LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

MASTER OF SCIENCE IN
FINANCE

MASTERS FINAL WORK
DISSERTATION

SIMPLE AND ROBUST TESTS OF THE QUADRATIC BREAK TREND HYPOTESHIS FOR $I(0)$, $I(1)$ AND $I(2)$ TIME SERIES.

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SUPERVISOR:

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Ao João.

À minha Família.

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Resumo

Esta tese teve como objetivo a construção de uma estatística de teste para a hipótese nula da não existência de quebra na tendência de uma série temporal unidimensional. A sua principal inovação foi o desenvolvimento de um teste robusto não só para a presença de erros $I(0)$ e $I(1)$ mas também para erros $I(2)$. Para isso, construiu-se um modelo quadrático que incluiu uma variável auxiliar, com a mesma ordem, e foram propostos dois testes distintos, um para uma data de quebra conhecida e o outro para uma data de quebra desconhecida. O primeiro é uma média ponderada pelas estatísticas de teste apropriadas para o caso em que os erros são $I(0)$, $I(1)$ ou $I(2)$. Esta estatística de teste tem uma distribuição normal padrão. O segundo é uma média ponderada que se obtém depois de encontrado o supremo sobre todas as possíveis datas de quebra, sujeitas a um parâmetro delimitador da amostra. Neste caso, os valores críticos foram calculados através de simulação de Monte Carlo. A metodologia de Harvey et al. (2009) foi seguida em ambos os cenários. Mais ainda, conceitos sobre convergência assintótica para processos com duas raízes unitárias foram revistos e algumas propriedades assintóticas de regressões, com uma ou duas raízes unitárias, foram derivadas. Os testes desenvolvidos têm aplicação no estudo de séries económicas e financeiras.

Palavras-chave: Quebra estrutural; processo integrado de ordem dois; tendência quadrática; teoria assintótica.

Abstract

The aim of this thesis was the construction of a test statistic for the null hypothesis of no break in trend in an univariate time series. The breakthrough was to make the test robust not only for the presence of $I(0)$ and $I(1)$ shocks but also for the $I(2)$ case scenario. For this reason, a quadratic trend break model and a quadratic dummy variable were designed. The assumption of known or unknown break date motivated the construction of two separate test statistics. The former is a weighted average of the appropriate t-statistics for the case of $I(0)$, $I(1)$ and $I(2)$ shocks and it was shown to have standard normal limiting distribution. The latter is a weighted average of the statistics formed as the supremum over all possible break dates, subject to a trimming parameter. In addition, the critical values for this test statistic were computed through Monte Carlo simulation. The general framework of Harvey et al. (2009) was adopted to test for the presence of a break under a known or unknown break date. At the same time, asymptotic theory for $I(2)$ processes was reviewed and simple asymptotic properties of second and first order auto-regressions were derived. The tests can be applied to the study of financial and economic time series.

Keywords: Structural break; double integrated process; quadratic trend; asymptotic theory.

Contents

List of Tables

List of Figures

Basic Notation

General

$\mathbb{1}(A)(x)$	indicator function; it is equal to 1 if x belongs to the set A and equal to 0, otherwise
$\lfloor x \rfloor$	integer part of a real number x
σ	standard deviation
μ	expected value
$N(\mu, \sigma^2)$	normal distribution with expected value μ and variance σ^2
T	dimension of the sample
x'	transpose of x
$x := y$	x is defined by y
$\equiv$	identical to
$\xrightarrow{p}$	convergence in probability
$\xrightarrow{d}$	convergence in distribution
$g(n) = o(f(n))$	means that $g(n)/f(n) \to 0$, that is, the function $g(n)$ is negligible compared to $f(n)$
$g(n) = O(f(n))$	means that $g(n)/f(n) \to c$, where c is a constant; that is, the functions grow at the same rate
$[.]_{jj}$	denotes the jj'th element of a matrix
$[.]_j$	denotes the j'th element of a vector
ξ	significance level
L	lag operator

Unit Root Functions

$I(d)$	integrated process of order d
$I(0)$	stationary process apart from the deterministic components
$I(1)$	unit root process
$I(2)$	two unit roots process
$AR(d)$	autoregressive process of order d
DTQ_t	quadratic dummy variable. D stands for dummy, T for trend and Q for quadratic
DTL_t	linear dummy variable. L stands for linear
DTU_t	step dummy variable
$\triangle$	first differences
$\triangle^2$	second differences
$W(r)$	standard Brownian motion

Acronyms

DGP	data generating process
OLS	ordinary least squares
LR	long run
$i.i.d.$	independent and identically distributed

The symbol □ denotes the end of a Proof. Each section is divided into subsections, with consecutive labelling of Equations, Lemmas, Propositions, Remarks and Theorems.

1 Introduction

Structural changes have been described as a widespread phenomenon in economics and finance. They are a consequence of a rare but outstanding historical event that changes permanently the behaviour of a standard economic time series. With no surprise, the existence of such events affects the statistical properties of estimators, compromising forecasts and statistical inference from the data. For that reason, the econometric literature has proposed a number of statistics to test for the presence of structural breaks in a given time series.

In recent years, with an added relevance to this thesis, there has been an upsurge of interest on devising tests for the breaking trend hypothesis which can be used regardless of whether the underlying shocks are $I(0)$ or $I(1)$ processes. The most relevant papers on the subject include Kwiatkowski et al. (1992), Sayginsoy and Vogelsang (2004), Harvey et al. (2009), Kejriwal and Perron (2010) and Perron and Yabu (2012). Sobreira and Nunes (2015) provided tests of the presence of multiple breaks in the trend function which are valid in the presence of stationary or unit root time series. Apart from the $I(0)$ and $I(1)$ shocks, some studies found statistical evidence for the presence of two unit roots in prices, wages, stock variables, among others (see Haldrup, 1998, for example). Therefore, it is perfectly viable that some financial and economic time series are better described by an $I(2)$ process rather than an $I(0)$ or $I(1)$ process. Hence, creating a statistical procedure that may be used to test the null hypothesis of no breaks in trend which accounts for the possibility of the errors being $I(2)$ constitutes an interesting path of research.

Thus, the purpose of this thesis is to construct a test statistic that can be used to test the null hypothesis of no break in the trend function against the alternative hypothesis of one break in the quadratic trend and which is robust as to whether the underlying shocks are stationary, a random walk or a process with two unit roots. The first test statistic proposed is valid under the assumption of a known break date. It is a weighted average of the optimal tests appropriate for the $I(0)$, $I(1)$ and $I(2)$ shocks. It is proved that the test statistic has standard normal distribution so that tabulated critical values can be used. The second test statistic is valid under the assumption of an unknown break date and follows the framework proposed by Andrews (1993) and Harvey et al. (2009). This hypothesis brings additional complexity to the testing procedure: the break date must be estimated through a statistic formed as the over all possible break dates, conditional to a trimming parameter. The critical values can be calculated through Monte Carlo simulation.

The outline of this thesis is as follows. Section 2 provides a literature review on structural changes and unit root hypothesis testing. Sections 3 and 4 establish the general framework. In section 5, the test statistics for the known and for the unknown break date are presented. Section 6 illustrates the critical values computed for the unknown break date test statistic. Section 7 is reserved for concluding comments and suggestions for future research. Mathematical proofs are systematized in appendix A and asymptotic theory is summarized in appendix B. Appendix C contains R source code for the unknown break date case.

2 Literature Review

This thesis is about simple and robust tests for a structural break in the trend function that includes the case of double integrated shocks in an univariate time series. For that reason, this literature review covers the most important concepts and theories related with structural breaks and shocks with two unit roots.

By robust it is meant that testing for the presence of a changing point does not require to test in advance if the series is stationary, a unit root or a two unit roots process. This does not mean, though, that we can disregard any information related with the nature of the shocks. On the contrary, a robust test is one that can assess several levels of information simultaneously. This type of test procedure could only be possible after significant discoveries in asymptotic theory as well as with the introduction of mathematical concepts from calculus, real analysis and stochastic processes.

Regarding time series with two unit roots, they have been reported as being rare within the economic and financial context. But how much confidence can be given to this result when existing unit root test do not even consider the existence of those time series? Will it be possible to detect more series that mimic an $I(2)$ process if a test statistic is taken without testing *a priori* the nature of the shocks? If this is somehow feasible, it can be extremely useful since researchers are less likely to make errors while investigating the deterministic or stochastic properties of the time series under analysis.

The remainder of this literature review provides the reader with a definition of structural break, shows the most import contributors to the field of structural break and unit roots testing, with special attention to the works of Perron (1989), Kwiatkowski et al. (1992) and Harvey et al. (2009). It also explains the interest of extending existing robust tests to the $I(2)$ case. This literature will not discuss in detail different approaches to estimate the break date nor multiple break tests since it is out of the scope of this thesis.

2.1 Structural Break Tests

The concept of structural break has not always been clear in the literature. Although there is no formal definition of structural break, in what follows it will be adopted the one of Hansen (2001) which says that "a structural break occurs if at least one of the parameters of a model changes at some date, the break date, in the sample period". In this thesis, the attention will be focused on the curvature of the trend function given by the quadratic term of the proposed model.

The first studies about structural changes that are considered worth mentioning, for the purpose of this thesis, are those of Quandt (1958) and Brown et al. (1975). Both provided rudimentary tests that were based on parameter instability. The test developed by Quandt (1958), which became known as the Q statistics, consisted in a maximization of the likelihood function, at a single known break date. Another feature of the model was that it only considered normally distributed disturbances. Nevertheless, the author did not question which parameters were responsible for a possible switching point nor the consequences of a loss in power of the test as the magnitude of the switch increased. Brown et al. (1975), on their turn, introduced the Cumulative Sum (CUSUM) statistics to detect instability in the level of the model and the Cumulative Sum of Least Squares (CUSQ) for testing error variance instability. Subsequent works of Ploberger and Krämer (1990) and Ploberger and Krämer (1992) applied the CUSUM test to ordinary least squares, instead of recursive residuals, and showed that the limit distribution under the null hypothesis of parameter stability could be expressed in terms of Brownian Bridges. Although innovative for the time, all the aforementioned tests had relevant limitations. Vogelsang (1999), while studying the causes of non monotonic power functions of standard tests, found that the magnitude of trend shift exacerbated the estimated variance of the errors. As a consequence, the power of the underlying test (extended version of Q and CUSUM statistics) dropped near to zero.

At the same time, another strand of literature came up with a different but relevant approach to test the null hypothesis of a unit root in the presence of a structural break. This major breakthrough in unit root testing in the presence of a structural break is attributed to Perron (1989). In fact, the author proved that the presence of a structural change could bias unit root tests, such as Dickey and Fuller (1979), towards the non rejection of the unit root null hypothesis. To overcome this issue, Perron suggested that before making any statistical inferences, the break should be "modelled", that is, the deterministic component of the time series should be split into two parts, each one containing the observations before and after the break. The preselected break dates coincided with relevant historical events. For example, it was assumed that the Great Crash (1929), the Oil

Price Shocks (1973) and the World Wars were responsible for the occurrence of breaks in U.S. macroeconomic data.

Surprisingly, with the innovations introduced, the author questioned the prominent idea that most time series had a unit root process. By properly modelling a break, the author found that important financial and economic time series, such as those initially studied by Nelson and Plosser (1982) did not have any unit root. This is an important fact because from this moment on, structural change tests could not be dissociated from unit root testing.

