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Relations between crossed modules of different algebras

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Relations between crossed modules of different algebras

Fdo.: Rafael Fernández Casado

Memoria para optar al grado de Doctor realizada en el Departamento de Álgebra de la Universidad de Santiago de Compostela bajo la dirección de los Profesores D. Emzar Khmaladze y D. Manuel Ladra González.

Santiago de Compostela, a 28 de septiembre de 2015.

Fdo.: Emzar Khmaladze

Fdo.: Manuel Ladra González

Relations between crossed modules of different algebras

Dr. Emzar Khmaladze y Dr. Manuel Ladra González,

AUTORIZAMOS la presentación de la Tesis Doctoral con título **Relations between crossed modules of different algebras**, realizada por D. Rafael Fernández Casado bajo nuestra dirección en el Departamento de Álgebra de la Universidad de Santiago de Compostela, para optar al grado de Doctor por la Universidad de Santiago de Compostela.

Santiago de Compostela, a 28 de septiembre de 2015.

Fdo.: Emzar Khmaladze

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Resumen de la Tesis Doctoral:

Relaciones entre módulos cruzados de diferentes álgebras

Resumen abreviado:

En el presente trabajo extendemos a módulos cruzados la adjunción entre el funtor liezación $\mathtt{Lie}_{\mathtt{As}}\colon \mathsf{As} \to \mathsf{Lie}$ y el funtor álgebra envolvente universal $\mathtt{U}\colon \mathsf{Lie} \to \mathsf{As}$. Además, probamos que existe un isomorfismo entre las categorías de módulos por la izquierda sobre un módulo cruzado de álgebras de Lie y módulos por la izquierda sobre su módulo cruzado envolvente universal. Asimismo, construimos una generalización a dimensión 2 de la adjunción entre el funtor $\mathtt{Lb}\colon \mathsf{Dias} \to \mathsf{Lb}$ y el funtor diálgebra envolvente universal $\mathtt{U}_{\mathtt{d}}\colon \mathsf{Lb} \to \mathsf{Dias}$. Debido a que esta última generalización involucra a los módulos cruzados de diálgebras, damos una definición adecuada de los mismos, basada en la definición general de módulos cruzados en categorías de interés. Además, definimos el concepto de 2-diálgebra estricta, por analogía con la noción de 2-álgebra asociativa estricta. Asimismo, probamos que las categorías de módulos cruzados de diálgebras y 2-diálgebras estrictas son equivalentes. También construimos la diálgebra de los tetramultiplicadores, que resultará ser el actor en la categoría de diálgebras bajo ciertas condiciones. Además, a partir de un módulo cruzado de álgebras de Leibniz, construimos un actor general para el mismo, que resultará ser el actor en ciertos casos particulares.

El concepto de módulo cruzado de grupos fue formulado por primera vez por Whitehead a finales de la década de los 40 [83]. Poco después, Mac Lane y Whitehead [70] probaron que los módulos cruzados pueden utilizarse como modelo algebraico para los CW-espacios conexos cuyos grupos de homotopía son triviales en dimensión mayor que 2. Los módulos cruzados generalizan al mismo tiempo los conceptos de subgrupo normal y módulo sobre un grupo. Desde su introducción han jugado un papel muy importante en diversas áreas de las matemáticas, en particular en teoría de homotopía. Por ejemplo, aparecen en varios problemas de clasificación de tipos homotópicos en dimensión baja y en las generalizaciones del Teorema de van Kampen.

Más allá de su valor como herramienta para la teoría de homotopía, los módulos cruzados han sido estudiados como objetos algebraicos de propio derecho. Por ejemplo, Norrie extendió a módulos cruzados algunos conceptos y estructuras propias de la teoría de grupos en su tesis doctoral [72]. En particular, construyó el actor de un módulo cruzado de grupos e introdujo la noción de centro de un módulo cruzado, así como los conceptos de módulo cruzado completo y perfecto.

Los módulos cruzados de otras estructuras, no sólo de grupos, también han sido estudiados. Por ejemplo, en [60] Kassel y Loday usan módulos cruzados de álgebras de Lie como herramienta computacional para dar una interpretación de la tercera cohomología relativa de Chevalley-Eilenberg de álgebras de Lie. Guin [51] desarrolló la cohomología no abeliana en dimensiones bajas de álgebras de Lie con coeficientes en módulos cruzados de álgebras de Lie, la cual fue extendida posteriormente a dimensiones superiores [57]. La homología interna (cotriple) y la homología de Chevalley-Eilenberg de módulos cruzados de álgebras de Lie son objeto de investigación en [23, 35]. Además, los módulos cruzados de álgebras de Lie también aparecen en el problema de "categorificación" de la teoría de álgebras de Lie [4] como una formulación equivalente de las 2-álgebras de Lie estrictas.

Los módulos cruzados de álgebras asociativas no han sido estudiados con tanta profundidad como sus análogos para grupos y álgebras de Lie. Sin embargo, en los trabajos de Dedecker y Lue [33, 67] tienen un papel relevante como coeficientes para la cohomología no abeliana en dimensiones bajas. Además, en la tesis doctoral de Shammu [80] se estudia su estructura algebraica y categórica. Baues y Minian [6] hicieron uso de ellos para representar la cohomología de Hochschild de álgebras asociativas y, en el reciente artículo [34], se estudia la homología de Hochschild y la homología cíclica (cotriple) de los módulos cruzados de álgebras asociativas.

En lo que respecta a módulos cruzados de álgebras de Leibniz, una generalización no antisimétrica de las álgebras de Lie, introducida por Bloh en [8], estos fueron usados por primera vez por Loday y Pirashvili [66] para estudiar la cohomología de las álgebras de Leibniz. En una situación no conmutativa, en la que las álgebras de Lie son substituidas por las álgebras de Leibniz, el papel de las álgebras asociativas es jugado por las álgebras diasociativas (o simplemente diálgebras), introducidas y estudiadas por Loday [65]. Hasta la presentación de este trabajo, los módulos cruzados de diálgebras no habían sido estudiados.

Brown y Spencer demostraron en [13] que las categorías internas en la categoría de grupos son equivalentes a los módulos cruzados de grupos. En su artículo se menciona que este hecho ya era conocido por Verdier en 1965 y fue utilizado por Duskin en [36]. El trabajo de Brown y Spencer sirvió a Porter de inspiración para investigar para qué otras categorías la equivalencia entre módulos cruzados y categorías internas sigue siendo cierta, aunque el trabajo resultante no fue publicado. Pocos años después, Loday definió en [63] el concepto de cat^n-grupo, que es equivalente a la noción de categoría interna de orden n en grupos. Loday dio también la definición de cuadrado cruzado, la generalización a dimensión 2 de un módulo cruzado. En 1984, [37] extendió a dimensión arbitraria los resultados no publicados de Porter y demostró que, dada una categoría de Ω-grupos $\mathcal{C}$, las siguientes estructuras son equivalentes:

(i) Categorías internas de orden n en $\mathcal{C}$,

(ii) cat^n-objetos en $\mathcal{C}$,

(iii) n-cubos cruzados en $\mathcal{C}$,

(iv) n-objetos simpliciales en $\mathcal{C}$ cuyos complejos normales tengan longitud 1,

(v) Módulos cruzados de orden n en $\mathcal{C}$.

Los principales resultados presentados en esta tesis tienen a los módulos cruzados sobre las categorías de grupos, de álgebras de Lie, de álgebras asociativas, de álgebras de Leibniz y de diálgebras como principales protagonistas. Estas cinco categorías son categorías de interés (puede consultarse, por ejemplo, en [71]), las cuales son a su vez un caso particular de categorías de Ω-grupos. Por este motivo, comenzamos el primer capítulo recordando la equivalencia entre los módulos cruzados y las categorías internas en una categoría de Ω-grupos (Sección 1.1).

Los cat^1-objetos juegan un papel fundamental en las demostraciones de algunos de nuestros principales resultados. Por ello, en la Sección 1.2 recordamos la equivalencia entre módulos cruzados y cat^1-objetos para cada una de las cinco categorías particulares consideradas en esta tesis, es decir, grupos **Gr**, álgebras de Lie **Lie**, álgebras asociativas **As**, álgebras de Leibniz **Lb** y diálgebras **Dias**. Además, en las cinco subsecciones presentes en esta sección hacemos un breve repaso de las características fundamentales de los módulos cruzados en dichas categorías. Las correspondientes categorías de módulos cruzados son denotadas por **XGr**, **XLie**, **XAs**, **XLb** y **XDias** respectivamente.

Aunque la equivalencia entre módulos cruzados de diálgebras y cat^1-diálgebras es consecuencia de la misma equivalencia para categorías de Ω-grupos, la descripción explícita no había sido hecha con anterioridad a esta tesis, pues el concepto de módulo cruzado de diálgebras no había sido siquiera considerado.

Recordamos aquí la definición de diálgebra asociativa, dada por primera vez por Loday [65].

Definición 1.2.47. *Una diálgebra asociativa (o simplemente diálgebra), es un K-módulo D equipado con dos aplicaciones K-lineales*

$$\dashv, \vdash : D \otimes D \to D,$$

llamadas producto por la izquierda y por la derecha respectivamente, que satisfacen los siguientes axiomas:

$$\begin{aligned}
(x \dashv y) \dashv z &= x \dashv (y \vdash z),\\
(x \dashv y) \dashv z &= x \dashv (y \dashv z),\\
(x \vdash y) \dashv z &= x \vdash (y \dashv z),\\
(x \dashv y) \vdash z &= x \vdash (y \vdash z),\\
(x \vdash y) \vdash z &= x \vdash (y \vdash z),
\end{aligned}$$

para todo $x, y, z \in D$. Un morfismo de diálgebras es una aplicación K-lineal que conserva el producto por la izquierda y por la derecha.

Teniendo en cuenta las definiciones de módulo cruzado y cat^1-objeto en las categorías de grupos con operaciones, damos las siguientes definiciones de módulo cruzado de diálgebras y cat^1-diálgebra:

Definición 1.2.52. *Un módulo cruzado de diálgebras (L, D, μ) es un morfismo de diálgebras $\mu : L \to D$ junto con una acción de D sobre L tal que*

$$\mu(x * a) = x * \mu(a) \quad y \quad \mu(a * x) = \mu(a) * x,$$
$$\mu(a_1) * a_2 = a_1 * a_2 = a_1 * \mu(a_2).$$

para todo $x \in D$, $a_1, a_2 \in L$.

Nótese que en las identidades de la definición anterior, $* = \dashv$ y $* = \vdash$.

Definición 1.2.59. *Una cat^1-diálgebra (D_1, D_0, σ, τ) es una diálgebra D_1 junto con una subdiálgebra D_0 y dos morfismos de diálgebras $\sigma, \tau : D_1 \to D_0$ tales que*

$$\sigma|_{D_0} = \tau|_{D_0} = \mathrm{id}_{D_0},$$
$$\operatorname{Ker}\sigma * \operatorname{Ker}\tau = 0 = \operatorname{Ker}\tau * \operatorname{Ker}\sigma.$$

Denotamos por $\mathsf{C^1Dias}$ la categoría de cat^1-diálgebras. Por analogía con la situación general para categorías de Ω-grupos, obtenemos el siguiente resultado:

Proposición 1.2.61. *Las categorías* XDias *y* $\mathsf{C^1Dias}$ *son equivalentes.*

Los módulos cruzados de grupos, álgebras de Lie y álgebras asociativas son equivalentes a los 2-grupos estrictos, las 2-álgebras de Lie estrictas y las 2-álgebras asociativas estrictas, respectivamente. En la última sección del primer capítulo damos las siguientes definiciones de 2-álgebra de Leibniz estricta y 2-diálgebra estricta.

Definición 1.3.9. *Una 2-álgebra de Leibniz estricta es un 2-módulo L, junto con un funtor bilineal, el corchete $[-,-]\colon L \times L \to L$, tal que*

$$[[x,y]_i, z]_i = [x, [y,z]_i]_i + [[x,z]_i, y]_i.$$

para todo $x, y, z \in L_i$, $i = 0, 1$. Un morfismo de 2-álgebras de Leibniz estrictas, $F\colon L \to L'$, es un funtor lineal tal que

$$F_i([x,y]_i) = [F_i(x), F_i(y)]'_i.$$

para todo $x, y \in L_i$, $i = 0, 1$.

Definición 1.3.10. *Una 2-diálgebra estricta es un 2-módulo D, junto con dos funtores bilineales, los productos por la izquierda y por la derecha $-\dashv-\colon D \times D \to D$ y*

$- \vdash -\colon D \times D \to D$, tales que

$$\begin{aligned}
(x \dashv_i y) \dashv_i z &= x \dashv_i (y \vdash_i z),\\
(x \dashv_i y) \dashv_i z &= x \dashv_i (y \dashv_i z),\\
(x \vdash_i y) \dashv_i z &= x \vdash_i (y \dashv_i z),\\
(x \dashv_i y) \vdash_i z &= x \vdash_i (y \vdash_i z),\\
(x \vdash_i y) \vdash_i z &= x \vdash_i (y \vdash_i z).
\end{aligned}$$

para todo $x, y, z \in D_i$, $i = 0, 1$. Un morfismo de 2-diálgebras estrictas, $F\colon D \to D'$, es un funtor lineal tal que

$$F_i(x \dashv_i y) = F_i(x) \dashv'_i F_i(y) \quad y \quad F_i(x \vdash_i y) = F_i(x) \vdash'_i F_i(y)$$

para todo $x, y \in D_i$, $i = 0, 1$.

La primera de las definiciones concuerda con la que aparece en [81], donde las 2-álgebras de Leibniz estrictas se definen como un caso particular de las 2-álgebras de Leibinz semiestrictas. La segunda definición es original y está construida por analogía con la definición de las 2-álgebras asociativas estrictas, dada por Khmaladze en [61]. A partir de ambas definiciones conseguimos el siguiente resultado y su corolario, con los que cerramos el primer capítulo.

Teorema 1.3.11. *Las categorías* **IDias** *(respectivamente* **ILb***) y* **S2Dias** *(respectivamente* **S2Lb***) son isomorfas.*

Corolario 1.3.12. *Las categorías* **XDias** *(respectivamente* **XLb***) y* **S2Dias** *(respectivamente* **S2Lb***) son equivalentes.*

Es un hecho conocido que el actor de un grupo está dado por su grupo de automorfismos. En el caso de las álgebras de Lie, el papel de actor está desempeñado por el álgebra de Lie de las derivaciones. Para álgebras asociativas y álgebras de Leibniz, el actor no siempre existe, como se demuestra en [19, 20]. Sin embargo, bajo ciertas condiciones, está dado por el álgebra de los bimultiplicadores y el álgebra de Leibniz de las biderivaciones, respectivamente. Teniendo en cuenta lo anterior, en la Subsección 2.1.1, damos la siguiente definición:

Definición 2.1.9. *Sea L una diálgebra. Denotamos por* $\mathrm{Tetra}(L)$ *el conjunto de los tetramultiplicadores de L, cuyos elementos son cuartetos $t = (l, r, \tilde{l}, \tilde{r})$ de aplicaciones K-lineales de L en L tales que*

(1) $l(a \vdash b) = l(a) \dashv b$,

(2) $l(a \dashv b) = l(a) \dashv b$,

(3) $\tilde{l}(a \dashv b) = \tilde{l}(a) \dashv b$,

(4) $\tilde{l}(a \vdash b) = l(a) \vdash b$,

(5) $\tilde{l}(a \vdash b) = \tilde{l}(a) \vdash b$,

(6) $r(a) \dashv b = a \dashv \tilde{l}(b)$,

(7) $r(a) \dashv b = a \dashv l(b)$,

(8) $\tilde{r}(a) \dashv b = a \vdash l(b)$,

(9) $r(a) \vdash b = a \vdash \tilde{l}(b)$,

(10) $\tilde{r}(a) \vdash b = a \vdash \tilde{l}(b)$,

(11) $r(a \dashv b) = a \dashv \tilde{r}(b)$,

(12) $r(a \dashv b) = a \dashv r(b)$,

(13) $r(a \vdash b) = a \vdash r(b)$,

(14) $\tilde{r}(a \dashv b) = a \vdash \tilde{r}(b)$,

(15) $\tilde{r}(a \vdash b) = a \vdash \tilde{r}(b)$,

para todo $a, b \in L$.

Demostramos que $\mathrm{Tetra}(L)$ es en general no vacío y lo dotamos de una estructura de diálgebra:

Proposición 2.1.11. *Sea L una diálgebra. Entonces* $\mathrm{Tetra}(L)$ *es una diálgebra con los productos por la izquierda y la derecha dados por*

$$t_1 \dashv t_2 = (l_1\tilde{l}_2, r_2r_1, \tilde{l}_1\tilde{l}_2, r_2\tilde{r}_1),$$
$$t_1 \vdash t_2 = (\tilde{l}_1l_2, r_2r_1, \tilde{l}_1\tilde{l}_2, \tilde{r}_2r_1)$$

para todo $t_1 = (l_1, r_1, \tilde{l}_1, \tilde{r}_1), t_2 = (l_2, r_2, \tilde{l}_2, \tilde{r}_2) \in \mathrm{Tetra}(L)$.

Además, demostramos que $\mathrm{Tetra}(L)$ es un actor general para la categoría de diálgebras, el cual, bajo ciertas condiciones, se convierte en el actor de dicha categoría.

Teorema 2.1.13. $\mathrm{Tetra}(L)$ *es un actor general de la diálgebra L.*

Proposición 2.1.14. *Sea L una diálgebra tal que* $\mathrm{Ann}(L) = 0$ *o* $L \dashv L = L = L \vdash L$. *Entonces* $\mathrm{Tetra}(D)$ *es el actor de L.*

El objeto actor ha sido extendido a módulos cruzados para los casos particulares de grupos [73] y álgebras de Lie [27]. En las Subsecciones 2.2.1 y 2.2.2 recordamos los pasos fundamentales de dichas construcciones. La relativa facilidad de estas dos extensiones y sus buenas propiedades nos hicieron plantearnos la posibilidad de extender a módulos cruzados el álgebra de Leibniz de las biderivaciones. Es importante tener en cuenta que dicha álgebra de Leibniz es el actor sólo bajo determinadas hipótesis, así que lo esperable es que su generalización a módulos cruzados tenga un comportamiento parecido. De esta forma, en la Subsección 2.2.3, dado un módulo cruzado de álgebras de Leibniz $(\mathfrak{n}, \mathfrak{q}, \mu)$, damos la definición del conjunto de las biderivaciones de $\mathfrak{q}$ en $\mathfrak{n}$, denotado por $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$, y del conjunto de las biderivaciones del módulo cruzado $(\mathfrak{n}, \mathfrak{q}, \mu)$, denotado por $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$. Además, dotamos a ambos conjuntos de una estructura de álgebra de Leibniz y construimos entre ellos un morfismo de álgebras de Leibniz:

Proposición 2.2.15. *La aplicación K-lineal* $\Delta\colon \mathrm{Bider}(\mathfrak{q}, \mathfrak{n}) \to \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ *dada por* $\Delta(d, D) = ((d\mu, D\mu), (\mu d, \mu D))$*, es un morfismo de álgebras de Leibniz.*

Lo siguiente que hacemos es definir una acción de $\mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ sobre $\mathrm{Bider}(\mathfrak{q},\mathfrak{n})$, la cual aparece descrita en el siguiente resultado.

Teorema 2.2.16. *Existe una acción de Leibniz de* $\mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ *sobre* $\mathrm{Bider}(\mathfrak{q},\mathfrak{n})$ *dada por:*

$$[((\sigma_1,\theta_1),(\sigma_2,\theta_2)),(d,D)] = (\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2),$$
$$[(d,D),((\sigma_1,\theta_1),(\sigma_2,\theta_2))] = (d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D),$$

para todo $((\sigma_1,\theta_1),(\sigma_2,\theta_2)) \in \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu)$, $(d,D) \in \mathrm{Bider}(\mathfrak{q},\mathfrak{n})$. *El morfismo* Δ *de la proposición anterior junto con esta acción es un módulo cruzado de álgebras de Leibniz.*

Una vez conseguido el objeto $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}),\mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu),\Delta)$ en la categoría de módulos cruzados de álgebras de Leibniz, el cual denotamos por $\overline{\mathrm{Act}}(\mathfrak{n},\mathfrak{q},\mu)$, nuestro siguiente paso fue dar una caracterización en términos de ecuaciones de la existencia de un morfismo de módulos cruzados de álgebras de Leibniz entre un módulo cruzado $(\mathfrak{m},\mathfrak{p},\eta)$ y $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}),\mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu),\Delta)$. Para ello fue necesario demostrar primero el siguiente lema:

Lema 2.2.17.

(i) Sea $\mathfrak{q}$ *un álgebra de Leibniz y* $(\sigma,\theta),(\sigma',\theta') \in \mathrm{Bider}(\mathfrak{q})$. *Si* $\mathrm{Ann}(\mathfrak{q}) = 0$ *o* $[\mathfrak{q},\mathfrak{q}] = \mathfrak{q}$, *entonces*

$$\theta\sigma'(q) = \theta\theta'(q)$$

para todo $q \in \mathfrak{q}$.

(ii) Sea $(\mathfrak{n},\mathfrak{q},\mu)$ *un módulo cruzado de álgebras de Leibniz,* $((\sigma_1,\theta_1),(\sigma_2,\theta_2)) \in \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ *y* $(d,D) \in \mathrm{Bider}(\mathfrak{q},\mathfrak{n})$. *Si* $\mathrm{Ann}(\mathfrak{n}) = 0$ *o* $[\mathfrak{q},\mathfrak{q}] = \mathfrak{q}$, *entonces*

$$D\sigma_2(q) = D\theta_2(q),$$
$$\theta_1 d(q) = \theta_1 D(q),$$

para todo $q \in \mathfrak{q}$.

Apoyados en este lema, conseguimos el siguiente resultado:

Teorema 2.2.18. *Sean* $(\mathfrak{m},\mathfrak{p},\eta)$ *y* $(\mathfrak{n},\mathfrak{q},\mu)$ *módulos cruzados de álgebras de Leibniz. Si las siguientes condiciones se cumplen, entonces existe un morfismo de módulos cruzados de* $(\mathfrak{m},\mathfrak{p},\eta)$ *en* $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}),\mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu),\Delta)$.

(i) Existen acciones del álgebra de Leibniz $\mathfrak{p}$ *(y por tanto de* $\mathfrak{m}$*) sobre las álgebras de Leibniz* $\mathfrak{n}$ *y* $\mathfrak{q}$. *El morfismo* μ *es* $\mathfrak{p}$*-equivariante, es decir,*

$$\mu([p,n]) = [p,\mu(n)],$$
$$\mu([n,p]) = [\mu(n),p],$$

y las acciones de $\mathfrak{p}$ y $\mathfrak{q}$ sobre $\mathfrak{n}$ son compatibles, es decir,

$$\begin{aligned}
[n,[p,q]] &= [[n,p],q] - [[n,q],p],\\
[p,[n,q]] &= [[p,n],q] - [[p,q],n],\\
[p,[q,n]] &= [[p,q],n] - [[p,n],q],\\
[n,[q,p]] &= [[n,q],p] - [[n,p],q],\\
[q,[n,p]] &= [[q,n],p] - [[q,p],n],\\
[q,[p,n]] &= [[q,p],n] - [[q,n],p],
\end{aligned}$$

para todo $n \in \mathfrak{n}$, $p \in \mathfrak{p}$ y $q \in \mathfrak{q}$.

(ii) Existen dos aplicaciones K-bilineales $\xi_1 \colon \mathfrak{m} \times \mathfrak{q} \to \mathfrak{n}$ y $\xi_2 \colon \mathfrak{q} \times \mathfrak{m} \to \mathfrak{n}$ tales que

$$\begin{aligned}
\mu\xi_2(q,m) &= [q,m],\\
\mu\xi_1(m,q) &= [m,q],\\
\xi_2(\mu(n),m) &= [n,m],\\
\xi_1(m,\mu(n)) &= [m,n],\\
\xi_2(q,[p,m]) &= \xi_2([q,p],m) - [\xi_2(q,m),p],\\
\xi_1([p,m],q) &= \xi_2([p,q],m) - [p,\xi_2(q,m)],\\
\xi_2(q,[m,p]) &= [\xi_2(q,m),p] - \xi_2([q,p],m),\\
\xi_1([m,p],q) &= [\xi_1(m,q),p] - \xi_1(m,[q,p]),\\
\xi_2(q,[m,m']) &= [\xi_2(q,m),m'] - [\xi_2(q,m'),m],\\
\xi_1([m,m'],q) &= [\xi_1(m,q),m'] - [m,\xi_2(q,m')],\\
\xi_2([q,q'],m) &= [\xi_2(q,m),q'] + [q,\xi_2(q',m)],\\
\xi_1(m,[q,q']) &= [\xi_1(m,q),q'] - [\xi_1(m,q'),q],\\
[q,\xi_1(m,q')] &= -[q,\xi_2(q',m)],\\
\xi_1(m,[p,q]) &= -\xi_1(m,[q,p]),\\
[p,\xi_1(m,q)] &= -[p,\xi_2(q,m)],
\end{aligned}$$

para todo $m, m' \in \mathfrak{m}$, $n \in \mathfrak{n}$, $p \in \mathfrak{p}$, $q, q' \in \mathfrak{q}$.

Además, si al menos una de las siguientes condiciones se cumple, el enunciado recíproco también es cierto.

$$\begin{aligned}
&\mathrm{Ann}(\mathfrak{n}) = 0 = \mathrm{Ann}(\mathfrak{q}),\\
&\mathrm{Ann}(\mathfrak{n}) = 0 \quad y \quad [\mathfrak{q},\mathfrak{q}] = \mathfrak{q},\\
&[\mathfrak{n},\mathfrak{n}] = \mathfrak{n} \quad y \quad [\mathfrak{q},\mathfrak{q}] = \mathfrak{q}.
\end{aligned}$$

Para poder confirmar que la colección de ecuaciones del teorema anterior define un conjunto de acciones derivadas de un módulo cruzado de álgebras de Leibniz sobre otro, tenemos que comprobar que a partir de dichas ecuaciones podemos definir el producto semidirecto de los dos módulos cruzados correspondientes.

Dados $(\mathfrak{m}, \mathfrak{p}, \eta)$ y $(\mathfrak{n}, \mathfrak{q}, \mu)$ dos módulos cruzados de álgebras de Leibniz tales que las condiciones (i) y (ii) del Teorema 2.2.18 se cumplen, al existir acciones de $\mathfrak{m}$ sobre $\mathfrak{n}$ y de $\mathfrak{p}$ sobre $\mathfrak{q}$, es posible considerar los productos semidirectos de álgebras de Leibniz $\mathfrak{n} \rtimes \mathfrak{m}$ y $\mathfrak{q} \rtimes \mathfrak{p}$. Además, tenemos el siguiente resultado.

Teorema 2.2.21. *Existe una acción del álgebra de Leibniz $\mathfrak{q} \rtimes \mathfrak{p}$ sobre el álgebra de Leibniz $\mathfrak{n} \rtimes \mathfrak{m}$, dada por*

$$[(q, p), (n, m)] = ([q, n] + [p, n] + \xi_2(q, m), [p, m]),$$
$$[(n, m), (q, p)] = ([n, q] + [n, p] + \xi_1(m, q), [m, p]),$$

para todo $(q, p) \in \mathfrak{q} \rtimes \mathfrak{p}$, $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$, con ξ_1 y ξ_2 definidas como en el Teorema 2.2.18. Además, el morfismo de álgebras de Leibniz $(\mu, \eta)\colon \mathfrak{n} \rtimes \mathfrak{m} \to \mathfrak{q} \rtimes \mathfrak{p}$, dado por

$$(\mu, \eta)(n, m) = (\mu(n), \eta(m)),$$

para todo $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$ es un módulo cruzado de álgebras de Leibniz junto con la acción anterior.

Este último resultado nos permite definir el producto semidirecto de módulos cruzados de álgebras de Leibniz $(\mathfrak{n}, \mathfrak{q}, \mu)$ y $(\mathfrak{m}, \mathfrak{p}, \eta)$ como el módulo cruzado $(\mathfrak{n} \rtimes \mathfrak{m}, \mathfrak{q} \rtimes \mathfrak{p}, (\mu, \eta))$. Además, estamos en condiciones de escribir la siguiente definición:

Definición 2.2.23. *Si $(\mathfrak{m}, \mathfrak{p}, \eta)$ y $(\mathfrak{n}, \mathfrak{q}, \mu)$ son dos módulos cruzados de álgebras de Leibniz y se verifica al menos una de las siguientes condiciones,*

1. $\operatorname{Ann}(\mathfrak{n}) = 0 = \operatorname{Ann}(\mathfrak{q})$,
2. $\operatorname{Ann}(\mathfrak{n}) = 0$ *y* $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$,
3. $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}$ *y* $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$,

entonces una acción del módulo cruzado $(\mathfrak{m}, \mathfrak{p}, \eta)$ sobre $(\mathfrak{n}, \mathfrak{q}, \mu)$ es un morfismo de módulos cruzados de álgebras de Leibniz de $(\mathfrak{m}, \mathfrak{p}, \eta)$ en $\overline{\operatorname{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$. En otras palabras, bajo una de esas tres condiciones, $\overline{\operatorname{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ es el actor de $(\mathfrak{n}, \mathfrak{q}, \mu)$.

Los pasos en la construcción de $(\operatorname{Bider}(\mathfrak{q}, \mathfrak{n}), \operatorname{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$ sugieren candidatos claros para la extensión a módulos cruzados del álgebra de los bimultiplicadores y de la diálgebra de los tetramultiplicadores. Estas generalizaciones serán consideradas en futuros trabajos.

Cerramos el segundo capítulo con las definiciones de módulos por la izquierda sobre un módulo cruzado de álgebras de Lie y de álgebras asociativas, ya que una de las consecuencias de uno de nuestros resultados principales, que aparece en el último capítulo, será el isomorfismo entre las categorías de módulos por la izquierda sobre un módulo cruzado de álgebras de Lie y módulos por la izquierda sobre su módulo cruzado envolvente universal.

En el reciente artículo [22], los autores construyen un par de funtores adjuntos entre las categorías de módulos cruzados de grupos y álgebras asociativas unitarias. Estos funtores son una generalización natural de la adjunción clásica entre el funtor grupo de las unidades y el funtor álgebra de grupo. Esta última adjunción tiene un análogo para las categorías de álgebras de Lie y álgebras asociativas, dado por la adjunción entre el funtor liezación, que le da a cada álgebra A estructura de álgebra de Lie mediante el corchete $[a, b] = ab - ba$, $a, b \in A$, y el funtor que asigna a cada álgebra de Lie $\mathbb{p}$ su álgebra envolvente universal $\mathtt{U}(\mathbb{p})$. Además, existe una adjunción entre el funtor que asigna a cada diálgebra D el corchete de Leibniz dado por $[x, y] = x \dashv y - y \vdash x$ para todo $x, y \in D$, y el funtor diálgebra envolvente universal (ver [65]).

Comenzamos el último capítulo recordando la construcción de la extensión a módulos cruzados de la adjunción entre el funtor grupo de las unidades y el funtor álgebra de grupo. Además, en la Subsección 3.1.2, demostramos que la generalización a módulos cruzados del segundo de esos funtores no tiene un comportamiento natural con su versión en dimensión 1, en el sentido de que el siguiente diagrama de categorías y funtores, donde $\mathtt{E}_1(G) = (G, G, \mathrm{id}_G)$ y $\mathtt{E}'_1(A) = (A, A, \mathrm{id}_A)$,

$$\begin{tikzcd} \mathbf{Gr} \arrow[r, "\mathtt{E}_1"] \arrow[d, "\mathtt{K}"'] & \mathbf{XGr} \arrow[d, "\mathtt{XK}"] \\ \mathbf{As}^1 \arrow[r, "\mathtt{E}'_1"'] & \mathbf{XAs}^1 \end{tikzcd} .$$

no es conmutativo, ni siquiera salvo isomorfismo.

En las Secciones 3.2 y 3.3 presentamos la generalización a módulos cruzados de las adjunciones entre las categorías **Lie** vs **As** y **Lb** vs **Dias** mencionadas anteriormente, cumpliendo así uno de los objetivos principales de este trabajo. En ambos casos, primero construimos las correspondientes extensiones a módulos cruzados y después comprobamos el buen comportamiento de las mismas a través de los siguientes resultados. Es importante tener en cuenta que en la generalización de los funtores álgebra envolvente universal y diálgebra envolvente universal, los cat^1-objetos juegan un papel fundamental.

Para **XLie** vs **XAs**:

Teorema 3.2.5. *El funtor* $\mathtt{XU}$ *es adjunto por la izquierda del funtor* $\mathtt{XLie}_{\mathtt{As}}$.

Teorema 3.2.6. *Los cuadrados interiores y exteriores de los siguientes diagramas son conmutativos o conmutan salvo isomorfismo para $i = 0, 1$.*

$$
\begin{tikzcd}
\textsf{As} \arrow[r, shift right, "\mathtt{Lie_{As}}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathtt{I}'_i"] \arrow[d, phantom, "\dashv"] & \textsf{Lie} \arrow[l, shift right, "\mathtt{U}"'] \arrow[d, shift right, "\mathtt{I}_i"'] \arrow[d, phantom, "\vdash"] \\
\textsf{XAs} \arrow[u, shift left, "\Phi'_i"] \arrow[r, shift left, "\mathtt{XLie_{As}}"] \arrow[r, phantom, "\top"] & \textsf{XLie} \arrow[l, shift left, "\mathtt{XU}"] \arrow[u, shift right, "\Phi_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\textsf{As} \arrow[r, shift right, "\mathtt{Lie_{As}}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\mathtt{I}'_i"'] \arrow[d, phantom, "\dashv"] & \textsf{Lie} \arrow[l, shift right, "\mathtt{U}"'] \arrow[d, shift left, "\mathtt{I}_i"] \arrow[d, phantom, "\vdash"] \\
\textsf{XAs} \arrow[u, shift right, "\Phi'_{i+1}"'] \arrow[r, shift left, "\mathtt{XLie_{As}}"] \arrow[r, phantom, "\top"] & \textsf{XLie} \arrow[l, shift left, "\mathtt{XU}"] \arrow[u, shift left, "\Phi_{i+1}"]
\end{tikzcd}
$$

Además, tenemos el siguiente resultado:

Teorema 3.2.8. *Sea $(\mathbb{m}, \mathbb{p}, \nu)$ un módulo cruzado de álgebras de Lie. Entonces, las categorías de $(\mathbb{m}, \mathbb{p}, \nu)$-módulos por la izquierda y $\mathtt{XU}(\mathbb{m}, \mathbb{p}, \nu)$-módulos por la izquierda son isomorfas.*

Para **XLb** vs **XDias**:

Teorema 3.3.4. *El funtor $\mathtt{XU_d}$ es adjunto por la izquierda del funtor $\mathtt{XLb}$.*

Teorema 3.3.5. *Los cuadrados interiores y exteriores de los siguientes diagramas son conmutativos o conmutan salvo isomorfismo para $i = 0, 1$.*

$$
\begin{tikzcd}
\textsf{Dias} \arrow[r, shift right, "\mathtt{Lb}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathtt{J}'_i"] \arrow[d, phantom, "\dashv"] & \textsf{Lb} \arrow[l, shift right, "\mathtt{U_d}"'] \arrow[d, shift right, "\mathtt{J}_i"'] \arrow[d, phantom, "\vdash"] \\
\textsf{XDias} \arrow[u, shift left, "\Psi'_i"] \arrow[r, shift left, "\mathtt{XLb}"] \arrow[r, phantom, "\top"] & \textsf{XLb} \arrow[l, shift left, "\mathtt{XU_d}"] \arrow[u, shift right, "\Psi_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\textsf{Dias} \arrow[r, shift right, "\mathtt{Lb}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\mathtt{J}'_i"'] \arrow[d, phantom, "\dashv"] & \textsf{Lb} \arrow[l, shift right, "\mathtt{U_d}"'] \arrow[d, shift left, "\mathtt{J}_i"] \arrow[d, phantom, "\vdash"] \\
\textsf{XDias} \arrow[u, shift right, "\Psi'_{i+1}"'] \arrow[r, shift left, "\mathtt{XLb}"] \arrow[r, phantom, "\top"] & \textsf{XLb} \arrow[l, shift left, "\mathtt{XU_d}"] \arrow[u, shift left, "\Psi_{i+1}"]
\end{tikzcd}
$$

Finalmente, en la última sección del último capítulo, completamos los siguientes diagramas, formados por cuadrados interiores y exteriores conmutativos (o que conmutan salvo isomorfismo) para $i = 0, 1$:

$$
\begin{tikzcd}
\textsf{As} \arrow[r, shift right, "\subset"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathtt{I}'_i"] \arrow[d, phantom, "\dashv"] & \textsf{Dias} \arrow[l, shift right, "\mathtt{As}"'] \arrow[d, shift right, "\mathtt{J}'_i"'] \arrow[d, phantom, "\vdash"] \\
\textsf{XAs} \arrow[u, shift left, "\Phi'_i"] \arrow[r, shift left, "\subset"] \arrow[r, phantom, "\top"] & \textsf{XDias} \arrow[l, shift left, "\mathtt{XAs}"] \arrow[u, shift right, "\Psi'_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\textsf{As} \arrow[r, shift right, "\subset"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\mathtt{I}'_i"'] \arrow[d, phantom, "\dashv"] & \textsf{Dias} \arrow[l, shift right, "\mathtt{As}"'] \arrow[d, shift left, "\mathtt{J}'_i"] \arrow[d, phantom, "\vdash"] \\
\textsf{XAs} \arrow[u, shift right, "\Phi'_{i+1}"'] \arrow[r, shift left, "\subset"] \arrow[r, phantom, "\top"] & \textsf{XDias} \arrow[l, shift left, "\mathtt{XAs}"] \arrow[u, shift left, "\Psi'_{i+1}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathbf{Lie} \arrow[r, shift right, "\subset"'] \arrow[d, shift left, "\mathtt{I}_i"] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathbf{Lb} \arrow[l, shift right, "\mathtt{Lie}_{\mathtt{Lb}}"'] \arrow[d, shift right, "\mathtt{J}_i"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{XLie} \arrow[u, shift left, "\Phi_i"] \arrow[r, shift left, "\subset"] \arrow[r, phantom, "\top"] & \mathbf{XLb} \arrow[l, shift left, "\mathtt{XLie}_{\mathtt{Lb}}"] \arrow[u, shift right, "\Psi_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large, row sep=large]
\mathbf{Lie} \arrow[r, shift right, "\subset"'] \arrow[d, shift right, "\mathtt{I}_i"'] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathbf{Lb} \arrow[l, shift right, "\mathtt{Lie}_{\mathtt{Lb}}"'] \arrow[d, shift left, "\mathtt{J}_i"] \arrow[d, phantom, "\vdash"] \\
\mathbf{XLie} \arrow[u, shift right, "\Phi_{i+1}"'] \arrow[r, shift left, "\subset"] \arrow[r, phantom, "\top"] & \mathbf{XLb} \arrow[l, shift left, "\mathtt{XLie}_{\mathtt{Lb}}"] \arrow[u, shift left, "\Psi_{i+1}"]
\end{tikzcd}
$$

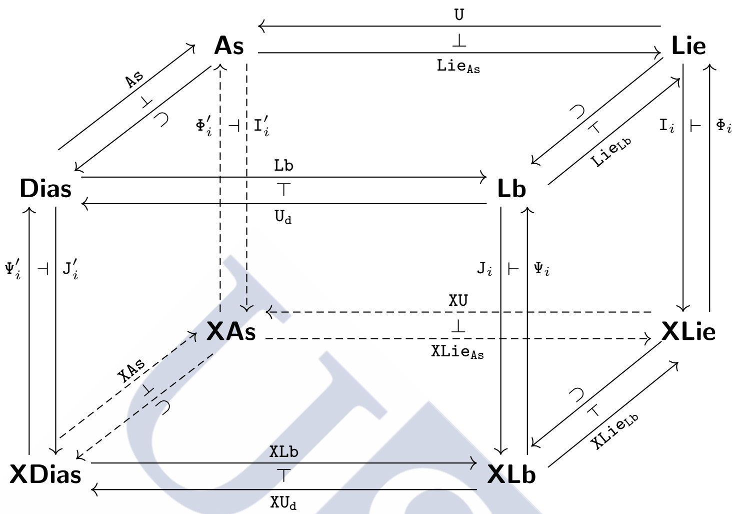
$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathbf{XAs} \arrow[r, shift right, "\mathtt{XLie}_{\mathtt{As}}"'] \arrow[d, shift left, "\subset"] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathbf{XLie} \arrow[l, shift right, "\mathtt{XU}"'] \arrow[d, shift right, "\subset"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{XDias} \arrow[u, shift left, "\mathtt{XAs}"] \arrow[r, shift left, "\mathtt{XLb}"] \arrow[r, phantom, "\top"] & \mathbf{XLb} \arrow[l, shift left, "\mathtt{XU}_{\mathtt{d}}"] \arrow[u, shift right, "\mathtt{XLie}_{\mathtt{Lb}}"']
\end{tikzcd}
$$

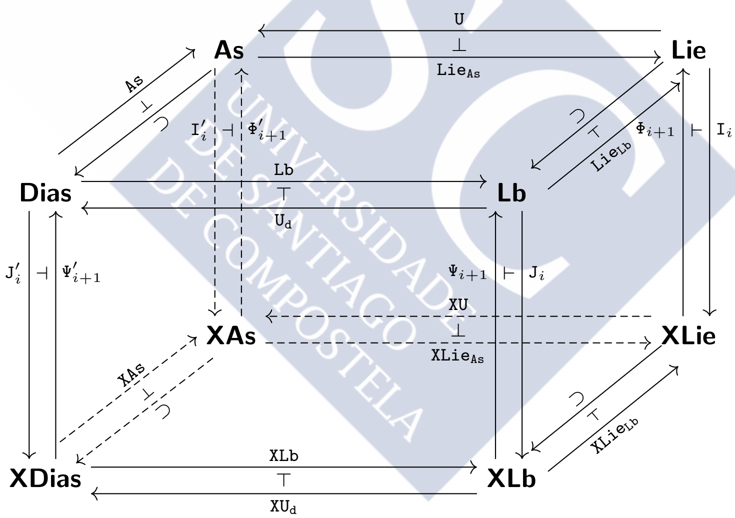
Estos diagramas, junto al ya conocido

$$
\begin{tikzcd}[column sep=large, row sep=large]
\mathbf{As} \arrow[r, shift right, "\mathtt{Lie}_{\mathtt{As}}"'] \arrow[d, shift left, "\subset"] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \mathbf{Lie} \arrow[l, shift right, "\mathtt{U}"'] \arrow[d, shift right, "\subset"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{Dias} \arrow[u, shift left, "\mathtt{As}"] \arrow[r, shift left, "\mathtt{Lb}"] \arrow[r, phantom, "\top"] & \mathbf{Lb} \arrow[l, shift left, "\mathtt{U}_{\mathtt{d}}"] \arrow[u, shift right, "\mathtt{Lie}_{\mathtt{Lb}}"']
\end{tikzcd}
$$

nos permiten construir los cuatro paralelepípedos protagonistas del teorema que cierra la tesis:

Teorema 3.4.3. *En los siguientes paralelepípedos de categorías y funtores*





todos los cuadrados interiores y exteriores de funtores adjuntos son conmutativos o conmutan salvo isomorfismo para $i = 0, 1$. Es importante tener en cuenta que en cada una de las caras de los paralelepípedos, los adjuntos por la izquierda forman los cuadrados exteriores y los adjuntos por la derecha los cuadrados interiores.

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Introduction

Aims and background

The concept of crossed module of groups was formulated for the first time by Whitehead in the late 1940s [83]. Soon after, Mac Lane and Whitehead [70] proved that crossed modules work as an algebraic model for path-connected CW-spaces whose homotopy groups are trivial in dimensions greater than 2. They are algebraic objects with a rich structure and provide a generalization of both the concepts of normal subgroup and module over a group. From the very beginning, crossed modules of groups have played an important role in several areas of mathematics, particularly in homotopy theory. For example, they appear in various classification problems for low-dimensional homotopy types and the derivation of van Kampen theorem generalizations (see the survey by Brown [12]).

Beyond their value as a tool for homotopy theory, crossed modules have been studied as algebraic objects in their own right. For instance, Norrie generalized some group theoretic concepts and structures to crossed modules in her PhD thesis [72]. In particular, she defined the actor of a crossed module, as well as the notions of centre of a crossed module, complete and perfect crossed modules.

The study of (co)homological properties of crossed modules of groups has been the subject of several papers. We point out two (co)homology theories of crossed modules of groups, one introduced and investigated in the works of Baues [5] and Ellis [39] via classifying spaces, and the other defined by Carrasco, Cegarra and R.-Grandjeán [15] as cotriple (co)homology. Later, R.-Grandjeán, Ladra and Pirashvili [49] found a relation between these two homology theories.

Crossed modules of different algebraic objects, not only groups, have also been studied. For instance, in [60] Kassel and Loday used Lie crossed modules as computational tools in order to give an interpretation of the third relative Chevalley-Eilenberg cohomology of Lie algebras. Internal (cotriple) homology and Chevalley-Eilenberg homology theories of Lie crossed modules were investigated in [23, 35]. Lie crossed modules also occur in the "categorification" problem of the theory of Lie algebras [4] as an equivalent formulation of strict Lie 2-algebras.

From the 1960s, many authors have attempted to answer the question of what we should mean by non-abelian (co)homology, that is (co)homology with non-abelian coefficients, of various algebraic structures (see [30, 32, 33, 67, 79]). A convincing answer for groups and Lie algebras was given by Guin in [50, 51], where coefficients are taken to be crossed modules of groups and Lie algebras. Guin's results were later extended to higher dimensions in [54, 55] for groups and in [56, 57] for Lie algebras. The new non-abelian (co)homology theory differs from that by Serre [79] and from the setting of various papers on non-abelian (co)homology of groups [14, 28, 32].

Crossed modules of associative algebras have not been so deeply studied as their Lie and group analogues. However, in the works of Dedecker and Lue [33, 67] they play a central role as coefficients for low-dimensional non-abelian cohomology. Besides, in Shammu's PhD thesis [80] the algebraic and categorical structure of crossed modules of algebras was studied. Baues and Minian [6] used them to represent the Hochschild cohomology of associative algebras and, in the recent article [34], the Hochschild and (cotriple) cyclic homologies of crossed modules of associative algebras were constructed and investigated.

