A Note on Stability of Impulsive Scalar Delay Differential Equations

Dedicated to Professor Tibor Krisztin, on the occasion of his 60th birthday

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Abstract

For a class of scalar delay differential equations with impulses and satisfying a Yorke-type condition, criteria for the global asymptotic stability of the zero solution are established. These equations possess a non-delayed feedback term, which will be used to refine the general results on stability presented in recent literature. The usual requirements on the impulses are also relaxed. As an application, sufficient conditions for the global attractivity of a periodic solution for an impulsive periodic model are given.

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1 Introduction

In this paper, we consider a family of scalar non-autonomous delay differential equations (DDEs) with impulses, and establish a criterion for the global asymptotic stability of its trivial solution. In order to establish stability results, the basic approach is to control the growth of the delayed terms by imposing a Yorke-type condition coupled with limitations on the amplitude of the delays. Since the classic works of Wright [8], Yorke [13] and Yoneyama [12], this procedure has been often used by many authors, and has led to notable generalized versions of the so-called "$\frac{3}{2}$-conditions", see e.g. Liz et al. [7]. Some historical conjectures, such as Wright's conjecture, remain open, in spite of the long-time investigation by some mathematicians; we refer the reader to the recent work by Bánhelyi et al. [1]. On the other hand, to deal with the impulsive character of the equation, assumptions on lower and upper bounds for the jump discontinuities at the instants of impulses are prescribed here.

This note is a continuation of the research recently conducted by the authors in [3]. The family of impulsive DDEs under consideration possesses a non-delayed feedback term, which will be used to refine the general criterion for stability in [3].

Although we are mostly concerned with models with bounded, time-varing delays, the present approach encompasses DDEs with unbounded delays. We shall consider a very general setting for our method, not presenting however any theoretical results about existence and global continuation of solutions, since this has been the topic of a variety of papers; see some references given below. Nevertheless, we need to introduce some notation.

For $[\alpha,\beta]\subset\mathbb{R}$, denote by $B([\alpha,\beta];\mathbb{R})$ the space of bounded functions $\varphi:[\alpha,\beta]\to\mathbb{R}$ and by $PC([\alpha,\beta];\mathbb{R})$ the subspace of $B([\alpha,\beta];\mathbb{R})$ of functions which are piecewise continuous on $[\alpha,\beta]$ and left continuous on $(\alpha, \beta]$, endowed with the supremum norm. Define the space $PC = PC((-\infty, 0]; \mathbb{R})$ as the space of functions $\varphi : (-\infty, 0] \to \mathbb{R}$ whose restriction to each compact interval $[\alpha, \beta] \subset (-\infty, 0]$ is in the closure of $PC([\alpha, \beta]; \mathbb{R})$ in $B([\alpha, \beta]; \mathbb{R})$. Thus, each $\varphi \in PC$ is continuous everywhere except at most for an enumerable number of isolated points s for which $\varphi(s^-)$, $\varphi(s^+)$ exist and $\varphi(s^-) = \varphi(s)$. Denote by BPC the subspace of all bounded functions in PC, $BPC = \{\varphi \in PC : \varphi$ is bounded on $(-\infty, 0]\}$, with the supremum norm $\|\varphi\| = \sup_{s \le 0} |\varphi(s)|$.

Consider now a finite set of continuous delay functions $\tau_i : [0, \infty) \to [0, \infty)$, $i = 1, \ldots, m$, such that $\lim_{t \to \infty} (t - \tau_i(t)) = \infty$. The functions $d_i(t) = \inf_{s \ge t} (s - \tau_i(s))$ and $d(t) = \min_{1 \le i \le m} d_i(t)$ are continuous and non-decreasing. In what follows, and without loss of generality, we shall suppose that the functions $t \mapsto t - \tau_i(t)$ are non-decreasing; otherwise, they can be replaced by $d_i(t)$. For $t \ge 0$, we set

$$\tau(t) = \max_{1 \le i \le m} \tau_i(t), \; d(t) = t - \tau(t), \; d^2(t) = d(d(t)) \quad \text{for} \quad t \ge 0.$$

For each $t \ge 0$, the spaces $PC^i(t) = PC([-\tau_i(t), 0]; \mathbb{R})\, (1 \le i \le m)$ and $PC(t) = PC([-\tau(t), 0]; \mathbb{R})$ are taken as subspaces of BPC, with $PC^i(t) \subset PC(t) \subset PC$ for all i. For $x(t)$ defined on $(-\infty, a]$ and $\sigma \le a$, we denote by x_σ the function defined by $x_\sigma(s) = x(s + \sigma)$ for $s \le 0$.

Consider a family of scalar impulsive DDEs of the form

$$\begin{aligned} x'(t) + a(t)x(t) &= \sum_{i=1}^{m} f_i(t, x_t^i), \quad 0 \le t \ne t_k, \\ \Delta(x(t_k)) &:= x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad k \in \mathbb{N}, \end{aligned} \tag{1.1}$$

where: $x'(t)$ is the left-hand derivative of $x(t)$; $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$; $a : [0, \infty) \to [0, \infty)$ is piecewise continuous and $I_k : \mathbb{R} \to \mathbb{R}$ continuous, $k = 1, 2, \ldots$; $\tau_i : [0, \infty) \to [0, \infty)$ are continuous with $d_i(t) := t - \tau_i(t)$ non-decreasing, for $i = 1, \ldots, m$, and let $\tau(t) = \max_{1 \le i \le m} \tau_i(t)$; x_t^i denotes the restriction of $x(t)$ to the interval $[t - \tau_i(t), t]$, so that $f_i(t, x_t^i) = f_i(t, x_{|[t - \tau_i(t), t]})$, with $x_t^i \in PC^i(t)$ given by

$$x_t^i(\theta) = x(t + \theta) \quad \text{for} \quad -\tau_i(t) \le \theta \le 0;$$

$f_i(t, \varphi)$ is a functional defined for $t \ge 0$ and $\varphi \in PC^i(t)$ with some regularity discussed below. We shall also assume that $f(t, 0) = 0$ for $t \ge 0$ and $I_k(0) = 0$ for $k \in \mathbb{N}$, thus $x \equiv 0$ is a solution of (1.1). For the impulsive DDE (1.1), we consider initial conditions of the form $x_{t_0} = \varphi$, or in other words

$$x(t_0 + s) = \varphi(s), \qquad s \le 0, \tag{1.2}$$

with $t_0 \ge 0$ and $\varphi \in BPC$.