That paper also highlighted the most prominent ideas about asymptotic theory of the time. Phillips (1986) exploited the concept of spurious regressions by studying linear regressions with integrated random processes and Phillips (1987) and Phillips and Perron (1988) proposed new tests for detecting the presence of a unit root where the models were drawn to discriminate unit root non stationarity and stationarity about a deterministic trend. Park and Phillips (1989) developed asymptotic theory of regressions for multivariate linear models that included integrated processes of different orders, nonzero means, drifts, time trends, among others. All these authors provide mandatory reading for those seeking to understand the mathematics behind unit root testing and asymptotic theory.

Nevertheless, Perron (1989) received a lot of criticism. Christiano (1992) refused to accept the assumption of a known break date. In his opinion, the time of break should always be estimated in order to obtain a conclusion robust to subjective reasoning and data mining. Another paper that highlighted the importance of testing under the assumption of an unknown break date is Montañés (1997).

As an example, the author studied a macroeconomic indicator from Spain. The country joined the EU in 1986. At that time, it was expected that such event would have an impact on the exports and that a structural break would be observed in that year. Instead, what was found was that the structural break took place an year earlier. This raised the question of how Perron's test would behave if the break time date was misspecified. The author found that such a small difference would not entail a loss of power, at least asymptotically. However, for a finite sample, a loss of power would be observed.

In fact, a huge effort was made to propose powerful structural break tests which allowed for a structural change at an unknown date. Andrews (1993) and Andrews and Ploberger (1994) are among the most cited authors in this field. In particular, Andrews (1993) proposed the Wald, Lagrange Multiplier and Likelihood Ratio tests. The innovation of that paper was to consider parameter instability which allowed for a time structural change at an unknown date. Andrews and Ploberger (1994) studied the distributional properties of the break date estimates. Vogelsang (1997) proposed a Wald-Type test based on the mean and exponential statistics of Andrews and Ploberger (1994) and the supremum statistics of Andrews (1993). Bai and Perron (1998) addressed the problem of testing multiple structural changes.

2.2 Trend Break Tests Robust to I(0) and I(1) Shocks

Standard unit root tests have a key role in structural hypothesis testing. These can be classified into two different categories depending on the null hypothesis to be tested. On one hand, the unit root null hypothesis can be tested. As an example, there are the Dickey-Fuller, Augmented Dickey-Fuller and Phillips-Perron tests. On the other hand, it can be tested the null hypothesis of stationarity. Kwiatkowski et al. (1992) provided straightforward tests of the null hypothesis of stationarity against the alternative of a unit root, while allowing for error autocorrelation.

Nevertheless, as it has already been seen by the Nelson and Plosser (1982) data, it is not straightforward to determine if a time series has a unit root or not. As a consequence, the lack of confidence about the output provided by an unit root test can bias structural change tests. This has motivated Perron and Yabu (2012) to propose a robust type of structural change test. Another relevant paper in the field is given to Harvey et al. (2009). The authors were not only interested in testing or searching for the existence of a structural break but also in how to do it without pre-testing the nature of the shocks. The tests are a weighted average of the optimal tests appropriate for the $I(0)$ and $I(1)$ case scenarios. The weighting function employed was based on the KPSS stationary statistics applied to the levels and growth rate date (Kwiatkowski et al., 1992). But once more, their analysis failed to include the $I(2)$ case. However, the work of Harvey et al. (2009) proved to be useful for a large set of this investigation. Sobreira et al. (2014) and Sobreira and Nunes (2015) proposed a test for multiple structural changes in the trend function which did not require pre-specifying whether the underlying time series (the per capita GDP) was $I(0)$ or $I(1)$. Sobreira et al. (2014) used the proposed econometric methodology to classify countries according to the growth path that best describes the behaviour of their real per capita GDP.

This thesis takes one step further and proposes an econometric methodology to test for the presence of a break in the curvature of the trend function. Another innovation is that the proposed test statistic is also robust to the presence of $I(2)$ shocks in the DGP.

2.3 Shocks with Two Unit Roots

The literature about errors with two unit roots is not extensive. One reason for that is the belief that most time series are well categorized as stationary or integrated of order one. Nevertheless, some macroeconomic time series like prices, wages, stock variables, among others, are potentially integrated of order two (Haldrup, 1998). Hence, this possibility should be taken into account when conducting empirical research and it is the reason why this thesis is concerned in making the testing procedure robust to $I(2)$-ness.

Time series integrated of order two often look smoother and more slowly changing than known variables integrated of order one. This can be pointed out as an argument towards their existence. The reason why researchers care so much about unit root testing is that the number of unit roots presented in a time series determines the correct approach to render the series stationary. On one hand, if the process is a random walk or, equally saying, is a single summation of shocks, then first differences to the original equation should be taken. On the other hand, if the time series has two unit roots, equally saying, is a double summation of shocks, then this process requires to be differenced twice in order to become stationary[1].

The most important works concerned with univariate testing for $I(2)$ are Dickey and Pantula (1987), Dickey and Fuller (1979) and Haldrup (1998). Asymptotic theory related with two unit roots can be found in Park and Phillips (1989). Their findings were essential to give the proof of Proposition B.3 in appendix B.

Dickey and Pantula (1987) showed that the null distribution of traditional augmented Dickey-Fuller tests was affected in the presence of two unit roots, after observing substantial size distortions. Furthermore, an interesting paper which has motivated this thesis is Haldrup and Lildholdt (2002). The authors proposed themselves to test the behaviour of standard unit root tests for a single unit root when there was evidence that double integrated processes were present in the data generating process. They examined the robustness of Dickey-Fuller and Phillips-Perron tests for a unit root and concluded that when the underlying series was doubly integrated it was likely to give rise to excessive rejection of the unit root null hypothesis in favour to the explosive alternative because the test statistic would have a non-similar distribution, caused by the extra unit root.

A further instance for the existence of time series with two unit roots is given by the work of Sen and Dickey (1987) who suggested that U.S. population is a plausible candidate for an $I(2)$ variable. Georgoutsos and Kouretas (2004) made the same observation for some nominal price indices in the context of the purchasing power parity. Banerjee et al. (2001) made an $I(2)$ analysis of Australian inflation and found that the levels of prices and costs were best characterized as integrated processes of order two.

[1] Please check appendix B.2.

3 Assumptions and Methodology

This thesis followed the general framework of Harvey et al. (2009) who proposed a test statistic to test for the presence of a break in the slope of a linear trend without knowing *a priori* if the errors were $I(0)$ or $I(1)$. In spite of being a central paper to this thesis, the linear model and the overall assumptions did not provide a comprehensive tool to deal with the $I(2)$ case scenario.

In this context, the first innovation of this thesis was to introduce a quadratic trend to the general framework. Along with that, a quadratic dummy variable was designed to model a structural break in the presence of an integrated process of order two. This prevents the appearance of an impulse dummy variable after taking two differences to render the time series stationary. The implication of an impulse dummy variable is that the null hypothesis of no break in trend would be tested with the information provided from a single observation - an infeasible test to carry out. On the contrary, the suggested quadratic dummy variable ensues an indicator function which equals zero before the time break and one afterwards. Furthermore, the quadratic DGP was expected to bring more flexibility to the testing procedure. As expressed by Harvey et al. (2011), a quadratic model offers a reasonable degree of local non-linearity in the deterministic trend function, motivating the choice of a polynomial model.

Furthermore, it was assumed throughout this work that there is only one break date, which occurs instantly.

4 The Quadratic Trend Break Model

Consider a time-series $\{y_t\}_{t=1}^{T}$ given by the following trend break data generating process:

$$y_t = \alpha + \beta t + \frac{1}{2}\vartheta t^2 + \gamma DTQ_t(\tau^*) + u_t \quad t = 1, \ldots, T, \tag{1}$$

$$u_t = (\alpha_1 + \alpha_2)u_{t-1} - \alpha_1\alpha_2 u_{t-2} + \varepsilon_t \quad u_1 = \varepsilon_1 \quad t = 2, \ldots, T. \tag{2}$$

In equation (1), α is a constant, β is the coefficient of the linear trend and $(1/2)\vartheta$ is the coefficient of the quadratic trend. Furthermore, γ is the coefficient of the quadratic dummy variable $DTQ_t(\tau^*)$. The quadratic dummy variable can be defined as follows:

$$DTQ_t(\tau^*) := \mathbb{1}(t > T_b^*)\left[\frac{1}{2}(t - T_b^*)^2\right] \quad t = 1, \ldots, T, \tag{3}$$

where $T_b^* := \lfloor \tau^* T \rfloor$ is the trend break date which is the integer part of the product of the break fraction, $\tau^* \in]0, 1[$, multiplied by the sample's size, T. Concerning the existence of a break, only two case scenarios were considered. On the one hand, there may not exist a break in trend. This is expected to happen whenever the coefficient γ in equation (1) is zero. This means that the time series is well described by the equation $y_t = \alpha + \beta t + (1/2)\vartheta t^2 + u_t$ for $t = 1, \ldots, T$. In other words, the coefficients α, β and ϑ are constant throughout the sample period. On the other hand, if γ in equation (1) is different from zero then a break in trend is expected to occur. In this case, for the points ranging between $[1, T_b^*]$ the level is simply equal to α, the coefficient of the linear trend is β and the coefficient of the quadratic trend is $(1/2)\vartheta$. For all points in the time interval $[T_b^* + 1, T]$, the level changes from α to $\alpha + (1/2)\gamma(T_b^*)^2$, the coefficient of the linear trend changes from β to $\beta - \gamma T_b^*$ and the quadratic trend changes from $(1/2)\vartheta$ to $(1/2)(\vartheta + \gamma)$.

In equation (2), the disturbance term u_t, or shock, is assumed to have an AR(2) representation. The constants α_1 and α_2, where $\alpha_1, \alpha_2 \in [0, 1]$, determine the number of unit roots that the autoregressive polynomial can have. If $\alpha_1 = 0$ and $|\alpha_2| < 1$ or $|\alpha_1| < 1$ and $\alpha_2 = 0$ then u_t is stationary. If $\alpha_1 = 0$ and $\alpha_2 = 1$ or $\alpha_1 = 1$ and $\alpha_2 = 0$ then u_t is integrated of order one. Finally, if both $\alpha_1 = \alpha_2 = 1$ then u_t is integrated of order two[2]. As a benchmark, we naturally have that u_t is stationary when $\alpha_1 = \alpha_2 = 0$. Regarding the process $\{\varepsilon_t\}$, two different approaches can be taken. The most conservative one assumes the stochastic process to be $i.i.d.$, that is:

Assumption 4.1.a. The process $\{\varepsilon_t\}$ is $i.i.d$ with mean zero and constant variance σ_ϵ^2.

Nevertheless, this is an unrealistic assumption since the structural break tests are typically applied to series which are highly dependent over time (Kwiatkowski et al., 1992). For that reason, consider Assumption 1 of Sayginsoy and Vogelsang (2004), which states a weaker assumption about the errors:

Assumption 4.1.b. The process $\{\varepsilon_t\}$ is such that $\varepsilon_t = c(L)\eta_t$, where $c(L) = \sum_{i=0}^{\infty} c_i L^i$, with $[c(1)]^2 > 0$ and $\sum_{i=0}^{\infty} i|c_i| < \infty$, and where $\{\eta_t\}$ is a martingale difference sequence with unit conditional variance and $\sup_t E(\eta_t^4) < \infty$.

[2] Please check Table II, columns one to four in appendix A.

Remark 4.1. Under the conditions of Assumption 4.1.b, the LR variance of $\{\varepsilon_t\}$ is given by $\omega_\varepsilon^2 := \lim_{T\to\infty} T^{-1} E\Big(\sum_{t=1}^{T} \varepsilon_t\Big)^2 = c(1)^2$. Furthermore, when u_t is $I(0)$, with $|\alpha_2| < 1$, the LR variance of u_t is given by $\omega_u^2 := \lim_{T\to\infty} T^{-1} E\Big(\sum_{t=1}^{T} u_t\Big)^2$, such that $\omega_u^2 = \omega_\varepsilon^2/(1-\alpha_2)^2$.