Regarding Leibniz algebras, a non-antisymmetric generalization of Lie algebras introduced by Bloh [8] and Loday [64], their crossed modules were used for the first time by Loday and Pirashvili [66] to study the cohomology of Leibniz algebras. Crossed modules of Leibniz algebras were also used as coefficients for non-abelian (co)homology of Leibniz algebras in [25, 47].

Brown and Spencer proved in [13] that internal categories within the category of groups are equivalent to crossed modules of groups. They mention in their article that this fact was already known by Verdier in 1965 and used by Duskin in [36]. Porter used this work as an inspiration to investigate for which categories the equivalence between crossed modules of the appropriate type and internal categories still holds, although the resulting work was never published. A few years later, Loday introduced in [63] the concept of cat^n-groups, which are equivalent to internal n-fold categories in groups (see also [38]). Besides, he gave the notion of crossed square, the 2-dimensional version of a crossed module. In 1984, Ellis [37] extended to arbitrary dimension the unpublished results achieved by Porter and proved that, given a category of Ω-groups $\mathcal{C}$, the following structures are equivalent:

(i) n-fold internal categories in $\mathcal{C}$,

(ii) cat^n-objects in $\mathcal{C}$,

(iii) Crossed n-cubes in $\mathcal{C}$,

(iv) n-simplicial objects in $\mathcal{C}$ whose normal complexes are of length 1,

(v) n-fold crossed modules in $\mathcal{C}$.

The main results presented in this work are done for crossed modules of groups, Lie algebras, Leibniz algebras, associative algebras and associative dialgebras, all of which constitute categories of interest (see [71], for instance). Categories of interest are a particular case of categories of groups with operations, which are themselves a particular case of the more general notion of variety of universal algebra, whose definition can be found in [69].

Another equivalent description of crossed modules of groups is that of strict 2-groups, which are defined by Baez [3] as strict monoidal categories for which every morphism is invertible and every object has an inverse. Baez himself introduced the notion of strict Lie 2-algebra [4], while Khmaladze gave a definition of strict associative 2-algebras in [61], both equivalent to their corresponding crossed modules. In the present work we recall the concept of Leibniz strict 2-algebra (see [81]) and define strict associative 2-dialgebras. Additionally, we prove that they are equivalent to their corresponding crossed modules.

Recently, in [22], the authors constructed a pair of adjoint functors between the categories of crossed modules of groups and associative unital algebras. This adjunction is a natural generalization of the classical adjunction between the unit group functor and the group algebra functor. Furthermore, they established an isomorphism between the categories of modules over a crossed module of groups and its respective crossed module of associative algebras.

It is also a classical fact that associative algebras and Lie algebras are related by a pair of well-known adjoint functors: the Liezation functor, which makes every associative algebra A into a Lie algebra via the bracket $[a,b] = ab - ba$, $a, b \in A$, is right adjoint to the functor that assigns to every Lie algebra $\mathbb{p}$ its universal enveloping algebra $\mathtt{U}(\mathbb{p})$.

In the non-commutative framework, when Lie algebras are replaced by Leibniz algebras, the analogous objects to associative algebras are diassociative algebras (or dialgebras, for short), introduced and studied by Loday [65]. In the same article, Loday proves that there is an adjunction between the functor that assigns to every dialgebra D the Leibniz bracket given by $[x,y] = x \dashv y - y \vdash x$ for all $x, y \in D$, and the universal enveloping dialgebra functor. The previous adjunctions are part of the following diagram

$$\begin{tikzcd}
\mathsf{\textbf{As}} \arrow[r, shift right, "\mathtt{Lie_{As}}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\subset"] \arrow[d, phantom, "\dashv"] & \mathsf{\textbf{Lie}} \arrow[l, shift right, "\mathtt{U}"'] \arrow[d, shift right, "\subset"'] \arrow[d, phantom, "\vdash"] \\
\mathsf{\textbf{Dias}} \arrow[u, shift left, "\mathtt{As}"] \arrow[r, shift left, "\mathtt{Lb}"] \arrow[r, phantom, "\top"] & \mathsf{\textbf{Lb}} \arrow[l, shift left, "\mathtt{U_d}"] \arrow[u, shift right, "\mathtt{Lie_{Lb}}"']
\end{tikzcd}$$

where the inner square is commutative and the outer square commutes up to isomorphism. In the present work we give an explicit definition of crossed modules of dialgebras and explore some of their basic properties in order to extend the previous diagram to crossed modules. Then we use the resulting diagrams to construct four parallelepipeds of categories and functors in which, for every face, the inner and the outer squares are commutative or commute up to isomorphism.

Additionally, we establish an isomorphism between the categories of modules over a Lie crossed module and its universal enveloping crossed module, for which the actor of a Lie crossed module, introduced by Casas and Ladra [27] plays and essential role.

It is well-known that the actor of a group is given by its group of automorphisms. In the case of Lie algebras, the role of actor is played by the Lie algebra of derivations. For associative and Leibniz algebras the actors do not always exist (see [19, 20]). Nevertheless, under certain conditions, they are given by the algebra of bimultipliers and the Leibniz algebra of biderivations, respectively. On account of these facts, we give a construction of the dialgebra of tetramultipliers of a given dialgebra, which works as its actor under certain conditions.

As we have previously stated, the actor has been extended to crossed modules for the particular cases of groups (see [73]) and Lie algebras (see [27]). The ease of those generalizations led us to construct a general actor crossed module of a Leibniz crossed module which becomes the actor in some cases.

Structure, methodology and main results

Every chapter begins with a brief description of what is contained in it, but here we present the basic structure of the work and the main results.

Since one of the objectives was to make this work as self-contained as possible, in the first chapter we gather some essential definitions and results. We begin by recalling the notions of internal category and crossed module in categories of Ω-groups, together with the equivalence between them as proved by Porter [78].

Crossed modules in categories of groups with operations can alternatively be described as cat^1-objects, which play an essential role in our main results in Chapter 3. That is the reason for us to include a detailed description of the equivalence between crossed modules and cat^1-objects for the cases of groups, Lie, Leibniz and associative algebras and associative dialgebras. A meticulous reader might find the subsections contained in Section 1.2 a bit repetitive, considering that the core of the different proofs is very similar independently of the base category. However we decided to leave the details in the final version to facilitate the reading, since although most of the steps follow by analogy, there are slight differences.

In the last section of Chapter 1 we recall the notion of strict Leibniz 2-algebra and describe analogously the concept of strict associative 2-dialgebra. Besides we prove that they are equivalent to their corresponding crossed modules.

Chapter 2 is divided in three parts. Firstly, we remind the reader of the notion of actor in a category of interest and recall the actor for groups and for Lie algebras, that is the group of automorphisms and the Lie algebra of derivations, respectively. On account of that and the fact that the algebra of bimultipliers plays the role of actor of an associative algebra under certain conditions, we construct the dialgebra of tetramultipliers and prove that it works as the actor under some particular circumstances.

In the next subsection we briefly describe the actor crossed module of groups and Lie algebras, as presented in [73] and [27] respectively. Bearing that in mind, we give, for any given Leibniz crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$, a construction of the Leibniz crossed module $(\mathrm{Bider}(\mathfrak{q}, \mathfrak{n}), \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$ and prove that it is the actor under certain conditions. Although the 2-dimensional generalizations of the algebra of bimultipliers and the dialgebra of tetramultipliers are not achieved, the steps in the construction in the Leibniz case suggest a possible candidate for each situation.

The final section in Chapter 2 is dedicated to defining modules over crossed modules of groups, Lie algebras and associative algebras. These definitions together with one of the main results from the last chapter allow us to establish an isomorphism of categories of modules over a Lie crossed module and its universal enveloping crossed module.

Chapter 3 begins with the construction of the extension to crossed modules of the adjunction between the unit group functor and the group algebra functor by Casas, Inassaridze, Khmaladze and Ladra [22]. It is appropriate to include it for two reasons: On one hand, it was a catalyst for us to construct the 2-dimensional generalizations of the corresponding adjunctions for **Lie** vs **As** and **Lb** vs **Dias**, presented in Sections 3.2 and 3.3 respectively. On the other hand, we prove that the functor $\mathtt{XK}$ that assigns to any crossed module of groups (H, G, ∂) the crossed module of unital algebras $(\operatorname{Ker} \overline{\mathtt{K}}(s), \mathtt{K}(G), \overline{\mathtt{K}}(t)|_{\operatorname{Ker} \overline{\mathtt{K}}(s)})$ does not commute with the group algebra functor $\mathtt{K}$ and one of the two obvious ways of regarding a group and an associative algebra as a crossed module of groups and associative algebras, respectively. This is an interesting result, since the analogous diagram for Lie and associative algebras (respectively Leibniz algebras and dialgebras) does commute.

In the last section of Chapter 3 we assemble all the commutative squares of categories and functors into four parallelepipeds containing the original adjunctions and their natural generalizations. For every face in those parallelepipeds the inner and outer squares are commutative or commute up to isomorphism.

Finally, in Chapter 4 we survey the accomplished results and sketch some possible further directions in which our research could continue. Some of the results of this thesis are presented in [17] and [18].

Notation

Notation has been thoroughly selected with the purpose of facilitating the reading and making it clear what algebraic structures are considered in the different parts of the work.

We denote by K a commutative ring with unit. The category of groups is denoted by **Gr**. Lie and Leibniz algebras are considered over K and their categories are denoted by **Lie** and **Lb** respectively. Algebras are (not necessarily unital) associative algebras over K and their category is denoted by **As**. The subcategory of associative unital algebras is denoted by $\mathbf{As}^1$. Regarding the category of dialgebras, it is denoted by

Dias.

The corresponding categories of crossed modules are denoted by adding the prefix **X** to the base category, that is **XGr**, **XLie**, **XLb**, **XAs** (or $\mathbf{XAs}^1$ when necessary) and **XDias**. Additionally, we have tried to use an specific notation in order to relate every object to its corresponding category. In this way, crossed modules are denoted by (H, G, ∂), $(\mathbb{m}, \mathbb{p}, \nu)$, (B, A, ρ), $(\mathfrak{m}, \mathfrak{p}, \eta)$ and (D, L, μ) for groups, Lie algebras, associative algebras, Leibniz algebras and dialgebras, respectively.

Particular labels are used to make reference to equivariance and the Peiffer identity. Namely, (XGr1) and (XGr2), (XLie1) and (XLie2), and so on, depending on the base category. However, in Chapter 2, due to the amount of labelled equations in use, we chose to call them by their name, independently of the category.

We have tried to respect the existing notation and terminology for the known concepts. Regarding the new ones, we have named them by analogy to the notions that inspired them.

The group operation is denoted additively in Section 1.1 out of respect for the notation used by Orzech [74] and Porter [78]. However, we use multiplicative notation for groups with no other operation in the rest of the work to avoid confusions with the commutative group operation of the underlying K-module of a Lie algebra, associative algebra, Leibniz algebra or dialgebra. Concerning actions, they are denoted with the same symbol used for the corresponding operations, except for groups, in order to avoid confusions with equivariance and the Peiffer identity.

Contents

Chapter 1

Crossed modules and equivalent structures

In Section 1.1 we recall the well-known equivalence between crossed modules and internal categories in a category of groups with operations. We gather some indispensable notions stated by Orzech [74] and the equivalence itself, proved by Porter [78].

In Section 1.2 we sketch the well-known equivalence between cat^1-objects and internal categories in a category of groups with operations. In the five contained subsections we give some basic definitions and explore essential properties of crossed modules in the five particular categories considered in this thesis. Furthermore we give an explicit description of the equivalence between crossed modules and cat^1-objects for the five different situations, since the aforementioned equivalence is essential for a proper comprehension of the proofs in Chapter 3. All the results were known prior to this thesis, although crossed modules of dialgebras had not been explicitly described so far.

Finally, in Section 1.3 we introduce the notion of strict 2-dialgebra by analogy to the concept of strict associative 2-algebra by Khmaladze [61] and the one of strict Lie 2-algebra by Baez [4]. Additionally, we prove that strict 2-dialgebras and strict Leibniz 2-algebras are equivalent to crossed modules of dialgebras and Leibniz algebras respectively.

1.1 Crossed modules and internal categories

1.1.1 Crossed modules of Ω-groups

The next definition can be found in [19, 20, 71, 74, 78] (for additive notation) and [76] (for multiplicative notation). It is based on the more general notion of category of groups with multiple operators introduced by Higgins [52].

Definition 1.1.1. *A category of groups with operations (or Ω-groups) is a category $\mathcal{C}$ whose objects are groups with a set of operations Ω and with a set of identities* E, *such that* E *includes the group laws and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then:*

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$.

(b) The group operations (written using additive notation: $0, -, +$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega'_1 = \Omega_1 \setminus \{-\}$ and $\Omega'_2 = \Omega_2 \setminus \{+\}$. If $* \in \Omega'_2$, then $*^\circ \in \Omega'_2$, with $x_1 *^\circ x_2 = x_2 * x_1$. Besides, $\Omega_0 = \{0\}$.

(c) For any $* \in \Omega'_2$, E *contains the identity $x_1 * (x_2 + x_3) = (x_1 * x_2) + (x_1 * x_3)$.*

(d) For any $\omega \in \Omega'_1$ and $* \in \Omega'_2$, E *contains the identities:*

$$\omega(x_1 + x_2) = \omega(x_1) + \omega(x_2),$$
$$\omega(x_1 * x_2) = \omega(x_1) * x_2.$$

A morphism of Ω-groups is a set map which preserves all the operations.

Remark 1.1.2. *It is important to note that the group operation is not necessarily commutative, hence $-(x_1 + x_2) = -x_2 - x_1$ for all $x_1, x_2 \in C$, with C an object in category of groups with operations, in contrast to the first identity from (d). Besides, the fact that $*^\circ \in \Omega'_2$ for any $* \in \Omega'_2$ will allow us to disregard the right sided version of many identities involving operations in Ω'_2, as they will follow immediately from the left sided version.*

The following lemma is an immediate consequence of the group structure and the axiom (c) from Definition 1.1.1.

Lemma 1.1.3. *Let $\mathcal{C}$ be a category of groups with operations and C an object in $\mathcal{C}$. Then,*

(i) $x * 0 = 0$,

(ii) $-(x_1 * x_2) = -x_1 * x_2$,

(iii) $x_1 * x_2 + x_3 * x_4 = x_3 * x_4 + x_1 * x_2$,

for all $x, x_1, x_2, x_3, x_4 \in C$, $* \in \Omega'_2$.

In [74], Orzech introduced the notion of category of interest, which is no more than a category of groups with operations that verifies some extra conditions.

Definition 1.1.4. *A category of interest is a category of groups with operations which satisfies two additional axioms:*

(1) $x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1$.

(2) For any ordered pair $(*, \bar{*}) \in \Omega_2' \times \Omega_2'$, *there is a word* W *such that,*

$$\begin{aligned}(x_1 * x_2)\bar{*}x_3 = W(&x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1,\\ &x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),\end{aligned}$$

where each juxtaposition represents an operation in Ω_2'.

The reason for us to include the previous definition is that our principal results establish relations between crossed modules in categories that satisfy the foregoing axioms. Categories of groups **Gr**, Lie algebras **Lie**, associative algebras **As**, Leibniz algebras **Lb** and associative dialgebras **Dias** can be found among the examples of categories of interest provided in [71], along with the counterexample of Jordan algebras, for which axiom (2) fails.

Let $\mathcal{C}$ be a category of groups with operations. It is possible to define the notions of action and semidirect product in such a category.

Definition 1.1.5 ([74])**.** *Let* A *and* B *be objects in* $\mathcal{C}$. *An extension of* B *by* A *is a sequence*

$$0 \longrightarrow A \xrightarrow{\iota} C \xrightarrow{\sigma} B \longrightarrow 0$$

in which σ *is surjective and* ι *is the kernel of* σ. *We say that an extension is split if there is a morphism* $\varepsilon\colon B \to C$ *such that* $\sigma\varepsilon = \mathrm{id}_B$. *A split extension of* B *by* A *is called* B*-structure on* A. *An extension is singular if* A *is singular, that is, if* A *is abelian as a group and* $a_1 * a_2 = 0$ *for all* $a_1, a_2 \in A$, $* \in \Omega_2'$.

Given a B-structure on A, there is an induced set of actions of B on A, one for each operation in Ω_2. If we assume $A \subset C$, with ι the inclusion, the definition of those actions is as follows:

$$\begin{aligned}{}^{b}a &= \varepsilon(b) + a - \varepsilon(b),\\ b * a &= \varepsilon(b) * a,\end{aligned}$$

for all $a \in A$, $b \in B$. Actions arising from split extensions are called derived actions in [74], where Orzech proves the following result:

Theorem 1.1.6 ([74])**.** *Let* A *and* B *be objects in* $\mathcal{C}$. *Given a set of actions of* B *on* A *(one for each operation in* Ω_2*), the semidirect product* $A \rtimes B$, *which consists of* $A \times B$ *as a set, with the operations:*

$$\omega(a, b) = (\omega(a), \omega(b)), \tag{1.1.1}$$

$$(a, b) + (a', b') = (a + {}^{b}a', b + b'), \tag{1.1.2}$$

$$(a, b) * (a', b') = (a * a' + b * a' + b' *^{\circ} a, b * b'), \tag{1.1.3}$$

for all $a, a' \in A$, $b, b' \in B$, $\omega \in \Omega_1'$, $* \in \Omega_2'$, *is an object in* $\mathcal{C}$ *if and only if the set of actions of* B *on* A *is a set of derived actions.*

Remark 1.1.7. *Observe that* $-(a,b) = ({}^{-b}(-a), -b)$, *immediately from the definition of the addition in* $A \rtimes B$ *together with the identities (1), (2) and (3) from the next lemma.*

In the categories of groups, Lie algebras, associative algebras, Leibniz algebras and associative dialgebras, derived actions will be called simply actions. However, in Chapter 2 we will use again the adjective "derived" whenever we need to stress that a set of actions is indeed induced by a split extension. In any case, the context will always make the difference clear.

Due to the way a set of derived actions is defined, one can easily check that the identities in the following lemma hold.

Lemma 1.1.8. *Let* A *and* B *be objects in* $\mathcal{C}$, *together with a* B*-structure on* A. *Then:*

(1) ${}^{0}a = a$,

(2) ${}^{b}(a_1 + a_2) = {}^{b}a_1 + {}^{b}a_2$,

(3) ${}^{(b_1+b_2)}a = {}^{b_1}({}^{b_2}a)$,

(4) $b * (a_1 + a_2) = b * a_1 + b * a_2$,

(5) $(b_1 + b_2) * a = b_1 * a + b_2 * a$,

(6) ${}^{(b_1 * b_2)}(a_1 * a_2) = a_1 * a_2$,

(7) ${}^{(b_1 * b_2)}(b * a) = b * a$,

(8) $a_1 * ({}^{b}a_2) = a_1 * a_2$,

(9) $b_1 * ({}^{b_2}a) = b_1 * a$,

(10) $\omega({}^{b}a) = {}^{\omega(b)}\omega(a)$,

(11) $\omega(b * a) = \omega(b) * a = b * \omega(a)$,

(12) $x_1 * x_2 + x_3 * x_4 = x_3 * x_4 + x_1 * x_2$,

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $x_1, x_2, x_3, x_4 \in A \cup B$, $\omega \in \Omega_1'$, $* \in \Omega_2'$.

Proof. All the equalities can be easily proved by using the definition of the derived actions along with axioms (b), (c), (d) from Definition 1.1.1, Lemma 1.1.3 (ii) and the fact that ε (as in the definition of a B-structure) is a morphism. □

Remark 1.1.9. *In [31], the previous equalities are proved to be not only necessary conditions but also sufficient to define a set of derived actions in a category of* Ω*-groups. That is the reason why we include all the equalities, although we will only make use of a few of them. Nevertheless, in the next section, for every specific category, we will define derived actions in terms of equations by using the description given in [20, p. 91] for the particular case of categories of interest, in which conditions (6) and (7) are replaced by*

$$
\begin{aligned}
{}^{b}(a_1 * a_2) &= a_1 * a_2, \\
{}^{b_1}(b_2 * a) &= b_2 * a, \\
{}^{(b_1 * b_2)}a &= a,
\end{aligned}
$$

and condition (12) is replaced by $a_1 + (b * a_2) = (b * a_2) + a_1$ *(as in axiom (1) from Definition 1.1.4) for all* $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $* \in \Omega_2'$. *Besides, an additional condition is required, described as axiom (2) from Definition 1.1.4, but with* $x_1, x_2, x_3 \in A \cup B$.

Now we can define crossed modules in terms of actions and operations.

Definition 1.1.10. *A crossed module in a category of groups with operations* $\mathcal{C}$ *is a triple* (A, B, μ)*, where* μ *is a morphism between the objects* A *and* B*, together with a* B*-structure on* A*, such that for all* $a, a_1, a_2 \in A$, $b \in B$, $* \in \Omega_2'$,

$$\begin{aligned} \mu({}^{b}a) &= b + \mu(a) - b, \\ \mu(b * a) &= b * \mu(a), \end{aligned} \tag{CM1}$$

$$\begin{aligned} {}^{\mu(a_1)}a_2 &= a_1 + a_2 - a_1, \\ \mu(a_1) * a_2 &= a_1 * a_2. \end{aligned} \tag{CM2}$$

Note that Porter [78] includes the identities $\mu(a * b) = \mu(a) * b$ in (CM1) and $a_1 * \mu(a_2) = a_1 * a_2$ in (CM2), but those follow from $\mu(b*a) = b*\mu(a)$ and $\mu(a_1)*a_2 = a_1 * a_2$ respectively, due to condition (b) in Definition 1.1.1 along with the way of defining a set of derived actions from a B-structure.

Remark 1.1.11. *The second axiom is usually called* Peiffer identity*, while the first one is sometimes referred to as* equivariance*. Both* (CM1) *and* (CM2) *will have specific descriptions and tags for the five particular categories considered (see Subsections 1.2.1–1.2.5). However, we will sometimes write simply equivariance or Peiffer identity since the context itself will clarify the category in use. A* precrossed module *is a triple* (A, B, μ)*, together with a* B*-structure on* A*, that only satisfies the equivariance condition.*

Definition 1.1.12. *Given two crossed modules* (A, B, μ) *and* (A', B', μ')*, a morphism of crossed modules is a pair* (φ, ψ) *of morphisms in* $\mathcal{C}$, $\varphi\colon A \to A'$ *and* $\psi\colon B \to B'$*, such that* $\mu'\varphi = \psi\mu$ *and:*

$$\varphi({}^{b}a) = {}^{\psi(b)}\varphi(a), \tag{1.1.4}$$

$$\varphi(b * a) = \psi(b) * \varphi(a), \tag{1.1.5}$$

for all $a \in A$, $b \in B$.

Composition of morphisms of crossed modules is defined component-wise and the identity morphism is given by $(\mathrm{id}_A, \mathrm{id}_B)$ for any crossed module (A, B, μ). We will denote by $\mathbf{XMod}(\mathcal{C})$ the category of crossed modules and morphisms of crossed modules in $\mathcal{C}$. However, in Section 1.2 we will present specific notation for the five particular categories considered in this thesis, together with several examples and essential properties.

1.1.2 Internal categories

Internal categories were introduced by Ehresmann [40, 41] although a more accessible description can be found in [3, 4, 9, 44, 76]. Categories within a category can be defined in a more general context than that of Ω-groups, so let us assume for the rest of this subsection that $\mathcal{C}$ is simply a category with pullbacks.

Definition 1.1.13. *An internal category C in $\mathcal{C}$ consists of an object of objects, C_0, an object of arrows, C_1, and the diagram:*

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\kappa"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend right, "e"'] \end{tikzcd}$$

where s, t are the source and target maps, e is the identity-assigning map, κ is the composition map, $C_1 \times_{C_0} C_1$ is the pullback:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd} \tag{1.1.6}$$

and the following diagrams commute, expressing the usual category laws:

$$\begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "\mathrm{id}_{C_0}"'] & C_1 \arrow[d, "s"] \\ & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_0 \arrow[r, "e"] \arrow[dr, "\mathrm{id}_{C_0}"'] & C_1 \arrow[d, "t"] \\ & C_0 \end{tikzcd} \tag{1.1.7}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_1"] \arrow[d, "\kappa"'] & C_1 \arrow[d, "s"] \\ C_1 \arrow[r, "s"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\kappa"'] & C_1 \arrow[d, "t"] \\ C_1 \arrow[r, "t"'] & C_0 \end{tikzcd} \tag{1.1.8}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "\kappa \times_{C_0} \mathrm{id}_{C_1}"] \arrow[d, "\mathrm{id}_{C_1} \times_{C_0} \kappa"'] & C_1 \times_{C_0} C_1 \arrow[d, "\kappa"] \\ C_1 \times_{C_0} C_1 \arrow[r, "\kappa"'] & C_1 \end{tikzcd} \tag{1.1.9}$$

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[r, "e \times_{C_0} \mathrm{id}_{C_1}"] \arrow[dr, "\pi_2"'] & C_1 \times_{C_0} C_1 \arrow[d, "\kappa"] & C_1 \times_{C_0} C_0 \arrow[l, "\mathrm{id}_{C_1} \times_{C_0} e"'] \arrow[dl, "\pi_1"] \\ & C_1 & \end{tikzcd} \tag{1.1.10}$$

Note that $C_0 \times_{C_0} C_1$ and $C_1 \times_{C_0} C_0$ are respectively the pullbacks

$$\begin{tikzcd} C_0 \times_{C_0} C_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "s"] \\ C_0 \arrow[r, "\mathrm{id}_{C_0}"'] & C_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_1 \times_{C_0} C_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C_0 \arrow[d, "\mathrm{id}_{C_0}"] \\ C_1 \arrow[r, "t"'] & C_0. \end{tikzcd}$$

Whenever we want to consider an internal category, we will simply write a sextuple $C = (C_1, C_0, s, t, e, \kappa)$. Furthermore, since objects have elements in the categories we will use, those in C_1 will be denoted by letters typically used for morphisms in order to establish a complete analogy with the classical notion of category. Note that if one thinks of the elements in C_1 as maps, the disposition of π_1 and π_2 in diagrams (1.1.8) implies that $\kappa(f, g) = g \circ f$ in standard notation.

Definition 1.1.14. *Let $C = (C_1, C_0, s, t, e, \kappa)$ and $C' = (C_1', C_0', s', t', e', \kappa')$ be two internal categories. An internal functor $F\colon C \to C'$ consists of a pair (F_1, F_0) of morphisms in $\mathcal{C}$, $F_1 : C_1 \to C_1'$, $F_0 : C_0 \to C_0'$, such that the following diagrams commute:*

$$\begin{tikzcd} C_1 \arrow[r, "s"] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"] \\ C_1' \arrow[r, "s'"'] & C_0' \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "t"] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"] \\ C_1' \arrow[r, "t'"'] & C_0' \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[d, "F_1"'] & C_0 \arrow[l, "e"'] \arrow[d, "F_0"] \\ C_1' & C_0' \arrow[l, "e'"] \end{tikzcd} \tag{1.1.11}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "F_1 \times_{F_0} F_1"] \arrow[d, "\kappa"'] & C_1' \times_{C_0'} C_1' \arrow[d, "\kappa'"] \\ C_1 \arrow[r, "F_1"'] & C_1 \end{tikzcd} \tag{1.1.12}$$

where $F_1 \times_{F_0} F_1$ is given by:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[rr, "\pi_2"] \arrow[dr, dashed, "F_1 \times_{F_0} F_1"] \arrow[dd, "\pi_1"'] & & C_1 \arrow[d, "F_1"] \\ & C_1' \times_{C_0'} C_1' \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] & C_1' \arrow[d, "s'"] \\ C_1 \arrow[r, "F_1"'] & C_1' \arrow[r, "t'"'] & C_0'. \end{tikzcd} \tag{1.1.13}$$

Composition of internal functors is defined in the obvious way. We will denote by $\mathsf{ICat}(\mathcal{C})$ the category of internal categories and internal functors in $\mathcal{C}$. It is possible to introduce the notion of *internal natural transformation* between two internal functors in $\mathcal{C}$ and prove that internal categories, functors and natural transformations form a 2-category (see [4, Section 2]).

1.1.3 Equivalence between crossed modules and internal categories

The equivalence between $\mathsf{ICat}(\mathcal{C})$ and $\mathsf{XMod}(\mathcal{C})$ was proved by Porter in [78] for $\mathcal{C}$ a category of groups with operations. Note that this equivalence has been recently extended to the more general situation of semiabelian categories (see [58, 59]). Nevertheless, for the purposes of the present work, we will recall Porter's construction. Let us assume for rest of this subsection that $\mathcal{C}$ is a category of Ω-groups.

Lemma 1.1.15. *Let $C = (C_1, C_0, s, t, e, \kappa)$ be an internal category in $\mathcal{C}$. Then, $C_1 \times_{C_0} C_1 = \{(f, g) \mid f, g \in C_1, t(f) = s(g)\}$, with the operations given by:*

$$\omega(f, g) = (\omega(f), \omega(g)),$$
$$(f, g) * (f', g') = (f * f', g * g'),$$

for all $(f, g), (f', g') \in C_1 \times_{C_0} C_1$, $\omega \in \Omega_1$, $* \in \Omega_2$. Besides,

$$\kappa((f * f'), (g * g')) = \kappa(f, g) * \kappa(f', g'),$$

for all $(f, g), (f', g') \in C_1 \times_{C_0} C_1$, $* \in \Omega_2$. These last identities are called the interchange laws.

Proof. It is straightforward to check that $\{(f, g) \mid f, g \in C_1, t(f) = s(g)\}$ is an object in $\mathcal{C}$. Besides, it is immediate to prove that it is the pullback (1.1.6) with the projections π_1 and π_2 defined in the obvious way.

On the other hand, $\kappa \colon C_1 \times_{C_0} C_1 \to C_1$ is a morphism in $\mathcal{C}$. Therefore, given $(f, g), (f', g') \in C_1 \times_{C_0} C_1$, $* \in \Omega_2$,

$$\kappa(f * f', g * g') = \kappa((f, g) * (f', g')) = \kappa(f, g) * \kappa(f', g').$$

$\square$

We will say that the elements in $C_1 \times_{C_0} C_1$ are pairs of "composable arrows". Furthermore, internal categories in a category of groups with operations "have all their arrows invertible" in the following sense:

Definition 1.1.16. *An internal groupoid is an internal category $C = (C_1, C_0, s, t, e, \kappa)$ for which, given any $f \in C_1$, there is $f' \in C_1$ such that $\kappa(f, f') = es(f)$ and $\kappa(f', f) = et(f)$.*

Theorem 1.1.17. *Every internal category in a category of Ω-groups is an internal groupoid.*

Proof. Given $f \in C_1$, we define $f^{-1} = et(f) - f + es(f)$. It is clear that $(f, f^{-1}) \in C_1 \times_{C_0} C_1$, since $s(f^{-1}) = s(et(f) - f + es(f)) = t(f) - s(f) + s(f) = t(f)$, directly from the fact that s is a morphism and the commutativity of the first diagram in (1.1.7). Analogously, by using the fact that t is a morphism and the commutativity of the second diagram in (1.1.7), we get that $(f^{-1}, f) \in C_1 \times_{C_0} C_1$.

It is evident that $(f, et(f)), (es(f), f), (es(f), es(f)) \in C_1 \times_{C_0} C_1$, so we can write the following:

$$\begin{aligned}\kappa(f, f^{-1}) = \kappa(f - es(f) + es(f), et(f) - f + es(f)) = \kappa(f, et(f)) - \kappa(es(f), f)\\ + \kappa(es(f), es(f)) = f - f + es(f) = es(f),\end{aligned}$$

due to the interchange laws and the commutativity of diagram (1.1.10). One can similarly prove that $\kappa(f^{-1}, f) = et(f)$. □

Directly from the interchange laws, we get the following lemma:

Lemma 1.1.18. *Let $C = (C_1, C_0, s, t, e, \kappa)$ be an internal category in $\mathcal{C}$. Then:*

(i) $\kappa(f, g) = f - es(g) + g = g - es(g) + f$ for all $(f, g) \in C_1 \times_{C_0} C_1$,

(ii) $f + g = g + f$ for all $f, g \in C_1$ such that $t(f) = 0 = s(g)$,

(iii) $f * g = 0$ for all $f, g \in C_1$ such that $t(f) = 0 = s(g)$, $* \in \Omega_2'$.

Proof. Let $(f, g) \in C_1 \times_{C_0} C_1$, that is $t(f) = s(g)$. It is clear that $(f, es(g))$, $(et(f), es(g))$, $(et(f), g)$ are pairs of composable arrows. Then we can write the following:

$$\begin{aligned}\kappa(f, g) &= \kappa(f - et(f) + et(f), es(g) - es(g) + g)\\ &= \kappa(f, es(g)) - \kappa(et(f), es(g)) + \kappa(et(f), g)\\ &= f - es(g) + g,\end{aligned}$$

as a consequence of the interchange laws (see Lemma 1.1.15) and the commutativity of diagram (1.1.10). Similarly, we can write:

$$\begin{aligned}\kappa(f, g) &= \kappa(et(f) - et(f) + f, g - es(g) + es(g))\\ &= \kappa(et(f), g) - \kappa(et(f), es(g)) + \kappa(f, es(g))\\ &= g - es(g) + f,\end{aligned}$$

so (i) holds.

Let us show that (ii) follows immediately from (i). If $t(f) = 0 = s(g)$, it is clear that $(f, g) \in C_1 \times_{C_0} C_1$ and $es(g) = 0$. Due to (i), $\kappa(f, g) = f - es(g) + g = f + g$, but also $\kappa(f, g) = g - es(g) + f = g + f$. Hence, $f + g = g + f$.

We can use a similar technique in order to prove (iii):

$$0 = \kappa(0,0) = \kappa(f * e(0), e(0) * g) = \kappa(f, e(0)) * \kappa(e(0), g) = f * g.$$

Note that we made use of Lemma 1.1.3 (i) as well as the interchange laws and the commutativity of diagram (1.1.10). $\square$

Remark 1.1.19. *Observe that the identity $g * f = 0$ for all $f, g \in C_1$ such that $t(f) = 0 = s(g)$, $* \in \Omega_2'$, follows immediately from Lemma 1.1.18 (iii) and axiom (b) from Definition 1.1.1.*

Theorem 1.1.20 ([78])**.** *Given a category of groups with operations $\mathcal{C}$, the categories* $\mathbf{ICat}(\mathcal{C})$ *and* $\mathbf{XMod}(\mathcal{C})$ *are equivalent.*

Proof. Below we define two functors, $\mathtt{I_{cat}}\colon \mathbf{XMod}(\mathcal{C}) \to \mathbf{ICat}(\mathcal{C})$ and $\mathtt{X_{mod}}\colon \mathbf{ICat}(\mathcal{C}) \to \mathbf{XMod}(\mathcal{C})$, and prove that $\mathtt{X_{mod}} \circ \mathtt{I_{cat}} \cong 1_{\mathbf{XMod}(\mathcal{C})}$ and $\mathtt{I_{cat}} \circ \mathtt{X_{mod}} \cong 1_{\mathbf{ICat}(\mathcal{C})}$.

Let us begin with the definition of $\mathtt{I_{cat}}$ on objects. Let (A, B, μ) be a crossed module in $\mathcal{C}$. By definition, there is a B-structure on A, so $A \rtimes B$ is an object in $\mathcal{C}$ (see Theorem 1.1.6). Consider the following diagram:

$$\begin{tikzcd}
(A \rtimes B) \times_B (A \rtimes B) \arrow[r, "\kappa"] & A \rtimes B \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & B \arrow[l, bend right=60, "e"']
\end{tikzcd}$$

with $s(a,b) = b$, $t(a,b) = \mu(a)+b$, $e(b) = (0,b)$ and $\kappa((a,b),(a',\mu(a)+b)) = (a'+a,b)$ for all $a, a' \in A$, $b \in B$. Note that if $((a,b),(a',b')) \in (A \rtimes B) \times_B (A \rtimes B)$, $b' = s(a',b') = t(a,b) = \mu(a)+b$, so the definition of κ makes sense. It is necessary to prove that s, t, e and κ are morphisms in $\mathcal{C}$, that is, they preserve all the operations in Ω. Since 0 is clearly preserved by all of them and it is obvious that s and e preserve the rest of the operations, we will focus on sketching how to prove that t and κ preserve the operations in Ω_1 and Ω_2. Calculations are quite long, so we will not include them, although we will point out the crucial ideas required to complete them. Observe that the operations in $A \rtimes B$ are described in (1.1.1), (1.1.2), (1.1.3).

Regarding t, it preserves all the operations in Ω_1 directly from the fact that μ preserves them and the first identity in axiom (d) from Definition 1.1.1. Furthermore, (CM1) and the fact that μ is a morphism are the key to prove that t preserves all the operations in Ω_2.

Concerning κ, note that the elements in $(A \rtimes B) \times_B (A \rtimes B)$ are of the form $((a,b),(a',\mu(a)+b))$, with $a, a' \in A$, $b \in B$. Proving that κ preserves every operation in Ω_1 is as simple as proving it for t. Immediately below we show the calculations required to prove that κ preserves $+$ as an example. Let $((a_i,b_i),(a_i',\mu(a_i)+b_i)) \in$

$(A \rtimes B) \times_B (A \rtimes B)$ for $i = 1, 2$. On one hand we have that

$$\begin{aligned}
&\kappa(((a_1, b_1), (a'_1, \mu(a_1) + b_1)) + ((a_2, b_2), (a'_2, \mu(a_2) + b_2)))\\
&= \kappa((a_1, b_1) + (a_2, b_2), (a'_1, \mu(a_1) + b_1) + (a'_2, \mu(a_2) + b_2))\\
&= \kappa((a_1 + {}^{b_1}a_2, b_1 + b_2), (a'_1 + {}^{(\mu(a_1)+b_1)}a'_2, \mu(a_1) + b_1 + \mu(a_2) + b_2))\\
&= (a'_1 + {}^{(\mu(a_1)+b_1)}a'_2 + a_1 + {}^{b_1}a_2, b_1 + b_2)\\
&= (a'_1 + {}^{\mu(a_1)}({}^{b_1}a'_2) + a_1 + {}^{b_1}a_2, b_1 + b_2)\\
&= (a'_1 + a_1 + {}^{b_1}a'_2 - a_1 + a_1 + {}^{b_1}a_2, b_1 + b_2)\\
&= (a'_1 + a_1 + {}^{b_1}a'_2 + {}^{b_1}a_2, b_1 + b_2),
\end{aligned}$$

by making use of the definition of $+$ in $(A \rtimes B) \times_B (A \rtimes B)$ and $A \rtimes B$, the definition of κ, Lemma 1.1.8 (3) and the first identity in (CM2). Note that $\mu(a_1 + {}^{b_1}a_2) + b_1 + b_2 = \mu(a_1) + b_1 + \mu(a_2) - b_1 + b_1 + b_2 = \mu(a_1) + b_1 + \mu(a_2) + b_2$, due to the first identity in (CM1), so the third line in the previous calculations makes sense. On the other hand,

$$\begin{aligned}
&\kappa((a_1, b_1), (a'_1, \mu(a_1) + b_1)) + \kappa((a_2, b_2), (a'_2, \mu(a_2) + b_2))\\
&= (a'_1 + a_1, b_1) + (a'_2 + a_2, b_2) = (a'_1 + a_1 + {}^{b_1}(a'_2 + a_2), b_1 + b_2)\\
&= (a'_1 + a_1 + {}^{b_1}a'_2 + {}^{b_1}a_2, b_1 + b_2),
\end{aligned}$$

by making use of the definition of κ, the addition in $A \rtimes B$ and Lemma 1.1.8 (2). Hence, κ preserves $+$. Calculations for $* \in \Omega'_2$ are similar, but involving Lemma 1.1.8 (4), (5) and (12), axiom (c) from Definition 1.1.1 and the second identity in (CM2).

Commutativity of the diagrams in (1.1.7) and (1.1.8) follows directly from the definitions of s, t, e and κ and the fact that μ is a morphism. Commutativity of (1.1.9) and (1.1.10) is easy to prove bearing in mind that the elements in $(A \rtimes B) \times_B (A \rtimes B) \times_B (A \rtimes B)$ are of the form $((a, b), (a', \mu(a) + b), (a'', \mu(a') + \mu(a) + b))$, with $a, a', a'' \in A$, $b \in B$, while those in $B \times_B (A \rtimes B)$ and $(A \rtimes B) \times_B B$ are of the form $(b, (a, b))$ and $((a, b), \mu(a) + b)$ respectively.

Defining $\mathtt{I_{cat}}$ on morphisms is quite obvious. Given a morphism of crossed modules (φ, ψ) between (A, B, μ) and (A', B', μ'), its corresponding internal functor is given by $(\varphi, \psi)\colon A \rtimes B \to A' \rtimes B'$ and $\psi\colon B \to B'$, where $(\varphi, \psi)(a, b) = (\varphi(a), \psi(b))$. The map (φ, ψ) preserves the operations in Ω_2 due to (1.1.4) and (1.1.5), while it obviously preserves 0 and operations in Ω_1. Hence, (φ, ψ) is a morphism. Commutativity of the diagrams in (1.1.11) and (1.1.12) follows from the definitions of s, s', t, t', e, e', κ and κ', along with the equality $\psi\mu = \mu'\varphi$.

$\mathtt{I_{cat}}$ is clearly a functor with the previous assignments for objects and morphisms. Now let us define the functor $\mathtt{X_{mod}}$. Let $C = (C_1, C_0, s, t, e, \kappa)$ be an internal category in $\mathcal{C}$. Consider $\operatorname{Ker} s$ and the morphism $t|_{\operatorname{Ker} s}\colon \operatorname{Ker} s \to C_0$. We will write t in order to ease notation. The sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} s \arrow[r, hook] & C_1 \arrow[r, "s"'] & C_0 \arrow[l, bend right, "e"'] \arrow[r] & 0
\end{tikzcd}$$

defines a C_0-structure on $\operatorname{Ker} s$ due to the commutativity of the first diagram in (1.1.7). It only remains to prove that $(\operatorname{Ker} s, C_0, t|_{\operatorname{Ker} s})$ satisfies (CM1) and (CM2). Given $x \in C_0$ and $f \in \operatorname{Ker} s$,

$$\begin{aligned} t({}^{x}f) &= t(e(x) + f - e(x)) = te(x) + t(f) - te(x) = x + t(f) - x, \\ t(x * f) &= t(e(x) * f) = te(x) * t(f) = x * t(f), \end{aligned}$$

so (CM1) holds. Note that we made use of the definition of the set of derived actions induced by the C_0-structure on $\operatorname{Ker} s$, the fact that f is a morphism and the commutativity of the second diagram in (1.1.7).

Regarding (CM2), let $f_1, f_2 \in \operatorname{Ker} s$. We know that $t(-et(f_1) + f_1) = -t(f_1) + t(f_1) = 0$, due to the commutativity of the second diagram in (1.1.7). Therefore we can apply Lemma 1.1.18 (ii) to $-et(f_1) + f_1$ and f_2, that is

$$f_2 - et(f_1) + f_1 = -et(f_1) + f_1 + f_2.$$

Hence,

$${}^{t(f_1)}f_2 = et(f_1) + f_2 - et(f_1) = f_1 + f_2 - f_1.$$

Furthermore, if we apply Lemma 1.1.18 (iii) to $et(f_1) - f_1$ and f_2, we get that

$$0 = (et(f_1) - f_1) * f_2 = et(f_1) * f_2 - f_1 * f_2.$$

Thus,

$$t(f_1) * f_2 = et(f_1) * f_2 = f_1 * f_2.$$

Therefore, (CM2) holds.

Defining $\mathtt{X}_{\mathtt{mod}}$ on morphisms is also quite obvious. Let $C = (C_1, C_0, s, t, e, \kappa)$ and $C' = (C'_1, C'_0, s', t', e', \kappa')$ be two internal categories in $\mathcal{C}$ and $F\colon C \to C'$ an internal functor, with $F_1\colon C_1 \to C'_1$ and $F_0\colon C_0 \to C'_0$. Its corresponding morphism of crossed modules is given by $(F_1|_{\operatorname{Ker} s}, F_0)$. The identity $t'F_1|_{\operatorname{Ker} s} = F_0 t$ follows from the commutativity of the second diagram in (1.1.11). Moreover, given $x \in C_0$ and $f \in \operatorname{Ker} s$,

$$\begin{aligned} F_1({}^{x}f) &= F_1(e(x) + f - e(x)) = F_1e(x) + F_1(f) - F_1e(x) \\ &= e'F_0(x) + F_1(f) - e'F_0(x) = {}^{F_0(x)}F_1(f) \end{aligned}$$

and

$$F_1(x * f) = F_1(e(x) * f) = F_1e(x) * F_1(f) = e'F_0(x) * F_1(f) = F_0(x) * F_1(f),$$

due to the fact that F_1 is a morphism and the commutativity of the third diagram in (1.1.11). It is immediate to check that, with the previous assignments, $\mathtt{X}_{\mathtt{mod}}$ is indeed a functor.

$\mathtt{I}_{\mathtt{cat}}$ and $\mathtt{X}_{\mathtt{mod}}$ establish an equivalence between the categories $\mathbf{XMod}(\mathcal{C})$ and $\mathbf{ICat}(\mathcal{C})$ with the natural isomorphisms $\alpha\colon 1_{\mathbf{XMod}(\mathcal{C})} \to \mathtt{X}_{\mathtt{mod}} \circ \mathtt{I}_{\mathtt{cat}}$ and $\beta\colon 1_{\mathbf{ICat}(\mathcal{C})} \to \mathtt{I}_{\mathtt{cat}} \circ \mathtt{X}_{\mathtt{mod}}$, given respectively, for a fixed crossed module (A, B, μ) and a fixed internal category $C = (C_1, C_0, s, t, e, \kappa)$, by:

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\mathrm{id}_B"] \\ A \rtimes \{0\} \arrow[r, "\mu"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_{C_1}"'] & C_0 \arrow[l, bend right=60, "e"'] \arrow[d, "\mathrm{id}_{C_0}"] \\ \operatorname{Ker} s \rtimes C_0 \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right, "\tilde{t}"'] & C_0 \arrow[l, bend left=60, "\tilde{e}"] \end{tikzcd}$$

respectively, where $\alpha_A(a) = (a, 0)$ for every $a \in A$, $\beta_{C_1}(f) = (f - es(f), s(f))$ for every $f \in C_1$. Observe that $\mathtt{X}_{\mathtt{mod}}(\mathtt{I}_{\mathtt{cat}}(A, B, \mu)) = (A \rtimes \{0\}, \mu, B)$. It is clear that $(\alpha_A, \mathrm{id}_B)$ is an isomorphism of crossed modules and naturality of α is obvious.

