14 + 4 + 2 = 14 + 2 + 2 + 2 = 8 + 8 + 4 = 8 + 8 + 2 + 2\\ &= 8 + 4 + 4 + 4 = 8 + 4 + 4 + 2 + 2 = 8 + 4 + 2 + 2 + 2 + 2\\ &= 8 + 2 + 2 + 2 + 2 + 2 + 2 = 4 + 4 + 4 + 4 + 4 = 4 + 4 + 4 + 4 + 2 + 2\\ &= 4 + 4 + 4 + 2 + 2 + 2 + 2 = 4 + 4 + 2 + 2 + 2 + 2 + 2 + 2\\ &= 4 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2,\\ 18_r &= 16_r + 2_r = 14_r + 4_r = 12_r + 6_r = 12_r + 4_r + 2_r = 10_r + 8_r\\ &= 10_r + 6_r + 2_r = 8_r + 6_r + 4_r.\end{aligned}$$

Theorem 6.4 *Let S denote the set consisting of one copy of the positive integers that are multiples of 3 and one additional copy of those positive integers that are multiples of 5. Let T denote the set consisting of one copy of the positive integers that are not multiples of 3 or 5. Let $A(N)$ be the number of partitions of $2N + 1$ into odd parts in S or into even parts that are not multiples of 3 or 5, let $B(N)$ be the number of partitions of $2N$ into even parts in S or into even parts that are not multiples of 3 or 5, let $C(N)$ be the number of partitions of N into odd parts in S and let $D(N)$ and $E(N)$ be the number of partitions of $2N$ into, respectively, odd parts in T and odd number of even parts in T. Then,*

(i) *for $N \geq 1$, $A(N) = B(N)$,*
(ii) *for $N \geq 0$, $C(2N + 1) = E(N)$ and $C(2N) = D(N)$.*

Proof From [6, p. 377, Entry 9(vii)], we note that

$$\Big(\psi(q^3)\psi(q^5) + q\psi(q)\psi(q^{15})\Big)\varphi(-q)\varphi(-q^{15}) = f(-q)f(-q^3)f(-q^5)f(-q^{15}). \tag{6.18}$$

Transcribing (6.18) into q-products, we deduce that

$$\frac{(q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty} + q\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q^6;q^6)_\infty(q^{10};q^{10})_\infty} = (q^3;q^6)_\infty(q^5;q^{10})_\infty. \tag{6.19}$$

Replacing q by $-q$ in (6.19), we obtain

$$\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty} - q\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q^6;q^6)_\infty(q^{10};q^{10})_\infty} = (-q^3;q^6)_\infty(-q^5;q^{10})_\infty. \tag{6.20}$$

Subtracting (6.19) from (6.20), we obtain the equality

$$\begin{aligned}(-q^3;q^6)_\infty(-q^5;q^{10})_\infty - (q^3;q^6)_\infty(q^5;q^{10})_\infty &= -2q\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}\\ &+ \left\{\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty} - \frac{(q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty}\right\}.\end{aligned} \tag{6.21}$$

Employing (6.8) in (6.21), we deduce that

$$\begin{aligned}&\left\{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty\right\}\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}\\&=2q(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty\frac{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}-2q.\end{aligned}\tag{6.22}$$

Suppose that the moduli α, β, γ, and δ are of degree 1, 3, 5, and 15, respectively. If $m = z_1/z_3$ and $m' = z_5/z_{15}$. Then [6, p. 384, Entry 11(ix)],

$$-\sqrt{\frac{m}{m'}}=\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}.\tag{6.23}$$

Transforming (6.23) into q-products. Then invoking in (6.21), we find that

$$\begin{aligned}&(-q^3\;;q^6)_\infty(-q^5;q^{10})_\infty-(q^3;q^6)_\infty(q^5;q^{10})_\infty\\&\quad=q\left\{\frac{(-q^2;q^2)_\infty(-q^{30};q^{30})_\infty}{(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty}-\frac{(q^2;q^2)_\infty(q^{30};q^{30})_\infty}{(q^6;q^6)_\infty(q^{10};q^{10})_\infty}\right\}.\end{aligned}\tag{6.24}$$

On the other hand adding (6.19) and (6.20), we find that

$$\begin{aligned}&(-q^3\;;q^6)_\infty(-q^5;q^{10})_\infty+(q^3;q^6)_\infty(q^5;q^{10})_\infty\\&\quad=\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}+\frac{(q;q^2)_\infty(q^{15};q^{30})_\infty}{(q^3;q^6)_\infty(q^5;q^{10})_\infty}.\end{aligned}\tag{6.25}$$