For $t \ge 0, \varphi \in PC^i(t)$ and $i \in \{1, \ldots, m\}$, we take the extension $\tilde{\varphi} \in BPC$ of φ which is $\varphi(-\tau_i(t))$ on $(-\infty, -\tau_i(t)]$. In this way, each function f_i can be regarded as the restriction of some function $F_i : [0, \infty) \times PC \to \mathbb{R}$, with $f_i(t, \varphi) = f_i(t, L_i(t, \tilde{\varphi})) =: F_i(t, \tilde{\varphi})$, where $L_i(t, \tilde{\varphi}) = \tilde{\varphi}_{|[-\tau_i(t), 0]}$. In view of our purposes, we assume that these extensions F_i of f_i are continuous or piecewise continuous (for simplicity, we abuse the language and refer to f_i as being continuous or piecewise continuous as well), but in fact less regularity could be prescribed. It is important to mention that these conditions together with the set of assumptions imposed in the next section imply that the initial value problem (1.1)-(1.2) has a unique solution $x(t)$ defined on $[t_0, \infty)$, which will be denoted by $x(t, t_0, \varphi)$, see e.g. [2, 5, 11].

We should emphasize that many authors restrict their analysis to impulsive DDEs with impulses given by linear functions $I_k(u) = b_k u$ $(k \in \mathbb{N})$, whereas we treat the more general case of impulses given by functions I_k satisfying $b_k u \le I_k(u) \le a_k u$, and prescribe some behaviour for the sequences $(b_k), (a_k)$. Our method to study the stability of the zero solution of (1.1) improves several results in the latest literature. Clearly, it is applicable to the study of the global attractivity of other solutions, such as periodic solutions, as illustrated in Section 3 with an example.

2 Preliminaries

In what follows, we denote

$$f(t,x_t)=\sum_{i=1}^{m} f_i(t,x_t^i), \quad t\geq 0,\ x_t\in BPC, \tag{2.3}$$

where $f_i(t,x_t^i)=f_i(t,x_{|_{[t-\tau_i(t),t]}}), \tau_i(t)\ (1\leq i\leq m)$ are as in (1.1)

In a previous paper [3], the authors gave sufficient conditions for the stability and global attractivity of the trivial solution of (1.1), relative to solutions with initial conditions (1.2) in BPC. The main assumptions in [3], where either hypotheses (H2) or (H3) were adopted (but not both simultaneously), are the following:

(H1) there exist positive sequences (a_k) and (b_k) such that

$$b_k x^2\leq x[x+I_k(x)]\leq a_k x^2, \quad x\in\mathbb{R},\ k\in\mathbb{N};$$

(H2) (i) the sequence $P_n=\prod_{k=1}^{n} a_k$ is bounded; (ii) $\int_0^\infty a(u)\,du=\infty$;

(H3) (i) the sequence $P_n=\prod_{k=1}^{n} a_k$ is convergent;

(ii) if $w:[0,\infty)\to\mathbb{R}$ is a bounded, non-oscillatory and piecewise differentiable function with $w'(t)w(t)\leq 0$ on (t_k,t_k+1), $k\in\mathbb{N}$, and $\lim_{t\to\infty} w(t)=c\neq 0$, then

$$\int_0^\infty f(s,w_s)\,ds=-\mathrm{sgn}(c)\infty;$$

(H4) there exist piecewise continuous functions $\lambda_{1,i},\lambda_{2,i}:[0,\infty)\to[0,\infty)$ such that

$$-\lambda_{1,i}(t)\mathcal{M}_t^i(\varphi)\leq f_i(t,\varphi_{|_{[-\tau_i(t),0]}})\leq \lambda_{2,i}(t)\mathcal{M}_t^i(-\varphi), \quad t\geq 0,\ \varphi\in PC(t), \tag{2.4}$$

where $\mathcal{M}_t^i(\varphi)=\max\{0,\sup_{\theta\in[-\tau_i(t),0]}\varphi(\theta)\}$, for $i=1,\ldots,m$;

(H5) there exists $T>0$ with $d(T)\geq 0$ such that

$$\alpha_1^*\alpha_2^*<1,$$

where the coefficients $\alpha_j^*:=\alpha_j^*(T)$ are given by

$$\alpha_j^*(T)=\sup_{t\geq T}\int_{t-\tau(t)}^{t}\sum_{i=1}^{m}\lambda_{j,i}(s)e^{-\int_s^t a(u)du}B_i(s)\,ds, \quad j=1,2, \tag{2.5}$$

and

$$B_i(t):=\max_{\theta\in[-\tau_i(t),0]}\left(\prod_{k:t+\theta\leq t_k<t} b_k^{-1}\right),\ i=1,\ldots,m. \tag{2.6}$$

The above hypotheses (H1) and (H4) imply that $I_k(0)=0$ and $f_i(t,0)=0$ for $k\in\mathbb{N}$, $t\geq 0, 1\leq i\leq m$, thus $x=0$ is an equilibrium point of (1.1). In (2.6), the standard convention that a product $B_i(t)$ is equal to one if the number of factors is zero is used. We recall here some usual definitions for stability.

Definition 2.1. Let $S \subset BPC$ be a set of initial conditions. The solution $x = 0$ of (1.1) is said to be **stable** in S if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$||\varphi|| < \delta \Rightarrow |x(t, t_0, \varphi)| < \varepsilon, \quad \text{for} \quad t \geq t_0,\, \varphi \in S.$$

We say that $x = 0$ of (1.1) is **globally attractive** in S if all solutions of (1.1) with initial conditions in S tend to zero as $t \to \infty$. We say that $x = 0$ is **globally asymptotically stable** if it is stable and global attractive. If either $S = BPC$ or it is clear which set S we are dealing with, we omit the reference to it.

In what concerns the stability of (1.1), some of the main results from [3] are summarized below (see [3, Theorems 2.1, 2.2, and 2.3]).

Theorem 2.2. *(i) Assume (H1), (H4), either (H2) or (H3), and $\alpha_1^* \alpha_2^* \leq 1$, where α_1^*, α_2^* are as in* (2.5)*. Then all solutions of* (1.1) *are defined and bounded on* $[0, \infty)$ *and the trivial solution of* (1.1) *is uniformly stable.*

(ii) Assume (H1), (H4), (H5), and either (H2) or (H3). Then the zero solution of (1.1) *is globally asymptotically stable.*

3 Asymptotic Stability

In this section, we claim that the assertions in Theorem 2.2 remain valid if (H5) is replaced by a weaker hypothesis and the other ones are kept unchanged. Instead of (H5), we shall impose:

(H5*) there exists $T > 0$ with $d^2(T) \geq 0$ such that

$$l\left(\alpha_1, \alpha_1^*\right) l\left(\alpha_2, \alpha_2^*\right) < 1, \tag{3.7}$$

where $l : \left\{(z, w) \in \mathbb{R}^2 : z \geq w \geq 0\right\} \to \mathbb{R}$ is defined by

$$l(z, w) = \begin{cases} w \min\left\{1, z - \frac{w}{2}\right\}, & w \leq 1 \\ \min\left\{w, z - \frac{1}{2}\right\}, & w > 1 \end{cases}, \tag{3.8}$$

and the coefficients $\alpha_j := \alpha_j(T)$ and $\alpha_j^* := \alpha_j^*(T)$ are given by

$$\alpha_j(T) = \sup_{t \geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{\int_{t-\tau(t)}^{s} a(u)\, du} B_i(s)\, ds, \tag{3.9}$$

$$\alpha_j^*(T) = \sup_{t \geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{-\int_{s}^{t} a(u)\, du} B_i(s)\, ds, \tag{3.10}$$

with $B_i(t)$ given by (2.6), for $i = 1, \ldots, m$ and $j = 1, 2$.