It is the order of integration of the shocks that determines the number of differences that must be taken to the initial model for it to become stationary. For the purpose of this thesis, if the time series is stationary then regressions in levels should be used. If the shocks are a random walk, then the correct approach is to model the first-differences of equation (1). That is:

$$\Delta y_t = \theta + \vartheta t + \gamma DTL_t(\tau^*) + \Delta u_t \quad t = 2, \ldots, T, \tag{4}$$

$$DTL_t(\tau^*) := \mathbb{1}(t > T_b^*)(t - T_b^* - 1/2) \quad t = 2, \ldots, T, \tag{5}$$

where $\theta = \beta - (1/2)\vartheta$ is a constant.

In the same fashion, if the shocks have two unit roots then second differences must be taken to render the series stationary. This is given by the equations:

$$\Delta^2 y_t = \vartheta + \gamma DTU_t(\tau^*) + \Delta^2 u_t \quad t = 3, \ldots, T, \tag{6}$$

$$DTU_t(\tau^*) := \begin{cases} 0 & \text{if } t < T_b^* + 1, \\ 0.5 & \text{if } t = T_b^* + 1, \\ 1 & \text{if } t > T_b^* + 1. \end{cases} \tag{7}$$

The first and second differences of the quadratic trend break model along with the dummy variables $DTQ_t(\tau^*)$, $DTL_t(\tau^*)$ and $DTU_t(\tau^*)$ provide the appropriate regressors to estimate γ via OLS.

5 Tests for a Break in Trend

The null hypothesis of interest is $H_0 : \gamma = 0$ against the two sided alternative $H_A : \gamma \neq 0$. In contrast with traditional testing procedures, in this thesis it is not needed to pretest the nature of the shocks. For that reason, the appropriate statistics to test for the presence of a structural break in the quadratic trend are made robust for the presence of $I(0)$, $I(1)$ and $I(2)$ shocks. Following the same line of reasoning from Harvey et al. (2009), two different test statistics are presented, one for the known break date and the other for the unknown break date.

5.1 Known Break Fraction

The proposed test statistic for the known break fraction hypothesis is a weighted average of the optimal tests for $I(0)$, $I(1)$ and $I(2)$ shocks. However, to build such test statistic it is mandatory to find the right weights that allow only one of the t-tests $t_0(\tau^*)$, $t_1(\tau^*)$ or $t_2(\tau^*)$, defined immediately above, to be chosen in the end. For the sake of brevity, in this section, it is only presented the t-test for each optimal case.

Firstly, consider the case scenario where u_t in equation (2) is $I(0)$ and Assumption 4.1.a holds. The optimal test of H_0 against H_A rejects for large values of the absolute value of the t-ratio appropriated for γ when equation (1) is estimated by OLS. That is $|t_0(\tau^*)|$ where:

$$t_0(\tau^*) \overset{H_0}{:=} \frac{\hat{\gamma}_0(\tau^*)}{\sqrt{\hat{\sigma}_0^2(\tau^*)\left[\{\sum_{t=1}^T x_{DTQ,t}(\tau^*)x_{DTQ,t}(\tau^*)'\}^{-1}\right]_{44}}}, \quad (8)$$

$$\hat{\gamma}_0(\tau^*) := \left[\left\{\sum_{t=1}^T x_{DTQ,t}(\tau^*)x_{DTQ,t}(\tau^*)'\right\}^{-1}\sum_{t=1}^T x_{DTQ,t}(\tau^*)y_t\right]_4, \quad (9)$$

with $x_{DTQ,t}(\tau^*) := \{1, t, t^2, DTQ_t(\tau^*)\}'$, $\hat{\sigma}_0^2(\tau^*) := T^{-1}\sum_{t=1}^T \hat{u}_t(\tau^*)^2$ is the OLS estimate of the residual variance and $\hat{u}_t(\tau^*) := y_t - \hat{\alpha} - \hat{\beta}t - \hat{\varphi}t^2 - \hat{\gamma}_0(\tau^*)DTQ_t(\tau^*)$ and $\varphi = (1/2)\vartheta$.

Secondly, admit that u_t is $I(1)$. The optimal test of H_0 against H_A rejects for large values of the absolute value of the t-ratio appropriated for γ when equation (1) is estimated via OLS in first differenced form. The optimal test is $|t_1(\tau^*)|$ and $t_1(\tau^*)$ is given by:

$$t_1(\tau^*) \overset{H_0}{:=} \frac{\hat{\gamma}_1(\tau^*)}{\sqrt{\hat{\sigma}_1^2(\tau^*)\left[\{\sum_{t=2}^T x_{DTL,t}(\tau^*)x_{DTL,t}(\tau^*)'\}^{-1}\right]_{33}}}, \quad (10)$$

$$\hat{\gamma}_1(\tau^*) := \left[\left\{\sum_{t=2}^T x_{DTL,t}(\tau^*)x_{DTL,t}(\tau^*)'\right\}^{-1}\sum_{t=2}^T x_{DTL,t}(\tau^*)\Delta y_t\right]_3, \quad (11)$$

with $x_{DTL,t}(\tau^*) := \{1, t, DTL_t(\tau^*)\}'$, $\hat{\sigma}_1^2(\tau^*) := (T-1)^{-1}\sum_{t=2}^T \hat{v}_t(\tau^*)^2$ and $\hat{v}_t(\tau^*) := \Delta y_t - \hat{\theta} - \hat{\vartheta}t - \hat{\gamma}_1(\tau^*)DTL_t(\tau^*)$.

Finally, if u_t is assumed to be $I(2)$ then the appropriate inference method for testing H_0 against H_A is to consider the t-ratio test associated with γ when equation (1) is estimated via OLS in second differenced form. That is $|t_2(\tau^*)|$ where:

$$t_2(\tau^*) \overset{H_0}{:=} \frac{\hat{\gamma}_2(\tau^*)}{\sqrt{\hat{\sigma}_2^2(\tau^*)\left[\{\sum_{t=3}^{T} x_{DTU,t}(\tau^*)x_{DTU,t}(\tau^*)'\}^{-1}\right]_{22}}}, \tag{12}$$

$$\hat{\gamma}_2(\tau^*) := \left[\Big\{\sum_{t=3}^{T} x_{DTU,t}(\tau^*)x_{DTU,t}(\tau^*)'\Big\}^{-1}\sum_{t=3}^{T} x_{DTU,t}(\tau^*)\Delta^2 y_t\right]_2, \tag{13}$$

with $x_{DTU,t}(\tau^*) := \{1, DTU_t(\tau^*)\}'$, $\hat{\sigma}_2^2(\tau^*) := (T-2)^{-1}\sum_{t=3}^{T}\hat{k}_t(\tau^*)^2$ and $\hat{k}_t(\tau^*) := \Delta^2 y_t - \hat{\vartheta} - \hat{\gamma}_2(\tau^*)DTU_t(\tau^*)$.

In order to deal with more general $I(0)$, $I(1)$ and $I(2)$ processes for u_t the OLS estimates of residual variance $\hat{\sigma}_i^2(\tau^*)$, for $i = 0, 1, 2$, can be replaced by the corresponding non-parametric long run variance $\hat{\omega}_i^2(\tau^*)$, for $i = 0, 1, 2$. As pointed out by Kwiatkowski et al. (1992), Assumption 4.1.a can bias conclusions since most economic and financial time series to which the stationary tests can be applied are usually time dependent. To this end, the LR variance estimators for the $I(0)$, $I(1)$ and $I(2)$ are defined as follows:

$$\hat{\omega}_0^2(\tau^*) := \hat{\sigma}_0^2(\tau^*) + 2\sum_{j=1}^{T-1} h(j/l)\hat{\gamma}_{j,0}(\tau^*); \hat{\gamma}_{j,0}(\tau^*) := T^{-1}\sum_{t=j+1}^{T}\hat{u}_t(\tau^*)\hat{u}_{t-j}(\tau^*), \tag{14}$$

$$\hat{\omega}_1^2(\tau^*) := \hat{\sigma}_1^2(\tau^*) + 2\sum_{j=1}^{T-2} h(j/l)\hat{\gamma}_{j,1}(\tau^*); \hat{\gamma}_{j,1}(\tau^*) := (T-1)^{-1}\sum_{t=j+2}^{T}\hat{v}_t(\tau^*)\hat{v}_{t-j}(\tau^*), \tag{15}$$

$$\hat{\omega}_2^2(\tau^*) := \hat{\sigma}_2^2(\tau^*) + 2\sum_{j=1}^{T-3} h(j/l)\hat{\gamma}_{j,2}(\tau^*); \hat{\gamma}_{j,2}(\tau^*) := (T-2)^{-1}\sum_{t=j+3}^{T}\hat{k}_t(\tau^*)\hat{k}_{t-j}(\tau^*). \tag{16}$$

The function $h(j/l)$ is called the Bartlett window and $h(j/l) := 1 - j/(l+1)$, where l is the bandwidth parameter and $l = O(T^{1/4})$. In this thesis, the same kernel and bandwidth parameters of Harvey et al. (2009) were used. From now on, any reference to $t_0(\tau^*)$, $t_1(\tau^*)$ or $t_2(\tau^*)$ will mean that they are based on the LR variance estimators presented in equations (14), (15) and (16).

The Theorem bellow summarizes the asymptotic behaviour of the $|t_0(\tau^*)|$, $|t_1(\tau^*)|$ and $|t_2(\tau^*)|$ statistics under the presence of $I(0)$, $I(1)$ and $I(2)$ shocks.

Theorem 5.1. *Let the time series process* $\{y_t\}$ *be generated according to equations* (1) *and* (2) *and let Assumption 4.1.b hold.*

(i) If u_t *in* (2) *is* $I(0)$ *then: (a)* $|t_0(\tau^*)| \xrightarrow{d} |L_{00}(r,\tau^*)|$ *where*

$$L_{00}(r,\tau^*) = \frac{\int_0^1 RTQ(r,\tau^*)dW(r)}{\sqrt{\int_0^1 RTQ(r,\tau^*)^2 dr}},$$

(b) $|t_1(\tau^*)| = O_p(l/T)^{1/2}$ *and (c)* $|t_2(\tau^*)| = O_p(l/T)^{1/2}$*;*

(ii) If u_t *in* (2) *is* $I(1)$ *then: (a)* $|t_0(\tau^*)| = O_p(T/l)^{1/2}$*, (b)* $|t_1(\tau^*)| \xrightarrow{d} |L_{11}(r,\tau^*)|$ *where*

$$L_{11}(r,\tau^*) = \frac{\int_0^1 RTL(r,\tau^*)dW(r)}{\sqrt{\int_0^1 RTL(r,\tau^*)^2 dr}},$$

and (c) $|t_2(\tau^*)| = O_p(l/T)^{1/2}$*;*

(iii) If u_t *in* (2) *is* $I(2)$ *then: (a)* $|t_0(\tau^*)| = O_p(T^3/l)^{1/2}$*, (b)* $|t_1(\tau^*)| = O_p(T/l)^{1/2}$ *and (c)* $|t_2(\tau^*)| \xrightarrow{d} |L_{22}(r,\tau^*)|$ *where*

$$L_{22}(r,\tau^*) = \frac{\int_0^1 RTU(r,\tau^*)dW(r)}{\sqrt{\int_0^1 RTU(r,\tau^*)^2 dr}}.$$

Proof. The proof of items (i.b) and (iii.a) are only valid under a set of conjectures that require careful examination in the future. The proof of the remaining items can be found in appendix A. □

Here, $W(r)$ is a standard Brownian motion on $[0,1]$ and $RTQ(r,\tau^*)$ is the continuous time residual from the projection of $\mathbb{1}(r > \tau^*)(r-\tau^*)^2$ into the space spanned by $\{1, r, r^2\}$, $RTL(r,\tau^*)$ is the residual from a projection of $\mathbb{1}(r>\tau^*)(r-\tau^*)$ into the space spanned by $\{1, r\}$ and $RTU(r,\tau^*)$ is the residual from a projection of $\mathbb{1}(r>\tau^*)$ into the space spanned by $\{1\}$.

