UNIVERSIDADE DE LISBOA
INSTITUTO SUPERIOR DE ECONOMIA E GESTÃO

MASTER'S THESIS

APPROXIMATIONS TO RUIN PROBABILITIES IN INFINITE TIME USING A LÉVY PROCESS

BY YACINE KOUCHA

SUPERVISED BY

PROFESSOR ALFREDO DUARTE EGÍDIO DOS REIS

In Partial Fulfillment of the Requirements for the Degree of...

MSC ACTUARIAL SCIENCE

NOVEMBER 2016

ACKNOWLEDGEMENTS

Do you remember the first day you set foot into the playground? Can you close your eyes and simply imagine the most incredible place to you? If so, then you are already a remarkable writer. You have the gift to create an outstanding project.

During the last nine months on this Masters (not a pregnancy speech), thinking for great ideas, developing them, or failing them, what I found difficult is constructing this paper. For a long time, I have been purposelessly gazing at blank pages, seeking for creativeness that was coming in slower than my capacity to digest red meat. I cannot count how many people were conscientiously involved with me on this project, and to simply "thank" them would be belittling their consistent support, encouragement and direction during this thesis. I wish to thank the following:

- *Professor Alfredo Egídio dos Reis*, my thesis advisor extraordinaire, for encouraging me as well as entrusting me to complete this pretentious project. Without his help & exceptional expertise, this paper would still be fifty shades of white.

- *Professor Maria de Lourdes Centeno*, for her advice, feedback and influential insights throughout the thesis writing process, as well as pushing me to succeed in this Masters.

- *My family*: my mother, *Aicha*; my father, *Abdelhakim*; my sister, *Yasmin*, and my brother, *Sofiane*, for the immense support they have provided for me in this Section of my life.

- *My new amigos* at ISEG, Portuguese & international, for being with me during these two amazing years. I had arrived to this country as an inexperienced traveller. With their help, I have grown and developed.

- *The Almighty*.

ABSTRACT

In this thesis, we work with prominence to a key area in actuarial science, namely ruin theory. The Cramér-Lundberg model of collective risk theory is adapted for the perturbed model, by adding a Lévy (α-stabled) process to the compound Poisson process, which allows us to consider uncertainty to the premium income, fluctuations of the interest rates, changes to the number of policyholders, without neglecting all other assumptions.

On the way, we present new approximation techniques, built for the perturbed model in infinite time, and recall a remarkable family of well-known approximations by DE VYLDER (1996), DUFRESNE AND GERBER (1989), POLLACZEK-KHINCHINE and PADÉ (see AVRAM *ET. AL* (2001) and JOHNSON AND TAAFFE (1989)), obtained by fitting one, two, three or four (we also attempt five) ordinary moments of the claim amount distribution, and thus significantly generalising these approximations. Finding such approximation which fit the Laplace transform of the ruin probability would also be quite valuable, see FURRER (1998).

We test the accuracy of the approximations using a mixture of light and heavy tailed distributions for the individual claim amount. We evaluate the ultimate ruin probability and illustrate in detail some numerical results.

KEYWORDS

Lévy process, α-stable process, Pollaczek-Khinchine, Ruin theory, (Collective) risk theory, perturbed model, Padé approximation, De Vylder approximation, ruin probability approximations.

RESUMO

Esta dissertação aborda especificamente problemas da área da teoria da ruína, sub-área da teoria do risco para a atividade seguradora. Em particular, estudamos a probabilidade de ruína eventual. Adaptamos o modelo de risco coletivo de Cramér-Lundberg, estendendo para o modelo perturbado. Adicionamos ao modelo de Poisson composto uma componente representativa de um processo de Lévy (alfa estável). Esta componente adicional permite-nos incorporar incertezas decorrentes de, por exemplo, flutuações de taxas de juro, alterações no número de apólices na carteira, em quaisquer dos casos mantendo as hipóteses tradicionais.

Com o objetivo de cálculo da probabilidade de ruína no modelo perturbado, apresentamos novas técnicas, recuperando e generalizando modelos de aproximação bem conhecidos, tais como os de DE VYLDER (1996), DUFRESNE AND GERBER (1989), POLLACZEK-KHINCHINE, PADÉ (ver AVRAM *ET AL*. (2001) e JOHNSON AND TAAFFE (1989)), obtidas ajustando um, dois, três ou quatro momentos ordinários da distribuição dos montantes das indemnizações. Para além disso, considerámos também importante que as aproximações ajustassem a transformada de Laplace (para a probabilidade de ruína), veja-se FURRER (1998).

Avaliamos a qualidade das aproximações estudadas exemplificando para um conjunto de distribuições de cauda leve e de cauda pesada. Ilustramos com detalhe com alguns resultados numéricos.

PALAVRAS-CHAVE

Processo de Lévy, processo de alfa estável, Pollaczek-Khinchine, a teoria da ruína, Teoria do Risco, modelo perturbado, Padé, De Vylder, aproximações ruína de probabilidade.

List of Mathematical Notations and Abbreviations

r.v.	random variable.
$U(t)$	Cramér-Lundberg model, surplus at time t.
u	initial surplus or initial reserve.
c	premium rate.
$S(t)$	aggregate claim amounts occurred in $(0, t]$
$N(t)$	number of claims up to time t.
X_i	individual claim amount i.
λ	Poisson parameter: intensity rate.
i.i.d.	independent and identically distributed.
pdf	probability density function.
CDF	cumulative distribution function.
$F_X(.)$	CDF of X.
μ_k	k-th raw moment of X.
θ	safety positive loading coefficient.
$V(t)$	Extension of $U(t)$, inclusion of a diffusion component.
σ	drift parameter.
$Z_\alpha(t)$	α-stable Lévy process.
$W(t)$	standard Brownian motion (special case when $\alpha = 2$).
$\Psi(u)$	ultimate ruin probability at initial surplus.
$\phi(u)$	ultimate survival probability at initial surplus.
R	adjustment coefficient.
MGF	moment generating function.
CGF	cumulants generating function.
LT	Laplace transform.
PK	Pollaczek-Khinchine.
L	aggregate loss distribution.
$L_i^{(j)}$	record highs due to $j = 1$ (oscillation) or $j = 2$ (claims) and part of L.
$F_L^{*n}(.)$	n-fold transform distribution of L.
M	number of records distribution.
q	geometric parameter representing the number of failures under M.
ρ	profit rate.
η_k	Lévy moments.
$\breve{\mu}_k$	Factorial reduced moments of L.
IDD	infinitely divisible distribution.
4MGDV	4-moment gamma De Vylder.
5MGDV	5-moment gamma De Vylder.

CONTENTS

List of Figures

List of Tables

1 INTRODUCTION

> If I were again beginning my studies, I would follow the advice of Plato and start with mathematics.
>
> Galileo Galilei

Ruin theory (collective risk theory) is a field of mathematics that is an important part of actuarial education, as it uses mathematical models to explain an insurer's level on vulnerability to ruin. Risk theory, on the other hand, has its origins in the early 20th century, when FILIP LUNDBERG (1903) published his initial ideas on the classical surplus process. SPARRE ANDERSEN (1957) adapted Lundberg's process to allow for other claim inter-arrival times. As such, key quantities of interest are the ruin probabilities, distribution of surplus immediately prior to ruin, and the deficit at the time of ruin.

The Cramér-Lundberg model of collective risk theory is adapted for the perturbed model, by adding a Lévy (α-stabled) process to the compound Poisson process, which allows us to consider uncertainty to the premium income, fluctuations of the interest rates, changes to the number of policyholders, without neglecting all other assumptions.

The aim of this project is to obtain approximations for the probability of the process falling into ruin (i.e., becoming negative) in infinite time. The idea of approximating empirical data in the form of ordinary moments is like bread and butter of classical statistics & probability.

Some well-known approximations used in modern risk theory is discussed here, such as an explicit Pollaczek formula for the Laplace transform, as well as approximations in the form of Renyi (generalisation of Beekman-Bowers) [see GRANDELL (2000)], DE VYLDER (1996), DUFRENSE AND GERBER (1989) and PADÉ [see AVRAM *ET AL.* (2001)], which all fit a high number of ordinary moments of the claim amount distribution. Of course, since input data are usually linked to uncertainty, it is interesting to develop approximations based on finitely many ordinary moments, i.e. formed by the Laplace transform power series expansion around zero.

Luckily, there exists an extensive literature for further reading, with many contributions in this field, especially with recent developments to: (i) the Compound-Poisson risk models with constant/stochastic interest, (ii) the Brownian motion, (iii) the α-stabled process and (iv) the general diffusion risk processes; for instance, see THORIN (1974).

Outlining this thesis, we proceed as follows: firstly, SECTION 2 will commence with a comprehensive outlook of the perturbed model. SECTION 3 will focus on the approximation techniques used for later discussion. Numerical approximations computed via *MS Excel*, *Wolfram Alpha* and *Mathematica* are presented under SECTION 4, for illustrative purposes, to test our hypothesis on the validity and accuracy of each approximation method. We present an alternative method of sorts to the Dufresne and Gerber's upper and lower bounds found in SILVA (2006), by using convolution techniques, as well as a new approximation under the classical exponential/gamma De Vylder case (see SECTION 4.1 and 4.2). In the latter half of this Section, we also incorporate first order Padé approximation of the Pollaczek-Khinchine transform under Renyi and De Vylder here [see AVRAM *ET AL.* (2001)], and an "update" to a second order Padé approximation (to ensure it works for claims having several moments). Finally, some conclusions and recommendations are present in SECTION 5.

The reader must be familiar with works on ruin theory and risk theory. As such, key fields of interest are stochastic calculus, statistics, renewal theory and probability theory. Excellent first references are KLUGMAN *ET AL.* (2012).

---

2 THE INSURANCE MODEL

> Mathematics is the most beautiful and most powerful creation of the human spirit.
>
> Stefan Benach

2.1 Introduction

In this Section, we introduce a different model with another source of randomness attached, i.e. the Lévy process, named after the French mathematician Paul Lévy, see in APPLEBAUM (2014b). This is a stochastic process and is the continuous-time equivalent of a *random walk*. We consider this Section as a *spiritual* sequel to the work presented by SEIXAS AND EGÍDIO DOS REIS (2013), where $W(t)$ is a standard Brownian motion with drift (a well-known example of Lévy processes).

Lévy processes are becoming much more relevant, since they can describe the observed reality of financial markets with greater accuracy than a model based on Brownian motion. After a thorough review of the literature, it seems that Lévy processes has been restricted to a Brownian motion & α-stable process. In practice, there seems to be two approaches when it comes to applying Lévy processes, that is by:

i. replacing the aggregate claim process, or
ii. using Lévy process as perturbation to the classical model.

Of course, we are interested in the second approach. Other general perturbation processes considered in the past were in GERBER (1970).

2.2 Cramér-Lundberg Risk Process

We start this thesis by defining some basic assumptions, explaining how ruin theory is applied, and thus giving an overview over the results collected in this thesis. We present the standard model given by BOWERS *ET AL.* (1986), as the Cramér-Lundberg (classical) model in many actuarial journals and textbooks:

$$U(t) = u + ct - S(t), \qquad t \geq 0,$$

where, $U(t)$ is the surplus at time t, $u = U(0) \geq 0$ is the initial surplus or initial reserve, c is the rate at which premiums are received, $S(t) = \sum_{i=0}^{N_t} X_i$ is the aggregate claim amounts occurred in $(0, t]$, $N(t)$ is the number of claims up to time t and X_i is the individual claim amount i.

We denote the counting process $\{N(t); t \geq 0\}$ as a Poisson process with intensity rate $\lambda > 0$. The sequence $\{X_i\}_{i \geq 1}$ are independent and identically distributed (i.i.d.)

random variables, with cumulative distribution function (CDF), $F_X(\,.\,)$, such that $F_X(0) = 0$ and the k-th ordinary moment $\mu_k = E[X^k]$, which we assume to exist if we set $X_0 \equiv 0$.

The model assumes that $\{X_i\}_{i\geq 1}$ and $\{N(t); t \geq 0\}$ are independent, and, we can assume there exists some positive safety loading, such that $\theta = c(\lambda\mu_1)^{-1} - 1 > 0$ is the strictly positive loading coefficient. This is positive, otherwise $c < \lambda\mu_1$ and so this risk business would be negative with a probability of one in infinite time.

For mathematical purposes, we conclude that the aggregate claim amounts process $\{S(t); t \geq 0\}$ is a compound Poisson process.

2.3 Perturbed Risk Process

We present the perturbed risk model, $\{V(t)\}_{t\geq 0}$, where it is assumed that the process is independent of $U(t)$ and a Lévy process, and so, the model at time t is given by:

$$V(t) = U(t) + \sigma Z_\alpha(t), \qquad t \geq 0, \tag{2.3.1}$$

where, $V(t)$ is an extension to the Cramér-Lundberg model with the inclusion of a drift parameter $\sigma > 0$ (infinitesimal variance $\sigma/2$) and perturbed by an α-stable Lévy process $Z_\alpha(t)$ such that $\alpha \in (0,2]$. In this thesis, we will consider cases when $\alpha = 2$, then $Z_2(t) = W(t)$, where $W(t)$ is a standard Brownian motion [see APPENDIX B for properties]. The diffusion term $\sigma Z_\alpha(t)$ in (2.3.1) expressing an additional uncertainty to the aggregate claims, allows us to consider uncertainty to the premium income, fluctuations of the interest rates, changes to the number of policyholders, without neglecting all other assumptions.

2.4 Ruin Probability

In this section, we introduce common elements and definitions for the model presented in equation (2.3.1), namely, the ultimate ruin and survival probabilities, the Lundberg's inequality, the adjustment coefficient, a simple upper bound, maximal aggregate loss and some asymptotic results. From here on, all stochastic quantities are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.4.1 Common Ruin Terminology

For simplicity, consider the standard Brownian case, where $Z_2(t) = W(t)$. Now consider ruin probability in infinite time; we use definitions found in CENTENO (2015):

DEFINITION 1. *Let T_u be the random variable representing the time when ruin occurs, from initial surplus u, and so:*

$$T_u = \inf\{t \geq 0 \; and \; V(t) < 0\}, \quad u \geq 0, \tag{2.4.1}$$

otherwise, $T_u = \infty$ if ruin doesn't occur when $V(t) \geq 0$ for all t.

DEFINITION 2. *Suppose we let $G(u, y) = P(V(T) \in (-y, 0) \; and \; T_u < \infty \mid V(0) = u)$ be the probability that ruin occurs with initial surplus u and the deficit immediately after ruin occurs is at most y, then as $y \to \infty$, we obtain:*

$$\lim_{y \to \infty} G(u, y) = \Psi(u) = P(T_u < \infty \mid V(0) = u), \quad u \geq 0, \tag{2.4.2}$$

where, $\Psi(u)$ is the ultimate ruin probability in continuous time and infinite time horizon.

Using equations (2.4.1) and (2.4.2), we define $\phi(u) = 1 - \Psi(u)$ as the survival or non-ruin probability, i.e. the probability that ruin never occurs from initial surplus u. Now to guarantee that $\phi(u) \neq 0$ for all $u \geq 0$, we must assume the *net profit condition*

$$c - \mu_1 \lambda > 0, \tag{2.4.3}$$

which means that for each unit of time, the premium income exceeds the expected aggregate claim amount. If this condition fails, then $\phi(u) = 0 \Longrightarrow \Psi(u) = 1$ for all $u \geq 0$. Condition (2.4.3) brings economic sense to the classical model, and therefore it is convenient to write $c = (1 + \theta)\mu_1 \lambda$ for $\theta > 0$. In the next Section, we consider finding an explicit formula for $\Psi(u)$ for the event described in (2.4.2) for $G(u, y)$, by using an *integro-differential equation*, see APPENDIX F and SECTION 3.4 for properties.

2.4.2 A Defective Renewal Approach for a Brownian Motion

In the model in (2.3.1) with $\alpha = 2$, DUFRESNE AND GERBER (1991) introduced important ruin probabilities caused by oscillation, $\Psi_d(u)$ and caused by claim occurrence, $\Psi_c(u)$ and thus, led to the derivation of defective renewal equations for $\Psi(u), \Psi_d(u)$ and $\Psi_c(u)$.