Some comments about our new hypothesis (H5*) are useful (for a discussion of the other ones, see [3]). The coefficients α_j^* are the ones in the former assumption (H5). Since $a(t) \geq 0$, for

$$\gamma_j^*(t) = \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{-\int_{s}^{t} a(u)\, du} B_i(s)\, ds, \gamma_j(t) = \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{\int_{t-\tau(t)}^{s} a(u)\, du} B_i(s)\, ds,$$

we have $\gamma_j(t) = \gamma_j^*(t) e^{\int_{t-\tau(t)}^{t} a(u)\, du}$, thus $\alpha_j^* \leq \alpha_j$ for $t \geq 0, j = 1, 2$. For $a : [0, \infty) \to [0, \infty)$ piecewise continuous and not identically zero, with the possible exception of a countable set of points, then $\gamma_j^*(t) < \gamma_j(t)$, $j = 1, 2$, for $t > 0$. As we shall see, (3.7) is satisfied if either $\alpha_1^*(T)\alpha_2^*(T) < 1$ or

$\alpha_1(T)\alpha_2(T) < 9/4$ for some $T \geq 0$. As a consequence, (H5) is strictly stronger than (H5*) for $a(t)$ as in (1.1) and such that $\liminf_{t\to\infty} \int_{t-\tau(t)}^{t} a(u)\,du > 0$. In some situations, which depend on the values of $a(t)$, one might have $\alpha_1^*(t)\alpha_2^*(t) > 1$ for all $t > 0$ and $\alpha_1(T)\alpha_2(T) < 9/4$ for some $T > 0$, in which case Theorem 2.2 is not applicable, but (H5*) is fulfilled. For a comparison with alternative hypotheses in the literature, see Remark 3.1, as well as [3] and references therein.

The proof of our main result, stated below, will be given in appendix.

Theorem 3.1. *(i) Assume (H1), (H4), either (H2) or (H3), and* $l(\alpha_1, \alpha_1^*)\, l(\alpha_2, \alpha_2^*) \leq 1$, *where* l, α_j, α_j^* $(j = 1, 2)$ *are defined by formulae* (3.8)-(3.10). *Then all solutions of* (1.1) *are defined and bounded on* $[0, \infty)$ *and the trivial solution of* (1.1) *is uniformly stable.*

(ii) Assume (H1), (H4), (H5), and either (H2) or (H3). Then the zero solution of* (1.1) *is globally asymptotically stable.*

In applications, it is useful to have criteria to easily check wether (3.7) is satisfied or not.

Theorem 3.2. *For* $l(z, w)$ *as in* (3.8) *and* α_j, α_j^*, $j = 1, 2$, *as in* (3.9),(3.10), *the estimate* (3.7) *is satisfied if one of the following conditions holds:*

(i) $\alpha_1^*\alpha_2^* < 1$;

(ii) $\alpha_1\alpha_2 < (3/2)^2$;

(iii) $L(\alpha_1)L(\alpha_2) < 1$, *where* $L(z) := l(z, z) = \begin{cases} \dfrac{z^2}{2}, & z \leq 1 \\ z - \dfrac{1}{2}, & z > 1 \end{cases}$.

Proof. Since $l(z, w) \leq w$ for $(z, w) \in \mathrm{dom}\,(l) = \{(x, y) \in \mathbb{R}^2 : x \geq y \geq 0\}$, condition $\alpha_1^*\alpha_2^* < 1$ implies (3.7). Now, we show that the generalized "$\frac{3}{2}$-type condition" (ii) is more restrictive than (iii). In fact, assuming that $0 < \alpha_1\alpha_2 < 9/4$, we have: if $\max\{\alpha_1, \alpha_2\} \leq 1$, obviously $L(\alpha_1)L(\alpha_2) < 1$; if $\min\{\alpha_1, \alpha_2\} > 1$, then $L(\alpha_1)L(\alpha_2) = \frac{(2\alpha_1-1)(2\alpha_2-1)}{4} \leq (2\alpha_1 - 1)\big(\frac{9}{2\alpha_1} - 1\big)\frac{1}{4} = \frac{-4\alpha_1^2+20\alpha_1-9}{8\alpha_1} < 1$; if $\alpha_1 \leq 1 < \alpha_2$ (similarly if $\alpha_2 \leq 1 < \alpha_1$), we get $L(\alpha_1)L(\alpha_2) = \frac{\alpha_1^2}{2}\big(\alpha_2 - \frac{1}{2}\big) \leq \frac{\alpha_1^2}{2}\big(\frac{9}{4\alpha_1} - \frac{1}{2}\big) = \frac{\alpha_1(9-2\alpha_1)}{8} < 1$.

Finally, we deduce that (iii) implies (3.7). It is sufficient to show that $l(z, w) \leq l(z, z)$ for any (z, w) with $z \geq w > 0$. For the case $z \leq 1$, we have $l(z, w) - l(z, z) \leq w(z - \frac{w}{2}) - \frac{z^2}{2} = -\frac{1}{2}(z - w)^2 \leq 0$. If $z > 1$ and $w \geq 1$, then clearly $l(z, w) \leq z - \frac{1}{2} = l(z, z)$. For $0 < w < 1 < z$, we have: if $z - \frac{w}{2} \geq 1$, then $l(z, w) - l(z, z) = w - (z - \frac{1}{2}) = \frac{w+1}{2} - z < 0$; if $z - \frac{w}{2} < 1$, then $l(z, w) - l(z, z) = -\frac{1}{2}(w^2 - 2wz + 2z - 1) < 0$. □

