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DISSERTAÇÃO

FRACTIONAL PROCESSES: AN APPLICATION TO FINANCE

FRANCISCO DE CASTILHO MONTEIRO GIL SERRANO

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ORIENTAÇÃO:

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Abstract

In this work it is presented an extensive mathematical description oriented to financial modelling based on three main fractional processes: the fractional Brownian motion and both fractional Lévy processes. It is shown how these processes were originated. The concept of self-similarity is explored and we present some notions of fractional calculus. It is discussed the opportunity of these processes in pricing financial derivatives and we present a new approach for simulation of the fractional Lévy process, which allows a Monte Carlo method for pricing financial derivatives.

Keywords fractional processes; fractional Brownian motion; fractional Lévy process; simulation; fractional financial models; mixed models; option pricing; Monte Carlo.

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Contents

1 Introduction

In the following few pages, we aim to introduce the subject of fractional stochastic processes that may be considered a possible and original approach to the issue of pricing financial derivatives. This is actually one of the final causes of this study. On a different (and greatly secondary, but not unimportant at all) perspective, with this work we also hope to reach a phenomenologic or heuristic approach to some understanding of the notion of mathematics itself: arguing that the mathematics has always a beginning and an end necessarily in reality itself as a main claim, even for the most abstract mathematical subject, or even if these endings are not known. One consequence is the refutation of the existence of a “world” of mathematical objects as well as the notion of mathematics as just a (cruel) chess game. And this is still possible when reaffirming the non-real proper existence of the mathematical objects: there is no ontological difference between a real number and an imaginary number. Somehow, both objects are imaginary. But they are the result of the same creativity that creates a portrait which is not the thing that represents, just like it is mentioned by Magritte in his “La trahison des images”. But this creativity is always locked up in reality in some sense. There is nothing new, in its absolute sense, as humanity will not ever create anything. Mathematics is art.

Somehow, water was the first element to deliver a clue on one of the historically most important blocks of financial mathematical modelling. This piece of knowledge is called the Brownian motion, named after Robert Brown. In Brown (1828) the author describes the irregular movement of particles of pollen on the surface of water. This description was later formally defined by Wiener, which was the cause of the name of the stochastic process (Wiener process) whose trajectories are Brownian motions. Nevertheless, it is commonly accepted to call Brownian motion to the process itself. The Wiener process is a stochastic process with almost surely continuous paths in time, but these are almost surely not differentiable in each instant. Also, the increments of this process are stationary, independent and Gaussian distributed.

The use of this process, whose historical origin is a natural phenomena in water, represented something quite important and frequent in mathematics. We can call it the pursuit of simplicity. Since the prices of financial assets (such as stocks) have mostly human causes, its variation and future value may be influenced by a not easy composition of a quite large quantity of deterministic and non-deterministic free human actions. Even if we admit a rational behaviour of economic agents in a perfect market (which is a way to determine free human actions), we would still be left with a great problem to determine a theoretical appropriate model, in a first place, followed by the huge problem to observe variables that (almost them) cannot be observed.

At a first glance, with stochastic modelling, the deterministic complexity was substituted by simple randomness.

Here we will present a mathematical phenomenon similar to the simplification earlier described. The irregularity found in the particles of pollen surrounded by water was a first approximation to the irregularity observed in prices of stocks. But this irregularity is different from chaos. There are real physical causes for the movements of the those particles and there are real human causes for the oscillations of the prices in a market, even if we do not know them completely. This may be one of the reasons to abandon the term irregularity (a close neighbour of chaos, which is the absence of causes) for another one, for instance, roughness, a term proposed by Mandelbrot, which is a word often used to describe the surfaces of fractals. On one hand, fractals may suggests chaos, but on the other hand, fractals are definitely not irregular, since it is a result of a quite formally determined pattern.

The trajectories of the Brownian motion are self-explanatory, which is something found in fractal geometry. Somehow, each part is (in some sense) the same as the whole. Graphically, a particular zoom in into one of the trajectories of the Brownian motion looks the same as the initial path. Financially, this would mean that given a path of a price of an asset, we would not be able to distinguish whether it is a time interval of a month, or two months, or even a day or five minutes. This self-explanatory pattern is an heuristic to the statistical property of self-similarity, which is a quite important detail in the Brownian motion. Actually, every fixed-scale zoom in a Brownian motion is itself a Brownian motion with probability one. And so, the self-similarity can be understood as a statistically fractal property.

Anyway, in Hurst (1951), the author found a self-explanatory movement in the level of water in a river. But in this case this movement was modelled, not only with roughness but with dependence, which is something not covered by the Brownian motion case, since its increments are independent.

This stochastic process, described by Hurst, was later formalized in Mandelbrot and van Ness (1968), and then in Molchan and Golosov (1969) and also by Kolmogorov (1940). The name of this process was introduced in Mandelbrot and van Ness (1968) as the fractional Brownian motion.

In some manner, the Lévy process corresponded to the passage from the Brownian process to a family of processes that are continuous in probability (a weaker form of continuity which allows "jumps" in its paths) and moreover its increments may assume different distributions than the Gaussian, resulting in a more theoretical and empirical conformation of the stochastic process with the object that is being modelled.

The first fractional process - the fractional Brownian motion - came from a generalization of the Wiener process in the sense of self-similarity. And can be seen as a possible answer for a question such as: "What are the stochastic processes closer to the Brownian motion but still self-similar?". The answer is a zero mean self-similar Gaussian process with stationary increments, which is a proper generalization, since the standard Brownian motion is itself a fractional Brownian motion.

The jump from Brownian motion to Lévy processes still ensures the statistical independence of increments. With the fractional Brownian motion we still have the Gaussian distribution as long with the path's continuity in time of each path of the process, but we gain a dependence structure that allows an approximation to the concept of memory. The possibility that future changes in prices are related and, in some cases, even caused by present price shocks or oscillations is not to be discarded. For instance in

Mandelbrot (1997a) and Shiryaev (1999) we can find several theoretical and empiric arguments towards this possibility. In (Mandelbrot, 1997b, page 418) it is suggested a simple approach to this present-future relation claiming that "large changes tend to be followed by large changes - of either sign - and small changes tend to be followed by small changes". Statistically, this can be read as a positive correlation between the increments of the process which will be in charge of model a financial asset's price.

One of the main points of this work is to present the definition of the fractional Brownian motion and its properties, along with a mathematical argument towards its mathematical formalization. This step turns out to be the preamble of the fractional Lévy process, that comes directly from one of the constructions of the fractional Brownian motion.

Here we argue that fractional Lévy process is not a generalization of the fractional Brownian motion, in the same sense of the generalization of the standard Brownian motion to the fractional Brownian motion nor to the Lévy process. From the fractional Brownian motion to the fractional Lévy process we will not only loose some properties, but we will get with a different kind of family of stochastic processes. It will be a process with the same dependence structure of the fractional Brownian motion, i.e. we can still model "memory", but the trajectories of this process are not self-explained. Meaning that we obtain a non-self-similar stochastic process.

Curiously, the financial modelling with the last process is much simpler (in some sense) when compared to the use of the fractional Brownian motion. And this seems to be a contradiction, since it is a much complex instrument, and we gain a distance to the belief of Mandelbrot on the fractal geometry of the financial prices trajectories that, somehow, inspired the appearance of fractional processes.

This work is organized in three chapters. The introduction of the fractional Brownian motion with its three possible definitions and statistical properties, emphasising the concept of self-similarity and a possible way of formalizing the idea of memory as it was suggested by Mandelbrot. In a second step and by a similar method used in the previous process, we introduce the fractional Lévy process in its two possible constructions proposing two numerical methods to its simulation, of them is an innovation, as far as we know. Finally we illustrate a numerical use of the fractal processes in financial modelling and we present a method to price an European call option whose underlying is modelled by a mixed model of a fractional Lévy process and a standard Brownian motion, where a result of no arbitrage is possible.

2 Fractional Brownian Motion

In this section we will present the definitions as well as the main properties of the fractional Brownian motion (abbreviated fBm). As we will see, this is a simple example of a process which is neither a semimartingale nor a Markov process. It corresponds to the first step into fractional stochastic processes and it is the origin of the fractional Lévy process, which will be introduced in the succeeding section.

Whenever it is needed we will always assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an increasing and right-continuous filtration $\mathbf{F} = (\mathcal{F}_t)_{0 \le t \le T}$.

Given a stochastic process X, we will use the notation $X(t)$ when possible, and X_t, with the same meaning, whenever it makes the reading easier.

2.1 Definition

One can define a one-sided fBm as presented below.

Definition 2.1 (Fractional Brownian Motion). *We call fBm to the zero mean Gaussian process* $B^H = \{B_t^H, t \ge 0\}$, *with* Hurst index $H \in (0,1)$ *which verifies* $B_0^H = 0$ *a.s. and*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{2.1}$$

for each $t, s \ge 0$.

We can also refer to this process as H-fBm.

This process does exist and it is well defined. In order to define a Gaussian process, it is only needed to refer its first two moments, the mean and covariance function. The good definition of this process will end up in a good definition of the covariance function (2.1), which must be a non-negative function. Actually, the expression (2.1) is well defined if and only if $H \in (0,1]$. A proof can be found, for instance, in Sottinen (2003).

It is easy to see, however, that the case $H = 1$ results on the process $B_t^1 = tZ$, where $Z \sim N(0,1)$ (it is enough to compute the covariance function of tZ, and compare with the covariance function (2.1) with $H = 1$). This trivial case is excluded from the definition of fBm.

Given that the standard Brownian motion is a centered Gaussian process with covariance function $\mathbb{E}\left[X_t X_s\right] = min\{t, s\}$, we can actually define it by a fBm with Hurst index of $\frac{1}{2}$. And we can conclude that, in fact, the Brownian motion is a particular case of the fBm ($\frac{1}{2}$-fBm).

The main properties of this process will be presented some sections ahead. Nevertheless, we may have a fully understanding of the previous definition, as well as some insight on the particulars of the fBm, with the following topic.

2.1.1 Self-similarity

One of the well-known properties of the standard Brownian motion is the *self-similarity*. Graphically, a zoom-in into a trajectory of the Brownian motion is itself a Brownian motion. This is what caught the interest of Mandelbrot. In some way, this property ends up in a "probability-fractal" property, since a self-similar process is a stochastic process that is invariant in distribution, given some stretch on space and time variables. As we will see, this is the cause of the form of covariance function of the fBm process. Most of the proofs of the following results as well as a more detailed survey on self-similarity can be found in Lamperti (1962).

We will write $\{X(t), t \geq 0\} \overset{d}{=} \{Y(t), t \geq 0\}$ to indicate that all finite-dimensional distributions of the processes X and Y are the same.

In literature, it is common to define *self-similarity* for a quite restrained class of stochastic processes. But, in a more general way, we can define it as follows.

Definition 2.2 (Self-similar)**.** *We say that* $\{X(t), t \geq 0\}$ *is* self-similar *if, for any positive* a *there exists a positive* b *such that*

$$\{X(at), t \geq 0\} \overset{d}{=} \{bX(t), t \geq 0\}. \tag{2.2}$$

Definition 2.3. *The process* $\{X(t), t \geq 0\}$ *is* continuous in probability *(or* stochastically continuous *) at* t *if, for any positive* ϵ *we have*

$$\lim_{h \to 0} P\left(|X(t+h) - X(t)| > \epsilon\right) = 0. \tag{2.3}$$

If we have a process continuous in probability, the Definition 2.2 of a self-similar process can be slightly modified with the following theorem, whose proof can be found in Lamperti (1962) (proof of Theorem 1).

Theorem 2.1. *Let* $X(t)$ *be a self-similar, non-trivial stochastic process continuous in probability at* $t = 0$*. There exists an unique* $H \geq 0$ *such that, for all positive* a*, we have*

$$\{X(at), t \geq 0\} \overset{d}{=} \{a^H X(t), t \geq 0\}. \tag{2.4}$$

Moreover, $H > 0$ *if and only if we have* $X(0) = 0$ *a.s. .*

The parameter H is usually referred as *Hurst index*, and we say that the process $\{X(t), t \geq 0\}$ is *trivial* whenever the probability law of $X(t)$ is a Dirac measure for each t.

Proposition 2.1. *The H-fBm is self-similar with Hurst index H.*

Proof. Since B_t^H is continuous in probability at $t = 0$ (one can prove it by proving the continuity in mean square, using the covariance structure (2.1)) it is enough to get a result similar to (2.4). Let us define $Y(t) = B_{at}^H$, for some positive a. We have to show that

$$Y(t) \overset{d}{=} a^H B_t^H. \tag{2.5}$$

Since both processes are Gaussian with zero mean, we only have to check whether the covariance functions coincide (Lemma 11.1 (i) in Kallenberg (1997)). For the first one we have

$$\mathbb{E}\left[Y(t)Y(s)\right] = \mathbb{E}\left[B^H_{at}B^H_{as}\right], \tag{2.6}$$

which can be written as

$$\frac{a^{2H}}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

and it corresponds to the covariance of the right-hand side of (2.5). □

2.1.2 An apology for the covariance structure

In order to justify the covariance structure of the fBm (2.1), we will present the definition of one of the key features of this process.