DEFINITION 3. *Let the ultimate ruin probability from initial surplus u be given by the total ruin caused due to oscillation and claim occurrence, thus:*

$$\Psi(u) = P(T_u < \infty \mid V(0) = u) = \Psi_d(u) + \Psi_c(u), \tag{2.4.4}$$

where,

$$\Psi_d(u) = P(T_u < \infty \; and \; V(T) < 0 \mid V(0) = u),$$
$$\Psi_c(u) = P(T_u < \infty \; and \; V(T) = 0 \mid V(0) = u).$$

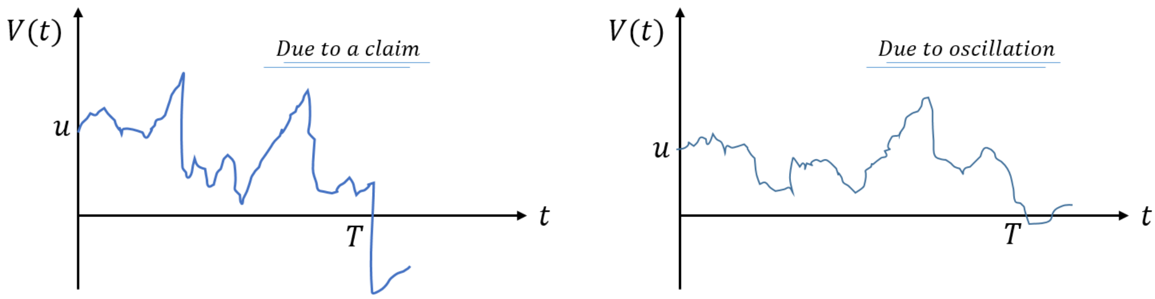


FIGURE 1 – TWO TYPES OF RUIN WITH ONE DUE TO A CLAIM AND ANOTHER DUE TO OSCILLATION

Due to the diffusive nature of the process, we deduce that $\Psi_c(0) = 0$ and $\Psi_d(0) = \Psi(0) = 1$. Applying standard renewal theory techniques, DUFRESNE AND GERBER (1991) arrived to a generalization that for $u \geq 0$:

$$\Psi(u) = q\big(1 - H_1(u)\big) + (1-q)\big(H_1(u) - [H_1 * H_2](u)\big) + (1-q)\int_0^u \Psi(u-x)[h_1 * h_2](x)\,dx,$$

where, $[h_1 * h_2](z) = \int_0^z h_1(x)h_2(z-x)\,dx$ is the convolution concentrated over a finite range $(0, z)$, with density functions $h_1(\,.\,)$ and $h_2(\,.\,)$ defined by:

$$h_1(x) = \tau e^{-\tau x} \sim exponential\left(\tau = \frac{2c}{\sigma^2}\right), \tag{2.4.5}$$
$$h_2(x) = \mu_1^{-1}[1 - F_X(x)], \; x > 0. \tag{2.4.6}$$

Thus,

$$\Psi_d(u) = 1 - H_1(u) + (1-q)\int_0^u \Psi_d(u-x)[h_1 * h_2](x)\,dx,$$

$$\Psi_c(u) = (1-q)\big(H_1(u) - [H_1 * H_2](u)\big) + (1-q)\int_0^u \Psi_c(u-x)[h_1 * h_2](x)\,dx.$$

These renewal applications have provided insight into the theory, and it has been shown in DUFRESNE AND GERBER (1991) that numerical solutions can be obtained for the integral equation encompassing $1 - \Psi(u)$ above.

2.5 *Lundberg's equation*

The adjustment coefficient is the strictly positive coefficient, denoted by R, which gives a measure of risk for a surplus process. Assuming (2.4.3) is satisfied, then $r = R$ is the only solution to [see ROLSKI *ET AL*. (2008)]:

$$cr - \frac{1}{2}\sigma^2 r^2 - \lambda(M_X(r) - 1) = 0, \qquad r < \eta, \tag{2.5.1}$$

where, $\eta = \sup\{ r : M_X(r) < \infty\}$ and $M_X(r) = E[e^{rX}]$ is the moment generating function of the claim amount distribution, if it exists, $M_X(0) = 1, \eta > 0$ and $M_X(r) \to \infty$ when $r \to \eta$. The process $e^{-R\{V(t)-u\}}$ is a martingale with mean one.

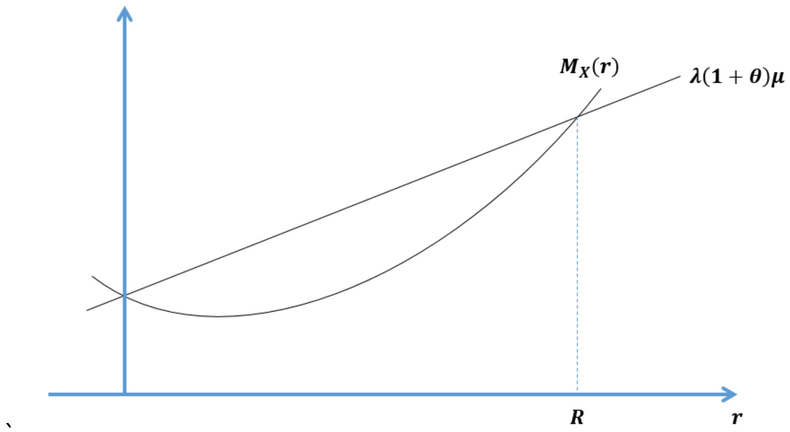


FIGURE 2 – ADJUSTMENT COEFFICIENT.

We can see that equation (2.5.1) has only one positive root, that is, if we let (2.5.1) equal to $h(r)$ with $h(0) = 0$. Now taking derivatives, we conclude that

$$h'(r) = c - \sigma^2 r - \lambda M_X'(r) \Longrightarrow h'(0) = c - \lambda\mu_1 > 0,$$
$$h''(r) = -\sigma^2 - \lambda M_X''(r) < 0,$$

the function is a maximum and concave down. The concavity implies that the $\lim_{r\to\eta} h(r) \to -\infty$, hence, (2.5.1) has two roots, $R > 0$ and a trivial solution. In numerical analysis (see Section 4), we could use the Newton-Raphson method to find successively better approximations, such that $r = R$ to the real-valued function, $h(r)$. Note that $R < r_0 = \frac{2\theta\mu_1}{E(X^2)}$ is a good "guess" to start with to calculate the adjustment coefficient, see CENTENO (2015). Thus, using the iterative process (assuming functions h and h' are defined over the real numbers r), we can obtain a sufficiently accurate value, given by:

$$r_{n+1} = r_n - \frac{h(r_n)}{h'(r_n)}, \qquad n = 0,1,2,\ldots$$

Thus, we can derive a simple upper bound using Lundberg's inequality, which is:

$$\Psi(u) = \Psi_d(u) + \Psi_c(u) < e^{-Ru}, \quad u > 0. \tag{2.5.2}$$

We develop interesting results in for sections 2.6 and 3.3 for a mixture of discrete $\Psi(s)$, where we apply the LT to decompose a PK formula for the ultimate ruin. See APPENDIX C and D for convolution and LT properties, respectively.

2.6 Maximal Aggregate Loss

We present the maximal aggregate loss variable. Let us first define some key components for the later sections of this thesis:

DEFINITION 4. *Let L be the random variable representing the maximal aggregate loss, such that the process $\{L(t); t \geq 0\}$ with $L(t) = u - V(t)$ with probability distribution function*:

$$F_L(u) = P(L \leq u) = P(L(t) \leq u, \forall t \geq 0) = P(V(t) \geq 0, \forall t \geq 0) = \phi(u). \tag{2.6.1}$$

The decomposition of $L(t)$ can be found in DUFRESNE AND GERBER (1991) and updated in SEIXAS AND EGÍDIO DOS REIS (2013), and contrary to the Cramer-Lungberg model, the survival probability $\phi(u)$ at the point $u = 0$ is equal to $\phi(0) = F_L(0) = q = \theta/(1+\theta)$.

Working with the perturbed Brownian process in (2.3.1) and setting $\sigma = 1$ for simplicity sakes, the decomposition yields:

$$L = \max\{u - V(t)\} = \max\{S(t) - ct - W(t)\} = L_0^{(1)} + \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right), \tag{2.6.2}$$

where, $L_i^{(1)}$ and $L_i^{(2)}$ are record highs due to oscillation and claim occurrences and are part of the maximal aggregate loss distribution, and M is the number of records of L_t that are due to a claim and follows a geometric distribution, with parameter q and probability function $m_k = P(M = k) = q(1-q)^k,\ k = 0,1,2,\ldots$

THEOREM 1. *The sequences, $\left\{L_i^{(1)}\right\}_{i\geq1}$ and $\left\{L_i^{(2)}\right\}_{i\geq1}$ are $i.i.d.$ random variable, with common distribution functions $H_1(\,.\,)$ and $H_2(\,.\,)$, respectively, and defined by:*

$$L_i^{(1)} = \max(L_t, t < t_{i+1}) - L_{t_i}, \;\; i = 0,1,\dots,M, \tag{2.6.4}$$

$$L_i^{(2)} = \left(L_{t_i} - L_{t_{i-1}}\right) - L_{i-1}^{(1)}, \;\; i = 0,1,\dots,M. \tag{2.6.5}$$

Density functions for $L_i^{(1)}$ and $L_i^{(2)}$ are given by $h_1(\,.\,)$ and $h_2(\,.\,)$, respectively.

PROOF. See SEIXAS AND EGÍDIO DOS REIS (2013).

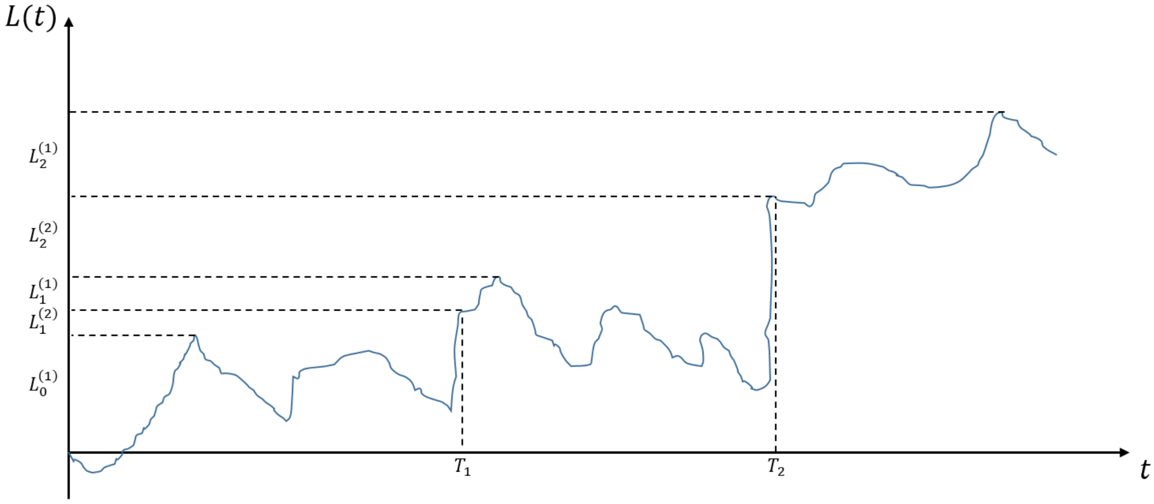


FIGURE 3 – DECOMPOSITION OF THE MAXIMAL AGGREGATE LOSS

REMARK 1. *Obviously, the maximal aggregate loss random variable, L is a compound geometric distribution, with CDF*

$$F_L(u) = \phi(u) = \textstyle\sum_{k=0}^{\infty} q(1-q)^k H_1^{*(k+1)} * H_2^{*k}(u). \tag{2.6.6}$$

THEOREM 2. *Let the moment generation function of L be given by*

$$M_L(r) = \frac{r\tau(c-\lambda\mu_1)}{c\tau(r(\tau-r)-\lambda M_X(r)-1)}, \;\; \sigma > 0. \tag{2.6.7}$$

PROOF. See APPENDIX E. ■

The central moments of L can be deduced (if they exist) from the special properties of compound distributions and can be seen in APPENDIX B, also for instance, see SEIXAS AND EGÍDIO DOS REIS (2013) or JACINTO (2008).

We extend this in another way, that is, by using the PK formula, i.e. the same formula in (2.6.6), to compute the ultimate ruin, where $F_L^{*n}(u)$ is the n-fold transform of the aggregate loss distribution (see SECTION 3.3).

2.6.1 *A standard α-stable Lévy Process*

The biggest drawback in the original perturbed model by DUFRESNE AND GERBER (1991) was that the Brownian motion ($\alpha = 2$) was not adequate to model large fluctuations and variations, hence a further generalization was proposed by FURRER (1998), refer to (2.3.1). By modifying the process and adding a new parameter α, one can then change the variability of the process. Hence the smaller the α, the greater the changes in fluctuations, and so, the nicer it behaves. For the process in (2.3.1), the subsequent formula was proved in FURRER (1998) for $\phi(u)$, that is:

THEOREM 3. *Let the risk process in (2.3.1) and CDF of L in (2.6.6) have parameters* $\alpha \in (0,2)$ *and* $\beta = -1$ *(allows for negative jumps), then the ultimate non-ruin probability satisfies:*

$$\phi(u) = \sum_{k=0}^{\infty} q(1-q)^k F_L^{*n} * H_1^{*(n+1)}(u)\,, \quad u \geq 0 \tag{2.6.9}$$

where, β *is the jump parameter and* F_L^{*n} *is* n*-fold transform of the aggregate loss distribution and belongs to a class of sub-exponential distributions with CDF,* $F_L^{*n}(u) = H_2(u) = \mu_1^{-1} \int_0^u [1 - F_X(t)]\, dt$.

The choice of β guarantees that the process avoids positive jumps, and this, makes the perturbed process (2.3.1) a negative Lévy process for which "passage times have nicer expressions".

THEOREM 4. *By taking LT on the ultimate survival probability in (2.6.9), and applying the defective renewal technique in (2.4.4),* FURRER (1998) *deduced that:*

$$H_1(u) = 1 - \sum_{n=0}^{\infty} \left[u^{(\alpha-1)n} \left(-\frac{c}{\eta^{\alpha}}\right)^n \cdot [\Gamma(1 + (\alpha-1)n)]^{-1} \right],\ u \geq 0, \tag{2.6.10}$$

$$H_2(u) = \frac{\int_0^u [1-F_X(t)]dt}{\int_0^{\infty} [1-F_X(t)]dt} = \mu_1^{-1} \left(\int_0^u [1 - F_X(t)]\, dt \right), \quad u \geq 0. \tag{2.6.11}$$

where, $\Gamma(\,.\,)$ *is the gamma function.*

Note, when $\alpha = 2$ and $n \to \infty$, then (2.6.10) and (2.6.11) reduce to the Brownian definition. Moreover, results above agree with ZOLOTAREV (1964) in the following theorem:

THEOREM 5. *Let Y be an α-stable process with no jumps in the positive direction, $\gamma = E[Y(1)] \geq 0$ and $\Psi(u) = P(\inf\{Y(t) < -u\})$, then:*

$$\Psi^*(s) = \int_0^\infty e^{-st}\Psi(t)\,dt = \frac{1}{s}\left(1 - \frac{\gamma}{\Psi(s)}\right), \quad s, t \geq 0. \qquad (2.6.12)$$

The integral part of the equation encompassing $\Psi(u)$ in (2.6.12) is nothing other than a Laplace transform [see APPENDIX D] and can be manipulated to obtain (2.6.9). In a ruin context, (2.6.9) represents the perturbed aggregate claim minus the premium components, in mathematical language:

$$Y(t) = S(t) - \eta Z_\alpha(t) - ct, \quad t \geq 0. \qquad (2.6.13)$$

Moreover, SCHMIDLI (2001) noted that the perturbed process for an α-stabled Lévy process from the work in FURRER (1998) follows the same decomposition as (2.6.2), (2.6.4) and (2.6.5). He then showed that part of the ladder height due to the perturbation ($L^{(1)}$) becomes an exponential distribution when $\alpha = 2$, and part of the ladder height due to a claim ($L^{(2)}$) has the same distribution $F_L(x) = \phi(x)$ irrespective of the perturbation method applied. This fully agrees with the deductions made in DUFRESNE AND GERBER (1991).