Regarding β_{C_1}, note that $\mathtt{I}_{\mathtt{cat}}(\mathtt{X}_{\mathtt{mod}}(C)) = (\operatorname{Ker} s \rtimes C_0, C_0, \tilde{s}, \tilde{t}, \tilde{e}, \tilde{\kappa})$, with $\tilde{s}(f, x) = x$, $\tilde{t}(f, x) = t(f) + x$, $\tilde{e}(x) = (0, x)$ and $\tilde{\kappa}((f, x), (f', t(f) + x)) = (f' + f, x)$ for all $f, f' \in \operatorname{Ker} s$, $x \in C_0$. Calculations in order to prove that β_{C_1} is a morphism in $\mathcal{C}$ can be easily completed by using the definition of the operations in $\operatorname{Ker} s \rtimes C_0$ (see Theorem 1.1.6), the definition of the actions of C_0 on $\operatorname{Ker} s$ and Lemma 1.1.8 (2), (4) and (12).

In order to prove that $(\beta_{C_1}, \mathrm{id}_{C_0})$ is an internal functor, it is necessary to check the commutativity of the following diagrams:

$$\begin{tikzcd} C_1 \arrow[r, "s"] \arrow[d, "\beta_{C_1}"'] & C_0 \arrow[d, "\mathrm{id}_{C_0}"] \\ \operatorname{Ker} s \rtimes C_0 \arrow[r, "\tilde{s}"'] & C_0 \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r, "t"] \arrow[d, "\beta_{C_1}"'] & C_0 \arrow[d, "\mathrm{id}_{C_0}"] \\ \operatorname{Ker} s \rtimes C_0 \arrow[r, "\tilde{t}"'] & C_0 \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[d, "\beta_{C_1}"'] & C_0 \arrow[l, "e"'] \arrow[d, "\mathrm{id}_{C_0}"] \\ \operatorname{Ker} s \rtimes C_0 & C_0 \arrow[l, "\tilde{e}"] \end{tikzcd}$$

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "(\beta_{C_1}, \beta_{C_1})"] \arrow[d, "\kappa"'] & (\operatorname{Ker} s \rtimes C_0) \times_{C_0} (\operatorname{Ker} s \rtimes C_0) \arrow[d, "\tilde{\kappa}"] \\ C_1 \arrow[r, "\beta_{C_1}"'] & \operatorname{Ker} s \rtimes C_0. \end{tikzcd}$$

For the first three diagrams it follows directly from the definitions of β_{C_1}, $\tilde{s}$, $\tilde{t}$, $\tilde{e}$ and the commutativity of the diagrams in (1.1.7) corresponding to the internal category C. As for the fourth diagram, let $(f, g) \in C_1 \times_{C_0} C_1$. We have that

$$\tilde{\kappa}((f - es(f), s(f)), (g - es(g), s(g))) = (g - es(g) + f - es(f), s(f)).$$

Note that $\tilde{t}(f - es(f), s(f)) = t(f - es(f)) + s(f) = t(f) - s(f) + s(f) = t(f) = s(g)$, so the previous composition makes sense. On the other hand,

$$\begin{aligned} \beta_{C_1}(\kappa(f, g)) &= (\kappa(f, g) - es(\kappa(f, g)), s(\kappa(f, g))) \\ &= (\kappa(f, g) - es(f), s(f)) = (g - es(g) + f - es(f), s(f)), \end{aligned}$$

due to the commutativity of the first diagram in (1.1.8) and Lemma 1.1.18 (i)

The inverse of β_{C_1} is given by $\beta_{C_1}^{-1}(f,x) = f + e(x)$, for all $(f,x) \in \operatorname{Ker} s \rtimes C_0$. Naturality of β is a matter of simple calculations which follow easily from the commutativity of the first and the third diagrams in (1.1.11) for any internal functor F between two internal categories C and C'. $\square$

1.2 Crossed modules and cat^1-objects

Let $\mathcal{C}$ be a category of groups with operations. Bearing in mind Lemma (1.1.18) (i), the following question arises naturally: Given the diagram

$$\begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend right, "e"'] \end{tikzcd}$$

with objects and morphisms in $\mathcal{C}$, such that $se = \mathrm{id}_{C_0} = te$ (see (1.1.7)), is the previous data, together with the composition given by $\kappa(f,g) = f - es(g) + g$ for all $(f,g) \in C_1 \times_{C_0} C_1$, an internal category in $\mathcal{C}$?

Firstly, we should check if κ is a morphism in $\mathcal{C}$. Let $(f,g) \in C_1 \times_{C_0} C_1$, that is $t(f) = s(g)$. It is obvious that κ preserves $0 \in \Omega_0$. Let us now consider $\omega \in \Omega_1'$.

$$\begin{aligned} \kappa(\omega(f,g)) = \kappa(\omega(f),\omega(g)) &= \omega(f) - es(\omega(g)) + \omega(g) \\ &= \omega(f - es(g) + g) = \omega(\kappa(f,g)), \end{aligned}$$

due to axiom (d) from Definition 1.1.1 and the fact that e and s are morphisms. Concerning the other operation in Ω_1, preserving $-$ would follow immediately from preserving $+$, so we can move forward to Ω_2. Let $(f_1,g_1),(f_2,g_2) \in C_1 \times_{C_0} C_1$. Then

$$\begin{aligned} \kappa(f_1+f_2, g_1+g_2) &= f_1 + f_2 - es(g_1+g_2) + g_1 + g_2 \\ &= f_1 + f_2 - es(g_2) - es(g_1) + g_1 + g_2. \end{aligned}$$

On the other hand,

$$\kappa(f_1,g_1) + \kappa(f_2,g_2) = f_1 - es(g_1) + g_1 + f_2 - es(g_2) + g_2.$$

Observe that $-es(g_1)+g_1 \in \operatorname{Ker} s$ and $f_2 - es(g_2) \in \operatorname{Ker} t$, so if we add Lemma 1.1.18 (ii) as a condition for our data, that is, $[\operatorname{Ker} s, \operatorname{Ker} t] = 0$, with $[\operatorname{Ker} s, \operatorname{Ker} t]$ the commutator of $\operatorname{Ker} s$ and $\operatorname{Ker} t$, then κ preserves the group operation. Recall that, given an internal category, composition is not only determined by the group operation, but there is also some kind of commutativity in its definition (see Lemma 1.1.18 (i)). Let us show that if we ask for the commutator of $\operatorname{Ker} s$ and $\operatorname{Ker} t$ to be zero, we get the same result. Given $(f,g) \in C_1 \times_{C_0} C_1$, we can write $g - f = g - es(g) + es(g) - f$, with

$g - es(g) \in \operatorname{Ker} s$, $es(g) - f \in \operatorname{Ker} t$. Therefore, rearranging the previous identity, we get

$$\begin{aligned} g - f &= es(g) - f + g - es(g) \\ -es(g) + g - f &= -f + g - es(g) \\ f - es(g) + g - f &= g - es(g) \\ f - es(g) + g &= g - es(g) + f. \end{aligned}$$

It remains to check the behaviour of κ with the rest of the operations in Ω_2. Let $(f_1, g_1), (f_2, g_2) \in C_1 \times_{C_0} C_1$ and $* \in \Omega_2'$. Then

$$\kappa(f_1 * f_2, g_1 * g_2) = f_1 * f_2 - es(g_1) * es(g_2) + g_1 * g_2. \tag{1.2.1}$$

Observe that we made use of Lemma 1.1.3 (ii) and the fact that e and s are morphisms in $\mathcal{C}$. On the other hand,

$$\kappa(f_1, g_1) * \kappa(f_2, g_2) = (f_1 - es(g_1) + g_1) * (f_2 - es(g_2) + g_2). \tag{1.2.2}$$

We would get 9 addends from (1.2.2), none of which would cancel. Since using Lemma 1.1.18 (ii) as a condition solved the problem with addition, it seems natural to request that (C_1, C_0, s, t, e) satisfies Lemma 1.1.18 (iii) in order to solve the problem with the operations in Ω_2'. Therefore, let us assume that $\operatorname{Ker} s * \operatorname{Ker} t = 0$ for any $* \in \Omega_2'$. Under this hypothesis we can rearrange (1.2.2), considering that $-es(g_1) + g_1 \in \operatorname{Ker} s$ and $f_2 - es(g_2) \in \operatorname{Ker} t$:

$$\begin{aligned} (1.2.2) &= f_1 * (f_2 - es(g_2)) + f_1 * g_2 + (-es(g_1) + g_1) * g_2 \\ &= f_1 * f_2 - f_1 * es(g_2) + f_1 * g_2 - es(g_1) * g_2 + g_1 * g_2 \\ &= f_1 * f_2 - f_1 * es(g_2) + f_1 * g_2 - es(g_1) * g_2 \\ &\quad + es(g_1) * es(g_2) - es(g_1) * es(g_2) + g_1 * g_2 \\ &= f_1 * f_2 + (f_1 - es(g_1)) * (g_2 - es(g_2)) - es(g_1) * es(g_2) + g_1 * g_2 \\ &= f_1 * f_2 - es(g_1) * es(g_2) + g_1 * g_2. \end{aligned}$$

Note that we made use of Lemma 1.1.3 (ii) and (iii) as well as the fact that $f_1 - es(g_1) \in \operatorname{Ker} t$, $g_2 - es(g_2) \in \operatorname{Ker} s$. Hence (1.2.1) = (1.2.2) and κ preserves $*$.

As a result of using Lemma (1.1.18) (ii) and (iii) as hypotheses, κ is guaranteed to be a morphism in $\mathcal{C}$. Besides, commutativity of the diagrams (1.1.8)–(1.1.10) follows easily from the definition of κ and the commutativity of (1.1.7). On account of all the previous verifications, we can write the following result:

Theorem 1.2.1. *Let $\mathcal{C}$ be a category of groups with operations. Consider the diagram in $\mathcal{C}$*

$$\begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend right, "e"'] \end{tikzcd}$$

such that

$$\begin{aligned} se &= \mathrm{id}_{C_0} = te, \\ [\operatorname{Ker} s, \operatorname{Ker} t] &= 0, \\ \operatorname{Ker} s * \operatorname{Ker} t &= 0, \end{aligned}$$

for all $* \in \Omega_2'$, *where* $[\operatorname{Ker} s, \operatorname{Ker} t]$ *is the commutator of* $\operatorname{Ker} s$ *and* $\operatorname{Ker} t$. *Then* $C = (C_1, C_0, s, t, e, \kappa)$ *is an internal category in* $\mathcal{C}$, *with* κ *given by*

$$\kappa(f, g) = f - es(g) + g$$

for all $(f, g) \in C_1 \times_{C_0} C_1$.

A diagram satisfying the hypotheses of the previous theorem is usually called cat^1-object in $\mathcal{C}$, alternatively described in the same way but with C_0 a subobject of C_1. One can easily derive from this result and Lemma (1.1.18) the well-known equivalence between $\mathsf{ICat}(\mathcal{C})$ and the category of cat^1-objects in $\mathcal{C}$. Observe that we have not given a definition for morphisms of cat^1-objects, but it is fairly obvious.

The cat^1-object structure will be essential in the proofs of the main results in Chapter 3. Hence, although the equivalence with crossed modules holds in general for any category of groups with operations, we will show the equivalence for the five particular cases considered in this thesis.

Although we will not mention it for every particular case, all the definitions in the subsequent subsections agree with their corresponding general version for categories of groups with operations. Observe that it is not our intention to present a thorough review on crossed modules, but to introduce basic notions and tools required for the main results in this thesis.

1.2.1 The case of groups

Crossed modules of groups were first described by Whitehead [83] in the late 1940s, as algebraic models for path-connected CW-spaces whose homotopy groups are trivial in dimensions > 2. Since their first appearance, crossed modules have become an important tool in different areas of mathematics such as homotopy theory, group (co)homology [16] or K-theory.

Observe that we will use multiplicative notation for the group operation, in contrast to the notation used in Section 1.1.

Let us first recall what an action of a group on another group means in terms of equations.

Definition 1.2.2. *An action of a group* G *on a group* H *is a map* $G \times H \to H$ $(g, h) \mapsto {}^{g}h$, *such that:*

(1) ${}^{1}h = h$,

(2) $^{g'}(^{g}h) = {}^{(g'g)}h$,

(3) $^{g}(hh') = {}^{g}h{}^{g}h'$,

for all $g, g' \in G$, $h, h' \in H$.

Given an action of a G on H it is possible to define the semidirect product $H \rtimes G$ as the underlying set of $H \times G$ equipped with the operation given by:

$$(h, g)(h', g') = (h{}^{g}h', gg')$$

for every $(h, g), (h', g') \in H \times G$. The identity element is $(1, 1)$ and for any $(h, g) \in H \rtimes G$, its inverse is given by $({}^{(g^{-1})}(h^{-1}), g^{-1})$.

Definition 1.2.3. *A crossed module of groups* (H, G, ∂) *is a group homomorphism* $\partial \colon H \to G$ *together with an action of* G *on* H *such that*

$$\partial({}^{g}h) = g\partial(h)g^{-1}, \tag{XGr1}$$

$$^{\partial(h)}h' = hh'h^{-1}. \tag{XGr2}$$

for all $h, h' \in H$, $g \in G$. *The first axiom is sometimes called equivariance, while the second one is usually known as Peiffer identity. If* (H, G, ∂) *satisfies* (XGr1) *but not necessarily* (XGr2)*, it is called precrossed module.*

Directly from the definition, we get the following well-known result:

Lemma 1.2.4. *Given a crossed module* (H, G, ∂)*,*

(i) $\operatorname{Ker}\partial$ *is a normal subgroup of* H *and* $\operatorname{Im}\partial$ *is a normal subgroup of* G.

(ii) $\operatorname{Ker}\partial \subset \mathrm{Z}(H)$*, where* $\mathrm{Z}(H)$ *is the centre of* H.

Proof. (i) The kernel of a group homomorphism is always a normal subgroup. Regarding $\operatorname{Im}\partial$, let $g' \in \operatorname{Im}\partial$ and $g \in G$. There is $h \in H$ such that $\partial(h) = g'$. As a consequence of equivariance, $gg'g^{-1} = g\partial(h)g^{-1} = \partial({}^{g}h) \in \operatorname{Im}\partial$.

(ii) Let $h \in \operatorname{Ker}\partial$ and $h' \in H$. Then, due to the Peiffer identity, $h' = {}^{1}h' = {}^{\partial(h)}h' = hh'h^{-1}$. Therefore $h'h = hh'$ and $\operatorname{Ker}\partial \subset \mathrm{Z}(H)$. □

Example 1.2.5. *We recall some generic examples, which can be found, for instance, in [45, 72]. Let* G *be a group.*
(i) G *acts on any normal subgroup* $N \lhd G$ *by conjugation and the inclusion* $i \colon N \hookrightarrow G$ *together with that action is a crossed module.* $\{1\}$ *and* G *are normal subgroups of* G*, so any group* G *can be regarded as a crossed module in two obvious ways:* $(\{1\}, G, 1)$*, where* 1 *is the trivial map, or* (G, G, id_G).
(ii) $(G, \{1\}, 1)$ *with the trivial action is a precrossed module. It satisfies the Peiffer identity if and only if* G *is an abelian group.*

(iii) Given a G-module M, that is an abelian group M together with an action of G on M, $(M, G, 1)$ is a crossed module.

(iv) If $1 \to N \to H \xrightarrow{\partial} G \to 1$ is a central extension, that is a short exact sequence with $N \subset \mathrm{Z}(H)$, then (H, G, ∂) is a crossed module, with the action of G on H given by ${}^g h = h_g h h_g^{-1}$ for all $g \in G$, $h \in H$, where h_g is an element in H such that $\partial(h_g) = g$, .

The next one is a specially interesting example:

Example 1.2.6. *Let H be a group. The morphism $\alpha\colon H \to \mathrm{Aut}(H)$, where $\alpha(h)(h') = hh'h^{-1}$, is a crossed module, together with the action of $\mathrm{Aut}(H)$ on H defined by ${}^\varphi h = \varphi(h)$ for all $\varphi \in \mathrm{Aut}(H)$, $h \in H$.*

The interesting idea behind $\mathrm{Aut}(H)$ is not just $(H, \mathrm{Aut}(H), \alpha)$ being a crossed module, but also the fact that for every action of a group G on H there is a unique group homomorphism $\beta\colon G \to \mathrm{Aut}(H)$ with ${}^g h = {}^{\beta(g)} h$. Therefore it would be possible to define a group action of G on H as a group homomorphism from G to $\mathrm{Aut}(H)$. The search for an analogous object in the categories of Lie algebras, Leibniz algebras and associative algebras led to the idea of *actor* in a category of interest [19, 20]. See Section 2.1 for more details.

Definition 1.2.7. *A morphism of crossed modules of groups $(\varphi, \psi)\colon (H, G, \partial) \to (H', G', \partial')$ is a pair of group homomorphisms, $\varphi\colon H \to H'$ and $\psi\colon G \to G'$, such that*

$$\psi\partial = \partial'\varphi, \tag{1.2.3}$$

$$\varphi({}^g h) = {}^{\psi(g)}\varphi(h), \tag{1.2.4}$$

for all $g \in G$, $h \in H$.

Example 1.2.8. *Let (H, G, ∂) be a crossed module of groups and N a group:*
(i) Given a morphism of groups $\psi\colon G \to N$ such that $\psi\partial = 0$, $(1, \psi)\colon (H, G, \partial) \to (\{1\}, N, 1)$ is a morphism of crossed modules.
(ii) Given a morphism of groups $\psi\colon N \to G$, $(1, \psi)\colon (\{1\}, N, 1) \to (H, G, \partial)$ is a morphism of crossed modules.
(iii) Given a morphism of groups $\psi\colon G \to N$, $(\psi\partial, \psi)\colon (H, G, \partial) \to (N, N, \mathrm{id}_N)$ is a morphism of crossed modules. In particular, $(\partial, \mathrm{id}_G)\colon (H, G, \partial) \to (G, G, \mathrm{id}_G)$ is a morphism of crossed modules.
(iv) Given a morphism of groups $\varphi\colon N \to H$, $(\varphi, \partial\varphi)\colon (N, N, \mathrm{id}_N) \to (H, G, \partial)$ is a morphism of crossed modules. In particular, $(\mathrm{id}_H, \partial)\colon (H, H, \mathrm{id}_H) \to (H, G, \partial)$ is a morphism of crossed modules.
(v) Considering $(G, \mathrm{Aut}(G), \alpha)$ as in Example 1.2.6, $(\partial, \alpha)\colon (H, G, \partial) \to (G, \mathrm{Aut}(G), \alpha)$ is a morphism of crossed modules.

Composition of morphisms of crossed modules is defined component-wise and the identity morphism is given by $(\mathrm{id}_H, \mathrm{id}_G)$ for any crossed module (H, G, ∂). We will denote by **XGr** the category of crossed modules of groups and morphisms of crossed modules.

Bearing in mind Example 1.2.5 (i), it is possible to define the full embeddings $\mathtt{E}_0\colon \mathbf{Gr} \to \mathbf{XGr}$ and $\mathtt{E}_1\colon \mathbf{Gr} \to \mathbf{XGr}$, where $\mathtt{E}_0(G) = (\{1\}, G, 1)$ and $\mathtt{E}_1(G) = (G, G, \mathrm{id}_G)$ for all $G \in \mathbf{Gr}$. Given a morphism of groups $\alpha\colon G \to G'$, $\mathtt{E}_0(\alpha) = (1, \alpha)$ and $\mathtt{E}_1(\alpha) = (\alpha, \alpha)$.

Additionally, let us define the functors Υ_0, Υ_1 and Υ_2, from **XGr** to **Gr**, given by $\Upsilon_0(H, G, \partial) = G/\partial(H)$, $\Upsilon_1(H, G, \partial) = G$ and $\Upsilon_2(H, G, \partial) = H$, for any crossed module of groups (H, G, ∂). Given a morphism of crossed modules $(\varphi, \psi)\colon (H, G, \partial) \to (H', G', \partial')$, $\Upsilon_0(\varphi, \psi) = \overline{\psi}$, $\Upsilon_1(\varphi, \psi) = \psi$ and $\Upsilon_2(\varphi, \psi) = \varphi$, where $\overline{\psi}$ is the morphism from $G/\partial(H)$ to $G'/\partial'(H')$ induced by ψ. Note that $\overline{\psi}$ is well defined, since given $g_1, g_2 \in G$ such that $\overline{g_1} = \overline{g_2}$ in $G/\partial(H)$, $g_1 g_2^{-1} \in \partial(H)$, so there is $h \in H$ for which $\partial(h) = g_1 g_2^{-1}$. Due to (1.2.3), $\psi(g_1 g_2^{-1}) = \psi(\partial(h)) = \partial'\varphi(h) \in \partial'(H')$.

Proposition 1.2.9. *Υ_0 is left adjoint to $\mathtt{E}_0$, $\mathtt{E}_0$ is left adjoint to Υ_1, Υ_1 is left adjoint to $\mathtt{E}_1$ and $\mathtt{E}_1$ is left adjoint to Υ_2.*

Proof. It is fairly easy to construct the corresponding natural bijections in order to prove each of the adjunctions. Actually, there are explicit descriptions of them in Example 1.2.8 (i)–(iv). For instance, for the fist adjunction:

Let (H, G, ∂) be a crossed module and N a group. Given $\alpha \in \mathrm{Hom}_{\mathbf{Gr}}(G/\partial(H), N)$, we can define the morphism of crossed modules

$$\begin{tikzcd} H \arrow[r, "\partial"] \arrow[d, "1"'] & G \arrow[d, "\alpha\pi"] \\ \{1\} \arrow[r, "1"'] & N \end{tikzcd}$$

where π is the projection from G to $G/\partial(H)$. Note that $(1, \alpha\pi)$ is a particular case of Example 1.2.8 (i). Conversely, given $(1, \psi) \in \mathrm{Hom}_{\mathbf{XGr}}((H, G, \partial), (\{1\}, N, 1))$, due to (1.2.3), $\psi\partial = 1$. Hence, $\tilde{\psi}\colon G/\partial(H) \to N$, given by $\tilde{\psi}(\overline{g}) = \psi(g)$ is well defined. Naturality is obvious.

The other three adjunctions can be proved similarly. $\square$

Definition 1.2.10. *A* cat^1*-group* (G_1, G_0, s, t) *consists of a group* G_1 *together with a subgroup* G_0 *and structural the morphisms* $s, t\colon G_1 \to G_0$ *such that*

$$s|_{G_0} = t|_{G_0} = \mathrm{id}_{G_0}, \tag{CGr1}$$

$$[\mathrm{Ker}\, s, \mathrm{Ker}\, t] = 1, \tag{CGr2}$$

where $[\mathrm{Ker}\, s, \mathrm{Ker}\, t]$ *is the commutator of* $\mathrm{Ker}\, s$ *and* $\mathrm{Ker}\, t$.

Definition 1.2.11. *A morphism of* cat^1*-groups* $\gamma\colon (G_1, G_0, s, t) \to (G'_1, G'_0, s', t')$ *is a group homomorphism* $\gamma\colon G_1 \to G'_1$ *such that* $\gamma(G_0) \subseteq G'_0$ *and* $s'\gamma = \gamma|_{G_0} s$, $t'\gamma = \gamma|_{G_0} t$.

Composition of morphisms of cat^1-groups is obvious. We will denote by $\mathsf{C^1Gr}$ the category of cat^1-groups and morphisms of cat^1-groups.

Proposition 1.2.12. *The categories* XGr *and* $\mathsf{C^1Gr}$ *are equivalent.*

Proof. Given a crossed module of groups (H, G, ∂), the corresponding cat^1-group is $(H \rtimes G, G, s, t)$, where $s(h, g) = g$ and $t(h, g) = \partial(h)g$ for all $(h, g) \in H \rtimes G$. It is clear that s is a group homomorphism, directly from the group operation in $H \rtimes G$ and the definition of s. Regarding t, given $(h_1, g_1), (h_2, g_2) \in H \rtimes G$,

$$\begin{aligned} t((h_1, g_1)(h_2, g_2)) &= t(h_1{}^{g_1}h_2, g_1g_2) = \partial(h_1{}^{g_1}h_2)g_1g_2 = \partial(h_1)g_1\partial(h_2)g_1^{-1}g_1g_2 \\ &= \partial(h_1)g_1\partial(h_2)g_2 = t(h_1, g_1)t(h_2, g_2), \end{aligned}$$

due to (XGr1). Note that G can be regarded as a subgroup of $H \rtimes G$ via the monomorphism $g \mapsto (1, g)$. It is clear that $s|_G = t|_G = \mathrm{id}_G$. Besides, $\operatorname{Ker} s = \{(h, 1) \mid h \in H\}$ and $\operatorname{Ker} t = \{(h, \partial(h^{-1})) \mid h \in H\}$. Let $h_1, h_2 \in H$. Directly from (XGr2), we have that $h_1{}^{\partial(h_1^{-1})}h_2 = h_2h_1$. Therefore,

$$(h_2, 1)(h_1, \partial(h_1^{-1})) = (h_2h_1, \partial(h_1^{-1})) = (h_1{}^{\partial(h_1^{-1})}h_2, \partial(h_1^{-1})) = (h_1, \partial(h_1^{-1}))(h_2, 1),$$

Hence $[\operatorname{Ker} s, \operatorname{Ker} t] = 1$ and $(H \rtimes G, G, s, t)$ is a cat^1-group.

Additionally, given a morphism of crossed modules $(\varphi, \psi)\colon (H, G, \partial) \to (H', G', \partial')$, the corresponding morphism of cat^1-groups is defined by $f_{\varphi,\psi}(h, g) = (\varphi(h), \psi(g))$ for any $(h, g) \in H \rtimes G$. It is clear that $f_{\varphi,\psi}$ is a group homomorphism, directly from (1.2.4) and the fact that φ and ψ are group homomorphisms. On the other hand, it is immediate that $f_{\varphi,\psi}(G) \subseteq G'$, and the squares

$$\begin{tikzcd} H \rtimes G \arrow[r, "s"] \arrow[d, "f_{\varphi,\psi}"'] & G \arrow[d, "f_{\varphi,\psi}|_G"] \\ H' \rtimes G' \arrow[r, "s'"'] & G' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} H \rtimes G \arrow[r, "t"] \arrow[d, "f_{\varphi,\psi}"'] & G \arrow[d, "f_{\varphi,\psi}|_G"] \\ H' \rtimes G' \arrow[r, "t'"'] & G' \end{tikzcd}$$

are commutative, the first one directly from the definition of the morphisms involved; the second one due to (1.2.3). The previous assignments clearly define a functor from XGr to $\mathsf{C^1Gr}$, which will be denoted by $\mathtt{cat}_{\mathtt{Gr}}$.

Conversely, given a cat^1-group (G_1, G_0, s, t), the corresponding crossed module is $t|_{\operatorname{Ker} s}\colon \operatorname{Ker} s \to G_0$, with the action of G_0 on $\operatorname{Ker} s$ given by conjugation. Sometimes we will write simply t for ease of notation. The action is obviously well defined. Regarding (XGr1), it follows immediately from (CGr1), specifically from the identity $t|_{G_0} = \mathrm{id}_{G_0}$:

$$t({}^{y}x) = t(yxy^{-1}) = t(y)t(x)t(y^{-1}) = yt(x)y^{-1}$$

for all $y \in G_0$, $x \in \operatorname{Ker} s$. Now, let $x_1, x_2 \in \operatorname{Ker} s$. Since $t|_{G_0} = \mathrm{id}_{G_0}$, it is clear that $t(x_1^{-1})x_1 \in \operatorname{Ker} t$. Then, by (CGr2), $x_2 t(x_1^{-1})x_1 = t(x_1^{-1})x_1 x_2$. Therefore

$$^{t(x_1)}x_2 x_1 = t(x_1)x_2 t(x_1^{-1})x_1 = t(x_1)t(x_1^{-1})x_1 x_2 = x_1 x_2,$$

hence

$$^{t(x_1)}x_2 = x_1 x_2 x_1^{-1},$$

so (XGr2) holds.

Moreover, given a morphism of cat^1-groups $\gamma\colon (G_1, G_0, s, t) \to (G_1', G_0', s', t')$, its corresponding morphism of crossed modules is given by

$$\begin{tikzcd}
\operatorname{Ker} s \arrow[r, "t|_{\operatorname{Ker} s}"] \arrow[d, "\gamma|_{\operatorname{Ker} s}"'] & G_0 \arrow[d, "\gamma|_{G_0}"] \\
\operatorname{Ker} s' \arrow[r, "t'|_{\operatorname{Ker} s'}"'] & G_0'.
\end{tikzcd}$$

Note that $\gamma(\operatorname{Ker} s) \subset \operatorname{Ker} s'$, directly from the identity $s'\gamma = \gamma|_{G_0} s$. The commutativity of the previous diagram follows from the identity $t'\gamma = \gamma|_{G_0} t$. Besides, given $y \in G_0$, $x \in \operatorname{Ker} s$, the identity $\gamma|_{\operatorname{Ker} s}(^{y}x) = {}^{\gamma|_{G_0}(y)}\gamma(x)$ follows from the definition of the action of G_0 on $\operatorname{Ker} s$ and the fact that γ is a group homomorphism. The previous assignments clearly define a functor from $\mathbf{C^1Gr}$ to $\mathbf{XGr}$, which will be denoted by $\mathtt{Xm}_{\mathtt{Gr}}$.

$\mathtt{cat}_{\mathtt{Gr}}$ and $\mathtt{Xm}_{\mathtt{Gr}}$ establish an equivalence between the categories $\mathbf{XGr}$ and $\mathbf{C^1Gr}$, with the natural isomorphisms $\alpha\colon 1_{\mathbf{XGr}} \to \mathtt{Xm}_{\mathtt{Gr}} \circ \mathtt{cat}_{\mathtt{Gr}}$ and $\beta\colon 1_{\mathbf{C^1Gr}} \to \mathtt{cat}_{\mathtt{Gr}} \circ \mathtt{Xm}_{\mathtt{Gr}}$ given, for a fixed (H, G, ∂) in $\mathbf{XGr}$ and a fixed (G_1, G_0, s, t) in $\mathbf{C^1Gr}$, by:

$$\begin{tikzcd}
H \arrow[r, "\partial"] \arrow[d, "\alpha_H"'] & G \arrow[d, "\mathrm{id}_G"] \\
H \rtimes \{1\} \arrow[r, "\partial"'] & G
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_{G_1}"'] & G_0 \arrow[d, "\mathrm{id}_{G_0}"] \\
\operatorname{Ker} s \rtimes G_0 \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right, "\tilde{t}"'] & G_0
\end{tikzcd}$$

respectively, where $\alpha_H(h) = (h, 1)$ for every $h \in H$, $\beta_{G_1}(g) = (gs(g^{-1}), s(g))$ for every $g \in G_1$. It is clear that $(\alpha_H, \mathrm{id}_G)$ is an isomorphism of crossed modules and the naturality of α is obvious.

Concerning β_{G_1}, observe that $\mathtt{cat}_{\mathtt{Gr}}(\mathtt{Xm}_{\mathtt{Gr}}(G_1, G_0, s, t)) = (\operatorname{Ker} s \rtimes G_0, G_0, \tilde{s}, \tilde{t})$, with $\tilde{s}(x, y) = y$ and $\tilde{t}(x, y) = t(x)y$ for all $x \in \operatorname{Ker} s$, $y \in G_0$. It is easy to check that β_{G_1} is a group homomorphism just by using the definition of the group operation in $\operatorname{Ker} s \rtimes G_0$ and the action of G_0 on $\operatorname{Ker} s$. Besides, given $y \in G_0$, $\beta_{G_1}(y) = (1, y)$, since $s|_{G_0} = \mathrm{id}_{G_0}$. Calculations in order to check the identities $\tilde{s}\beta_{G_1} = s$ and $\tilde{t}\beta_{G_1} = t$ are obvious. The inverse of β_{G_1} is given by $\beta_{G_1}^{-1}(x, y) = xy$, for all $(x, y) \in \operatorname{Ker} s \rtimes G_0$. Naturality of β can be readily checked by using the identity $s'\gamma = \gamma|_{G_0} s$ for any morphism γ between two given cat^1-groups (G_1, G_0, s, t) and (G_1', G_0', s', t'). $\square$

1.2.2 The case of Lie algebras

Lie crossed modules have been investigated by various authors. Namely, in [60] Kassel and Loday use Lie crossed modules as computational tools in order to give an interpretation of the third relative Chevalley-Eilenberg cohomology of Lie algebras. Guin [51] developed the low-dimensional non-abelian cohomology of Lie algebras with coefficients in Lie crossed modules, which later was extended to higher dimensions in [57]. Internal (cotriple) homology and Chevalley-Eilenberg homology theories of Lie crossed modules were investigated in [23, 35]. Lie crossed modules also occur in the "categorification" problem of the theory of Lie algebras [4] as an equivalent formulation of strict Lie 2-algebras (see Section 1.3).

Recall that a *Lie algebra* $\mathbb{p}$ over K is a K-module together with a bilinear operation $[\ ,\ ]\colon \mathbb{p} \times \mathbb{p} \to \mathbb{p}$, called the Lie bracket, such that

$$\begin{aligned}&[p,p] = 0,\\ &[p_1,[p_2,p_3]] + [p_2,[p_3,p_1]] + [p_3,[p_1,p_2]] = 0,\end{aligned}$$

for all $p, p_1, p_2, p_3 \in \mathbb{p}$. The second equality is usually called the *Jacobi identity*. A *morphism of Lie algebras* is a K-linear map that preserves the bracket. We will denote by **Lie** the category of Lie algebras and morphisms of Lie algebras.

Definition 1.2.13. *Let $\mathbb{m}$ and $\mathbb{p}$ be two Lie algebras. An action of $\mathbb{p}$ on $\mathbb{m}$ is a bilinear map $\mathbb{p} \times \mathbb{m} \to \mathbb{m}$, $(p,m) \mapsto [p,m]$ such that*

(1) $[[p_1,p_2],m] = [p_1,[p_2,m]] - [p_2,[p_1,m]]$,

(2) $[p,[m_1,m_2]] = [[p,m_1],m_2] + [m_1,[p,m_2]]$,

for all $m, m_1, m_2 \in \mathbb{m}$ and $p, p_1, p_2 \in \mathbb{p}$.

We will say that $\mathbb{p}$ *acts trivially* on $\mathbb{m}$ if $[p,m] = 0$ for all $m \in \mathbb{m}$, $p \in \mathbb{p}$. Observe that we denote the action by the same symbol used for the multiplication in $\mathbb{m}$ and $\mathbb{p}$, by analogy to the notation used for Ω-groups. This means no ambiguity, since the arguments of the bracket will always determine the only possible choice. Note that the two identities in the definition of a Lie action can be obtained from the Jacobi identity by taking two elements in $\mathbb{p}$ and one in $\mathbb{m}$ (first identity), and two elements in $\mathbb{m}$ and one in $\mathbb{p}$ (second identity).

Given a Lie action of $\mathbb{p}$ on $\mathbb{m}$ we can form the *semidirect product* Lie algebra, $\mathbb{m} \rtimes \mathbb{p}$, with the underlying K-module $\mathbb{m} \oplus \mathbb{p}$ and the Lie bracket given by

$$[(m_1,p_1),(m_2,p_2)] = ([m_1,m_2] + [p_1,m_2] - [p_2,m_1], [p_1,p_2]),$$

for all $(m_1,p_1),\ (m_2,p_2) \in \mathbb{m} \oplus \mathbb{p}$.

Definition 1.2.14. *A crossed module of Lie algebras (or Lie crossed module)* $(\mathbb{m}, \mathbb{p}, \nu)$ *is a Lie homomorphism* $\nu\colon \mathbb{m} \to \mathbb{p}$ *together with an action of* $\mathbb{p}$ *on* $\mathbb{m}$ *such that*

$$\nu([p,m]) = [p, \nu(m)], \tag{XLie1}$$

$$[\nu(m_1), m_2] = [m_1, m_2]. \tag{XLie2}$$

for all $m, m_1, m_2 \in \mathbb{m}$ *and* $p \in \mathbb{p}$.

For the sake of coherence, (XLie1) will be called *equivariance* and (XLie2) *Peiffer identity*. If $(\mathbb{m}, \mathbb{p}, \nu)$ satisfies (XLie1) but not necessarily (XLie2), it is called *precrossed module*. Moreover, we have the following result:

Lemma 1.2.15. *Given a Lie crossed module* $(\mathbb{m}, \mathbb{p}, \nu)$*,*

(i) $\operatorname{Ker}\nu$ *is an ideal of* $\mathbb{m}$ *and* $\operatorname{Im}\nu$ *is an ideal of* $\mathbb{p}$.

(ii) $\operatorname{Ker}\nu \subset \operatorname{Ann}(\mathbb{m})$*, where* $\operatorname{Ann}(\mathbb{m})$ *is the annihilator of* $\mathbb{m}$.

Proof. $\operatorname{Ker}\nu$ is an ideal for any Lie homomorphism ν. The calculations for the other two statements can be easily completed by using (XLie1) and (XLie2) respectively. □

Example 1.2.16. *Let* $\mathbb{p}$ *be a Lie algebra.*
(i) The Lie bracket in $\mathbb{p}$ *yields an action of* $\mathbb{p}$ *on any ideal* $\mathbb{q}$ *of* $\mathbb{p}$*. The inclusion* $i\colon \mathbb{q} \hookrightarrow \mathbb{p}$ *together with that action is a Lie crossed module.* $\{0\}$ *and* $\mathbb{p}$ *are ideals of* $\mathbb{p}$*, so any Lie algebra* $\mathbb{p}$ *can be regarded as a crossed module in two obvious ways:* $(\{0\}, \mathbb{p}, 0)$*, where* 0 *is the trivial map, or* $(\mathbb{p}, \mathbb{p}, \mathrm{id}_{\mathbb{p}})$*.*
(ii) $(\mathbb{p}, \{0\}, 0)$ *with the trivial action is a precrossed module. It satisfies the Peiffer identity if and only if* $\mathbb{p}$ *is abelian, that is, if the bracket is trivial.*
(iii) If $0 \to \mathbb{q} \to \mathbb{m} \xrightarrow{\nu} \mathbb{p} \to 0$ *is a short exact sequence with* $\mathbb{q} \subset \operatorname{Ann}(\mathbb{m})$*, then* $(\mathbb{m}, \mathbb{p}, \nu)$ *is a Lie crossed module, with the action of* $\mathbb{p}$ *on* $\mathbb{m}$ *given by* $[p,m] = [m_p, m]$ *for all* $p \in \mathbb{p}$*,* $m \in \mathbb{m}$*, where* m_p *is an element in* $\mathbb{m}$ *such that* $\nu(m_p) = p$*.*

The role played by $\operatorname{Aut}(H)$ for any group H (see Example 1.2.6) is played by $\operatorname{Der}(\mathbb{m})$, the Lie algebra of derivations, for any Lie algebra $\mathbb{m}$. Note that an element $d \in \operatorname{Der}(\mathbb{m})$ is a K-linear map from $\mathbb{m}$ to $\mathbb{m}$ such that $d[m_1, m_2] = [d(m_1), m_2] + [m_1, d(m_2)]$. The Lie structure in $\operatorname{Der}(\mathbb{m})$ is given by $[d_1, d_2] = d_1 d_2 - d_2 d_1$ for all $d_1, d_2 \in \operatorname{Der}(\mathbb{m})$.

Example 1.2.17. *The Lie homomorphism* $\alpha\colon \mathbb{m} \to \operatorname{Der}(\mathbb{m})$*, where* $\alpha(m)(m') = [m, m']$*, together with the action of* $\operatorname{Der}(\mathbb{m})$ *on* $\mathbb{m}$ *defined by* $[\varphi, m] = \varphi(m)$ *for all* $\varphi \in \operatorname{Der}(\mathbb{m})$*,* $m \in \mathbb{m}$*, is a Lie crossed module.*

Every action of a Lie algebra $\mathbb{p}$ on a Lie algebra $\mathbb{m}$ yields a unique morphism of Lie algebras $\beta\colon \mathbb{p} \to \operatorname{Der}(\mathbb{m})$, such that $[\beta(p), m] = [p, m]$. Hence, it would be possible to define a Lie action of $\mathbb{p}$ on $\mathbb{m}$ as a Lie homomorphism from $\mathbb{p}$ to $\operatorname{Der}(\mathbb{m})$. See Section 2.1 for more details.

Definition 1.2.18. *A morphism of Lie crossed modules* $(\varphi,\psi)\colon(\mathbb{m},\mathbb{p},\nu)\to(\mathbb{m}',\mathbb{p}',\nu')$ *is a pair of Lie homomorphisms,* $\varphi\colon\mathbb{m}\to\mathbb{m}'$ *and* $\psi\colon\mathbb{p}\to\mathbb{p}'$*, such that*

$$\psi\nu=\nu'\varphi, \tag{1.2.5}$$

$$\varphi([p,m])=[\psi(p),\varphi(m)], \tag{1.2.6}$$

for all $m\in\mathbb{m}$*,* $p\in\mathbb{p}$*.*

Example 1.2.19. *Let* $(\mathbb{m},\mathbb{p},\nu)$ *be a Lie crossed module and* $\mathbb{q}$ *a Lie algebra.*
(i) Given a Lie homomorphism $\psi\colon\mathbb{p}\to\mathbb{q}$ *such that* $\psi\nu=0$*,* $(0,\psi)\colon(\mathbb{m},\mathbb{p},\nu)\to(\{0\},\mathbb{q},0)$ *is a morphism of Lie crossed modules.*
(ii) Given a Lie homomorphism $\psi\colon\mathbb{q}\to\mathbb{p}$*,* $(0,\psi)\colon(\{0\},\mathbb{q},0)\to(\mathbb{m},\mathbb{p},\nu)$ *is a morphism of Lie crossed modules.*
(iii) Given a Lie homomorphism $\psi\colon\mathbb{p}\to\mathbb{q}$*,* $(\psi\nu,\psi)\colon(\mathbb{m},\mathbb{p},\nu)\to(\mathbb{q},\mathbb{q},\mathrm{id}_{\mathbb{q}})$ *is a morphism of Lie crossed modules. In particular,* $(\nu,\mathrm{id}_{\mathbb{p}})\colon(\mathbb{m},\mathbb{p},\nu)\to(\mathbb{p},\mathbb{p},\mathrm{id}_{\mathbb{p}})$ *is a morphism of Lie crossed modules.*
(iv) Given a Lie homomorphism $\varphi\colon\mathbb{q}\to\mathbb{m}$*,* $(\varphi,\nu\varphi)\colon(\mathbb{q},\mathbb{q},\mathrm{id}_{\mathbb{q}})\to(\mathbb{m},\mathbb{p},\nu)$ *is a morphism of Lie crossed modules. In particular,* $(\mathrm{id}_{\mathbb{m}},\nu)\colon(\mathbb{m},\mathbb{m},\mathrm{id}_{\mathbb{m}})\to(\mathbb{m},\mathbb{p},\nu)$ *is a morphism of Lie crossed modules.*
(v) $(\nu,\alpha)\colon(\mathbb{m},\mathbb{p},\nu)\to(\mathbb{p},\mathrm{Der}(\mathbb{p}),\alpha)$ *is a morphism of Lie crossed modules, with* $(\mathbb{p},\mathrm{Der}(\mathbb{p}),\alpha)$ *as in Example 1.2.17.*

Composition of morphisms of Lie crossed modules is defined component-wise and the identity morphism is given by $(\mathrm{id}_{\mathbb{m}},\mathrm{id}_{\mathbb{p}})$ for any crossed module $(\mathbb{m},\mathbb{p},\nu)$. We will denote by **XLie** the category of Lie crossed modules and morphisms of Lie crossed modules.

Just like in the case of crossed modules of groups, it is possible to define the full embeddings $\mathtt{I}_0\colon$ **Lie** $\to$ **XLie** and $\mathtt{I}_1\colon$ **Lie** $\to$ **XLie**, with $\mathtt{I}_0(\mathbb{p})=(\{0\},\mathbb{p},0)$ and $\mathtt{I}_1(\mathbb{p})=(\mathbb{p},\mathbb{p},\mathrm{id}_{\mathbb{p}})$ for any Lie algebra $\mathbb{p}$. Given a morphism of Lie algebras $\alpha\colon\mathbb{p}\to\mathbb{p}'$, $\mathtt{I}_0(\alpha)=(0,\alpha)$ and $\mathtt{I}_1(\alpha)=(\alpha,\alpha)$.

Continuing with the analogy, let us define the functors Φ_0, Φ_1 and Φ_2, from **XLie** to **Lie**, given by $\Phi_0(\mathbb{m},\mathbb{p},\nu)=\mathbb{p}/\nu(\mathbb{m})$, $\Phi_1(\mathbb{m},\mathbb{p},\nu)=\mathbb{p}$ and $\Phi_2(\mathbb{m},\mathbb{p},\nu)=\mathbb{m}$ for any Lie crossed module $(\mathbb{m},\mathbb{p},\nu)$. Given a morphism of Lie crossed modules $(\varphi,\psi)\colon(\mathbb{m},\mathbb{p},\nu)\to(\mathbb{m}',\mathbb{p}',\nu')$, $\Phi_0(\varphi,\psi)=\overline{\psi}$, $\Phi_1(\varphi,\psi)=\psi$ and $\Phi_2(\varphi,\psi)=\varphi$, where $\overline{\psi}$ is the morphism from $\mathbb{p}/\nu(\mathbb{m})$ to $\mathbb{p}'/\nu'(\mathbb{m}')$ induced by ψ.