The results in Theorem 5.1 show that $t_0(\tau^*) \xrightarrow{d} N(0,1)$ if u_t is $I(0)$, $t_1(\tau^*) \xrightarrow{d} N(0,1)$ if u_t is $I(1)$, while $t_2(\tau^*) \xrightarrow{d} N(0,1)$ if u_t is $I(2)$. As a consequence, the appropriate two-sided test can be implemented using critical values from the standard normal distribution if the time of break is assumed to be known.

Remark 5.1. From part (i) of Theorem 5.1 it can be seen that if u_t is $I(0)$ then $|t_0(\tau^*)|$ attains the Gaussian distribution asymptotically, while $|t_1(\tau^*)|$ and $|t_2(\tau^*)|$ converges in probability to zero. Similarly, from part (ii) of the same theorem, if u_t is $I(1)$ then $|t_0(\tau^*)|$ diverges, $|t_1(\tau^*)|$ attains the Gaussian distribution asymptotically while $|t_2(\tau^*)|$ converges in probability to zero. Finally, from part (iii), if u_t is $I(2)$ then $|t_0(\tau^*)|$ and $|t_1(\tau^*)|$ diverge, though at a different rate, while $|t_2(\tau^*)|$ converges to the Gaussian distribution asymptotically.

From the results above and given that the order of integration of u_t is not known *a priori* three auxiliary functions are proposed to ensure that the statistic $|t_0(\tau^*)|$ of (8) is selected when u_t is $I(0)$, $|t_1(\tau^*)|$ of (10) is selected when u_t is $I(1)$ while $|t_2(\tau^*)|$ of (12) is selected when u_t is $I(2)$, thereby ensuring that the asymptotically optimal test is chosen in the limit, following the same reasoning from Harvey et al. (2009). To that end, consider this new test statistic for the presence of a break in the quadratic trend which is based on a weighted average of $|t_0(\tau^*)|$, $|t_1(\tau^*)|$ and $|t_2(\tau^*)|$ and is robust as to whether the shocks are an $I(0)$, $I(1)$ or an $I(2)$

process:

$$\begin{aligned} t^*_{\lambda,2} &= \lambda(S_0(\tau^*), S_1(\tau^*)) \times |t_0(\tau^*)| + [\lambda(S_1(\tau^*), S_2(\tau^*)) - \lambda(S_0(\tau^*), S_1(\tau^*))] \times |t_1(\tau^*)| \\ &\quad + [1 - \lambda(S_1(\tau^*), S_2(\tau^*))] \times |t_2(\tau^*)|. \end{aligned} \tag{17}$$

In (17), let $\lambda : \mathbb{R}^2 \to \mathbb{R}$ be a weight function that can either converge to unity or zero. For the proposed method to work, the function $\lambda(S_0(\tau^*), S_1(\tau^*))$ should converge to unity when u_t is $I(0)$ and to zero when u_t is $I(1)$ or $I(2)$. On the other hand, $\lambda(S_1(\tau^*), S_2(\tau^*))$ should converge to unity when u_t is $I(0)$ or $I(1)$ and to zero when u_t is $I(2)$. For that reason it is now necessary to choose the appropriate auxiliary statistics $S_i(\tau^*)$ for $i = 0, 1, 2$, and the weight functions $\lambda(S_0(\tau^*), S_1(\tau^*))$ and $\lambda(S_1(\tau^*), S_2(\tau^*))$.

The auxiliary statistics were build based on the stationary LM statistics of Kwiatkowski et al. (1992). They are calculated from the residuals $\{\hat{u}_t(\tau^*)\}_{t=1}^T$, $\{\hat{v}_t(\tau^*)\}_{t=2}^T$ and $\{\hat{k}_t(\tau^*)\}_{t=3}^T$, each of each invariant to the values of α, β and γ and $\hat{\omega}_i^2(\tau^*)$ for $i = 0, 1, 2$ is the long run variance estimator.

$$S_0(\tau^*) := \frac{\sum_{t=1}^T (\sum_{i=1}^t \hat{u}_i(\tau^*))^2}{T^2 \hat{\omega}_0^2(\tau^*)}, \tag{18}$$

$$S_1(\tau^*) := \frac{\sum_{t=2}^T (\sum_{i=2}^t \hat{v}_i(\tau^*))^2}{(T-1)^2 \hat{\omega}_1^2(\tau^*)}, \tag{19}$$

$$S_2(\tau^*) := \frac{\sum_{t=3}^T (\sum_{i=3}^t \hat{k}_i(\tau^*))^2}{(T-2)^2 \hat{\omega}_2^2(\tau^*)}. \tag{20}$$

The relevant large sample properties of these three statistics are established in the following Lemma:

Lemma 5.1. *Let the conditions of Theorem 5.1 hold:*

(i) If u_t is $I(0)$ then: (a) $S_0(\tau^*) = O_p(1)$, (b) $S_1(\tau^*) = O_p(l/T)$ and (c) $S_2(\tau^*) = O_p(l/T)$;

(ii) If u_t is $I(1)$ then: (a) $S_0(\tau^*) = O_p(T/l)$, (b) $S_1(\tau^*) = O_p(1)$ and (c) $S_2(\tau^*) = O_p(l/T)$;

(iii) If u_t is $I(2)$ then: (a) $S_0(\tau^*) = O_p(T^3/l)$, (b) $S_1(\tau^*) = O_p(T/l)$ and (c) $S_2(\tau^*) = O_p(1)$.

Proof. The proof is similar to that of Lemma 1 in Harvey et al. (2009). □

The previous Lemma suggests two weight functions $\lambda : \mathbb{R}^2 \to \mathbb{R}$ such that:

$$\lambda(S_0(\tau^*), S_1(\tau^*)) := e^{[-(g_1 S_0(\tau^*) S_1(\tau^*))^{g_2}]}, \quad (21)$$

$$\lambda(S_1(\tau^*), S_2(\tau^*)) := e^{[-(g_3 S_1(\tau^*) S_2(\tau^*))^{g_4}]}, \quad (22)$$

where g_i for $i = 1, 2, 3, 4$ are positive constants in order to keep the convergence in the limit. For both functions, the large sample convergence is achieved at an exponential rate. From Theorem 5.1 and Lemma 5.1 the following corollary can be stated:

Corollary 5.1. *Let the conditions of Theorem 5.1 hold.*

(i) If u_t is $I(0)$ then (a) $\lambda(S_0(\tau^*), S_1(\tau^*)) \xrightarrow{p} 1$, (b) $\lambda(S_1(\tau^*), S_2(\tau^*)) \xrightarrow{p} 1$ and $t^*_{\lambda,2} = |t_0(\tau^*)| + o_p(1) \xrightarrow{d} N(0,1)$;

(ii) If u_t is $I(1)$ then (a) $\lambda(S_0(\tau^*), S_1(\tau^*)) \xrightarrow{p} 0$, (b) $\lambda(S_1(\tau^*), S_2(\tau^*)) \xrightarrow{p} 1$ and $t^*_{\lambda,2} = |t_1(\tau^*)| + o_p(1) \xrightarrow{d} N(0,1)$;

(iii) If u_t is $I(2)$ then (a) $\lambda(S_0(\tau^*), S_1(\tau^*)) \xrightarrow{p} 0$, (b) $\lambda(S_1(\tau^*), S_2(\tau^*)) \xrightarrow{p} 0$ and $t^*_{\lambda,2} = |t_2(\tau^*)| + o_p(1) \xrightarrow{d} N(0,1)$.

Remark 5.2. The results in Corollary 5.1 show that if u_t is $I(0)$ then $t^*_{\lambda,2}$ is asymptotically equivalent to $|t_0(\tau^*)|$. Equivalently, if u_t is $I(1)$ then $t^*_{\lambda,2}$ is asymptotically equivalent to $|t_1(\tau^*)|$. Finally, if u_t is $I(2)$ then $t^*_{\lambda,2}$ is asymptotically equivalent to $|t_2(\tau^*)|$. That is, $t^*_{\lambda,2}$ achieves the appropriate limit distribution independently of the nature of the shocks.

When the break fraction, τ^*, can be arbitrarily chosen it was proved that $t^*_{\lambda,2} \xrightarrow{d} N(0,1)$, irrespective of whether the shocks are $I(0)$, $I(1)$ or $I(2)$. To allow that test to be robust to $I(2)$ shocks, both the auxiliary statistic $S_2(\tau^*)$ and the weight function $\lambda(S_1(\tau^*), S_2(\tau^*))$ had to be created and added to existing robust tests for the $I(0)$ and $I(1)$ processes, after significant transformations. To move forward for the unknown break fraction the same methodology was applied though the break date had to be estimated through a maximization process.

5.2 Unknown Break Fraction

This is the most realistic case scenario. As pointed out by Montañés (1997) the time of break does not always coincide with the announcement of an economic event nor with its realization. It can take place further back in time or ahead. For that reason, a data dependent algorithm is needed to select the date after which perceptible changes occur in the quadratic time trend. The same methodology of Harvey et al. (2009) and Andrews (1993) was followed to estimate the break date for the $I(0)$, $I(1)$ and $I(2)$ shocks.

The appropriate statistics to test for the existence of a break in trend is given by the maxima of the sequence of statistics $\{|t_0(\tau)|, \tau \in \Lambda\}$, $\{|t_1(\tau)|, \tau \in \Lambda\}$, and

$\{|t_2(\tau)|, \tau \in \Lambda\}$ where $\Lambda = [\tau_L, \tau_U]$, and $0 < \tau_L < \tau_U < 1$. The quantities τ_L and τ_U will be referred as the *lower* and *upper trimming* parameters, respectively. Furthermore, consider the set $\Lambda^* := \{\lfloor \tau_L T \rfloor, \ldots, \lfloor \tau_U T \rfloor\}$ and the true break fraction $\tau^* \in \Lambda$. The aforementioned statistics are given by the following set of equations:

$$t_i^* := \sup_{s \in \Lambda^*} \left| t_i\left(\frac{s}{T}\right)\right| \quad \text{for } i = 0, 1, 2 \text{ and } T > 0, \tag{23}$$

with associated breakpoint estimators of τ^* given by $\hat{\tau}_i := \arg\sup_{s \in \Lambda^*} \left|t_i\left(\frac{s}{T}\right)\right|$ for $i = 0, 1, 2$ and $T > 0$, such that $t_0^* \equiv |t_0(\hat{\tau}_0)|$, $t_1^* \equiv |t_1(\hat{\tau}_1)|$ and $t_2^* \equiv |t_2(\hat{\tau}_2)|$. Furthermore, the dummy variables in section 5.1 must be redefined. In the case of unknown break fraction $DTQ_t(\hat{\tau}_0) := \mathbb{1}(t > T_0^*)[(1/2)(t - T_0^*)^2]$, $DTL_t(\hat{\tau}_1) := \mathbb{1}(t > T_1^*)(t - T_1^* - 1/2)$ and

$$DTU_t(\hat{\tau}_2) = \begin{cases} 0 & \text{if } t < T_2^*, \\ 0.5 & \text{if } t = T_2^*, \\ 1 & \text{if } t > T_2^*, \end{cases} \quad \text{for all } t \in \Lambda^*,$$

where $T_0^* := \lfloor \hat{\tau}_0 T \rfloor$, $T_1^* := \lfloor \hat{\tau}_1 T \rfloor$ and $T_2^* := \lfloor \hat{\tau}_2 T \rfloor$. Similar considerations must be adopted to the stationary statistics of Kwiatkowski et al. (1992).