2.7 Cramer's Asymptotic Result for Ruin Probabilities

FURRER *ET. AL* (1997) spent considerable periods of time concerning finite ruin probabilities results for α-stable processes. In the end, they could present the following theorem:

THEOREM 6. *Let Z_α be an α-stable process with parameter $|\beta| < 1$, and as $u \to \infty$, then the real-world probability measure*

$$\mathbb{P}[u + cs - \sigma^{-\alpha} Z_\alpha(s) \leq t] \sim \frac{\lambda t}{2} C_\alpha (1+\beta)(u + ct)^{-\alpha},$$

where, $C_\alpha = (1-\alpha)\left[\Gamma(2-\alpha)\cos\left(\frac{1}{2}\pi\alpha\right)\right]^{-1}$ *is a constant.*

THEOREM 6 leads us to upper bounds for the ruin probability under an α-stable process.

THEOREM 7. *Let $\alpha = 2$ (a Gaussian), we can obtain an asymptotic expression by renewal reasoning from (2.5.2), such that*

$$\psi_d(u) \sim C_d e^{-Ru}, \qquad \psi_c(u) \sim C_c e^{-Ru}, \qquad u \to \infty,$$

where, C_d and C_c are constants, with $C = C_d + C_c \leq 1$. The notation $f_1(x) \sim f_2(x), x \to \infty$, means $\lim_{x\to\infty} f_1(x)/f_2(x) = 1$. For significantly large values of initial surplus, u, we obtain:

$$C_d = \left[\int_0^\infty e^{Rx}[1 - H_1(x)]\,dx\right] \times \left[(1-q)\int_0^\infty xe^{Rx} h_1 * h_2\,dx\right]^{-1},$$

$$C_c = \left[\int_0^\infty e^{Rx}(1-q)[H_1(x) - H_1 * H_2(x)]\,dx\right] \times \left[(1-q)\int_0^\infty xe^{Rx} h_1 * h_2\,dx\right]^{-1}.$$

Proof. This is fully developed in Cai and Garrido (2002) and utilises the key renewal theorem with defective equations for $\psi(u)$, $\psi_d(u)$ and $\psi_c(u)$. ■

2.8 Infinitely Divisible Distributions & Lévy Characterization

This final section will be purely conceptual, and will provide some further answers and developments to the α-stabled process introduced in (2.3.1). Definitions and concepts here will be discussed and applied in Sections 3 and 4.

Consider two well-known distributions that are infinitely divisible (ID): The Poisson and the Gaussian (see Appendix B). We can verify this using the following definition:

Definition 5. *Let the law ρ be called ID, such that for any positive integer n, there exists a probability measure ρ_n such that*

$$\varphi_\rho(u) = \left[\varphi_{\rho_n}(u)\right]^n \tag{2.8.1}$$

where, ρ is the n-th convolution power of ρ_n.

We present the following theorem thanks to Lévy which fully characterises the family of ID distributions.

Theorem 8. *For every ID distribution ρ, then* Ito (1969) *deduces that equation (2.8.1) can be expressed by*

$$\varphi_\rho(u) = \exp\left\{-\widetilde{\Psi}_\rho(u)\right\}, \quad u \in \mathbb{R}, \tag{2.8.2}$$

where, $\widetilde{\Psi}_{\rho}(u) = iau + \frac{b^2}{2}u^2 + \int_{\mathbb{R}}\left[1 - e^{ius} + ius \cdot I(u)\right]\nu(ds),\ a \in \mathbb{R}, b^2 > 0,\ \nu$ *is a measure on* $\mathbb{R}_0 = \mathbb{R} - \{0\}$*, i is the imaginary number* $\sqrt{-1}$ *and I is the indicator function. The component* ν *is defined as the Lévy measure.*

ZOLOTAREV (1986) discussed that these ID α-stabled distributions are achieved as limits of normalized sums of $i.i.d.$ random variables. As defined in APPENDIX A, α-stabled distributions do not have a closed form density function (unless α takes values 0.5, 1 or 2), but it's $\widetilde{\Psi}$ is found to be:

$$\widetilde{\Psi}(u) = \mathrm{c}|u|^{\alpha}[1 - i\beta \mathrm{sgn}(u)\tan(\alpha\pi/2)] + imu,\ \ \alpha \in (0,1) \cup (1,2), \quad (2.8.3)$$

with (2.8.3) satisfying (2.8.2), where $\mathrm{sgn}(u) \coloneqq \{-1 \text{ if } u < 0; 0 \text{ if } u = 0; 1 \text{ if } u > 0\}$. As the initial reserve $u > 0$, then we can conclude that $\mathrm{sgn}(u) = 1$. See APPENDIX B for full characterisation when $\alpha = 1$ or $\alpha = 2$.

3 APPROXIMATION TECHNIQUES

Mathematics, rightly viewed, possesses not only truth, but supreme beauty.

Bertrand Russell

3.1 *Constructing Bounds*

SILVA (2006) and SEIXAS AND EGÍDIO DOS REIS (2013) defines key random variables from the maximal aggregate loss random variables using (2.6.2), such that:

$$L^j = L_0^{j,(1)} + \sum_{i=1}^{M} \left(L_i^{j,(1)} + L_i^{j,(2)} \right),$$

with $L_0^{j,(1)} \equiv L^j$ if $M = 0$, and $j = \{-,+\}$. $L_i^{j,(k)}$ must be concentrated on a lattice $\vartheta\mathbb{N}_0 = \{0, \vartheta, 2\vartheta, 3\vartheta, \dots\}$, where the lattice width $\vartheta > 0$. In this application, $L_i^{-,(k)} = \vartheta\left\lfloor L_i^{(k)}/\vartheta \right\rfloor$, $L_i^{+,(k)} = \vartheta\left\lfloor \left(L_i^{(k)} + \vartheta\right)/\vartheta \right\rfloor$ for $\{k = 1, i = 0, \dots, M\}$ and $\{k = 2, i = 0, \dots, M\}$. Hence, each summand of L approximates the lower and upper multiples of ϑ, such that $L^- < L < L^+$. This leads to

$$\Psi^-(v) \le \Psi(v) \le \Psi^+(v), \qquad v = 0,1,2, \dots,$$

where $\Psi^j(v) = 1 - P\left(L^j \le v\right)$ for $j = \{-,+\}$.

3.1.1 *Constructing an Arithmetic Distribution*

We need to derive the density functions of the random variable $L_i^{j,(1)} + L_i^{j,(2)}$, denoted by $p_n^j(\,.\,)$, with $j = \{-,+\}$. In actuarial practice, discretization of claims (i.e. lower and upper) are useful, and hence, denoting $L_i^{j,(3)} = L_i^{j,(1)} + L_i^{j,(2)}$ the lower and upper difference, with a suitably small ϑ, are given by

$$p_n^- = P\left(L_i^{-,(3)} = \vartheta n\right) = \begin{cases} [H_1 * H_2](\vartheta), & n = 0, \\ [H_1 * H_2]\left(\vartheta(n+1)\right) - [H_1 * H_2](\vartheta n), & n = 1,2, \dots, \end{cases}$$

$$p_n^+ = P\left(L_i^{+,(3)} = \vartheta n\right) = \begin{cases} 0, & n = 0, \\ [H_1 * H_2](\vartheta n) - [H_1 * H_2]\left(\vartheta(n-1)\right), & n = 1,2, \dots, \end{cases}$$

where, the convolution $[H_1 * H_2](x) = \int_0^x H_1(x-t)h_2(t)\,dt = \int_0^x h_1(t)H_2(x-t)\,dt$ is concentrated on positive numbers, i.e. from $(0, +\infty)$, see APPENDIX C (since taking limits from $(-\infty, +\infty)$ will result in divergence). Note that the CDF of $L_i^{j,(3)}$ is suitably arithmetic. The pdf for L^- and L^+, can be deduced by using Panjer's recursion formula, for a compound geometric distribution, such that $g_n^j(\vartheta n) = P(L^j = \vartheta n)$ for $n = 0,1,2,...$ and $j = \{-,+\}$, see for instance: HESS AND SCHMIDT (2002).

3.1.2 *The Recursion Formula*

DEFINITION 6 (Panjer's recursion). *Since the frequency distribution M is a member of the $(a, b, 0)$ class of distributions, i.e. a Geometric with parameter q, and $L_i^{j,(3)}$ takes values on the non-negative integers, then the pdf of L^j satisfies*

$$g_n^j = \frac{1-q}{1-(1-q)p_0^j}\sum_{i=1}^{n} p_i^j g_{n-i}^j, \qquad n = 1,2,...$$

$$g_0^j = P_M(p_0^j) = \frac{qp_0^j}{1-(1-q)p_0^j},$$

where, $P_M(p_0^j)$ is the probability generating function of M at the point $p_0^j = P\left(L_i^{j,(3)} = 0\right)$, for more details see PANJER (1981).

Recall from (2.6.6) that L has cdf $\sum_{k=0}^{\infty} g_k$, by applying Panjer's recursion for $n \geq 0$ (see SILVA (2006) and KLUGMAN *ET. AL* (2012)), we obtain the following compound probabilities:

LOWER BOUND.

$$g_0^- = \frac{qp_0^-}{1-(1-q)p_0^-}, \qquad g_n^- = \frac{1-q}{1-(1-q)p_0^-}\sum_{i=1}^{n} p_i^- g_{n-i}^-, \qquad n = 1,2,...$$

UPPER BOUND.

$$g_0^+ = 0, \qquad g_n^+ = (1-q)\sum_{i=1}^{n} p_i^+ g_{n-i}^+, \qquad n = 1,2,...$$

Evaluating at these bounds prove to be useful when testing the accuracy of other approximations for the cases where we do not have exact results for $\psi(u)$. Regarding key formulae and assumptions from KLUGMAN *ET AL.* (2012), we conclude with the following representation $\Psi^{-}_{D\&G}(\vartheta m) \leq \Psi(\vartheta m) \leq \Psi^{+}_{D\&G}(\vartheta m)$ for $m = 0,1,2,\ldots$

3.2 A classical De Vylder approximation

3.2.1 Four-moment exponential approximation

The main idea behind De Vylder's approximation technique was to replace the classical risk process (characterized as $U(t)$) with a new process (and new parameters) by using a three-moment exponential approximation (say, $U_{3E}(t)$), with mean $1/\beta$. See more in DE VYLDER (1978). Now consider a new perturbed process, characterized by replacing $V(t)$ with a four-moment approximation, $V_{4E}(t)$. For convenience sake, it may be better to work with an aggregate loss process, $\{S_{4E}(t): t \geq 0\}$, where $S_{4E}(t)$ is a new compound Poisson process with parameter λ_* and the claim amount distribution $X_{4E}(t)$, follows an exponential with mean $1/\beta$. The central moments of $V(t)$, given by ν_k, are found in Appendix B and the ordinary moments of an exponential are found by:

$$E\left[X_{4E}^k(t)\right] = \frac{\Gamma(1+k)}{\beta^k}, \qquad \text{for } k = 1,2,\ldots,n,$$

where, $\Gamma(.)$ is the gamma function. Matching the central moments by the relationship $\nu_k = \nu_{k,4E}$, for $k = 1,2,3,4$; simplifying expressions, we have a system of equations that must satisfy:

$$ct - \lambda t\mu_1 = c_* t - \frac{\lambda_* t}{\beta}, \qquad \sigma^2 t + \lambda t\mu_2 = \sigma_*^2 t + \frac{2\lambda_* t}{\beta^2}, \qquad -\lambda t\mu_3 = \frac{-6\lambda_* t}{\beta^3},$$

$$6t\sigma^4 + 6t\lambda\sigma^2\mu_2 + 3t\lambda^2\mu_2^2 + \lambda\mu_4 = 6t\sigma_*^4 + \frac{12t\lambda_*\sigma_*^2}{\beta^2} + \frac{12t\lambda_*^2}{\beta^4} + \frac{24\lambda_*}{\beta^4}.$$

We find that the solutions here match those from SEIXAS AND EGÍDIO DOS REIS (2013):

$$\lambda_* = \frac{32}{3}\lambda \cdot \frac{\mu_3^4}{\mu_4^3}, \qquad c_* = \lambda\left(\frac{8}{3}\cdot\frac{\mu_3^3}{\mu_4^2} + \theta\mu_1\right),$$

$$\beta = 4\cdot\frac{\mu_3}{\mu_4}, \qquad \sigma_* = \sqrt{\lambda\left(\mu_2 - \frac{4\mu_3^2}{3\mu_4}\right) + \sigma^2}.$$

Now, we will apply the theorem presented by DUFRESNE AND GERBER (1991) on calculating ultimate ruin probability formula:

THEOREM 9. *If the claim amount distribution is from a combination of a family of exponentials with pdf $f(x) = \sum_{k=1}^{n} A_k f_X(x)$, with parameters β_k, where $\sum_{k=1}^{n} A_k = 1$, then the exact ruin probability in infinite time is given by*

$$\Psi_{4E}(u) = \sum_{k=1}^{n+1} C_k e^{-r_k u}, \; u \geq 0, \tag{3.2.1}$$

where,

$$C_k = \prod_{j=1}^{n} (r_k - \beta_j)/\beta_j \cdot \prod_{\substack{j=1 \\ j \neq k}}^{n+1} r_j/(r_k - r_j),$$

for $k = 1, \dots, n+1$ with $\sum_{k=1}^{n} C_k = 1$ and $r_1, r_2, \dots, r_n$ being the solutions of

$$\frac{A_1}{\beta_1 - r} + \frac{A_2}{\beta_2 - r} + \cdots + \frac{A_n}{\beta_n - r} = \frac{2c_* - \sigma_*^2 r}{2\lambda_*}.$$

PROOF. See DUFRESNE AND GERBER (1991).

REMARK 2. Setting $n = 1$ implies that the claim size is an exponential with parameter β and $A = 1$. Hence, for a mixture of exponentials, our approximation (also the exact) is given by $\Psi_{4E}(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u}$, where,

$$C_1 = \frac{r_1 - \beta}{\beta} \frac{r_2}{r_1 - r_2}, \qquad C_2 = \frac{r_2 - \beta}{\beta} \frac{r_1}{r_2 - r_1}, \qquad \frac{\lambda_*}{\beta - r} = c_* - \frac{1}{2}\sigma_*^2 r,$$

leading to:

$$r_{1,2} = \frac{2c_* + \beta\sigma_*^2 \pm \sqrt{4(c_*^2 - \beta c_* \sigma_*^2 + 2\lambda_* \sigma_*^2) + \beta^2 \sigma_*^4}}{2\sigma_*^2}.$$

REMARK 3. Setting $n = 3$ satisfies the claim distribution presented in SECTION 4.3 by CRAMÉR (1955) to explain a distribution for a Swedish non-industry fire insurance, where we a mixture of exponentials with parameters β_i and $\sum A_i = 1$ for $i = 1,2,3$. Hence, our ruin probability approximation is given by:

$$\Psi_{4E'}(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u} + C_3 e^{-r_3 u} + C_4 e^{-r_4 u}, \tag{3.2.2}$$

where,

$$C_k = \frac{(r_k - \beta_1)(r_k - \beta_2)(r_k - \beta_3)}{\beta_1 \beta_2 \beta_3} \times \prod_{\substack{j=1 \\ j \neq k}}^{4} \frac{r_j}{r_k - r_j}, \qquad \frac{A_1}{\beta_1 - r} + \frac{A_2}{\beta_2 - r} + \frac{A_3}{\beta_3 - r} = \frac{2c_* - \sigma_*^2 r}{2\lambda_*}.$$

Solutions for r cannot be deduced analytically hence we will solve these numerically.

THEOREM 10. *If the claim amount distribution is of the same form as the pdf $f(x) = \sum_{k=1}^{n} A_k f_X(x)$, with paramenters γ_k and β_k where $\sum_{k=1}^{n} A_k = 1$, then the exact ruin probability due to oscillation must clearly be*

$$\Psi_{4E,d}(u) = \sum_{k=1}^{n} C_k^d e^{-r_k u}, \; u \geq 0, \tag{3.2.3}$$

where $C_j^d = \frac{2(1+\bar{\theta})\mu_1 \bar{\lambda}}{\bar{\theta}(1+\bar{\theta})\bar{\sigma}^2} \cdot C_j r_j$ for $j = 1, \dots, n+1$. Then the exact ruin probability due to a claim can be computed by subtracting (3.2.3) from (3.2.1).