Remark 3.1. In [14], Zhang studied the stability of system (1.1) only for the situation $a(t) \equiv 0$ and $m = 1$. The global attractivity of the zero solution was proven assuming that the impulsive functions I_k satisfy (H1) with $a_k = 1$ for all $k \in \mathbb{N}$, the function $f = f_1$ satisfies (H3)(ii) and (H4), and that the "$\frac{3}{2}$-type condition" $\alpha_1\alpha_2 < (3/2)^2$ holds. As observed, with $a(t) = 0$ for all $t \geq 0$, then $\alpha_j = \alpha_j^*$, $j = 1, 2$, and condition $l(\alpha_1, \alpha_1^*)l(\alpha_2, \alpha_2^*) < 1$ reads as $L(\alpha_1)L(\alpha_2) < 1$, thus our Theorem 3.1 generalizes the stability result in [14, Theorem 2.2]. On the other hand, in [10], Yan considered (1.1) with $m = 1$ and obtained the global attractivity of its zero solution assuming a set of more restrictive hypotheses: again the impulsive functions I_k are required to satisfy (H1) with $a_k = 1$ for all $k \in \mathbb{N}$, the Yorke condition (H4) for $f = f_1$ in (2.3) was assumed with a unique function $\lambda_1(t) = \lambda_2(t) =: \lambda(t)$ providing the left and right growth control of f in (2.4), and the "$\frac{3}{2}$-type condition"

$$\alpha := \sup_{t\geq 0} \int_{t-\tau(t)}^{t} \lambda(s) e^{\int_{t-\tau(t)}^{s} a(u)\,du} B(s)\,ds < \frac{3}{2}, \tag{3.11}$$

with $B(t) = B_1(t)$ as in (2.6), was imposed. In the case $\lambda_1(t) = \lambda_2(t) = \lambda(t)$, it is clear that $\alpha_1 = \alpha_2 =: \alpha$ and $\alpha_1^* = \alpha_2^* =: \alpha^*$ and the inequality $l(\alpha_1, \alpha_1^*)l(\alpha_2, \alpha_2^*) < 1$ reads simply as

$$l(\alpha, \alpha^*) < 1. \tag{3.12}$$

If $\alpha^* \geq 1$, then inequalities (3.11) and (3.12) (with $T = 0$) are equivalent; however, if $\alpha^* < 1$, condition (3.11) is more restrictive than (3.12). In conclusion, our Theorem 3.1 also improves the stability result in [10, Theorem 4.2].

Remark 3.2. As $l(z, w)$ is a continuous function and condition (3.7) is a strict inequality, the definitions of α_j and α_j^*, $j = 1, 2$, given in (3.9) and (3.10) can be replaced by, respectively,

$$\alpha_i = \limsup_{t\to+\infty} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{\int_{t-\tau(t)}^{s} a(u)\,du} B_i(s)\,ds, \quad j = 1, 2, \tag{3.13}$$

$$\alpha_i^* = \limsup_{t\to+\infty} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{-\int_{s}^{t} a(u)du} B_i(s)\,ds, \quad j = 1, 2. \tag{3.14}$$

For the situation without impulses, we obtain the following criterion:

Corollary 3.1. *For $a, \tau_i : [0, +\infty) \to [0, +\infty)$ and $f_i(t, \varphi)$ as in* (1.1)*, and $\tau(t) = \max_{1\leq i\leq m} \tau_i(t)$, consider the scalar DDE*

$$x'(t) + a(t)x(t) = \sum_{i=1}^{m} f_i(t, x_t^i), \quad t \geq 0, \tag{3.15}$$

and assume either (H2)(ii) or (H3)(ii), the Yorke condition (H4), and $l(\alpha_1, \alpha_1^*)l(\alpha_2, \alpha_2^*) < 1$, where $l(\cdot, \cdot)$ is defined by (3.8) *and*

$$\alpha_j = \sup_{t\geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{\int_{t-\tau(t)}^{s} a(u)\,du}\,ds, \;\; \alpha_j^* = \sup_{t\geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) e^{-\int_{s}^{t} a(u)\,du}\,ds,$$

$j = 1, 2$, for some $T > 0$. Then the zero solution of (3.15) *is globally asymptotically stable.*

Example 3.1. Consider a periodic Lasota-Wazewska model with impulses and time independent delays multiple of the period (see e.g. [4, 6, 9]):

$$\begin{aligned} &N'(t) + a(t)N(t) = \sum_{i=1}^{n} b_i(t) e^{-\beta_i(t)N(t-m_i\omega)}, \quad 0 \leq t \neq t_k, \\ &\Delta N(t_k) := N(t_k^+) - N(t_k) = I_k(N(t_k)), \quad k = 1, 2, \dots, \end{aligned} \tag{3.16}$$

where $0 < t_1 < t_2 < \cdots < t_k < \cdots$ with $t_k \to \infty$, and

(f_0) the functions $a(t), b_i(t), \beta_i(t)$ are continuous, positive and ω-periodic and $m_i \in \mathbb{N}$, for some constant $\omega > 0$ and for $1 \leq i \leq n, t \in \mathbb{R}$;

(i_0) the functions $I_k : [0, \infty) \to \mathbb{R}$ are continuous with $I_k(0) = 0$, $u + I_k(u) > 0$ for $u > 0$, $k \in \mathbb{N}$; moreover, there is a positive integer p such that

$$t_{k+p} = t_k + \omega, \quad I_{k+p}(u) = I_k(u), \quad k \in \mathbb{N}, u > 0;$$

(i_1) there exist constants $a_1, \dots, a_p$ and $b_1, \dots, b_p$, with $b_k > -1$, and such that

$$b_k \leq \frac{I_k(x) - I_k(y)}{x - y} \leq a_k, \quad x, y \geq 0, x \neq y, k = 1, \dots, p;$$

(i_2) $\displaystyle\prod_{k=1}^{p}(1 + a_k) \leq 1$.

To fix our setting, and without loss of generality, we suppose that there are exactly p instants $t_1, t_2, \ldots, t_p$ of impulses on the interval $[0, \omega]$. In view of the biological interpretation of the model, only positive solutions of (3.16) are to be considered.

The existence a positive ω-periodic solution $N^*(t)$ of (3.17) has been established by some authors (see e.g. [4, 6]), under some severe additional restrictions, both on the impulses and on the delays. Here, we assume that such an ω-periodic solution $N^*(t)$ exists, and effect the change of variables $x(t) = N(t) - N^*(t)$. Eq. (3.16) is transformed into

$$\begin{aligned} &x'(t) + a(t)x(t) = \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N^*(t)}\Big[e^{-\beta_i(t)x(t-m_i\omega)} - 1\Big], \quad 0 \le t \ne t_k, \\ &\Delta x(t_k) = \tilde{I}_k(x(t_k)), \quad k \in \mathbb{N}, \end{aligned} \tag{3.17}$$

where

$$\tilde{I}_k(u) = I_k\big(N^*(t_k) + u\big) - I_k\big(N^*(t_k)\big), \quad k = 1, \ldots, p.$$

For (3.17), we take $S = \{\varphi \in PC([-\bar{m}\omega, 0]; \mathbb{R}) : \varphi(\theta) \ge -N^*(\theta) \text{ for } -\bar{m}\omega \le \theta < 0, \varphi(0) > -N^*(0)\}$, where $\bar{m} = \max_{1\le i\le n} m_i$, as the set of admissible initial conditions; the spaces $PC^i(t)$ in (1.1) are replaced here by $S^i(t) = \{\varphi \in PC^i(t) : \varphi(\theta) \ge -N^*(t-\theta) \text{ for } -m_i\omega \le \theta \le 0\}$.