Definition 2.4. *The stochastic process $\{X(t), t \geq 0\}$ has stationary increments if the distribution of the increment process $\{X(t+h) - X(t), t \geq 0\}$, does not depend on h, for any non-negative h. Or*

$$\{X(t+h) - X(h), t \geq 0\} \stackrel{d}{=} \{X(t) - X(0), t \geq 0\}, \tag{2.7}$$

for any $h \geq 0$.

Proposition 2.2. *The fBm has stationary increments.*

Proof. In order to check the relation in (2.7) for fBm, it is only required to see that

$$\{B^H_{t+h} - B^H_h, t \geq 0\} \stackrel{d}{=} \{B^H_t, t \geq 0\},$$

for each positive h. And this is done again by comparing the first two moments of both processes, since they are both Gaussian. Both means are zero, and the variance is given by t^{2H}, when applying the covariance structure of the fBm (2.1).

□

The distribution of the increments of the fBm is a zero mean Gaussian with variance given by

$$\mathbb{E}\left[\left(B^H_t - B^H_s\right)^2\right] = |t-s|^{2H},$$

and so we have for each natural k

$$\mathbb{E}\left[\left(B^H_t - B^H_s\right)^{2k}\right] = \frac{(2k)!}{k!2^k}|t-s|^{2Hk}. \tag{2.8}$$

Now, for a more general stochastic process satisfying the main previous properties, self-similarity and stationary on its increments, we have the following result concerning its second moment structure.

Theorem 2.2. *Given a non-trival stochastic process $\{X(t), t \geq 0\}$ self-similar with Hurst index H satisfying* (2.4) *with stationary increments, we have, for $t, s \geq 0$,*

$$\mathbb{E}\left[X(t)X(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)\mathbb{E}\left[X(1)^2\right], \tag{2.9}$$

Assuming that $\mathbb{E}\left[|X(1)|^2\right] < \infty$.

Proof. Without loss of generality, let us assume that $s < t$. By the stationarity of the distributions of the process $\{X(t), t \geq 0\}$, we can easily see that $X(t) - X(s) \stackrel{d}{=} X(t-s)$ (Recall that $X(0) = 0$ a.s. in Theorem 2.1). Moreover, by self-similarity, for any positive t we have

$$X(t) \stackrel{d}{=} t^H X(1).$$

Therefore,

$$\mathbb{E}\left[\left(X(t) - X(s)\right)^2\right] = (t-s)^{2H}\mathbb{E}\left[X(1)^2\right].$$

The left-hand side of the previous equation leads us to

$$t^{2H}\mathbb{E}\left[X(1)^2\right] - 2\mathbb{E}\left[X(t)X(s)\right] + s^{2H}\mathbb{E}\left[X(1)^2\right],$$

and the final result follows immediately.

□

And this justifies the equivalent definition of the fBm given by Sottinen (2003):

Definition 2.5 (fractional Brownian motion)**.** *The fBm B_t^H is the unique zero mean self-similar Gaussian process with Hurst index $H \in (0,1)$ and stationary increments verifying $\mathbb{E}\left[\left(B_1^H\right)^2\right] = 1$.*

2.2 Properties

The previous equivalent definition for fBm already emphasises two of the main properties of this process: self-similarity and the stationarity of increments. The main difference between the fBm and the simple Brownian motion is that this process no longer has independence of increments. In this section we mainly will see the importance of the H parameter for the dependence structure of the process.

2.2.1 Increments and Correlation

Given two mutually exclusive time intervals on the real line, let us denote $t_1 < t_2 < t_3 < t_4$ the positive instants of the endings of those intervals. The increments $B_{t_2}^H - B_{t_1}^H$ and $B_{t_4}^H - B_{t_3}^H$ are Gaussian variables with zero mean and variance given by $(t_2 - t_1)^{2H}$ and $(t_4 - t_3)^{2H}$. This is an immediate corollary of the stationarity of the increments of the fBm as well as some simple logical steps, given that the sum of Gaussian variables is itself a Gaussian variable. So, the covariance between both increments is given by

$$\mathbb{E}\left[\left(B_{t_2}^H - B_{t_1}^H\right)\left(B_{t_4}^H - B_{t_3}^H\right)\right],$$

which will lead us to the following expression

$$\frac{1}{2}\left((t_3 - t_2)^{2H} + (t_4 - t_1)^{2H} - (t_4 - t_2)^{2H} - (t_3 - t_1)^{2H}\right). \tag{2.10}$$

Moreover, for any two disjoint increments, this expression is zero if and only if $H = \frac{1}{2}$. So the standard Brownian motion is the unique fBm with independent increments (Lemma 11.1 (ii) in Kallenberg (1997)). On the other hand, the Proposition 11.7 in Kallenberg (1997) allows us to conclude that the Brownian motion is the unique fBm which is a Markov process.

Now, given that fBm is a process with stationary increments, we can consider the unit increments of B_t^H for the instants 0 to 1 and from the instants n to $n+1$, and denote by $\gamma(n)$ the covariance function between those increments which is given by, for $n \geq 1$,

$$\gamma(n) = \mathbb{E}\left[\left(B_{n+1}^H - B_n^H\right)\left(B_1^H - B_0^H\right)\right].$$

Recalling that $B_0^H = 0$ a.s., by expression (2.10) we can write

$$\gamma(n) = \frac{1}{2}\left((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right). \tag{2.11}$$

Again, for $H = \frac{1}{2}$, $\gamma(n) = 0$ for $n \geq 1$. But, for $H \in (0,1)$ and $H \neq \frac{1}{2}$ we have

$$\lim_{n\to\infty} \frac{\frac{1}{2}\left((n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right)}{H(2H-1)n^{2H-2}} = 1,$$

or simply,

$$\gamma(n) \sim H(2H-1)n^{2H-2} \quad \text{when} \quad n \to \infty. \tag{2.12}$$

From the previous expressions (2.11) and (2.12), we conclude that in the case of $H < \frac{1}{2}$, the function $\gamma(n) < 0$ for each $n > 1$ and $\sum_{n=1}^{\infty} |\gamma(n)| < \infty$, which indicates that the increments are *negatively correlated* and have a *short-range dependence*.

On the other hand, however, if $H > \frac{1}{2}$, the function $\gamma(n)$ function is positive and the series $\sum_{n=1}^{\infty} \gamma(n)$ does not converge. In this case we say that the fBm presents increments with *positive correlation* and *long-range dependence*. The last case is the most interesting in finance, and it is the mathematical version of the claim of Mandelbrot in (Mandelbrot, 1997b, page 418): “large changes tend to be followed by large changes - of either sign - and small changes tend to be followed by small changes”.

Summarizing, the Hurst parameter will control the dependence structure of the process. And, except for the standard Brownian motion, the fBm is never either a Markov process nor a process with independent increments.

2.2.2 Sample paths

The formulation of the Kolmogorov criterion in Øksendal (2000, Theorem 2.2.3) states that a stochastic process X_t has a version with continuous sample paths whenever, for each $T > 0$ there exist positive constants α, β and D such that

$$\mathbb{E}\left[|X_t - X_s|^\alpha\right] \leq D|t-s|^{1+\beta},$$

for $0 \leq s, t \leq T$.

From equation (2.8) we can easily conclude that, for all $H \in (0,1)$, there exists a version of B_t^H with continuous trajectories. It is enough to choose a $k > \frac{1}{2H}$.

Moreover, one can define β-Hölder continuous functions as follows.

Definition 2.6. *A function f defined in interval $[a,b]$ in $\mathbb{R}$ is said to be β-Hölder continuous if there exist non-negative constants β and C such that*

$$\forall_{x,y\in[a,b]} \quad |f(x) - f(y)| \leq C|x-y|^\beta.$$

The Hölder continuity is stronger than simple continuity. And β parameter somehow classifies the regularity of the function. Note that the case $\beta = 1$ corresponds to the Lipschitz condition. For β-Hölder continuous stochastic processes we can have the following definition:

Definition 2.7. *The process X_t is β-Hölder continuous stochastic process in $[a,b]$ if it verifies*

$$\forall_{s,t\in[a,b]} \quad |X_s - X_t| \leq Y|s-t|^\beta, \tag{2.13}$$

for some finite random variable Y.

The proof of the following theorem can be found in Sottinen (2003, proof of Proposition 3.2).

Theorem 2.3. *The H-fBm has a version with β-Hölder continous sample paths if and only if $\beta \in (0, H)$.*

So we can conclude that the Hurst parameter controls not only all the dependence structure of the fBm, but also the regularity of its sample paths. Note that for greater values of H we get more regularity (the excluded case with $H = 1$ illustrates how regular this process becomes with higher values of H). And so, the long-range dependence case ($\frac{1}{2} < H < 1$) ends up to be more regular than the standard Brownian motion. Nevertheless, we can state the following result, which has a different interesting proof in Mandelbrot and van Ness (1968).

Theorem 2.4. *For any possible $H \in (0,1)$ the sample paths of the fBm are not differentiable with probability one.*

Proof. Since the process B_t^H is stationary, it is enough to show that it cannot be differentiable at $t = 0$. Thus, supposing that it is the case, we would have some positive ϵ and a finite random variable B_0' such that for all s in $(0, \epsilon)$ the following regularity condition holds

$$|B_s^H - B_0^H| \leq |s|\left(\epsilon + B_0'\right).$$

But then, by inequality (2.13) we can see that, in this case, B_t^H would be 1-Hölder continuous at $t = 0$, which is a contradiction with Theorem 2.3. $\square$

2.2.3 p-variation

An important property of the standard Brownian motion is that its sample paths have finite quadratic variation. The set of stochastic processes (semimartingales) that verify this property is often referred has the *natural* class of processes in which we can define a stochastic integral. In Papapantoleon (2007) we can get a possible definition for semimartingales.

Definition 2.8. *A stochastic process X_t is said to be a* semimartingale *if it can be decomposed as*

$$X_t = X_0 + M_t + A_t,$$

where X_0 is finite, M_t is a local martingale and A_t is an adapted finite variation process. Moreover $M_0 = A_0 = 0$.

The local martingale is a local version of the martingale property, every martingale is a local martingale. For more details see Kallenberg (1997, chapter 15).

The fBm, however, will not verify this property, as we will see.

Following Sottinen (2003) we now introduce the concept of p-variation. Given a partition of the interval $[a, b]$, with $0 \leq a < b$, we can write $\pi = \{t_k : \quad a = t_0 < t_1 < \cdots < t_n = b\}$. The diameter of the partition is the value $|\pi|$ which is given by $\max_{t_k \in \pi} \Delta t_k$, where $\Delta t_k = t_k - t_{k-1}$.

Definition 2.9. *Given a function f defined in the interval $[a, b]$, we call* p-variation of f along the partition π *to the value*

$$var_p(f; \pi) = \sum_{k=1}^{n} |\Delta f(t_k)|^p,$$

for $p \in [1, \infty)$ and given that $\Delta f(t_k) = f(t_k) - f(t_{k-1})$.

Given this base concept, we can define the following.

Definition 2.10. *Given a function f defined in the interval $[a, b]$, we say that f has* finite p-variation *if*

$$var_p^0(f) = \lim_{|\pi| \to 0} var_p(f; \pi),$$

exists.

On the other hand, we say that f has bounded p-variation *if the following is finite*

$$var_p(f) = \sup_{\pi} var_p(f; \pi).$$

Moreover, we call the variation index *of f to*

$$var(f) = \inf\{p > 0 : \quad var_p(f) < \infty\}.$$

The previous concepts are applied analogously to the stochastic process X_t if they are applied path-wise ω by ω a.s. . In Sottinen (2003, Proposition 3.8) it is presented and proved the following important result regarding the p-variation of the fBm.

Theorem 2.5. *If $p > \frac{1}{H}$ then the fBm has a.s. finite p-variation, moreover $var_p^0(B_t^H) = 0$. For $p < \frac{1}{H}$ then the fBm has unbounded p-variation and $var_p^0(B_t^H)$ does not exist. Also, the variation index of the fBm is given by $var(B_t^H) = \frac{1}{H}$.*

So, one can conclude the following.

Corollary 2.1. *The long-range dependence case of the fBm ($H > \frac{1}{2}$) has sample-paths with zero quadratic variation. On the other hand, the short-range dependence case of the fBm has infinite quadratic variation.*

Moreover, fBm has unbounded total variation (or 1-variation).

Corollary 2.2. *Except for the standard Brownian motion case, the fBm is not a semimartingale.*

And from this result it follows that the standard stochastic Itô integral is not possible for the fBm (except for the $H = \frac{1}{2}$ case). A proof can be found in Embrechts and Maejima (2002, Theorem 4.2.1).