PROOF. Again, see DUFRESNE AND GERBER (1991).

3.2.2 Gamma approximation

Now we present a five-moment gamma, where we replace the risk process with another one for which the ultimate ruin probability is explicit. Like the exponential case, our process is characterized by replacing $V(t)$ (standard Brownian motion case $\alpha = 2$) with $V_{5G}(t)$. Other new processes include, $S_{5G}(t)$ with parameter λ_* and the claim amount distribution $X_{5G}(t)$, follows a gamma with shape and scale parameters, γ and β, respectively. The central moments of $V(t)$, given by v_k, are found in APPENDIX B and the ordinary moments of a gamma can be found by:

$$E\left[X_{5G}^k(t)\right] = \frac{\gamma(1+\gamma)(2+\gamma)\dots(k-1+\gamma)}{\beta^k}, \qquad \text{for } k = 1,2,\dots,n.$$

Matching the moments by the relationship $v_k = v_{k,5G}$, for $k = 1,2,\dots,5$; simplifying expressions, we have a system of equations, see TABLE 1.

k	v_k	$v_{k,5G}$
1	$\theta\lambda t\mu_1$	$\theta_*\lambda_* t\beta^{-1}\gamma$
2	$\lambda t\mu_2 + \sigma^2 t$	$\lambda_* t\beta^{-2}\gamma(1+\gamma) + \sigma_*^2 t$
3	$-\lambda t\mu_3$	$-\lambda_* t\beta^{-3}\gamma(1+\gamma)(2+\gamma)$
4	$\lambda t\mu_4 + 3\sigma^4 t^2 + 3(\lambda t\mu_2 + \sigma_*^2 t)^2$	$\lambda_* t\beta^{-4}\gamma(1+\gamma)(2+\gamma)(3+\gamma) + 3\sigma_*^4 t^2 + 3(\lambda_* t\beta^{-2}\gamma(1+\gamma) + \sigma_*^2 t)^2$
5	$-\lambda t\mu_5 - 10\lambda t\mu_3(\lambda t\mu_2 + \sigma^2 t)$	$-\lambda_* t\beta^{-5}\gamma(1+\gamma)\times \ldots \times(4+\gamma) - 10\lambda_* t\beta^{-3}\gamma(1+\gamma)(2+\gamma)(\lambda_* t\beta^{-2}\gamma(1+\gamma) + \sigma_*^2 t)$

TABLE 1 – MOMENTS FOR 5MGDV APPROXIMATION.

As the solution is very hard to obtain analytically, we must modify some parameters. Note that estimating σ_* required a fifth moment and equation. Nevertheless, we will still evaluate the ruin probability for a 5-moment gamma numerically, for instance, SECTION 3.2 for techniques used to obtain the ruin probability approximation. In the sequel, we now match the first four moments and assume σ is fixed, hence our system of equations now satisfies $v_k = v_{k,4G}$, see TABLE 2.

k	v_k	$v_{k,4G}$
1	$\theta\lambda t\mu_1$	$\theta_*\lambda_* t\beta^{-1}\gamma$
2	$\lambda t\mu_2$	$\lambda_* t\beta^{-2}\gamma(1+\gamma)$
3	$-\lambda t\mu_3$	$-\lambda_* t\beta^{-3}\gamma(1+\gamma)(2+\gamma)$
4	$\lambda t\mu_4 + 3(\lambda t\mu_2 + \sigma^2 t)^2$	$\lambda_* t\beta^{-4}\gamma(1+\gamma)(2+\gamma)(3+\gamma) + 3(\lambda_* t\beta^{-2}\gamma(1+\gamma) + \sigma^2 t)^2$

TABLE 2 – MOMENTS FOR 4MGDV APPROXIMATION.

Solving this system analytically yields (*Case 1*):

$$\theta_* = \pm\frac{\theta\mu_1(2\mu_3^2 - \mu_2\mu_4)}{\mu_2^2\mu_3}, \qquad \lambda_* = \frac{\lambda\mu_2^3\mu_3^2}{6\mu_3^4 - 7\mu_2\mu_3^2\mu_4 + 2\mu_2^2\mu_4^2},$$

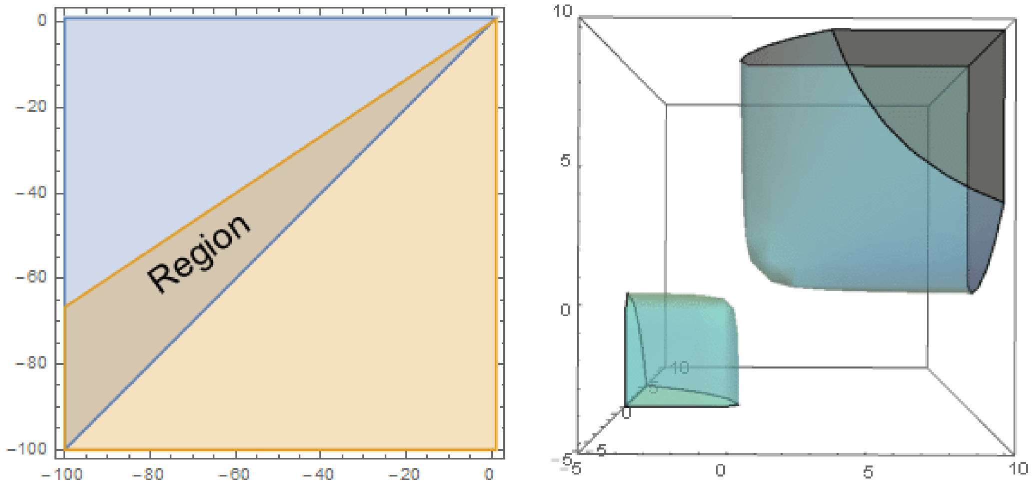
$$\gamma = \frac{2\mu_2\mu_4 - 3\mu_3^2}{\mu_3^2 - \mu_2\mu_4}, \qquad \beta = \frac{\mu_2\mu_3}{\mu_2\mu_4 - \mu_3^2},$$

with the assumptions $\mu_3^2 > \mu_2\mu_4$, $2\mu_2\mu_4 > 3\mu_3^2$ and $\mu_2\mu_3 < 0$ to ensure that $\gamma, \beta > 0$. If this condition cannot be fulfilled, then simply we do not calculate μ_4 and solve up to the the third moment. This leads to $\lambda_* = \lambda$, and (*Case I1*):

$$\theta_* = \frac{\theta\mu_1(\mu_3 \pm \sqrt{8\mu_2^3 + \mu_3^2})}{4\mu_2^2}, \quad \gamma = \frac{-4\mu_2^3 + \mu_3(\mu_3 \mp \sqrt{8\mu_2^3 + \mu_3^2})}{2(\mu_2^3 - \mu_3^2)},$$

$$\beta = \frac{\mu_2(-3\mu_3 \mp \sqrt{8\mu_2^3 + \mu_3^2})}{2(\mu_2^3 - \mu_3^2)},$$

with the strict assumption that $\mu_2^3 > \mu_3^2$.



(a) (b)

FIGURE 4 – CASE I. REGION OF ACCEPTABLE VALUES FOR 4MGDV PARAMETERS.

THEOREM 11. *If the claim amount distribution is from an exponential or gamma distribution, then the 4-moment gamma De Vylder approximation is given by:*

$$\Psi_{4G}(u) = \frac{\theta_* \left(1 - \frac{R}{\gamma}\right) \exp\left\{-\frac{\beta}{\gamma} Ru\right\}}{1 + (1 + \theta_*)R - (1 + \theta_*)\left(1 - \frac{R}{\gamma}\right)} + \frac{\gamma\theta_* \sin(\gamma\pi)}{\pi} \cdot I,$$

where $I = \int_0^\infty \frac{x^\gamma \exp\{-(x+1)\beta u\}}{[x^\gamma\{1+\gamma(1+\theta_*)(x+1)\} - \cos(\gamma\pi)]^2 + \sin^2(\gamma\pi)} dx$ *and* R *is the adjustment coefficient defined in Section 2.*

PROOF. See an extensive proof in GRANDELL AND SEGERDAHL (1971).

GRANDELL (2000) demonstrated that the method above is said to give the exact ruin probability for exponential or gamma claims, and very good approximations for other distributions with the first four moments being finite. Furthermore, BURNECKI *ET. AL* (2003) used numerical illustrations to prove that this method gives a slight improvement to the ruin probability proposed by De Vylder, famous for being the "best" among usual approximation techniques. In many cases, it may be difficult to numerically solve I for certain distributions, so it is usually best to use (3.2.1). Under the same conditions, we could also obtain the ruin probability due to oscillation.

3.3 Pollaczek-Khinchine approximations

The PK formula was first published in POLLACZEK (1930), where a major study on queueing theory was conducted; this formula explains the relationship between the queue length and a time distribution, by taking Laplace transforms for an M/G/1 queue (i.e. where jobs follow a Poisson process). In ruin theory, this formula is sought to calculate the ultimate ruin probability, and so, one can use this formula to derive explicit solutions for $X(t)$, see PANJER AND WILLMOT (1993). In this section, we consider a new derivation of De Vylder by considering the first two moments (obtained from a Padé approximation) and a Renyi approximation (based of Beekman-Bower's approximation), see AVRAM *ET. AL* (2011).

3.3.1 Preliminaries

We begin with some preliminaries. First, consider taking the LT of our perturbed model in equation (2.3.1). This yields:

$$V^*(s) = s\left(c - \lambda\big(1 - F^*(s)\big) + \frac{\sigma^2}{2}\right) \tag{3.3.1}$$

where, the LT of $Z_2(t) = W(t)$ (in the standard Brownian motion case) is $s\frac{\sigma^2}{2}$ and $F^*(s)$ is the LT of the claims distribution. Recall that the superscript $(*)$ denotes the LT. Hence, the PK formula can be deduced by taking the LT of the Kolmogorov equation for $\psi(u)$ for perturbed model, and thus yields:

$$\Psi^*(s) = \frac{1}{s} - \frac{V^{*\prime}(0)}{V^*(s)} = \frac{(1-q)\big(1-f_e^*(s)\big)+\frac{s\sigma^2}{2c}}{s\left(1-(1-q)f_e^*(s)+\frac{s\sigma^2}{2c}\right)} = \frac{1}{s}\big(1 - \psi^*(s)\big), \tag{3.3.2}$$

where, $1 - F_L(s) = \Psi(s)$ is the survival function of the aggregate loss L, $f_e(x) = h_2(x) = \mu_1^{-1}\big(1 - F(x)\big)$ is the equilibrium density for X and $\breve{\psi}(.)$ is the ruin density function. The survival LT function $1 - F^*(s)$ emphasizes that the result in the perturbed case depends only on $f_e(x)$ for X. A slight modification of result from (2.6.8) that replaces r with $-s$ to form a LT coinciding with the following PK formula for the transformed density $\psi^*(s)$:

$$\psi^*(s) = E[e^{-sL}] = E\left[\exp\left\{-s\left(L_0^{(1)} + \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right)\right)\right\}\right] = 1 - s\Psi^*(s)$$

$$\Leftrightarrow \psi^*(s) = s\frac{\mathcal{L}'(0)}{\mathcal{L}(s)} = \frac{q}{1-(1-q)f_e^*(s)+\frac{s\sigma^2}{2c}}. \tag{3.3.3}$$

Letting $\sigma = 0$ gives a beautiful rendition of the non-perturbed PK where (3.3.3) can be expanded into a geometric series, yielding:

$$\psi^*(s) = \frac{q}{1-(1-q)f_e^*(s)} = q\sum_{k=0}^{\infty}[(1-q)f_e^*(s)]^k. \tag{3.3.4}$$

The rationalization behind this is that $\psi^*(s)$ is revealed to be the LT of a geometric sum on convolutions of the equilibrium distribution (see APPENDICES C and E), which agrees with the theory in SECTION 2.6. Similarly, we get an identical result to (2.6.2) for the "ladder decomposition" L and M is the number of records of L, which follows a geometric random variable, $m_k = P(M = k) = q(1-q)^k,\ k = 0,1,2,\ldots$ (see APPENDIX E).

REMARK 4. *Using results from* SECTION 2.6 *(maximal aggregate loss as a mixture of a discrete mass of* $\Psi(0) = \psi^*(\infty) = q$*) and the LT of* $\psi(u)$ *in (3.1.3), we can conclude the following decomposition:*

$$\psi^*(s) = q + (1-q)\breve{\psi}^*(s) \Leftrightarrow \psi(s) = q\sigma_0(s) + (1-q)\breve{\psi}(s)$$

$$\Leftrightarrow \frac{\psi^*(s)-\psi^*(\infty)}{\psi^*(s)-\psi^*(0)} = \frac{qf_e^*(s)}{1-(1-q)f_e^*(s)} = \psi^*(s)f_e^*(s), \tag{3.3.5}$$

where $\sigma_0(s)$ *represents the drift function in terms of LTs and* $\breve{\psi}^*(s)$ *is the LT of the density ruin function defined in (3.3.2). The behavior of plus infinity differentiates between the perturbed* $(\sigma > 0)$ *and non-perturbed* $(\sigma = 0)$ *case:*

$$\lim_{s\to\infty}\psi^*(s) = \lim_{s\to\infty}[1 - s\Psi^*(s)] = \begin{cases} q, & \sigma = 0 \\ 0, & \sigma > 0 \end{cases}$$

Hence, considering ruin theory, the LT of ψ, i.e. ψ^*, is a fundamental quantity in the theory of Lévy processes.

Henceforth, we will assume a standard Brownian perturbation ($\alpha = 2$), and we will directly approximate the aggregate loss distribution L. As the moments for L are already defined in APPENDIX E, we will go one step further and obtain factorial reduced moments, which are found by normalizing with respect to the exponential moments of L. Please see APPENDIX E again for deduction of these new moments.

In the next sub-sections, we move on to the approximation techniques under Renyi, and De Vylder, where we make some observations that (3.3.1) is a case of Padé approximant of Laplace transforms.

DEFINITION 7 (Padé Approximant). *Let the Padé approximant be given by*

$$(\psi^*)^{(n)}(s) = \wp_{(m,n)}(\psi^*)(s) = \wp_{(m,n)}([f(s)]_N), \tag{3.3.6}$$

where $\wp_{(m,n)}$ denotes the classical Padé approximation based on the Taylor series around zero, with integers $m \geq 0$ and $n \geq 1$, and $[f(s)]_N$ is the truncated formal power series where, Padé approximants can be applied to divergent summated series, with $N = m + n$, see more in APPENDIX E *and* AVRAM *ET. AL* (2011).

The purpose behind this approximant is that the Renyi and De Vylder approximations, are assumed to be the one point Padé approximation of $\Psi^*(s)$ around the "zero-th" Taylor point, of orders $(n-1, n)$ at $n = 1$. We move on to simple cases for our perturbed model.

3.3.2 Renyi approximation

Consider a one-moment Renyi exponential approximation, which is from the family of Ramsay-type approximations of $h_2(x) = f_e(x)$. Since we can consider this as a $\wp_{(n-1,n)}$ approximation of the aggregate loss pdf at $n = 1$, as defined in (3.3.6), which also satisfies the limiting behavior from REMARK 4, then we can obtain an approximation for the Laplace ruin transform:

$$\Psi^*(s) \approx \frac{1-q}{s+c_0} \Leftrightarrow f_e^*(s) \approx \frac{c_0/q}{s+c_0/q},$$

where, $f_e^*(s)$ is the LT of the equilibrium density function of the claims distribution (see 3.3.1) and c_0 is a constant. By matching the first factorial moment where, $\breve{\mu}_k = \frac{\mu_{k+1}}{(k+1)\mu_1}$ are factorial reduced equilibrium moments, (these factorial moments are fully defined in APPENDIX E), this yields $\breve{\mu}_1 = \frac{\mu_2}{2\mu_1} = \frac{q}{c_0} \Leftrightarrow c_0 = q/\breve{\mu}_1$, and so $\Psi^*(s) \approx \frac{1-q}{s+q/\breve{\mu}_1} \Leftrightarrow \Psi_R(u) \approx (1-q)e^{-uq/\breve{\mu}_1}$.