Proposition 1.2.20. Φ_0 *is left adjoint to* $\mathtt{I}_0$*,* $\mathtt{I}_0$ *is left adjoint to* Φ_1*,* Φ_1 *is left adjoint to* $\mathtt{I}_1$ *and* $\mathtt{I}_1$ *is left adjoint to* Φ_2*.*

Proof. The corresponding natural bijections can be readily described by using Example 1.2.19 (i)–(iv). ☐

Definition 1.2.21. *A* cat^1*-Lie algebra* $(\mathbb{p}_1,\mathbb{p}_0,s,t)$ *consists of a Lie algebra* $\mathbb{p}_1$ *together with a Lie subalgebra* $\mathbb{p}_0$ *and the structural morphisms* $s,t\colon\mathbb{p}_1\to\mathbb{p}_0$ *such* *that*

$$s|_{\mathbb{p}_0} = t|_{\mathbb{p}_0} = \mathrm{id}_{\mathbb{p}_0}, \tag{CLie1}$$

$$[\operatorname{Ker} s, \operatorname{Ker} t] = 0. \tag{CLie2}$$

Definition 1.2.22. *A morphism of* cat^1*-Lie algebras* $\gamma\colon (\mathbb{p}_1, \mathbb{p}_0, s, t) \to (\mathbb{p}'_1, \mathbb{p}'_0, s', t')$ *is a Lie homomorphism* $\gamma\colon \mathbb{p}_1 \to \mathbb{p}'_1$ *such that* $\gamma(\mathbb{p}_0) \subseteq \mathbb{p}'_0$ *and* $s'\gamma = \gamma|_{\mathbb{p}_0} s$*,* $t'\gamma = \gamma|_{\mathbb{p}_0} t$*.*

Composition of morphisms of cat^1-Lie algebras is obvious. We will denote by $\mathbf{C^1Lie}$ the category of cat^1-Lie algebras and morphisms of cat^1-Lie algebras.

Proposition 1.2.23. *The categories* $\mathbf{XLie}$ *and* $\mathbf{C^1Lie}$ *are equivalent.*

Proof. Given a crossed module of Lie algebras $(\mathbb{m}, \mathbb{p}, \nu)$, the corresponding cat^1-Lie algebra is $(\mathbb{m} \rtimes \mathbb{p}, \mathbb{p}, s, t)$, with $s(m,p) = p$ and $t(m,p) = \nu(m)+p$ for all $(m,p) \in \mathbb{m} \rtimes \mathbb{p}$. It is evident that s is a Lie homomorphism, while t preserves the bracket due to (XLie1), the fact that ν is a Lie homomorphism and the bilinearity and antisymmetry of the bracket in $\mathbb{p}$. Note that $\mathbb{p}$ can be regarded as a Lie subalgebra of $\mathbb{m} \rtimes \mathbb{p}$ via the morphism $p \mapsto (0,p)$. It is obvious that $s|_{\mathbb{p}} = t|_{\mathbb{p}} = \mathrm{id}_{\mathbb{p}}$. Directly from the definition of s and t, we get that $\operatorname{Ker} s = \{(m,0) \mid m \in \mathbb{m}\}$ and $\operatorname{Ker} t = \{(m, -\nu(m)) \mid m \in \mathbb{m}\}$. Given $m_1, m_2 \in \mathbb{m}$, due to (XLie2) and the antisymmetry of the bracket in $\mathbb{m}$, we know that $-[m_1, m_2] = [m_2, m_1] = [\nu(m_2), m_1]$. Hence,

$$[(m_1, 0), (m_2, -\nu(m_2))] = ([m_1, m_2] + [\nu(m_2), m_1]\,, 0) = (0,0).$$

Therefore, $[\operatorname{Ker} s, \operatorname{Ker} t] = 0$ and $(\mathbb{m} \rtimes \mathbb{p}, \mathbb{p}, s, t)$ is a cat^1-Lie algebra.

Additionally, given a morphism of Lie crossed modules (φ, ψ) from $(\mathbb{m}, \mathbb{p}, \nu)$ to $(\mathbb{m}', \mathbb{p}', \nu')$, the corresponding morphism of cat^1-Lie algebras is defined by $f_{\varphi,\psi}(m,p) = (\varphi(m), \psi(p))$ for all $(m,p) \in \mathbb{m} \rtimes \mathbb{p}$. One can easily check that $f_{\varphi,\psi}$ is a Lie homomorphism by making use of (1.2.6) and the fact that φ and ψ are Lie homomorphisms. It is clear that $f_{\varphi,\psi}(\mathbb{p}) \subseteq \mathbb{p}'$. Besides, the identity $s'\gamma = \gamma|_{\mathbb{p}_0} s$ follows from the definition of the morphisms involved, while $t'\gamma = \gamma|_{\mathbb{p}_0} t$ is an immediate consequence of (1.2.5). The previous assignments clearly define a functor from $\mathbf{XLie}$ to $\mathbf{C^1Lie}$, which will be denoted by $\mathtt{cat}_{\mathtt{Lie}}$.

Conversely, given a cat^1-Lie algebra $(\mathbb{p}_1, \mathbb{p}_0, s, t)$, the corresponding Lie crossed module is $t|_{\operatorname{Ker} s}\colon \operatorname{Ker} s \to \mathbb{p}_0$, with the action of $\mathbb{p}_0$ on $\operatorname{Ker} s$ induced by the bracket in $\mathbb{p}_1$. We will write simply t instead of $t|_{\operatorname{Ker} s}$. (XLie1) follows directly from the fact that t is a Lie homomorphism and (CLie1), specifically from the identity $t|_{\mathbb{p}} = \mathrm{id}_{\mathbb{p}}$.

Now, let $x_1, x_2 \in \operatorname{Ker} s$. It is clear that $t(x_1) - x_1 \in \operatorname{Ker} t$, since t is linear and $t|_{\mathbb{p}_0} = \mathrm{id}_{\mathbb{p}_0}$. Therefore, due to (CLie2) and the bilinearity of the bracket in $\mathbb{p}_1$, we have that

$$0 = [t(x_1) - x_1, x_2] = [t(x_1), x_2] - [x_1, x_2]\,.$$

Hence, $(\operatorname{Ker} s, \mathbb{p}_0, t|_{\operatorname{Ker} s})$ satisfies (XLie2) and it is a Lie crossed module.

Moreover, given a morphism of cat^1-Lie algebras $\gamma\colon (\mathbb{p}_1, \mathbb{p}_0, s, t) \to (\mathbb{p}'_1, \mathbb{p}'_0, s', t')$, its corresponding morphism of Lie crossed modules is given by

$$\begin{tikzcd} \operatorname{Ker} s \arrow[r, "t|_{\operatorname{Ker} s}"] \arrow[d, "\gamma|_{\operatorname{Ker} s}"'] & \mathbb{p}_0 \arrow[d, "\gamma|_{\mathbb{p}_0}"] \\ \operatorname{Ker} s' \arrow[r, "t'|_{\operatorname{Ker} s'}"'] & \mathbb{p}'_0. \end{tikzcd}$$

Note that $\gamma(\operatorname{Ker} s) \subset \operatorname{Ker} s'$, directly from the identity $s'\gamma = \gamma|_{\mathbb{p}_0} s$. Furthermore, the identity $t'\gamma = \gamma|_{\mathbb{p}_0} t$ implies the commutativity of the previous diagram. Besides, (1.2.6) follows from the definition of the action of $\mathbb{p}_0$ on $\operatorname{Ker} s$ and the fact that γ is a Lie homomorphism. The previous assignments clearly define a functor from $\mathbf{C^1Lie}$ to $\mathbf{XLie}$, which will be denoted by $\mathtt{Xm_{Lie}}$.

$\mathtt{cat_{Lie}}$ and $\mathtt{Xm_{Lie}}$ establish an equivalence between the categories $\mathbf{XLie}$ and $\mathbf{C^1Lie}$, with the natural isomorphisms $\alpha\colon 1_{\mathbf{XLie}} \to \mathtt{Xm_{Lie}} \circ \mathtt{cat_{Lie}}$ and $\beta\colon 1_{\mathbf{C^1Lie}} \to \mathtt{cat_{Lie}} \circ \mathtt{Xm_{Lie}}$ given, for a fixed $(\mathbb{m}, \mathbb{p}, \nu)$ in $\mathbf{XLie}$ and a fixed $(\mathbb{p}_1, \mathbb{p}_0, s, t)$ in $\mathbf{C^1Lie}$, by:

$$\begin{tikzcd} \mathbb{m} \arrow[r, "\nu"] \arrow[d, "\alpha_{\mathbb{m}}"'] & \mathbb{p} \arrow[d, "\mathrm{id}_{\mathbb{p}}"] \\ \mathbb{m} \rtimes \{0\} \arrow[r, "\nu"'] & \mathbb{p} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{p}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_{\mathbb{p}_1}"'] & \mathbb{p}_0 \arrow[d, "\mathrm{id}_{\mathbb{p}_0}"] \\ \operatorname{Ker} s \rtimes \mathbb{p}_0 \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right, "\tilde{t}"'] & \mathbb{p}_0 \end{tikzcd}$$

respectively, where $\alpha_{\mathbb{m}}(m) = (m, 0)$ for every $m \in \mathbb{m}$, $\beta_{\mathbb{p}_1}(p) = (p - s(p), s(p))$ for every $p \in \mathbb{p}_1$. It is clear that $(\alpha_{\mathbb{m}}, \mathrm{id}_{\mathbb{p}})$ is an isomorphism of Lie crossed modules and the naturality of α is obvious.

Concerning $\beta_{\mathbb{p}_1}$, observe that $\mathtt{cat_{Lie}}(\mathtt{Xm_{Lie}}(\mathbb{p}_1, \mathbb{p}_0, s, t)) = (\operatorname{Ker} s \rtimes \mathbb{p}_0, \mathbb{p}_0, \tilde{s}, \tilde{t})$, where $\tilde{s}(x, y) = y$ and $\tilde{t}(x, y) = t(x) + y$ for all $x \in \operatorname{Ker} s$, $y \in \mathbb{p}_0$. It is easy to check that $\beta_{\mathbb{p}_1}$ is a Lie homomorphism simply by using the definition of the Lie bracket in $\operatorname{Ker} s \rtimes \mathbb{p}_0$ and the action of $\mathbb{p}_0$ on $\operatorname{Ker} s$. Besides, given $y \in \mathbb{p}_0$, $\beta_{\mathbb{p}_1}(y) = (0, y)$, since $s|_{\mathbb{p}_0} = \mathrm{id}_{\mathbb{p}_0}$. Calculations in order to check the identities $\tilde{s}\beta_{\mathbb{p}_1} = s$ and $\tilde{t}\beta_{\mathbb{p}_1} = t$ are obvious. The inverse of $\beta_{\mathbb{p}_1}$ is given by $\beta_{\mathbb{p}_1}^{-1}(x, y) = x + y$, for all $(x, y) \in \operatorname{Ker} s \rtimes \mathbb{p}_0$. Naturality of β can be readily checked by using the identity $s'\gamma = \gamma|_{\mathbb{p}_0} s$ for any morphism γ between two given cat^1-Lie algebras $(\mathbb{p}_1, \mathbb{p}_0, s, t)$ and $(\mathbb{p}'_1, \mathbb{p}'_0, s', t')$. $\square$

1.2.3 The case of associative algebras

Crossed modules of associative algebras have not been so deeply studied as their Lie and group analogues. However, in the works of Dedecker and Lue [33, 67], crossed modules of associative algebras play a central role as coefficients for low-dimensional non-abelian cohomology. Besides, in Shammu's PhD thesis [80] the author investigates the algebraic and categorical structure of crossed modules of algebras. Baues and Minian [6] used them to represent the Hochschild cohomology of associative algebras and, in the recent article [34], the Hochschild and (cotriple) cyclic homologies of crossed modules of associative algebras were constructed and investigated.

Recall that an associative algebra (or simply algebra) A over K is a K-module together with an associative bilinear map $A \times A \to A$, $(a_1, a_2) \mapsto a_1 \cdot a_2$. A morphism of algebras is a K-linear map that preserves the product. We will denote by **As** the category of algebras over K and morphisms of algebras.

Definition 1.2.24. *An algebra A acts on another algebra B if there are two bilinear maps, $A \times B \to B$, $(a, b) \mapsto a \cdot b$ and $B \times A \to B$, $(b, a) \mapsto b \cdot a$ such that*

(1) $a \cdot (b_1 \cdot b_2) = (a \cdot b_1) \cdot b_2$,

(2) $(b_1 \cdot a) \cdot b_2 = b_1 \cdot (a \cdot b_2)$,

(3) $(b_1 \cdot b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,

(4) $b \cdot (a_1 \cdot a_2) = (b \cdot a_1) \cdot a_2$,

(5) $(a_1 \cdot b) \cdot a_2 = a_1 \cdot (b \cdot a_2)$,

(6) $(a_1 \cdot a_2) \cdot b = a_1 \cdot (a_2 \cdot b)$,

for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$.

For instance, if A is a subalgebra of an algebra B and I is an ideal in B, multiplication in B yields an action of A on I. Observe that we denote the action by the same symbol used for the multiplication in A and B, by analogy to the notation used for Ω-groups.

Remark 1.2.25. *From now on, in many occasions we will omit the symbol $\cdot$ when we refer to the multiplication of an associative algebra or an algebra action. However, it will appear whenever it is considered to facilitate the text comprehension.*

Given an action of A on B, it is possible to consider the *semidirect product algebra* $B \rtimes A$, which consists of the underlying K-module $B \oplus A$ endowed with the multiplication given by

$$(b_1, a_1)(b_2, a_2) = (b_1 b_2 + a_1 b_2 + b_1 a_2, a_1 a_2)$$

for all $(b_1, a_1), (b_2, a_2) \in B \oplus A$.

Definition 1.2.26. *A crossed module of algebras (B, A, ρ) is an algebra homomorphism $\rho\colon B \to A$ together with an action of A on B such that*

$$\rho(ab) = a\rho(b) \quad \textit{and} \quad \rho(ba) = \rho(b)a, \tag{XAs1}$$

$$\rho(b_1)b_2 = b_1 b_2 = b_1 \rho(b_2), \tag{XAs2}$$

for all $a \in A$, $b_1, b_2 \in B$.

For the sake of coherence, (XAs1) will be called *equivariance* and (XAs2) *Peiffer identity.* If (B, A, ρ) satisfies (XAs1) but not necessarily (XAs2), it is called *precrossed module.* Moreover, we have the following result:

Lemma 1.2.27. *Given a crossed module of algebras* (B, A, ρ)*,*

(i) $\operatorname{Ker}\rho$ *is an ideal of* B *and* $\operatorname{Im}\rho$ *is an ideal of* A*.*

(ii) $\operatorname{Ker}\rho \subset \operatorname{Ann}(B)$*, where* $\operatorname{Ann}(B)$ *is the two-sided annihilator of* B*.*

The concept of crossed module of algebras generalizes simultaneously the concepts of ideal and bimodule.

Example 1.2.28. *Let* A *be an algebra.*
(i) Given an ideal I *of* A*, the inclusion* $i\colon I \hookrightarrow A$ *together with action of* A *on* I *induced by the product in* A *is a crossed module of algebras.* $\{0\}$ *and* A *are ideals of* A*, so any algebra* A *can be regarded as a crossed module in two obvious ways:* $(\{0\}, A, 0)$*, where* 0 *is the trivial map, or* (A, A, id_A)*.*
(ii) $(A, \{0\}, 0)$ *with the trivial action is a precrossed module. It satisfies the Peiffer identity if and only if the product in* A *is trivial.*
(iii) Any surjective morphism of algebras $B \twoheadrightarrow A$ *with its kernel in the two-sided annihilator of* B *is a crossed module, with the action of* A *on* B *given by* $ab = \tilde{b}b$ *and* $ba = b\tilde{b}$ *for any* $a \in A$*,* $b \in B$*, where* $\tilde{b}$ *is any element in the preimage of* a*.*
(iv) A DG-algebra concentrated in degrees 0 *and* 1*,* $A = \{A_1 \xrightarrow{d} A_0\}$*, with* A_0 *acting on* A_1 *by multiplication in* A*, is a crossed module.*

The analogue to $(H, \operatorname{Aut}(H), \alpha)$ in **Gr** and $(\mathfrak{m}, \operatorname{Der}(\mathfrak{m}), \alpha)$ in **Lie** (see Examples 1.2.6 and 1.2.17) does not always exist in **As**. We will give more details about this construction in Section 2.1.

Definition 1.2.29. *A morphism of crossed modules of algebras* $(\varphi, \psi)\colon (B, A, \rho) \to (B', A', \rho')$ *is a pair of algebra homomorphisms,* $\varphi\colon B \to B'$ *and* $\psi\colon A \to A'$*, such that*

$$\psi\rho = \rho'\varphi, \tag{1.2.7}$$

$$\varphi(ba) = \varphi(b)\psi(a) \quad \text{and} \quad \varphi(ab) = \psi(a)\varphi(b), \tag{1.2.8}$$

for all $b \in B$*,* $a \in A$*.*

Example 1.2.30. *Let* (B, A, ρ) *be a crossed module of algebras and* C *an algebra.*
(i) Given a morphism of algebras $\psi\colon A \to C$ *such that* $\psi\rho = 0$*,* $(0, \psi)\colon (B, A, \rho) \to (\{0\}, C, 0)$ *is a morphism of crossed modules.*
(ii) Given a morphism of algebras $\psi\colon C \to A$*,* $(0, \psi)\colon (\{0\}, C, 0) \to (B, A, \rho)$ *is a morphism of crossed modules.*
(iii) Given a morphism of algebras $\psi\colon A \to C$*,* $(\psi\rho, \psi)\colon (B, A, \rho) \to (C, C, \mathrm{id}_C)$ *is a morphism of crossed modules. In particular,* $(\rho, \mathrm{id}_A)\colon (B, A, \rho) \to (A, A, \mathrm{id}_A)$ *is a morphism of crossed modules.*
(iv) Given a morphism of algebras $\varphi\colon C \to B$*,* $(\varphi, \rho\varphi)\colon (C, C, \mathrm{id}_C) \to (B, A, \rho)$ *is a morphism of crossed modules. In particular,* $(\mathrm{id}_B, \rho)\colon (B, B, \mathrm{id}_B) \to (B, A, \rho)$ *is a morphism of crossed modules.*

Composition of morphisms of crossed modules is defined component-wise and the identity morphism is given by $(\mathrm{id}_B, \mathrm{id}_A)$ for any crossed module (B, A, ρ). We will denote by $\mathbf{XAs}$ the category of crossed modules of algebras and morphisms of crossed modules. Furthermore, $\mathbf{XAs}^1$ will denote the subcategory of $\mathbf{XAs}$ of crossed modules of unital algebras, whose objects are crossed modules (B, A, ρ) with A a unital algebra such that $1 \cdot b = b = b \cdot 1$ for all $b \in B$, and whose morphisms are crossed module homomorphisms (φ, ψ) with ψ a morphism of unital algebras.

Just like in the case of crossed modules of groups, it is possible to define the full embeddings $\mathtt{I}'_0 \colon \mathbf{As} \to \mathbf{XAs}$ and $\mathtt{I}'_1 \colon \mathbf{As} \to \mathbf{XAs}$, with $\mathtt{I}'_0(A) = (\{0\}, A, 0)$ and $\mathtt{I}'_1(A) = (A, A, \mathrm{id}_A)$ for any algebra A. Given a morphism of algebras $\alpha \colon A \to A'$, $\mathtt{I}'_0(\alpha) = (0, \alpha)$ and $\mathtt{I}'_1(\alpha) = (\alpha, \alpha)$.

Furthermore, let us define the functors Φ'_0, Φ'_1 and Φ'_2, from $\mathbf{XAs}$ to $\mathbf{As}$, given by $\Phi'_0(B, A, \rho) = A/\rho(B)$, $\Phi'_1(B, A, \rho) = A$ and $\Phi_2(B, A, \rho) = B$ for any crossed module of algebras (B, A, ρ). Given a morphism of crossed modules of algebras $(\varphi, \psi) \colon (B, A, \rho) \to (B', A', \rho')$, $\Phi'_0(\varphi, \psi) = \overline{\psi}$, $\Phi'_1(\varphi, \psi) = \psi$ and $\Phi'_2(\varphi, \psi) = \varphi$, where $\overline{\psi} \colon A/\rho(B) \to A'/\rho'(B')$ is the morphism induced by ψ.

Proposition 1.2.31. *Φ'_0 is left adjoint to $\mathtt{I}'_0$, $\mathtt{I}'_0$ is left adjoint to Φ'_1, Φ'_1 is left adjoint to $\mathtt{I}'_1$ and $\mathtt{I}'_1$ is left adjoint to Φ'_2.*

Proof. The corresponding natural bijections can be readily described by using Example 1.2.30 (i)–(iv). □

Definition 1.2.32. *A* cat^1*-algebra (A_1, A_0, σ, τ) consists of an algebra A_1 together with a subalgebra A_0 and the structural morphisms $\sigma, \tau \colon A_1 \to A_0$ such that*

$$\sigma|_{A_0} = \tau|_{A_0} = \mathrm{id}_{A_0}, \tag{CAs1}$$

$$\operatorname{Ker}\sigma \operatorname{Ker}\tau = 0 = \operatorname{Ker}\tau \operatorname{Ker}\sigma. \tag{CAs2}$$

Definition 1.2.33. *A morphism of* cat^1*-algebras $\gamma \colon (A_1, A_0, \sigma, \tau) \to (A'_1, A'_0, \sigma', \tau')$ is an algebra homomorphism $\gamma \colon A_1 \to A'_1$ such that $\gamma(A_0) \subseteq A'_0$ and $\sigma'\gamma = \gamma|_{A_0}\sigma$, $\tau'\gamma = \gamma|_{A_0}\tau$.*

Composition of morphisms of cat^1-algebras is obvious. We will denote by $\mathbf{C}^1\mathbf{As}$ the category of cat^1-algebras and morphisms of cat^1-algebras.

Proposition 1.2.34. *The categories $\mathbf{XAs}$ and $\mathbf{C}^1\mathbf{As}$ are equivalent.*

Proof. Given a crossed module of algebras (B, A, ρ), the corresponding cat^1-algebra is $(B \rtimes A, A, \sigma, \tau)$, where $\sigma(b, a) = a$, $\tau(b, a) = \rho(b) + a$ for all $(b, a) \in B \rtimes A$. It is clear that σ is an algebra homomorphism. As for τ, it is an algebra homomorphism due to (XAs1), the fact that ρ is an algebra homomorphism and the bilinearity of the product in A. Note that A can be regarded as a subalgebra of $B \rtimes A$ via the morphism $a \mapsto (0, a)$. It is clear that $\sigma|_A = \tau|_A = \mathrm{id}_A$. Besides, one can easily derive from the definition of σ and τ that $\operatorname{Ker}\sigma = \{(b,0) \mid b \in B\}$ and $\operatorname{Ker}\tau = \{(b,-\rho(b)) \mid b \in B\}$. Let $(b_1,0) \in \operatorname{Ker}\sigma$ and $(b_2,-\rho(b_2)) \in \operatorname{Ker}\tau$. Then

$$(b_1,0)(b_2,-\rho(b_2)) = (b_1b_2 - b_1\rho(b_2),0) = (0,0),$$
$$(b_2,-\rho(b_2))(b_1,0) = (b_2b_1 - \rho(b_2)b_1,0) = (0,0),$$

due to (XAs2). Therefore, $\operatorname{Ker}\sigma \operatorname{Ker}\tau = 0 = \operatorname{Ker}\tau \operatorname{Ker}\sigma$ and $(B \rtimes A, A, \sigma, \tau)$ is a cat^1-algebra.

Additionally, given a morphism of crossed modules of algebras (φ,ψ) from (B,A,ρ) to (B',A',ρ'), the corresponding morphism of cat^1-algebras is defined by $f_{\varphi,\psi}(b,a) = (\varphi(b),\psi(a))$ for all $(b,a) \in B \rtimes A$. One can easily check that $f_{\varphi,\psi}$ is an algebra homomorphism by making use of (1.2.8) and the fact that φ and ψ are morphisms as well. It is obvious that $f_{\varphi,\psi}(A) \subseteq A'$. The identity $\sigma'\gamma = \gamma|_{A_0}\sigma$ follows from the definition of the morphisms involved and $\tau'\gamma = \gamma|_{A_0}\tau$ is an immediate consequence of (1.2.7). The previous assignments clearly define a functor from $\mathbf{XAs}$ to $\mathbf{C^1As}$, which will be denoted by $\mathtt{cat}_{\mathtt{As}}$.

Conversely, given a cat^1-algebra (A_1,A_0,σ,τ), the corresponding crossed module is $\tau|_{\operatorname{Ker}\sigma}\colon \operatorname{Ker}\sigma \to A_0$, with the action of A_0 on $\operatorname{Ker}\sigma$ induced by the product in A_1. We will write simply τ instead of $\tau|_{\operatorname{Ker}\sigma}$. (XAs1) follows from the fact that τ is an algebra homomorphism and (CAs1), more precisely from the identity $\tau|_{A_0} = \mathrm{id}_{A_0}$.

Now, let $x_1, x_2 \in \operatorname{Ker}\sigma$. It is clear that $\tau(x_1) - x_1 \in \operatorname{Ker}\tau$, since τ is linear and $\tau|_{A_0} = \mathrm{id}_{A_0}$. Therefore, due to (CAs2) and the bilinearity of the product in A_1, we have that

$$0 = (\tau(x_1) - x_1)x_2 = \tau(x_1)x_2 - x_1x_2,$$
$$0 = x_2(\tau(x_1) - x_1) = x_2\tau(x_1) - x_2x_1.$$

Hence, $(\operatorname{Ker}\sigma, A_0, \tau|_{\operatorname{Ker}\sigma})$ verifies (XAs2) and it is a crossed module of algebras.

Moreover, given a morphism of cat^1-algebras $\gamma\colon (A_1,A_0,\sigma,\tau) \to (A_1',A_0',\sigma',\tau')$, its corresponding morphism of crossed modules of algebras is given by

$$\begin{tikzcd}
\operatorname{Ker}\sigma \arrow[r, "\tau|_{\operatorname{Ker}\sigma}"] \arrow[d, "\gamma|_{\operatorname{Ker}\sigma}"'] & A_0 \arrow[d, "\gamma|_{A_0}"] \\
\operatorname{Ker}\sigma' \arrow[r, "\tau'|_{\operatorname{Ker}\sigma'}"'] & A_0'.
\end{tikzcd}$$

Note that $\gamma(\operatorname{Ker}\sigma) \subset \operatorname{Ker}\sigma'$, directly from the identity $\sigma'\gamma = \gamma|_{A_0}\sigma$. The commutativity of the previous diagram follows from the identity $\tau'\gamma = \gamma|_{A_0}\tau$. Besides, (1.2.8) follows from the definition of the action of A_0 on $\operatorname{Ker}\sigma$ and the fact that γ is an algebra homomorphism. The previous assignments clearly define a functor from $\mathbf{C^1As}$ to $\mathbf{XAs}$, which will be denoted by $\mathtt{Xm}_{\mathtt{As}}$.

$\mathtt{cat}_{\mathtt{As}}$ and $\mathtt{Xm}_{\mathtt{As}}$ establish an equivalence between the categories $\mathbf{XAs}$ and $\mathbf{C^1As}$, with the natural isomorphisms $\alpha\colon \mathtt{1}_{\mathbf{XAs}} \to \mathtt{Xm}_{\mathtt{As}} \circ \mathtt{cat}_{\mathtt{As}}$ and $\beta\colon \mathtt{1}_{\mathbf{C^1As}} \to \mathtt{cat}_{\mathtt{As}} \circ \mathtt{Xm}_{\mathtt{As}}$

given, for a fixed (B, A, ρ) in **XAs** and a fixed (A_1, A_0, σ, τ) in $\mathbf{C^1As}$, by:

$$\begin{tikzcd} B \arrow[r, "\rho"] \arrow[d, "\alpha_B"'] & A \arrow[d, "\mathrm{id}_A"] \\ B \rtimes \{0\} \arrow[r, "\rho"'] & A \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A_1 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] \arrow[d, "\beta_{A_1}"'] & A_0 \arrow[d, "\mathrm{id}_{A_0}"] \\ \operatorname{Ker}\sigma \rtimes A_0 \arrow[r, shift left, "\tilde{\sigma}"] \arrow[r, shift right, "\tilde{\tau}"'] & A_0 \end{tikzcd}$$

respectively, where $\alpha_B(b) = (b, 0)$ for every $b \in B$, $\beta_{A_1}(a) = (a - \sigma(a), \sigma(a))$ for every $a \in A_1$. It is clear that $(\alpha_B, \mathrm{id}_A)$ is an isomorphism of crossed modules of algebras and the naturality of α is obvious.

Regarding β_{A_1}, observe that $\mathtt{cat}_{\mathtt{As}}(\mathtt{Xm}_{\mathtt{As}}(A_1, A_0, \sigma, \tau)) = (\operatorname{Ker}\sigma \rtimes A_0, A_0, \tilde{\sigma}, \tilde{\tau})$, with $\tilde{\sigma}(x, y) = y$ and $\tilde{\tau}(x, y) = \tau(x) + y$ for all $x \in \operatorname{Ker}\sigma$, $y \in A_0$. It is easy to check that β_{A_1} is an algebra homomorphism just by using the definition of the product in $\operatorname{Ker}\sigma \rtimes A_0$ and the action of A_0 on $\operatorname{Ker}\sigma$. Besides, given $y \in A_0$, $\beta_{A_1}(y) = (0, y)$, since $\sigma|_{A_0} = \mathrm{id}_{A_0}$. Calculations in order to check the identities $\tilde{\sigma}\beta_{A_1} = \sigma$ and $\tilde{\tau}\beta_{A_1} = \tau$ are obvious. The inverse of β_{A_1} is given by $\beta_{A_1}^{-1}(x, y) = x + y$, for all $(x, y) \in \operatorname{Ker}\sigma \rtimes A_0$. Naturality of β can be readily checked by using the identity $\sigma'\gamma = \gamma|_{A_0}\sigma$ for any morphism γ between two given cat^1-algebras (A_1, A_0, σ, τ) and $(A'_1, A'_0, \sigma', \tau')$. $\square$

1.2.4 The case of Leibniz algebras

A non-commutative, or more precisely non-antisymmetric, generalization of Lie algebras was first considered by Bloh [8], but it was not until almost thirty years later that Loday made Leibniz algebras popular [64]. Their crossed modules were defined for the first time in [66] in order to study the cohomology of Leibniz algebras. Later, they were extended to n-Leibniz algebras in [24].

Definition 1.2.35. *A Leibniz algebra $\mathfrak{p}$ over K is a K-module together with a bilinear operation $[\ ,\ ]\colon \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$, called the Leibniz bracket, which satisfies the Leibniz identity:*

$$[[p_1, p_2], p_3] = [p_1, [p_2, p_3]] + [[p_1, p_3], p_2],$$

for all $p_1, p_2, p_3 \in \mathfrak{p}$. A morphism of Leibniz algebras is a K-linear map that preserves the bracket.

Remark 1.2.36. *Directly from the Leibniz identity, one easily derives that*

$$[p_1, [p_2, p_3]] + [p_1, [p_3, p_2]] = 0,$$

for all $p_1, p_2, p_3 \in \mathfrak{p}$. Furthermore, as remarked in [65], this is in fact a right Leibniz algebra. For the opposite structure, that is, $[p_1, p_2]' = [p_2, p_1]$, the left Leibniz identity is

$$[p_1, [p_2, p_3]']' = [[p_1, p_2]', p_3]' + [p_2, [p_1, p_3]']',$$

for all $p_1, p_2, p_3 \in \mathfrak{p}$.

If the bracket of a Leibniz algebra $\mathfrak{p}$ happens to be anticommutative, the $\mathfrak{p}$ is a Lie algebra. Furthermore, every Lie algebra is a Leibniz algebra. We will denote by **Lb** the category of Leibniz algebras and morphisms of Leibniz algebras.

Definition 1.2.37. *Let $\mathfrak{p}$ and $\mathfrak{m}$ be two Leibniz algebras. An action of $\mathfrak{p}$ on $\mathfrak{m}$ consists of a pair of bilinear maps, $\mathfrak{p}\times\mathfrak{m} \to \mathfrak{m}$, $(p,m) \mapsto [p,m]$ and $\mathfrak{m}\times\mathfrak{p} \to \mathfrak{m}$, $(m,p) \mapsto [m,p]$, such that*

(1) $[p,[m_1,m_2]] = [[p,m_1],m_2] - [[p,m_2],m_1]$,

(2) $[m_1,[p,m_2]] = [[m_1,p],m_2] - [[m_1,m_2],p]$,

(3) $[m_1,[m_2,p]] = [[m_1,m_2],p] - [[m_1,p],m_2]$,

(4) $[m,[p_1,p_2]] = [[m,p_1],p_2] - [[m,p_2],p_1]$,

(5) $[p_1,[m,p_2]] = [[p_1,m],p_2] - [[p_1,p_2],m]$,

(6) $[p_1,[p_2,m]] = [[p_1,p_2],m] - [[p_1,m],p_2]$,

for all $m, m_1, m_2 \in \mathfrak{m}$ and $p, p_1, p_2 \in \mathfrak{p}$.

Note that, as an immediate consequence of (2) and (3),

$$[m_1,[p,m_2]] + [m_1,[m_2,p]] = 0$$

for all $m_1, m_2 \in \mathfrak{m}$ and $p \in \mathfrak{p}$. On the other hand, from (5) and (6),

$$[p_1,[m,p_2]] + [p_1,[p_2,m]] = 0$$

for all $m \in \mathfrak{m}$ and $p_1, p_2 \in \mathfrak{p}$.

We will say that $\mathfrak{p}$ *acts trivially* on $\mathfrak{m}$ if $[p,m] = 0 = [m,p]$ for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$. Observe that we denote the action by the same symbol used for the multiplication in $\mathfrak{m}$ and $\mathfrak{p}$, by analogy to the notation used for Ω-groups. Note that the six identities in the definition of a Leibniz action can be obtained from the Leibniz identity by taking two elements in $\mathfrak{p}$ and one in $\mathfrak{m}$ (three identities), and two elements in $\mathfrak{m}$ and one in $\mathfrak{p}$ (the other three identities).

For example, if $\mathfrak{p}$ is a Leibniz subalgebra of some Leibniz algebra $\mathfrak{q}$, and if $\mathfrak{m}$ is an ideal in $\mathfrak{q}$, then the Leibniz bracket in $\mathfrak{q}$ yields an action of $\mathfrak{p}$ on $\mathfrak{m}$.

Given a Leibniz action of $\mathfrak{p}$ on $\mathfrak{m}$ we can consider the *semidirect product* Leibniz algebra $\mathfrak{m} \rtimes \mathfrak{p}$, which consists of the K-module $\mathfrak{m}\oplus\mathfrak{p}$ together with the Leibniz bracket given by

$$[(m_1,p_1),(m_2,p_2)] = ([m_1,m_2] + [p_1,m_2] + [m_1,p_2],[p_1,p_2])$$

for all (m_1,p_1), $(m_2,p_2) \in \mathfrak{m} \oplus \mathfrak{p}$.

Definition 1.2.38. *A crossed module of Leibniz algebras (or Leibniz crossed module)* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *is a morphism of Leibniz algebras* $\eta\colon \mathfrak{m} \to \mathfrak{p}$ *together with an action of* $\mathfrak{p}$ *on* $\mathfrak{m}$ *such that*

$$\eta([p, m]) = [p, \eta(m)] \quad \textit{and} \quad \eta([m, p]) = [\eta(m), p], \tag{XLb1}$$
$$[\eta(m_1), m_2] = [m_1, m_2] = [m_1, \eta(m_2)]. \tag{XLb2}$$

for all $m, m_1, m_2 \in \mathfrak{m}$, $p \in \mathfrak{p}$.

For the sake of coherence, (XLb1) will be called *equivariance* and (XLb2) *Peiffer identity*. If $(\mathfrak{m}, \mathfrak{p}, \eta)$ satisfies (XLb1) but not necessarily (XLb2), it is called *precrossed module*. Moreover, we have the following result:

Lemma 1.2.39. *Given a Leibniz crossed module* $(\mathfrak{m}, \mathfrak{p}, \eta)$,

(i) $\operatorname{Ker}\eta$ *is an ideal of* $\mathfrak{m}$ *and* $\operatorname{Im}\eta$ *is an ideal of* $\mathfrak{p}$.

(ii) $\operatorname{Ker}\eta \subset \operatorname{Ann}(\mathfrak{m})$, *where* $\operatorname{Ann}(\mathfrak{m})$ *is the annihilator of* $\mathfrak{m}$.

Example 1.2.40. *Let* $\mathfrak{p}$ *be a Leibniz algebra.*
(i) The Leibniz bracket in $\mathfrak{p}$ *yields an action of* $\mathfrak{p}$ *on any ideal* $\mathfrak{q}$ *of* $\mathfrak{p}$. *The inclusion* $i\colon \mathfrak{q} \hookrightarrow \mathfrak{p}$ *together with that action is a Leibniz crossed module.* $\{0\}$ *and* $\mathfrak{p}$ *are ideals of* $\mathfrak{p}$, *so any Leibniz algebra* $\mathfrak{p}$ *can be regarded as a crossed module in two obvious ways:* $(\{0\}, \mathfrak{p}, 0)$, *where* 0 *is the trivial map, or* $(\mathfrak{p}, \mathfrak{p}, \mathrm{id}_{\mathfrak{p}})$.
(ii) $(\mathfrak{p}, \{0\}, 0)$ *with the trivial action is a precrossed module. It satisfies the Peiffer identity if and only if the bracket in* $\mathfrak{p}$ *is trivial.*
(iii) If $0 \to \mathfrak{q} \to \mathfrak{m} \xrightarrow{\eta} \mathfrak{p} \to 0$ *is a short exact sequence with* $\mathfrak{q} \subset \operatorname{Ann}(\mathfrak{m})$, *then* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *is a Leibniz crossed module, with the action of* $\mathfrak{p}$ *on* $\mathfrak{m}$ *given by* $[p, m] = [m_p, m]$ *for all* $m \in \mathfrak{m}$, $p \in \mathfrak{p}$, *where* m_p *is an element in* $\mathfrak{m}$ *such that* $\eta(m_p) = p$.

The analogue to $(H, \operatorname{Aut}(H), \alpha)$ in **Gr** and $(\mathfrak{m}, \operatorname{Der}(\mathfrak{m}), \alpha)$ in **Lie** (see Examples 1.2.6 and 1.2.17) does not always exist in **Lb**. We will give more details about this construction in Section 2.1.

Definition 1.2.41. *A morphism of Leibniz crossed modules* (φ, ψ) *from* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *to* $(\mathfrak{m}', \mathfrak{p}', \eta')$ *is a pair of Leibniz homomorphisms,* $\varphi\colon \mathfrak{m} \to \mathfrak{m}'$ *and* $\psi\colon \mathfrak{p} \to \mathfrak{p}'$, *such that*

$$\psi\eta = \eta'\varphi, \tag{1.2.9}$$
$$\varphi([p, m]) = [\psi(p), \varphi(m)] \quad \textit{and} \quad \varphi([m, p]) = [\varphi(m), \psi(p)] \tag{1.2.10}$$

for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$.

Example 1.2.42. *Let* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *be a Leibniz crossed module and* $\mathfrak{q}$ *a Leibniz algebra.*
(i) Given a Leibniz homomorphism $\psi\colon \mathfrak{p} \to \mathfrak{q}$ *such that* $\psi\eta = 0$, $(0, \psi)\colon (\mathfrak{m}, \mathfrak{p}, \eta) \to (\{0\}, \mathfrak{q}, 0)$ *is a morphism of Leibniz crossed modules.*

(ii) Given a Leibniz homomorphism $\psi\colon \mathfrak{q} \to \mathfrak{p}$, $(0,\psi)\colon (\{0\},\mathfrak{q},0) \to (\mathfrak{m},\mathfrak{p},\eta)$ *is a morphism of Leibniz crossed modules.*
(iii) Given a Leibniz homomorphism $\psi\colon \mathfrak{p} \to \mathfrak{q}$, $(\psi\eta,\psi)\colon (\mathfrak{m},\mathfrak{p},\eta) \to (\mathfrak{q},\mathfrak{q},\mathrm{id}_{\mathfrak{q}})$ *is a morphism of Leibniz crossed modules. In particular,* $(\eta,\mathrm{id}_{\mathfrak{p}})\colon (\mathfrak{m},\mathfrak{p},\eta) \to (\mathfrak{p},\mathfrak{p},\mathrm{id}_{\mathfrak{p}})$ *is a morphism of Leibniz crossed modules.*
(iv) Given a Leibniz homomorphism $\varphi\colon \mathfrak{q} \to \mathfrak{m}$, $(\varphi,\eta\varphi)\colon (\mathfrak{q},\mathfrak{q},\mathrm{id}_{\mathfrak{q}}) \to (\mathfrak{m},\mathfrak{p},\eta)$ *is a morphism of Leibniz crossed modules. In particular,* $(\mathrm{id}_{\mathfrak{m}},\eta)\colon (\mathfrak{m},\mathfrak{m},\mathrm{id}_{\mathfrak{m}}) \to (\mathfrak{m},\mathfrak{p},\eta)$ *is a morphism of Leibniz crossed modules.*

Composition of morphisms of Leibniz crossed modules is defined component-wise and the identity morphism is given by $(\mathrm{id}_{\mathfrak{m}},\mathrm{id}_{\mathfrak{p}})$ for any Leibniz crossed module $(\mathfrak{m},\mathfrak{p},\eta)$. We will denote by **XLb** the category of Leibniz crossed modules and morphisms of Leibniz crossed modules.

Just like in the case of crossed modules of groups, it is possible to define the full embeddings $\mathtt{J}_0\colon \mathbf{Lb} \to \mathbf{XLb}$ and $\mathtt{J}_1\colon \mathbf{Lb} \to \mathbf{XLb}$, with $\mathtt{J}_0(\mathfrak{p}) = (\{0\},\mathfrak{p},0)$ and $\mathtt{J}_1(\mathfrak{p}) = (\mathfrak{p},\mathfrak{p},\mathrm{id}_{\mathfrak{p}})$ for any Leibniz algebra $\mathfrak{p}$. Given a morphism of Leibniz algebras $\alpha\colon \mathfrak{p} \to \mathfrak{p}'$, $\mathtt{J}_0(\alpha) = (0,\alpha)$ and $\mathtt{J}_1(\alpha) = (\alpha,\alpha)$.

Besides, let us define the functors Ψ_0, Ψ_1 and Ψ_2, from **XLb** to **Lb**, given by $\Psi_0(\mathfrak{m},\mathfrak{p},\eta) = \mathfrak{p}/\eta(\mathfrak{m})$, $\Psi_1(\mathfrak{m},\mathfrak{p},\eta) = \mathfrak{p}$ and $\Psi_2(\mathfrak{m},\mathfrak{p},\eta) = \mathfrak{m}$ for any Leibniz crossed module $(\mathfrak{m},\mathfrak{p},\eta)$. Given a morphism of Leibniz crossed modules $(\varphi,\psi)\colon (\mathfrak{m},\mathfrak{p},\eta) \to (\mathfrak{m}',\mathfrak{p}',\eta')$, $\Psi_0(\varphi,\psi) = \overline{\psi}$, $\Psi_1(\varphi,\psi) = \psi$ and $\Psi_2(\varphi,\psi) = \varphi$, where $\overline{\psi}$ is the morphism from $\mathfrak{p}/\eta(\mathfrak{m})$ to $\mathfrak{p}'/\eta'(\mathfrak{m}')$ induced by ψ.

Proposition 1.2.43. Ψ_0 *is left adjoint to* $\mathtt{J}_0$, $\mathtt{J}_0$ *is left adjoint to* Ψ_1, Ψ_1 *is left adjoint to* $\mathtt{J}_1$ *and* $\mathtt{J}_1$ *is left adjoint to* Ψ_2.

Proof. The corresponding natural bijections can be readily described by using Example 1.2.42 (i)–(iv). □

Definition 1.2.44. *A* cat^1*-Leibniz algebra* $(\mathfrak{p}_1,\mathfrak{p}_0,s,t)$ *consists of a Leibniz algebra* $\mathfrak{p}_1$ *together with a Leibniz subalgebra* $\mathfrak{p}_0$ *and the structural morphisms* $s,t\colon \mathfrak{p}_1 \to \mathfrak{p}_0$ *such that*

$$s|_{\mathfrak{p}_0} = t|_{\mathfrak{p}_0} = \mathrm{id}_{\mathfrak{p}_0}, \tag{CLb1}$$
$$[\operatorname{Ker} s, \operatorname{Ker} t] = 0 = [\operatorname{Ker} t, \operatorname{Ker} s] \tag{CLb2}$$

Definition 1.2.45. *A homomorphism of* cat^1*-Leibniz algebras* γ *from* $(\mathfrak{p}_1,\mathfrak{p}_0,s,t)$ *to* $(\mathfrak{p}'_1,\mathfrak{p}'_0,s',t')$ *is a Leibniz homomorphism* $\gamma\colon \mathfrak{p}_1 \to \mathfrak{p}'_1$ *such that* $\gamma(\mathfrak{p}_0) \subseteq \mathfrak{p}'_0$ *and* $s'\gamma = \gamma|_{\mathfrak{p}_0}s$, $t'\gamma = \gamma|_{\mathfrak{p}_0}t$.

Composition of morphisms of cat^1-Lie algebras is obvious. We will denote by $\mathbf{C}^1\mathbf{Lb}$ the category of cat^1-Leibniz algebras and morphisms of cat^1-Leibniz algebras.