In the next theorem, for an ω-periodic real function $f : \mathbb{R} \to \mathbb{R}$, we use the notation $\overline{f} := \sup_{t\in[0,\omega]} f(t)$.

Theorem 3.3. *Consider* (3.16) *and set* $\bar{m} = \max_{1\le i\le n} m_i$. *Assume* (f_0), (i_0)–(i_2) *and that there is a positive* ω*-periodic solution* $N^*(t)$ *of system* (3.16). *If either* $\sigma < 1$ *or* $\sigma\, e^{\bar{m}\int_0^\omega a(u)\,du} < \frac{3}{2}$, *where*

$$\begin{aligned} \sigma = B^{\bar{m}} &\left(\overline{\beta}\, \overline{N^*}(e^{\overline{\beta N^*}} - 1)\right)^{\frac{1}{2}} \left(1 - e^{-\bar{m}\int_0^\omega a(u)\,du}\right) \\ &\cdot \left[1 - \left(1 - e^{-\int_0^\omega a(u)\,du}\right)^{-1} \sum_{k=1}^{p} \min(b_k, 0)\right], \end{aligned} \tag{3.18}$$

and $B = \max\limits_{1\le l,j\le p} \prod_{k=1}^{j}(1 + b_{l+k})^{-1}$, *then* $N^*(t)$ *attracts any positive solution* $N(t)$ *of* (3.16).

Proof. It was proven in [3, Theorem 3.3] that the assumptions (f_0), (i_0)–(i_2) imply that (3.17) satisfies (H1), (H2) and (H4), with $\lambda_{j,i}(t)$, for $j = 1, 2, i = 1, \ldots, n$, given by

$$\lambda_{1,i}(t) = \beta_i(t)b_i(t)e^{-\beta_i(t)N^*(t)}, \ \lambda_{2,i}(t) = \frac{1}{\overline{N^*}}(e^{\overline{\beta N^*}} - 1)b_i(t)e^{-\beta_i(t)N^*(t)}, \ 0 \le t \ne t_k.$$

Note that condition (i_2) implies $B \ge 1$ for B defined above, hence for $t \ge 0$ and $1 \le i \le n$ we have

$$B_i(t) := \max_{\theta\in[-m_i\omega, 0]} \left(\prod_{k: t+\theta \le t_k < t} (b_k + 1)^{-1} \right) \le B^{m_i} \le B^{\bar{m}}.$$

For the sake of simplicity, in what follows we suppose that the coefficients b_k in (i_1) satisfy $b_k \in (-1, 0]$ $(1 \le k \le p)$; otherwise we may replace b_k by $\min\{0, b_k\}$, as it appears in (3.18).

Since $N^*(t)$ is an ω-periodic solution of (3.16), for $t > 0, t \ne t_k$, it was derived in [3] that

$$\begin{aligned} \alpha_1^*(t) &:= \int_{t-\bar{m}\omega}^{t} \sum_{i=1}^{n} \lambda_{1,i}(s)B_i(s)e^{-\int_s^t a(u)\,du}\,ds \\ &\le B^{\bar{m}}\overline{\beta}\overline{N^*}\left(1 - e^{-\bar{m}\int_0^\omega a(u)\,du}\right)\left[1 - \left(1 - e^{-\int_0^\omega a(u)\,du}\right)^{-1}\sum_{k=1}^{p} b_k\right] =: \sigma_1\,, \end{aligned}$$

$$\alpha_2^*(t) := \int_{t-\bar{m}\omega}^{t} \sum_{i=1}^{n} \lambda_{2,i}(s) B_i(s) e^{-\int_s^t a(u)\,du}\, ds$$
$$\leq B^{\bar{m}} (e^{\overline{\beta N^*}} - 1)\Big(1 - e^{-\bar{m}\int_0^\omega a(u)\,du}\Big)\left[1 - \Big(1 - e^{-\int_0^\omega a(u)\,du}\Big)^{-1} \sum_{k=1}^{p} b_k\right] =: \sigma_2.$$

We have $\sigma_1\sigma_2 = \sigma^2$, for σ as in (3.18). Clearly, condition $\sigma_1\sigma_2 < 1$ is equivalent to $\sigma < 1$. On the other, for the present situation

$$\alpha_j(t) = \alpha_j^*(t) e^{\bar{m}\int_0^\omega a(u)\,du}, \quad j = 1, 2,$$

thus $\sigma\, e^{\bar{m}\int_0^\omega a(u)\,du} < 3/2$ implies $\alpha_1\alpha_2 < 3/2$. The result follows by Theorems 3.1 and 3.2. □

4 Appendix: proof of Theorem 3.1

The proof of Theorem 3.1 follows exactly along the lines of the arguments in [3], with the exception that assumptions $\alpha_1^*\alpha_2^* \leq 1$ and $\alpha_1^*\alpha_2^* < 1$ are replaced by the weaker conditions $l(\alpha_1, \alpha_1^*) l(\alpha_2, \alpha_2^*) \leq 1$ and $l(\alpha_1, \alpha_1^*) l(\alpha_2, \alpha_2^*) < 1$, respectively. Therefore, here we only present the part of the proof which has to be modified accordingly: to be more precise, this amounts to substitute Lemma 2.4 in [3] by Lemma 4.1 below.