This can also be regarded as a simplified version of the Beekman-Bowers approximation, again see GRANDELL (2000). Hence, this leads us to believe that this method is probably not as good as De Vylder's approximation in SECTION 3.1 since there we matched four moments and here we only matched two.

3.3.3 A new De Vylder approximation to the exponential case

In previous section, De Vylder's approximation was acquired by matching moments the first few moments of the classical risk process ($\sigma = 0$ and $\sigma > 0$) with a new process whose claims distribution followed (i) an exponential, and (ii) a gamma. Now for this case, using the factorial moments of the aggregate loss density (see APPENDIX E), we can show that this approximation coincides with the expansion series given by the $\wp_{(n-1,n)}$ approximation at $n = 1$, as defined in (3.3.6) of the LT, $\Psi^*(s)$. We start by expanding the PK formula deduced in (3.3.1) in power series:

$$\Psi^*(s) = \frac{\frac{\eta_{2,\sigma}}{2!} - s\frac{\eta_3}{3!} + s^2\frac{\eta_4}{4!} - \cdots}{\rho + s\frac{\eta_{2,\sigma}}{2!} - s^2\frac{\eta_3}{3!} + s^3\frac{\eta_4}{4!} - \cdots} \approx \frac{A_0}{s+\beta} \Leftrightarrow \Psi_{PKDV}(u) = A_0 e^{-\beta u},$$

where $\eta_k = \lambda\mu_k, k = 1,2,\ldots$ represent moments of a Lévy measure (with $\eta_{2,\sigma} = \lambda\mu_2 + \sigma^2$), A_0 is a constant and β is exponential parameter. Manipulating the equation above then yields:

$$A_0 s\left(\rho + s\frac{\eta_{2,\sigma}}{2!} - s^2\frac{\eta_3}{3!} + s^3\frac{\eta_4}{4!} - \cdots\right) \approx (s+\beta)\left(s\frac{\eta_{2,\sigma}}{2!} - s^2\frac{\eta_3}{3!} + s^3\frac{\eta_4}{4!} - \cdots\right).$$

Solving this consists of matching coefficients of order $s^k, k = 1,2$, then:

$$O(s):\ A_0\rho = \beta\frac{\eta_{2,\sigma}}{2!}, \qquad O(s^2):\ A_0\frac{\eta_{2,\sigma}}{2!} = \frac{\eta_{2,\sigma}}{2!} - \beta\frac{\eta_3}{3!}.$$

Hence, A_0 and β as a $f(\mu_k, \lambda, \sigma), k = 1,2,3$ yields

$$A_0^* = \frac{3(\lambda\mu_2 + \sigma^2)^2}{3\lambda^2\mu_2^2 + 6\lambda\mu_2\sigma^2 + 2\lambda\rho\mu_3 + 3\sigma^4}, \qquad \beta^* = \frac{6\rho(\lambda\mu_2 + \sigma^2)}{3\lambda^2\mu_2^2 + 6\lambda\mu_2\sigma^2 + 2\lambda\rho\mu_3 + 3\sigma^4}.$$

Thus, substituting terms back into $\Psi^*(s)$; using the Inverse LT yields the ultimate ruin probability.

REMARK 5. *Setting* $\eta_{2,\sigma} \approx \eta_2$ *, where* $\sigma = 0$*, reduces the above results to*

$$A_0^* = \frac{3\lambda\mu_2^2}{3\lambda\mu_2^2 + 2\rho\mu_3}, \qquad \beta^* = \frac{6\rho\mu_2}{3\lambda\mu_2^2 + 2\rho\mu_3}.$$

3.4 Two-point Padé approximation

In this scenario, we consider further information beyond aggregate moments from SECTION 3.3 using the two-point Padé approximations (with special attention brought to Lévy processes) and an integro-differential equation that was briefly touched up in SECTION 2.4. By keeping our notation consistent (from the previous sub-section), we now expand our LT to infinity:

$$\Psi^*(s) = \int_0^\infty e^{st} \sum_{k=0}^\infty \Psi^{(k)}(0) \frac{t^k}{k!}, \tag{3.4.1}$$

simplifying the RHS gives $\Psi^*(s) = \sum_{k=0}^\infty \Psi^{(k)}(0) s^{-(k+1)}$ where

$$\Psi(0) = \begin{cases} 1, & \sigma > 0 \\ \mu_1\lambda/c, & \sigma = 0 \end{cases}$$

is well defined. Now we consider finding an explicit formula for $\Psi(u)$.

3.4.1 Deriving $\Psi(u)$ from integro-differential equations

Consider the following theorems:

THEOREM 12. *The function from (2.4.2), for* $u \geq 0$*, satisfies the equation*

$$\frac{\partial}{\partial u} G(u,y) = \frac{\lambda}{c}\left[G(u,y) - \int_0^u G(u-x,y)dF(x) - \{F(u+y) - F(u)\}\right]. \tag{3.4.2}$$

THEOREM 13. *An explicit formula for* $G(0,y)$ *is given by*

$$G(0,y) = \frac{\lambda}{c}\int_0^y [1 - F(x)]\, dx, \;\; y \geq 0. \tag{3.4.3}$$

PROOF. Both theorems have been proven in the literature by many authors. See KLUGMAN *ET AL.* (2012) for a beautiful rendition of these proofs. ∎

THEOREM 14. *The probability of ultimate survival, defined in* (2.4.4)*, satisfies*

$$\phi'(u) = \frac{\lambda}{c}\left[\phi(u) - \int_0^u \phi(u-x)dF(x)\right], \quad u \geq 0. \tag{3.4.4}$$

PROOF. Again, see KLUGMAN *ET AL.* (2012) ■

The explicit solution is easy to derive by taking the second derivative of $\phi(u)$ and thus obtaining an equation in terms of $\phi'(u)$, which can be reformulated algebraically to remove the integrals in $\phi(u)$. The easiest case would be to let the severity distribution follow an exponential. See APPENDIX F for properties and derivatives of $\Psi(x)$ about $x = 0$ which are needed for the final act.

3.4.2 A two-point Padé-Ramsay approximation

THEOREM 15. *Let us consider a two-point Padé-Ramsay approximation, where we impose some formal limiting behavior as found in a similar fashion to (3.3.4), that is* $\lim_{s\to\infty} s\Psi^*(s) = 1 - q$ *with first derivative* $\Psi'(0) = -q(1-q)/\mu_1$ *(see* APPENDIX F*). This leads to a two-point* $\wp_{(n-1,n)}$ *Laplace ruin transform approximation at* $n = 2$,

$$f_e^*(s) \approx \frac{\beta_0 + \gamma_1 s}{\beta_0 + \beta_1 s + \beta_2 s^2} \Leftrightarrow \Psi^*(s) \approx \frac{(1-q)(\beta_2 s + \beta_1 - \gamma_1)}{q\beta_0 + (\beta_1 - (1-q)\gamma_1)s + \beta_2 s^2},$$

where $\gamma_1 = \beta_2/\mu_1$ *and coefficients* $\beta_i, i = 0,1,2$ *found by fitting (and matching) the first few aggregate loss moments (see* APPENDIX E*). Approximating each coefficient yields:*

$$\beta_0 = \mu_2 - 2\mu_1^2, \qquad \beta_1 = \frac{\mu_3}{3} - \mu_1\mu_2, \qquad \beta_2 = \frac{\mu_1\mu_3}{3} - \frac{\mu_2^2}{2}.$$

PROOF. Recall equation (3.3.2) and assume that the drift $\sigma \approx 0$, then

$$\Psi^*(s) \approx \frac{(1-q)\big(1 - f_e^*(s)\big)}{s\big(1 - (1-q)f_e^*(s)\big)}.$$

This satisfies the approximation of the form:

$$s\Psi^*(s) \approx \frac{(1-q)(\beta_2 s^2 + (\beta_1 - \gamma_1)s)}{\beta_2 s^2 + \big(\beta_1 - p(1-q)\big)s + q\beta_0},$$

which surely satisfies the limiting behavior $\lim_{s\to\infty} s\Psi^*(s) = 1 - q$ (can be shown by dividing the numerator and denominator by s^2). ■

We can solve the problem above by letting

$$\Psi^*(s) \approx \frac{(1-q)(\beta_2 s + \beta_1 - \gamma_1)}{q\beta_0 + (\beta_1 - (1-q)\gamma_1)s + \beta_2 s^2} = \frac{As + B}{Cs^2 + Ds + E},$$

where $A = (1-q)\beta_2$, $B = (1-q)(\beta_1 - \gamma_1)$, $C = \beta_2$, $D = \beta_1 - (1-q)\gamma_1$ and $E = q\beta_0$. The inverse LT can then be deduced in *Mathematica*, and thus an analytical solution for the above is found to be:

$$\Psi_{2PP}(u) = k_1 e^{-\varpi_1 u} + k_2 e^{-\varpi_2 u},$$

where, $k_{1,2} = \frac{A(\Delta \pm D) \mp 2BC}{2C\Delta}$, $\varpi_{1,2} = \frac{D \pm \Delta}{2C}$ and $\Delta = \sqrt{D^2 - 4CE}$, with real roots iff $\Delta > 0$.

THEOREM 16. *Let us consider another two-point Padé approximation, where we impose the same limiting behavior as found in (3.3.4), that is* $\lim_{s\to\infty} s\Psi^*(s) = 1 - q$ *with first derivative* $\Psi'(0) = -q(1-q)/\mu_1$ *(see* APPENDIX F*). This leads to a two-point* $\wp_{(n-1,n)}$ *Laplace ruin transform approximation at* $n = 2$*,*

$$f_e^*(s) \approx \frac{\beta_0 + \gamma_1 s}{\beta_0 + \beta_1 s + \beta_2 s^2} \Leftrightarrow \Psi^*(s) \approx \frac{(1-q)(\beta_2 s + \beta_1 - \gamma_1)}{\beta_2 s^2 + (\beta_1 - (1-q)\gamma_1)s + q\beta_0},$$

where, $\gamma_1 = \beta_1 - \breve{\mu}_1\beta_0$ *and coefficients* $\beta_i, i = 0,1,2$ *are found by fitting (and matching) the first few moments of L (see* APPENDIX E*). Approximating each coefficient yields:*

$$\beta_0 = \breve{\mu}_2 - \breve{\mu}_1^2 > 0, \qquad \beta_1 = \breve{\mu}_3 - \breve{\mu}_1\breve{\mu}_2, \qquad \beta_2 = \breve{\mu}_1\breve{\mu}_3 - \breve{\mu}_2^2 > 0,$$

where, $\breve{\mu}_k = \frac{\mu_{k+1}}{(k+1)\mu_1}$ *are factorial reduced equilibrium moments.*

4 NUMERICAL ILLUSTRATION

Riches are not from an abundance of worldly goods but from a contented mind.

Prophet Muhammad (PBUH)

We present numerous illustrations for four cases with perturbation. For SECTION 4.1 to 4.3, we assume that the claim amount distribution X has expectation $\mu_1 = 1$ and our upper and lower bounds has lattice width $\vartheta = 0.1$. Under the perturbed case, we set the diffusion component as $\sigma = 1$. This leads to $c = \lambda(1+\theta) > 0$ and $q = \theta/(1+\theta)$. With this, we modify our perturbed risk model from (2.3.1) and denote this updated one by $\tilde{V}(t)$, hence the model at time t is given by:

$$\tilde{V}(t) = u + \lambda(1+\theta)t - S(t) + Z_\alpha(t), \qquad t \geq 0. \tag{4.0.1}$$

This lead to $H_1(x) = 1 - e^{-2cx}$ and $h_2(x) = f_e(x) = 1 - F_X(x)$, where the equilibrium density $h_2(x)$ is the survival function for the claim distribution. The convolution CDF, if and only if the properties of a distribution is satisfied, is given by

$$[H_1 * H_2](x) = \int_0^x H_1(x-t)h_2(t)\,dt = \int_0^x \left(1 - e^{-2c(x-t)}\right)\left(1 - F_X(t)\right)dt, \tag{4.0.2}$$

and is under the assumption that their respective random variables are concentrated on $(0, x)$, see SEIXAS AND EGÍDIO DOS REIS (2013) and APPENDIX C. Approximations considered were De Vylder's exponential, Dufresne and Gerber's bounds, Pollaczek-Khinchine's One-Point (Renyi and new DV) and Two-Point Padé. All figures and illustrations were computed in *MS Excel*, *Wolfram Alpha* and *Mathematica*. Relative errors are useful here to determine the precision of these approximations. The formula is given by:

$$\Psi_{Error} = \left|\frac{\Psi_{Approx}(u) - \Psi_{Exact}(u)}{\Psi_{Exact}(u)}\right|. \tag{4.0.3}$$

4.1 *Exponential claim distribution*

Suppose that the claim distribution followed an $exponential(\beta)$ with Poisson parameter $\lambda = 1$, safety positive loading coefficient $\theta = 1\%$ and probability density function given by $f_X(x) = \beta e^{-\beta x} = h_2(x),\ x > 0$, where $\beta = 1$ is the scale parameter and the mean. Thus, the convolution CDF equation defined in (4.02) approximates to

$$[H_1 * H_2](x) \approx 0.980392(e^{-2.02x} - e^{-x}) + \sinh(x) - \cosh(x) + 1.$$

The rate of ruin under this distribution is proportional to its power or time, i.e. the ruin rate is constant over time. Notice that this distribution is light-tailed, and thus, the contribution of claims to ruin is expected to be less important than heavier-tailed distributions. Raw moments are easy to compute: $\mu_k = k!/\beta^k$.

Initial, u	**Exact,** $\Psi(u)$	**Lower Bound**	**DV 4 Moment Exponential**	**PK Renyi**	**PK De Vylder**	**Upper Bound**
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.5	0.99324	0.98790	0.99324	0.99506	0.99228	0.98901
1.0	0.98919	0.98406	0.98919	0.99015	0.98899	0.98577
1.5	0.98574	0.98132	0.98574	0.98526	0.98571	0.98337
2.0	0.98244	0.97717	0.98244	0.98039	0.98245	0.97972
2.5	0.97918	0.97439	0.97918	0.97555	0.97919	0.97729
3.0	0.97593	0.97025	0.97593	0.97073	0.97595	0.97365
3.5	0.97270	0.96749	0.97270	0.96594	0.97271	0.97123
4.0	0.96947	0.96337	0.96947	0.96117	0.96949	0.96761
4.5	0.96626	0.96064	0.96626	0.95642	0.96628	0.96520
5.0	0.96306	0.95655	0.96306	0.95170	0.96308	0.96160
5.5	0.95987	0.95383	0.95987	0.94700	0.95989	0.95921
6.0	0.95669	0.94977	0.95669	0.94232	0.95671	0.95564
6.5	0.95352	0.94707	0.95352	0.93767	0.95354	0.95326
7.0	0.95036	0.94304	0.95036	0.93304	0.95038	0.94971
7.5	0.94721	0.94036	0.94721	0.92843	0.94723	0.94735

TABLE 3 – RUIN PROBABILITIES, EXPONENTIAL CLAIMS $(\beta = 1), \lambda = 1, \theta = 1\%$.

Initial, u	Ψ_{Error} **Lower Bound**	Ψ_{Error} **PK-R**	Ψ_{Error} **PK-DV**	Ψ_{Error} **Upper Bound**
0.5	0.005378185	0.001831717	0.000972172	0.004256357
1.0	0.005183530	0.000970080	0.000201145	0.003457658
1.5	0.004484530	0.000490563	0.000029547	0.002406271
2.0	0.005367598	0.002082280	0.000008596	0.002765299
2.5	0.004883695	0.003701039	0.000017053	0.001927372
3.0	0.005822822	0.005323739	0.000018908	0.002339192
3.5	0.005350241	0.006945254	0.000019296	0.001510937
4.0	0.006291899	0.008564449	0.000019356	0.001925351
4.5	0.005820118	0.010181076	0.000019345	0.001097227
5.0	0.006761489	0.011795082	0.000019317	0.001511926
5.5	0.006289960	0.013406461	0.000019285	0.000683481
6.0	0.007230895	0.015015212	0.000019253	0.001098357
6.5	0.006759591	0.016621341	0.000019220	0.000269571
7.0	0.007700082	0.018224851	0.000019188	0.000684618
7.5	0.007229000	0.019825746	0.000019155	0.000144511

TABLE 4 – RELATIVE ERROR, EXPONENTIAL CLAIMS $(\beta = 1), \lambda = 1, \theta = 1\%$.