2.3 Integral representations

2.3.1 Mandelbrot-van Ness

The fBm has been presented throughout this work taking advantage of features or key properties of the fBm itself (for instance, self-similarity or the covariance structure). These are the most common definitions which are presented in the recent literature covering this topic. Although in Taqqu (2013) it is argued that Mandelbrot was the pioneer in this process giving it a consistent formalization and a definition (in Mandelbrot and van Ness (1968)). This definition however is not quite the same given before, but an equivalent one. It will be presented an alternative representation following Tikanmäki and Mishura (2011). These representations are also called *moving average representations.*

For the Mandelbrot-van Ness integral representation, let us define the following function (also known as *Mandelbrot-van Ness kernel*), were H is the Hurst parameter of the fBm, and s, t are reals.

$$f_H(t,s) = C_H \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right), \tag{2.14}$$

where $(x)_+$ represents $\max(x, 0)$ and C_H is a normalizing constant which may be represented by the following expression

$$C_H = \left(\int_0^\infty \left((1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} \right)^2 ds + \frac{1}{2H} \right)^{-\frac{1}{2}}.$$

The Mandelbrot-van Ness integral representation of the fBm is then given by, for $t \in \mathbb{R}$,

$$B_t^H \overset{d}{=} \int_{-\infty}^t f_H(t,s) dB_s, \tag{2.15}$$

where B_s is a two-sided Brownian motion, which can be defined by simply considering two independent standard Brownian motions, say $B_t^{(1)}$ and $B_t^{(2)}$, and imposing $B_t = B_t^{(1)}$ for $t \geq 0$ and $B_t = -B_{-t}^{(2)}$ for $t < 0$. Note that the resulting fBm is also a two-sided process.

The proof for this can be found directly in Mandelbrot and van Ness (1968). Since the kernel function is regular enough for the Itô integral and it is deterministic, the process in (2.15) is Gaussian, and it has zero mean. Therefore, the proof ends up on checking that, in fact, the second moment structure of the process coincides with the covariance function of the fBm given in (2.1).

The regularity of the kernel function can be seen in Marquardt (2006, Proposition 3.1). In particular we have $f_H(t,\cdot) \in L^p(\mathbb{R})$ for $p > (\frac{3}{2} - H)^{-1}$, and so $f_H(t,\cdot) \in L^2(\mathbb{R})$.

2.3.2 Molchan-Golosov

Besides the Mandelbrot-van Ness representation given previously, there are other equivalent ways to write the kernel function (2.14). Nevertheless, it is also possible to have the same fBm but with a different integral representation. This one was firstly presented in Molchan and Golosov (1969). For a more detailed survey on the relations between each representation we refer to Jost (2005).

Again, one of the possible forms to write the Molchan-Golosov integral representation of the fBm is the following, due to Tikanmäki and Mishura (2011):

$$B_t^H \stackrel{d}{=} \int_0^t z_H(t,s)dB_s \tag{2.16}$$

for $t \geq 0$.

The Molchan-Golosov kernel z_H, for $0 < s < t < \infty$, is given by

$$z_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} F\left(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, \frac{s-t}{s}\right), \tag{2.17}$$

and $z_H(t,s) = 0$ otherwise. The function F is the Gaussian hypergeometric function that can be defined as

$$F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

given that $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$, for $k \in \mathbb{N}$, and $(\alpha)_0 = 1$. And the constant c_H can be written as

$$c_H = \frac{1}{\Gamma(H+\frac{1}{2})}\left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

A detailed proof for the Molchan-Golosov integral representation can be found in Jost (2007). Note that in this case, we have an integral over a compact interval and we only need a one-sided Brownian motion. This can result in nicer numerical approximations, since we would not need to truncate the integral. For the case of the Hurst parameter $H > \frac{1}{2}$ there are simplifications that can be made in the integral expression (2.16) that can be found in Tikanmäki and Mishura (2011).

2.3.3 The Name and Fractional Calculus

Up to this point, one can see the fBm as a generalization of the standard Brownian motion in the sense of *self-similarity.* In fact, the fBm for each Hurst parameter $H \in (0,1)$ is a self-similar process. But now, excluding the case of the Brownian motion, the fBm does not have independent increments, but, on the other hand, they are always stationary. As it is claimed in Taqqu (2013), the formalization of this process by Mandelbrot was motivated by his interest in the self-similarity property. However the name given was *fractional* rather than similar or self-similar. We will present some different representations for Mandelbrot-van Ness and Molchan-Golosov definitions of the fBm, that will justify the name of the process.

Given constants a and b, $a < b$ and a continuous function f in $[a,b]$, by partial integration we can easily check by induction the so called iterated integral formula

$$\int_a^{t_n} \int_a^{t_{n-1}} \dots \int_a^{t_1} f(s) ds dt_1 \dots dt_{n-1} = \frac{1}{(n-1)!} \int_a^{t_n} \frac{f(s)}{(t_n - s)^{1-n}} ds \tag{2.18}$$

for $t_n \in [a,b]$, $n \geq 1$.

Note that the right-hand side of (2.18) can be written as

$$\frac{1}{\Gamma(n)} \int_a^{t_n} \frac{f(s)}{(t_n - s)^{1-n}} ds$$

where $\Gamma(x)$ denotes the Gamma function. And we would be able to extend this iteration to a non-integer step iteration.

Definition 2.11 (Fractional integral)**.** *If $f \in L^1[a,b]$ and given a parameter $\alpha > 0$ we call* (Riemann-Liouville) fractional integral of order α *to*

$$(I^\alpha_{a+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha)} \int_a^b f(s)(t-s)_+^{\alpha-1} ds$$

and

$$(I^\alpha_{b-} f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(s)(s-t)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha)} \int_a^b f(s)(s-t)_+^{\alpha-1} ds,$$

where in both cases $t \in (a,b)$. The first integral is referred as the left-sided integral *and the second as the* right-sided integral

Given this definition it is also possible to define an inverse operator, which can be understood as some *fractional derivative.* This operator is not as simple to define as the fractional integral. We present the definition of Fink and Scherr (2014). And from now on we will be only interested in the right-sided fractional integral.

Definition 2.12. *Let f be a function in $L^1([a,b])$ and $0 < \alpha < 1$ such that*

$$f(t) = \left(I^\alpha_{b-} \psi \right)(t), \quad a < t < b,$$

for some $\psi \in L^1([a,b])$.

The fractional (Riemann-Liouville) derivative of f of order α *is given by*

$$\left(D^{\alpha}_{b-}f\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha\int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}}ds\right),$$

for $a < t < b$.

For some $\alpha \in (0,1)$ we shall write $I^{-\alpha}_{b-} = D^{\alpha}_{b-}$ and also $I^{0}_{b-} = D^{0}_{b-} = Id$.

Instead of defining the fractional integral in compact intervals, we can define it analogously of functions defined in the real line.

Definition 2.13. *If* $0 < \alpha < 1$ *and* $f \in L^1(\mathbb{R})$ *the* (Riemann-Liouville) fractional integral of order α *is given by*

$$(I^{\alpha}_{-}f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^{\infty} f(s)(s-t)^{\alpha-1}ds,$$

for $t \in \mathbb{R}$.

For the real line case we will present a different version of the fractional derivative operator, following Fink (2011).

Definition 2.14. *Let* f *be a function in* $L^1(\mathbb{R})$ *and* $0 < \alpha < 1$ *such that*

$$f(t) = I^{\alpha}_{-}(\psi(\cdot))(t), \quad t \in \mathbb{R},$$

for some $\psi \in L^1(\mathbb{R})$.

The fractional (Marchaud) derivative of f of order α *is given by*

$$\left(D^{\alpha}_{-}f\right)(t) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\infty} \frac{f(t)-f(t+s)}{s^{\alpha+1}}ds,$$

for $t \in \mathbb{R}$.

Again we will use the convention $I^{-\alpha}_{-} = D^{\alpha}_{-}$ together with $I^{0}_{-} = D^{0}_{-} = Id$. For a complete survey on fractional calculus we refer to Samko et al. (1993).

We can now state the following result whose proof can be found in Fink (2011) (partially distributed by the proofs of Propositions 1.5.8 and 1.5.10 in the same thesis). We use $\propto$ to indicate proportionality.

Theorem 2.6. *The fBm defined by the Mandelbrot-van Ness representation* (2.15), B^H_t, *over the real line is a.s. given by*

$$B^H_t \overset{d}{\propto} \int_{-\infty}^{\infty} I^{H-\frac{1}{2}}_{-}\mathbf{1}_{[0,t)}(s)dB_s. \tag{2.19}$$

The fBm B^H_t, *with* $t \geq 0$, *resulting from the Molchan-Golosov representation* (2.16), *can be written a.s. by*

$$B^H_t \overset{d}{\propto} \int_0^T s^{-H+\frac{1}{2}} I^{H-\frac{1}{2}}_{T-}\left((\cdot)^{H-\frac{1}{2}}\mathbf{1}_{[0,t)}(\cdot)\right)(s)dB_s. \tag{2.20}$$

Each kernel (2.14) and (2.17) of the integral representations of the fBm can be written with fractional calculus. And note that for the short-range dependence case ($H < \frac{1}{2}$) the fBm results on the application of the fractional derivative, while for the long-range dependence we have the fractional integral. And this is the reason why, in Mandelbrot and van Ness (1968) the fBm (with Hurst index different than $\frac{1}{2}$) was labelled as the fractional integral or derivative of the usual Brownian motion.

More insights on the connections between fractional calculus and the fBm can be found in Doukhan and Taqqu (2003).

2.4 Integral

In this last section, we will provide two main approaches to the definition of an integral with respect to the fractional Brownian motion. These next concepts will be mainly useful to the last section of this work.

2.4.1 Path-wise

As it was already observed, it is not possible to define a Itô-type stochastic integral with respect to fBm. But a result from Young (1936) allows the definition of the Riemann-Stieltjes integral for well-behaved functions in terms of its Hölder continuity.

Theorem 2.7. *Given two real functions f, g defined in $[0, T]$ which are Hölder continuous of order p and q such that $p + q > 1$, then the Riemann-Stieltjes integral*

$$\int_0^T f(t)dg(t)$$

exists.

Recalling that the paths of fBm are Hölder continuous of any order in $(0, H)$, we can easily apply the previous result to the trajectories of the fBm.

Theorem 2.8. *Given a stochastic process u_t defined in $[0, T]$ with p-Hölder continuous trajectories, with $p > 1 - H$, then the Riemann-Stieltjes integral*

$$\int_0^T u_t dB_t^H$$

exists path-wise.

In Sottinen (2003, Corollary 6.3) the previous result is clarified by the following proposition.

Proposition 2.3. *In the conditions of the previous theorem, for $t \in (0, T)$ the integral*

$$\int_0^t u_s dB_s^H$$

is a limit of Riemann-Stieltjes sums, moreover it is a.s. β-Hölder continuous with any order $\beta < H$.

As it is referred in Nualart (2006, Section 3.2.1), it is possible to show that the expected value of the previous integral is not necessarily zero, as it happens in the Itô integral case, and its variance has not a easy calculation formula. These results are obtained by Malliavin calculus.

2.4.2 Wick integral

The Wick integral can be seen as an alternative to stochastic integral that cannot be defined in Itô's sense in the fBm case. It is an operator that was introduced in Malliavin calculus and it fits the fBm. The Wick integral is based on the concept of Wick product.

However, the details of the Wick product are a deep subject that fall out of the scope of this work. For more details we refer to Nunno et al. (2009). A simpler approach to this topic can be found in Nualart (2006). We briefly introduce its definition in Appendix A.

3 Fractional Lévy Process

The fractional Lévy process (abbreviated fLp) follows directly from the construction of the integral representations of the fBm. The Lévy processes generalize the Brownian motion allowing discontinuities on the sample paths (also referred as jumps) as well as different distributions apart from the Gaussian distribution, resulting in quite flexible models for financial data. The passage from the fBm to the fLp is the result of a technical difference in the use of an integral. And so, at a first glance, it is not a pursuit of some properties or a priori advantages that were to be gained with this technical change, as it happened in the Lévy process case. The almost "Let us see what happens" approach actually gave us a quite important type of processes in fractional financial modelling, when comparing to the previous case. This importance is greatly based on the semimartingale property that, in some cases, can be verified.

The fLp was firstly introduced in Benassi et al. (2004).

From now on, we will focus only on the long-memory case $H \in (\frac{1}{2}, 1)$.

3.1 Lévy process

We refer to Applebaum (2009) and Sato (1999) for standard reference texts on Lévy processes and proofs of the basic properties as well as for the standard definition of a stochastic integral with respect to a Lévy process.

3.1.1 Definitions

The basic definition can be presented as follows.

Definition 3.1. *The stochastic process $(L_t)_{t\geq 0}$ continuous in probability is a* Lévy process *if $L_0 = 0$ a.s., the increments of L_t are stationary and independent, and the sample paths of L_t are right-continuous with* *left limits a.s. (or the paths of L_t are* càdlág*).*

When comparing the previous definition to the definition of the Brownian motion, we do not have a Gaussian distribution for the increments anymore nor an imposition of continuity of the sample paths. Actually, the paths of a Lévy process may have discontinuities (*jumps*) and the probability distribution of the increments is not necessarily Gaussian. Nevertheless, the Brownian motion is a Lévy process.

An important concept which is closely related to Lévy processes is the concept of infinitely divisible distribution.

Definition 3.2. *A random variable X is said to have an* infinitely divisible distribution *if, for each $n \in \mathbb{N}$, there exist random variables i.i.d. Y_1^n, Y_2^n, up to Y_n^n, such that*

$$X \overset{d}{=} Y_1^n + Y_2^n + \cdots + Y_n^n.$$

A process $(X_t)_{t\geq 0}$ is infinitely divisible if, for each possible t, X_t is an infinitely divisible random variable.