Proposition 1.2.46. *The categories* **XLb** *and* $\mathbf{C}^1\mathbf{Lb}$ *are equivalent.*

Proof. Given a crossed module of Leibniz algebras $(\mathfrak{m},\mathfrak{p},\eta)$, the corresponding cat^1-Leibniz algebra is $(\mathfrak{m}\rtimes\mathfrak{p},\mathfrak{p},s,t)$, where $s(m,p)=p$ and $t(m,p)=\eta(m)+p$ for all $(m,p)\in\mathfrak{m}\rtimes\mathfrak{p}$. It is evident that s is a Leibniz homomorphism, while t is a Leibniz homomorphism due to (XLb1), the fact that η is a Leibniz homomorphism and the bilinearity of the bracket in $\mathfrak{p}$. Note that $\mathfrak{p}$ can be regarded as a Lie subalgebra of $\mathfrak{m}\rtimes\mathfrak{p}$ via the morphism $p\mapsto(0,p)$. It is clear that $s|_{\mathfrak{p}}=t|_{\mathfrak{p}}=\mathrm{id}_{\mathfrak{p}}$. Directly from the definition of s and t, we get that $\operatorname{Ker}s=\{(m,0)\mid m\in\mathfrak{m}\}$ and $\operatorname{Ker}t=\{(m,-\eta(m))\mid m\in\mathfrak{m}\}$. Let $(m_1,0)\in\operatorname{Ker}s$ and $(m_2,-\eta(m_2))\in\operatorname{Ker}t$:

$$[(m_1,0),(m_2,-\eta(m_2))]=([m_1,m_2]-[m_1,\eta(m_2)],0)=(0,0),$$

due to (XLb2). Analogously $[(m_2,-\eta(m_2)),(m_1,0)]=(0,0)$, so $[\operatorname{Ker}s,\operatorname{Ker}t]=0=[\operatorname{Ker}t,\operatorname{Ker}s]$ and $(\mathfrak{m}\rtimes\mathfrak{p},\mathfrak{p},s,t)$ is a cat^1-Leibniz algebra.

Additionally, given a morphism of Leibniz crossed modules (φ,ψ) from $(\mathfrak{m},\mathfrak{p},\eta)$ to $(\mathfrak{m}',\mathfrak{p}',\eta')$, the corresponding morphism of cat^1-Leibniz algebras is defined by $f_{\varphi,\psi}(m,p)=(\varphi(m),\psi(p))$, for all $(m,p)\in\mathfrak{m}\rtimes\mathfrak{p}$. One can easily check that $f_{\varphi,\psi}$ is a Leibniz homomorphism by making use of (1.2.10) and the fact that φ and ψ are Leibniz homomorphisms. It is clear that $f_{\varphi,\psi}(\mathfrak{p})\subseteq\mathfrak{p}'$. The identity $s'\gamma=\gamma|_{\mathfrak{p}_0}s$ follows from the definition of the morphisms involved and $t'\gamma=\gamma|_{\mathfrak{p}_0}t$ is an immediate consequence of (1.2.9). The previous assignments clearly define a functor from $\mathbf{XLb}$ to $\mathbf{C^1Lb}$, which will be denoted by $\mathtt{cat_{Lb}}$.

Conversely, given a cat^1-Leibniz algebra $(\mathfrak{p}_1,\mathfrak{p}_0,s,t)$, the corresponding Leibniz crossed module is $t|_{\operatorname{Ker}s}\colon \operatorname{Ker}s\to\mathfrak{p}_0$, with the action of $\mathfrak{p}_0$ on $\operatorname{Ker}s$ induced by the bracket in $\mathfrak{p}_1$. (XLb1) follows from the fact that t is a Leibniz homomorphism and (CLb1), specifically from the identity $t|_{\mathfrak{p}_0}=\mathrm{id}\,\mathfrak{p}$.

Now, let $x_1,x_2\in\operatorname{Ker}s$. It is clear that $t(x_1)-x_1\in\operatorname{Ker}t$, since t is linear and $t|_{\mathfrak{p}_0}=\mathrm{id}_{\mathfrak{p}_0}$. Therefore, due to (CLb2) and the bilinearity of the bracket in $\mathbb{p}_1$, we have that

$$0=[t(x_1)-x_1,x_2]=[t(x_1),x_2]-[x_1,x_2]\,.$$

Analogously, $[x_1,t(x_2)]=[x_1,x_2]$. Thus, $(\operatorname{Ker}s,\mathfrak{p}_0,t|_{\operatorname{Ker}s})$ is a crossed module of Leibniz algebras.

Given a morphism of cat^1-Leibniz algebras $\gamma\colon(\mathfrak{p}_1,\mathfrak{p}_0,s,t)\to(\mathfrak{p}'_1,\mathfrak{p}'_0,s',t')$, its corresponding morphism of Leibniz crossed modules is given by

$$\begin{tikzcd}
\operatorname{Ker}s \arrow[r, "t|_{\operatorname{Ker}s}"] \arrow[d, "\gamma|_{\operatorname{Ker}s}"'] & \mathfrak{p}_0 \arrow[d, "\gamma|_{\mathfrak{p}_0}"] \\
\operatorname{Ker}s' \arrow[r, "t'|_{\operatorname{Ker}s'}"'] & \mathfrak{p}'_0.
\end{tikzcd}$$

Note that $\gamma(\operatorname{Ker}s)\subset\operatorname{Ker}s'$, directly from the identity $s'\gamma=\gamma|_{\mathfrak{p}_0}s$. The commutativity of the previous diagram follows from the identity $t'\gamma=\gamma|_{\mathfrak{p}_0}t$. Besides, (1.2.10) follows from the definition of the action of $\mathfrak{p}_0$ on $\operatorname{Ker}s$ and the fact that γ is a Leibniz homomorphism. The previous assignments clearly define a functor from $\mathbf{C^1Lb}$ to $\mathbf{XLb}$, which will be denoted by $\mathtt{Xm_{Lb}}$.

$\mathtt{cat_{Lb}}$ and $\mathtt{Xm_{Lb}}$ establish an equivalence between the categories $\mathbf{XLb}$ and $\mathbf{C^1Lb}$, with the natural isomorphisms $\alpha\colon 1_{\mathbf{XLb}} \to \mathtt{Xm_{Lb}} \circ \mathtt{cat_{Lb}}$ and $\beta\colon 1_{\mathbf{C^1Lb}} \to \mathtt{cat_{Lb}} \circ \mathtt{Xm_{Lb}}$ given, for a fixed $(\mathfrak{m}, \mathfrak{p}, \eta)$ in $\mathbf{XLb}$ and a fixed $(\mathfrak{p}_1, \mathfrak{p}_0, s, t)$ in $\mathbf{C^1Lb}$, by:

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\eta"] \arrow[d, "\alpha_{\mathfrak{m}}"'] & \mathfrak{p} \arrow[d, "\mathrm{id}_{\mathfrak{p}}"] \\ \mathfrak{m} \rtimes \{0\} \arrow[r, "\eta"'] & \mathfrak{p} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathfrak{p}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_{\mathfrak{p}_1}"'] & \mathfrak{p}_0 \arrow[d, "\mathrm{id}_{\mathfrak{p}_0}"] \\ \operatorname{Ker} s \rtimes \mathfrak{p}_0 \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right, "\tilde{t}"'] & \mathfrak{p}_0 \end{tikzcd}$$

respectively, where $\alpha_{\mathfrak{m}}(m) = (m, 0)$ for every $m \in \mathfrak{m}$, $\beta_{\mathfrak{p}_1}(p) = (p - s(p), s(p))$ for every $p \in \mathfrak{p}_1$. It is clear that $(\alpha_{\mathfrak{m}}, \mathrm{id}_{\mathfrak{p}})$ is an isomorphism of Leibniz crossed modules and the naturality of α is obvious.

Concerning $\beta_{\mathfrak{p}_1}$, observe that $\mathtt{cat_{Lb}}(\mathtt{Xm_{Lb}}(\mathfrak{p}_1, \mathfrak{p}_0, s, t)) = (\operatorname{Ker} s \rtimes \mathfrak{p}_0, \mathfrak{p}_0, \tilde{s}, \tilde{t})$, with $\tilde{s}(x, y) = y$ and $\tilde{t}(x, y) = t(x) + y$ for all $x \in \operatorname{Ker} s$, $y \in \mathfrak{p}_0$. It is easy to check that $\beta_{\mathfrak{p}_1}$ is a Leibniz homomorphism just by using the definition of the Leibniz bracket in $\operatorname{Ker} s \rtimes \mathfrak{p}_0$ and the action of $\mathfrak{p}_0$ on $\operatorname{Ker} s$. Besides, given $y \in \mathfrak{p}_0$, $\beta_{\mathfrak{p}_1}(y) = (0, y)$, since $s|_{\mathfrak{p}_0} = \mathrm{id}_{\mathfrak{p}_0}$. Calculations in order to check the identities $\tilde{s}\beta_{\mathfrak{p}_1} = s$ and $\tilde{t}\beta_{\mathfrak{p}_1} = t$ are obvious. The inverse of $\beta_{\mathfrak{p}_1}$ is given by $\beta_{\mathfrak{p}_1}^{-1}(x, y) = x + y$, for all $(x, y) \in \operatorname{Ker} s \rtimes \mathfrak{p}_0$. Naturality of β can be readily checked by using the identity $s'\gamma = \gamma|_{\mathfrak{p}_0} s$ for any morphism γ between two given cat^1-Leibniz algebras $(\mathfrak{p}_1, \mathfrak{p}_0, s, t)$ and $(\mathfrak{p}'_1, \mathfrak{p}'_0, s', t')$. $\square$

1.2.5 The case of dialgebras

Associative dialgebras (or simply dialgebras), also known as diassociative algebras, were introduced by Loday [65] as an algebraic structure with a role with respect to Leibniz algebras analogous to the one that associative algebras play with respect to Lie algebras. Let us recall some basic definitions and elemental properties from [65].

Definition 1.2.47 ([65])**.** *An associative dialgebra (or simply dialgebra), is a K-module D equipped with two K-linear maps*

$$\dashv, \vdash\colon D \otimes D \to D,$$

called the left product and the right product respectively, satisfying the following axioms

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{Di1}$$
$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{Di2}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{Di3}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{Di4}$$
$$(x \vdash y) \vdash z = x \vdash (y \vdash z), \tag{Di5}$$

for all $x, y, z \in D$. A morphism of dialgebras is a K-linear map that preserves both the left and the right products.

Observe that here we break the "rule" of denoting elements with lower case letters after the name of the object (D in this particular case). This decision was made in order to prevent confusions with derivations. Likewise, whenever we consider a dialgebra L, its elements will not be denoted by l, since that symbol will be used for a specific type of maps in Subsection 2.1.1.

We will denote by **Dias** the category of dialgebras and morphisms of dialgebras.

Remark 1.2.48. *In some identities we may use $*$ to denote both $\vdash$ and $\dashv$, meaning that the corresponding equality is satisfied for $* = \vdash$ and $* = \dashv$.*

A *bar-unit* in a dialgebra D is an element $e \in D$ such that

$$x \dashv e = x = e \vdash x$$

for all $x \in D$. A bar-unit is not necessarily unique. The set of all bar-units is called *halo*. A *unital dialgebra* is a dialgebra with a specific bar-unit e. A morphism of dialgebras is said to be unital if the image of any bar-unit is a bar-unit.

Observe that if a dialgebra has a unit ϵ, i.e. an element such that $\epsilon \dashv x = x$ for all $x \in D$, from (Di1) we have $(\epsilon \dashv y) \dashv z = \epsilon \dashv (y \vdash z)$, that is $y \dashv z = y \vdash z$ for $y, z \in D$. Hence $\dashv = \vdash$ and D is merely an associative algebra with unit.

A dialgebra D is called *abelian* if both the left and the right products are trivial, that is $x \dashv y = x \vdash y = 0$ for all $x, y \in D$. Note that any K-module can be regarded as an abelian dialgebra. A submodule I of a dialgebra D is called an *ideal* of D if $x \dashv y, x \vdash y, y \dashv x, y \vdash x \in I$ for any $x \in I$ and $y \in D$. The *annihilator* of a dialgebra D is given by:

$$\mathrm{Ann}(D) = \{x \in D \mid x \dashv y = y \dashv x = x \vdash y = y \vdash x = 0, \text{ for all } y \in D\}.$$

It is immediate to check that $\mathrm{Ann}(D)$ is indeed an ideal of D.

Example 1.2.49 ([65]).
(i) Any algebra A with $a \dashv b = a \cdot b = a \vdash b$ for all $a, b \in A$ is a dialgebra.
(ii) Let (A, d) be a non-graded differential associative algebra. By hypothesis, $d(a \cdot b) = d(a) \cdot b + a \cdot d(b)$ and $d^2 = 0$. A is a dialgebra with the left and the right products given, for any $a, b \in A$, by

$$a \dashv b = a \cdot d(b) \qquad \textit{and} \qquad a \vdash b = d(a) \cdot b$$

(iii) Let A be an associative algebra and M be an A-bimodule. Let $f : M \to A$ be an A-bimodule map. M together with

$$m \dashv n = m \cdot f(n) \qquad \textit{and} \qquad m \vdash n = f(m) \cdot n$$

for all $m, n \in M$, is a dialgebra.
(iv) Let A be an associative algebra and put $D = A \oplus A$. The products

$$a_1 \oplus b_1 \dashv a_2 \oplus b_2 = a_1 \oplus b_1 \cdot a_2 \cdot b_2 \qquad \text{and} \qquad a_1 \oplus b_1 \vdash a_2 \oplus b_2 = a_1 \cdot b_1 \cdot a_2 \oplus b_2,$$

extended by linearity to $A \oplus A$, endow it with a dialgebra structure.

Another interesting dialgebra is studied by Lin and Zhang in [62]. Let $F[x, y]$ be the polynomial algebra over a field F of characteristic zero. Then $F[x, y]$ is a dialgebra with the products

$$f(x, y) \dashv g(x, y) = f(x, y) \cdot g(y, y) \quad \text{and} \quad f(x, y) \vdash g(x, y) = f(x, x) \cdot g(x, y),$$

for all $f(x, y), g(x, y) \in F[x, y]$. It is not mentioned in [62], but this structure can be extended to the polynomial algebra with n variables $F[x_1, \dots, x_n]$ by defining the products as

$$f(x_1, \dots, x_n) \dashv g(x_1, \dots, x_n) = f(x_1, \dots, x_n) \cdot g(x_n, \dots, x_n),$$
$$f(x_1, \dots, x_n) \vdash g(x_1, \dots, x_n) = f(x_1, \dots, x_1) \cdot g(x_1, \dots, x_n).$$

Definition 1.2.50. *Let D and L be dialgebras. An action of D on L consists of four linear maps, two of them denoted by the symbol $\dashv$ and the other two by $\vdash$,*

$$\dashv : D \otimes L \to L, \ \dashv : L \otimes D \to L,$$
$$\vdash : D \otimes L \to L, \ \vdash : L \otimes D \to L$$

such that the following 30 equalities hold:

(1) $(x \dashv a) \dashv b = x \dashv (a \vdash b)$,	*(11)* $(a \dashv b) \dashv x = a \dashv (b \vdash x)$,
(2) $(x \dashv a) \dashv b = x \dashv (a \dashv b)$,	*(12)* $(a \dashv b) \dashv x = a \dashv (b \dashv x)$,
(3) $(x \vdash a) \dashv b = x \vdash (a \dashv b)$,	*(13)* $(a \vdash b) \dashv x = a \vdash (b \dashv x)$,
(4) $(x \dashv a) \vdash b = x \vdash (a \vdash b)$,	*(14)* $(a \dashv b) \vdash x = a \vdash (b \vdash x)$,
(5) $(x \vdash a) \vdash b = x \vdash (a \vdash b)$,	*(15)* $(a \vdash b) \vdash x = a \vdash (b \vdash x)$,
(6) $(a \dashv x) \dashv b = a \dashv (x \vdash b)$,	*(16)* $(a \dashv x) \dashv y = a \dashv (x \vdash y)$,
(7) $(a \dashv x) \dashv b = a \dashv (x \dashv b)$,	*(17)* $(a \dashv x) \dashv y = a \dashv (x \dashv y)$,
(8) $(a \vdash x) \dashv b = a \vdash (x \dashv b)$,	*(18)* $(a \vdash x) \dashv y = a \vdash (x \dashv y)$,
(9) $(a \dashv x) \vdash b = a \vdash (x \vdash b)$,	*(19)* $(a \dashv x) \vdash y = a \vdash (x \vdash y)$,
(10) $(a \vdash x) \vdash b = a \vdash (x \vdash b)$,	*(20)* $(a \vdash x) \vdash y = a \vdash (x \vdash y)$,

(21) $(x \dashv a) \dashv y = x \dashv (a \vdash y)$, *(26)* $(x \dashv y) \dashv a = x \dashv (y \vdash a)$,

(22) $(x \dashv a) \dashv y = x \dashv (a \dashv y)$, *(27)* $(x \dashv y) \dashv a = x \dashv (y \dashv a)$,

(23) $(x \vdash a) \dashv y = x \vdash (a \dashv y)$, *(28)* $(x \vdash y) \dashv a = x \vdash (y \dashv a)$,

(24) $(x \dashv a) \vdash y = x \vdash (a \vdash y)$, *(29)* $(x \dashv y) \vdash a = x \vdash (y \vdash a)$,

(25) $(x \vdash a) \vdash y = x \vdash (a \vdash y)$, *(30)* $(x \vdash y) \vdash a = x \vdash (y \vdash a)$,

for all $x, y \in D$; $a, b \in L$. The action is called trivial if these four maps are trivial.

Note that the previous identities are obtained from the axioms (Di1)–(Di5) by taking one variable in D and two variables in L (15 equalities), and one variable in L and two variables in D (15 equalities). Observe that we denote the action by the same symbol used for the left and the right products in D and L, by analogy to the notation used for Ω-groups.

Let us show some examples of actions. Note that **Dias** is a category of interest and the first example agrees with the definition of a set of derived actions from a D-structure.

Examples 1.2.51. (i) *If $0 \to L \xrightarrow{\iota} E \xrightarrow{\sigma} D \to 0$ is a split short exact sequence of dialgebras, i.e. there exists a homomorphism of dialgebras $\varphi : D \to E$ such that $\sigma\varphi = \mathrm{id}_D$, then there is an action of the dialgebra D on L defined in the standard way by taking the left and the right products in the dialgebra E:*

$$x * a = \varphi(x) * \iota(a) \quad \textit{and} \quad a * x = \iota(a) * \varphi(x)$$

for any $x \in D$, $a \in L$.

(ii) *If D is a subdialgebra of a dialgebra E (maybe $D = E$) and I is an ideal in E, then the left and the right products in E yield an action of D on I.*

(iii) *Any morphism of dialgebras $D \to L$ induces an action of D on L in the standard way by taking images of elements of D and the left and the right products in L.*

(iv) *If $\mu : L \to D$ is an surjective morphism of dialgebras with* $\operatorname{Ker}\mu$ *in the annihilator of L, then there is an action of D on L defined in the standard way by taking pre-images of the elements of D and the left and the right products in L.*

(v) *If L is a bimodule over a dialgebra D (for the definition see [65, Subsection 2.3]), thought as an abelian dialgebra, then the bimodule structure defines an action of D on the (abelian) dialgebra L.*

Note that if a dialgebra D acts on a dialgebra L, then L, as a K-module, has a structure of bimodule over the dialgebra D .

Given an action of a dialgebra D on a dialgebra L we define the semidirect product dialgebra, $L \rtimes D$, with the underlying K-module $L \oplus D$ endowed with the left and the right products given by

$$(a_1, x_1) * (a_2, x_2) = (a_1 * a_2 + x_1 * a_2 + a_1 * x_2, x_1 * x_2).$$

for all $x_1, x_2 \in D$, $a_1, a_2 \in L$.

Definition 1.2.52. *A crossed module of dialgebras (L, D, μ) is a morphism of dialgebras $\mu : L \to D$ together with an action of D on L such that*

$$\mu(x * a) = x * \mu(a) \quad \textit{and} \quad \mu(a * x) = \mu(a) * x, \tag{XDi1}$$

$$\mu(a_1) * a_2 = a_1 * a_2 = a_1 * \mu(a_2). \tag{XDi2}$$

for all $x \in D$, $a_1, a_2 \in L$.

For the sake of coherence, (XDi1) will be called *equivariance* and (XDi2) *Peiffer identity*. If (L, D, μ) satisfies (XDi1) but not necessarily (XDi2), it is called *precrossed module*. Moreover, we have the following result:

Lemma 1.2.53. *Given a crossed module of dialgebras (L, D, μ),*

(i) $\operatorname{Ker}\mu$ *is an ideal of L and* $\operatorname{Im}\mu$ *is an ideal of D.*

(ii) $\operatorname{Ker}\mu \subset \operatorname{Ann}(L)$.

The first two examples immediately below show that the concept of crossed module of dialgebras generalizes both the concepts of ideal and bimodule of dialgebras.

Example 1.2.54. *Let D be a dialgebra.*
(i) The inclusion $L \hookrightarrow D$ of an ideal L of D is a crossed module, with the action of D on L is given by the left and the right products in D, as in Example 1.2.51 (ii). Conversely, if $\mu : L \to D$ is a crossed module of dialgebras which is injective, then by Lemma 1.2.53 (i), L is isomorphic to an ideal of D. $\{0\}$ and D are ideals of D, so any dialgebra D can be regarded as a crossed module in two obvious ways: $(\{0\}, D, 0)$, where 0 is the trivial map, or (D, D, id_D).

(ii) For any bimodule L over a dialgebra D the trivial map $0 : L \to D$ is a crossed module with the action of D on the (abelian) dialgebra L described in Example 1.2.51 (v). Conversely, if $0 : L \to D$ is a crossed module of dialgebras, then L is necessarily an abelian dialgebra and the action of D on L determines on L a bimodule structure over D.

(iii) Any morphism of dialgebras $\mu : L \to D$ with L abelian and $\operatorname{Im}\mu \subset \operatorname{Ann}(D)$ *is a crossed module together with the trivial action of D on L.*

(iv) Any surjective morphism of dialgebras $\mu\colon L \to D$ with $\operatorname{Ker}\mu \subset \operatorname{Ann}(L)$ *and the action of D on L described in Example 1.2.51 (iv) is a crossed module of dialgebras.*

The analogue to $(H, \operatorname{Aut}(H), \alpha)$ in **Gr** and $(\mathfrak{m}, \operatorname{Der}(\mathfrak{m}), \alpha)$ in **Lie** (see Examples 1.2.6 and 1.2.17) does not always exist in **Dias**. We will give more details about this construction in Section 2.1.

Definition 1.2.55. *A* morphism of crossed modules of dialgebras $(\varphi,\psi)\colon (L,D,\mu)\to (L',D',\mu')$ is a pair of dialgebra homomorphisms, $\varphi\colon L\to L'$ and $\psi\colon D\to \mathrm{D}'$, such that

$$\psi\mu = \mu'\varphi, \tag{1.2.11}$$
$$\varphi(x*a) = \psi(x)*\varphi(a) \quad \text{and} \quad \varphi(a*x) = \varphi(a)*\psi(x), \tag{1.2.12}$$

for all $x\in D$, $a\in L$.

Example 1.2.56. *Let (L,D,μ) be a crossed module of algebras and E a dialgebra.*
(i) Given a morphism of dialgebras $\psi\colon D\to E$ such that $\psi\mu=0$, $(0,\psi)\colon (L,D,\mu)\to (\{0\},E,0)$ is a morphism of crossed modules.
(ii) Given a morphism of dialgebras $\psi\colon E\to D$, $(0,\psi)\colon (\{0\},E,0)\to (L,D,\mu)$ is a morphism of crossed modules.
(iii) Given a morphism of dialgebras $\psi\colon D\to E$, $(\psi\mu,\psi)\colon (L,D,\mu)\to (E,E,\mathrm{id}_E)$ is a morphism of crossed modules. In particular, $(\mu,\mathrm{id}_D)\colon (L,D,\mu)\to (D,D,\mathrm{id}_D)$ is a morphism of crossed modules.
(iv) Given a morphism of dialgebras $\varphi\colon E\to L$, $(\varphi,\mu\varphi)\colon (E,E,\mathrm{id}_E)\to (L,D,\mu)$ is a morphism of crossed modules. In particular, $(\mathrm{id}_L,\mu)\colon (L,L,\mathrm{id}_L)\to (L,D,\mu)$ is a morphism of crossed modules.

Composition of morphisms of crossed modules of dialgebras is defined componentwise and the identity morphism is given by $(\mathrm{id}_L,\mathrm{id}_D)$ for any crossed module (L,D,μ). We will denote by **XDias** the category of crossed modules of dialgebras and morphisms of crossed modules.

Just like for groups, it is possible to define the full embeddings $\mathtt{J}'_0\colon$ **Dias** $\to$ **XDias** and $\mathtt{J}'_1\colon$ **Dias** $\to$ **XDias**, with $\mathtt{J}'_0(D)=(\{0\},D,0)$ and $\mathtt{J}'_1(D)=(D,D,\mathrm{id}_D)$ for any dialgebra D. Given a morphism of dialgebras $\alpha\colon D\to D'$, $\mathtt{J}'_0(\alpha)=(0,\alpha)$ and $\mathtt{J}'_1(\alpha)=(\alpha,\alpha)$.

Besides, let us define the functors Ψ'_0, Ψ'_1 and Ψ'_2, from **XDias** to **Dias**, given by $\Psi'_0(L,D,\mu)=D/\mu(L)$, $\Psi'_1(L,D,\mu)=D$ and $\Psi'_2(L,D,\mu)=L$ for any crossed module of dialgebras (L,D,μ). Given a morphism of crossed modules of dialgebras $(\varphi,\psi)\colon (L,D,\mu)\to (L',D',\mu')$, $\Psi'_0(\varphi,\psi)=\overline{\psi}$, $\Psi'_1(\varphi,\psi)=\psi$ and $\Psi'_2(\varphi,\psi)=\varphi$, where $\overline{\psi}$ is the morphism from $D/\mu(L)$ to $D'/\mu'(L')$ induced by ψ.

Proposition 1.2.57. *Ψ'_0 is left adjoint to $\mathtt{J}'_0$, $\mathtt{J}'_0$ is left adjoint to Ψ'_1, Ψ'_1 is left adjoint to $\mathtt{J}'_1$ and $\mathtt{J}'_1$ is left adjoint to Ψ'_2.*

Proof. The corresponding natural bijections can be readily described by using Example 1.2.56 (i)–(iv). $\square$

Proposition 1.2.58. *Let D and L be dialgebras such that D acts on L.*

(i) A morphism of dialgebras $\mu\colon L \to D$ is a crossed module if and only if the maps

$$(\mu, \mathrm{id}_D)\colon L \rtimes D \to D \rtimes D \tag{1.2.13}$$

and

$$(\mathrm{id}_L, \mu)\colon L \rtimes L \to L \rtimes D \tag{1.2.14}$$

are morphisms of dialgebras.

(ii) If $\mu\colon L \to D$ is a crossed module of dialgebras, then

$$L \rtimes D \to L \rtimes D, \quad (a, x) \mapsto (-a, \mu(a) + x) \tag{1.2.15}$$

is a morphism of dialgebras.

Proof. (1.2.13) (resp. (1.2.14)) is a morphism if and only if the condition (XDi1) (resp. (XDi2)) holds. On the other hand, (1.2.15) is a morphism due to the conditions (XDi1) and (XDi2). □

Definition 1.2.59. *A cat^1-dialgebra (D_1, D_0, σ, τ) consists of a dialgebra D_1 together with a subdialgebra D_0 and two morphisms of dialgebras $\sigma, \tau : D_1 \to D_0$ such that*

$$\sigma|_{D_0} = \tau|_{D_0} = \mathrm{id}_{D_0}, \tag{CDi1}$$

$$\operatorname{Ker}\sigma * \operatorname{Ker}\tau = 0 = \operatorname{Ker}\tau * \operatorname{Ker}\sigma. \tag{CDi2}$$

Definition 1.2.60. *A morphism of cat^1-dialgebras $\gamma\colon (D_1, D_0, \sigma, \tau) \to (D'_1, D'_0, \sigma', \tau')$ is a morphism of dialgebras $\gamma\colon D_1 \to D'_1$ such that $\gamma(D_0) \subseteq D'_0$ and $\sigma'\gamma = \gamma|_{D_0}\sigma$, $\tau'\gamma = \gamma|_{D_0}\tau$.*

Composition of morphisms of cat^1-dialgebras is obvious. We will denote by $\mathbf{C^1Dias}$ the category of cat^1-dialgebras and morphisms of cat^1-dialgebras.

Proposition 1.2.61. *The categories* $\mathbf{XDias}$ *and* $\mathbf{C^1Dias}$ *are equivalent.*

Proof. Given a crossed module of dialgebras (L, D, μ), its corresponding cat^1-dialgebra is $(L \rtimes D, D, \sigma, \tau)$, where $\sigma(a, x) = x$, $\tau(a, x) = \mu(a) + x$, for all $(a, x) \in L \rtimes D$. It is clear that σ is a dialgebra homomorphism. Concerning τ, it is a dialgebra homomorphism due to (XDi1), the fact that μ is a morphism of dialgebras and the bilinearity of the products in D. Note that D can be regarded as a subdialgebra of $L \rtimes D$ via the dialgebra homomorphism $x \mapsto (0, x)$. It is clear that $\sigma|_D = \tau|_D = \mathrm{id}_D$. Directly from the definition of σ and τ, we get that $\operatorname{Ker}\sigma = \{(a, 0) \mid a \in L\}$ and $\operatorname{Ker}\tau = \{(a, -\mu(a)) \mid a \in L\}$. Let $(a_1, 0) \in \operatorname{Ker}\sigma$ and $(a_2, -\mu(a_2)) \in \operatorname{Ker}\tau$. Then

$$(a_1, 0) * (a_2, -\mu(a_2)) = (a_1 * a_2 - a_1 * \mu(a_2), 0) = (0, 0),$$
$$(a_2, -\mu(a_2)) * (a_1, 0) = (a_2 * a_1 - \mu(a_2) * a_1, 0) = (0, 0),$$

due to (XDi2). Therefore, $\operatorname{Ker}\sigma * \operatorname{Ker}\tau = 0 = \operatorname{Ker}\tau * \operatorname{Ker}\sigma$ and $(L \rtimes D, D, \sigma, \tau)$ is a cat^1-dialgebra.

Besides, given a morphism of crossed modules of dialgebras $(\varphi, \psi)\colon (L, D, \mu) \to (L', D', \mu')$, the corresponding morphism of cat^1-dialgebras is defined by $f_{\varphi,\psi}(a, x) = (\varphi(a), \psi(x))$ for all $(a, x) \in L \rtimes D$. One can easily check that $f_{\varphi,\psi}$ is a dialgebra homomorphism by making use of (1.2.12) and the fact that φ and ψ are morphisms as well. It is clear that $f_{\varphi,\psi}(D) \subseteq D'$. The identity $\sigma'\gamma = \gamma|_{D_0}\sigma$ follows from the definition of the morphisms involved and $\tau'\gamma = \gamma|_{D_0}\tau$ is an immediate consequence of (1.2.11). The previous assignments clearly define a functor from **XDias** to **C^{1}Dias**, which will be denoted by $\mathtt{cat}_{\mathtt{Dias}}$.

Conversely, given a cat^1-dialgebra (D_1, D_0, σ, τ), the corresponding crossed module of dialgebras is $\tau|_{\operatorname{Ker}\sigma}\colon \operatorname{Ker}\sigma \to D_0$, with the action of D_0 on $\operatorname{Ker}\sigma$ induced by the left and the right products in D_1. We will write simply τ instead of $\tau|_{\operatorname{Ker}\sigma}$. (XDi1) follows from the fact that τ is a morphism of dialgebras and (CDi1), more precisely from the identity $\tau|_{D_0} = \mathrm{id}_{D_0}$.

Now, let $x_1, x_2 \in \operatorname{Ker}\sigma$. It is clear that $\tau(x_1) - x_1 \in \operatorname{Ker}\tau$, since τ is linear and $\tau|_{D_0} = \mathrm{id}_{D_0}$. Therefore, due to (CDi2) and the bilinearity of the products in D_1, we have that

$$\begin{aligned} 0 &= (\tau(x_1) - x_1) * x_2 = \tau(x_1) * x_2 - x_1 * x_2, \\ 0 &= x_2 * (\tau(x_1) - x_1) = x_2 * \tau(x_1) - x_2 * x_1. \end{aligned}$$

Hence, $(\operatorname{Ker}\sigma, D_0, \tau|_{\operatorname{Ker}\sigma})$ verifies (XDi2) and it is a crossed module of dialgebras.

Moreover, given a morphism of cat^1-dialgebras $\gamma\colon (D_1, D_0, \sigma, \tau) \to (D_1', D_0', \sigma', \tau')$, its corresponding morphism of crossed modules of dialgebras is given by

$$\begin{array}{ccc} \operatorname{Ker}\sigma & \xrightarrow{\tau|_{\operatorname{Ker}\sigma}} & D_0 \\ {\scriptstyle \gamma|_{\operatorname{Ker}\sigma}}\big\downarrow & & \big\downarrow{\scriptstyle \gamma|_{D_0}} \\ \operatorname{Ker}\sigma' & \xrightarrow[\tau'|_{\operatorname{Ker}\sigma'}]{} & D_0'. \end{array}$$

Note that $\gamma(\operatorname{Ker}\sigma) \subset \operatorname{Ker}\sigma'$, directly from the identity $\sigma'\gamma = \gamma|_{D_0}\sigma$. The commutativity of the previous diagram follows from the identity $\tau'\gamma = \gamma|_{D_0}\tau$. Besides, (1.2.12) follows from the definition of the action of D_0 on $\operatorname{Ker}\sigma$ and the fact that γ is a dialgebra homomorphism. The previous assignments clearly define a functor from **C^{1}Dias** to **XDias**, which will be denoted by $\mathtt{Xm}_{\mathtt{Dias}}$.

$\mathtt{cat}_{\mathtt{Dias}}$ and $\mathtt{Xm}_{\mathtt{Dias}}$ establish an equivalence between the categories **XDias** and **C^{1}Dias**, with the natural isomorphisms $\alpha\colon 1_{\mathbf{XDias}} \to \mathtt{Xm}_{\mathtt{Dias}} \circ \mathtt{cat}_{\mathtt{Dias}}$ and $\beta\colon 1_{\mathbf{C^1Dias}} \to \mathtt{cat}_{\mathtt{Dias}} \circ \mathtt{Xm}_{\mathtt{Dias}}$ given, for a fixed (L, D, μ) in **XDias** and a fixed (D_1, D_0, σ, τ) in

$\mathbf{C^1Dias}$, by:

$$\begin{tikzcd} L \arrow[r, "\mu"] \arrow[d, "\alpha_L"'] & D \arrow[d, "\mathrm{id}_D"] \\ L \rtimes \{0\} \arrow[r, "\mu"'] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D_1 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] \arrow[d, "\beta_{D_1}"'] & D_0 \arrow[d, "\mathrm{id}_{D_0}"] \\ \operatorname{Ker}\sigma \rtimes D_0 \arrow[r, shift left, "\tilde{\sigma}"] \arrow[r, shift right, "\tilde{\tau}"'] & D_0 \end{tikzcd}$$

respectively, where $\alpha_L(a) = (a, 0)$ for every $a \in L$, $\beta_{D_1}(x) = (x - \sigma(x), \sigma(x))$ for every $x \in D_1$. It is clear that $(\alpha_L, \mathrm{id}_D)$ is an isomorphism of crossed modules of dialgebras and the naturality of α is obvious.

Regarding β_{D_1}, observe that $\mathtt{cat}_{\mathtt{Dias}}(\mathtt{Xm}_{\mathtt{Dias}}(D_1, D_0, \sigma, \tau)) = (\operatorname{Ker}\sigma \rtimes D_0, D_0, \tilde{\sigma}, \tilde{\tau})$, with $\tilde{\sigma}(x, y) = y$ and $\tilde{\tau}(x, y) = \tau(x) + y$ for all $x \in \operatorname{Ker}\sigma$, $y \in D_0$. It is easy to check that β_{D_1} is a dialgebra homomorphism just by using the definition of the products in $\operatorname{Ker}\sigma \rtimes D_0$ and the action of D_0 on $\operatorname{Ker}\sigma$. Besides, given $y \in D_0$, $\beta_{D_1}(y) = (0, y)$, since $\sigma|_{D_0} = \mathrm{id}_{D_0}$. Calculations in order to check the identities $\tilde{\sigma}\beta_{D_1} = \sigma$ and $\tilde{\tau}\beta_{D_1} = \tau$ are obvious. The inverse of β_{D_1} is given by $\beta_{D_1}^{-1}(x, y) = x + y$, for all $(x, y) \in \operatorname{Ker}\sigma \rtimes D_0$. Naturality of β can be readily checked by using the identity $\sigma'\gamma = \gamma|_{D_0}\sigma$ for any morphism γ between two given cat^1-algebras (D_1, D_0, σ, τ) and $(D_1', D_0', \sigma', \tau')$. $\square$

1.3 Categorification of algebraic structures and crossed modules

In Ellis's PhD thesis [37] it is proved that, given a category of Ω-groups $\mathcal{C}$, crossed n-cubes, cat^n-objects, n-fold internal categories, n-fold crossed modules and n-simplicial objects in $\mathcal{C}$ whose normal complexes are of length 1 are equivalent structures. For the category of groups and $n = 1$, it is possible to consider one additional description: strict 2-groups. In [3], they are defined as strict monoidal categories where every morphism is invertible and every object has an inverse. Alternatively, it is possible to define a strict 2-group as a group object in **Cat**, where **Cat** denotes the category of all small categories (cf. [2]). Brown and Spencer [13] proved that the categories of crossed modules of groups and strict 2-groups are equivalent. See [44] for another proof via internal categories.

Baez's definition of strict 2-groups has been extended to Lie algebras (cf. [48]), associative algebras and Leibniz algebras. Furthermore, the analogous equivalence has been proved for crossed modules of Lie algebras and strict Lie 2-algebras (see [4]), and crossed modules of associative algebras and strict associative 2-algebras (see [61]). In this section we recall the definition of strict Leibniz 2-algebras [81] and define strict 2-dialgebras by analogy to the notion of strict associative 2-algebra by Khmaladze. Additionally, we prove that those structures are equivalent to Leibniz crossed modules and crossed modules of dialgebras, respectively.

The reader might have noticed that Baez's definition is beyond our requirements if our intention is to limit ourselves to the strict case. In fact, some authors define strict 2-groups and strict Lie 2-algebras directly as internal categories in **Gr** and **Lie**, respectively (see for instance [46, Definitions 7 and 13] and [77, Subsection 1.2]), instead of taking a detour through 2-vector spaces. The major benefit of Baez's definition is that it allows to consider the more general notion of semistrict Leibniz 2-algebra (see [81]) and semistrict 2-dialgebra (still not described as far as we know).

Note that we work over a commutative ring with unit instead of a field, so we have to consider 2-modules instead of 2-vector spaces, but the idea behind them is essentially the same. In order to explore the semistrict generalization it would be necessary to endow 2-modules with a 2-category structure by introducing internal natural transformations. However, we do not give the definition, since it is not a requirement for the work developed in this thesis. See [4, 9] for further insight.

Definition 1.3.1. *A* 2*-module is an internal category in* **Mod***, the category of* K*-modules. A linear functor is an internal functor in* **Mod***.*

Therefore, a 2-module M consists of a K-module of objects M_0, a K-module of arrows M_1 and four K-linear maps, namely the source and target maps, the identity-assigning map and the composition map,

$$
\begin{tikzcd}
M_1 \times_{M_0} M_1 \arrow[r, "\kappa"] & M_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M_0 \arrow[l, bend right, "e"']
\end{tikzcd}
$$

such that the diagrams (1.1.7)–(1.1.10) commute. A linear functor F between the 2-modules M and M' is a pair (F_1, F_0) of K-linear homomorphisms, $F_1 \colon M_1 \to M_1'$, $F_0 \colon M_0 \to M_0'$, such that (1.1.11) and (1.1.12) commute. We will denote by **2Mod** the category of 2-modules and linear functors.

In [4], Baez proves that one can omit all mention to composition in the definition of a 2-vector space (it would work also for 2-modules), by describing it in terms of what he calls the arrow part of an element in M_1:

Definition 1.3.2. *Let* M *be a* 2*-module. Given* $f \in M_1$*, the arrow part of* f *is* $\overrightarrow{f} = f - es(f)$*.*

A K-module is a commutative group with no other binary operation. The following propositions are particular cases of Lemma 1.1.18 and Theorem 1.2.1 respectively.

Proposition 1.3.3. *Let* $(M_1, M_0, s, t, e, \kappa)$ *be a* 2*-module. Then* $\kappa(f, g) = f - es(g) + g = f + \overrightarrow{g}$ *for all* $(f, g) \in M_1 \times_{M0} M_1$*.*

Remark 1.3.4. *Statements in Lemma 1.1.18 (ii) and (iii) are omitted because the group operation is commutative and* $\Omega_2' = \emptyset$ *for this particular case. Besides, note that our composition formula for* 2*-modules agrees with the one by Baez for* 2*-vector spaces, but it is presented differently. He writes* $\kappa(f, g) = (x, \overrightarrow{f} + \overrightarrow{g})$*, where* x *is the source of* f*. Nevertheless, we can rearrange our formula in order to look very similar, that is in terms of* $\overrightarrow{f}$*,* $\overrightarrow{g}$ *and the source of* f*:*

$$\kappa(f, g) = f + \overrightarrow{g} = \overrightarrow{f} + \overrightarrow{g} + es(f).$$

Proposition 1.3.5. *Given two K-modules M_1 and M_0 and three morphisms of K-modules*

$$\begin{tikzcd} M_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & M_0 \arrow[l, bend right, "e"'] \end{tikzcd}$$

such that $se = \mathrm{id}_{M_0} = te$, that is (1.1.7) *holds,* $(M_1, M_0, s, t, e, \kappa)$ *is a* 2*-module, with κ given by $\kappa(f, g) = f + \overrightarrow{g}$ for all $(f, g) \in M_1 \times_{M_0} M_1$.*

It is possible to prove that **2Mod** is equivalent to **2Term**, the category of 2-term chain complexes of K-modules and chain maps. Furthermore, it is possible to define linear natural transformations between linear functors and chain homotopies between chain maps in order to consider **2Mod** and **2Term** as 2-categories (see [4, Proposition 8 and Theorem 12]). Nevertheless, a 2-category structure is not necessary if we limit ourselves to the strict case. On the contrary, we do need to consider the direct sum of 2-modules.

Proposition 1.3.6. *Given a pair of* 2*-modules, $M = (M_1, M_0, s, t, e, \kappa)$ and $M' = (M_1', M_0', s', t', e', \kappa')$, there is a* 2*-module $M \oplus M'$, with*

(i) *$M_1 \oplus M_1'$ as object of arrows,*

(ii) *$M_0 \oplus M_0'$ as object of objects,*

(iii) *$s \oplus s'$ as source map,*

(iv) *$t \oplus t'$ as target map,*

(v) *$e \oplus e'$ as identity-assigning map,*

(vi) *$\kappa \oplus \kappa'$ as composition map.*

Proof. It is straightforward to check that $M = (M_1 \oplus M_1', M_0 \oplus M_0', s \oplus s', t \oplus t', e \oplus e')$ satisfies (1.1.7). Hence, by Proposition 1.3.5, $M = (M_1 \oplus M_1', M_0 \oplus M_0', s \oplus s', t \oplus t', e \oplus e', \tilde{\kappa})$ is a 2-module, with $\tilde{\kappa}$ given by

$$\tilde{\kappa}((f, f'), (g, g')) = (f, f') + (g, g') - (es(g), e's'(g')) = (f + \overrightarrow{g}, f' + \overrightarrow{g'}),$$

for all $((f, f'), (g, g')) \in (M_1 \oplus M_1') \times_{M_0 \oplus M_0'} (M_1 \oplus M_1')$. Note that $t(f) = s(g)$ and $t'(f') = s'(g')$. On the other hand, $\kappa \oplus \kappa'$ is given by

$$(\kappa \oplus \kappa')((f, g), (f', g')) = (f + \overrightarrow{g}, f' + \overrightarrow{g'}),$$

for all $((f, g), (f', g')) \in (M_1 \times_{M_0} M_1) \oplus (M_1' \times_{M_0'} M_1')$. Observe that $t(f) = s(g)$ and $t'(f') = s'(g')$. It is evident that $(M_1 \oplus M_1') \times_{M_0 \oplus M_0'} (M_1 \oplus M_1')$ and $(M_1 \times_{M_0} M_1) \oplus (M_1' \times_{M_0'} M_1')$ are isomorphic. Therefore, $M = (M_1 \oplus M_1', M_0 \oplus M_0', s \oplus s', t \oplus t', e \oplus e', \kappa \oplus \kappa')$ is a 2-module. $\square$

The next proposition shows that $M \oplus M'$ is correctly defined.

Proposition 1.3.7. *Given a pair of 2-modules M and M', $M \oplus M'$ is both the product and the coproduct of M and M'. Projections are given by the linear functors $\pi \colon M \oplus M' \to M$ and $\pi' \colon M \oplus M' \to M'$, with $\pi_i(x, y) = x$ and $\pi'_i(x, y) = y$ for all $(x, y) \in M_i \oplus M'_i$, $i = 0, 1$. Injections are given by the linear functors $\iota \colon M \to M \oplus M'$ and $\iota' \colon M' \to M \oplus M'$, with $\iota_i(x) = (x, 0)$ and $\iota'_i(y) = (0, y)$ for all $x \in M_i$, $y \in M'_i$, $i = 0, 1$.*

Proof. It is straightforward to check that π, π', ι and ι' are linear functors. Now, given a 2-module N and two linear functors $P \colon N \to M$ and $P' \colon N \to M'$, we can define the linear functor $H \colon N \to M \oplus M'$, with $H_i(x) = (P_i(x), P'_i(x))$ for all $x \in N_i$, $i = 0, 1$, such that $P = \pi H$ and $P' = \pi' H$. Correctness and uniqueness of H can be readily checked. The universal property of the coproduct can be proved analogously. $\square$

It is also possible to give a definition of the tensor product of a pair of 2-modules similar to the definition of the direct sum (see [4, Proposition 14]). Furthermore, we can categorify the ground ring K and the left and right multiplications by scalars, but those constructions are not essential in order to define strict Leibniz 2-algebras and strict 2-dialgebras.

On the contrary, it is indeed necessary to define what a bilinear functor is so we can endow 2-modules with a Leibniz (respectively dialgebra) structure, since we need to define a Leibniz bracket (respectively left and right products). Besides, due to the preceding proposition, we can write $M \times M'$ instead of $M \oplus M'$.