TABLES 3 and 4 show the exact, approximate ruin probabilities and relative errors. We see that all ruin approximations appear to be excellent for low levels of u. Relative errors for De Vylder's four moment exponential is not needed here since it is the exact ruin probability, hence we need not to worry for bounds. Excluding bounds, we see that Renyi performs worse than Pollaczek-Khinchine approximations of the first Padé order for DV, most likely since we only match a single moment.

The error for PK-DV converges slowly towards 0.0019% for larger values of u, and for most values of u, this leads to a better approximation for $\Psi(u)$. Conversely, Renyi's relative error increases as u increases, despite this, it's a better approximation for extremely low levels of capital, say $u < 1$.

4.2 Gamma claim distribution

4.2.1 Gamma with $\gamma = 0.1$ and $\beta = 10$

We consider conservative claims which has often appeared in actuarial literature, see RAMSAY (1992) and GRANDELL (2000), with intensity parameter $\lambda = 1$, safety positive loading coefficient $\theta = 10\%$ following a $Gamma(0.1, 10)$ with CDF and CGF equal to

$$F_X(x) = \int_0^x \frac{10^{0.1} e^{-10t}}{\Gamma(0.1) t^{0.9}} dt \approx 0.1051137[\Gamma(0.1) - \Gamma(0.1,10x)],$$
$$\varphi_X(z) = -0.1 \ln(1 - 10z),$$

respectively. Obtaining an analytical expression for $[H_1 * H_2](x)$ is difficult due to the presence of an incomplete gamma function, $\Gamma(0.1,10x)$. Likewise, calculating the area under a convolution by using the products of areas under factors (see APPENDIX C) is also difficult since the expression does not converge, hence for this reason, we will not consider bounds. TABLE 5 shows the results and relative errors for Pollaczek-Khinchine approximations of the first Padé order (Renyi & DV) and the Two-Point Padé approximation (2PP), with initial reserve increasing in increments of five. We see that all ruin approximations under Renyi appears to be extremely accurate with relative less than 2% for $u \leq 25$. The 2PP approximation is an odd case as the relative error tends to zero as $u \to 30$, then slowly worsens as the initial reserve increases beyond 30. Lastly, the error for PK-DV converges slowly towards 0.02% as u increass, and for most values of u, this leads to a better approximation for $\Psi(u)$.

Initial u	Exact, $\Psi(u)$	PK-R	PK-DV	2PP	Ψ_{Error} PK-R	Ψ_{Error} PK-DV	Ψ_{Error} 2PP
0	1.00000	1.00000	1.00000	1.00000	-	-	-
5	0.83623	0.83698	0.83788	0.89440	0.000892562	0.001963121	0.069557004
10	0.77428	0.77059	0.77711	0.81835	0.004765369	0.003661279	0.056922458
15	0.71821	0.70947	0.72076	0.74877	0.012181643	0.003543488	0.042547645
20	0.66623	0.65319	0.66849	0.68511	0.019572574	0.003395088	0.028336954
25	0.61801	0.60138	0.62001	0.62686	0.026908706	0.003246194	0.014319444
30	0.57327	0.55367	0.57505	0.57356	0.034189953	0.003097313	0.000493002
35	0.53178	0.50976	0.53335	0.52479	0.041416718	0.002948455	0.013144969
40	0.49329	0.46932	0.49467	0.48017	0.048589408	0.002799618	0.026597037
45	0.45759	0.43209	0.45880	0.43934	0.055708427	0.002650804	0.039865737
50	0.42447	0.39782	0.42553	0.40199	0.062774178	0.002502012	0.052953568

TABLE 5 – RUIN PROBABILITIES, RELATIVE ERROR, GAMMA CLAIMS ($\gamma = 0.1, \beta = 10$).

4.2.2 Gamma with $\gamma = 5/2$ and $\beta = 2/5$

Now we consider Padé-based Gamma $\left(\frac{5}{2}, \frac{2}{5}\right)$ claims with higher intensity parameter ($\lambda = 10$), safety positive loading coefficient $\theta = 10\%$. This is an interesting case, since not only is the parameters reciprocals of each other, but AVRAM *ET. AL* (2011) deduced using a similar example that moment-based Padé approximation of X should not result in valid distributions. This is elaborated further in AVRAM *ET. AL* (2011). However, we present approximations which indeed, lead to valid ultimate ruin probabilities, see TABLE 6.

Initial u	Exact, $\Psi(u)$	PK-R	PK-DV	2PP	Ψ_{Error} PK-R	Ψ_{Error} PK-DV	Ψ_{Error} 2PP
0	1.00000	1.00000	1.00000	1.00000	-	-	-
0.5	0.95917	0.90161	0.89661	0.96886	0.060010237	0.065219325	0.010103361
1	0.92999	0.89419	0.88989	0.96029	0.038496671	0.043120754	0.032586227
1.5	0.90851	0.88683	0.88321	0.95180	0.023869893	0.027848854	0.047643110
2	0.89214	0.87953	0.87659	0.94338	0.014134047	0.017429560	0.057434897
2.5	0.87914	0.87229	0.87002	0.93503	0.007795707	0.010384089	0.063572231
3	0.86840	0.86511	0.86349	0.92676	0.003782536	0.005649584	0.067210663
3.5	0.85915	0.85799	0.85701	0.91856	0.001344789	0.002482272	0.069157519
4	0.85090	0.85093	0.85059	0.91044	0.000032688	0.000370675	0.069967128
4.5	0.84334	0.84393	0.84421	0.90238	0.000699833	0.001032922	0.070015784
5	0.83623	0.83698	0.83788	0.89440	0.000892562	0.001963121	0.069557004

TABLE 6 – RUIN PROBABILITIES, RELATIVE ERROR, GAMMA CLAIMS ($\gamma = 5/2, \beta = 2/5$).

From TABLE 6, relative errors for ruin approximations under Renyi and PK-DV for initial reserve, $u \leq 5$ (with increments of 0.5) are in line with the exact, compared to 2PP which converges to an error of 7% by $u = 5$.

4.3 *Mixed exponential claim distribution*

The calculation of ruin probabilities when claims follow a mixture of exponentials has been heavily touched upon in actuarial and statistical literature by THORIN AND WIKSTAD (1977), and many others over recent years. Here we present the claim distribution presented here by CRAMÉR (1955), is an attempt by WIKSTAD (1971) to explain a distribution for a Swedish non-industry fire insurance, given by $f_X(x) = A_1 e^{-5.514588x} + A_2 e^{-0.190206x} + A_3 e^{-0.014631x}$, where, $A_1 = 0.8881815, A_2 = 0.1078392$ and $A_3 = 0.0039793$, and thus satisfies the constraints from (3.2.3), i.e. $\sum A_i = 1$ for $i = 1,2,3$. We set conservative parameters $\lambda = 1$ and $\theta = 10\%$. We find that the convolution distribution, i.e.,

$$[H_1 * H_2](x) \approx 0.0193851e^{-5.51459\,x} - 3.26287e^{-0.190206\,x} - 18.7136e^{-0.014631\,x} + 0.357962e^{-2.2\,x} + 0.00000230391x + 21.5991,$$

does not satisfy all properties of a distribution function, i.e. that $[H_1 * H_2](-\infty) \neq 0$ and $[H_1 * H_2](\infty) \neq 1$ but instead diverges off towards negative and positive infinity, respectively, hence bounds will once again not be considered. By using methods detailed in REMARK 3, our exact ruin probability is found to be:

$$\Psi(u) = 0.77979e^{-0.0035558u} + 0.1273167e^{-0.0779348u} + 0.0891495e^{-1.8496200u} + 0.0037434e^{-5.98831u}, \qquad u \geq 0, \qquad \Psi(0) = 1.$$

Raw moments of a mixed exponential distribution, for $k = 1,2,\dots,5$, can be evaluated by

$$\mu_k \approx \frac{d^{(k)}}{dt}\{4.3897955(5.514588 - t)^{-1} + 0.020512(0.190206 - t)^{-1} + 0.000058(0.014631 - t)^{-1}\}|_{t=0}.$$

Initial u	**Exact, $\Psi(u)$**	**PK-R**	**PK-DV**	**2PP**	**Ψ_{Error} PK-R**	**Ψ_{Error} PK-DV**	**Ψ_{Error} 2PP**
0	1.00000	1.00000	1.00000	1.00000	0.00000000	0.00000000	0.00000000
60	0.63116	0.70618	0.63842	0.75062	0.11885874	0.01149464	0.18926736
120	0.50895	0.54856	0.51495	0.57052	0.07781775	0.01177715	0.12096135
240	0.33217	0.33101	0.33503	0.32958	0.00349365	0.00860710	0.00778105

TABLE 7 – RUIN PROBABILITIES, RELATIVE ERROR, MIXED EXPONENTIAL, $\Lambda = 1, \Theta = 10\%$.

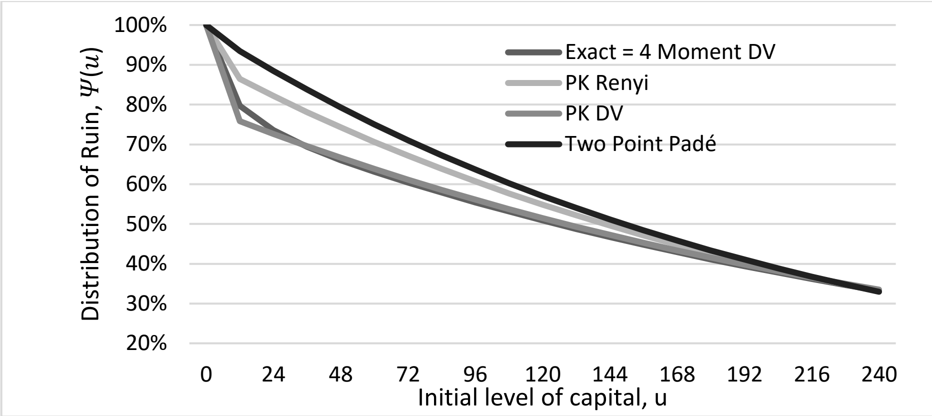


FIGURE 5 – RUIN PROBABILITIES, MIXED EXPONENTIAL, $\Lambda = 1, \Theta = 10\%$.

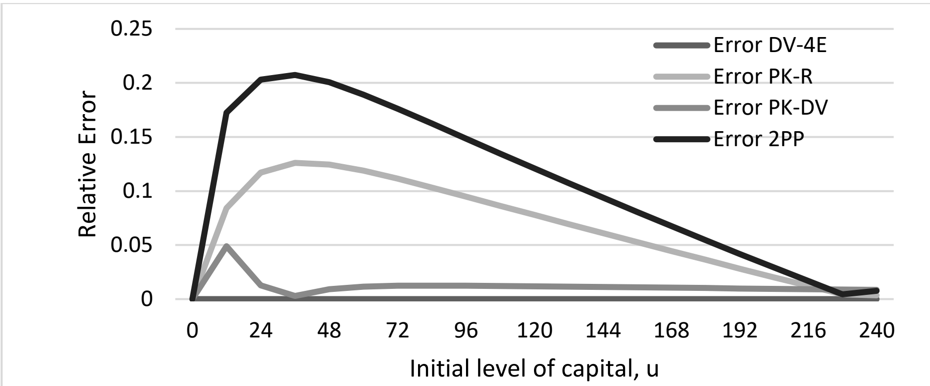


FIGURE 6 – RELATIVE ERROR, MIXED EXPONENTIAL, $\lambda = 1, \theta = 10\%$.

We recall that this distribution is highly skewed with variance ≈ 42 and skewness ≈ 28, with fourth and fifth raw moments floating in the millions. TABLE 7 shows the results and relative errors for De Vylder's classical four moment exponential (exact ruin), Pollaczek-Khinchine approximations of the first Padé order (Renyi & DV) and the Two-Point Padé approximation (2PP). For extremely high levels of initial u, all approximations reveal a good fit to this distribution with the majority having errors that reach a significant figure after 3 decimal points. Terrible results are expected for low levels of u due to the nature of heavily skewed data. FIGURES 5 and 6 supports this by illustrating unimpressive figures for small u, especially for Renyi and 2PP. Hence, we conclude that, for a large enough u these approximations would surely return great results due to the asymptotic nature of the curves (FIGURE 5).

5 CONCLUSION

> The task of the modern educator is not to cut down jungles, but to irrigate deserts.
>
> C.S. Lewis

Solving problems is a significant part of mathematics and many other sciences. Using a blend of statistics, business knowledge & mathematics, actuaries forecast possibilities and develop plans to manage financial risks, and thus serve as trusted financial and business advisors. Actuaries inform and make decisions that lead to profits, savings, stability and success. So, a question to consider, what makes these text book figures that are only known by certain formulae and theorems so pleasing to learn about? Well simply, behind those countless memorisations whose purpose only serves well for exams, there is potential for students of mathematics to carry these actuaries' work to something perhaps more satisfying. This project also provided the opportunity to tackle uncharted territories of insurance.

To summarize, we adapted the perturbed model to the Cramér-Lundberg model, by adding a Lévy (α-stabled) process to the compound Poisson process, which allows us to consider uncertainty to the premium income, fluctuations of the interest rates, changes to the number of policyholders, without neglecting all other assumptions. The results obtained seem to give us an indication that the diffusion parameter (σ) can have a significant effect in calculating the exact ruin probability, this is especially true for light tailed distributions (exponential,). The illustrations for different σ is not covered here, but once can find that the error is smaller for larger values of σ (at least for the mixed exponential claim distribution) and for most values of u, a far better approximation to $\Psi(u)$, see for instance AVRAM, *ET. AL* (2011).

The approximations and relative errors for De Vylder's classical four moment exponential, Pollaczek-Khinchine's One-Padé (Renyi & DV) and Two-Point Padé all appear capable of producing excellent results if the claim distribution is well parameterized. In other cases, Dufresne & Gerber's upper and lower bounds returned good approximations when the claim distribution was exponential. However, we can all agree that the Renyi approximation usually produces the poorest fit, regardless of the claim distribution used. Similar conclusions can be found in SEIXAS AND EGÍDIO DOS REIS (2013), with the Beekman-Bowers. Note: Renyi is a simplified version of the Beekman-Bowers approximations, see GRANDELL (2000).

Overall, this investigation went quite well. However, some problem was found to be notorious. For example, estimating bounds when the claim amount distribution is a gamma or mixture of exponentials proved to be difficult since obtaining an analytical expression for the convolution distribution, i.e. $[H_1 * H_2](x)$, was not feasible, or even the fact that it does not satisfy all properties of a distribution function. Likewise, calculating the area under a convolution by using the products of areas under factors was also difficult since these do not converge.

Nonetheless, other areas were touched upon briefly but not fully developed, or even conceived in this project at all. In future, I would aim to write a thesis that expands this current work and possibly, the work set out in BURNECKI *ET. AL* (2003) on a new De Vylder type (gamma) approximation of the ruin probability in infinite time (of course, by adapting a perturbed model), where they find that using a lognormal (heavy-tailed) and mixture of two exponentials (light-tailed) proves to provide much better results than the classical De Vylder case. In other words, the gamma approximation will always be significantly less than the error of the classical method.

APPENDIX

A Properties of the Perturbed Risk Process

A1 Lévy Process

DEFINITION 8 (Lévy Process). *A stochastic process,* $\{Z(t) : t \geq 0\}$ *is said to be a Lévy process, defined on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, if the following properties are satisfied:*

i. $Z(0) = 0$*, almost surely.*

ii. $\{Z(t) : t \geq 0\}$ *has independent increments for* $0 \leq t_1 < t_2 < \cdots < \infty$.

iii. $\{Z(t) : t \geq 0\}$ *has stationary increments for* $s < t$*, then* $X_t - X_s$ *is equal to* X_{t-s}.

iv. $\{Z(t) : t \geq 0\}$ *has right continuous sample paths.*

v. $\{Z(t) : t \geq 0\}$ *has infinite divisibility for any positive* ϵ *and* $t \geq 0$*. For a very small increment* h*, it holds that* $\lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon)$ *is equal to zero.*

We shall always use the measure $\mathbb{P}$ *(i.e. the real-world measure) when talking about a Lévy process, unless stated otherwise.*

REMARK 6. Properties (*ii*) and (*iii*) implies that, $\{Z(t) : t \geq 0\}$ is a Markov process.