Proposition 3.1. *Each Lévy process L_t is infinitely divisible.*

Proof. Since the increments of the Lévy process are independent and stationary, any partition of the interval $[0, t]$ with equal spaced intervals will define a finite number of i.i.d. random variables whose sum is equal in distribution to L_t. These random variables are the increments of L_t for each time-line partition. □

Theorem 3.1 (Lévy-Khintchine formula)**.** *The random variable X has an infinitely divisible distribution if and only if there exists $b \in \mathbb{R}$, a non-negative c and a measure ν satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty$, such that*

$$\mathbb{E}\left[e^{iuX}\right] = \exp\left(\eta(u)\right),$$

for $u \in \mathbb{R}$, where

$$\eta(u) = ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}]}\right) \nu(dx). \tag{3.1}$$

The triplet (b, c, ν) is often called the characteristic (or Lévy) triplet and ν is named Lévy measure. We call $\eta(u)$ the characteristic (or Lévy) exponent. The b parameter is related with the (deterministic) drift of the process, the c is the volatility associated with the Brownian component of L_t and the remaining integral is associated to the jump component of the Lévy process.

Now, by Proposition 3.1 as well as the previous theorem, for each positive t the characteristic function of the random variable L_t can be associated with a characteristic triplet. Although, the Lévy-Itô decomposition together with the Lévy-Khintchine theorem (that can be found for instance in Applebaum (2009), Sato (1999)) allow us to understand the relation between the characteristic triplet of each variable L_t of a Lévy process.

Theorem 3.2. *Given a Lévy process L_t, there exists an unique Lévy triplet (b, c, ν) such that the characteristic function of L_t can be written as*

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left(t\eta(u)\right), \tag{3.2}$$

for $u \in \mathbb{R}$, where $\eta(u)$ is the characteristic exponent associated with (b, c, ν).

On the other hand, given a characteristic triplet associated with a random variable with an infinitely divisible distribution, one can define a Lévy process that verifies (3.2)*.*

The Lévy triplet (b, c, ν) is the one associated with L_1, and it is usually referred as the characteristic triplet of the Lévy process.

3.1.2 Integral

From now on, we will assume that the Lévy process L_t has zero mean and finite second moment, moreover it does not have Brownian part and the Lévy measure satisfies:

$$\int_{|x|<1} |x|\nu(dx) < \infty. \tag{3.3}$$

In this case we assure the finite variation of the sample paths of L_t. A necessary and sufficient condition for the existence of the second moment of L_t is that

$$\int_{|x|>1} |x|^2\nu(dx) < \infty.$$

In this, case we have

$$var(L_t) = t\int_{\mathbb{R}} x^2\nu(dx),$$

and the Lévy exponent (3.1) is simplified into

$$\eta(u) = \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right)\nu(dx). \tag{3.4}$$

Given these restrictions, the process L_t can be also represented as the following integral

$$L_t = \int_0^t \int_{\mathbb{R}} x\tilde{N}(dx, ds), \quad t \geq 0,$$

where the compensated Poisson measure $\tilde{N}$ is given by $\tilde{N} = N(dx, ds) - \nu(dx)$. The random Poisson measure $N(t, A)$ can be defined as

$$N(t, A) = \#\{0 \leq s \leq t; \quad \Delta L_s \in A\},$$

where $\Delta L_s = L_s - L_{s-}$. For simplicity we will assume $N(t, A) = 0$ for any $A \in \mathcal{B}(\mathbb{R})$ such that $0 \in \bar{A}$, where $\bar{A}$ is the closure of A.

With this set up, we can define stochastic integrals with respect to a Lévy process. We will define it more generally with respect to a two-sided Lévy process, L_t with $t \in \mathbb{R}$ simply calling two i.i.d. Lévy processes $L_t^{(1)}$ and $L_t^{(2)}$ and defining $L_t = L_t^{(1)}$ for $t \geq 0$ and $L_t = -L^{(2)}_{-(t^-)}$ for $t < 0$.

Given a mensurable function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ verifying, for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(|f(t,s)x|^2 \wedge |f(t,s)x| \right) \nu(dx)ds,$$

the following stochastic integral is well defined as the limit in probability of integrals of simple (or step) functions whose limit is f,

$$X_t = \int_{\mathbb{R}} f(t,s)L(ds) = \int_{\mathbb{R}\times\mathbb{R}} f(t,s)x\tilde{N}(dx,ds). \tag{3.5}$$

We can also write,

$$X_t = \int_{\mathbb{R}} f(t,s)dL_s. \tag{3.6}$$

In Marquardt (2006), we find the following summary of the properties of X_t.

Proposition 3.2. *Given that the two-sided Lévy process L_t has the characteristic exponent* (3.4)*, the stochastic process X_t as it is defined in* (3.5) *has an infinitely divisible distribution and verifies*

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp\left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} \left(e^{iuf(t,s)x} - 1 - iuxf(t,s) \right) \nu(dx)ds \right\}.$$

Moreover, if, for each t, $f(t,\cdot) \in L^2(\mathbb{R})$ we have

$$\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[L_1^2\right] \|f(t,.)\|^2_{L^2(\mathbb{R})}.$$

Before we conclude this part, it is important to notice that, in fact, the Lévy process is a semimartingale, as it is emphasized in the following theorem.

Theorem 3.3. *Every Lévy process L_t is a semimartingale.*

A proof is an immediate consequence of the Lévy-Itô decomposition of any Lévy process and it can be found for instance in Applebaum (2009). And this fact allow us to get a slightly different approach to the integral (3.6), that will be used later in Section 3.4.2.

Corollary 3.1. *Within the conditions of the Proposition 3.2, considering an measurable function f and assuming the usual restrictions made in this section regarding the Lévy process L_t, the integral* (3.6) *is given by the following limit, a.s.*

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} f(t,s_i)(L_{s_{i+1}} - L_{s_i}),$$

where π is a partition of the interval $[0,T]$, $0 = s_0 < s_1 < \cdots < s_n = T$ and $|\pi| = \max_n\{s_{n+1} - s_n\}$.

A proof for this can be found in Bichteler (1981). And this is a possible way to define the stochastic integral with respect to a semimartingale.

3.2 Definitions

Now we are in conditions to define the fLp. In the case of the previous fractional process, there are two possible ways to define the fBm through an integral approach. But both approaches result on the same stochastic process. The fLp appears directly from these integral representations, but instead of an integral with respect to the Brownian motion, we have an integral driven by an appropriate Lévy process. In the present case, however, we will have two different processes. The following first definition of the fLp can be found for instance in Marquardt (2006), Tikanmäki (2012).

Definition 3.3 (Mandelbrot-van Ness fLp)**.** *The fLp resulting from the Mandelbrot-van Ness representation (MVN-fLp) is the two-sided stochastic process given by,*

$$X_t^H = \int_{\mathbb{R}} f_H(t,s)dL_s, \tag{3.7}$$

where f_H, for $H \in (0,1)$ is the Mandelbrot-van Ness kernel (2.14) *and L_s is a zero mean square integrable two-sided Lévy process without Brownian component.*

Similarly, as in Tikanmäki and Mishura (2011), we can define the fLp with the Molchan-Golosov integral representation.

Definition 3.4 (Molchan-Golosov fLp)**.** *The fLp resulting from the Molchan-Golosov representation (MG-fLp) is the stochastic process given by,*

$$Y_t^H = \int_0^t z_H(t,s)dL_s, \tag{3.8}$$

for $t \in \mathbb{R}_0^+$, where z_H, for $H \in (0,1)$, is the Molchan-Golosov kernel (2.17) *and L_s is a zero mean square integrable Lévy process without Brownian component.*

In both cases, we will often refer to the Lévy process L_s as the driving Lévy process. Again, since the kernels are the same used in the previous section, the name fractional is already justified, and one can see the fLp has a fractional integral or derivative of a Lévy process.

Theorem 3.4. *The conditions imposed on the beginning of this section regarding the driving Lévy process (finite second moment as well as the condition* (3.3) *on the Lévy measure) are sufficient and necessary conditions for the processes MVN-fLp and MG-fLp to be well defined, given an appropriate probability space.*

Proof. The conditions imposed to the driving Lévy process are necessary and sufficient conditions to the good definition of the integral of a measurable deterministic function with respect to a Lévy process.

So, it is enough to check that the kernels of the fractional processes are measurable, which was already done in the previous chapter. □

The fact that the two previous definitions are not equivalent is the main result proved in Tikanmäki and Mishura (2011). It claims the following.

Theorem 3.5. *In the conditions of the definition of both fLp, for the case $\frac{1}{2} < H < 1$,*

- *if $\mathbb{E}\left[|L_1|^3\right] < \infty$ and $\mathbb{E}\left[L_1^3\right] \neq 0$, then the MVN-fLp and the MG-fLp have different finite dimensional distributions.*
- *If $\mathbb{E}\left[L_1^4\right] < \infty$, then MVN-fLp and MG-fLp have different finite dimensional distributions.*

And so, we are in presence of two distinct stochastic processes.

3.3 Properties

As it was done in the case of the fBm process, we will now summarize the main properties of each fLp.

3.3.1 Mandelbrot-van Ness fractional Lévy process

The results in this subsection will mainly follow Marquardt (2006).

By observing the construction of the MVN-fLp, one can conclude that, for each $t \in \mathbb{R}$, the MVN-fLp X_t^H will depend on L_s for all $s \in \mathbb{R}$. Therefore, X_t^H is not adapted to the filtration generated by the driving Lévy process.

Given that the MVN kernel verifies $f_H(t, \cdot) \in L^2(\mathbb{R})$, by Proposition 3.2, it is immediate to conclude the following result.

Proposition 3.3. *The MVN-fLp X_t^H has a infinitely divisible distribution for each $t \in \mathbb{R}$, moreover*

$$\mathbb{E}\left[(X_t^H)^2\right] = \|f(t, \cdot)_H\|_{L^2(\mathbb{R})}^2 \, \mathbb{E}\left[L_1^2\right],$$

for $t \in \mathbb{R}$.

Proof. It is an immediate consequence of Proposition 3.2. □

Another important result is the fact that the MVN-fLp has an improper Riemann integral representation.

Proposition 3.4. *The MVN-fLp X_t^H has a version with the form*

$$\int_{\mathbb{R}} f(t, s) L_s ds, \quad t \in \mathbb{R}, \tag{3.9}$$

which is continuous in t.

For a proof see Marquardt (2006, proof of Theorem 3.4).

From now on, we will assume the continuous version of the MVN-fLp. In Fink (2011) and Marquardt (2006) we find the following two key features of this process along with the respective proofs.

Proposition 3.5. *The MVN-fLp X_t^H is a zero mean process, its increments are stationary and it is symmetric, in the sense of*

$$X_{-t}^H \stackrel{d}{=} -X_t^H, \quad t \in \mathbb{R}.$$

Furthermore, for $t, s \in \mathbb{R}$, X_t^H has, up to a constant, the same covariance structure as the fBm:

$$Cov\left(X_s^H, X_t^H\right) \propto \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

With this result we find an "approximation" to the fBm. In the case of the MNV-fLp, we still have a stochastic process with stationary increments built with a covariance structure that allows this process to have *memory*. The characterization made in subsection 2.2.1 can be applied to any process with the covariance of the form (2.1), so we conclude that the increments of MVN-fLp has long-range dependence for the present case. These details are emphasized in the following corollaries.

Corollary 3.2. *The MVN-fLp does not have independent increments.*

Moreover, the increments of the MVN-fLp have long-range memory and are positively correlated.

And so, the MVN-fLp cannot be a Lévy process.

On the other hand, the following theorem breaks the fundamental point that motivated the study of the fractional processes. A proof can be seen in Marquardt (2006, proof of Theorem 4.4).

Theorem 3.6. *The MVN-fLp X_t^H cannot be self-similar.*

Nevertheless, it is still possible to find in the MVN-fLp some clues on self-similarity. In Benassi et al. (2004) it is presented the concept of asymptotic self-similarity that can be found in the distribution of X_t^H. In Marquardt (2006) it is proved that, under some conditions, X_t^H is locally self-similar with parameter $\tilde{H} = H + \frac{1}{\alpha} - \frac{1}{2}$, were $\alpha \in (1, 2)$.

Sample paths properties

Proposition 3.6. *The sample paths of MVN-fLp X_t^H are a.s. locally Hölder continuous of any order $\beta < H - \frac{1}{2}$.*

The proof can be found in Marquardt (2006, Theorem 4.3).

The locally Hölder continuity of the trajectories of the MVN-fLp implies that these paths are Hölder-continuous in compacts.

Proposition 3.7. *If the Lévy measure ν associated to the Lévy process driving the MVN-fLp X_t^H is of finite activity ($\nu(\mathbb{R}) < \infty$), then the total variation of each sample path of X_t^H is finite on compacts $[a, b]$.*

An immediate consequence of the previous result (whose proof can be found in Marquardt (2006, proof of Theorem 4.6)) is the following.

Corollary 3.3. *The MVN-fLp X_t^H is a semimartingale with respect to any filtration it is adapted to if $\nu(\{\mathbb{R}\}) < \infty$.*

But it is also possible to find examples of MVN-fLp whose trajectories are not that regular.