For the unknown break date case the proposed test statistic is given by:

$$\begin{aligned} t_{\lambda,2} \;=\; & \lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1)) \times t_0^* + m_{\xi_1}[\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2)) - \lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1))] \times t_1^* \\ & + m_{\xi_2}[1 - \lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2))] \times t_2^*. \end{aligned} \tag{24}$$

with $\lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1)) := e^{[-(g_1 S_0(\hat{\tau}_0) S_1(\hat{\tau}_1))^{g_2}]}$ and $\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2)) := e^{[-(g_3 S_1(\hat{\tau}_1) S_2(\hat{\tau}_2))^{g_4}]}$. The associated break point estimator, $\hat{\tau}$, is going to be $\hat{\tau}_0$ if u_t is $I(0)$, $\hat{\tau}_1$ if u_t is

$I(1)$ or $\hat{\tau}_2$ if u_t is $I(2)$.

The large sample behaviour of the t_0^*, t_1^* and t_2^* statistics must be taken both under the null hypothesis $H_0 : \gamma = 0$ and the alternative $H_A : \gamma \neq 0$, when the shocks u_t are $I(0)$, $I(1)$ or $I(2)$. For the null hypothesis, it can be proved that:

Theorem 5.2. *Let the time series process* $\{y_t\}$ *be generated according to equations* (1) *and* (2) *under* $H_0 : \gamma = 0$ *and let the conditions of Theorem 5.1 hold.*

(i) If u_t *in* (2) *is* $I(0)$ *then: (a)* $t_0^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{00}(r, \tau^*)|$*, (b)* $t_1^* = O_p(l/T)^{1/2}$ *and (c)* $t_2^* = O_p(l/T)^{1/2}$*;*

(ii) If u_t *in* (2) *is* $I(1)$ *then: (a)* $t_0^* = O_p(T/l)^{1/2}$*, (b)* $t_1^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{11}(r, \tau^*)|$ *and (c)* $t_2^* = O_p(l/T)^{1/2}$*;*

(iii) If u_t *in* (2) *is* $I(2)$ *then: (a)* $t_0^* = O_p(T^3/l)^{1/2}$*, (b)* $t_1^* = O_p(T/l)^{1/2}$ *and (c)* $t_2^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{22}(r, \tau^*)|$.

Proof. The sketch of the proof can be found in the appendix A. □

Similarly to Lemma 5.1, the large sample behaviour of the $S_0(\hat{\tau}_0)$, $S_1(\hat{\tau}_1)$ and $S_2(\hat{\tau}_2)$ statistics can also be stated.

Lemma 5.2. *Let the conditions of Theorem 5.1 hold:*

(i) If u_t *is* $I(0)$ *then: (a)* $S_0(\hat{\tau}_0) = O_p(1)$*, (b)* $S_1(\hat{\tau}_1) = O_p(l/T)$ *and (c)* $S_2(\hat{\tau}_2) = O_p(l/T)$*;*

(ii) If u_t *is* $I(1)$ *then: (a)* $S_0(\hat{\tau}_0) = O_p(T/l)$*, (b)* $S_1(\hat{\tau}_1) = O_p(1)$ *and (c)* $S_2(\hat{\tau}_2) = O_p(l/T)$*;*

(iii) If u_t *is* $I(2)$ *then: (a)* $S_0(\hat{\tau}_0) = O_p(T^3/l)$*, (b)* $S_1(\hat{\tau}_1) = O_p(T/l)$ *and (c)* $S_2(\hat{\tau}_2) = O_p(1)$.

By using the same arguments as Harvey et al. (2009), regardless of whether H_0 or H_A holds, it is possible to show that $\lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1))$ and $\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2))$

converge in probability to zero or one, depending on the nature of the shocks. These results are summarized in Table III.

Although the following Corollary has not been formally proved, it is expected to describe with some accuracy the asymptotic null distribution of the statistic in equation (24), under $H_0 : \gamma = 0$.

Corollary 5.2. *Let the conditions of Theorem 5.1 hold. Let $H_0 : \gamma = 0$ hold. Then:*

(i) If u_t is $I(0)$: $t_{\lambda,2} = t_0^* + o_p(1) \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{00}(r, \tau^*)|$;

(ii) If u_t is $I(1)$: $t_{\lambda,2} = m_{\xi_1} t_1^* + o_p(1) \xrightarrow{d} m_{\xi_1} \sup_{\tau\in\Lambda} |L_{11}(r, \tau^*)|$;

(iii) If u_t is $I(2)$: $t_{\lambda,2} = m_{\xi_2} t_2^* + o_p(1) \xrightarrow{d} m_{\xi_2} \sup_{\tau\in\Lambda} |L_{22}(r, \tau^*)|$.

For a given significance level, ξ, under the null hypothesis H_0, the constants m_{ξ_1} and m_{ξ_2} can be calculated to ensure that, asymptotically, the critical values of the $m_{\xi_1} \sup_{\tau\in\Lambda} |L_{11}(r, \tau^*)|$ and $m_{\xi_2} \sup_{\tau\in\Lambda} |L_{22}(r, \tau^*)|$ coincide with the critical values of $\sup_{\tau\in\Lambda} |L_{00}(r, \tau^*)|$, similarly to Vogelsang (1998).

The consistency rates of the statistics t_i^* for $i = 0, 1, 2$ under a fixed alternative of the form $H_A : \gamma \neq 0$ are hard to obtain and the proof of those results are out of the scope of this thesis and are left for future investigation. Nevertheless, it is possible to obtain the asymptotic critical values of the test through Monte Carlo techniques by simulating the appropriate limiting distribution. The practical implementation of the test procedures is explained in section 6.

6 Practical Implementation of the Test Procedures

The asymptotic critical values of the test statistic $t_{\lambda,2}$ are provided in Table I, for $\xi = \{0.10, 0.05, 0.01\}$, along with the corresponding values $m_{\xi 1}$ and $m_{\xi 2}$. This figure was obtained under the null hypothesis of no break in trend, $H_0 : \gamma = 0$, against the two-sided alternative $H_A : \gamma \neq 0$. A 10% *trimming* parameter was used such that $\tau_L = 0.10$ and $\tau_U = 0.90$, similarly to Sayginsoy and Vogelsang (2004). By doing so, the searching range for the occurrence of a structural break is set to the restricted interval $[\lfloor \tau_L^* T \rfloor, \lfloor \tau_U^* T \rfloor]$ with length $\lfloor 0.8T + 1 \rfloor$.

The results were obtained by simulation of the appropriate limiting distribution with discrete approximations for $T = 100$ and 10000 replications using the *rnorm* normal random number generator of R version 0.99.879.

Table I: Asymptotic critical values, m_{ξ_1} and m_{ξ_2} for the $t_{\lambda,2}$ test statistic.

ξ	Critical value	m_{ξ_1}	m_{ξ_2}
0.10	2.300	1.086	1.159
0.05	2.695	1.096	1.187
0.01	3.489	1.113	1.181

Source: R output

To implement the test statistic it was also needed to specify the constants g_1,g_2, g_3 and g_4. Different constants can be assigned to these values, however, and as a rule of thumb, the ones to be picked are those that deliver the best overall performance after Monte Carlo simulations of the finite sample size and power. The same reasoning can be applied to choose the bandwidth parameter.

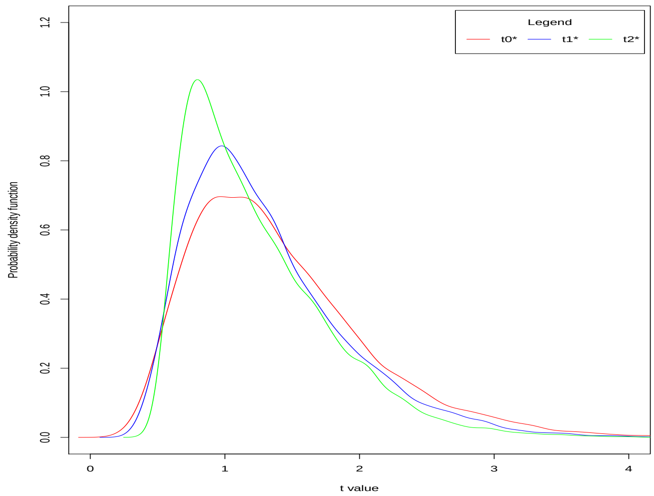

Figure 1: Probability density functions of t_0^*, t_1^* and t_2^*.

In this testing procedure, the recommended values of Harvey et al. (2009) were used such that $g_1 = g_3 = 500$, $g_2 = g_4 = 2$ and $l = \lfloor 4(T/100)^{\frac{1}{4}} \rfloor$. Figure 1 plots the probability density functions of t_0^* , t_1^* and t_2^* and it can be observed that for all cases the asymptotic distribution is skewed to the right.

The source code to apply the test statistic $t_{\lambda,2}$ to real-life data can be found in appendix C.

7 Conclusions

This thesis proposed new tests for the presence of a one time structural change in the trend level of a univariate time series which is valid regardless of the shocks being $I(0)$, $I(1)$ or $I(2)$. The $I(2)$ hypothesis was introduced to the subject of structural break testing to fulfil a long standing gap in the econometric literature.

With this idea in mind, two different frameworks were established depending on the information available about the trend break date. Under a known break date, the proposed test is a weighted average of the absolute values of three regression t-ratios, each one appropriate for the case where the data is generated by an $I(0)$, $I(1)$ or an $I(2)$ process. These statistics were proved to have standard normal limiting null distributions. For the opposite case scenario, of unknown break date, additional complexity was brought to the testing procedure. In fact, a *supremum* based approach had to be taken to determine the break point estimators of the true break date. The asymptotic critical values were computed through Monte Carlo simulation and provided a completely different picture when compared to the known break date hypothesis.

In the future, it would be useful to restate Theorem 5.1 to a wider scenario. The results in section 5.2 should also be given more attention. In particular, the consistency rates of the appropriate statistics under a fixed alternative $H_A : \gamma \neq 0$ should be established. Furthermore, size and power tests must be taken for different values of the constants g_i for $i = 1, 2, 3, 4$ to find out which combinations of values deliver the best overall result. It is also desirable to obtain asymptotic critical values for 50000 replications and $T = 1000$, which requires solving an optimization problem while implementing the Monte Carlo simulation. Finally, after overcoming all these technical issues, it would be interesting to provide tests for multiple breaks and to propose an empirical application to financial and economic data.

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A Mathematical Appendix

Theorem A.1 (Frisch-Waugh-Lovell Theorem (FWLT))**.** *Consider the model* $Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$ *where* $X = (X_1, X_2)$ *and* $\beta = (\beta_1, \beta_2)'$*. The OLS estimator of* β_2 *and the OLS residuals* $\hat{\varepsilon}$ *may be computed by the following algorithm:*

1. *Regress* Y *on* X_1*; obtain residuals* $\tilde{Y}$*;*
2. *Regress* X_2 *on* X_1*; obtain residuals* $\tilde{X}_2$*;*
3. *Regress* $\tilde{Y}$ *on* $\tilde{X}_2$*; obtain OLS estimates* $\hat{\beta}_2$ *and residuals* $\hat{\varepsilon}$*.*

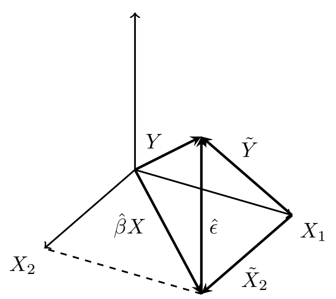

Figure 2: The Frisch-Waugh-Lovell Theorem with $X_1 \perp X_2$.