There are many well-known examples of Lévy processes the Brownian motion & the Poisson process.

THEOREM 17. *A Lévy process is said to be-:*

i. *A standard Brownian motion,* $W(t)$*, if the probability distribution of* $Z(t) - Z(s)$ *is normally distributed with mean zero and variance* $t - s$.

ii. *A Poisson process,* $N(t)$*, if the probability distribution of* $Z(t) - Z(s)$ *is Poisson distributed with mean and variance equal* $\lambda(t - s)$*, where* $\lambda > 0$ *is the intensity rate.*

A2 Alpha-Stabled Process

DEFINITION 9 (Standard α-stable Process). *A process,* $\{Z_\alpha(t) : t \geq 0\}$ *is said to be a standard* α*-stable process, defined on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, if:*

i. $Z_\alpha(0) = 0$, *almost surely*

ii. $\{Z_\alpha(t) : t \geq 0\}$ *has independent increments for* $0 \leq t_1 < t_2 < \cdots < \infty$.

iii. $Z_\alpha(t) - Z_\alpha(s) \sim S_\alpha(0, (t-s)^{-\alpha}, \beta)$ *for* $s < t$ *and for* $\alpha \in (0,2]$ *"is the rate of decay" , where* $S_n = S(n) = \sum_{i=1}^{n} X_i$ *(defined in Section 2.2),* $|\beta| \leq 1$ *is the jump parameter, and so we say that* $X \in$ *domain of an* α*-stable random variable* S_α*, if there exist some constants* $a_n \in \mathbb{R}, b_n > 0$*, s.t.:*

$$\frac{S_n - a_n}{b_n} \to S_\alpha,$$

in distribution, as $n \to \infty$*; this is the central limit theorem, see Iglehart (1969) and Furrer et. al (1997).*

iv. $Z_2(t) = W(t)$ *follows a Normal distribution.*

v. $Z_1(t)$ *follows a Cauchy distribution.*

vi. $Z_{0.5}(t)$ *follows a one-sided stable distribution.*

For $\{Z_\alpha(t) : t \geq 0\}$, there are usually four parameters, i.e. $S_\alpha(m, v, \beta)$, where m and v are usually set to 0 and 1 representing location & scale, respectively. The density function does not exist for this type of distribution, except for cases when α takes values 0.5, 1 or 2, see more in APPENDIX B.

A3 Poisson Process

DEFINITION 10 (Poisson Process). *A process,* $\{N(t) : t \geq 0\}$ *is said to be a Poisson process, defined on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with intensity* $\lambda > 0$*, if the following properties are satisfied:*

i. $N(0) = 0$.

ii. $\{N(t) : t \geq 0\}$ *has independent increments for* $0 \leq t_1 < t_2 < \cdots < \infty$.

iii. $\{N(t) : t \geq 0\}$ *has stationary increments for* $s < t$*, then* $N(t) - N(s)$ *is equal to* $N(t-s)$ *with Poisson parameter* $\lambda(t-s)$.

iv. $\mathbb{P}(N(h) = 1) = \lambda h + o(h)$.

v. $\mathbb{P}(N(h) \geq 2) = o(h)$.

REMARK 7. The function $o(h)$ means $\lim_{h \to 0} \frac{f(h)}{h} = o(h)$. Properties (*iii*) to (*v*) is useful to determine where, $\{N(t) : t \geq 0\}$ is a Poisson process.

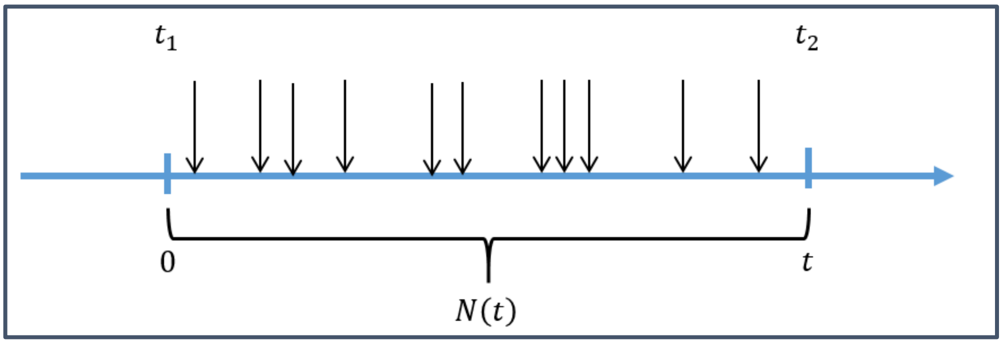


FIGURE 7 – POISSON PROCESS AS "DISCRETE ARRIVALS".

A4 Moments of V(t)

The moments of $S(t)$, the aggregate claims up to time t, are well defined from actuarial literature, the deduction of it can be seen in SIMAR (1976), and others, for instance APPLEBAUM (2004a), and so, the k-th central moments of $S(t)$ are computed by taking the k-th derivative of the CGF, $\varphi_{S(t)}(s) = \lambda t(e^s - 1)$ at $s = 0$; hence, these yield $E[(S(t) - E[S(t)])^k] = \lambda t \mu_k$, for all k. Similarly, the first five moments of $W(t)$ are: $E[W(t)] = 0$, $E[W^2(t)] = t$, $E[W^3(t)] = 0$, $E[W^4(t)] = 3t^2$ and $E[W^5(t)] = 0$.

By using generating functions and setting $\alpha = 2$ (for simplification), then for $V(t)$:

$$\begin{aligned}\varphi_{V(t)}(s) = \ln E\left[e^{sV(t)}\right] &= \ln E\left[e^{s(u+ct-S(t)+\sigma W(t))}\right] \\ &= s(u + ct) + \varphi_{S(t)}(-s) + \varphi_{W(t)}(\sigma s),\end{aligned}$$

where, $\varphi_{V(t)}(s)$ is the CGF of $V(t)$ at the point s. Note that $\varphi_{S(t)}(-s)$ is a LT (see APPENDIX C). Hence, taking derivatives with respect to s and setting $s = 0$ yields corresponding central moments $\nu_k = \varphi^{(k)}_{V(t)}(0) = E[(V(t) - E[V(t)])^k]$ for $k = 1,2,3$: $\nu_1 = u + ct - \lambda t \mu_1$, $\nu_2 = \lambda t \mu_2 + \sigma^2 t$ and $\nu_3 = -\lambda t \mu_3$, where, $\mu_k = E[X^k]$ are raw moments from the claim distribution. Note that higher-order cumulants from $k \geq 4$ are not the same as moments about the mean. Instead we need to use:

$$\begin{aligned}&k = 4 : \nu_4 = \varphi^{(4)}_{V(t)}(0) + 3\nu_2^2, \Leftrightarrow \nu_4 = \lambda t \mu_4 + 3\sigma^4 t^2 + 3(\lambda t \mu_2 + \sigma^2 t)^2, \\ &k = 5 : \nu_5 = \varphi^{(5)}_{V(t)}(0) + 10\nu_3\nu_2, \Leftrightarrow \nu_5 = -\lambda t \mu_5 - 10\lambda t \mu_3(\lambda t \mu_2 + \sigma^2 t).\end{aligned}$$

If the diffusion component σ is zero, then $E[Z(t)] = E[U(t)], \forall k$.

B Alpha Stable Distribution

We present here a basic review on the theory of Lévy process, specifically, infinitely divisible distributions, see ITO (1969) or the modern interpretation SATO (1999). For historical purposes, see Lévy (1954). Note that the characteristics exponent completely defines the Z_α spectrum of distributions between values 0 and 2. From (2.8.1), we could write the characteristic function by $\varphi_X(u) = M_X(iu) = E\left[e^{iuX}\right]$ where X is a random variable. Let us consider changes made to (2.8.3) when we choose values for α.

B1 Case $\alpha = 1$ (Cauchy Distribution)

When $\alpha = 1$, we have a Cauchy distribution (defined by the claim amount random variable X), where the density function of the standard Cauchy distribution is a solution to the following first order ODE:

$$(1 + x^2)f'(x) + 2xf(x) = 0, \qquad f(1) = \frac{1}{2\pi}.$$

The characteristic function is simply a Fourier transform of the pdf, see for instance PAPOULIS (1984). The original pdf can be expressed by using the inverse Fourier transform, i.e.

$$\varphi_X(u) = E\left[e^{iuX}\right] = \int_{-\infty}^{\infty} f(s)e^{ius}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi_X(s)e^{-ius}\,ds.$$

Note that moments can be obtained by $E[X^n] = \frac{d^n}{ds^n}\varphi_X(0)$. But, $\varphi_X(s)$ is not differentiable at zero, hence this implies that the Cauchy distribution is not well-defined for moments higher than zero. Then this distribution cannot be used when matching the moments in SECTION 3 since we require some moments to exist up to a certain order.

B2 Case $\alpha = 2$ (Gaussian Distribution)

When $\alpha = 2$, the distribution follows a Gaussian, and hence, $Z_2(t) = W(t)$ is a standard Brownian motion.

To conclude, the charm of these distributions lie within α. It controls the level of decay of the tail, as it decreases its value from a normal distribution (light-tailed) to a Cauchy distribution (heavy-tailed). Its Poisson component allows for haphazard changes in the development of the system. Hence these features make this model more flexible to work with.

C Convolution

DEFINITION 11 (Convolution). *This is an integral that is defined as an overlap of one function $f(.)$ as it is shifted towards another function $g(.)$, and is defined by over a finite range $(0, t)$, such that:*

$$[f * g](t) = \int_0^t f(x)g(t-x)\,dx,$$

where, the notation $[f * g](t)$ is the convolution of f and g at the point t.

Theoretically speaking, a convolution is simply the product of two functions that are within the algebra of Schwartz functions in $\mathbb{R}^n$. Usually, we apply the convolution over an infinite range, i.e. $(-\infty. +\infty)$, see more in BRACEWELL (2000).

DEFINITION 12 (Area Under Convolution). *Taking integrals under a convolution has a unique property, as it is defined by the product of areas under factors:*

$$\begin{aligned}\int_{-\infty}^{+\infty} [f * g](t)\,dt &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(x)g(t-x)\,dx \right) dt \\ &= \left(\int_{-\infty}^{+\infty} f(x)\,dx \right) \left(\int_{-\infty}^{+\infty} g(t)\,dt \right).\end{aligned}$$

THEOREM 18 (Stieltjes Integral). *Let X and Y be two independent, continuous random variables, with $CDFs$ $F_X(.)$ and $F_Y(.)$ and, $pdfs$ $f_X(.)$ and $f_Y(.)$, respectively. Then the CDF and pdf of a convolution, i.e. $[F_X * F_Y](t)$ and $[f_X * f_Y](t)$, are given by, respectively:*

$$[F_X * F_Y](t) = \int_0^t F_X(x)f_Y(t-x)\,dx, \qquad [f_X * f_Y](t) = \int_0^t f_X(x)f_Y(t-x)\,dx.$$

PROOF. Consider a new random variable $Z = X + Y$, such that:

$$F_Z(t) = \iint_{x+y\leq t} f_X(x)f_Y(y)\,dxdy = \int_{-\infty}^{+\infty}\int_{-\infty}^{t-y} f_X(x)f_Y(y)\,dxdy$$
$$= \int_0^t F_X(x)f_Y(t-x)\,dx.$$

By differentiating $F_Z(t)$ w.r.t t, we get:

$$f_Z(t) = \frac{d}{dt}\left[\int_0^t F_X(x)f_Y(t-x)\,dx\right] = \int_0^t f_X(x)f_Y(t-x)\,dx.$$

■

For a list of the many properties of the Stieltjes Integral, see DRESHER (1981).

REMARK 8. As noted in BRACEWELL (2000), taking the first derivative of a convolution yields $[f * g]'(t) = [f' * g](t) = [f * g'](t)$. These leads to the following pairs

$$[F_X * F_Y]'(t) = \left[f_X * F_y\right](t) = \int_0^t f_X(x)F_Y(t-x)\,dx,$$
$$[F_X * F_Y]'(t) = [F_X * f_Y](t) = \int_0^t F_X(x)f_Y(t-x)\,dx.$$

D Laplace Transform

DEFINITION 13 (Laplace Transform). *If $f_X(t)$ is a continuous function defined for $t \in [0, +\infty)$, then the Laplace transform (LT) of f, denoted as $f_X^*(s)$, which is a unilateral transform, is given by*

$$f_X^*(s) = \int_0^{+\infty} e^{-st} f_X(t)\,dt = E[e^{-st}], \qquad t \geq 0,$$

where, s is a complex number frequency parameter such that $s = a + ib$, with imaginary number $i = \sqrt{-1}$ and real numbers a, b. Similar results can be expected for a discrete case.

THEOREM 19. *If $f_X^*(s) = f_Y^*(s)$ exist for all s, then $f_X(t) = f_Y(t)$ for all t where both are continuous.*

THEOREM 20. *Let $f_X^*(s)$ and $f_Y^*(s)$ exist for all s, then assigning constants a and b to $f_X(t)$ and $f_Y(t)$, respectively, then its LT is*

$$\int_0^{+\infty} e^{-st}\big(af_X(t) + b\, f_Y(t)\big)\, dt = af_X^*(s) + bf_Y^*(s).$$

THEOREM 21. *If the cumulative distribution function is $F_X(t) = \int_0^t f_X(y)\, dy$, then its LT is given by*

$$F_X^*(s) = \int_0^{+\infty} e^{-st}\left(\int_0^t f_X(y)\, dy\right) dt = \frac{1}{s} f_X^*(s).$$

PROOF. Everything have been proven and discussed extensively in many textbooks: BRACEWELL (2000), FELLER (1971) and WILLIAMS (1973). ■

E Aggregate Loss Distribution

E1 Moment Generating Function Proof

THEOREM 22. *Let the moment generation function of L be given by*

$$M_L(r) = \frac{r\tau(c - \lambda\mu_1)}{c\tau(r(\tau - r) - \lambda M_X(r) - 1)}, \qquad \sigma > 0.$$

PROOF. By the expected value definition of a MGF, we find that

$$\begin{aligned} M_L(r) = E[e^{rL}] &= E\left[\exp r\left(L_0^{(1)} + \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right)\right)\right] \\ &= M_{L_0^{(1)}}(r) \cdot M_M\left[\ln M_{\left(L_i^{(1)}+L_i^{(2)}\right)}(r)\right] \\ &= \left(\frac{\tau}{\tau - r}\right) \cdot q\left[1 - (1-q)\exp\left(\ln M_{\left(L_i^{(1)}+L_i^{(2)}\right)}(r)\right)\right]^{-1} \end{aligned}$$

which is compound geometric distribution. Recall that the convolution $[h_1 * h_2](.)$ is the pdf of $L_i^{(1)} + L_i^{(2)}$, and so using the properties of convolutions, we have:

$$\begin{aligned} M_{\left(L_i^{(1)}+L_i^{(2)}\right)}(r) &= \int_0^\infty e^{rt}[h_1 * h_2](t)\, dt = \int_0^\infty \int_0^t e^{rt} h_1(t-s) h_2(s)\, dsdt \\ &= \frac{\tau}{\mu_1}\int_{t=0}^{\infty} e^{(r+\tau)t} \int_{s=0}^{t} e^{-\tau s}\,[1 - F_X(s)]dsdt. \end{aligned}$$

Using integration by parts twice and setting $u = 1 - F_X(s)$ and $dv/ds = e^{-\tau s}$, the integral encompassing $e^{-\tau s}[1 - F_X(s)]$ results to

$$\int_{s=0}^{t} e^{-\tau s}[1 - F_X(s)]\, dsdt = \frac{(F_X(t) + \tau - 1) + \tau(e^{-t\tau}F_X(t) - 1)(1 - e^{-t\tau})}{\tau(\tau - 1)(1 - e^{-t\tau})}.$$