Proposition 3.8. *Suppose that the Lévy measure ν associated to the Lévy process driving the MVN-fLp X_t^H is given by $\nu(dx) = g(x)dx$, where g is a positive measurable real function verifying*

$$g(x) \sim |x|^{-1-\alpha} \quad \textit{when} \quad x \to 0,$$

for some $\alpha \in (1, 2)$ and, for each $x \in \mathbb{R}$,

$$g(x) \leq C|x|^{-1-\alpha}.$$

In this case, the sample paths of the MVN-fLp have a.s. infinite variation in any compact interval.

Moreover, in this case, the MVN-fLp is not a semimartingale.

For a proof see the proofs of Theorems 4.5 and 4.7 in Marquardt (2006). On the other hand, see Tikanmäki and Mishura (2011, proof of Theorem 3.9) for a justification of the following result.

Proposition 3.9. *The MVN-fLp has zero quadratic variation.*

In Bender et al. (2012), the authors prove the following remarkable result regarding the MVN-fLp.

Proposition 3.10. *The MVN-fLp driven by the Lévy process with the usual assumptions has Lebesgue almost everywhere differentiable sample paths in any compact interval.*

Proved in Bender et al. (2012, proof of Theorem 2.1).

3.3.2 Molchan-Golosov fractional Lévy process

The results summarized in this subsection mainly follow Tikanmäki and Mishura (2011).

Proposition 3.11. *Each random variable of the MG-fLp, Y_t^H, has an infinitely divisible distribution.*

See proof of Propositin 3.10 in Tikanmäki and Mishura (2011).

While the MVN-fLp is not adapted to the filtration generated by the drivinng Lévy process, the MG-fLp is adapted to it, as we can see by its integral definition (3.8).

As the MVN-fLp, the MG-fLp also has the same covariance structure of the fBm.

Proposition 3.12. *The MG-fLp Y_t^H verifies, for $s, t \geq 0$,*

$$\mathbb{E}\left[Y_t^H\right] = 0,$$

and

$$\mathbb{E}\left[Y_t^H Y_s^H\right] = \frac{\mathbb{E}\left[L_1^2\right]}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Proof. For the proof of the first moment, we refer to the proof of Proposition 3.4 in Fink (2013), considering the case of $\mathbb{E}[L_1] = 0$.

Given that

$$\mathbb{E}\left[\left(Y_t^H + Y_s^H\right)^2\right] = \mathbb{E}\left[(Y_t^H)^2\right] + \mathbb{E}\left[(Y_s^H)^2\right] + 2\mathbb{E}\left[Y_t^H Y_s^H\right],$$

the L^2-isometry in Proposition 3.2 allows us to write

$$\mathbb{E}\left[(Y_t^H)^2\right] = t^{2H}\mathbb{E}\left[L_1^2\right]$$

and

$$E\left(Y_t^H + Y_s^H\right)^2 = |t-s|^{2H}\mathbb{E}\left[L_1^2\right],$$

for $t, s \geq 0$.

Thus, for $t, s \geq 0$,

$$\mathbb{E}\left[Y_t^H Y_s^H\right] = \frac{\mathbb{E}\left[L_1^2\right]}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

□

Corollary 3.4. *The MG-fLp does not have independent increments. Moreover, these increments have long-range memory and are positively correlated.*

An interesting observation is that the MG-fLp cannot be a Lévy process.

As it was done for the MVN-fLp, it is also possible to have a path-wise (or Riemman) integral for the MG-fLp, although in this case it requires a more restrict class of driving Lévy processes, as it is proved in Tikanmäki and Mishura (2011, Proposition 3.6).

Proposition 3.13. *Suppose L_t is a compound Poisson process with characteristic triplet $(0, 0, \nu)$. Then Y_t^H has a version that can be represented as follows*

$$\int_0^t \left(-\frac{d}{ds} z_H(t,s)\right) L_s ds.$$

The case of the MG-fLp results even more "distanced" from the fBm when compared to the MVN-fLp. A "first break" is related to the stationarity of distribution of its increments, and it is recorded in the following proposition.

Proposition 3.14. *There is at least one MG-fLp Y_t^H whose increments are not stationary.*

Proof. When the Lévy process driving the fLp is given by L_t with characteristic triplet $(0, 0, \nu)$, where $\nu = \frac{1}{2}(\delta_1 + \delta_{-1})$ (i.e. L_t is a sum of two compound Poisson processes with jumps lengths 1 and -1), it is shown in proof of Proposition 3.11 of Tikanmäki and Mishura (2011) that Y_t^H does not have stationary increments. □

But, as it happens to the MVN-fLp, the following result still holds for the MG-fLp.

Theorem 3.7. *The MG-fLp cannot be self-similar.*

See proof of Theorem 3.12 of Tikanmäki and Mishura (2011).

Sample paths properties

Proposition 3.15. *The MG-fLp Y_t^H has a.s. Hölder continuous paths of any order $\beta < H - \frac{1}{2}$*

For a proof check the demonstration of Proposition 3.7 in Tikanmäki and Mishura (2011).

(Note that for the short-memory case $H < \frac{1}{2}$ the MG-fLp and the MVN-fLp have discontinuities with positive probability as well as unbounded sample paths. See Tikanmäki and Mishura (2011) for more insights.)

Proposition 3.16. *The MG-fLp has a.s. zero quadratic variation.*

A proof can be found in proof of Theorem 3.8 in Tikanmäki and Mishura (2011).

Similarly to the case of MVN-fLp, the MG-fLp will depend greatly on the Lévy process in which it relies. There is not a complete characterization of Y_t^H in terms of the semimartingale property (more observations in Tikanmäki (2012)).

3.3.3 Summary and observations

The following table allows a quick summary of the properties each fLp.

Property	MVN-fLp	MG-fLp
Two-sided process	Yes	No
Covariance structure of fBm	Yes	Yes
Increments with positive correlation and long-memory dependence	Yes	Yes
Path-wise construction	Yes	In some cases
Adapted to natural filtration	No	Yes
Hölder continuous sample paths	Yes (locally)	Yes
Self-similarity	No	No
Stationarity of increments	Yes	Not necessarily
Infinitely divisible	Yes	Yes
Semimartingale	In some cases	Unknown

Table 1: Comparing the MVN-fLp and the MG-fLp

The “two-sided” definition of the MVN-fLp happens to be quite important, for the same reasons given to the integral representation of the fBm by Mandelbrot-van Ness. In this case, the simulation of the process must lead to truncate the integral somewhere.

It is not hard to understand the reasons which made the Brownian motion be gradually substituted by the Lévy process. The most famous arguments towards the use of the Lévy process within financial modelling are the following: the distribution of the increments of a Lévy process does not have to be the Gaussian distribution, enabling, for example, a non-symmetric distribution with “heavier tails”, as well as the possibility of admitting non-constant implied volatilities, and, finally, the sample paths of a Lévy process may have “jumps” or discontinuities. All these upgrades assume all the other advantages of the Brownian motion, besides it is more conform to empirical knowledge and more economically realistic. Yet, the Lévy process is more analytically complicated to deal with, and for most cases there is no closed forms solutions for important models based on Lévy processes, such as prices of options. The advance from the Brownian motion to the Lévy process also rise problems in the economic functioning of the financial models: the arbitrage issue may become a problem, and the hedging and completeness may not be possible in most cases. Some may argue that, in some cases, the models are badly specified. But the advances towards reality in modelling may also imply a badly and insufficient specification of the financial theory as well.

Curiously, when comparing both previous fractional processes to the Lévy process, in some sense, it does not seem to be an improvement, but actually a throwback. The trajectories of the fLp have no longer jumps, and in some cases, they are differentiable on compacts (see Proposition 3.10), which is not an expected trace on the movement of assets' prices. But we gain a memory perspective which was not contemplated within Lévy's family.

Nevertheless, except for the excess of regularity of the fLp, the driving Lévy process impress the fLp with more possibilities than the fBm. Actually the fLp is a family of processes, depending in the Lévy process in which it relies. The fLp is not a Gaussian process (it is an immediate consequence of Proposition 3.10 in Tikanmäki and Mishura (2011)). And not all fLp have stationary increments. On the other hand, the great advantage of the fLp, when compared with the fBm, is that it can be a semimartingale.

3.4 Simulation

In order to simulate the previous two fLp, we will present two main approaches.

3.4.1 Path-wise Riemann integral approach

In the present section, we propose a numerical method to simulate the previous fLp.

From Propositions 3.4 and 3.13, the problem of simulation of these processes is simply solved by a numerical integration, method given a simulation of the driving Lévy process. This is a perfect fit for the MVN-fLp, but not to the MG-fLp, since in this case, the Riemann integral representation is only available for a compound Poisson driving process.

Simulation of the MNV-fLp Within this approach, by Proposition 3.4, the MNV-fLp X_t^H is the result of the following integral

$$\int_{\mathbb{R}} g_t(s)ds, \tag{3.10}$$

where $g_t(s) = f_H(t,s)L(s)$ for a given $t \in \mathbb{R}$.

The simulation of X_t^H, within this approach, is summarized by the simulation of the driving Lévy process and by the application of a numerical scheme to approximate an Riemann integral.

However, the function g_t is not continuous in $\mathbb{R}$ but it is Lebesgue a.s. continuous. Let

$$\ldots, \tau_{-2}, \tau_{-1}, \tau_1, \tau_2, \ldots$$

be the instants of the jumps of the two-sided Lévy process.

So (3.10) can be written as

$$\sum_n \int_{\tau_{n-1}}^{\tau_n} g_t(s)ds. \tag{3.11}$$

If we choose, for instance, the composite trapezoidal scheme in m equally spaced partitions for the numerical approximation, each integral in (3.11) may be computed as

$$\lim_{\delta \to 0^+} \int_{\tau_{n-1}}^{\tau_n - \delta} g_t(s)ds \approx \frac{(\tau_n - \epsilon) - \tau_{n-1}}{m} \left(\frac{g_t(\tau_{n-1})}{2} + \sum_{k=1}^{m-1} g_t \left(\tau_{n-1} + \frac{(\tau_n - \epsilon) - \tau_{n-1}}{m} k \right) + \frac{g_t(\tau_n - \epsilon)}{2} \right),$$

with $\epsilon > 0$, where the ϵ corresponds to the approximation to the interval end where the function is not left-continuous. (Recall that the paths of Lévy process are right-continuous). The only point left is the truncation of the integral (3.10), which can be picked a priori substituting $\mathbb{R}$ by the compact interval $[-k, k]$.

With this background, we can follow up the following steps in order to simulate the MNV-fLp.

Algorithm 3.1 (Path-wise Riemann simulation of MVN-fLp)**.** *Using the trapezoidal rule for the integral's approximation, for each $t \in \mathbb{R}$, X_t^H may be simulated by the following.*

1. *Fix a truncation k, a number m of equally spaced partition intervals and truncation ϵ for each interval;*
2. *Simulate the Lévy process L_s for $s \in [-k, k]$, denote $\tau^*_{-n+1}, \ldots, \tau^*_{-1}, \tau^*_1, \ldots, \tau^*_{n-1}$ to the simulated jump instants, and call $\tau^*_{-n} = -k$ and $\tau^*_n = k$, moreover $\tau^*_0 = 0$;*
3. *For each interval $[\tau^*_{i-1}, \tau^*_i - \epsilon]$ define $s_k = \tau^*_{i-1} + \frac{\tau^*_i - \epsilon - \tau^*_{i-1}}{m} k$, for $k = 0, 1, \ldots, m$ and approximate the integral over $[\tau^*_{i-1}, \tau^*_i - \epsilon]$ with*
$$I_i := \frac{s_n - s_0}{m} \left(\frac{g_t(s_0)}{2} + \sum_{k=1}^{m-1} g_t(s_k) + \frac{g_t(s_m)}{2} \right);$$
4. *Sum the integrals I_i correspondent to each interval between the Lévy jumps.*

Simulation of the MG-fLp Analogously, the Proposition 3.13 ensures that, if L_s is a compound Poisson process, the MG-fLp Y_t^H is represented a.s. by the integral on a compact interval given by

$$\int_0^t h_t(s)ds, \tag{3.12}$$

where $h_t(s) = \left(-\frac{d}{ds} z_H(t,s)\right) L_s$.

After choosing an appropriate numerical method to compute the integral, it is only required to simulate a compound Poisson process and the get the values of the partial derivative of MG kernel in the desirable points. And we apply this program to each pretended time instant t of the MG-fLp process. Given the jump times of the compound Poisson process in $[0,t]$, $\tau_1, \tau_2, \ldots, \tau_{n-1}$, we can write (3.12) as

$$\sum_{i=1}^{n} \lim_{\delta \to 0^+} \int_{\tau_{i-1}}^{\tau_i - \delta} h_t(s)ds,$$

where $\tau_0 = 0$ and $\tau_n = t$. The reason for the limit is again the non-left-continuity of the compound Poisson process already explained. For $\epsilon > 0$ and $i = 0, 1, \ldots, n$, h_t is continuous in each interval $[\tau_{i-1}, \tau_i - \epsilon]$.