Proof of Theorem (5.1)*.* The proof of this Theorem follows from natural extension to Theorem 1 of Harvey et al. (2009) and is done in three steps. First, the FWLT is applied to estimate γ from equation (1). With this procedure, complex numerical calculations in inverting high order matrices are avoided. Then, results about weak convergence from asymptotic theory are applied to $t_0(\tau^*)$ under the assumptions that the errors might be either $I(0)$, $I(1)$ or $I(2)$. This procedure is repeated for equations (4) and (6). Due to the invariance of the statistics, α and

β can be set to zero. Table II bellow illustrates the disturbance term in equations (1), (4) and (6) under different assumptions about the stationary properties of the shocks and provides the error terms that appear in the proof of parts I, II and III below.

Table II: Error terms after first and second differences.

u_t	α_1	α_2	eq. (1)	eq. (4)	eq. (6)
$I(0)$	0	0	$u_t = \varepsilon_t$	$\Delta u_t = \Delta\varepsilon_t$	$\Delta^2 u_t = \Delta^2\varepsilon_t$
$I(1)$	0	1	$u_t = u_{t-1} + \varepsilon_t$	$\Delta u_t = \varepsilon_t$	$\Delta^2 u_t = \Delta\varepsilon_t$
$I(2)$	1	1	$u_t = 2u_{t-1} - u_{t-2} + \varepsilon_t$	$\Delta u_t = \Delta u_{t-1} + \varepsilon_t$	$\Delta^2 u_t = \varepsilon_t$

PART I: The regressor γ can be estimated from equation (1) and Theorem (A.1). To that end, consider equation (25) below:

$$Y_t = \gamma RTQ_t(\tau^*) + e_t \quad t = 1, \ldots, T, \tag{25}$$

where e_t is an error term, Y_t are the OLS residuals from regressing y_t into 1, t and t^2 and $RTQ_t(\tau^*)$ are the OLS residual from regressing $DTQ_t(\tau^*)$ into 1, t and t^2. It is possible to show that under H_0 the test statistic $t_0(\tau^*)$ can be determined by the following equation:

$$t_0(\tau^*) = \frac{\sum_{t=1}^{T} RTQ_t(\tau^*) u_t}{\sqrt{\hat{\omega}_0^2 \sum_{t=1}^{T} RTQ_t(\tau^*)^2}}. \tag{26}$$

(i.a) The shocks are $I(0)$: The proof of the weak convergence result is carried out by looking through Proposition B.1, items (e) and (g) and Remark 4.1

which states that $\hat{\omega}_0^2(\tau^*) \xrightarrow{p} \omega_u^2$. That is:

$$t_0(\tau^*) = \frac{T^{-5/2}\sum_{t=1}^{T} RTQ_t(\tau^*)\varepsilon_t}{\sqrt{T^{-5}\sum_{t=1}^{T} RTQ_t(\tau^*)^2}} \times \frac{1}{\sqrt{\hat{\omega}_0^2}} \xrightarrow{d} \frac{\int_0^1 RTQ(r,\tau^*)dW(r)}{\sqrt{\int_0^1 RTQ(r,\tau^*)^2 dr}},$$

where $W(r)$ is a standard Brownian motion on $[0,1]$ and $RTQ(r,\tau^*)$ is the continuous time residual from the projection of $\mathbb{1}(r>\tau^*)(r-\tau^*)^2$ into the space spanned by $\{1,r,r^2\}$.

(ii.a) The shocks are $I(1)$: Harvey et al. (2009) extended the results in Kwiatkowski et al. (1992) to establish the result that $(lT)^{-1}\hat{\omega}_0^2 \xrightarrow{d} \omega_\varepsilon^2 \int_0^1 H_1(r,\tau^*)^2 dr$, where $H_1(r,\tau^*)$is a continuous time residual from the projection of $W(r)$ into the space spanned by $\{1,r,r^2,\mathbb{1}(r>\tau^*)(r-\tau^*)^2\}$. By Proposition B.1, items (e), (f) and (g) it can be seen that:

$$\begin{aligned} T^{-7/2}\sum t^2 u_t &= T^{-7/2}\sum t^2 u_{t-1} + T^{-7/2}\sum t^2\varepsilon_t \\ &= T^{-7/2}\sum t^2 u_{t-1} + T^{-5/2}T^{-1}\sum t^2\varepsilon_t \\ &= T^{-7/2}\sum t^2 u_{t-1} + \frac{1}{T}T^{-5/2}\sum t^2\varepsilon_t \quad \text{as } T\to\infty \\ &\equiv T^{-7/2}\sum t^2 u_{t-1} + o_p(1). \end{aligned}$$

Finally:

$$(lT^{-1})^{1/2}t_0(\tau^*) = \frac{T^{-7/2}\sum_{t=1}^{T} RTQ_t(\tau^*)u_{t-1}}{\sqrt{T^{-5}\sum_{t=1}^{T} RTQ_t(\tau^*)^2}} \times \frac{1}{\sqrt{(lT)^{-1}\omega_0^2}} = O_p(1).$$

(iii.a) The shocks are $I(2)$: It can be conjectured from Lemma 5 of Haldrup and Lildholdt (2002) that $(lT)^{-1}\omega_0^2 \xrightarrow{d} \omega_\varepsilon^2 \int_0^1 H_2(r,\tau^*)^2 dr$, where $H_2(r,\tau^*)$is a continuous time residual from the projection of $W(r)$ into the space spanned by $\{1, r, r^2, \mathbb{1}(r > \tau^*)(r - \tau^*)^2\}$. This comes from the fact that one more unit root is present in $\{u_t\}$. By Proposition B.3, items (j) and Proposition B.1 item (g) the result follows:

$$(lT^{-3})^{1/2}t_0(\tau^*) = \frac{T^{-9/2}\sum_{t=1}^{T} RTQ_t(\tau^*)u_t}{\sqrt{T^{-5}\sum_{t=1}^{T} RTQ_t(\tau^*)^2}} \times \frac{1}{\sqrt{(lT)^{-1}\omega_0^2}} = O_p(1).$$

PART II: The regressor γ can be estimated from equation (4) and Theorem (A.1). To that end, consider following equation:

$$Y_t = \gamma RTL_t(\tau^*) + e_t \quad t = 2, \ldots, T, \tag{27}$$

where e_t is an error term, Y_t are the OLS residuals from regressing Δy_t into 1 and t and $RTL_t(\tau^*)$ are the OLS residual from regressing $DTL_t(\tau^*)$ into 1 and t. Using the FWLT, it is possible to show that under H_0 the test statistic $t_1(\tau^*)$ can be determined by the following equation:

$$t_1(\tau^*) = \frac{\sum_{t=2}^{T} RTL_t(\tau^*)\Delta u_t}{\sqrt{\hat{\omega}_1^2\sum_{t=2}^{T} RTL_t^2(\tau^*)}}. \tag{28}$$

(i.b) The shocks are $I(0)$: It is conjectured that $\sum RTL_t(\tau^*)\Delta\varepsilon_t \equiv O_p(T)$ and that $l\hat{\omega}_1^2(\tau^*) \xrightarrow{p} c$ where $c = -2\sum_{s=1}^{\infty} s\gamma_s'$ and $\gamma_s' = E(\Delta u_t \Delta u_{t-s})$ (see Leybourne et al., 2004) which implies that:

$$(Tl^{-1})^{1/2}t_1(\tau^*) = \frac{T^{-1}\sum_{t=2}^{T} RTL_t(\tau^*)\Delta\varepsilon_t}{\sqrt{T^{-3}\sum_{t=2}^{T} RTL_t^2(\tau^*)}} \times \frac{1}{\sqrt{l\hat{\omega}_1^2}} = O_p(1).$$

(ii.b) The shocks are $I(1)$: By Proposition B.1, items (c) and (g) and Remark 4.1 we know that $\hat{\omega}_1(\tau^*) \to \omega_\varepsilon^2$. Finally:

$$t_1(\tau^*) = \frac{T^{-\frac{3}{2}} \sum_{t=2}^{T} RTL_t(\tau^*)\varepsilon_t}{\sqrt{T^{-3} \sum_{t=2}^{T} RTL_t^2(\tau^*)}} \times \frac{1}{\sqrt{\hat{\omega}_1^2(\tau^*)}} \xrightarrow{d} \frac{\int_0^1 RTL(r,\tau^*)dW(r)}{\sqrt{\int_0^1 RTL(r,\tau^*)^2 dr}}.$$

(iii.b) The shocks are $I(2)$: $(lT)^{-1}\hat{\omega}_1^2 \to \omega_\varepsilon^2 \int_0^1 H_3(r,\tau^*)dr$, where $H_3(r,\tau^*)$is a continuous time residual from the projection of $W(r)$ into the space spanned by $\{1, r, \mathbb{1}(r > \tau^*)(r-\tau^*)\}$. According to Proposition B.3 item (g) and Proposition B.1 item (g):

$$(lT^{-1})^{1/2} t_1(\tau^*) = \frac{T^{-5/2} \sum_{t=2}^{T} RTL_t(\tau^*)\Delta u_t}{\sqrt{T^{-3} \sum_{t=2}^{T} RTL_t^2(\tau^*)}} \times \frac{1}{\sqrt{(lT)^{-1}\hat{\omega}_1^2(\tau^*)}} = O_p(1).$$

PART III: The regressor γ can be estimated from equation (6) and Theorem (A.1). To that end, consider following equation:

$$Y_t = \gamma RTU_t(\tau^*) + e_t \quad t = 3, \ldots, T,$$

where e_t is an error term, Y_t are the OLS residuals from regressing $\Delta^2 y_t$ into 1 and $RTU_t(\tau^*)$ are the OLS residuals from regressing $DTU_t(\tau^*)$ into 1. It is possible to show that under $H_0 : \gamma = 0$, equation (12) is equivalent to:

$$t_2(\tau^*) = \frac{\sum_{t=3}^{T} RTU_t(\tau^*)\Delta^2 u_t}{\sqrt{\hat{\omega}_2^2 \sum_{t=3}^{T} RTU_t^2(\tau^*)}}. \tag{29}$$

(i.c) The shocks are $I(0)$:The goal now is to determine the rate of convergence of $\sum_{t=3}^{T} RTU_t(\tau^*)\Delta^2 u_t$. The residuals $RTU_t(\tau^*)$ are equal to

$$RTU_t(\tau^*) = DTU_t(\tau^*) - \hat{\theta}_2 = \begin{cases} 0 - \overline{DTU}(\tau^*) & \text{if } t < T_b^* + 1, \\ 0.5 - \overline{DTU}(\tau^*) & \text{if } t = T_b^* + 1, \\ 1 - \overline{DTU}(\tau^*) & \text{if } t > T_b^* + 1, \end{cases}$$

where $\hat{\theta}_2 = \overline{DTU}(\tau^*)$. $RTU_t(\tau^*)$ is asymptotically equivalent to

$$RTU_t(\tau^*) = \begin{cases} \tau^* - 1 & \text{if } t < T_b + 1^*, \\ \tau^* - 0.5 & \text{if } t = T_b^* + 1. \\ \tau^* & \text{if } t > T_b^* + 1, \end{cases} \quad \text{since } \overline{DTU}(\tau^*) \to 1 - \tau^*.$$