Definition 1.3.8. *Let M, M' and N be 2-modules. A bilinear functor $F \colon M \times M' \to N$ consists of a pair (F_1, F_0) of bilinear maps, $F_1 \colon M_1 \to M'_1$ and $F_0 \colon M_0 \to M'_0$, such that the diagrams (1.1.11) and (1.1.12) commute.*

Now we meet all the requirements in order to define what strict Leibniz 2-algebras and strict 2-dialgebras are.

Definition 1.3.9. *A strict Leibniz 2-algebra is a 2-module L, together with a bilinear functor, the bracket $[-,-] \colon L \times L \to L$, such that*

$$[[x, y]_i, z]_i = [x, [y, z]_i]_i + [[x, z]_i, y]_i. \tag{1.3.1}$$

for all $x, y, z \in L_i$, $i = 0, 1$. A morphism of strict Leibniz 2-algebras, $F \colon L \to L'$, is a linear functor such that

$$F_i([x, y]_i) = [F_i(x), F_i(y)]'_i. \tag{1.3.2}$$

for all $x, y \in L_i$, $i = 0, 1$.

We will denote by **S2Lb** the category of strict Leibniz 2-algebras and the corresponding homomorphisms. Our definition agrees with the one in [81], where strict

Leibniz 2-algebras are defined as a particular case of semistrict Leibniz 2-algebras. In order to define this more flexible structure, it is necessary to endow **2Mod** with a 2-category structure, i.e. it is mandatory to consider internal natural transformations. By doing so, it is possible to define a trilinear natural isomorphism, the Jacobiator, along with the Jacobiator identity, which relates the two ways the Jacobiator can be used in order to rebracket $[[[x,y],z],w]$ for $x,y,z,w \in L_0$. The Jacobiator weakens the Leibniz condition (1.3.1), making it into a natural isomorphism instead of an identity. Moreover, in the semistrict framework, the identities 1.3.2 in the definition of a Leibniz 2-algebra homomorphism are replaced by a bilinear natural transformation.

Definition 1.3.10. *A strict 2-dialgebra is a 2-module D, together with two bilinear functors, the left and the right products $- \dashv -\colon D \times D \to D$ and $- \vdash -\colon D \times D \to D$, such that*

$$(x \dashv_i y) \dashv_i z = x \dashv_i (y \vdash_i z), \tag{1.3.3}$$
$$(x \dashv_i y) \dashv_i z = x \dashv_i (y \dashv_i z), \tag{1.3.4}$$
$$(x \vdash_i y) \dashv_i z = x \vdash_i (y \dashv_i z), \tag{1.3.5}$$
$$(x \dashv_i y) \vdash_i z = x \vdash_i (y \vdash_i z), \tag{1.3.6}$$
$$(x \vdash_i y) \vdash_i z = x \vdash_i (y \vdash_i z). \tag{1.3.7}$$

for all $x,y,z \in D_i$, $i = 0,1$. A morphism of strict 2-dialgebras, $F\colon D \to D'$, is a linear functor such that

$$F_i(x \dashv_i y) = F_i(x) \dashv_i' F_i(y) \quad \text{and} \quad F_i(x \vdash_i y) = F_i(x) \vdash_i' F_i(y) \tag{1.3.8}$$

for all $x,y \in D_i$, $i = 0,1$.

We will denote by **S2Dias** the category of strict 2-dialgebras and the corresponding homomorphisms. The notion of semistrict 2-dialgebra has not been explored as far as we now. The construction followed in the case of semistrict Leibniz 2-algebras suggests that it would be necessary to weaken (1.3.3)–(1.3.7) by introducing five trilinear natural isomorphisms, the associators, and the corresponding coherence laws in order to relate all the possible ways of using the associators to rebracket all the possible combinations involving four elements in D_0 and the bilinear functors $\vdash$ and $\dashv$. However, this approach is just a thought and it has not yet been proved to be valid. It is our intention to consider semistrict 2-dialgebras as a possible aim for further research.

Observe that the difference between the previous definitions and the corresponding internal categories is merely semantic. Let us show it for the dialgebra case. In the definition of an internal category $D = (D_1, D_0, s, e, t, \kappa)$ in **Dias**, the dialgebra structure of D_i is precisely given by two bilinear maps, $- \dashv_i -\colon D_i \times D_i \to D_i$ and $- \vdash_i -\colon D_i \times D_i \to D_i$, such that (1.3.3)–(1.3.7) are satisfied, for $i = 0,1$. On the other hand, the fact that s, t, e and κ preserve the dialgebra structure is equivalent to the commutativity of the diagrams

$$
\begin{tikzcd}
D_1 \oplus D_1 \arrow[r, "s\oplus s"] \arrow[d, "\dashv_1"'] & D_0 \oplus D_0 \arrow[d, "\dashv_0"] \\
D_1 \arrow[r, "s"'] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D_1 \oplus D_1 \arrow[r, "t\oplus t"] \arrow[d, "\dashv_1"'] & D_0 \oplus D_0 \arrow[d, "\dashv_0"] \\
D_1 \arrow[r, "t"'] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D_1 \oplus D_1 \arrow[d, "\dashv_1"'] & D_0 \oplus D_0 \arrow[l, "e\oplus e"'] \arrow[d, "\dashv_0"] \\
D_1 & D_0 \arrow[l, "e"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
(D_1 \oplus D_1) \times_{D_0 \oplus D_0} (D_1 \oplus D_1) \arrow[r, "(\dashv_1, \dashv_1)"] \arrow[d, "\kappa\oplus\kappa"'] & D_1 \times_{D_0} D_1 \arrow[d, "\kappa"] \\
D_1 \oplus D_1 \arrow[r, "\dashv_1"'] & D_1
\end{tikzcd}
$$

along with the analogous versions involving $\vdash_0$ and $\vdash_1$.

As for morphisms, in the definition of an internal functor F between two internal categories in **Dias**, it is obvious that condition (1.3.8) is equivalent to F_0 and F_1 being morphisms of dialgebras. This situation is completely analogous in the Leibniz case. Hence, we have the following theorem:

Theorem 1.3.11. *The categories* **IDias** *(respectively* **ILb***) and* **S2Dias** *(respectively* **S2Lb***) are isomorphic.*

Bearing in mind Theorem 1.1.20 we have:

Corollary 1.3.12. *The categories* **XDias** *(respectively* **XLb***) and* **S2Dias** *(respectively* **S2Lb***) are equivalent.*

Chapter 2

Actors and modules over crossed modules

In Section 2.1 we recall the construction of the algebra of bimultipliers and the Leibniz algebra of biderivations, which are, under certain conditions, the actor in the categories **As** and **Lb** respectively. Additionally we construct the dialgebra of tetramultipliers and prove that it is the actor in **Dias** under certain similar conditions.

In Section 2.2 we recall the construction of the actor crossed module in **XGr** and **XLie** and give the description of a general actor in **XLb** that becomes the actor under certain conditions.

In Section 2.3 we define the notion of left module over crossed modules of Lie algebras and recall the definitions of left modules over crossed modules of associative algebras and groups.

2.1 Actors in categories of interest

Given a group H, we denote by $\mathrm{Aut}(H)$ the group of automorphisms of H. The morphism $\alpha\colon H \to \mathrm{Aut}(H)$, where $\alpha(h)(h') = hh'h^{-1}$, together with the action of $\mathrm{Aut}(H)$ on H defined by ${}^{\varphi}h = \varphi(h)$ for all $\varphi \in \mathrm{Aut}(H)$, $h \in H$ is a crossed module. Furthermore, for every action of a group G on H there is a unique group homomorphism $\beta\colon G \to \mathrm{Aut}(H)$ with ${}^{g}h = {}^{\beta(g)}h$. Conversely, every group homomorphism from G to $\mathrm{Aut}(H)$ induces an action of G on H. Therefore it would be possible to define a group action of G on H as a group homomorphism from G to $\mathrm{Aut}(H)$.

The analogue to automorphisms of groups for a Lie algebra $\mathfrak{m}$ is $\mathrm{Der}(\mathfrak{m})$, the Lie algebra of derivations of $\mathfrak{m}$. Recall that an element in $\mathrm{Der}(\mathfrak{m})$ is a K-linear map d from $\mathfrak{m}$ to $\mathfrak{m}$ such that $d([m_1, m_2]) = [d(m_1), m_2] + [m_1, d(m_2)]$ for all $m_1, m_2 \in \mathfrak{m}$. The Lie structure is given by the bracket $[d_1, d_2] = d_1 d_2 - d_2 d_1$ for all $d_1, d_2 \in \mathrm{Der}(\mathfrak{m})$. The Lie homomorphism $\alpha\colon \mathfrak{m} \to \mathrm{Der}(\mathfrak{m})$, where $\alpha(m)(m') = [m, m']$, is a crossed module together with the action of $\mathrm{Der}(\mathfrak{m})$ on $\mathfrak{m}$ defined by $[\varphi, m] = \varphi(m)$ for all $\varphi \in \mathrm{Der}(\mathfrak{m})$, $m \in \mathfrak{m}$. Just like for groups, given an action of a Lie algebra $\mathfrak{p}$ on a Lie algebra $\mathfrak{m}$ there is a unique morphism of Lie algebras $\beta \colon \mathfrak{p} \to \mathrm{Der}(\mathfrak{m})$, such that $[p, m] = [\beta(p), m]$.

Bearing this in mind, Casas, Datuashvili and Ladra give the following definition in [19] (see also [11]).

Definition 2.1.1 ([19])**.** *Let $\mathcal{C}$ be a category of interest. For any object A in $\mathcal{C}$, an actor of A is a crossed module $(A, \mathrm{Act}(A), \alpha)$ such that for any object $C \in \mathcal{C}$ and an action of C on A, there is a unique morphism $\beta \colon C \to \mathrm{Act}(A)$ with ${}^{c}a = {}^{\beta(c)}a$, $c * a = \beta(c) * a$ for any $a \in A$, $c \in C$, $* \in \Omega_2'$.*

It follows immediately from the previous definition that, given an object A in $\mathcal{C}$, an actor $\mathrm{Act}(A)$ is a unique object up to isomorphism.

In [19, Definition 3.9] and the later paper [20, Proposition and Definition 3.1], Casas, Datuashvili and Ladra give an equivalent definition of the actor, in which the condition of the existence of a crossed module $(A, \mathrm{Act}(A), \alpha)$ is changed by simply asking for $\mathrm{Act}(A)$ to have a set of derived actions on A. In fact, given an object A in $\mathcal{C}$, there is always a set of derived actions of A on itself, given by conjugation for the group operation and simply by the operations themselves for Ω_2'. If the actor exists, there is a unique morphism $\beta \colon A \to \mathrm{Act}(A)$, which is a crossed module in $\mathcal{C}$ [20, Proposition 3.5 (a)]. This "upgraded" definition is equivalent to that of *split extension classifier* from [10].

It is also proved in [19, 20] that the actor in the case of associative algebras and Leibniz algebras does not exist unless some extra conditions are considered. Let us briefly recall the distinctive features of those particular cases.

The following definition is closely related to the notion of multiplication of a ring by Hochschild [53], called bimultiplication by Mac Lane [68].

Definition 2.1.2. *Let B be an associative algebra. A bimultiplier of B is a pair (l, r) of K-linear maps $l, r \colon B \to B$ such that*

$$\begin{aligned} l(b \cdot b') &= l(b) \cdot b', \\ r(b \cdot b') &= b \cdot r(b'), \\ b \cdot l(b') &= r(b) \cdot b'. \end{aligned}$$

for all $b, b' \in B$.

We will denote by $\mathrm{Bim}(B)$ the set of bimultipliers of B. Observe that these three conditions are a perfect match with the identities (1), (2) and (3) from Definition 1.2.24.

It is clear that, given an element $b \in B$, the pair (l_b, r_b), with $l_b(b') = b \cdot b'$ and $r_b(b') = b' \cdot b$ for all $b' \in B$, is a bimultiplier. The K-module structure of $\mathrm{Bim}(B)$ is obvious and its algebra structure is given by:

$$(l_1, r_1) \cdot (l_2, r_2) = (l_1 l_2, r_2 r_1)$$

for all $(l_1, r_1), (l_2, r_2) \in \mathrm{Bim}(B)$.

It is immediate to check that the map $\alpha\colon B \to \mathrm{Bim}(B)$, $b \mapsto \alpha(b) = (l_b, r_b)$, with l_b and r_b as defined previously, is a morphism of algebras. The problem is that the set of actions of $\mathrm{Bim}(B)$ on B does not satisfy all the axioms of what we call an action of an algebra, i.e. it is not a set of derived actions. That set of actions is given by

$$(l, r) \cdot b = l(b),$$
$$b \cdot (l, r) = r(b),$$

for all $(l, r) \in \mathrm{Bim}(B)$, $b \in B$. All the axioms from Definition 1.2.24 are satisfied, except number (5). Let $(l_1, r_1), (l_2, r_2) \in \mathrm{Bim}(B)$ and $b \in B$. Then

$$((l_1, r_1) \cdot b) \cdot (l_2, r_2) = l_1(b) \cdot (l_2, r_2) = r_2(l_1(b)), \tag{2.1.1}$$

but

$$(l_1, r_1) \cdot (b \cdot (l_2, r_2)) = (l_1, r_1) \cdot r_2(b) = l_1(r_2(b)). \tag{2.1.2}$$

In general $r_2 l_1(b)$ and $l_1 r_2(b)$ are not necessarily equal.

Recall that a category of interest is a category of groups with operations with two additional axioms (see Definition 1.1.4). In [19, 20], given a category of interest $\mathcal{C}$ with the set of identities E, it is denoted by E_G the subset of E that includes all the identities except those from the additional axioms satisfied by a category of interest. The category with the same set of operations and E_G as the set of identities is denoted by $\mathcal{C}_G$. It is immediate that there is a full inclusion functor $\mathcal{C} \hookrightarrow \mathcal{C}_G$.

The set of actions of $\mathrm{Bim}(B)$ on B is indeed a set of derived actions in As_G, since it satisfies the conditions from Lemma 1.1.8, that is bilinearity, since the group action is trivial due to the commutativity of the addition. Moreover, $(B, \mathrm{Bim}(B), \alpha)$ is a crossed module in As_G and given an action of an algebra A on B, there is a morphism of algebras $\beta\colon A \to \mathrm{Bim}(B)$ such that $a \cdot b = \beta(a) \cdot b$, for any $a \in A$, $b \in B$. In other words, $\mathrm{Bim}(B)$ is what Casas, Datuashvili and Ladra call a general actor of B:

Definition 2.1.3 ([19, 20]). *Let $\mathcal{C}$ be a category of interest and A an object in $\mathcal{C}$. A general actor object $\mathrm{GAct}(A)$ of A is an object in $\mathcal{C}_G$ that has a set of actions on A, which is a set of derived actions in $\mathcal{C}_G$, such that for any object C in $\mathcal{C}$ with a set of derived actions on A in $\mathcal{C}$, there exist a unique morphism in $\mathcal{C}_G$, $\beta\colon C \to \mathrm{GAct}(A)$, with ${}^{c}a = {}^{\beta(c)}a$, $c * a = \beta(c) * a$ for all $c \in C$, $a \in A$, $* \in \Omega_2'$.*

Observe that, unlike the actor, a general actor is not necessarily unique. In fact, in [19], for an object A in a category of interest $\mathcal{C}$, it is constructed a general actor, denoted $\mathfrak{B}(A)$ such that $\mathfrak{B}(A) \hookrightarrow \mathrm{Bim}(A)$ when $\mathcal{C} = \mathsf{As}$. In order to construct $\mathfrak{B}(A)$, Casas, Datuashvili and Ladra consider all the split extensions of A in $\mathcal{C}$, that is all the objects B in $\mathcal{C}$ with a set of derived actions on A in $\mathcal{C}$. Then, for any element b in any of those objects B they consider the maps from A to A which take an element a in A to the result of b acting on a, one for every action in the set of actions of B. Afterwards, they define a set of operations for all those action maps and denote by

$\mathfrak{B}(A)$ the set of all those maps and the new maps obtained as a result of operating the initial ones.

In [20], due to a recommendation by Z. Janelidze, the authors included the notions of strict action [20, Definition 3.2] and strict general actor (which is a general actor whose action on the corresponding object is strict), along with a condition on a general actor [20, Condition A, p. 100] in order to define the universal strict general actor. Note that $\mathrm{Bim}(B)$ (and $\mathrm{Bider}(\mathfrak{m})$, which is described below) is not necessarily a universal strict general actor, although it is a strict general actor.

$\mathfrak{B}(A)$ as constructed in [19] is in fact a universal strict general actor [20, Theorem 4.3], which is unique up to isomorphism if it is appropriate (in the sense of the equalities in Condition A) to the given presentation of the category of interest. Additionally, if an object A has an actor, then $\mathrm{Act}(A) = \mathfrak{B}(A)$ [20, Proposition 4.7].

Back to the particular situation of associative algebras, the major weakness of $\mathrm{Bim}(B)$ is that, in general, for any algebra A there are morphisms from A to $\mathrm{Bim}(B)$ that do not induce a set of derived actions of A on B. Nevertheless, in [19, 20] the authors give a particular case of associative algebras for which the actor is indeed the algebra of bimultipliers, which follows directly from the following result.

Lemma 2.1.4. *Let B be an associative algebra such that* $\mathrm{Ann}(B) = 0$ *or* $B^2 = B$. *Then, given* $(l_1, r_1), (l_2, r_2) \in \mathrm{Bim}(B)$,

$$r_2 l_1(b) = l_1 r_2(b).$$

for any $b \in B$.

Proof. Let us first assume that $\mathrm{Ann}(B) = 0$. Let $b, b' \in B$. Then

$$\begin{aligned}(l_1r_2(b) - r_2l_1(b)) \cdot b' &= l_1r_2(b) \cdot b' - r_2l_1(b) \cdot b' = l_1(r_2(b) \cdot b') - l_1(b) \cdot l_2(b') \\ &= l_1(b \cdot l_2(b')) - l_1(b) \cdot l_2(b') = l_1(b) \cdot l_2(b') - l_1(b) \cdot l_2(b') = 0,\end{aligned}$$

just by using the properties of the bimultipliers. By similar calculations, one can easily prove that $b' \cdot (l_1r_2(b) - r_2l_1(b)) = 0$. Hence, $l_1r_2(b) - r_2l_1(b)$ is an element of the annihilator and $l_1r_2(b) = r_2l_1(b)$.

If we consider $B^2 = B$ as the hypothesis, it would be sufficient to prove that $l_1r_2(b \cdot b') = r_2l_1(b \cdot b')$ for any pair of elements $b, b' \in B$, but that identity follows almost immediately from the properties of the bimultipliers. □

Observe that the only impediment for $\mathrm{Bim}(B)$ to be the actor of B was that in general (2.1.1) and (2.1.2) are not equal. Hence, directly from the previous lemma, we have the following result.

Proposition 2.1.5 ([19, 20])**.** *Let B be an associative algebra such that* $\mathrm{Ann}(B) = 0$ *or* $B^2 = B$. *Then* $\mathrm{Act}(B) = \mathrm{Bim}(B)$.

Concerning Leibniz algebras, the situation is quite similar. Given a Leibniz algebra $\mathfrak{m}$, the role of general actor is played by the Leibniz algebra of biderivations $\mathrm{Bider}(\mathfrak{m})$, described for the first time by Loday [64].

Definition 2.1.6. *Let $\mathfrak{m}$ be a Leibniz algebra. A biderivation of $\mathfrak{m}$ is a pair (d, D) of K-linear maps $d, D\colon \mathfrak{m} \to \mathfrak{m}$ such that*

$$d([m, m']) = [d(m), m'] + [m, d(m')], \tag{2.1.3}$$
$$D([m, m']) = [D(m), m'] - [D(m'), m], \tag{2.1.4}$$
$$[m, d(m')] = [m, D(m')], \tag{2.1.5}$$

for all $m, m' \in \mathfrak{m}$.

In other words, d is a derivation (first identity) and D is an anti-derivation (second identity), which additionally satisfy the third condition.

It is not difficult to check that, given an element $m \in \mathfrak{m}$, the pair $(\mathrm{ad}(m), \mathrm{Ad}(m))$, with $\mathrm{ad}(m)(m') = -[m', m]$ and $\mathrm{Ad}(m)(m') = [m, m']$ for all $m' \in \mathfrak{m}$, is a biderivation. Loday calls $(\mathrm{ad}(m), \mathrm{Ad}(m))$ the inner biderivation of m. The K-module structure of $\mathrm{Bider}(\mathfrak{m})$ is obvious and its Leibniz structure is given by

$$[(d_1, D_1), (d_2, D_2)] = (d_1 d_2 - d_2 d_1, D_1 d_2 - d_2 D_1) \tag{2.1.6}$$

for all $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{m})$. Checking that $[(d_1, D_1), (d_2, D_2)]$ is indeed a biderivation of $\mathfrak{m}$ is fairly simple, although the identity (2.1.5) is not completely straightforward. Nevertheless, it can be easily derived from the following result.

Lemma 2.1.7. *Let $\mathfrak{m}$ be a Leibniz algebra and $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{m})$. Then*

$$[D_1 d_2(m), m'] = [D_1 D_2(m), m'],$$
$$[m, D_1 d_2(m')] = [m, D_1 D_2(m')],$$

for all $m, m' \in \mathfrak{m}$.

Proof. Let $m, m' \in \mathfrak{m}$ and $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{m})$. According to the identity (2.1.5) for (d_2, D_2), $[m', d_2(m)] = [m', D_2(m)]$, so $D_1([m', d_2(m)]) = D_1([m', D_2(m)])$. Simply by using the fact that D_1 is an anti-derivation we get that

$$[D_1(m'), d_2(m)] - [D_1 d_2(m), m'] = [D_1(m'), D_2(m)] - [D_1 D_2(m), m'].$$

Therefore $[D_1 d_2(m), m'] = [D_1 D_2(m), m']$, since $[D_1(m'), d_2(m)] = [D_1(m'), D_2(m)]$ due to (2.1.5) for (d_2, D_2). Analogously, $d_1([m, d_2(m')]) = d_1([m, D_2(m')])$. If we use the fact that d_1 is a derivation,

$$[d_1(m), d_2(m')] + [m, d_1 d_2(m')] = [d_1(m), D_2(m')] + [m, d_1 D_2(m')].$$

Since $[d_1(m), d_2(m')] = [d_1(m), D_2(m')]$, we have that $[m, d_1 d_2(m')] = [m, d_1 D_2(m')]$. Hence, $[m, D_1 d_2(m')] = [m, D_1 D_2(m')]$ due to the identity (2.1.5) for (d_1, D_1). $\square$

It is easy to check that the map $\alpha\colon \mathfrak{m} \to \mathrm{Bider}(\mathfrak{m})$, $m \mapsto \alpha(m) = (\mathrm{ad}(m), \mathrm{Ad}(m))$ is a morphism of Leibniz algebras, but, analogously to what happens with the algebra of bimultipliers, there is a problem with the set of actions of $\mathrm{Bider}(\mathfrak{m})$ on $\mathfrak{m}$ (cf. [29]). That set of actions is given by

$$\begin{aligned}
[(d, D), m] &= D(m),\\
[m, (d, D)] &= -d(m),
\end{aligned}$$

for all $(d, D) \in \mathrm{Bider}(\mathfrak{m})$, $m \in \mathfrak{m}$. All the axioms from Definition 1.2.37 are satisfied, except number (6). Let $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{m})$ and $m \in \mathfrak{m}$. Then

$$[(d_1, D_1), [(d_2, D_2), m]] = [(d_1, D_1), D_2(m)] = D_1(D_2(m)),$$

but

$$\begin{aligned}
[[(d_1, D_1), (d_2, D_2)], m] - [[(d_1, D_1), m], (d_2, D_2)] &= [(d_1d_2 - d_2d_1, D_1d_2 - d_2D_1), m]\\
- [D_1(m), (d_2, D_2)] = D_1(d_2(m)) - d_2(D_1(m)) &+ d_2(D_1(m)) = D_1(d_2(m)).
\end{aligned}$$

In general $D_1D_2(m)$ and $D_1d_2(m)$ are not necessarily equal.

Similarly to the situation for associative algebras, there is a sufficient condition in order to guarantee that $\mathrm{Bider}(\mathfrak{m})$ is the actor of $\mathfrak{m}$, which follows directly from Lemma 2.1.7.

Proposition 2.1.8 ([19, 20])**.** *Let $\mathfrak{m}$ be a Leibniz algebra such that $\mathrm{Ann}(\mathfrak{m}) = 0$ or $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. Then $\mathrm{Act}(\mathfrak{m}) = \mathrm{Bider}(\mathfrak{m})$.*

Proof. Let $m \in \mathfrak{m}$ and $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{m})$. Recall that the issue that prevents $\mathrm{Bider}(\mathfrak{m})$ from being the actor of $\mathfrak{m}$ is that $D_1D_2(m)$ and $D_1d_2(m)$ are not equal in general. Due to Lemma 2.1.7, $D_1d_2(m) - D_1D_2(m) \in \mathrm{Ann}(\mathfrak{m})$. Hence, if we assume that $\mathrm{Ann}(\mathfrak{m}) = 0$, $D_1d_2(m) = D_1D_2(m)$.

Let us now work under the hypothesis $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. In this situation it is enough to prove that $D_1d_2([m, m']) = D_1D_2([m, m'])$ for any pair of elements $m, m' \in \mathfrak{m}$. By applying (2.1.3) and (2.1.4) we get that

$$\begin{aligned}
D_1D_2([m, m']) = [D_1D_2(m), m'] &- [D_1(m'), D_2(m)]\\
&- [D_1D_2(m'), m] + [D_1(m), D_2(m')],
\end{aligned}$$

$$\begin{aligned}
D_1d_2([m, m']) = [D_1d_2(m), m'] &- [D_1(m'), d_2(m)]\\
&+ [D_1(m), d_2(m')] - [D_1d_2(m'), m],
\end{aligned}$$

so $D_1d_2([m, m']) = D_1D_2([m, m'])$, due to Lemma 2.1.7 and the identity (2.1.5) for (d_2, D_2). $\square$

In the next subsection we construct an object in **Dias** analogous to the algebra of bimultipliers in **As** and the Leibniz algebra of biderivations in **Lie**.

2.1.1 The actor in the category of dialgebras

In [19, 20] there is a description of a method to construct a general actor of an object in a category of interest. However, the dialgebra presented here is constructed by analogy to the algebra of bimultipliers.

Definition 2.1.9. *Let L be a dialgebra. We denote by* $\mathrm{Tetra}(L)$ *the set of tetramultipliers of L, whose elements are quadruples $t = (l, r, \tilde{l}, \tilde{r})$ of K-linear maps from L to L such that*

(1) $l(a \vdash b) = l(a) \dashv b$,

(2) $l(a \dashv b) = l(a) \dashv b$,

(3) $\tilde{l}(a \dashv b) = \tilde{l}(a) \dashv b$,

(4) $\tilde{l}(a \vdash b) = l(a) \vdash b$,

(5) $\tilde{l}(a \vdash b) = \tilde{l}(a) \vdash b$,

(6) $r(a) \dashv b = a \dashv \tilde{l}(b)$,

(7) $r(a) \dashv b = a \dashv l(b)$,

(8) $\tilde{r}(a) \dashv b = a \vdash l(b)$,

(9) $r(a) \vdash b = a \vdash \tilde{l}(b)$,

(10) $\tilde{r}(a) \vdash b = a \vdash \tilde{l}(b)$,

(11) $r(a \dashv b) = a \dashv \tilde{r}(b)$,

(12) $r(a \dashv b) = a \dashv r(b)$,

(13) $r(a \vdash b) = a \vdash r(b)$,

(14) $\tilde{r}(a \dashv b) = a \vdash \tilde{r}(b)$,

(15) $\tilde{r}(a \vdash b) = a \vdash \tilde{r}(b)$,

for all $a, b \in L$.

Note that the aim is to construct an object which can be used to describe every action on L. Therefore it makes sense to consider elements that respect the axioms of a dialgebra action (see conditions (1)–(15)) from Definition 1.2.50.

Lemma 2.1.10. *Let L be a dialgebra. Given $a \in L$, the quadruple $(l_a, r_a, \tilde{l}_a, \tilde{r}_a)$, with*

$$\begin{aligned} l_a(a') &= a \dashv a', & r_a(a') &= a' \dashv a, \\ \tilde{l}_a(a') &= a \vdash a', & \tilde{r}_a(a') &= a' \vdash a, \end{aligned}$$

for all $a' \in L$, is an element in $\mathrm{Tetra}(L)$.

Proof. Given $a \in L$, $(l_a, r_a, \tilde{l}_a, \tilde{r}_a)$ verifies conditions (1)–(15) from Definition 2.1.9 directly from the five di-associativity axioms in L (see Definition 1.2.47). □

Let us now define left and right products in $\mathrm{Tetra}(L)$.

Proposition 2.1.11. *Let L be a dialgebra. Then* $\mathrm{Tetra}(L)$ *is a dialgebra with the left and the right products defined by*

$$t_1 \dashv t_2 = (l_1\tilde{l}_2, r_2r_1, \tilde{l}_1\tilde{l}_2, r_2\tilde{r}_1),$$
$$t_1 \vdash t_2 = (\tilde{l}_1 l_2, r_2r_1, \tilde{l}_1\tilde{l}_2, \tilde{r}_2 r_1)$$

for all $t_1 = (l_1, r_1, \tilde{l}_1, \tilde{r}_1), t_2 = (l_2, r_2, \tilde{l}_2, \tilde{r}_2) \in \mathrm{Tetra}(L)$.

Proof. The K-module structure of $\mathrm{Tetra}(D)$ is obvious and follows from the bilinearity of the left and the right products in L.

The 30 identities that we have to check in order to prove that $t_1 \dashv t_2$ and $t_1 \vdash t_2$ are elements in $\mathrm{Tetra}(L)$ follow easily from the 15 identities satisfied by t_1 and t_2. For instance, identity (1) for $t_1 \dashv t_2$ follows from (5) for t_2 and (1) for t_1:

$$l_1\tilde{l}_2(a \vdash b) \overset{(5)}{=} l_1(\tilde{l}_2(a) \vdash b) \overset{(1)}{=} l_1\tilde{l}_2(a) \dashv b.$$

The rest of the identities can be proved similarly.

Proving that $\dashv$ and $\vdash$ satisfy the five di-associativity axioms is also a matter of routine calculations. Let $t_1, t_2, t_3 \in \mathrm{Tetra}(L)$. Then

$$(t_1 \dashv t_2) \dashv t_3 = (l_1\tilde{l}_2, r_2r_1, \tilde{l}_1\tilde{l}_2, r_2\tilde{r}_1) \dashv t_3 = (l_1\tilde{l}_2\tilde{l}_3, r_3r_2r_1, \tilde{l}_1\tilde{l}_2\tilde{l}_3, r_3r_2\tilde{r}_1)$$

and

$$t_1 \dashv (t_2 \vdash t_3) = t_1 \dashv (\tilde{l}_2 l_3, r_3r_2, \tilde{l}_2\tilde{l}_3, \tilde{r}_3r_2) = (l_1\tilde{l}_2\tilde{l}_3, r_3r_2r_1, \tilde{l}_1\tilde{l}_2\tilde{l}_3, r_3r_2\tilde{r}_1).$$

The four remaining identities are left to the reader. □

Lemma 2.1.12. *The map* $\alpha\colon L \to \mathrm{Tetra}(L)$, $a \mapsto \alpha(a) = (l_a, r_a, \tilde{l}_a, \tilde{r}_a)$ *is a morphism of dialgebras.*

Proof. α is K-linear due to the bilinearity of the left and the right products in L. Let us show that it also preserves $\dashv$ and $\vdash$. Let $a, b \in L$. Then

$$\alpha(a \dashv b) = (l_{a\dashv b}, r_{a\dashv b}, \tilde{l}_{a\dashv b}, \tilde{r}_{a\dashv b}),$$
$$\alpha(a) \dashv \alpha(b) = (l_a, r_a, \tilde{l}_a, \tilde{r}_a) \dashv (l_b, r_b, \tilde{l}_b, \tilde{r}_b) = (l_a\tilde{l}_b, r_br_a, \tilde{l}_a\tilde{l}_b, r_b\tilde{r}_a).$$

Given $c \in L$,

$$l_{a\dashv b}(c) = (a \dashv b) \dashv c = a \dashv (b \vdash c) = l_a(b \vdash c) = l_a\tilde{l}_b(c),$$
$$r_{a\dashv b}(c) = c \dashv (a \dashv b) = (c \dashv a) \dashv b = r_b(c \dashv a) = r_br_a(c),$$
$$\tilde{l}_{a\dashv b}(c) = (a \dashv b) \vdash c = a \vdash (b \vdash c) = \tilde{l}_a(b \vdash c) = \tilde{l}_a\tilde{l}_b(c),$$
$$\tilde{r}_{a\dashv b}(c) = c \vdash (a \dashv b) = (c \vdash a) \dashv b = r_b(c \vdash a) = r_b\tilde{r}_a(c),$$

due to the di-associativity axioms in L. Analogously, $\alpha(a \vdash b) = \alpha(a) \vdash \alpha(b)$. □

In general, the set of actions of $\mathrm{Tetra}(L)$ on L, given by:

$$(l,r,\tilde{l},\tilde{r}) \dashv a = l(a), \qquad a \dashv (l,r,\tilde{l},\tilde{r}) = r(a),$$
$$(l,r,\tilde{l},\tilde{r}) \vdash a = \tilde{l}(a), \qquad a \vdash (l,r,\tilde{l},\tilde{r}) = \tilde{r}(a),$$

for any $(l,r,\tilde{l},\tilde{r}) \in \mathrm{Tetra}(L)$ and $a \in L$, is not a set of derived actions in **Dias**. In other words, although the four maps are clearly bilinear, not all the identities from Definition 1.2.50 hold. Specifically, identities (20)–(25) and (27) are the ones not satisfied in general. Let $a \in L$ and $t_1, t_2 \in \mathrm{Tetra}(L)$. Then

$$(a \vdash t_1) \vdash t_2 = \tilde{r}_2\tilde{r}_1(a) \neq \tilde{r}_2 r_1(a) = x \vdash (t_1 \vdash t_2), \tag{20}$$
$$(t_1 \dashv a) \dashv t_2 = r_2 l_1(a) \neq l_1 \tilde{r}_2(a) = t_1 \dashv (x \vdash t_2), \tag{21}$$
$$(t_1 \dashv a) \dashv t_2 = r_2 l_1(a) \neq l_1 r_2(a) = t_1 \dashv (x \dashv t_2), \tag{22}$$
$$(t_1 \vdash a) \dashv t_2 = r_2 \tilde{l}_1(a) \neq \tilde{l}_1 r_2(a) = t_1 \vdash (x \dashv t_2), \tag{23}$$
$$(t_1 \dashv a) \vdash t_2 = \tilde{r}_2 l_1(a) \neq \tilde{l}_1 \tilde{r}_2(a) = t_1 \vdash (x \vdash t_2), \tag{24}$$
$$(t_1 \vdash a) \vdash t_2 = \tilde{r}_2 \tilde{l}_1(a) \neq \tilde{l}_1 \tilde{r}_2(a) = t_1 \vdash (x \vdash t_2), \tag{25}$$
$$(t_1 \dashv t_2) \dashv a = l_1 \tilde{l}_2(a) \neq l_1 l_2(a) = t_1 \dashv (t_2 \dashv a). \tag{27}$$

Observe that the set of actions of $\mathrm{Tetra}(L)$ on L is indeed a set of derived actions in $\mathbf{Dias}_G$ and the morphism described in Lemma 2.1.12 is a crossed module in $\mathbf{Dias}_G$. Let $t \in \mathrm{Tetra}(L)$ and $a \in L$. Then $\alpha(t \dashv a) = \alpha(l(a)) = (l_{l(a)}, r_{l(a)}, \tilde{l}_{l(a)}, \tilde{r}_{l(a)})$, while $t \dashv \alpha(a) = (l\tilde{l}_a, r_a r, \tilde{l}\tilde{l}_a, r_a\tilde{r})$. Given $b \in L$, directly from the conditions satisfied by t (see Definition 2.1.9), we get that

$$l_{l(a)}(b) = l(a) \dashv b \overset{(1)}{=} l(a \vdash b) = l\tilde{l}_a(b),$$
$$r_{l(a)}(b) = b \dashv l(a) \overset{(7)}{=} r(b) \dashv a = r_a r(b),$$
$$\tilde{l}_{l(a)}(b) = l(a) \vdash b \overset{(4)}{=} \tilde{l}(a \vdash b) = \tilde{l}\tilde{l}_a(b),$$
$$\tilde{r}_{l(a)}(b) = b \vdash l(a) \overset{(8)}{=} \tilde{r}(b) \dashv a = r_a \tilde{r}(b).$$

Hence, $\alpha(t \dashv a) = t \dashv \alpha(a)$. It can be proved analogously that $\alpha(t \vdash a) = t \vdash \alpha(a)$, $\alpha(a \dashv t) = \alpha(a) \dashv t$ and $\alpha(a \vdash t) = \alpha(a) \vdash t$. Therefore $(L, \mathrm{Tetra}(L), \alpha)$ satisfies the equivariance condition. Concerning the Peiffer identity, it follows immediately from the definition of α. Let $a, b \in L$. Then

$$\alpha(a) \dashv b = (l_a, r_a, \tilde{l}_a, \tilde{r}_a) \dashv b = l_a(b) = a \dashv b,$$
$$\alpha(a) \vdash b = (l_a, r_a, \tilde{l}_a, \tilde{r}_a) \vdash b = \tilde{l}_a(b) = a \vdash b,$$
$$a \dashv \alpha(b) = a \dashv (l_b, r_b, \tilde{l}_b, \tilde{r}_b) = r_b(a) = a \dashv b,$$
$$a \vdash \alpha(b) = a \vdash (l_b, r_b, \tilde{l}_b, \tilde{r}_b) = \tilde{r}_b(a) = a \vdash b.$$

Additionally, given a dialgebra D with a set of derived actions on L in **Dias**, that is verifying all the identities from Definition 1.2.50, there exists a unique morphism of dialgebras, $\beta\colon D \to \mathrm{Tetra}(L)$, given by $\beta(x) = (l_x, r_x, \tilde{l}_x, \tilde{r}_x)$, with

$$\begin{aligned} l_x(a) &= x \dashv a, & r_x(a) &= a \dashv x,\\ \tilde{l}_x(a) &= x \vdash a, & \tilde{r}_x(a) &= a \vdash x, \end{aligned}$$

for all $x \in D$, $a \in L$, such that

$$\begin{aligned} x \dashv a &= \beta(x) \dashv a, & a \dashv x &= a \dashv \beta(x),\\ x \vdash a &= \beta(x) \vdash a, & a \vdash x &= a \vdash \beta(x). \end{aligned}$$

Note that β is well defined, since the conditions (1)–(15) for $(l_x, r_x, \tilde{l}_x, \tilde{r}_x)$ coincide with conditions (1)–(15) from Definition 1.2.50. The linearity of β follows from the bilinearity of the maps that define the action of D on L. Concerning the preservation of $\dashv$ (respectively $\vdash$), it follows from conditions (17), (18), (26) and (29) (respectively (16), (19), (28) and (30)) from Definition 1.2.50. For instance, given $x, y \in D$, let us show that $\beta(x \dashv y) = \beta(x) \dashv \beta(y)$. By definition, $\beta(x \dashv y) = (l_{x\dashv y}, r_{x\dashv y}, \tilde{l}_{x\dashv y}, \tilde{r}_{x\dashv y})$ and $\beta(x) \dashv \beta(y) = (l_x\tilde{l}_y, r_y r_x, \tilde{l}_x\tilde{l}_y, r_y\tilde{r}_x)$. Let $a \in L$. Then,

$$\begin{aligned} l_{x\dashv y}(a) &= (x \dashv y) \dashv a \overset{(26)}{=} x \dashv (y \vdash a) = l_x\tilde{l}_y(a),\\ r_{x\dashv y}(a) &= a \dashv (x \dashv y) \overset{(17)}{=} (a \dashv x) \dashv y = r_y r_x(a),\\ \tilde{l}_{x\dashv y}(a) &= (x \dashv y) \vdash a \overset{(29)}{=} x \vdash (y \vdash a) = \tilde{l}_x\tilde{l}_y(a),\\ \tilde{r}_{x\dashv y}(a) &= a \vdash (x \dashv y) \overset{(18)}{=} (a \vdash x) \dashv y = r_y\tilde{r}_x(a). \end{aligned}$$

Analogously $\beta(x \vdash y) = \beta(x) \vdash \beta(y)$. Therefore, we can write the following result:

Theorem 2.1.13. $\mathrm{Tetra}(L)$ *is a general actor of a dialgebra* L.

Recall that $\mathfrak{B}(L) \hookrightarrow \mathrm{Tetra}(L)$ for any dialgebra L, but in general, $\mathfrak{B}(L) \neq \mathrm{Tetra}(L)$, with $\mathfrak{B}(L)$ as described in [19].

Furthermore, we can give sufficient conditions on L in order to guarantee that $\mathrm{Tetra}(L)$ is its actor:

Proposition 2.1.14. *Let* L *be an associative dialgebra such that* $\mathrm{Ann}(L) = 0$ *or* $L \dashv L = L = L \vdash L$. *Then* $\mathrm{Tetra}(D)$ *is the actor of* L.

Proof. Let us assume that $\mathrm{Ann}(L) = 0$. Since we know that $\mathrm{Tetra}(L)$ is a general actor of L, it is only necessary to check that under this new hypothesis the failing conditions (20)–(25) and (27) do not fail any longer. Given $t_1, t_2 \in \mathrm{Tetra}(L)$ and $a \in L$, since $\mathrm{Ann}(L) = 0$ it would be sufficient to prove that

$$\begin{aligned} &\tilde{r}_2\tilde{r}_1(a) - \tilde{r}_2 r_1(a),\ r_2 l_1(a) - l_1\tilde{r}_2(a),\ r_2 l_1(a) - l_1 r_2(a),\ r_2\tilde{l}_1(a) - \tilde{l}_1 r_2(a),\\ &\tilde{r}_2 l_1(a) - \tilde{l}_1\tilde{r}_2(a),\ \tilde{r}_2\tilde{l}_1(a) - \tilde{l}_1\tilde{r}_2(a)\ \text{ and }\ l_1\tilde{l}_2(a) - l_1 l_2(a) \end{aligned}$$

are elements in $\mathrm{Ann}(L)$. Calculations, although long, follow easily from the conditions satisfied by t_1 and t_2 (see Definition 2.1.9). As an example, we show here that $r_2\tilde{l}_1(a) - \tilde{l}_1 r_2(a) \in \mathrm{Ann}(L)$ and leave the rest to the reader. Let $b \in L$. Then

$$\begin{aligned}
& r_2\tilde{l}_1(a) \dashv b - \tilde{l}_1 r_2(a) \dashv b \\
\overset{(7)}{=}\; & \tilde{l}_1(a) \dashv l_2(b) - \tilde{l}_1 r_2(a) \dashv b \\
\overset{(3)}{=}\; & \tilde{l}_1(a) \dashv l_2(b) - \tilde{l}_1(r_2(a) \dashv b) \\
\overset{(7)}{=}\; & \tilde{l}_1(a) \dashv l_2(b) - \tilde{l}_1(a \dashv l_2(b)) \\
\overset{(3)}{=}\; & \tilde{l}_1(a) \dashv l_2(b) - \tilde{l}_1(a) \dashv l_2(b) = 0,
\end{aligned}
\qquad
\begin{aligned}
& r_2\tilde{l}_1(a) \vdash b - \tilde{l}_1 r_2(a) \vdash b \\
\overset{(9)}{=}\; & \tilde{l}_1(a) \vdash \tilde{l}_2(b) - \tilde{l}_1 r_2(a) \vdash b \\
\overset{(5)}{=}\; & \tilde{l}_1(a) \vdash \tilde{l}_2(b) - \tilde{l}_1(r_2(a) \vdash b) \\
\overset{(9)}{=}\; & \tilde{l}_1(a) \vdash \tilde{l}_2(b) - \tilde{l}_1(a \vdash \tilde{l}_2(b)) \\
\overset{(5)}{=}\; & \tilde{l}_1(a) \vdash \tilde{l}_2(b) - \tilde{l}_1(a) \vdash \tilde{l}_2(b) = 0,
\end{aligned}$$

$$\begin{aligned}
& b \dashv r_2\tilde{l}_1(a) - b \dashv \tilde{l}_1 r_2(a) \\
\overset{(6)}{=}\; & b \dashv r_2\tilde{l}_1(a) - r_1(b) \dashv r_2(a) \\
\overset{(12)}{=}\; & r_2(b \dashv \tilde{l}_1(a)) - r_1(b) \dashv r_2(a) \\
\overset{(6)}{=}\; & r_2(r_1(b) \dashv a) - r_1(b) \dashv r_2(a) \\
\overset{(12)}{=}\; & r_1(b) \dashv r_2(a) - r_1(b) \dashv r_2(a) = 0,
\end{aligned}
\qquad
\begin{aligned}
& b \vdash r_2\tilde{l}_1(a) - b \vdash \tilde{l}_1 r_2(a) \\
\overset{(9)}{=}\; & b \vdash r_2\tilde{l}_1(a) - r_1(b) \vdash r_2(a) \\
\overset{(13)}{=}\; & r_2(b \vdash \tilde{l}_1(a)) - r_1(b) \vdash r_2(a) \\
\overset{(9)}{=}\; & r_2(r_1(b) \vdash a) - r_1(b) \vdash r_2(a) \\
\overset{(13)}{=}\; & r_1(b) \vdash r_2(a) - r_1(b) \vdash r_2(a) = 0.
\end{aligned}$$

On the other hand, if we assume that $L \dashv L = L = L \vdash L$, any element a in L can be expressed either as a linear combination of left products $b \dashv c$ or right products $b \vdash c$ in L. Bearing that in mind, it would be sufficient to show that the identities

$$\begin{aligned}
& \tilde{r}_2\tilde{r}_1(a) = \tilde{r}_2 r_1(a),\ r_2 l_1(a) = l_1\tilde{r}_2(a),\ r_2 l_1(a) = l_1 r_2(a),\ r_2\tilde{l}_1(a) = \tilde{l}_1 r_2(a), \\
& \tilde{r}_2 l_1(a) = \tilde{l}_1\tilde{r}_2(a),\ \tilde{r}_2\tilde{l}_1(a) = \tilde{l}_1\tilde{r}_2(a) \text{ and } l_1\tilde{l}_2(a) = l_1 l_2(a)
\end{aligned}$$

hold when a is either of the form $b \dashv c$ or $b \vdash c$, given $t_1, t_2 \in \mathrm{Tetra}(L)$. Actually, straightforward calculations, using the conditions satisfied by t_1 and t_2, show that:

$$\begin{aligned}
& \tilde{r}_2\tilde{r}_1(b \dashv c) = \tilde{r}_2 r_1(b \dashv c),\ r_2 l_1(b * c) = l_1\tilde{r}_2(b * c),\ r_2 l_1(b * c) = l_1 r_2(b * c), \\
& r_2\tilde{l}_1(b * c) = \tilde{l}_1 r_2(b * c),\ \tilde{r}_2 l_1(b * c) = \tilde{l}_1\tilde{r}_2(b * c),\ \tilde{r}_2\tilde{l}_1(b * c) = \tilde{l}_1\tilde{r}_2(b * c) \\
& \text{and } l_1\tilde{l}_2(b \vdash c) = l_1 l_2(b \vdash c),
\end{aligned}$$

but in general $\tilde{r}_2\tilde{r}_1(b \vdash c) \neq \tilde{r}_2 r_1(b \vdash c)$ and $l_1\tilde{l}_2(b \dashv c) = l_1 l_2(b \dashv c)$, for all $b, c \in L$. Therefore the result would not be necessarily true under the hypothesis $L \dashv L = L$ or $L \vdash L = L$. $\square$

2.2 Actor crossed module

Since crossed modules of groups can be regarded as 2-dimensional groups, it makes sense to generalize some results and constructions from **Gr** to **XGr**. As Norrie states in [73], where she constructs the actor crossed module of groups, the 2-dimensional analogue to the group of automorphisms of groups, it is surprising the ease with which the theory transcribes.