We propose to approximate (3.12) by

$$\sum_{i=1}^{n} I_i,$$

where I_i is the result of a numeric approximation of each integral

$$\int_{\tau_{i-1}}^{\tau_i - \epsilon} h_t(s)ds.$$

As it was done to the MVN-fLp, we can follow the following steps to numerically approximate the MG-fLp using the trapezoidal scheme.

Algorithm 3.2 (Path-wise Riemann simulation of MG-fLp)**.** *Using the trapezoidal rule for the integral's approximation the computation of the MG-fLp may follow the following steps.*

For each fixed $t > 0$, we get an approximation to Y_t^H by

1. *Fix a truncation ϵ for the integral, and a number n of equally spaced intervals;*
2. *Simulate the compound Poisson L_s for $s \in [0,t]$ and represent the jump instants by $\tau_1^*, \tau_2^*, \ldots, \tau_{n-1}^*$. We will assume $\tau_0^* = \delta$ and $\tau_n^* = t$, where $\delta > 0$ (the partial derivative of z_H may not be defined when $s = 0$);*
3. *For each interval $[\tau_{i-1}^*, \tau_i^* - \epsilon]$ define $s_k = \tau_{i-1}^* + \frac{\tau_i^* - \epsilon - \tau_{i-1}^*}{n} k$ for $k = 0, 1, \ldots, n$ and approximate the integral over $[\tau_{i-1}^*, \tau_i^* - \epsilon]$ with*

$$I_i := \frac{s_n - s_0}{n} \left(\frac{h_t(s_0)}{2} + \sum_{k=1}^{i-1} h_t(s_k) + \frac{h_t(s_i)}{2} \right);$$

4. *Sum the integrals I_n correspondent to each interval within the compound Poisson jumps.*

3.4.2 Semimartingale stochastic integral approach

The MVN-fLp can be simulated by the previous method independently of the choice of the driving Lévy process (within the conditions of its definition). On the other hand, the other method is only applicable in the case of the MG-fLp if its driving Lévy process is a compound Poisson process.

The Corollary 3.1 gives a possible approximation for both processes X_t^H and Y_t^H that is valid for any driving Lévy process. This approach is possible mainly because of the semimartingale property of the Lévy process which allows the definition of a stochastic integral, and so, the definition of the fLp itself. And, for the best of our knowledge, it is an innovation.

Simulation of MG-fLp By Corollary 3.1, for each $t > 0$, the MG-fLp Y_t^H can be written a.s. as

$$\lim_{|\pi|\to 0} \sum_{i=0}^{n-1} z_H(t, s_i)(L_{s_{i+1}} - L_{s_i}),$$

where π is a partition of the interval $[0, t]$, $0 = s_0 < s_1 < \cdots < s_n = t$ and $|\pi| = \max_n\{s_{n+1} - s_n\}$.

The new numerical approximation that we propose is a simple choice of a partition π. Considering a partition in intervals with the same size, an algorithm may be the following.

Algorithm 3.3. *Using the previous reasoning for the integral approximation, the computation of the MG-fLp may follow the following steps.*

For each fixed t, we get an approximation to Y_t^H by

1. *Fix the number of equally spaced intervals n.*

   The partition of the interval $[0, t]$ is given by the nodes s_i, for $i = 0, 1, \ldots, n$, where $s_0 = 0$ and $s_i = \frac{t}{n} i$.

2. *Compute the MG kernel function $z_H(t, s)$ from the point (t, s_0) to (t, s_n);*

3. *Simulate the Lévy process L_s and get its trajectory values $L^*_{s_0}$ up to $L^*_{s_n}$;*

4. *Finally, the approximation of the simulated value of Y_t^H is given by the following sum:*

$$\sum_{i=0}^{n-1} z_h(t, s_i)(L^*_{s_{i+1}} - L^*_{s_i}).$$

Simulation of MNV-fLp The peculiarity of this approach regarding the MNV-fLp is that the Corollary 3.1 does not cover integrals over $\mathbb{R}$. But a generalization is assumed.

We propose the following simple approach to the approximation of the simulated value of X^H at instant t, given a positive truncation constant k.

$$\sum_{i=-n}^{n-1} f_H(t, s_i)(L^*_{s_{i+1}} - L^*_{s_i}),$$

where $-k = s_{-n} < s_{-n+1} < \cdots < s_n = k$ corresponds to a partition of $[-k, k]$.

A very similar process used in the previous algorithm may be used to simulate these values.

Algorithm 3.4. *For each fixed t, we get an approximation to X_t^H by*

1. *Fix the truncation k and the number of equally spaced intervals n.*

   The partition of the interval $[-k, k]$ is given by the nodes s_i, for $i = 0, 1, \ldots, n$ where $s_0 = 0$ and $s_i = -k + \frac{2k}{n} i$.

2. *Compute the MVN kernel function $f_H(t, s)$ from the point (t, s_0) to (t, s_n);*

3. *Simulate the Lévy process L_s and get its trajectory values $L^*_{s_0}$ up to $L^*_{s_n}$;*

4. *Finally, the approximation of the simulated value of X_t^H is given by the following sum:*

$$\sum_{i=0}^{n-1} z_h(t, s_i)(L^*_{s_{i+1}} - L^*_{s_i}).$$

See Appendix B to see some numerical examples of these algorithms.

4 Financial Fractional Models

In this last chapter we briefly expose some financial models whose prices of risky assets are modelled by the previously studied processes. The self-similarity of the trajectories of the fBm is an interesting approach to describe the irregularity of prices, by considering it not completely irregular but using some measure of roughness.

But the main argument towards the use of fractional processes in the financial modelling is the long-memory dependence structure of these processes. It is not hard to accept that a shock in the price will have an influence on the future behaviour of the price trajectory. A complete survey on the arguments towards the use of memory can be found in Mandelbrot (1997a) and Shiryaev (1999).

For more insights on the mathematics behind these models we refer to Bender et al. (2008, 2011) and Fink (2011).

Note: in this section we still consider only the long memory case $H > \frac{1}{2}$.

4.1 Fractional Brownian motion

4.1.1 Fractional B-S model

As it was already discussed, the fBm B_t^H with its properties (mainly the long-memory dependence structure, the stationarity of its increments as well as the self-similarity of its paths) may happen to be some upgrade to the financial modelling, comparing to the previous case of the standard Brownian motion.

A first fractional model to be considered covering this process is the Fractional Black-Scholes (B-S) model. This is a simple generalization of the previous model, in the sense of considering the fBm instead of the Brownian motion in the dynamics of the price of the risky asset.

In this case, the price of the underlying asset is a fractional geometric Brownian motion with the following expression, under the natural (or *objective*) probability measure $\mathbb{P}$,

$$S_t = S_0 \exp\left(\mu t + \sigma B_t^H - \frac{\sigma^2}{2} t^{2H}\right), \tag{4.1}$$

where μ corresponds to the rate of return of the asset, and σ to its volatility. As usual, the price of the non-risky asset with interest rate r is given by e^{rt}, at instant t. The expression in (4.1) is obtained by considering the dynamics of the risky asset as $dS_t = \mu S_t dt + \sigma S_t dB_t^H$.

Denoting a_t and b_t the number of non-risky and risky assets detained by an investor at time t, respectively, then the value of the portfolio at time t is given by

$$V_t = a_t e^{rt} + b_t S_t.$$

Moreover, the portfolio is said to be self-financing if the following is verified (see Nualart (2006)), for each t,

$$V_t = V_0 + r \int_0^t a_s e^{rs} ds + \int_0^t b_s dS_s. \tag{4.2}$$

Now, we notice that the last integral in (4.2) is an integral with respect to a process involving the fBm. The two integral types presented in Section 2.4 can be used, but with different results.

Path-wise integral If we use the path-wise Riemann-Stieltjes integral, as it was introduced in Subsection 2.4.1, then the model allows arbitrage opportunities. The main cause for this is the fact that the fBm is not (except for the standard Brownian case) a semimartingale.

There are simple self-financed strategies within the current set up that allow arbitrage. For more details we refer to Nualart (2006) and Cheridito (2001b).

But interestingly, when we consider a more realistic model also involving a geometric fBm type along with transaction costs, it is possible to get a model without arbitrage opportunities. For more insights, see Guasoni (2006).

Wick product Björk and Hult (2005) criticize a possible non-arbitrage approach to this model. The key solution would be the substitution of the path-wise integral by the Wick integral mentioned in Subsection 2.4.2. A possible way (there is more than one) to get an arbitrage-free model by the use of Wick integral is assuming that the asset value S_t verifies

$$dS_t = rS_t dt + \sigma S_t \diamond dB_t^H.$$

and the self-financed condition as

$$V_t = V_0 + \int_0^t (ra_s e^{rs} + rb_s S_s)\, ds + \sigma \int_0^t b_s S_s \diamond dB_s^H.$$

The non-arbitrage conclusion is taken by several authors. This can be seen for instance in Biagini et al. (2002), Elliott and Van der Hoek (2003).

The main critics in Björk and Hult (2005) are due to the lack of economic reasoning in the mathematical formulation of both Wick-new self-financed condition as well as for the new formulation of the concept of arbitrage used in some of the articles based on this approach, mainly in Elliott and Van der Hoek (2003).

4.1.2 Mixed Model

A different approach to insert the fBm into a financial model is by considering both standard Brownian motion as well as the fBm. This is a possible way to get a model involving the fBm along with an economically meaningful attempt.

In this case the stock price process S_t can be written as

$$S_t = S_0 \exp\left(\mu t + \sigma B_t + \nu B_t^H\right).$$

In this case, instead of a simple substitution of the standard Brownian motion, it is added the fBm in the original geometric Brownian motion. As we will see, in this case we can avoid some of the economically not meaningful problems verified in the earlier case. The Brownian motion will correspond to the short term market modifications, and the fBm will model the long term fluctuations, determining the memory of the process. A complete introduction and basic proofs within this model can be found in Cheridito (2001a).

On one hand, it is proved in Cheridito (2001a) that the mixed fBm process $M_t^{H,\alpha} = B_t + \alpha B_t^H$ is still not a semimartingale, for most cases. In particular, it is not a semimartingale for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$ (Corollaries 2.3 and 2.6 of Cheridito (2001a)). Yet, for the remaining values of H ($H \in (\frac{3}{4}, 1)$) the mixed process $M^{H,\alpha}$ is not only a semimartingale, but it is equivalent in probability to the standard Brownian motion (see proof of Theorem 1.7 in Cheridito (2001a).)

So, for the non-semimartingale case, we are in front of the same problem as the previous model, and for the semimartingale case, we are back to the old geometric Brownian motion, which is a singularity in the fractional financial models.

4.2 Fractional Lévy process

For the fLp financial application, we will only consider a mixed (geometric) fractional model, where we will mix the model involving the fLp with a standard Brownian motion.

4.2.1 Mixed Model

In Tikanmäki and Mishura (2011), the authors present a mixed model involving the fLp together with a non-arbitrage opportunity result.

Given a fLp L_t^H, that can be either the MVN-fLp or the MG-fLp, the mixed stochastic process S_t that models the asset price movement along time can be determined by

$$\ln S_t = \sigma L_t^H + \epsilon B_t, \tag{4.3}$$

where B_t is the standard Brownian motion and $\sigma, \epsilon > 0$.

4.2.2 Arbitrage

The so called doubling strategies are investor strategies which allow, under some circumstances, positive profit in a finite time interval (that can happen to be long) with probability one if one is able to spend an unbounded amount of cash. One of the conditions for the famous Black-Scholes model to be an arbitrage-free model is the assumption of the impossibility of doubling strategies.

The main arbitrage result regarding the absence of arbitrage opportunities in model (4.3) is also based on a restriction of the possible trading strategies in the market. The formulation of the following theorem and its proof can be found in Tikanmäki and Mishura (2011, Theorem 6.1).

Theorem 4.1. *Given a filtered probability space (we may call $\mathcal{F}_t$ the filtration generated by the asset price process S_t), and assuming only* stopping-smooth trading strategies*, the market modelled by* (4.3) *has no arbitrage opportunities.*

The assumed trading strategies in the previous theorem are deeply detailed in Bender et al. (2008). These authors present the following definition of stopping-smooth trading strategies.

Definition 4.1. *We say that Φ is a* stopping-smooth trading strategy *if it can be written as*

$$\Phi_t = \sum_{k=1}^{n} \Phi_t^{(k)} \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t), \tag{4.4}$$

where $\Phi^{(k)}$ are smooth and τ_k are stopping times locally continuous.

Φ_t represents the number of stocks held by an investor at instant t, and so, given an initial capital v_0, the value of the investment in the asset whose price is represented by S_t can be written as

$$V_t = v_0 + \int_0^t \Phi_s dS_s.$$

Now, the *smoothness* of the trading strategy has a quite technical description in Bender et al. (2008, Definition 4.3). Basically and summarily, Φ_t is smooth if each $\Phi^{(k)}$ in (4.4) is a predicable smooth strategy, meaning that $\Phi^{(k)}$ contains all past information up to instant τ_k (or simply, $\Phi^{(k)}$ is $\mathcal{F}_{\tau_k}$-mensurable).

Moreover, $\Phi^{(k)}$ does not allow doubling-strategies and it is of class C^1 in all its variables (which can be of three types: time, the price asset process S_t, and functions of both t and S_t). The no-doubling strategy detail can be formally described as

$$\int_0^t \Phi_s dS_s \geq -a$$

for a positive constant a, almost surely.