It is seen that (6.24), (6.25), and (6.22) have partition theoretic interpretation given in Theorem 6.4 (ii) and (i), respectively. □

Example 16 Let $N = 7$. Then $A(7) = B(7) = 14$, we have the representations

$$\begin{aligned}15_r&=15_b=9_r+4+2=9_r+2+2+2=8+5_b+2=8+4+3_r=8+3_r+2+2\\&=5_b+4+4+2=5_b+4+2+2+2=5_b+2+2+2+2+2=4+4+4+3_r\\&=4+4+3_r+2+2=4+3_r+2+2+2+2=3_r+2+2+2+2+2+2,\\14&=12_r+2=10_b+4=10_b+2+2=8+6_r=8+4+2=8+2+2+2\\&=6+4+4=6+4+2+2=6+2+2+2+2=4+4+4+2\\&=4+4+2+2+2=4+2+2+2+2+2=2+2+2+2+2+2+2.\end{aligned}$$

7 Partitions and Theta Constant Identities of Degree 63

Theorem 7.1 *Let $A(N)$ denote the number of partitions of $2N+1$ into parts congruent to $\pm 6 \pmod{18}$ or into distinct odd parts that are not multiples of 7 or 9 and let $B(N)$ denote the number of partitions of $2N+1$ into parts congruent to $\pm 14 \pmod{42}$ or into two different colors, with one color appearing at most once and only in odd parts and the other color, also appearing at most once and only in odd multiples of 63. Then, for $N \geq 0$,*

$$A(N) = B(N).$$

Proof From [6, p. 426, Entry 19(ii)], we note that

$$\psi(q^7)\psi(q^9)-q^6\psi(q)\psi(q^{63})=f(-q^6)f(-q^{63}).\tag{7.1}$$

Multiplying (7.1) by $1/\psi(q)\psi(q^{63})$, we obtain the equality

$$\frac{\psi(q^7)\psi(q^9)}{\psi(q)\psi(q^{63})} - q^6 = \frac{f(-q^6)f(-q^{63})}{\psi(q)\psi(q^{63})}. \tag{7.2}$$

Replacing q by $-q$ in (7.2). Then subtracting (7.2) from the resulting identity, we find the following equivalent q-product identity

$$\begin{aligned} &\left\{\frac{(-q;q^2)_\infty(-q^{63};q^{126})_\infty}{(-q^7;q^{14})_\infty(-q^9;q^{18})_\infty} - \frac{(q;q^2)_\infty(q^{63};q^{126})_\infty}{(q^7;q^{14})_\infty(q^9;q^{18})_\infty}\right\}\frac{1}{(q^6,q^{12};q^{18})_\infty} \\ &= \frac{(-q;q^2)_\infty(-q^{63};q^{126})_\infty - (q;q^2)_\infty(q^{63};q^{126})_\infty}{(q^{14},q^{28};q^{42})_\infty}. \end{aligned} \tag{7.3}$$

It is now easy to see that (7.3) gives partition-theoretic interpretation given in the Theorem 7.1. □

Example 17 Let $N = 12$. Then $A(12) = B(12) = 14$, we have the representations

$$\begin{aligned} 25 &= 24+1 = 19+6 = 19+5+1 = 17+5+3 = 15+6+3+1 = 13+12 = 13+11+1 \\ &= 13+6+6 = 13+6+5+1 = 12+12+1 = 12+6+6+1 = 11+6+5+3 \\ &= 6+6+6+6+1, \\ 25 &= 21+3+1 = 19+5+1 = 17+7+1 = 17+5+3 = 15+9+1 = 15+7+3 \\ &= 14+11 = 14+7+3+1 = 13+11+1 = 13+9+3 = 13+7+5 \\ &= 11+9+4+1 = 9+7+5+3+1. \end{aligned}$$

Remark 1 Identities (3.56), (3.57), (4.14), (4.19), (6.24), and (6.25) are special cases of Warnaar's identities [20], a bijective proof for the same is given by Sun Kim in [14]. Also, Identities (3.57), (4.14), (6.12), and (6.25) were proved by N. D. Baruah and B. C. Berndt in their paper [2] using Ramanujan's modular equations.

Acknowledgments The authors would like to thank anonymous referee for his/her valuable comments. The first author would like to thank DST for financial support through project no. SR/S4/MS:739/11, the second author would like to thank UGC for providing JRF, ref. no. F.17-58/2008(SA-I), and the third author would like to thank CSIR for providing JRF, ref. no. 09/039(0111)/2014-EMR-I.

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