Finally, by successive substitutions, the result follows:

$$\begin{aligned} \sum_{t=3}^{T} RTU_t(\tau^*)\Delta^2 u_t &= \sum_{t=3}^{T_b^*} RTU_t(\tau^*)\Delta^2 u_t + [\tau^* - 0.5]\Delta^2 u_{T_b^*+1} + \sum_{t=T_b^*+2}^{T} RTU_t(\tau^*)\Delta^2 u_t \\ &= (\tau^* - 1)\sum_{t=3}^{T_b^*} \Delta^2 u_t + [\tau^* - 0.5]\Delta^2 u_{T_b^*+1} + \tau^* \sum_{t=T_b^*+2}^{T} \Delta^2 u_t \\ &= (\tau^* - 1)(\Delta^2 u_3 + \ldots + \Delta^2 u_{T_b^*}) + [\tau^* - 0.5]\Delta^2 u_{T_b^*+1} + \\ &+ \tau^*(\Delta^2 u_{T_b^*+2} + \ldots + \Delta^2 u_T) \\ &= (1 - \tau^*)\Delta u_2 - 0.5(\Delta u_{T_b^*} + \Delta u_{T_b^*+1}) + (\tau^*)\Delta u_T, \end{aligned}$$

which implies that $\sum_{t=3}^{T} RTU_t(\tau^*)\Delta^2 u_t = O_p(1)$, since u_t is $I(0)$. Furthermore, it is assumed that $l\hat{\omega}_2(\tau^*) \overset{p}{\to} c$ where $c = -2\sum_{s=1}^{\infty} s\gamma_s'$ and $\gamma_s' = E(\Delta^2 u_t \Delta^2 u_{t-s})$. As a result:

$$(l^{-1}T)^{1/2} t_2(\tau^*) = \frac{\sum_{t=3}^{T} RTU_t(\tau^*)\Delta^2 u_t}{\sqrt{T^{-1}\sum_{t=3}^{T} RTU_t^2(\tau^*)}} \times \frac{1}{\sqrt{l\hat{\omega}_2(\tau^*)^2}} = O_p(1).$$

(ii.c) The shocks are $I(1)$: Similarly to the result established above, it is possible to show that $\sum_{t=3}^{T} RTU_t \Delta\varepsilon_t = (1 - \tau^*)\varepsilon_2 - 0.5(\varepsilon_{T_b^*} + \varepsilon_{T_b^*+1}) + (\tau^*)\varepsilon_T = O_p(1)$

since ε_t is a white noise process. Hence:

$$(l^{-1}T)^{1/2}t_2(\tau^*) = \frac{\sum_{t=3}^{T} RTU_t(\tau^*)\Delta^2 u_t}{\sqrt{T^{-1}\sum_{t=3}^{T} RTU_t^2(\tau^*)}} \times \frac{1}{\sqrt{l\hat{\omega}_2(\tau^*)^2}} = O_p(1).$$

(iii.c) The shocks are $I(2)$: The result follows immediately from Proposition B.1 item (a).

$$t_2(\tau^*) = \frac{T^{-1/2}\sum_{t=3}^{T} RTU_t(\tau^*)\varepsilon_t}{\sqrt{T^{-1}\sum_{t=3}^{T} RTU_t^2(\tau^*)}} \times \frac{1}{\sqrt{\hat{\omega}_2^2}} \xrightarrow{d} \frac{\int_0^1 RTU(r,\tau^*)dW(r)}{\sqrt{\int_0^1 RTU(r,\tau^*)^2 dr}}.$$

□

Proof of Theorem (5.2). The proof is similar to that of Theorem 5.1 but requires the application of the Continuous Mapping Theorem and the fact that the *sup* function is continuous in the time interval under study. It is a natural extension of the proof of Theorem 2 in (Harvey et al., 2009), which is omitted for the interest of brevity. □

Table III: Convergence in probability of $\lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1))$ and $\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2))$.

$u_t \sim I(0)$	$u_t \sim I(1)$	$u_t \sim I(2)$
$\lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1)) \xrightarrow{p} 1$	$\lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1)) \xrightarrow{p} 0$	$\lambda(S_0(\hat{\tau}_0), S_1(\hat{\tau}_1)) \xrightarrow{p} 0$
$\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2)) \xrightarrow{p} 1$	$\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2)) \xrightarrow{p} 1$	$\lambda(S_1(\hat{\tau}_1), S_2(\hat{\tau}_2)) \xrightarrow{p} 0$

B Mathematical Appendix

Phillips (1987) was the first to use the Functional Central Limit to study the asymptotic distribution of statistics constructed from a unit root process. Its main findings can be found in Proposition 17.1 of Hamilton (1994). In the case of $I(1)$ errors and known break date the asymptotic distribution of $t_i, i = 0, 1, 2$ can be derived by using Proposition B.1 bellow. Park and Phillips (1989) and Haldrup (1998) gave their contribution to the analysis of the asymptotic properties of second order autoregressive functions. Proposition B.2 and B.3 were used to prove the asymptotic distribution of $|t_i(\tau^*)|$ for $i = 0, 1, 2$ in the case of $I(2)$ errors presented in Theorem 5.1.

B.1 Asymptotic Properties of a First-Order Autoregression

Proposition B.1. *Suppose that $\{y_t\}$ follows a random walk without drift, that is:*

$$y_t = y_{t-1} + u_t, \tag{30}$$

where $y_0 = 0$ and $\{u_t\}$ is an i.i.d.sequence with zero mean and variance σ^2. Then:

(a) $T^{-1/2} \sum_{t=1}^{T} u_t \xrightarrow{d} \sigma W(1)$

(b) $T^{-1} \sum_{t=1}^{T} y_{t-1} u_t \xrightarrow{d} (1/2)\sigma^2\{[W(1)]^2 - 1\} = \sigma^2 \int_0^1 W(r) dW(r)$

(c) $T^{-3/2} \sum_{t=1}^{T} t u_t \xrightarrow{d} \sigma[W(1) - \int_0^1 W(r) dr] = \sigma \int_0^1 r dW(r)$

(d) $T^{-5/2} \sum_{t=1}^{T} t y_{t-1} \xrightarrow{d} \sigma \int_0^1 r W(r) dr$

(e) $T^{-5/2} \sum_{t=1}^{T} t^2 u_t \xrightarrow{d} \sigma[r W(1) - \int_0^1 r W(r) dr] = \sigma \int_0^1 r^2 dW(r)$

(f) $T^{-7/2} \sum_{t=1}^{T} t^2 y_{t-1} \xrightarrow{d} \sigma \int_0^1 r^2 W(r) dr$

(g) $T^{-(v+1)} \sum_{t=1}^{aT} t^v \to \frac{a}{(v+1)}$ *for* $v = 0, 1, \ldots$

Proof. The proof of results (a),(b),(c), (d) and (g) can be found in Hamilton (1994) and the proof of items (e) and (f) are done in what follows. From item (c) and the

Continuous Mapping Theorem (CMT) it is possible to show that:

$$
\begin{aligned}
T^{-5/2}\sum_{t=1}^{T} t^2 u_t &= T^{-1}T^{-3/2}\sum_{t=1}^{T} t^2 u_t = T^{-3/2}\sum_{t=1}^{T} \frac{t}{T} t u_t = T^{-3/2}\sum_{t=1}^{T} r t u_t \\
&\xrightarrow{d} \sigma \int_0^1 r^2 dW(r),
\end{aligned}
$$

where $r = t/T$ is the same defined in (Hamilton, 1994, pag. 486). From item (d) and the CMT it is possible to prove that:

$$
T^{-7/2}\sum_{t=1}^{T} t^2 y_{t-1} = T^{-1}T^{-5/2}\sum_{t=1}^{T} t^2 y_{t-1} = T^{-5/2}\sum_{t=1}^{T} r t y_{t-1} \xrightarrow{d} \sigma \int_0^1 r^2 W(r) dr.
$$

□

B.2 Asymptotic Properties of a Second-Order Autoregression

It is a well established result that equation (30), a random walk without drift, is equal to $y_t = \sum_{i=1}^{t} u_i$, with initial condition $y_0 = 0$ and where $\{u_t\}$ is an $i.i.d$ random variable with zero mean and constant variance. This shows that for a random walk, the shocks to the series occurring in the past will persist and have a influence on the levels of the series. Interestingly, a process with two unit roots given by the equation $y_t = 2y_{t-1} - y_{t-2} + u_t$, with initial conditions $y_{-1} = y_0 = 0$ and $y_1 = u_1$, is equal to a double summation of shocks, that is, $y_t = \sum_{i=1}^{t}\sum_{j=1}^{i} u_j$. The double summation of shocks is one of the main reasons that explains the smoothness of a graph of an $I(2)$ time series when compared to the graph of an $I(1)$ time series. But is there a meaning to a single summation of shocks in the context of an $I(2)$ process? In this appendix, it is given the answer to this question and it is shown how it can be useful to derive the asymptotic properties of a second order process from Proposition B.1.

Proposition B.2. *Consider the process with two unit roots $y_t = 2y_{t-1} - y_{t-2} + u_t$, for all $t = 2, \ldots, T$, with initial conditions $y_{-1} = y_0 = 0$ and $y_1 = u_1$. Then:*

$$\Delta y_t = \sum_{i=1}^{t} u_i. \tag{31}$$

Proof. By Mathematical Induction. For $t = 1$, Δy_1 is trivially equal to u_1. If equation (31) is true then:

$$\Delta y_{t+1} = \sum_{i=1}^{t} u_i + u_{t+1} \Leftrightarrow \Delta y_{t+1} \overset{eq.(31)}{=} \Delta y_t + u_{t+1}$$
$$\Leftrightarrow \Delta y_{t+1} - \Delta y_t = u_{t+1}$$
$$\Leftrightarrow u_{t+1} = u_{t+1}.$$

□

The main purpose of this Proposition is to establish a parallel between the theory developed for random walks and a process with two unit roots. This appendix follows the same methodology of (Hamilton, 1994, chap. 17) but details are omitted in the interest of brevity.

Consider the stochastic function $X_T^*(r)$, constructed from the sample mean of the first r^{th} fraction of observations from a sample of size T, where $r \in [0, 1]$, and that is defined by[3]:

$$X_T^*(r) \equiv \frac{1}{T} \sum_{t=1}^{\lfloor Tr \rfloor^*} u_t. \tag{32}$$

For any given realization, $X_T^*(r)$ is a step function in r. By equation (31):

$$X_T^*(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq 1/T, \\ u_1/T = \Delta y_1/T & \text{for } 1/T \leq r < 2/T, \\ (u_1 + u_2)/T = \Delta y_2/T & \text{for } 2/T \leq r < 3/T, \\ \vdots & \\ (u_1 + u_2 + \cdots + u_T)/T = \Delta y_t/T & \text{for } r = 1. \end{cases} \tag{33}$$

Figure 3 plots $X_T^*(r)$ as a function of r. The area under the step function is the sum of T rectangles. The t^{th} rectangle has height $\Delta y_{t-1}/T$, width $1/T$ and area $\Delta y_{t-1}/T^2$. The steps become increasingly smaller as $T \to \infty$, so that in the limit $X_T^*(r)$ is a continuous function of r.