Moreover, in Bender et al. (2008) is argued that this type of trading strategies has an economic reasoning that can be found in the most common financial derivatives, such as the case of European call options, which is the important case for the rest of our application.

4.2.3 Arbitrage-free option price simulation

Recalling the process S_t as it is described in (4.3) and assuming the conditions presented in the previous section, we end up with a no-arbitrage scenario which will be the final set-up to get an approach to the issue of pricing of a derivative whose underling involves a fractional Lévy process.

Since there is absence of arbitrage opportunities, it is possible to have an equivalent measure $\mathbb{Q}$ under which the actualized asset price process is a martingale. And we can get a "risk-neutral" pricing formula for the contingent T-claim X of the form

$$\Pi(t; X) = e^{-r(T-t)}\mathbb{E}^Q\left[X|\mathcal{F}_t\right],$$

admitting a flat term structure interest rate at r between $[0, T]$ and assuming again the $\mathcal{F}_t$ as being the filtration generated by the asset price process.

Supposing that the present T-claim is an European call option with exercise price K and that we want the price at $t = 0$, the previous pricing formula can be written as

$$\Pi_0 = \mathbb{E}^Q\left[e^{-rT}\delta(S(T))\right], \tag{4.5}$$

where $\delta(S(T)) = \max\{S_T - K, 0\}$.

If the price process under the martingale measure is still a fLp mixed model as the solution of (4.3), we can use the algorithms in Section 3.4 to get option prices with Monte Carlo.

By Monte Carlo method, obtain the arbitrage-free price of an European call option, we just follow the following steps, given a choice of model parameters and processes (mainly the driven Lévy process of the fLp). A numerical illustration is collected in Appendix C.

1. Simulations of the Brownian motion and the fLp, using the methods proposed in Section 3.4;

2. Computation of the present values of the call payoff;

3. Computation of the mean value of the previous values, whose result is an approximation to the pretended price (4.5).

5 Conclusions

Independently of the possible uses of fractional processes in mathematical modelling, with this work we did not pretend to propose an improved approach to model some dimension of the financial phenomenon, namely the price of an asset. As it was already mentioned in Subsection 3.3.3, the Lévy process has greater flexibility when compared to the fLp, when it is used to describe these objects. The main point was to introduce the concept of "memory" in this description. This is an acceptable detail in a price of an asset by both empirical and economical reasoning, but also with common sense, which may well be the key reason.

This "memory" feature followed the very simple propose of Mandelbrot already cited in Introduction: assuming a positive correlation on the increments of the process. The fact that large changes tend to be followed by large changes, and analogously for small changes, is a quite poor description of what we may call "memory". This was the basic detail in common to every fractional process considered in these pages. One possible way to enrich this "memory" model is to make this trend in price changes more dynamic by allowing the Hurst parameter to vary along time. This would take out the self-similarity of the fBm, though it is not a problem for those who accepted the fLp.

The fBm successfully generalised the Brownian motion, and we ended up with a self-similar family of processes with correlated increments. But in this case, we do not have a semimartingale, and the great pressures of financial requirements - mainly the absence of non-arbitrage results, or an economical meaningless attempt with the Wick integral - resulted in its gave up.

Of course, this was not the cause for the appearance of the fLp. We claim that it was a result of a "Let's see what happens" if one substitutes the Brownian motion by a suitable Lévy process in the integral definitions of the fBm. The result is non-self-similar process with correlated increments that, in some cases, can actually be a semimartingale. And besides this detail, it is possible a non-arbitrage result for a simple mixed model of a Brownian motion and a fLp. This mixture happens to solve the problem raised by fLp when it comes to model an asset price: it is too regular, since it is differentiable Lebesgue almost everywhere. In this sense, it is justified that the jump from fBm to the fLp actually ended up in a quite simpler instrument in financial mathematical modelling.

A possible future study object would be the attempt to generalize the non-arbitrage result regarding the mixed model of a fLp with a Brownian motion to a mixed model of a fLp with a Lévy process. This problem may admit a first solution when we restrain the fLp to its smaller family verifying the semimartingale property. In this case, the mixture is a semimartingale and the problem is simplified.

Given the simulation methods to the fLp, it would be interesting to empirically find out the most adapted parameters of the mixed fLp model to a real European call option.

6 References

David Applebaum. *Lévy Processes and Stochastic Calculus.* Cambridge University Press, 2nd edition, 2009.

Albert Benassi, Serge Cohen, and Jacques Istas. On roughness indices for fractional fields. *Bernoulli*, 10 (2):357–373, 2004.

Christian Bender, Tommi Sottinen, and Esko Valkeila. Pricing by hedging and no-arbitrage beyond semimartingales. *Finance Stoch*, 12:441–468, 2008.

Christian Bender, Tommi Sottinen, and Esko Valkeila. Fractional processes as models in stochastic finance. In *Advanced Mathematical Methods for Finance*, chapter 3. Springer, 2011.

Christian Bender, Alexander Lindner, and Markus Schicks. Finite variation of fractional Lévy processes. *Journal of Theoretical Probability*, 25(2):594–612, 2012.

Francesca Biagini, Yaozhong Huand Bernt Øksendal, and Agnès Sulem. A stochastic maximum principle for processes driven by fractional Brownian motion. *Stochastic processes and their applications*, 100(1): 233–253, 2002.

Klaus Bichteler. Stochastic integration and L^p-theory of semimartingales. *The Annals of Probability*, pages 49–89, 1981.

Tomas Björk and Henrik Hult. A note on Wick products and the fractional black-schole model. *Finance Stochas*, 9:197–209, 2005.

Robert Brown. Xxvii. a brief account of microscopical observations made in the months of june, july and august 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies. *Philosophical Magazine Series 2*, 4(21):161–173, 1828.

Patrick Cheridito. Mixed fractional Brownian motion. *Bernoulli*, pages 913–934, 2001a.

Patrick Cheridito. *Regularizing fractional Brownian motion with a view towards stock price modelling.* PhD thesis, Swiss Federal Institute of Technology Zurich, 2001b.

Paul Doukhan and George Oppenheim Murad Taqqu. *Theory and Applications of Long-Range Dependence.* Springer Science & Business Media, 2003.

Robert J. Elliott and John Van der Hoek. A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13(2):301–330, 2003.

Paul Embrechts and Makoto Maejima. *Selfsimilar Processes.* Princeton Series in Applied Mathematics. Princeton University Press, 2002.

Holger Fink. *Stochastic processes beyond semimartingales with application to interest rates, credit risk and volatility modeling.* PhD thesis, Technischen Universitat Munchen, 2011.

Holger Fink. Conditional characteristic functions of Molchan-Golosov fractional Lévy processes with application to credit risk. *Journal of Applied Probability*, 50(4):983–1005, 2013.

Holger Fink and Christian Scherr. CDS pricing with long memory via fractional Lévy process. *Journal of Financial Engineering*, 1(4), 2014.

Paolo Guasoni. No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Mathematical Finance*, 16(3):569–582, 2006.

Harold Edwin Hurst. Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers*, 116:770–808, 1951.

Céline Jost. A note on the connection between molchan–golosov and mandelbrot–van ness representation of fractional Brownian motion, preprint 424, department of mathematics and statistics, university of helsinki. 2005. Available from http://www.arxiv.org under Id math.PR/0602356.

Céline Jost. *Integral Transformations of Volterra Gaussian Processes.* PhD thesis, Faculty of Science of the University of Helsinki, 2007.

Olav Kallenberg. *Foundations of modern probability.* Springer, New York, 1997.

Andrei Nikolaevich Kolmogorov. Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C. R. (Doklady) Acad. URSS (N.S.)*, 26:115–118, 1940.

John Lamperti. Semi-stable processes. *Trans. Amr. Math*, 104:62–78, 1962.

Benoît Mandelbrot. *Fractals and Scaling in Finance. Discontinuity, Concentration, Risk.* Springer, New York, 1997a.

Benoît Mandelbrot. The variation of certain speculative prices. In *Fractals and Scaling in Finance*, pages 371–418. Springer, 1997b.

Benoît Mandelbrot and John van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Review*, (4), 1968.

Tina Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, (6):1099–1126, 2006.

George Molchan and Yu Golosov. Gaussian stationary processes with asymptotic power spectrum. *Soviet Math. Dokl.*, 10(1), 1969.

David Nualart. Fractional Brownian motion: stochastic calculus and applications. In *International Congress of Mathematicians*, volume 3, pages 1541–1562, 2006.

Giulia Di Nunno, Bernt Øksendal, and Frank Proske. *Malliavin calculus for Lévy processes with applications to finance.* Springer, 2009.

Bernt Øksendal. *Stochastic Differential Equations, An introduction with applications.* Springer, fifth edition, 2000.

Antonis Papapantoleon. *Applications of semimartingales and Lévy processes in finance: duality and valuation.* PhD thesis, Citeseer, 2007.

Stefan G. Samko, Anatoly A. Kilbas, and Oleg I. Marichev. *Fractional Integrals and Derivatives. Theory and applications.* Gordon and Breach Science Publishers, Yverdon, 1993.

Ken-Iti Sato. *Lévy Processes and infinitely divisible distributions.* Cambridge University Press, 1999.

Albert Shiryaev. *Essentials of Stochastic Finance. Facts, models, theory.* World Scientific, 1999.

Tommi Sottinen. *Fractional Brownian Motion in Finance and Queueing.* PhD thesis, University of Helsinki, 2003.

Murad Taqqu. Benoit Mandelbrot and fractional Brownian motion. *Statistical Science*, (1), 2013.

Heikki Tikanmäki. *Fractional processes, pathwise stochastic analysis and finance.* PhD thesis, Aalto University, 2012.

Heikki Tikanmäki and Yuliya Mishura. Fractional Lévy processes as a result of compact interval integral transformation. *Stochastic Analysis and Applications*, 29:1081–1101, 2011.

Laurence C. Young. An inequality of the hölder type connected with stieltjes integration. *Acta Math*, 67: 251–282, 1936.

A Wick integral

In Sottinen (2003), we find the following definition for the Wick product.

Definition A.1. *Given a centred Gaussian random variable ξ, its* Wick exponential *is defined as*

$$e^{\diamond\xi} = e^{\xi - \frac{1}{2}\mathbb{E}[\xi^2]}.$$

A random variable $X \diamond Y$ is the Wick product *of X and Y if*

$$\mathbb{E}\left[(X \diamond Y)e^{\diamond W(\phi)}\right] = \mathbb{E}\left[Xe^{\diamond W(\phi)}\right]\mathbb{E}\left[Ye^{\diamond W(\phi)}\right], \tag{A.1}$$

for all $\phi \in L^2([0,1])$.

The $W(\phi)$ denotes a Gaussian random variable, and this is a quite standard notation in Malliavin calculus. For a given element ϕ of $L^2([0,1])$, $W(\phi)$ is the result of the following integral

$$W(\phi) = \int_0^1 \phi_t dB_t.$$

One may claim that the definition of the Wick product in (A.1) is an attempt to *create* independence by substitution of the usual multiplication by some artificial and - one may say - meaningless (at least in finance) operation.

The definition of Wick integral will not be presented. In Nualart (2006, Proposition 3.3) it is justified the name "integral" for this operator, since for a specified class of well-behaved functions we actually have a relation between the Wick integral and the Riemann-Stieltjes integral with respect to the fBm.

Proposition A.1. *For $T > 0$, let $F \in C^1([0,T])$ such that both F and F' verify, for positive constants c and $\lambda < \frac{1}{4T^{2H}}$,*

$$|F(x)| \leq ce^{\lambda x^2}.$$

Then the Wick integral with respect to the fBm whit Hurst parameter $H > \frac{1}{2}$ can be written as

$$\int_0^T F(B_t^H) \diamond dB_t^H = \int_0^T F(B_t^H)dB_t^H - H\int_0^T F'(B_t^H)t^{2H-1}dt.$$

B Numerical simulation of fLp

In this Appendix, we present some simulations resulting from the algorithms presented in Section 3.4 and the use of *Mathematica*.

On simulation of a Lévy process

The simulation of the fLp actually relies on the simulation of the Lévy process, which is its only "source of randomness".

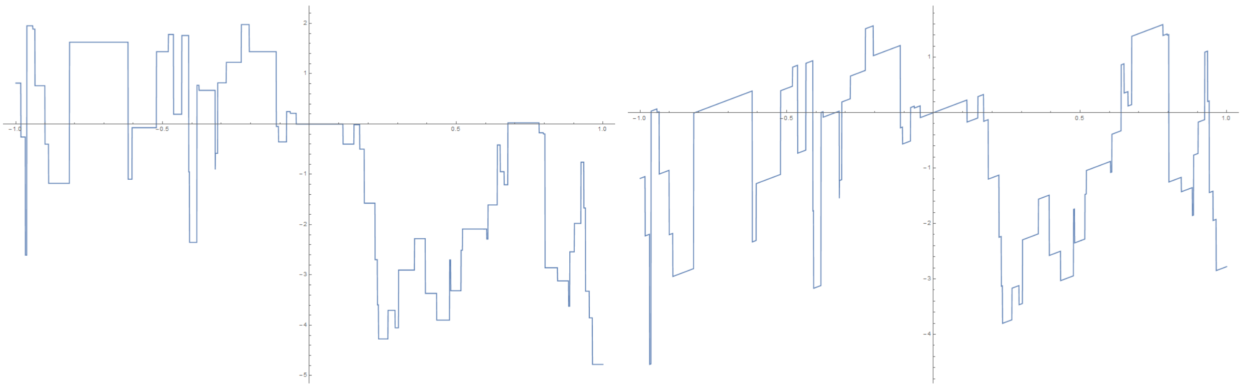
Summarizing the results in Chapter 3, there are three main Lévy processes that can be used to "build" the fLp. The Lévy process may be a jump process with or without drift component. In both cases, the jump component may be of finite or infinite activity.