Hence, $M_{\left(L_i^{(1)}+L_i^{(2)}\right)}(r)$ is equal to

$$\frac{1}{\mu_1(\tau - 1)}\left[\int_{t=0}^{\infty} \frac{e^{(r+\tau)t}(F_X(t) + \tau - 1)}{(1 - e^{-t\tau})}dt + \int_{t=0}^{\infty} \tau e^{(r+\tau)t}(e^{-t\tau}F_X(t) - 1)\, dt\right],$$

which results to $M_{\left(L_i^{(1)}+L_i^{(2)}\right)}(r) = \left(\frac{\tau}{\tau-r}\right)(\tau\mu_1)^{-1}[M_X(r) - 1]$ [see SEIXAS AND EGÍDIO DOS REIS (2013)], where, $h_1(\,.\,)$ and $h_2(\,.\,)$ are defined in (2.4.5) and (2.4.6), respectively, so:

$$M_L(r) = \frac{r\tau(c - \lambda\mu_1)}{cr(\tau - r) - \lambda\tau(M_X(r) - 1)}. \qquad \blacksquare$$

E2 Moments of the Geometric Compound Distribution

Let $L = L_0^{(1)} + L_M^{(*)}$ be a compound geometric distribution, with MGFs given by:

$$M_{L_0^{(1)}}(z) = \frac{\tau}{\tau - z}, \quad M_{L_i^{(3)}}(z) = \left(\frac{\tau}{\tau - r}\right)(\tau\mu_1)^{-1}[M_X(r) - 1], \quad M_M(z) = \frac{qe^z}{1 - (1 - q)e^z},$$

where, $L_M^{(*)} = \sum_{i=1}^{M}\left(L_i^{(1)} + L_i^{(2)}\right)$ and $L_i^{(3)} = L_i^{(1)} + L_i^{(2)}$. From this, we obtain moments:

$$E[M] = \frac{1 - q}{q}, \qquad Var[M] = \frac{1 - q}{q^2}, \qquad E\left[L_0^{(1)}\right] = \frac{1}{\tau}, \qquad Var\left[L_0^{(1)}\right] = \frac{1}{\tau^2}.$$

Thus, we can then obtain the first two central moments in the form:

$$E[L] = E\left[L_0^{(1)} + L_M^{(*)}\right] = E\left[L_0^{(1)}\right] + E[M] \cdot E\left[L_i^{(3)}\right],$$
$$Var[L] = Var\left[L_0^{(1)} + L_M^{(*)}\right] = Var\left[L_0^{(1)}\right] + E[M] \cdot Var\left[L_i^{(3)}\right] + Var[M] \cdot E\left[L_i^{(3)}\right]^2,$$

which leads to:

$$E[L] = \frac{1}{\tau q^2}\left(q^2 + \tau(1 - q)\left(\frac{\mu_1}{2\mu_2} + \frac{1}{\tau}\right)\right),$$

$$Var[L] = \frac{1}{\tau^2 q^3}\left(q^3 + \tau^2(1-q)\left(\left(\frac{\mu_1}{2\mu_2}+\frac{1}{\tau}\right)^2 - \frac{\mu_2^2}{4\mu_1^2} + \frac{\mu_3}{3\mu_1} + \frac{1}{\tau^2}\right)\right).$$

Higher moments can be obtained by methods defined in APPENDIX A. See more details in SEIXAS AND EGÍDIO DOS REIS (2013).

E3 Factorial Moments (Power Series Expansion)

We start by expanding (and simplifying) the Pollaczek-Khinchine formula in (3.3.1) and (3.3.3), and using LT geometric expansion, we obtain the following series:

$$\Psi^*(s) = \frac{(1-q)\big(1-f_e^*(s)\big) + \frac{s\sigma^2}{2c}}{s\left(1-(1-q)f_e^*(s) + \frac{s\sigma^2}{2c}\right)} = \frac{\frac{\eta_{2,\sigma}}{2!} - s\frac{\eta_3}{3!} + s^2\frac{\eta_4}{4!} - \cdots}{\rho + s\frac{\eta_{2,\sigma}}{2!} - s^2\frac{\eta_3}{3!} + s^3\frac{\eta_4}{4!} - \cdots} \Leftrightarrow$$

$$\psi^*(s) = \frac{\rho}{\rho + s\frac{\eta_{2,\sigma}}{2!} - s^2\frac{\eta_3}{3!} + s^3\frac{\eta_4}{4!} - \cdots}$$

$$= \frac{q}{1-(1-q)\left[1-\left(\frac{\mu_2}{\mu_1}+\frac{\sigma^2}{2\lambda\mu_1}\right)\frac{s}{2!} - \left(\frac{\mu_3}{\mu_1}\right)\frac{s^2}{3!} + \left(\frac{\mu_4}{\mu_1}\right)\frac{s^3}{4!} + \cdots\right]}$$

where, $\rho = cq > 0$ is the profit rate, $\eta_k = \lambda\mu_k$, for $k = 1,2,\ldots$ represent moments of a Lévy measure (with $\eta_{2,\sigma} = \lambda\mu_2 + \sigma^2$). Letting $\sigma = 0$ gives another beautiful rendition of the expansion:

$$f_e^*(s)|_{\sigma=0} = 1 - \left(\frac{\mu_2}{\mu_1}\right)\frac{s}{2!} - \left(\frac{\mu_3}{\mu_1}\right)\frac{s^2}{3!} + \left(\frac{\mu_4}{\mu_1}\right)\frac{s^3}{4!} + \cdots = \sum_{k=1}^{\infty}\left(\frac{\mu_k}{\mu_1}\right)\frac{s^{k-1}}{k!} = E[e^{-sL_i} \mid \{L_i > 0\}],$$

where L_i is identified as a well-known equilibrium variable given by X_i, with density $f_e(x) = h_2(x) = \mu_1^{-1}\big(1-F(x)\big)$, and excess (factorial) moments $\breve{\mu}_k = \frac{\mu_{k+1}}{(k+1)\mu_1}$.

REMARK 9. *Let $\zeta_k = \frac{1}{k!}E[L^k(t)], k = 0,1,2,\ldots$ denote factorial reduced moments, which are found by normalizing with respect to the exponential moments of L. Hence, expanding equation above in power series, with ζ_k up to order 4 (from zero):*

$$\zeta_0 = P(L > 0) = \frac{1}{1+\theta}, \qquad \zeta_1 = \frac{\breve{\mu}_1}{\theta} + \frac{\sigma^2}{p}, \qquad \zeta_2 = \frac{\breve{\mu}_2}{2!\,\theta} + \zeta_1^2,$$

$$\zeta_3 = \frac{\breve{\mu}_3}{3!\,\theta} + \frac{\breve{\mu}_1\breve{\mu}_2}{\theta^2} + \left(\frac{\breve{\mu}_1}{\theta}\right)^3, \qquad \zeta_4 = \frac{\breve{\mu}_4}{4!\,\theta} + \frac{(1/3)\breve{\mu}_1\breve{\mu}_3 + (\breve{\mu}_2/2)^2}{\theta^2} + \frac{(3/2)\breve{\mu}_1^2\breve{\mu}_2}{\theta^3} + \left(\frac{\breve{\mu}_1}{\theta}\right)^4.$$

REMARK 10. *We can also obtain factorial reduced moments for the conditional random variable, $L \mid L > 0$ by dividing by $1 - q = (1+\theta)^{-1}$ from L.*

F Integro-differential equations

F1 Definition and example

DEFINITION 14 (Integro-differential equation). *Let $f(x)$ be a continuous function, such that $f(x_0) = f_0$ with $x_0 \geq 0$, then a general first-order, linear integro-differential equation takes the form*

$$\frac{d}{dx}f(x) + \int_{x_0}^{x} g\big(t, f(t)\big)\,dt = h\big(x, f(x)\big),$$

where, a general solution can be derived by applying the inverse Laplace transform to the solution of this integro-differential equation.

EXAMPLE. Consider a first-order, linear integro-differential equation in the form

$$f'(t) + f(t) + 2\int_0^t f(y)\,dy = 1, \qquad f(0) = 0.$$

By taking the Laplace transform (see APPENDIX D) for each term, the integro-differential equation transforms into

$$\big(sf^*(s) - f(0)\big) + \big(f^*(s)\big) + \left(\frac{2}{s}f^*(s)\right) = \frac{1}{s}.$$

Thus, $f^*(s) = \frac{1}{s^2+s+2}$, which can be inverted using contour integral methods (or *Mathematica*), to give $f(t) = \frac{2}{\sqrt{7}}e^{-t/2}\sin\left(\frac{\sqrt{7}}{2}t\right)$.

F2 Derivatives for $\Psi(x)$.

Differentiating the integro-differential equation recursively for $\Psi(0)$ (with $\Psi(0) = q$) yields:

$$\Psi'(0) = -\frac{\lambda}{c}\left(1 - \frac{\mu_1\lambda}{c}\right) = \frac{q(q-1)}{\mu_1},$$

$$\Psi''(0) = \frac{\lambda}{c}\left(1 - \frac{\mu_1\lambda}{c}\right)\left(f_X(0) - \frac{\lambda}{c}\right),$$

$$\Psi'''(0) = \frac{\lambda}{c}\left(1 - \frac{\mu_1\lambda}{c}\right)\left(f_X'(0) + 2\frac{\lambda}{c}f_X(0) - \left(\frac{\lambda}{c}\right)^2\right),$$

$$\Psi^{(4)}(0) = \frac{\lambda}{c}\left(1 - \frac{\mu_1\lambda}{c}\right)\left(f_X''(0) + 2\frac{\lambda}{c}f_X'(0) - \frac{\lambda}{c}[f_X(0)]^2 + 3\left(\frac{\lambda}{c}\right)^2 f_X(0) - \left(\frac{\lambda}{c}\right)^3\right).$$

REFERENCES

APPLEBAUM, D. (2004a) *Lévy Processes and Stochastic Calculus*, 2nd ed., Cambridge: Cambridge University Press.

APPLEBAUM, D. (2004b) Lévy Processes: From Probability to Finance and Quantum Groups, *Notices of the American Mathematical Society*, 51(11), pp. 1336-1347.

AVRAM, F., CHEDOM, D.F. AND HORVÁTH, A. (2011) On moments based Padé approximations of ruin probabilities, *Journal of computational and applied mathematics*, 235(10), pp. 3215–3228.

BENES, V.E. (1957) On queues with Poisson arrivals, *Annals of Mathematical Statistics* 28(1), pp. 670–677.

BOWERS, N.L., JR., GERBER, H.U., HICKMAN, J.C., JONES, D. A. AND NESBITT, C.J. (1986). *Actuarial Mathematics*, Itasca, Illinois: Society of Actuaries.

BRACEWELL, R.N. (2000) *The Fourier Transform and Its Applications,* 3rd ed., Boston: McGraw-Hill

BURNECKI, M., MISTA, P. AND WERON, A. (2003) *A new De Vylder type approximation of the ruin probability in infinite time*, Wroclaw, Poland: Wroclaw University of Technology.

CAI, J. AND GARRIDO, J. (2002) Asymptotic forms and bounds for tails of convolutions of compound geometric distributions, with applications. *Recent Advances in Statistical Methods,* Imperial College Press, London, pp. 114-131.

CENTENO, M.L. (2015) Lecture Notes, *Risk Theory.* Lisbon, Portugal: Universidade de Lisboa, 4 May.

DE VYLDER, F. (1996) *Advanced Risk Theory. A Self-Contained Introduction.* Editions de l'Université de Bruxelles.

DUFRESNE, F. AND GERBER, H.U. (1989) Three methods to calculate the probability of ruin, *ASTIN Bulletin*, 19(1), pp. 71-90.

DUFRESNE, F. AND GERBER, H.U. (1991) Risk theory for the compound Poisson process that is perturbed by diffusion, *Insurance: Mathematics and Economics,* 10(1), pp. 51-59.

EVANS, M., HASTINGS, N., AND PEACOCK, B. (2000) *Statistical Distributions,* 3rd ed., New York: Wiley.

FURRER, H.J., MICHNA, Z. AND WERON, A. (1997) Stable Lévy motion approximation in collective risk theory, *Insurance: Mathematics and Economics* 20(2), pp. 97–114.

FURRER, H.J. (1998) Risk processes perturbed by α-stable Lévy motion, *Scandinavian Actuarial Journal,* 1998(1), pp. 59-74.

GERBER, H.U. (1970) An extension of the renewal equation and its application in the collective theory of risk, *Scandinavian Actuarial Journal,* 1970(3-4), pp. 205-210.

GRANDELL, J. AND SEGERDAHL, C.O. (1971) A comparison of some approximations of ruin probabilities, *Scandinavian Actuarial Journal,* 1971(3-4), pp. 143-158.

GRANDELL, J. (2000) Simple approximations of ruin probabilities, *Insurance: Mathematics and Economics,* 26(2), pp. 157-173.

HESS, K.T. AND SCHMIDT, K.D. (2002) A comparison of models for the chain-ladder method, *Insurance: Mathematics and Economics,* 31(3), pp. 351-364.

JACINTO, A.C.F.S (2008) *Approximations to ruin probabilities in the perturbed model*. Master thesis, ISEG Technical University of Lisbon.

JOHNSON, M.A. AND TAAffE, M.R. (1989) Matching moments to phase distributions: Mixtures of erlang distributions of common order, *Stochastic Models*, 5(4), pp. 711–743.

KLUGMAN, S.A., PANJER, H.H. AND WILLMOT G.E. (2012) *Loss Models: From Data to Decisions*, 4th ed., Hoboken, N.J.: John Wiley & Sons.

PAPOULIS, A. (1984) *Probability, Random Variables, and Stochastic Processes,* 2nd ed., New York: McGraw-Hill.

PANJER, H.H. (1981). Recursive evaluation of a family of compound distributions, *ASTIN Bulletin*, 12(1), pp. 22-26.

PANJER, H.H. AND WILLMOT, G.E. (1993) *Insurance Risk Models*, Schaumberg, Illinois: The Society of Actuaries.

POLLACZEK, F. (1930) Über eine Aufgabe der Wahrscheinlichkeitstheorie, *Mathematische Zeitschrift*, 32(1), pp. 64-100. [German]

RAMSAY, C.M. (1992) A practical algorithm for approximating the probability of ruin, *Transactions of the Society of Actuaries*, 44(1), pp. 443-461.

ROLSKI, T., SCHMIDLI, H., SCHMIDT, V. & TEUGELS, J.L. (2008) *Stochastic Processes for Insurance & Finance,* Hoboken, New Jersey: Wiley.

SEIXAS, M.J.M. AND EGIDIO DOS REIS, A.D. (2013) *Some simple and classical approximations to ruin probabilities applied to the perturbed model*, AFMathConf2013 Proceedings of the Actuarial and Financial Mathematics Conference, Interplay between Finance and Insurance, Brussels.

SCHMIDLI (2001) Optimal proportional reinsurance policies in a dynamic setting, *Scandinavian Actuarial Journal,* 2001(1), pp. 55-68.

SIMAR, L. (1976) Maximum Likelihood Estimation of a Compound Poisson Process, *Annals of Mathematical Statistics,* 4(6), pp. 1200-1209.

SILVA, M.D.V. (2006) Um Processo de Risco Perturbado: Aproximações Numéricas Probabilidade de Runa, Master Thesis, Faculty of Science University of Oporto,

THORIN, O. (1974) Some comments on the Sparre Andersen model in the risk theory, *ASTIN Bulletin,* 8(1), pp. 104-125.

THORIN, O. AND WIKSTAD, N. (1977) Calculation of Ruin Probabilities when the Claim Distribution is Lognormal, *ASTIN Bulletin,* 9(1-2), pp. 231-246.

ZOLOTAREV, V.M. (1964) The First Passage Time of a Level and the Behaviour at Infinity for a Class of Processes with Independent Increments, *Theory of Probability & Its Applications,* 9(4), pp. 653-662.

ZOLOTAREV, V.M. (1986) One-dimensional stable distributions, *Bulletin of the American Mathematical Society,* 20(2), pp. 270-277.