A similar construction of the actor crossed module of Lie algebras is given in [27]. This crossed module plays in **XLie** the role played by the Lie algebra of derivations in **Lie**. See also [1] for the case of crossed modules of commutative algebras.

In this section we recall the basics of both generalizations and introduce the construction of the actor crossed module of Leibniz algebras under certain conditions. Note that we will make reference to the axioms satisfied by crossed modules of groups, Lie and Leibniz algebras simply as *equivariance* and *Peiffer identity*, which of course have different, although equivalent, meanings depending on the context. See Section 1.2 for the corresponding definitions.

2.2.1 Actor crossed module of groups

The main ideas in this subsection are taken from [22, 73]. Let (M, P, μ) be a crossed module of groups. Consider $\mathrm{Der}(P, M)$, the set of all derivations from P to M, whose elements are maps $d\colon P \to M$ such that

$$d(pp') = d(p)^{p}d(p') \tag{2.2.1}$$

for all $p, p' \in P$. Immediately from the definition, given $d \in \mathrm{Der}(P, M)$, $d(1) = 1$ and $d(p)^{-1} = {}^{p}d(p^{-1})$. Whitehead [83] defined an operation $\circ$ in $\mathrm{Der}(P, M)$ given by

$$(d_1 \circ d_2)(p) = d_1\mu d_2(p)d_2(p)d_1(p) \tag{2.2.2}$$

for all $d_1, d_2 \in \mathrm{Der}(P, M)$, $p \in P$, which turns $\mathrm{Der}(P, M)$ into a monoid. Observe that the Peiffer identity and the equivariance are necessary in order to prove that $d_1 \circ d_2 \in \mathrm{Der}(P, M)$. As for associativity, it can be proved by direct calculations, using (2.2.1), the fact that μ is a group homomorphism and the Peiffer identity. The identity element is given by the derivation that maps every element of P to the identity element of M. *The Whitehead group* $\mathrm{D}(P, M)$ is the group of units of $\mathrm{Der}(P, M)$.

The other important group for the definition of the actor crossed module is $\mathrm{Aut}(\mu)$ (denoted by $\mathrm{Aut}(M, P, \mu)$ in [73]), the group of automorphisms of (M, P, μ) in the category **XGr**, i.e. its elements are morphisms of crossed modules (σ, θ), with σ and θ automorphisms of M and P respectively. Therefore,

$$\mu\sigma = \theta\sigma \qquad \text{and} \qquad \sigma({}^{p}m) = {}^{\theta(p)}\sigma(m)$$

for all $m \in M$, $p \in P$. The group operation in $\mathrm{Aut}(\mu)$ is given by the usual composition of morphisms of crossed modules.

The next step in the construction is to define a morphism of groups $\Delta\colon\ \mathrm{D}(P, M) \to \mathrm{Aut}(\mu)$, given by

$$\Delta(d) = (\sigma_d, \theta_d)$$

for all $d \in \mathrm{D}(P, M)$, where $\sigma_d(m) = d\mu(m)m$ and $\theta_d(p) = \mu d(p)p$ for all $m \in M$, $p \in P$. The Peiffer identity guarantees that σ_d and θ_d are endomorphisms of M and P respectively, for any $d \in \mathrm{Der}(P, M)$. Besides, if d belongs to the Whitehead group, the inverses of σ_d and θ_d are given by $\sigma_{d'}$ and $\theta_{d'}$ respectively, where d' is the inverse of d in $\mathrm{D}(P, M)$. The identity $\theta_d\mu = \mu\sigma_d$ follows directly from the definition of σ_d and θ_d. Regarding the identity $\sigma_d({}^{p}m) = {}^{\theta_d(p)}\sigma_d(m)$ for all $m \in M$, $p \in P$, it can be checked by routine calculations, making use, again, of the Peiffer identity, which is also the key to prove that Δ is a group homomorphism.

There is an action of $\mathrm{Aut}(\mu)$ on $\mathrm{D}(P, M)$, defined by

$$ {}^{(\sigma,\theta)}d = \sigma d\theta^{-1} $$

for any $(\sigma, \theta) \in \mathrm{Aut}(\mu)$, $d \in \mathrm{D}(P, M)$. Given $(\sigma, \theta) \in \mathrm{Aut}(\mu)$ and $d \in \mathrm{D}(P, M)$, $\sigma d\theta^{-1}$ is a derivation due to the identity $\sigma({}^{p}m) = {}^{\theta(p)}\sigma(m)$. The identity $\theta\mu = \mu\sigma$ is necessary to prove that ${}^{(\sigma,\theta)}d$ has an inverse in $\mathrm{Der}(P, M)$, which is given by $\sigma d'\theta^{-1}$, where d' is the inverse of d. The two first axioms in the definition of a group action (see Definition (1.2.2)) are obvious and the third one follows from the identity $\theta\mu = \mu\sigma$. Straightforward calculations show that Δ together with this action is a crossed module of groups, called the actor crossed module, which is denoted by $\mathrm{Act}(M, P, \mu)$.

Definition 2.2.1 ([73])**.** *An action of a crossed module (H, G, ∂) on another crossed module (M, P, μ) is a morphism of crossed modules from (H, G, ∂) to $\mathrm{Act}(M, P, \mu)$.*

As a first example, Norrie proves that there is an action of a crossed module (M, P, μ) on itself, given by the morphism of crossed modules $(\varphi, \psi)\colon (M, P, \mu) \to \mathrm{Act}(M, P, \mu)$, defined as follows. Given $m \in M$, $\varphi(m)(p) = m{}^{p}m^{-1}$ for all $p \in P$. Furthermore, given $p \in P$, $\psi(p) = (\sigma_p, \theta_p)$, with $\sigma_p(m) = {}^{p}m$ and $\theta_p(p') = pp'p^{-1}$ for all $m \in M$, $p' \in P$.

In [22] the authors give an equivalent description of actions of crossed modules of groups in terms of equations (see also [82]).

Proposition 2.2.2 ([22])**.** *Let (H, G, ∂) and (M, P, μ) be crossed modules of groups. Then there is an action of (H, G, ∂) on (M, P, μ) if and only if the following conditions hold:*

(i) The group G (and so H) acts on M and P, μ is a G-equivariant homomorphism, that is $\mu({}^{g}m) = {}^{g}\mu(m)$ and the action of P on M is a G-equivariant action, that is ${}^{g}({}^{p}m) = {}^{({}^{g}p)}({}^{g}m)$ for all $g \in G$, $m \in M$, $p \in P$.

(ii) There is a map $\xi\colon H \times P \to M$ such that

$$ \mu\xi(h, p) = {}^{\partial(h)}pp^{-1}, \tag{GrM1} $$

$$ \xi(h, \mu(m)) = {}^{\partial(h)}mm^{-1}, \tag{GrM2} $$

$$ {}^{g}\xi(h, p) = \xi({}^{g}h, {}^{g}p), \tag{GrM3} $$

$$ \xi(hh', p) = {}^{\partial(h)}\xi(h', p)\,\xi(h, p), \tag{GrM4} $$

$$ \xi(h, pp') = \xi(h, p)\,{}^{p}\xi(h, p') \tag{GrM5} $$

for all $h, h' \in H$, $p, p' \in P$, $m \in M$, $g \in G$.

Proof. Let us first suppose that (H, G, ∂) acts on (M, P, μ), that is there is a morphism of crossed modules

$$\begin{tikzcd} H \arrow[r, "\partial"] \arrow[d, "\varphi"'] & G \arrow[d, "\psi"] \\ \mathrm{D}(P, M) \arrow[r, "\Delta"'] & \mathrm{Aut}(\mu) \end{tikzcd} \tag{2.2.3}$$

The previous diagram is commutative and $\varphi({}^g h) = {}^{\psi(g)}\varphi(h)$ for all $h \in H$, $g \in G$. We will denote $\psi(g)$ by (σ_g, θ_g) for any $g \in G$. There is an action of G on M (respectively P) given by ${}^g m = \sigma_g(m)$ (respectively ${}^g p = \theta_g(p)$) for all $g \in G$, $m \in M$ (respectively $p \in P$), which induces an action of H on M (respectively P) via ∂. The identities $\mu({}^g m) = {}^g \mu(m)$ and ${}^g({}^p m) = {}^{({}^g p)}({}^g m)$ follow from $\mu\sigma_g = \theta_g\mu$ and $\sigma_g({}^p m) = {}^{\theta_g(p)}\sigma_g(m)$ respectively, for all $g \in G$, $m \in M$, $p \in P$. Therefore (i) holds.

Regarding (ii), we can define $\xi(h, p) = \varphi(h)(p)$ for any $h \in H$, $p \in P$. In this way, (GrM1) and (GrM2) follow from the commutativity of (2.2.3). (GrM3) follows from the identity $\varphi({}^g h) = {}^{\psi(g)}\varphi(h)$ for all $h \in H$, $g \in G$, and the definition of the action of $\mathrm{Aut}(\mu)$ on $\mathrm{D}(P, M)$. (GrM4) is an immediate consequence of φ being a morphism of groups and the definition of $\circ$ in $\mathrm{D}(P, M)$. Finally, (GrM5) is easy to prove by using that $\varphi(h)$ is a derivation for all $h \in H$.

The converse statement is rather obvious. If we assume that (i) and (ii) hold, it is possible to define a morphism of crossed modules (φ, ψ) from (H, G, ∂) to $\mathrm{Act}(M, P, \mu)$ as follows. Given $h \in H$, $\varphi(h)(p) = \xi(h, p)$ for all $p \in P$. For any $g \in G$, $\psi(g) = (\sigma_g, \theta_g)$, with $\sigma_g(m) = {}^g m$ and $\theta_g(p) = {}^g p$ for all $m \in M$, $p \in P$.

Given $h \in H$, (GrM5) guarantees that $\varphi(h)$ is a derivation. Furthermore, $\varphi(h^{-1})$ is the inverse of $\varphi(h)$ in $\mathrm{D}(P, M)$ and φ is a group homomorphism due to (GrM2) and (GrM4). Concerning ψ, given $g \in G$, σ_g and θ_g are automorphisms as a direct consequence of the three identities satisfied by the group actions of G on M and P respectively. Additionally (σ_g, θ_g) is a morphism of crossed modules due to the identities $\mu({}^g m) = {}^g \mu(m)$ and ${}^g({}^p m) = {}^{({}^g p)}({}^g m)$ for all $g \in G$, $m \in M$, $p \in P$.

The identity $\Delta\varphi = \psi\partial$ follows immediately from (GrM1) and (GrM2). Finally, the fact that $\varphi({}^g h) = {}^{\psi(g)}\varphi(h)$ for all $h \in H$, $g \in G$ is a consequence of (GrM3). $\square$

Let (H, G, ∂) be a crossed module of groups acting on the crossed module (M, P, μ). By Proposition 2.2.2, G acts on P and H acts on M, so it makes sense to consider the semidirect products of groups $P \rtimes G$ and $M \rtimes H$. There is an action of $P \rtimes G$ on $M \rtimes H$ given by ${}^{(p,g)}(m, h) = ({}^p({}^g m)(\xi({}^g h, p^{-1})), {}^g h)$ for all $(p, g) \in P \rtimes G$, $(m, h) \in M \rtimes H$. The morphism of groups $(\mu, \partial)\colon M \rtimes H \to P \rtimes G$, $(m, h) \mapsto (\mu(m), \partial(h))$ is a crossed module together with that action.

$(M \rtimes H, P \rtimes G, (\mu, \partial))$ is called the semidirect product of the crossed modules (M, P, μ) and (H, G, ∂). Note that the semidirect product determines an obvious split extension of (H, G, ∂) by (M, P, μ)

$$(0,0,0) \longrightarrow (M, P, \mu) \longrightarrow (M \rtimes H, P \rtimes G, (\mu, \partial)) \rightleftarrows (H, G, \partial) \longrightarrow (0,0,0)$$

Conversely, any split extension of (H, G, ∂) by (M, P, μ) is isomorphic to their semidirect product, where the action of (H, G, ∂) on (M, P, μ) is induced by the splitting morphism. Therefore, the definition of an action of a crossed module of groups on another crossed module of groups agrees with the general notion of derived action in a category of Ω-groups. Recall that **XGr** is indeed a category of interest (see [75]).

2.2.2 Actor crossed module of Lie algebras

It is possible to construct the actor of a Lie crossed module (see [27]), following a similar procedure to the one described for groups. Let us recall that construction in order to appreciate the slight differences. Note that in [27] K is considered a field, but the procedure still works for K a commutative unital ring.

Let $(\mathfrak{n}, \mathfrak{q}, \mu)$ be a Lie crossed module. Consider $\operatorname{Der}(\mathfrak{q}, \mathfrak{n})$, the K-module of all derivations from $\mathfrak{q}$ to $\mathfrak{n}$, that is all the K-linear maps $d \colon \mathfrak{q} \to \mathfrak{n}$ such that

$$d([q, q']) = [d(q), q'] + [q, d(q')] \tag{2.2.4}$$

for all $q, q' \in \mathfrak{q}$. There is a Lie bracket in $\operatorname{Der}(\mathfrak{q}, \mathfrak{n})$ given by

$$[d_1, d_2] = d_1 \mu d_2 - d_2 \mu d_1$$

for all $d_1, d_2 \in \operatorname{Der}(\mathfrak{q}, \mathfrak{n})$. Just like for groups, the Peiffer identity and the equivariance are essential ir order to prove that the result of this bracket lies within $\operatorname{Der}(\mathfrak{q}, \mathfrak{n})$. The antisymmetry and the Jacobi identity follow directly from the definition of the bracket. If we consider the crossed module $(\mathfrak{q}, \mathfrak{q}, \mathrm{id}_{\mathfrak{q}})$, $\operatorname{Der}(\mathfrak{q}, \mathfrak{q})$ is the Lie algebra of derivations of $\mathfrak{q}$, denoted simply by $\operatorname{Der}(\mathfrak{q})$.

The other Lie algebra required to define the actor crossed module is $\operatorname{Der}(\mathfrak{n}, \mathfrak{q}, \mu)$, the Lie algebra of derivations of the crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$, whose elements are all pairs (σ, θ), with $\sigma \in \operatorname{Der}(\mathfrak{n})$ and $\theta \in \operatorname{Der}(\mathfrak{q})$, such that

$$\theta\mu = \mu\sigma \qquad \text{and} \qquad \sigma([q, n]) = [q, \sigma(n)] + [\theta(q), n] \tag{2.2.5}$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$. The Lie structure of $\operatorname{Der}(\mathfrak{n}, \mathfrak{q}, \mu)$ is given by

$$\begin{aligned} (\sigma_1, \theta_1) + (\sigma_2, \theta_2) &= (\sigma_1 + \sigma_2, \theta_1 + \theta_2), \\ \lambda(\sigma, \theta) &= (\lambda\sigma, \lambda\theta), \\ [(\sigma_1, \theta_1), (\sigma_2, \theta_2)] &= ([\sigma_1, \sigma_2], [\theta_1, \theta_2]), \end{aligned}$$

for all $(\sigma_1, \theta_1), (\sigma_2, \theta_2), (\sigma, \theta) \in \operatorname{Der}(\mathfrak{n}, \mathfrak{q}, \mu)$, $\lambda \in K$. Recall that the bracket in the Lie algebra of derivations of a Lie algebra is described just before Example 1.2.17 and in the last paragraph from page 51.

There is a Lie homomorphism $\Delta\colon \operatorname{Der}(\mathfrak{q},\mathfrak{n}) \to \operatorname{Der}(\mathfrak{n},\mathfrak{q},\mu)$, given by

$$\Delta(d) = (d\mu, \mu d) \tag{2.2.6}$$

for all $d \in \operatorname{Der}(\mathfrak{q},\mathfrak{n})$. Observe that the Peiffer identity of $(\mathfrak{n},\mathfrak{q},\mu)$ guarantees that $d\mu$ is a derivation of $\mathfrak{n}$, while μd is a derivation of $\mathfrak{q}$ due to the equivariance. As for the condition (2.2.5), the first identity is obvious and the second one:

$$d\mu([q,n]) = d([q,\mu(n)]) = [d(q),\mu(n)] + [q, d\mu(n)] = [\mu d(q), n] + [q, d\mu(n)],$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$, follows from the equivariance, (2.2.4) and the Peiffer identity.

Furthermore, there is a Lie action of $\operatorname{Der}(\mathfrak{n},\mathfrak{q},\mu)$ on $\operatorname{Der}(\mathfrak{q},\mathfrak{n})$ defined by

$$[(\sigma,\theta), d] = \sigma d - d\theta. \tag{2.2.7}$$

for all $(\sigma,\theta) \in \operatorname{Der}(\mathfrak{n},\mathfrak{q},\mu)$, $d \in \operatorname{Der}(\mathfrak{q},\mathfrak{n})$. It is a matter of routine calculations to check that $\sigma d - d\theta$ is indeed a derivation from $\mathfrak{q}$ to $\mathfrak{n}$ and the identities

$$\begin{aligned}
[[(\sigma_1,\theta_1),(\sigma_2,\theta_2)], d] &= [(\sigma_1,\theta_1),[(\sigma_2,\theta_2),d]] - [(\sigma_2,\theta_2),[(\sigma_1,\theta_1),d]],\\
[(\sigma,\theta),[d_1,d_2]] &= [[(\sigma,\theta),d_1],d_2] + [d_1,[(\sigma,\theta),d_2]]
\end{aligned}$$

hold for all $(\sigma,\theta),(\sigma_1,\theta_1),(\sigma_2,\theta_2) \in \operatorname{Der}(\mathfrak{n},\mathfrak{q},\mu)$, $d, d_1, d_2 \in \operatorname{Der}(\mathfrak{q},\mathfrak{n})$. Moreover, we have that

$$\begin{aligned}
\Delta([(\sigma,\theta),d]) &= \Delta(\sigma d - d\theta) = ((\sigma d - d\theta)\mu, \mu(\sigma d - d\theta))\\
&= (\sigma d\mu - d\theta\mu, \mu\sigma d - \mu d\theta) = (\sigma d\mu - d\mu\sigma, \theta\mu d - \mu d\theta)\\
&= [(\sigma,\theta),(d\mu,\mu d)] = [(\sigma,\theta),\Delta(d)],
\end{aligned}$$

$$[\Delta(d_1), d_2] = [(d_1\mu, \mu d_1), d_2] = d_1\mu d_2 - d_2\mu d_1 = [d_1, d_2],$$

so $(\operatorname{Der}(\mathfrak{q},\mathfrak{n}), \operatorname{Der}(\mathfrak{n},\mathfrak{q},\mu), \Delta)$ is a Lie crossed module, called the actor crossed module of $(\mathfrak{n},\mathfrak{q},\mu)$ and denoted by $\operatorname{Act}(\mathfrak{n},\mathfrak{q},\mu)$.

Definition 2.2.3 ([27]). *An action of a Lie crossed module $(\mathfrak{m},\mathfrak{p},\nu)$ on another Lie crossed module $(\mathfrak{n},\mathfrak{q},\mu)$ is a morphism of Lie crossed modules from $(\mathfrak{m},\mathfrak{p},\nu)$ to $\operatorname{Act}(\mathfrak{n},\mathfrak{q},\mu)$.*

Example 2.2.4. *There is an action of a Lie crossed module $(\mathfrak{m},\mathfrak{p},\nu)$ on itself given by the morphism $(\varphi_\nu,\psi_\nu)\colon (\mathfrak{m},\mathfrak{p},\nu) \to \operatorname{Act}(\mathfrak{m},\mathfrak{p},\nu)$, where $\varphi_\nu(m)(p) = -[p,m]$ and $\psi_\nu(p) = (\sigma_p,\theta_p)$ with $\sigma_p(m) = [p,m]$ and $\theta_p(p') = [p,p']$ for all $m \in \mathfrak{m}$, $p,p' \in \mathfrak{p}$ (see [27] for more details).*

By analogy to Proposition 2.2.2, we give an equivalent description of an action of a Lie crossed module on another Lie crossed module in terms of equations.

Proposition 2.2.5. *Let $(\mathbb{m}, \mathbb{p}, \nu)$ and $(\mathbb{n}, \mathbb{q}, \mu)$ be Lie crossed modules. There is an action of $(\mathbb{m}, \mathbb{p}, \nu)$ on $(\mathbb{n}, \mathbb{q}, \mu)$ if and only if the following conditions hold:*

(i) There are actions of the Lie algebra $\mathbb{p}$ (and so $\mathbb{m}$) on the Lie algebras $\mathbb{n}$ and $\mathbb{q}$; μ is a $\mathbb{p}$-equivariant homomorphism, that is

$$\mu([p,n]) = [p, \mu(n)] \tag{LieEQ}$$

and the actions of $\mathbb{p}$ and $\mathbb{q}$ on $\mathbb{n}$ are compatible, that is

$$[[p,q],n] = [p,[q,n]] - [q,[p,n]] \tag{LieCOM}$$

for all $p \in \mathbb{p}$, $q \in \mathbb{q}$ and $n \in \mathbb{n}$.

(ii) There is a K-bilinear map $\xi \colon \mathbb{m} \times \mathbb{q} \to \mathbb{n}$ such that

$$\mu\xi(m,q) = [m,q], \tag{LieM1}$$
$$\xi\big(m, \mu(n)\big) = [m,n], \tag{LieM2}$$
$$[p, \xi(m,q)] = \xi([p,m],q) + \xi(m,[p,q]), \tag{LieM3}$$
$$\xi([m,m'],q) = [m, \xi(m',q)] - [m', \xi(m,q)], \tag{LieM4}$$
$$\xi(m,[q,q']) = [q, \xi(m,q')] - [q', \xi(m,q)], \tag{LieM5}$$

for all $m, m' \in \mathbb{m}$, $q, q' \in \mathbb{q}$, $n \in \mathbb{n}$, $p \in \mathbb{p}$.

Proof. Let us first assume that $(\mathbb{m}, \mathbb{p}, \nu)$ acts on $(\mathbb{n}, \mathbb{q}, \mu)$, that is there is a morphism of crossed modules

$$\begin{tikzcd} \mathbb{m} \arrow[r, "\nu"] \arrow[d, "\varphi"'] & \mathbb{p} \arrow[d, "\psi"] \\ \mathrm{Der}(\mathbb{q}, \mathbb{n}) \arrow[r, "\Delta"'] & \mathrm{Der}(\mathbb{n}, \mathbb{q}, \mu) \end{tikzcd} \tag{2.2.8}$$

Given $p \in \mathbb{p}$, let us denote $\psi(p)$ by (σ_p, θ_p), with $\sigma_p \in \mathrm{Der}(\mathbb{n})$, $\theta_p \in \mathrm{Der}(\mathbb{q})$ such that

$$\theta_p \mu = \mu \sigma_p \qquad \text{and} \qquad \sigma_p([q,n]) = [q, \sigma_p(n)] + [\theta_p(q), n]. \tag{2.2.9}$$

Due to (2.2.6), the commutativity of (2.2.8) can be expressed by the identity

$$(\varphi(m)\mu, \mu\varphi(m)) = (\sigma_{\nu(m)}, \theta_{\nu(m)}), \tag{2.2.10}$$

for all $m \in \mathbb{m}$. There is an action of $\mathbb{p}$ on $\mathbb{n}$ (respectively $\mathbb{q}$) given by $[p,n] = \sigma_p(n)$ (respectively $[p,q] = \theta_p(q)$) for all $p \in \mathbb{p}$, $n \in \mathbb{n}$ (respectively $q \in \mathbb{q}$), which induces an action of $\mathbb{m}$ on $\mathbb{n}$ (respectively $\mathbb{q}$) via ν. (LieEQ) and (LieCOM) follow from the first and the second identity in (2.2.9) respectively. Therefore (i) holds.

Concerning (ii), we can define $\xi(m,q)=\varphi(m)(q)$ for all $m \in \mathfrak{m}$, $q \in \mathfrak{q}$. Let us show that (LieM1), (LieM2) and (LieM4) follow from (2.2.10). Given $m, m' \in \mathfrak{m}$, $q \in \mathfrak{q}$ and $n \in \mathfrak{n}$,

$$\mu\xi(m,n)=\mu(\varphi(m)(q))=\theta_{\nu(m)}(q)=[\nu(m),q]=[m,q],$$

$$\xi(m,\mu(n))=\varphi(m)(\mu(n))=\sigma_{\nu(m)}(n)=[\nu(m),n]=[m,n],$$

$$\begin{aligned}\xi([m,m'],q)&=\varphi([m,m'])(q)=[\varphi(m),\varphi(m')](q)=(\varphi(m)\mu\varphi(m')-\varphi(m')\mu\varphi(m))(q)\\ &=\sigma_{\nu(m)}(\varphi(m')(q))-\sigma_{\nu(m')}(\varphi(m)(q))=[m,\xi(m',q)]-[m',\xi(m,q)].\end{aligned}$$

Regarding (LieM3), since (φ,ψ) is a morphism of Lie crossed modules, we know that $\varphi([p,m])=[(\sigma_p,\theta_p),\varphi(m)]$. Hence,

$$\begin{aligned}\xi([p,m],q)+\xi(m,[p,q])&=[(\sigma_p,\theta_p),\varphi(m)](q)+\varphi(m)(\theta_p(q))\\ &=\sigma_p\varphi(m)(q)-\varphi(m)\theta_p(q)+\varphi(m)\theta_p(q)=[p,\xi(m,q)],\end{aligned}$$

for any $m \in \mathfrak{m}$, $p \in \mathfrak{p}$, $q \in \mathfrak{q}$. Finally, (LieM5) follows easily from the fact that $\varphi(m)$ is a derivation from $\mathfrak{q}$ to $\mathfrak{n}$ for any $m \in \mathfrak{m}$.

Now, let us prove the converse statement. From (i), $\mathfrak{p}$ acts on $\mathfrak{n}$ and $\mathfrak{q}$, that is there are two bilinear maps $\mathfrak{p}\times\mathfrak{n}\to\mathfrak{n}$, $(p,n)\mapsto[p,n]$ and $\mathfrak{p}\times\mathfrak{q}\to\mathfrak{q}$, $(p,q)\mapsto[p,q]$ such that

$$[[p,p'],n]=[p,[p',n]]-[p',[p,n]], \tag{2.2.11}$$
$$[p,[n,n']]=[[p,n],n']+[n,[p,n']], \tag{2.2.12}$$

and

$$[[p,p'],q]=[p,[p',q]]-[p',[p,q]], \tag{2.2.13}$$
$$[p,[q,q']]=[[p,q],q']+[q,[p,q']], \tag{2.2.14}$$

for all $n,n' \in \mathfrak{n}$, $p,p' \in \mathfrak{p}$, $q,q' \in \mathfrak{q}$. It is possible to define a morphism of crossed modules (φ,ψ) from $(\mathfrak{m},\mathfrak{p},\nu)$ to $\mathrm{Act}(\mathfrak{n},\mathfrak{q},\mu)$ as follows. Given $m \in \mathfrak{m}$, $\varphi(m)(q)=\xi(m,q)$ for all $q \in \mathfrak{q}$. For any $p \in \mathfrak{p}$, $\psi(p)=(\sigma_p,\theta_p)$, with $\sigma_p(n)=[p,n]$ and $\theta_p(q)=[p,q]$ for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$.

It follows directly from (LieM5) that $\varphi(m)$ is a derivation from $\mathfrak{q}$ to $\mathfrak{n}$ for all $m \in \mathfrak{m}$. Moreover, given $m,m' \in \mathfrak{m}$ and $q \in \mathfrak{q}$,

$$\begin{aligned}[\varphi(m),\varphi(m')](q)&=\varphi(m)\mu\varphi(m')(q)-\varphi(m')\mu\varphi(m)(q)\\ &=\xi(m,\mu\xi(m',q))-\xi(m',\mu\xi(m,q))\\ &=[m,\xi(m',q)]-[m',\xi(m,q)]\\ &=\xi([m,m'],q)=\varphi([m,m'])(q),\end{aligned}$$

due to (LieM2) and (LieM4). Hence, φ is a morphism of Lie algebras. As for ψ, given $p \in \mathbb{p}$, σ_p (respectively θ_p) is a derivation of $\mathbb{n}$ (respectively $\mathbb{q}$) due to (2.2.12) (respectively (2.2.14)). (LieEQ) and (LieCOM) guarantee that the pair (σ_p, θ_p) satisfies the identities (2.2.9). Also, it can be readily checked that ψ is a Lie homomorphism by using (2.2.11) and (2.2.13).

Recall that $\Delta\varphi(m) = (\varphi(m)\mu, \mu\varphi(m))$ and $\psi\nu(m) = (\sigma_{\nu(m)}, \theta_{\nu(m)})$ for any $m \in \mathbb{m}$. By making use of (LieM1), (LieM2) and the fact that $\mathbb{m}$ acts on $\mathbb{n}$ and $\mathbb{q}$ via ν,

$$\varphi(m)\mu(n) = \xi(m, \mu(n)) = [m, n] = [\nu(m), n] = \sigma_{\nu(m)}(n),$$
$$\mu\varphi(m)(q) = \mu\xi(m, q) = [m, q] = [\nu(m), q] = \theta_{\nu(m)}(q),$$

for all $n \in \mathbb{n}$, $q \in \mathbb{q}$. Therefore, $\Delta\varphi = \psi\nu$. Furthermore, given $m \in \mathbb{m}$ and $p \in \mathbb{p}$, due to (2.2.7), $[\psi(p), \varphi(m)] = [(\sigma_p, \theta_p), \varphi(m)] = \sigma_p\varphi(m) - \varphi(m)\theta_p$. On the other hand, by using (LieM3), we get that

$$\varphi([p, m])(q) = \xi([p, m], q) = [p, \xi(m, q)] - \xi(m, [p, q]) = (\sigma_p\varphi(m) - \varphi(m)\theta_p)(q),$$

for any $q \in \mathbb{q}$. Hence, $\varphi([p, m]) = [\psi(p), \varphi(m)]$ for all $m \in \mathbb{m}$, $p \in \mathbb{p}$. □

Let $(\mathbb{m}, \mathbb{p}, \nu)$ be a Lie crossed module acting on a Lie crossed module $(\mathbb{n}, \mathbb{q}, \mu)$. By Proposition 2.2.5 there are Lie actions of $\mathbb{m}$ on $\mathbb{n}$ and of $\mathbb{p}$ on $\mathbb{q}$, so it makes sense to consider the semidirect products of Lie algebras $\mathbb{n} \rtimes \mathbb{m}$ and $\mathbb{q} \rtimes \mathbb{p}$. Furthermore, we have the following result.

Lemma 2.2.6. *There is an action of the Lie algebra $\mathbb{q} \rtimes \mathbb{p}$ on the Lie algebra $\mathbb{n} \rtimes \mathbb{m}$, given by*

$$[(q, p), (n, m)] = ([q, n] + [p, n] - \xi(m, q), [p, m]) \tag{2.2.15}$$

for all $(q, p) \in \mathbb{q} \rtimes \mathbb{p}$, $(n, m) \in \mathbb{n} \rtimes \mathbb{m}$, with ξ as in Proposition 2.2.5. Moreover, the Lie homomorphism $(\mu, \nu)\colon \mathbb{n} \rtimes \mathbb{m} \to \mathbb{q} \rtimes \mathbb{p}$, given by

$$(\mu, \nu)(n, m) = \big(\mu(n), \nu(m)\big)$$

for all $(n, m) \in \mathbb{n} \rtimes \mathbb{m}$, is a Lie crossed module together with the previous action.

Proof. Firstly, it is necessary to prove that (2.2.15) describes a Lie action, that is the identities

$$[[(q, p), (q', p')], (n, m)] = [(q, p), [(q', p'), (n, m)]] - [(q', p'), [(q, p), (n, m)]],$$
$$[(q, p), [(n, m), (n', m')]] = [[(q, p), (n, m)], (n', m')] + [(n, m), [(q, p), (n', m')]],$$

for all $(q, p), (q', p') \in \mathbb{q} \rtimes \mathbb{p}$, $(n, m), (n', m') \in \mathbb{n} \rtimes \mathbb{m}$. It is easy to check the first identity by making use of (LieCOM), (LieM3), (LieM5) and the analogous identity for the actions of $\mathbb{p}$ on $\mathbb{n}$ and $\mathbb{q}$ on $\mathbb{n}$. Regarding the second identity, straightforward calculations give rise to

$$\begin{aligned}
[(q,p),[(n,m),(n',m')]] = (&\underbrace{[q,[n,n']]}_{(1)}\underbrace{+[q,[m,n']]}_{(2)}\underbrace{-[q,[m',n]]}_{(3)}\underbrace{+[p,[n,n']]}_{(4)}\\
&\underbrace{+[p,[m,n']]}_{(5)}\underbrace{-[p,[m',n]]}_{(6)}\underbrace{-\xi([m,m'],q)}_{(7)},\underbrace{[p,[m,m']]}_{(8)}),\\
[[(q,p),(n,m)],(n',m')] = (&\underbrace{[[q,n],n']}_{(1')}\underbrace{-[\xi(m,q),n']}_{(2')}\underbrace{-[m',[q,n]]}_{(3')}\underbrace{+[[p,n],n']}_{(4')}\\
&\underbrace{+[[p,m],n']}_{(5')}\underbrace{-[m',[p,n]]}_{(6')}\underbrace{+[m',\xi(m,q)]}_{(7')},\underbrace{[[p,m],m']}_{(8')}),\\
[(n,m),[(q,p),(n',m')]] = (&\underbrace{[n,[q,n']]}_{(1'')}\underbrace{+[m,[q,n']]}_{(2'')}\underbrace{-[n,\xi(m',q)]}_{(3'')}\underbrace{+[n,[p,n']]}_{(4'')}\\
&\underbrace{+[m,[p,n']]}_{(5'')}\underbrace{-[[p,m'],n]}_{(6'')}\underbrace{-[m,\xi(m',q)]}_{(7'')},\underbrace{[m,[p,m']]}_{(8'')}).
\end{aligned}$$

It follows easily that $(i) = (i') + (i'')$ for $i = 1, 4, 8$, due to the action of $\mathbb{q}$ on $\mathbb{n}$ and the actions of $\mathbb{p}$ on $\mathbb{n}$ and $\mathbb{m}$. For $i = 7$, the identity follows from (LieM4). For $i = 2, 3$ it is important to note that

$$\begin{aligned}
[q,[m,n']] = [q,\xi(m,\mu(n'))] &= \xi(m,[q,\mu(n')]) + [\mu(n'),\xi(m,q)]\\
&= \xi(m,\mu([q,n'])) + [n',\xi(m,q)] = [m,[q,n']] - [\xi(m,q),n'],
\end{aligned}$$

immediately from (LieM2), (LieM5), the antisymmetry of the bracket in $\mathbb{n}$ and the equivariance and the Peiffer identity of $(\mathbb{n},\mathbb{q},\mu)$. Finally, for $i = 5, 6$, bearing in mind that the action of $\mathbb{m}$ on $\mathbb{n}$ is induced by the action of $\mathbb{p}$ on $\mathbb{n}$ via ν, we have that

$$\begin{aligned}
[p,[m,n']] = [p,[\nu(m),n']] &= [[p,\nu(m)],n'] + [\nu(m),[p,n']]\\
&= [\nu([p,m]),n'] + [m,[p,n']] = [[p,m],n'] + [m,[p,n']],
\end{aligned}$$

directly from the equivariance of $(\mathbb{m},\mathbb{p},\nu)$. Checking that (μ,ν) is a Lie homomorphism that satisfies equivariance and the Peiffer identity is also a matter of routine calculations. □

The Lie crossed module $\big(\mathbb{n} \rtimes \mathbb{m}, \mathbb{q} \rtimes \mathbb{p}, (\mu,\nu)\big)$ is called the semidirect product of the Lie crossed modules $(\mathbb{n},\mathbb{q},\mu)$ and $(\mathbb{m},\mathbb{p},\nu)$. Note that the semidirect product determines an obvious split extension of $(\mathbb{m},\mathbb{p},\nu)$ by $(\mathbb{n},\mathbb{q},\mu)$

$$(0,0,0) \longrightarrow (\mathbb{n},\mathbb{q},\mu) \longrightarrow \big(\mathbb{n} \rtimes \mathbb{m}, \mathbb{q} \rtimes \mathbb{p}, (\mu,\nu)\big) \rightleftarrows (\mathbb{m},\mathbb{p},\nu) \longrightarrow (0,0,0)$$

Conversely, any split extension of $(\mathbb{m}, \mathbb{p}, \nu)$ by $(\mathbb{n}, \mathbb{q}, \mu)$ is isomorphic to their semidirect product, where the action of $(\mathbb{m}, \mathbb{p}, \nu)$ on $(\mathbb{n}, \mathbb{q}, \mu)$ is induced by the splitting morphism (see [26]).

2.2.3 Actor crossed module of Leibniz algebras

In the previous two subsections we have seen that the actor in the categories of groups and Lie algebras has its corresponding 2-dimensional analogue. Given a Leibniz algebra $\mathfrak{m}$, $\mathrm{Bider}(\mathfrak{m})$ is not necessarily the actor of $\mathfrak{m}$, since the set of actions of $\mathrm{Bider}(\mathfrak{m})$ on $\mathfrak{m}$ is not a set of derived actions in general. In fact, the actor of a Leibniz algebra does not always exist. Nevertheless, under certain conditions, $\mathrm{Bider}\,\mathfrak{m}$ is indeed the actor of $\mathfrak{m}$. In this section we will show that under similar conditions, the actor crossed module can also be constructed.

Let us assume for the rest of this subsection that $(\mathfrak{n}, \mathfrak{q}, \mu)$ is a Leibniz crossed module.

Definition 2.2.7. *The set of biderivations from* $\mathfrak{q}$ *to* $\mathfrak{n}$*, denoted by* $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$*, consists of all the pairs* (d, D) *of* K*-linear maps,* $d, D\colon \mathfrak{q} \to \mathfrak{n}$*, such that*

$$d([q, q']) = [d(q), q'] + [q, d(q')], \tag{2.2.16}$$

$$D([q, q']) = [D(q), q'] - [D(q'), q], \tag{2.2.17}$$

$$[q, d(q')] = [q, D(q')], \tag{2.2.18}$$

for all $q, q' \in \mathfrak{q}$.

Analogously to the Lie case, we translate the notion of a biderivation of a Leibniz algebra into a biderivation between two Leibniz algebras via the action.

Given $n \in \mathfrak{n}$, the pair of K-linear maps $(\mathrm{ad}(n), \mathrm{Ad}(n))$, where $\mathrm{ad}(n)(q) = -[q, n]$ and $\mathrm{Ad}(n)(q) = [n, q]$ for all $q \in \mathfrak{q}$, is clearly a biderivation from $\mathfrak{q}$ to $\mathfrak{n}$, so $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ is not an empty set. Observe that if we consider the crossed module $(\mathfrak{q}, \mathfrak{q}, \mathrm{id}_{\mathfrak{q}})$, $\mathrm{Bider}(\mathfrak{q}, \mathfrak{q})$ is exactly the set of biderivations of $\mathfrak{q}$.

Directly from the definition and the fact that $(\mathfrak{n}, \mathfrak{q}, \mu)$ is a Leibniz crossed module, we get the following result.

Lemma 2.2.8. *Let* $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$*. Then* $(d\mu, D\mu) \in \mathrm{Bider}(\mathfrak{n})$ *and* $(\mu d, \mu D) \in \mathrm{Bider}(\mathfrak{q})$*.*

Proof. Let us show that $(d\mu, D\mu) \in \mathrm{Bider}(\mathfrak{n})$. It is obvious that $d\mu$ and $D\mu$ are K-linear maps from $\mathfrak{n}$ to $\mathfrak{n}$. Furthermore, given $n, n' \in \mathfrak{n}$,

$$\begin{aligned}
d\mu([n, n']) &= d([\mu(n), \mu(n')]) = [d\mu(n), \mu(n')] + [\mu(n), d\mu(n')] \\
&= [d\mu(n), n'] + [n, d\mu(n')], \\
D\mu([n, n']) &= D([\mu(n), \mu(n')]) = [D\mu(n), \mu(n')] - [D\mu(n'), \mu(n)] \\
&= [D\mu(n), n'] - [D\mu(n'), n], \\
[n, d\mu(n')] &= [\mu(n), d\mu(n')] = [\mu(n), D\mu(n')] = [n, D\mu(n')],
\end{aligned}$$

as a straightforward consequence of the Peiffer identity, the properties of the biderivation (d, D) and the fact that μ is morphism of Leibniz algebras. In order to prove that $(\mu d, \mu D) \in \mathrm{Bider}(\mathfrak{q})$ it is necessary to make use of the equivariance of $(\mathfrak{n}, \mathfrak{q}, \mu)$ instead of the Peiffer identity, but the procedure is quite similar. $\square$

Also from Definition 2.2.7 we get the following result.

Lemma 2.2.9. *Let* (d_1, D_1), $(d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. *Then*

$$[D_1\mu d_2(q), q'] = [D_1\mu D_2(q), q'],$$
$$[q, D_1\mu d_2(q')] = [q, D_1\mu D_2(q')],$$

for all $q, q' \in \mathfrak{q}$.

Proof. Let $q, q' \in \mathfrak{q}$ and $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. According to the identity (2.2.18) for (d_2, D_2), $[q', d_2(q)] = [q', D_2(q)]$, so $D_1\mu([q', d_2(q)]) = D_1\mu([q', D_2(q)])$. Due to (2.2.17) and the equivariance of $(\mathfrak{q}, \mathfrak{n}, \mu)$, one can easily derive that

$$[D_1(q'), \mu d_2(q)] - [D_1\mu d_2(q), q'] = [D_1(q'), \mu D_2(q)] - [D_1\mu D_2(q), q'].$$

By the Peiffer identity and (2.2.18) for (d_2, D_2), $[D_1(q'), \mu d_2(q)] = [D_1(q'), \mu D_2(q)]$. Therefore $[D_1\mu d_2(q), q'] = [D_1\mu D_2(q), q']$.

Analogously, $d_1\mu([q, d_2(q')]) = d_1\mu([q, D_2(q')])$. By the equivariance and (2.2.16) we get that

$$[d_1(q), \mu d_2(q')] + [q, d_1\mu d_2(q')] = [d_1(q), \mu D_2(q')] + [q, d_1\mu D_2(q')].$$

Due to the Peiffer identity and (2.2.18) for (d_2, D_2), $[d_1(q), \mu d_2(q')] = [d_1(q), \mu D_2(q')]$, so $[q, d_1\mu d_2(q')] = [q, d_1\mu D_2(q')]$ and using (2.2.18) again, this time for (d_1, D_1), we get that $[q, D_1\mu d_2(q')] = [q, D_1\mu D_2(q')]$. $\square$

Observe that Lemma 2.1.7 follows directly from the previous lemma for the particular case of the crossed module $(\mathfrak{m}, \mathfrak{m}, \mathrm{id}_{\mathfrak{m}})$.

$\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ has an obvious K-module structure. Regarding its Leibniz structure, it is described in the next proposition.

Proposition 2.2.10. $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ *is a Leibniz algebra with the bracket given by*

$$[(d_1, D_1), (d_2, D_2)] = (d_1\mu d_2 - d_2\mu d_1, D_1\mu d_2 - d_2\mu D_1) \tag{2.2.19}$$

for all $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$.

Proof. Let $(d_1, D_1), (d_2, D_2) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. In the first place, we have to confirm that $[(d_1, D_1), (d_2, D_2)]$ is a biderivation from $\mathfrak{q}$ to $\mathfrak{n}$. Linearity is obvious. The identities (2.2.16) and (2.2.17) for $[(d_1, D_1), (d_2, D_2)]$ follow easily from the same identities for (d_1, D_1) and (d_2, D_2) along with the equivariance and the Peiffer identity of $(\mathfrak{n}, \mathfrak{q}, \mu)$. Concerning (2.2.18), it is an immediate consequence of the second identity in Lemma 2.2.9 together with (2.2.18) for both (d_1, D_1) and (d_2, D_2). Checking the Leibniz identity is just a matter of routine calculations. $\square$

As an analogue to the Lie algebra of derivations of a crossed module, we state the following definition.