In this work, the numerical results are seen as just examples or illustrations, and, somehow they are a "proof" that some of the previous proposed methods for simulations are in fact attainable. So, we will only present the numerical methods corresponding to the stochastic integral approach presented in Subsection 3.4.2. Mainly because it is a simpler and cleaner procedure, it admits a greater class of Lévy processes, and also, as far as we know, this approach was firstly presented in this work.

We will only present two main Lévy processes: a two-sided compound Poisson (with and without drift) and a standard compound Poisson process (with and without drift). These are the base blocks for the construction of the MVN-fLp and the MG-fLp, respectively.

We will consider a λ-parameter of 0.03 for the exponential distribution modelling the time jumps of the Lévy process and a standard Gaussian distributions for the lengths of jumps. When a drift component is considered, we assume a slope of 2 units.

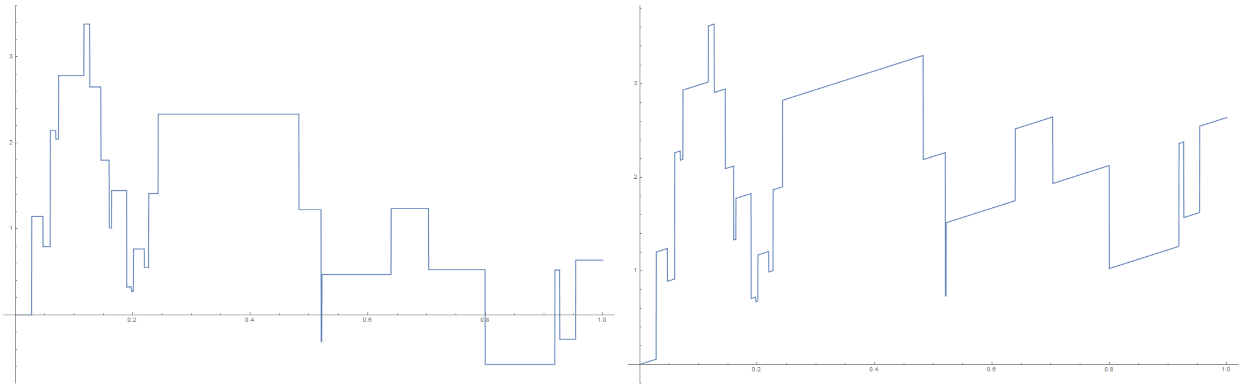
In Figure 1, we have a simulation for both cases of the two-sided process.



(a) Two-sided compound Poisson process without drift (b) Two-sided compound Poisson process with drift

Figure 1: Two-sided compound Poisson process, $\lambda = 0.03$ and jump length is modelled by standard Gaussian distribution

On the other hand, in Figure 2, we find a path for both cases of the one-sided process. Again the drift is given by a slope of 2 units.



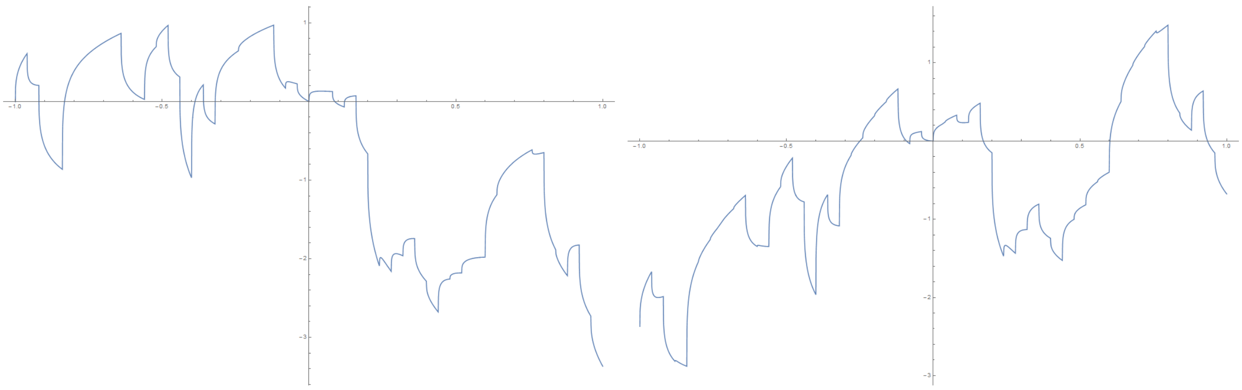
(a) Compound Poisson process without drift (b) Compound Poisson process with drift

Figure 2: One-sided compound Poisson process, $\lambda = 0.03$ and jump length is modelled by standard Gaussian distribution

B.1 On simulation of MVN-fLp

Based on Algorithm 3.4, the simulated trajectories of the MVN-fLp built on the Lévy paths in Figure 1 may look as it is shown on the following set of graphics.

In the present case, we consider a truncation in $[-2, 2]$ and 100 equally spaced intervals for the composite trapezoidal rule.



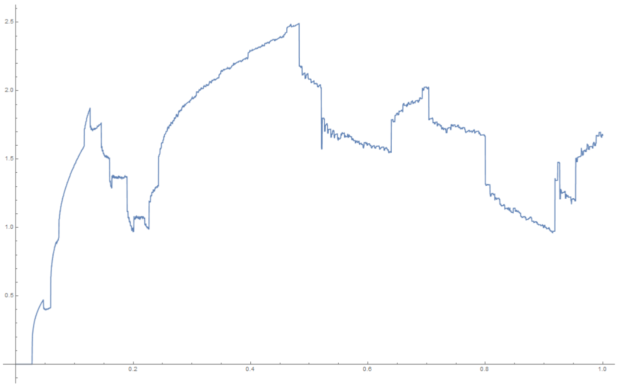
(a) MVN-fLp based on process in Figure 1a (b) MVN-fLp on process in Figure 1b

Figure 3: MVN-fLp with $H = 0.75$

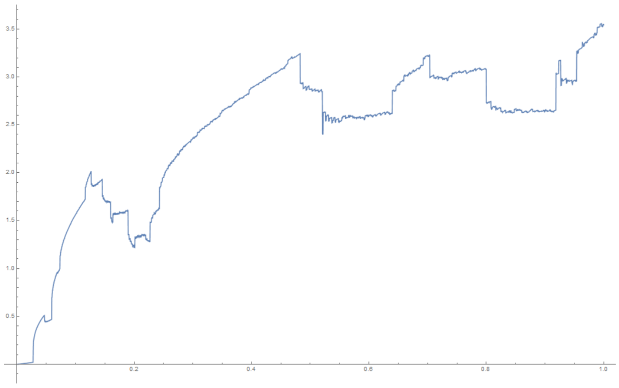
B.2 On simulation of MG-fLp

Now, based on Algorithm 3.3, the simulated trajectories of the MVN-fLp built on the Lévy paths in Figure 2 may look as it appears on the following figures.

In this case, we consider 100 equally spaced intervals for the composite trapezoidal rule.



(a) MG-fLp without drift, based on Figure 2a

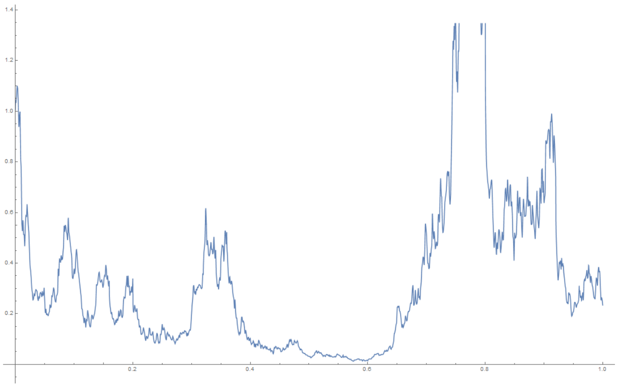


(b) MG-fLp with drift, based on Figure 2b

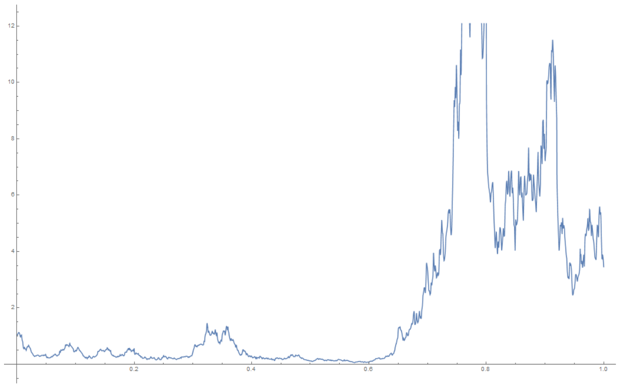
Figure 4: MG-fLp with $H = 0.75$

C Numerical simulation of European call option price under mixed fLp

In the mixed fLp model (4.3), if we consider the specific values of 1 and 0.1 for σ and ϵ, respectively, the simulated trajectories of the price process S_t based on the four different previous fLp are presented below.

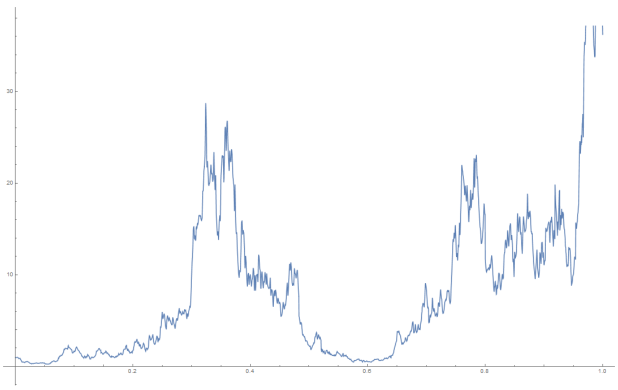


(a) Based in MVN-fLp in Figure 3a

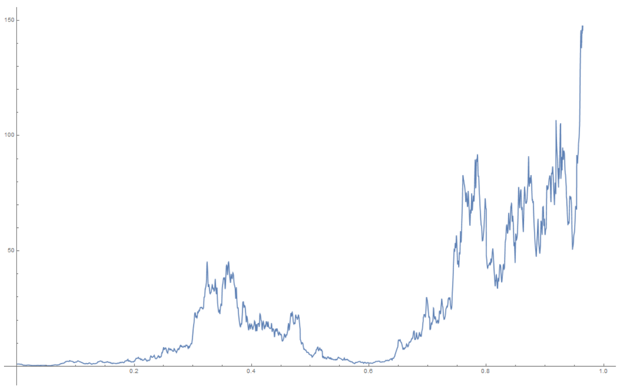


(b) Based in MVN-fLp in Figure 3b

Figure 5: mixed fLp model with parameters $\sigma = 1$ and $\epsilon = 0.1$ for MVN-fLp



(a) Based in MG-fLp in Figure 4a



(b) Based in MG-fLp in Figure 4b

Figure 6: mixed fLp model with parameters $\sigma = 1$ and $\epsilon = 0.1$ for MG-fLp

Now, the Monte Carlo method is the repetition of all this process for a sufficient number of trials, and approximate the expected value of the payoff for each option. We will apply this for each of the four processes.

For each case, we will simulate 1000 prices for $T = 1$, let us say one year, as it was done previously. The parameters of the model are unchanged, and so $\sigma = 1$ and $\epsilon = 0.1$, and the conditions for the simulation of the fLp are the same used in the Appendix B.

After one year, the average price of each asset (driven by different fLp) over 1000 trials is resumed in the following table. (Recall that the initial price of each asset is 1).

fLp	Expected price
MVN-fLp (based on a dritfless compound Poisson process)	13.36
MVN-fLp (based on a compound Poisson process with drift)	230.08
MG-fLp (based on a dritfless compound Poisson process)	228.47
MG-fLp (based on a compound Poisson process with drift)	775.64

Table 2: Expected prices an asset modelled with each fLp considered

For a given strike price K, the payoff of the European call is $\max\{S_T - K, 0\}$. The previous results state that for any strike price above the expected price, the price of the call option will be zero.

To end this illustration, we may consider an interest rate fixed at $r = 10\%$. And, for each strike price, we get the following price of the European call option.

fLp	Strike price	Call price
MVN-fLp (based on a dritfless compound Poisson process)	10	3.40
MVN-fLp (based on a compound Poisson process with drift)	200	27.22
MG-fLp (based on a dritfless compound Poisson process)	200	25.75
MG-fLp (based on a compound Poisson process with drift)	700	68.44

Table 3: Prices of European call options for fLp mixed model