Recall the definition of $f(t, x_t)$ given in (2.3). A standard change of variables introduced in [10] is useful: let $x(t)$ be a solution of (1.1) on $[0, \infty)$, and define $y(t)$ by

$$y(t) = \prod_{k: 0 \leq t_k < t} J_k(x(t_k)) x(t), \tag{4.19}$$

where

$$J_k(u) := \frac{u}{u + I_k(u)}, \quad u \in \mathbb{R} \setminus \{0\}, k \in \mathbb{N}. \tag{4.20}$$

From (H1), we have

$$a_k^{-1} \leq J_k(u) \leq b_k^{-1} \quad \text{for} \quad u \neq 0,\ k \in \mathbb{N}. \tag{4.21}$$

In [10], Yan showed that $y(t)$ is a continuous function satisfying

$$y'(t) + a(t) y(t) = \prod_{k: 0 \leq t_k < t} J_k(x(t_k)) f(t, x_t), \quad t \geq 0,\ t \neq t_k. \tag{4.22}$$

Note that (H4) implies that $f_i(t, \varphi^i) \leq 0$ if $\varphi^i \geq 0$ and $f_i(t, \varphi^i) \geq 0$ if $\varphi^i \leq 0$, for $t \geq 0, \varphi^i \in PC^i(t), 1 \leq i \leq m$. This condition and either (H2) or (H3), jointly with hypothesis (H1), which enables us to control the impulses, were used in [3] to derived that all non-oscillatory solutions converge to zero as $t \to \infty$. To deal with oscillatory solutions, hypotheses (H1), (H4) and (H5) were imposed: some essential estimates on the amplitude of solutions were deduced in [3, Lemma 2.4], and further used to show that all oscillatory solutions go to zero as $t \to \infty$. As announced, we prove a lemma which asserts that the estimates given in [3, Lemma 2.4] remain true with $\alpha_1^*\alpha_2^* \leq 1$ replaced by the weaker hypothesis

$$l(\alpha_1, \alpha_1^*) l(\alpha_2, \alpha_2^*) \leq 1. \tag{4.23}$$

Lemma 4.1. *Assume (H1), (H4), and* (4.23) *for some $\alpha_j = \alpha_j(T) < \infty$ and $\alpha_j^* = \alpha_j^*(T)$ as in* (3.9) *and* (3.10) *respectively, $j = 1, 2$. Let $x(t)$ be a solution of* (1.1) *on $[0, \infty)$ and $y(t)$ defined by* (4.19)*. Then, for any $\eta > 0$ and $t_0 > T$ such that $d^2(t_0) > 0$ and $y(t_0) = 0$, the following conditions hold:*

(i) *If $-\eta \leq y(t) \leq \eta l(\alpha_2, \alpha_2^*)$ for $t \in [d^2(t_0), t_0]$, then $-\eta \leq y(t) \leq \eta l(\alpha_2, \alpha_2^*)$ for all $t \geq t_0$;*

(ii) *If* $-\eta l(\alpha_1,\alpha_1^*)\le y(t)\le\eta$ *for* $t\in[d^2(t_0),t_0]$, *then* $-\eta l(\alpha_1,\alpha_1^*)\le y(t)\le\eta$ *for all* $t\ge t_0$.

Proof. For simplicity of exposition, we consider the case $m=1$ in (1.1), so that (1.1) reads as

$$\begin{aligned}&x'(t)+a(t)x(t)=f(t,x_t),\quad 0\le t\ne t_k,\\&\Delta(x(t_k)):=x(t_k^+)-x(t_k)=I_k(x(t_k)),\quad k=1,2,\ldots,\end{aligned}\tag{4.24}$$

where $f(t,\varphi)$ is defined for $t\ge 0$ and $\varphi\in PC(t)$: in fact, a careful reading of this proof shows that the arguments are carried out in a straightforward way to the situation of $m>1$.

With $m=1$, condition (2.4) reads as

$$-\lambda_1(t)\mathcal{M}_t(\varphi)\le f(t,\varphi)\le\lambda_2(t)\mathcal{M}_t(-\varphi),\quad t\ge 0,\ \varphi\in PC(t),\tag{4.25}$$

for some piecewise continuous functions $\lambda_1,\lambda_2:[0,\infty)\to[0,\infty)$. We shall use the notation

$$A(t)=\int_0^t a(u)du,\quad t\ge 0.\tag{4.26}$$

Let $x(t)$ be a solution of (4.24), and recall that $y(t)$ given by (4.19) satisfies (4.22). We now prove (i); the proof of (ii) is similar, so we omit it.

If the assertion (i) is false, there exists $T_0>t_0$ such that either $y(T_0)>\eta l(\alpha_2,\alpha_2^*)$ or $y(T_0)<-\eta$. We consider these two situations separately.

Case 1. Suppose that $y(T_0)>\eta l(\alpha_2,\alpha_2^*)$ for some $T_0>t_0$, with $-\eta\le y(t)<y(T_0)$ for $t\in[d^2(t_0),T_0)$.

We first prove that there is $\xi_0\in[T_0-\tau(T_0),T_0]$ such that $y(\xi_0)=0$. Otherwise, we obtain necessarily that $y(t)>0$ for $t\in[T_0-\delta-\tau(T_0-\delta),T_0]$ and some small $\delta>0$ (recall that $y(t)$ and $\tau(t)$ are continuous), and from (4.22) and (4.25) it follows that

$$y'(t)\le -a(t)y(t)+\prod_{k:0\le t_k<t}J_k(x(t_k))\,\lambda_2(t)\mathcal{M}_t(-x_t)\le 0,\quad t\in[T_0-\delta,T_0],$$

implying that $y(T_0-\delta)\ge y(T_0)$, which contradicts the definition of T_0.

Choose $\xi_0\in[T_0-\tau(T_0),T_0]$ such that $y(\xi_0)=0$. We may suppose that $y(t)>0$ for $\xi_0<t<T_0$, thus $t_0\le\xi_0$. Let $A(t)$ be given by (4.26). By (4.19), (4.21), (4.22) and (4.25), for $s\in[\xi_0-\tau(\xi_0),T_0]\setminus\{t_k\}$ we obtain

$$\begin{aligned}\left(e^{A(s)}y(s)\right)'&=\prod_{k:0\le t_k<s}J_k(x(t_k))e^{A(s)}f(s,x_s)\le e^{A(s)}\lambda_2(s)\prod_{k:0\le t_k<s}J_k(x(t_k))\mathcal{M}_s(-x_s)\\&=e^{A(s)}\lambda_2(s)\prod_{k:0\le t_k<s}J_k(x(t_k))\\&\qquad\cdot\max\left\{0,\sup_{\theta\in[-\tau(s),0]}\left(-y(s+\theta)\prod_{k:0\le t_k<s+\theta}J_k(x(t_k))^{-1}\right)\right\}\\&=e^{A(s)}\lambda_2(s)\max\left\{0,\sup_{\theta\in[-\tau(s),0]}\left(-y(s+\theta)\prod_{k:s+\theta\le t_k<s}J_k(x(t_k))\right)\right\}\\&\le e^{A(s)}\lambda_2(s)B(s)\mathcal{M}_s(-y_s),\end{aligned}\tag{4.27}$$

with $B(s)=B_1(s)$ as in (2.6). Now, as $y_s(\theta)\ge-\eta$ for $s\in[\xi_0-\tau(\xi_0),T_0]$ and $\theta\in[-\tau(s),0]$, we have

$$\left(e^{A(s)}y(s)\right)'\le\eta e^{A(s)}\lambda_2(s)B(s),\quad\forall s\in[\xi_0-\tau(\xi_0),T_0]\setminus\{t_k\}.\tag{4.28}$$