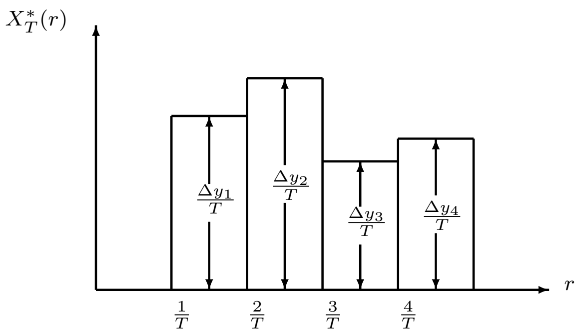

Figure 3: Plot of the stochastic function $X_T^*(r)$ as a function of r.

[3]Please notice the differences between eq. (32) and eq. (17.3.2) from (Hamilton, 1994).

The sequence of stochastic functions $\{\sqrt{T}X_T^*(\cdot)/\sigma\}_{T=1}^{\infty}$ has an asymptotic probability law[4] that is described by standard Brownian motion $W(\cdot)$:

$$\sqrt{T}X_T^*(\cdot)/\sigma \xrightarrow{d} W(\cdot). \tag{34}$$

Proposition B.3. *Suppose that $\{y_t\}$ is a random sequence with two unit roots*

$$y_t = 2y_{t-1} - y_{t-2} + u_t \quad t = 2, \ldots, T,$$

with initial conditions $y_{-1} = y_0 = 0$ and $y_1 = u_1$, satisfying the regulatory conditions of Phillips (1987), (Assumption 2.1), then as $T \to \infty$:

(a) $T^{-\frac{3}{2}}y_{\lfloor Tr \rfloor} \xrightarrow{d} \sigma \int_0^r W(s)ds \equiv \sigma\overline{W}(r)$

(b) $T^{-\frac{1}{2}}\Delta y_{\lfloor Tr \rfloor} \xrightarrow{d} \sigma W(r)$

(c) $T^{-4}\sum_{t=1}^T y_t^2 \xrightarrow{d} \sigma^2 \int_0^1 \overline{W}(r)^2 dr$

(d) $T^{-2}\sum_{t=1}^T \Delta y_t^2 \xrightarrow{d} \sigma^2 \int_0^1 W(r)^2 dr$

(e) $T^{-3}\sum_{t=1}^T y_{t-1}\Delta y_t \xrightarrow{d} \sigma^2 \int_0^1 \overline{W}(r)W(r)dr = \frac{\sigma^2}{2}\overline{W}(1)^2$

(f) $T^{-3}\sum_{t=1}^T t\Delta y_t^2 \xrightarrow{d} \sigma^2 \int_0^1 rW(r)^2 dr$

(g) $T^{-5/2}\sum_{t=1}^T t\Delta y_t \xrightarrow{d} \sigma \int_0^1 rW(r)dr$

(h) $T^{-7/2}\sum_{t=1}^T t^2\Delta y_t \xrightarrow{d} \sigma \int_0^1 r^2W(r)dr$

(i) $T^{-7/2}\sum_{t=1}^T ty_t \xrightarrow{d} \sigma \int_0^1 r\overline{W}(r)dr$

(j) $T^{-9/2}\sum_{t=1}^T t^2y_t \xrightarrow{d} \sigma \int_0^1 r^2\overline{W}(r)dr$

[4]The proof follows from trivial extensions to the results in Hamilton (1994, chap. 17).

Proof. The results (a), (b), (c) and (d) can be deduced from Lemma 2.1 of Park and Phillips (1989). Equation (e) is proven in Lemma 4 of Haldrup and Lildholdt (2002). Finally, equations (f) until (j) are proven in what follows.

From item (d) and the CMT is is possible to show that:

$$T^{-3}\sum_{t=1}^{T} t\Delta y_t^2 = T^{-1}T^{-2}\sum_{t=1}^{T} t\Delta y_t^2 = T^{-2}\sum_{t=1}^{T}\frac{t}{T}\Delta y_t^2 \xrightarrow{d} \sigma^2\int_0^1 rW(r)^2 dr.$$

The proof of item (h) follows from Lemma 2.1 (a.ii) of Park and Phillips (1989). Notice that in that papers' notation $x_{2t} \equiv \Delta y_t$. By the CMT and since x_{2t} is a process with one unit root, the result can be stated:

$$T^{-5/2}\sum_{t=1}^{T} t\Delta y_t = T^{-1}T^{-3/2}\sum_{t=1}^{T} t\Delta y_t = T^{-3/2}\sum_{t=1}^{T} r\Delta y_t \xrightarrow{d} \sigma\int_0^1 rW(r)dr.$$

By the aforementioned Lemma and the CMT it is possible to show that:

$$T^{-7/2}\sum_{t=1}^{T} t^2\Delta y_t = T^{-2}T^{-3/2}\sum_{t=1}^{T} t^2\Delta y_t = T^{-3/2}\sum_{t=1}^{T} r^2\Delta y_t \xrightarrow{d} \sigma\int_0^1 r^2W(r)dr.$$

Finally, the proof of item (j) follows from Lemma 2.1 (b.i) of Park and Phillips (1989). Notice that in the authors' notation x_{3t} is a process with two unit roots.

$$T^{-9/2}\sum_{t=1}^{T} t^2 y_t = T^{-1}T^{-7/2}\sum_{t=1}^{T} t^2 y_t = T^{-7/2}\sum_{t=1}^{T} rty_t \xrightarrow{d} \sigma\int_0^1 r^2\overline{W}(r)dr.$$

□

C Unknown Break Date Source Code

This appendix provides a computational framework to test for the existence of a structural break in the presence of an unknown break fraction. The software used was R version 0.99.879. This program requires the package “matrixcalc”.

C.1 Input

```
y
```

C.2 Auxiliary Constants and Lists

```
n=length(y); trm=0.1; tb_l=floor(trm*n); tb_u=n-tb_l
range=tb_u-tb_l+1
const=rep(1,n); tl=seq.int(1,n); tsqr=tl^2
bart_lag=floor(4*(n/100)^(1/4))
m1_10=1.086; m1_05=1.096; m1_01=1.113;
m2_10=1.159; m2_05=1.187; m2_01=1.181;
g1=500;  g2=2;   g3=500;  g4=2;
t0aux=numeric(range); t1aux=numeric(range); t2aux=numeric(range)
v0=numeric(nsim); v1=numeric(nsim); v2=numeric(nsim)
```

C.3 Auxiliary Functions

```
kpss=function(residuos,s2){(sum(cumsum(residuos)^2))/(s2*length(
    residuos)^2)}
lrvariance=function(residuos,l){
lth=length(residuos); aux1=numeric(lth-1); h=numeric(lth-1)
for (j in 1:(lth-1)) { h[j]<-1-j/(l+1)
```

```
for (t in (j+1):lth) { aux1[j]=aux1[j]+residuos[t]*residuos[t-j]
t=t+1}j=j+1}(sum(residuos^2) + 2*sum(h*aux1))/(lth)}
```

C.4 Dummy Variables

```
mtx_dtq= matrix(0.5*seq.int(0,n,1)^2, n, tb_u)
mtx_dtq[upper.tri(mtx_dtq, diag=TRUE)]=0
mtx_dtq=mtx_dtq[,-c(1:tb_l-1)]

mtx_dtl=matrix(seq.int(0,n,1)-0.5,n,tb_u)
mtx_dtl[upper.tri(mtx_dtl, diag=TRUE)]=0
mtx_dtl=mtx_dtl[,-c(1:tb_l-1)]

mtx_dtu=matrix(rep(1,n),n,tb_u)
mtx_dtu[upper.tri(mtx_dtu, diag=TRUE)]=0
mtx_dtu=mtx_dtu[,-c(1:tb_l-1)]
```

C.5 Time of Break for the I(0), I(1) and I(2) Case Scenarios

```
tb=tb_l
while (tb<=tb_u) {
mtx_stationary=matrix(c(const,tl,tsqr,mtx_dtq[,tb-tb_l+1]),n,4)
transpose_stationary=t(mtx_stationary)
inv_stationary=solve(t(mtx_stationary)%*%mtx_stationary)
B_stationary=inv_stationary%*%transpose_stationary%*%y
res_stationary=y-(mtx_stationary%*%B_stationary)
t0aux[tb-tb_l+1]=abs(B_stationary[4]/(sqrt(s2(res_stationary,bart_lag)*
    inv_stationary[4,4])))

```

```
mtx_oneroot=matrix(c(const,tl,mtx_dtl[,tb-tb_l+1]),n,3)[-1,]
transpose_oneroot=t(mtx_oneroot)
inv_oneroot=solve(transpose_oneroot%*%mtx_oneroot)
y1=diff(y,lag=1,differences = 1)
B_oneroot=inv_oneroot%*%(transpose_oneroot%*%y1)
res_oneroot=y1-mtx_oneroot%*%B_oneroot
t1aux[tb-tb_l+1]=abs(B_oneroot[3]/(sqrt(lrvariance(res_oneroot,bart_lag
    )*inv_oneroot[3,3])))

mtx_tworoot=matrix(c(const,mtx_dtu[,tb-tb_l+1]),n,2)[-(1:2),]
transpose_tworoot=t(mtx_tworoot)
inv_tworoot=solve(transpose_tworoot%*%mtx_tworoot)
y2=diff(y,lag=1,differences = 2)
B_tworoot=inv_tworoot%*%(transpose_tworoot%*%y2)
res_tworoot=y2-mtx_tworoot%*%B_tworoot
t2aux[tb-tb_l+1]=abs(B_tworoot[2]/(sqrt(lrvariance(res_tworoot,bart_lag
    )*inv_tworoot[2,2])))
tb=tb+1}
t0=max(t0aux); t1=max(t1aux); t2=max(t2aux)
```

C.6 Test Statistics for the Estimated Break Dates

```
time_stationary=which(t0==t0aux)
mtx_stationary=matrix(c(const,tl,tsqr,mtx_dtq[,time_stationary]),n,4)
transpose_stationary=t(mtx_stationary)
inv_stationary=solve(transpose_stationary%*%mtx_stationary)
B_stationary=inv_stationary%*%(transpose_stationary%*%y)
res_stationary=y-(mtx_stationary%*%B_stationary)
```

```
lrv_stationary=lrvariance(res_stationary,bart_lag)
S0=kpss(res_stationary,lrv_stationary)

time_oneroot=which(t1==t1aux)
mtx_oneroot=matrix(c(const,tl,mtx_dtl[,time_oneroot]),n,3)[-1,]
transpose_oneroot=t(mtx_oneroot)
inv_oneroot=solve(transpose_oneroot%*%mtx_oneroot)
y1=diff(y,lag=1,differences = 1)
B_oneroot=inv_oneroot%*%(transpose_oneroot%*%y1)
res_oneroot=y1-(mtx_oneroot%*%B_oneroot)
lrv_oneroot=lrvariance(res_oneroot,bart_lag)
S1=kpss(res_oneroot,lrv_oneroot)

time_tworoot=which(t2==t2aux)
mtx_tworoot=matrix(c(const,mtx_dtu[,time_tworoot]),n,2)[-(1:2),]
transpose_tworoot=t(mtx_tworoot)
inv_tworoot=solve(transpose_tworoot%*%mtx_tworoot)
y2=diff(y,lag=1,differences = 2)
B_tworoot=inv_tworoot%*%(transpose_tworoot%*%y2)
res_tworoot=y2-(mtx_tworoot%*%B_tworoot)
lrv_tworoot=lrvariance(res_tworoot,bart_lag)
S2=kpss(res_tworoot,lrv_tworoot)
```

C.7 Test Statistic

```
lam_kpss01=exp(-(g1*S0*S1)^g2); lam_kpss12=exp(-(g3*S1*S2)^g4)
test_kpss=lam_kpss01*t0+m1_10*(lam_kpss12-lam_kpss01)*t1
+m2_10*(1-lam_kpss12)*t2 # for a 10% significance level
```