Integrating over $[\xi_0, T_0]$, we get

$$y(T_0) \le \eta e^{-A(T_0)} \int_{\xi_0}^{T_0} e^{A(s)} \lambda_2(s) B(s)\, ds = \eta \int_{\xi_0}^{T_0} e^{-\int_s^{T_0} a(u)\, du} \lambda_2(s) B(s)\, ds \le \eta \alpha_2^*,$$

and deduce that

$$y(T_0) \le \eta \alpha_2^*. \tag{4.29}$$

From (4.28) and integrating over $[s, \xi_0]$, with $s \in [\xi_0 - \tau(\xi_0), \xi_0]$, we obtain

$$-y(s) \le \eta e^{-A(s)} \int_s^{\xi_0} e^{A(r)} \lambda_2(r) B(r)\, dr = \eta \int_s^{\xi_0} \lambda_2(r) e^{\int_s^r a(u)\, du} B(r)\, dr,$$

which implies that

$$y(s) \ge -\eta \int_s^{\xi_0} \lambda_2(r) e^{\int_s^r a(u)\, du} B(r)\, dr, \quad \forall s \in [\xi_0 - \tau(\xi_0), \xi_0]. \tag{4.30}$$

For $s \in [\xi_0, T_0]$ and $\theta \in [-\tau(s), 0]$, we therefore have $y(s+\theta) > 0$ if $s + \theta \in (\xi_0, T_0]$, and $y(s+\theta) \ge -\eta \int_{s+\theta}^{\xi_0} \lambda_2(r) B(r) e^{\int_{s+\theta}^r a(u)du} dr$ if $s + \theta \in [\xi_0 - \tau(\xi_0), \xi_0]$. Now, for $s \in [\xi_0, T_0] \setminus \{t_k\}$, we have

$$\begin{aligned}
\left(e^{A(s)} y(s)\right)' &\le e^{A(s)} \lambda_2(s) \max\left\{0, \sup_{\theta \in [-\tau(s), 0]} \left(-y(s+\theta) \prod_{k: s+\theta \le t_k < s} J_k(x(t_k)) \right)\right\} \\
&\le e^{A(s)} \lambda_2(s) \cdot \max\left\{0, \sup_{\theta \in [-\tau(s), 0]} \left(\eta \int_{s+\theta}^{\xi_0} \lambda_2(r) B(r) e^{\int_{s+\theta}^r a(u)du} dr \prod_{k: s+\theta \le t_k < s} J_k(x(t_k)) \right)\right\} \\
&\le \eta e^{A(s)} \lambda_2(s) B(s) \int_{s-\tau(s)}^{\xi_0} \lambda_2(r) B(r) e^{\int_{s-\tau(s)}^r a(u)du} dr.
\end{aligned} \tag{4.31}$$

From (4.28) and (4.31), for all $s \in [\xi_0, T_0] \setminus \{t_k\}$ we have

$$\left(e^{A(s)} y(s)\right)' \le \eta e^{A(s)} \lambda_2(s) B(s) \min\left\{1, \int_{s-\tau(s)}^{\xi_0} \lambda_2(r) B(r) e^{\int_{s-\tau(s)}^r a(u)du} dr\right\}. \tag{4.32}$$

For $\Lambda_2 := \int_{\xi_0}^{T_0} \lambda_2(r) B(r) e^{-\int_r^{T_0} a(u)du} dr$, clearly $\Lambda_2 \le \alpha_2^*$. Integrating over $[\xi_0, T_0]$, we obtain

$$\begin{aligned}
y(T_0) &\le \eta \int_{\xi_0}^{T_0} \lambda_2(s) B(s) e^{-\int_s^{T_0} a(u)du} \left(\int_{s-\tau(s)}^{\xi_0} \lambda_2(r) B(r) e^{\int_{s-\tau(s)}^r a(u)du} dr \right) ds \\
&= \eta \int_{\xi_0}^{T_0} \lambda_2(s) B(s) e^{-\int_s^{T_0} a(u)du} \\
&\quad \cdot \left(\int_{s-\tau(s)}^{s} \lambda_2(r) B(r) e^{\int_{s-\tau(s)}^r a(u)du} dr - \int_{\xi_0}^{s} \lambda_2(r) B(r) e^{\int_{s-\tau(s)}^r a(u)du} dr \right) ds \\
&\le \eta \left(\alpha_2 \int_{\xi_0}^{T_0} \lambda_2(s) B(s) e^{-\int_s^{T_0} a(u)du} ds \right. \\
&\quad \left. - \int_{\xi_0}^{T_0} \lambda_2(s) B(s) e^{-\int_s^{T_0} a(u)du} \int_{\xi_0}^{s} \lambda_2(r) B(r) e^{\int_{s-\tau(s)}^r a(u)du} dr ds \right).
\end{aligned} \tag{4.33}$$

Using the definition of Λ_2, (4.33) yields the estimate

$$\begin{aligned} y(T_0) &\leq \eta\left(\alpha_2\Lambda_2 - \int_{\xi_0}^{T_0} \lambda_2(s)B(s)e^{-\int_s^{T_0} a(u)du} \int_{\xi_0}^{s} \lambda_2(r)B(r)e^{-\int_r^{T_0} a(u)du} dr ds\right) \\ &= \eta\left(\alpha_2\Lambda_2 - \frac{1}{2}\left(\int_{\xi_0}^{T_0} \lambda_2(s)B(s)e^{-\int_s^{T_0} a(u)du} ds\right)^2\right) \\ &= \eta\left(\alpha_2\Lambda_2 - \frac{1}{2}\Lambda_2^2\right). \qquad (4.34) \end{aligned}$$

Since $\Lambda_2 \leq \alpha_2^* \leq \alpha_2$ and the function $x \mapsto \alpha_2 x - x^2/2$ is increasing on $(-\infty, \alpha_2]$, we obtain

$$y(T_0) \leq \eta\alpha_2^*\left(\alpha_2 - \frac{\alpha_2^*}{2}\right) \qquad (4.35)$$

and, from (4.29) and (4.35), we conclude that

$$y(T_0) \leq \eta\alpha_2^* \min\left\{1, \alpha_2 - \frac{\alpha_2^*}{2}\right\}. \qquad (4.36)$$

If $\alpha_2^* \leq 1$, then we obtain $y(T_0) \leq \eta l(\alpha_2, \alpha_2^*)$, which is not possible by definition of T_0, thus we have $\alpha_2^* > 1$.

If $\Lambda_2 \leq 1$, again by (4.34) we get

$$y(T_0) \leq \eta\left(\alpha_2 - \frac{1}{2}\right). \qquad (4.37)$$

If $\Lambda_2 > 1$, then there exists $\zeta \in (\xi_0, T_0)$ such that

$$\int_{\zeta}^{T_0} \lambda_2(r)B(r)e^{-\int_s^{T_0} a(u)du} dr = 1,$$

and from (4.32) we have

$$\begin{aligned} \int_{\xi_0}^{T_0} \left(e^{A(s)}y(s)\right)' ds &\leq \int_{\xi_0}^{\zeta} \eta e^{A(s)}\lambda_2(s)B(s) ds \\ &+ \int_{\zeta}^{T_0} \eta e^{A(s)}\lambda_2(s)B(s) \int_{s-\tau(s)}^{\xi_0} \lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r} a(u)du} dr ds, \end{aligned}$$

which implies

$$\begin{aligned}
y(T_0) &\leq \eta\left[\int_{\xi_0}^{\zeta}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}ds\right.\\
&\quad\left.+\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\left(\int_{s-\tau(s)}^{\xi_0}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr\right)ds\right]\\
&\leq \eta\left[\int_{\xi_0}^{\zeta}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}ds+\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\right.\\
&\quad\left.\cdot\left(\int_{s-\tau(s)}^{\zeta}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr-\int_{\xi_0}^{\zeta}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr\right)ds\right]\\
&\leq \eta\left[\int_{\xi_0}^{\zeta}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}ds\right.\\
&\quad-\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\left(\int_{\xi_0}^{\zeta}\lambda_2(r)B(r)e^{-\int_r^{T_0}a(u)du}dr\right)ds\\
&\quad\left.+\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\left(\int_{s-\tau(s)}^{\zeta}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr\right)ds\right]\\
&= \eta\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\left(\int_{s-\tau(s)}^{\zeta}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr\right)ds,
\end{aligned}$$

because $\Lambda_2 > 1$. This yields

$$\begin{aligned}
y(T_0) &\leq \eta\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\left(\int_{s-\tau(s)}^{s}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr\right.\\
&\quad\left.-\int_{\zeta}^{s}\lambda_2(r)B(r)e^{\int_{s-\tau(s)}^{r}a(u)du}dr\right)ds\\
&\leq \eta\int_{\zeta}^{T_0}\lambda_2(s)B(s)e^{-\int_s^{T_0}a(u)du}\left(\alpha_2-\int_{\zeta}^{s}\lambda_2(r)B(r)e^{-\int_r^{T_0}a(u)du}dr\right)ds\\
&= \eta\left[\alpha_2-\frac{1}{2}\left(\int_{\zeta}^{T_0}\lambda_2(r)B(r)e^{-\int_r^{T_0}a(u)du}dr\right)^2\right]=\eta\left(\alpha_2-\frac{1}{2}\right). \qquad (4.38)
\end{aligned}$$

Thus, from (4.29), (4.37) and (4.38), we obtain

$$y(T_0)\leq \eta\min\left\{\alpha_2^*,\alpha_2-\frac{1}{2}\right\}, \qquad (4.39)$$

and, since $\alpha_2^* > 1$, we conclude that

$$y(T_0)\leq \eta l(\alpha_2,\alpha_2^*),$$

which again contradicts the definition of T_0.

Case 2. Suppose that $y(T_0) < -\eta$ for some $T_0 > t_0$, with $y(T_0) < y(t) \leq \eta l(\alpha_2, \alpha_2^*)$ for all $t \in [t_0, T_0)$.

Reasoning as above, we deduce that there is $\xi_0 \in [t_0, T_0) \cap [T_0 - \tau(T_0), T_0)$ such that $y(\xi_0) = 0$ and $y(t) < 0$ for $\xi_0 < t \le T_0$. Since $y_s(\theta) \le \eta l(\alpha_2, \alpha_2^*)$ for $s \in [\xi_0 - \tau(\xi_0), T_0]$ and $\theta \in [-\tau(s), 0]$, by "dual" arguments, for all $s \in [\xi_0, T_0] \setminus \{t_k\}$ we now obtain

$$\left(e^{A(s)}y(s)\right)' \ge -\eta l(\alpha_2, \alpha_2^*)e^{A(s)}\lambda_1(s)B(s)\min\left\{1, \int_{s-\tau(s)}^{\xi_0} \lambda_1(r)B(r)e^{\int_{s-\tau(s)}^{r} a(u)du}dr\right\} \tag{4.40}$$

instead of (4.32). On one hand, integration over $[\xi_0, T_0]$ leads to

$$\begin{aligned} y(T_0) &\ge -\eta l(\alpha_2, \alpha_2^*)e^{-A(T_0)} \int_{\xi_0}^{T_0} e^{A(s)}\lambda_1(s)B(s)\, ds \\ &= -\eta l(\alpha_2, \alpha_2^*) \int_{\xi_0}^{T_0} e^{-\int_s^{T_0} a(u)\, du}\lambda_1(s)B(s)\, ds \ge -\eta l(\alpha_2, \alpha_2^*)\alpha_1^*, \end{aligned}$$

which implies

$$y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*)\alpha_1^*. \tag{4.41}$$

On the other hand, defining $\Lambda_1 := \int_{\xi_0}^{T_0} \lambda_1(r)B(r)e^{-\int_r^{T_0} a(u)du}dr$, and again integrating (4.40) over $[\xi_0, T_0]$, by dual arguments as in (4.34), we derive

$$y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*)\left(\alpha_1\Lambda_1 - \frac{1}{2}\Lambda_1^2\right).$$

Since $\Lambda_1 \le \alpha_1^* \le \alpha_1$ and the function $x \mapsto \alpha_1 x - x^2/2$ is increasing on $(-\infty, \alpha_1]$, we obtain

$$y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*)\alpha_1^*\left(\alpha_1 - \frac{\alpha_1^*}{2}\right), \tag{4.42}$$

and from (4.41) and (4.42) we conclude that

$$y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*)\alpha_1^* \min\left\{1, \alpha_1 - \frac{\alpha_1^*}{2}\right\}. \tag{4.43}$$

If $\alpha_1^* \le 1$, then, using the inequality (4.23), we obtain $y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*)l(\alpha_1, \alpha_1^*) \ge -\eta$, which is not possible. Thus we have $\alpha_1^* > 1$.

By similar arguments to the ones in (4.37), (4.38) and from (4.41), we now obtain

$$y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*) \min\left\{\alpha_1^*, \alpha_1 - \frac{1}{2}\right\}, \tag{4.44}$$

with $\alpha_1^* > 1$. Consequently, using again the inequality (4.23), we deduce

$$y(T_0) \ge -\eta l(\alpha_2, \alpha_2^*)l(\alpha_1, \alpha_1^*) \ge -\eta,$$

which is a contradiction. This ends the proof. □

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