Definition 2.2.11. *The set of biderivations of the Leibniz crossed module* $(\mathfrak{n}, \mathfrak{q}, \mu)$*, denoted by* $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$*, consists of all the quadruples* $((\sigma_1, \theta_1), (\sigma_2, \theta_2))$ *such that*

$$(\sigma_1, \theta_1) \in \mathrm{Bider}(\mathfrak{n}) \quad \textit{and} \quad (\sigma_2, \theta_2) \in \mathrm{Bider}(\mathfrak{q}), \tag{2.2.20}$$
$$\mu\sigma_1 = \sigma_2\mu \quad \textit{and} \quad \mu\theta_1 = \theta_2\mu, \tag{2.2.21}$$
$$\sigma_1([q, n]) = [\sigma_2(q), n] + [q, \sigma_1(n)], \tag{2.2.22}$$
$$\sigma_1([n, q]) = [\sigma_1(n), q] + [n, \sigma_2(q)], \tag{2.2.23}$$
$$\theta_1([q, n]) = [\theta_2(q), n] - [\theta_1(n), q], \tag{2.2.24}$$
$$\theta_1([n, q]) = [\theta_1(n), q] - [\theta_2(q), n], \tag{2.2.25}$$
$$[q, \sigma_1(n)] = [q, \theta_1(n)], \tag{2.2.26}$$
$$[n, \sigma_2(q)] = [n, \theta_2(q)], \tag{2.2.27}$$

for all $n \in \mathfrak{n}$*,* $q \in \mathfrak{q}$*.*

Note that (2.2.22)–(2.2.27) are very similar to (2.2.16)–(2.2.18), but here the two maps that define the action of $\mathfrak{q}$ on $\mathfrak{n}$ have to be considered, which is the reason why the identities appear duplicated.

Given $q \in \mathfrak{q}$, it can be readily checked that $((\sigma_1^q, \theta_1^q), (\sigma_2^q, \theta_2^q))$, where

$$\sigma_1^q(n) = -[n, q], \qquad \theta_1^q(n) = [q, n],$$
$$\sigma_2^q(q') = -[q', q], \qquad \theta_2^q(q') = [q, q'],$$

is a biderivation of the crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$.

The following result is similar to Lemmas 2.1.7 and 2.2.9, but it combines elements in $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ and $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$.

Lemma 2.2.12. *Let* $((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2')) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ *and* $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$*. Then*

$$\begin{array}{lll}
[D\sigma_2(q), q'] = [D\theta_2(q), q'], & [D\sigma_2(q), n] = [D\theta_2(q), n], & [\theta_1\sigma_1'(n), q] = [\theta_1\theta_1'(n), q], \\
[q, D\sigma_2(q')] = [q, D\theta_2(q')], & [n, D\sigma_2(q)] = [n, D\theta_2(q)], & [q, \theta_1\sigma_1'(n)] = [q, \theta_1\theta_1'(n)], \\
[\theta_1 d(q), q'] = [\theta_1 D(q), q'], & [\theta_1 d(q), n] = [\theta_1 D(q), n], & [\theta_2\sigma_2'(q), n] = [\theta_2\theta_2'(q), n], \\
[q, \theta_1 d(q')] = [q, \theta_1 D(q')], & [n, \theta_1 d(q)] = [n, \theta_1 D(q)], & [n, \theta_2\sigma_2'(q)] = [n, \theta_2\theta_2'(q)],
\end{array}$$

for all $n \in \mathfrak{n}$*,* $q, q' \in \mathfrak{q}$*.*

Proof. Let us begin with the first column. Let $q, q' \in \mathfrak{q}$, $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ and $((\sigma_1, \theta_1), (\sigma_2, \theta_2)) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$. Since (σ_2, θ_2) is a biderivation of $\mathfrak{q}$, we have that $[q', \sigma_2(q)] = [q', \theta_2(q)]$. Therefore $D([q', \sigma_2(q)]) = D([q', \theta_2(q)])$. Directly from (2.2.17), we get that

$$[D(q'), \sigma_2(q)] - [D\sigma_2(q), q'] = [D(q'), \theta_2(q)] - [D\theta_2(q), q'].$$

Recall that $D(q') \in \mathfrak{n}$. Therefore, due to (2.2.27), $[D(q'), \sigma_2(q)] = [D(q'), \theta_2(q)]$. Hence, $[D\sigma_2(q), q'] = [D\theta_2(q), q']$.

Similarly, $d([q, \sigma_2(q')]) = d([q, \theta_2(q')])$. By applying (2.2.16), one can easily derive that

$$[d(q), \sigma_2(q')] + [q, d\sigma_2(q')] = [d(q), \theta_2(q')] + [q, d\theta_2(q')].$$

Due to (2.2.27), $[d(q), \sigma_2(q')] = [d(q), \theta_2(q')]$. Therefore $[q, d\sigma_2(q')] = [q, d\theta_2(q')]$, and applying (2.2.18), we get that $[q, D\sigma_2(q')] = [q, D\theta_2(q')]$.

Regarding the third identity, $[q', d(q)] = [q', D(q)]$ due to (2.2.18). Consequently $\theta_1([q', d(q)]) = \theta_1([q', D(q)])$. By making use of (2.2.24), we get that

$$[\theta_2(q'), d(q)] - [\theta_1 d(q), q'] = [\theta_2(q'), D(q)] - [\theta_1 D(q), q'].$$

Due to (2.2.18), $[\theta_2(q'), d(q)] = [\theta_2(q'), D(q)]$. Therefore $[\theta_1 d(q), q'] = [\theta_1 D(q), q']$.

In order to check last identity in the first column, we have to apply σ_1 in both sides of the identity $[q, d(q')] = [q, D(q')]$. In this way, directly from (2.2.22) we get that

$$[\sigma_2(q), d(q')] + [q, \sigma_1 d(q')] = [\sigma_2(q), D(q')] + [q, \sigma_1 D(q')],$$

but $[\sigma_2(q), d(q')] = [\sigma_2(q), D(q')]$ due to (2.2.18), so $[q, \sigma_1 d(q')] = [q, \sigma_1 D(q')]$. Finally, due to (2.2.26), $[q, \theta_1 d(q')] = [q, \theta_1 D(q')]$.

The identities in the second column follow immediately from the ones in the first column and the Peiffer identity. Regarding the third column, the procedure in order to prove that those identities hold is very similar to the one used for the first column, but involving only properties of the biderivations $((\sigma_1, \theta_1), (\sigma_2, \theta_2))$ and $((\sigma_1', \theta_1'), (\sigma_2', \theta_2'))$. $\square$

Remark 2.2.13. *Observe that the identities in the right column of the previous lemma involve the action of $\mathfrak{q}$ on $\mathfrak{n}$. Of course, given $((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2')) \in$ $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$, (σ_1, θ_1) and (σ_1', θ_1') (respectively (σ_2, θ_2) and (σ_2', θ_2')) also verify the identities from Lemma 2.1.7, since they are biderivations of $\mathfrak{n}$ (respectively $\mathfrak{q}$).*

The K-module structure of $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ is evident, while its Leibniz structure is described in the proposition immediately below.

Proposition 2.2.14. $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ *is a Leibniz algebra with the bracket given by*

$$\begin{aligned}[]
[((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2'))] &= ([(\sigma_1, \theta_1), (\sigma_1', \theta_1')], [(\sigma_2, \theta_2), (\sigma_2', \theta_2')]) \\
&= ((\sigma_1\sigma_1' - \sigma_1'\sigma_1, \theta_1\sigma_1' - \sigma_1'\theta_1), (\sigma_2\sigma_2' - \sigma_2'\sigma_2, \theta_2\sigma_2' - \sigma_2'\theta_2)) \quad (2.2.28)
\end{aligned}$$

for all $((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2')) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$.

Proof. Firstly, we need to verify that the bracket is well defined, that is (2.2.28) has to satisfy (2.2.20)–(2.2.27) for all $((\sigma_1, \theta_1), (\sigma_2, \theta_2)), ((\sigma_1', \theta_1'), (\sigma_2', \theta_2')) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$. The identities in (2.2.21) are obvious, while (2.2.20) follows from the definition of the brackets in $\mathrm{Bider}(\mathfrak{n})$ and $\mathrm{Bider}(\mathfrak{q})$ (see (2.1.6)). The identities (2.2.22)–(2.2.25) are fairly straightforward, so we leave them to the reader.

Regarding (2.2.26), given $n \in \mathfrak{n}$ and $q \in \mathfrak{q}$, due to (2.2.26) for (σ_1, θ_1), we have that $[q, \sigma_1\sigma_1'(n)] = [q, \theta_1\sigma_1'(n)]$. Also, as a consequence of (2.2.26) for (σ_1', θ_1') and the second identity in the third column from Lemma 2.2.12, $[q, \sigma_1'\sigma_1(n)] = [q, \sigma_1'\theta_1(n)]$. Hence, $[q, (\sigma_1\sigma_1' - \sigma_1'\sigma_1)(n)] = [q, (\theta_1\sigma_1' - \sigma_1'\theta_1)(n)]$. (2.2.27) can be proved analogously.

The Leibniz identity follows immediately from the homonymous identity satisfied by the brackets in $\mathrm{Bider}(\mathfrak{n})$ and $\mathrm{Bider}(\mathfrak{q})$. $\square$

The next step is to construct a Leibniz algebra homomorphism from $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ to $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$.

Proposition 2.2.15. *The K-linear map $\Delta\colon \mathrm{Bider}(\mathfrak{q}, \mathfrak{n}) \to \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$, $(d, D) \mapsto ((d\mu, D\mu), (\mu d, \mu D))$ is a morphism of Leibniz algebras.*

Proof. Let $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. In the first place, it is necessary to prove that $\Delta(d, D)$ is indeed a biderivation of the crossed module $(\mathfrak{n}, \mathfrak{q}, \mu)$, that is we need to check that $((d\mu, D\mu), (\mu d, \mu D))$ satisfies the conditions from Definition 2.2.11. Condition (2.2.20) is an immediate consequence of Lemma 2.2.8, while (2.2.21) is obvious. Conditions (2.2.22)–(2.2.25) can be easily checked by making use of the equivariance, the Peiffer identity and the properties of (d, D). As an example, let us show how to prove (2.2.24). Given $n \in \mathfrak{n}$ and $q \in \mathfrak{q}$ we have that

$$D\mu([q, n]) = D([q, \mu(n)]) = [D(q), \mu(n)] - [D\mu(n), q] = [\mu D(q), n] - [D\mu(n), q].$$

Regarding (2.2.26), it follows immediately from the fact that (d, D) is a biderivation from $\mathfrak{q}$ to $\mathfrak{n}$, while it is necessary to use once again the Peiffer identity in order to prove that (2.2.27) holds. Namely:

$$[n, \mu d(q)] = [\mu(n), d(q)] = [\mu(n), D(q)] = [n, \mu D(q)],$$

for any $n \in \mathfrak{n}$, $q \in \mathfrak{q}$.

Checking that $\Delta([(d_1, D_1), (d_2, D_2)]) = [\Delta(d_1, D_1), \Delta(d_2, D_2)]$ only requires patience and a correct usage of both of the brackets in $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ and $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$. $\square$

Since we aspire to make Δ into a Leibniz crossed module, we need to define an action of $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ on $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$.

Theorem 2.2.16. *There is a Leibniz action of $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ on $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ given by:*

$$[((\sigma_1, \theta_1), (\sigma_2, \theta_2)), (d, D)] = (\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2), \tag{2.2.29}$$

$$[(d, D), ((\sigma_1, \theta_1), (\sigma_2, \theta_2))] = (d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D), \tag{2.2.30}$$

for all $((\sigma_1, \theta_1), (\sigma_2, \theta_2)) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$, $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. The Leibniz homomorphism Δ (see Proposition 2.2.15) together with the above action is a Leibniz crossed module.

Proof. Firstly, it is necessary to prove that the definition of the action makes sense. Let $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ and $((\sigma_1, \theta_1), (\sigma_2, \theta_2)) \in \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$. Linearity is not a problem, due to the nature of the maps involved. Checking that both $(\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2)$ and $(d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D)$ satisfy conditions (2.2.16) and (2.2.17) requires the combined use of the properties satisfied by the elements in $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ and (d, D), but calculations are fairly straightforward. As an example, we show how to prove that $(\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2)$ verifies (2.2.16). Let $q, q' \in \mathfrak{q}$. Then

$$\begin{aligned}
(\sigma_1 d - d\sigma_2)([q, q']) &= \sigma_1([d(q), q'] + [q, d(q')]) - d([\sigma_2(q), q'] + [q, \sigma_2(q')]) \\
&= [\sigma_1 d(q), q'] + [d(q), \sigma_2(q')] + [\sigma_2(q), d(q')] + [q, \sigma_1 d(q')] \\
&\quad - [d\sigma_2(q), q'] - [\sigma_2(q), d(q')] - [d(q), \sigma_2(q')] - [q, d\sigma_2(q')] \\
&= [(\sigma_1 d - d\sigma_2)(q), q'] + [q, (\sigma_1 d - d\sigma_2)(q')].
\end{aligned}$$

As for condition (2.2.18), in the case of $(\sigma_1 d - d\sigma_2, \theta_1 d - d\theta_2)$, it follows from (2.2.26), the identity (2.2.18) for (d, D) and the second identity in the first column from Lemma 2.2.12. Namely,

$$\begin{aligned}
[q, (\sigma_1 d - d\sigma_2)(q')] &= [q, \sigma_1 d(q')] - [q, d\sigma_2(q')] = [q, \theta_1 d(q')] - [q, D\sigma_2(q')] \\
&= [q, \theta_1 d(q')] - [q, D\theta_2(q')] = [q, \theta_1 d(q')] - [q, d\theta_2(q')],
\end{aligned}$$

for all $q, q' \in \mathfrak{q}$. A similar procedure allows to prove that $(d\sigma_2 - \sigma_1 d, D\sigma_2 - \sigma_1 D)$ satisfies condition (2.2.18) as well.

The next step is to check that these maps satisfy the axioms of a Leibniz action (see Definition 1.2.37). Routine calculations show that all the identities follow directly from (2.2.29) and (2.2.30) together with the definition of the brackets in $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ and $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. Additionally, for those identities involving one element in $\mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ and two elements in $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$, that is (1), (2) and (3) from Definition 1.2.37, it is necessary to use (2.2.21) in order to cancel (identity (1)) or relate (identities (2) and (3)) the corresponding addends.

It only remains to prove that Δ satisfies the equivariance and the Peiffer identity. It is immediate to check that

$$\begin{aligned}
\Delta([((\sigma_1, \theta_1), (\sigma_2, \theta_2)), (d, D)]) = ((&\sigma_1 d\mu - d\sigma_2\mu, \theta_1 d\mu - d\theta_2\mu), \\
&(\mu\sigma_1 d - \mu d\sigma_2, \mu\theta_1 d - \mu d\theta_2)),
\end{aligned} \tag{2.2.31}$$

while

$$\begin{aligned}
[((\sigma_1, \theta_1), (\sigma_2, \theta_2)), \Delta(d, D)] = ((&\sigma_1 d\mu - d\mu\sigma_1, \theta_1 d\mu - d\mu\theta_1), \\
&(\sigma_2\mu d - \mu d\sigma_2, \theta_2\mu d - \mu d\theta_2)).
\end{aligned} \tag{2.2.32}$$

Condition (2.2.21) guarantees that (2.2.31) = (2.2.32). Concerning the identity $\Delta([(d, D), ((\sigma_1, \theta_1), (\sigma_2, \theta_2))]) = [\Delta(d, D), ((\sigma_1, \theta_1), (\sigma_2, \theta_2))]$, it can be checked similarly. The Peiffer identity follows immediately from (2.2.29) and (2.2.30) along the definition of Δ and the bracket in $\mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$. $\square$

Bearing in mind the ease of the generalization of the actor in **Gr** (respectively **Lie**) to **XGr** (respectively **XLie**), together with the role of $\mathrm{Bider}(\mathfrak{m})$ in regard to any Leibniz algebra $\mathfrak{m}$, it makes sense to consider $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}), \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu), \Delta)$ as a candidate for general actor in **XLb**, or even actor under certain conditions (see Proposition 2.1.8). However, it would be reckless to define an action of a Leibniz crossed module $(\mathfrak{m},\mathfrak{p},\eta)$ on $(\mathfrak{n},\mathfrak{q},\mu)$ as a morphism from $(\mathfrak{m},\mathfrak{p},\eta)$ to the Leibniz crossed module $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}), \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu), \Delta)$, since we cannot ensure that the mentioned morphism induces a set of derived actions of $(\mathfrak{m},\mathfrak{p},\eta)$ on $(\mathfrak{n},\mathfrak{q},\mu)$, in the sense of the definition given by Orzech [74].

In [22, Proposition 2.1] the authors give an equivalent description of an action of a crossed module of groups in terms of equations (it appears in Subsection 2.2.1 as Proposition 2.2.2). Additionally, we give an analogous description of an action of a Lie crossed module (see Proposition 2.2.5). Furthermore, a closer look at the proofs of Propositions 2.2.2 and 2.2.5 reveals that the set of equations is no more than a rearrangement of the conditions satisfied by all the elements involved in the definition of a morphism from the corresponding crossed module to the corresponding actor.

The previous issues determined our approach to the problem. We considered a morphism from a Leibniz crossed module $(\mathfrak{m},\mathfrak{p},\eta)$ to $(\mathrm{Bider}(\mathfrak{q},\mathfrak{n}), \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu), \Delta)$, which will be denoted by $\overline{\mathrm{Act}}(\mathfrak{n},\mathfrak{q},\mu)$ from now on, and unravelled all the properties satisfied by the mentioned morphism, transforming them into a set of equations, similar to those from Propositions 2.2.2 and 2.2.5. Then we checked that the existence of that set of equations is equivalent to the existence of a morphism of Leibniz crossed modules from $(\mathfrak{m},\mathfrak{p},\eta)$ to $\overline{\mathrm{Act}}(\mathfrak{n},\mathfrak{q},\mu)$ only under certain conditions. Finally we proved that those equations indeed describe a set of derived actions by constructing the associated semidirect product, which is an object in **XLb**. We have tried to sum up all that process with the rest of the results in this subsection.

Observe that the first part of the following lemma could have been included in Section 2.1, but here we will make a better use of it.

Lemma 2.2.17.

(i) Let $\mathfrak{q}$ *be a Leibniz algebra and* $(\sigma,\theta),(\sigma',\theta') \in \mathrm{Bider}(\mathfrak{q})$. *If* $\mathrm{Ann}(\mathfrak{q}) = 0$ *or* $[\mathfrak{q},\mathfrak{q}] = \mathfrak{q}$,

$$\theta\sigma'(q) = \theta\theta'(q) \tag{2.2.33}$$

for all $q \in \mathfrak{q}$.

(ii) Let $(\mathfrak{n},\mathfrak{q},\mu)$ *be a Leibniz crossed module,* $((\sigma_1,\theta_1),(\sigma_2,\theta_2)) \in \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu)$ *and* $(d,D) \in \mathrm{Bider}(\mathfrak{q},\mathfrak{n})$. *If* $\mathrm{Ann}(\mathfrak{n}) = 0$ *or* $[\mathfrak{q},\mathfrak{q}] = \mathfrak{q}$,

$$D\sigma_2(q) = D\theta_2(q), \tag{2.2.34}$$

$$\theta_1 d(q) = \theta_1 D(q), \tag{2.2.35}$$

for all $q \in \mathfrak{q}$.

Proof. Calculations in order to prove (i) can be found in the proof of Proposition 2.1.8. Regarding (ii), $D\sigma_2(q) - D\theta_2(q)$ and $\theta_1 d(q) - \theta_1 D(q)$ are elements in $\mathrm{Ann}(\mathfrak{n})$, immediately from the identities in the second column from Lemma 2.2.12. Therefore, if $\mathrm{Ann}(\mathfrak{n}) = 0$, it is clear that (2.2.34) and (2.2.35) hold.

Let us now assume that $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}$. Given $q, q' \in \mathfrak{q}$, directly from the fact that $(\sigma_2, \theta_2) \in \mathrm{Bider}(\mathfrak{q})$ and $(d, D) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$, we get that

$$D\theta_2([q, q']) = [D\theta_2(q), q'] - [D(q'), \theta_2(q)] - [D\theta_2(q'), q] + [D(q), \theta_2(q')],$$
$$D\sigma_2([q, q']) = [D\sigma_2(q), q'] - [D(q'), \sigma_2(q)] + [D(q), \sigma_2(q')] - [D\sigma_2(q'), q].$$

Therefore, due to (2.2.27) and the first identity in the first column from Lemma 2.2.12, $D\theta_2([q, q']) = D\sigma_2([q, q'])$. By hypothesis, every element in $\mathfrak{q}$ can be expressed as a liner combination of elements of the form $[q, q']$. This fact, together with the linearity of D, σ_2 and θ_2, guarantees that $D\theta_2(q) = D\sigma_2(q)$ for all $q \in \mathfrak{q}$. (2.2.35) can be checked similarly by making use of (2.2.24), (2.2.25), (2.2.18) and the third identity in the first column from Lemma 2.2.12. □

The following theorem is the analogue to Propositions 2.2.2 and 2.2.5.

Theorem 2.2.18. *Let $(\mathfrak{m}, \mathfrak{p}, \eta)$ and $(\mathfrak{n}, \mathfrak{q}, \mu)$ be Leibniz crossed modules. If the following conditions hold, there exist a morphism of crossed modules from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $(\mathrm{Bider}(\mathfrak{q}, \mathfrak{n}), \mathrm{Bider}(\mathfrak{n}, \mathfrak{q}, \mu), \Delta)$.*

(i) There are actions of the Leibniz algebra $\mathfrak{p}$ (and so $\mathfrak{m}$) on the Leibniz algebras $\mathfrak{n}$ and $\mathfrak{q}$. The homomorphism μ is $\mathfrak{p}$-equivariant, that is

$$\mu([p, n]) = [p, \mu(n)], \tag{LbEQ1}$$
$$\mu([n, p]) = [\mu(n), p], \tag{LbEQ2}$$

and the actions of $\mathfrak{p}$ and $\mathfrak{q}$ on $\mathfrak{n}$ are compatible, that is

$$[n, [p, q]] = [[n, p], q] - [[n, q], p], \tag{LbCOM1}$$
$$[p, [n, q]] = [[p, n], q] - [[p, q], n], \tag{LbCOM2}$$
$$[p, [q, n]] = [[p, q], n] - [[p, n], q], \tag{LbCOM3}$$
$$[n, [q, p]] = [[n, q], p] - [[n, p], q], \tag{LbCOM4}$$
$$[q, [n, p]] = [[q, n], p] - [[q, p], n], \tag{LbCOM5}$$
$$[q, [p, n]] = [[q, p], n] - [[q, n], p], \tag{LbCOM6}$$

for all $n \in \mathfrak{n}$, $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$.

(ii) There are two K-bilinear maps $\xi_1\colon \mathfrak{m}\times\mathfrak{q}\to\mathfrak{n}$ and $\xi_2\colon \mathfrak{q}\times\mathfrak{m}\to\mathfrak{n}$ such that

$$\mu\xi_2(q,m) = [q,m], \tag{LbM1a}$$
$$\mu\xi_1(m,q) = [m,q], \tag{LbM1b}$$
$$\xi_2(\mu(n),m) = [n,m], \tag{LbM2a}$$
$$\xi_1(m,\mu(n)) = [m,n], \tag{LbM2b}$$
$$\xi_2(q,[p,m]) = \xi_2([q,p],m) - [\xi_2(q,m),p], \tag{LbM3a}$$
$$\xi_1([p,m],q) = \xi_2([p,q],m) - [p,\xi_2(q,m)], \tag{LbM3b}$$
$$\xi_2(q,[m,p]) = [\xi_2(q,m),p] - \xi_2([q,p],m), \tag{LbM3c}$$
$$\xi_1([m,p],q) = [\xi_1(m,q),p] - \xi_1(m,[q,p]), \tag{LbM3d}$$
$$\xi_2(q,[m,m']) = [\xi_2(q,m),m'] - [\xi_2(q,m'),m], \tag{LbM4a}$$
$$\xi_1([m,m'],q) = [\xi_1(m,q),m'] - [m,\xi_2(q,m')], \tag{LbM4b}$$
$$\xi_2([q,q'],m) = [\xi_2(q,m),q'] + [q,\xi_2(q',m)], \tag{LbM5a}$$
$$\xi_1(m,[q,q']) = [\xi_1(m,q),q'] - [\xi_1(m,q'),q], \tag{LbM5b}$$
$$[q,\xi_1(m,q')] = -[q,\xi_2(q',m)], \tag{LbM5c}$$
$$\xi_1(m,[p,q]) = -\xi_1(m,[q,p]), \tag{LbM6a}$$
$$[p,\xi_1(m,q)] = -[p,\xi_2(q,m)], \tag{LbM6b}$$

for all $m,m'\in\mathfrak{m}$, $n\in\mathfrak{n}$, $p\in\mathfrak{p}$, $q,q'\in\mathfrak{q}$.

Additionally, if one of the following conditions holds the converse statement is also true.

$$\operatorname{Ann}(\mathfrak{n}) = 0 = \operatorname{Ann}(\mathfrak{q}), \tag{CON1}$$
$$\operatorname{Ann}(\mathfrak{n}) = 0 \quad \textit{and} \quad [\mathfrak{q},\mathfrak{q}] = \mathfrak{q}, \tag{CON2}$$
$$[\mathfrak{n},\mathfrak{n}] = \mathfrak{n} \quad \textit{and} \quad [\mathfrak{q},\mathfrak{q}] = \mathfrak{q}. \tag{CON3}$$

Proof. Let us suppose that (i) and (ii) hold. Therefore $\mathfrak{p}$ acts on $\mathfrak{n}$ and $\mathfrak{q}$, that is there are four bilinear maps $\mathfrak{p}\times\mathfrak{n}\to\mathfrak{n}$, $(p,n)\mapsto[p,n]$; $\mathfrak{n}\times\mathfrak{p}\to\mathfrak{n}$, $(n,p)\mapsto[n,p]$; $\mathfrak{p}\times\mathfrak{q}\to\mathfrak{q}$, $(p,q)\mapsto[p,q]$ and $\mathfrak{q}\times\mathfrak{p}\to\mathfrak{q}$, $(q,p)\mapsto[q,p]$ such that

$$[p,[n,n']] = [[p,n],n'] - [[p,n'],n], \tag{2.2.36}$$
$$[n,[p,n']] = [[n,p],n'] - [[n,n'],p], \tag{2.2.37}$$
$$[n,[n',p]] = [[n,n'],p] - [[n,p],n'], \tag{2.2.38}$$
$$[n,[p,p']] = [[n,p],p'] - [[n,p'],p], \tag{2.2.39}$$
$$[p,[n,p']] = [[p,n],p'] - [[p,p'],n], \tag{2.2.40}$$
$$[p,[p',n]] = [[p,p'],n] - [[p,n],p'], \tag{2.2.41}$$

and

$$[p,[q,q']] = [[p,q],q'] - [[p,q'],q], \tag{2.2.42}$$
$$[q,[p,q']] = [[q,p],q'] - [[q,q'],p], \tag{2.2.43}$$
$$[q,[q',p]] = [[q,q'],p] - [[q,p],q'], \tag{2.2.44}$$
$$[q,[p,p']] = [[q,p],p'] - [[q,p'],p], \tag{2.2.45}$$
$$[p,[q,p']] = [[p,q],p'] - [[p,p'],q], \tag{2.2.46}$$
$$[p,[p',q]] = [[p,p'],q] - [[p,q],p'], \tag{2.2.47}$$

for all $n, n' \in \mathfrak{n}$, $p, p' \in \mathfrak{p}$, $q, q' \in \mathfrak{q}$. Recall that directly from (2.2.37), (2.2.38), (2.2.40) and (2.2.41),

$$[n,[p,n']] = -[n,[n',p]], \tag{2.2.48}$$
$$[p,[n,p']] = -[p,[p',n]], \tag{2.2.49}$$

for all $n, n' \in \mathfrak{n}$, $p, p' \in \mathfrak{p}$. Analogously,

$$[q,[p,q']] = -[q,[q',p]], \tag{2.2.50}$$
$$[p,[q,p']] = -[p,[p',q]], \tag{2.2.51}$$

for all $q, q' \in \mathfrak{q}$, $p, p' \in \mathfrak{p}$.

It is possible to define a morphism of crossed modules (φ, ψ) from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $\overline{\mathrm{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ as follows. Given $m \in \mathfrak{m}$, $\varphi(m) = (d_m, D_m)$, with

$$d_m(q) = -\xi_2(q,m), \qquad D_m(q) = \xi_1(m,q),$$

for all $q \in \mathfrak{q}$. On the other hand, for any $p \in \mathfrak{p}$, $\psi(p) = ((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$,

$$\sigma_1^p(n) = -[n,p], \qquad \theta_1^p(n) = [p,n],$$
$$\sigma_2^p(q) = -[q,p], \qquad \theta_2^p(q) = [p,q],$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$. It follows directly from (LbM5a)–(LbM5c) that $(d_m, D_m) \in \mathrm{Bider}(\mathfrak{q}, \mathfrak{n})$ for all $m \in \mathfrak{m}$. Besides, φ is clearly K-linear and given $m, m' \in \mathfrak{m}$,

$$[\varphi(m), \varphi(m')] = [(d_m, D_m), (d_{m'}, D_{m'})] = [d_m \mu d_{m'} - d_{m'} \mu d_m, D_m \mu d_{m'} - d_{m'} \mu D_m].$$

For any $q \in \mathfrak{q}$,

$$\begin{aligned} d_m \mu d_{m'}(q) - d_{m'} \mu d_m(q) &= -\xi_2(\mu d_{m'}(q), m) + \xi_2(\mu d_m(q), m') \\ &= -[d_{m'}(q), m] + [d_m(q), m'] \\ &= [\xi_2(q, m'), m] - [\xi_2(q,m), m'] \\ &= -\xi_2(q, [m, m']) = d_{[m,m']}(q), \end{aligned}$$

due to (LbM2a) and (LbM4a). Analogously, it can be easily checked the identity $(D_m\mu d_{m'} - d_{m'}\mu D_m)(q) = D_{[m,m']}(q)$ by making use of (LbM2a), (LbM2b) and (LbM4b). Hence, φ is a morphism of Leibniz algebras.

As for ψ, it is necessary to prove that $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$ satisfies all the axioms from Definition 2.2.11 for any $p \in \mathfrak{p}$. The fact that (σ_1^p, θ_1^p) (respectively (σ_2^p, θ_2^p)) is a biderivation of $\mathfrak{n}$ (respectively $\mathfrak{q}$) follows directly from (2.2.36), (2.2.38) and (2.2.48) (respectively (2.2.42), (2.2.44) and (2.2.50)). The identities $\mu\theta_1^p = \theta_2^p\mu$ and $\mu\sigma_1^p = \sigma_2^p\mu$ are an immediate consequence of (LbEQ1) and (LbEQ2) respectively.

Observe that the combinations of the identities (LbCOM1) and (LbCOM4) and the identities (LbCOM5) and (LbCOM6) yield the equalities

$$-[n,[q,p]] = [n,[p,q]] \qquad \text{and} \qquad -[q,[n,p]] = [q,[p,n]].$$

These together with (LbCOM2)–(LbCOM5) allow to prove that $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$ does satisfy conditions (2.2.22)–(2.2.27) from Definition 2.2.11. Therefore, ψ is well defined, while it is obviously K-linear.

Concerning the preservation of the Leibniz bracket by ψ, due to (2.2.28) we know that

$$[\psi(p), \psi(p')] = ((\sigma_1^p\sigma_1^{p'} - \sigma_1^{p'}\sigma_1^p, \theta_1^p\sigma_1^{p'} - \sigma_1^{p'}\theta_1^p), (\sigma_2^p\sigma_2^{p'} - \sigma_2^{p'}\sigma_2^p, \theta_2^p\sigma_2^{p'} - \sigma_2^{p'}\theta_2^p)),$$

and by definition

$$\psi([p,p']) = ((\sigma_1^{[p,p']}, \theta_1^{[p,p']}), (\sigma_2^{[p,p']}, \theta_2^{[p,p']})).$$

One can easily check that the corresponding components are equal by making use of (2.2.39), (2.2.40), (2.2.45) and (2.2.46). Hence, ψ is a morphism of Leibniz algebras.

Recall that

$$\begin{aligned}
\Delta\varphi(m) &= ((d_m\mu, D_m\mu), (\mu d_m, \mu D_m)),\\
\psi\eta(m) &= ((\sigma_1^{\eta(m)}, \theta_1^{\eta(m)}), (\sigma_2^{\eta(m)}, \theta_2^{\eta(m)})),
\end{aligned}$$

for any $m \in \mathfrak{m}$, but

$$\begin{aligned}
d_m\mu(n) &= -\xi_2(\mu(n), m) = -[n,m] = -[n,\eta(m)] = \sigma_1^{\eta(m)}(n),\\
D_m\mu(n) &= \xi_1(m, \mu(n)) = [m,n] = [\eta(m), n] = \theta_1^{\eta(m)}(n),\\
\mu d_m(q) &= -\mu\xi_2(q,m) = -[q,m] = -[q,\eta(m)] = \sigma_2^{\eta(m)}(q),\\
\mu D_m(q) &= \mu\xi_1(m,q) = [m,q] = [\eta(m), q] = \theta_2^{\eta(m)}(q),
\end{aligned}$$

for all $n \in \mathfrak{n}$, $q \in \mathfrak{q}$, due to (LbM1a), (LbM1b), (LbM2a), (LbM2b) and the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ via η. Therefore, $\Delta\varphi = \psi\eta$.

It only remains to check the behaviour of (φ,ψ) regarding the action of $\mathfrak{p}$ on $\mathfrak{m}$. Let $m \in \mathfrak{m}$ and $p \in \mathfrak{p}$. Due to (2.2.29) and (2.2.30),

$$[\psi(p),\varphi(m)] = (\sigma_1^p d_m - d_m\sigma_2^p, \theta_1^p d_m - d_m\theta_2^p),$$
$$[\varphi(m),\psi(p)] = (d_m\sigma_2^p - \sigma_1^p d_m, D_m\sigma_2^p - \sigma_1^p D_m).$$

On the other hand, by definition, we know that

$$\varphi([p,m]) = (d_{[p,m]}, D_{[p,m]}),$$
$$\varphi([m,p]) = (d_{[m,p]}, D_{[m,p]}).$$

Directly from (LbM3a), (LbM3b), (LbM3c) and (LbM3d) one can easily confirm that the required identities between components hold. Let us write, as an example, the calculations in order to prove that the second component of $[\varphi(m),\psi(p)]$ equals the second component of $\varphi([m,p])$. For any $q \in \mathfrak{q}$,

$$\begin{aligned}(D_m\sigma_2^p - \sigma_1^p D_m)(q) &= -D_m([q,p]) - \sigma_1^p(\xi_1(m,q))\\ &= -\xi_1(m,[q,p]) + [\xi_1(m,q),p],\end{aligned}$$

but according to (LbM3d), $-\xi_1(m,[q,p]) + [\xi_1(m,q),p] = \xi_1([m,p],q) = D_{[m,p]}(q)$. Hence, we can finally ensure that (φ,ψ) is a morphism of Leibniz crossed modules.

Now let us show that it is necessary that at least one of the conditions (CON1)–(CON3) holds in order to prove the converse statement. Let us suppose that there is a morphism of crossed modules

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\eta"] \arrow[d, "\varphi"'] & \mathfrak{p} \arrow[d, "\psi"] \\ \mathrm{Bider}(\mathfrak{q},\mathfrak{n}) \arrow[r, "\Delta"'] & \mathrm{Bider}(\mathfrak{n},\mathfrak{q},\mu) \end{tikzcd} \tag{2.2.52}$$

Given $m \in \mathfrak{m}$ and $p \in \mathfrak{p}$, let us denote $\varphi(m)$ by (d_m, D_m) and $\psi(p)$ by $((\sigma_1^p,\theta_1^p),(\sigma_2^p,\theta_2^p))$, which satisfy conditions (2.2.16)–(2.2.18) from Definition 2.2.7 and conditions (2.2.20)–(2.2.27) from Definition 2.2.11 respectively. Also, due to the definition of Δ (see Proposition 2.2.15), the commutativity of (2.2.52) can be expressed by the identity

$$((d_m\mu, D_m\mu),(\mu d_m, \mu D_m)) = ((\sigma_1^{\eta(m)},\theta_1^{\eta(m)}),(\sigma_2^{\eta(m)},\theta_2^{\eta(m)})) \tag{2.2.53}$$

for all $m \in \mathfrak{m}$. It is possible to define four bilinear maps, all of them denoted by $[-,-]$, from $\mathfrak{p}\times\mathfrak{n}$ to $\mathfrak{n}$, $\mathfrak{n}\times\mathfrak{p}$ to $\mathfrak{n}$, $\mathfrak{p}\times\mathfrak{q}$ to $\mathfrak{q}$ and $\mathfrak{q}\times\mathfrak{p}$ to $\mathfrak{q}$, given by

$$[p,n] = \theta_1^p(n), \qquad [n,p] = -\sigma_1^p(n),$$
$$[p,q] = \theta_2^p(q), \qquad [q,p] = -\sigma_2^p(q),$$

for all $n \in \mathfrak{n}$, $p \in \mathfrak{p}$, $q \in \mathfrak{q}$. In order to prove that those maps define Leibniz actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$, it is necessary to check that conditions (2.2.36)–(2.2.41) and (2.2.42)–(2.2.47) are satisfied by the corresponding maps. Conditions (2.2.36)–(2.2.38) (respectively (2.2.42)–(2.2.44)) follow easily from the fact that (σ_1^p, θ_1^p) (respectively (σ_2^p, θ_2^p)) is a biderivation of $\mathfrak{n}$ (respectively $\mathfrak{q}$).

Since ψ is a Leibniz homomorphism, we know that, given $p, p' \in \mathfrak{p}$, $\psi([p,p']) = [\psi(p), \psi(p')]$, that is

$$((\sigma_1^{[p,p']}, \theta_1^{[p,p']}), (\sigma_2^{[p,p']}, \theta_2^{[p,p']})) = ((\sigma_1^p\sigma_1^{p'} - \sigma_1^{p'}\sigma_1^p, \theta_1^p\sigma_1^{p'} - \sigma_1^{p'}\theta_1^p), \\ (\sigma_2^p\sigma_2^{p'} - \sigma_2^{p'}\sigma_2^p, \theta_2^p\sigma_2^{p'} - \sigma_2^{p'}\theta_2^p)).$$

The identities between the first and the second (respectively the third and the fourth) components in those quadruples allow to confirm that (2.2.39) and (2.2.40) (respectively (2.2.45) and (2.2.46)) hold.

As for conditions (2.2.41) and (2.2.47), it is fairly straightforward to check that

$$[[p,p'],n] - [[p,n],p'] = \theta_1^p\sigma_1^{p'}(n),$$
$$[[p,p'],q] - [[p,q],p'] = \theta_2^p\sigma_2^{p'}(q),$$

while

$$[p,[p',n]] = \theta_1^p\theta_1^{p'}(n),$$
$$[p,[p',q]] = \theta_2^p\theta_2^{p'}(q),$$

for all $n \in \mathfrak{n}$, $p, p' \in \mathfrak{p}$, $q \in \mathfrak{q}$. However, if at least one of the conditions (CON1)–(CON3) holds, due to Lemma 2.2.17 (i), $\theta_1^p\sigma_1^{p'}(n) = \theta_1^p\theta_1^{p'}(n)$ and $\theta_2^p\sigma_2^{p'}(q) = \theta_2^p\theta_2^{p'}(q)$. Therefore, we can ensure that there are Leibniz actions of $\mathfrak{p}$ on both $\mathfrak{n}$ and $\mathfrak{q}$, which induce actions of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ via η.

The reader might have noticed that a fourth possible condition on $(\mathfrak{n}, \mathfrak{q}, \mu)$ could have been considered in order to guarantee the existence of the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$ from the existence of the morphism of Leibniz crossed modules (φ, ψ). In fact, if $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}$ and $\operatorname{Ann}(\mathfrak{q}) = 0$, the problem with conditions (2.2.41) and (2.2.47) could have been solved the same way. Nevertheless, this fourth condition does not guarantee that (ii) holds, as we will prove immediately below.

Regarding (LbEQ1) and (LbEQ2), they follow directly from (2.2.21) (observe that, by hypothesis, $((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$ is a biderivation of $(\mathfrak{n}, \mathfrak{q}, \mu)$ for any $p \in \mathfrak{p}$). Similarly, (LbCOM1)–(LbCOM6) follow almost immediately from (2.2.22)–(2.2.27). Hence, (i) holds.

Concerning (ii), we can define $\xi_1(m,q) = D_m(q)$ and $\xi_2(q,m) = -d_m(q)$ for any $m \in \mathfrak{m}$, $q \in \mathfrak{q}$. In this way, ξ_1 and ξ_2 are clearly bilinear. (LbM1a), (LbM1b), (LbM2a) and (LbM2b) follow immediately from the identity (2.2.53) and the fact that the actions of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ are induced by the actions of $\mathfrak{p}$ via η.

Identities (LbM5a), (LbM5b) and (LbM5c) are a direct consequence of (2.2.16)–(2.2.18) (recall that, by hypothesis, (d_m, D_m) is a biderivation from $\mathfrak{q}$ to $\mathfrak{n}$ for any $m \in \mathfrak{m}$).

Note that φ is a Leibniz homomorphism, so $\varphi([m,m']) = [\varphi(m), \varphi(m')]$ for $m, m' \in \mathfrak{m}$, that is

$$(d_{[m,m']}, D_{[m,m']}) = (d_m \mu d_{m'} - d_{m'} \mu d_m, D_m \mu d_{m'} - d_{m'} \mu D_m).$$

This identity, together with (LbM2a) and (LbM2b), allows to easily prove that (LbM4a) and (LbM4b) hold.

Note that, since (φ, ψ) is a morphism of Leibniz crossed modules, $\varphi([p,m]) = [\psi(p), \varphi(m)]$ and $\varphi([m,p]) = [\varphi(m), \psi(p)]$ for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$. Due to the definition of the action of $\operatorname{Bider}(\mathfrak{n}, \mathfrak{q}, \mu)$ on $\operatorname{Bider}(\mathfrak{q}, \mathfrak{n})$ (see Theorem 2.2.16), we can write

$$\begin{aligned}
(d_{[p,m]}, D_{[p,m]}) &= (\sigma_1^p d_m - d_m \sigma_2^p, \theta_1^p d_m - d_m \theta_2^p),\\
(d_{[m,p]}, D_{[m,p]}) &= (d_m \sigma_2^p - \sigma_1^p d_m, D_m \sigma_2^p - \sigma_1^p D_m).
\end{aligned}$$

(LbM3a), (LbM3b), (LbM3c) and (LbM3d) follow immediately from the previous identities.

Regarding (LbM6a) and (LbM6b), directly from the definition of ξ_1, ξ_2 and the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{q}$, we have that

$$\begin{aligned}
\xi_1(m,[p,q]) &= D_m \theta_2^p(q), & [p, \xi_1(m,q)] &= \theta_1^p D_m(q),\\
-\xi_1(m,[q,p]) &= D_m \sigma_2^p(q), & -[p, \xi_2(q,m)] &= \theta_1^p d_m(q),
\end{aligned}$$

for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$, $q \in \mathfrak{q}$. Nevertheless, if at least one of the conditions (CON1)–(CON3) holds, due to Lemma 2.2.17 (ii), $D_m \theta_2^p(q) = D_m \sigma_2^p(q)$ and $\theta_1^p D_m(q) = \theta_1^p d_m(q)$. Hence, (ii) holds. $\square$

Remark 2.2.19. *A closer look at the proof of the previous theorem shows that neither conditions* (LbM6a) *and* (LbM6b)*, nor the identities* (2.2.41) *and* (2.2.47) *(which correspond to the sixth axiom satisfied by the actions of* $\mathfrak{p}$ *on* $\mathfrak{n}$ *and* $\mathfrak{q}$ *respectively) are necessary in order to prove the existence of a morphism of crossed modules* (φ, ψ) *from* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *to* $\overline{\operatorname{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$*, under the hypothesis that (i) and (ii) hold. Actually, if we remove those conditions from (i) and (ii), the converse statement would be true for any Leibniz crossed module* $(\mathfrak{n}, \mathfrak{q}, \mu)$*, even if it does not satisfy any of the conditions* (CON1)–(CON3)*. An early version of Theorem 2.2.18 did not include conditions* (LbM6a) *and* (LbM6b)*, although* (2.2.41) *and* (2.2.47) *were considered from the very beginning, since it does not seem natural to ask for* $\mathfrak{p}$ *to "almost" act on* $\mathfrak{n}$ *and* $\mathfrak{q}$*. The problem is that* (LbM6a)*,* (LbM6b)*,* (2.2.41) *and* (2.2.47) *are essential in order to prove that (i) and (ii) as in Theorem 2.2.18 describe a set of derived actions of* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *on* $(\mathfrak{n}, \mathfrak{q}, \mu)$*, as we will show immediately below. This agrees with the idea of* $\overline{\operatorname{Act}}(\mathfrak{n}, \mathfrak{q}, \mu)$ *not being "good enough" to be the actor of* $(\mathfrak{n}, \mathfrak{q}, \mu)$ *in general, just as* $\operatorname{Bider}(\mathfrak{m})$ *is not always the actor of a Leibniz algebra* $\mathfrak{m}$*.*

Example 2.2.20. *Given a Leibniz crossed module* $(\mathfrak{m}, \mathfrak{p}, \eta)$*, there is a morphism* $(\varphi, \psi) \colon (\mathfrak{m}, \mathfrak{p}, \eta) \to \overline{\mathrm{Act}}(\mathfrak{m}, \mathfrak{p}, \eta)$*, with* $\varphi(m) = (d_m, D_m)$ *and* $\psi(p) = ((\sigma_1^p, \theta_1^p), (\sigma_2^p, \theta_2^p))$*, where*

$$d_m(p) = -[p, m], \qquad D_m(p) = [m, p],$$

and

$$\begin{aligned} \sigma_1^p(m) &= -[m, p], \qquad & \theta_1^p(m) &= [p, m], \\ \sigma_2^p(p') &= -[p', p], \qquad & \theta_2^p(p') &= [p, p'], \end{aligned}$$

for all $m \in \mathfrak{m}$*,* $p, p' \in \mathfrak{p}$*. Calculations in order to prove that* (φ, ψ) *is indeed a morphism of Leibniz crossed modules are fairly straightforward only by making use of the axioms satisfied by the actions of* $\mathfrak{p}$ *on* $\mathfrak{m}$ *together with the equivariance and the Peiffer identity. Of course, this morphism does not necessarily define a set of derived actions. Theorem 2.2.18, along with the result immediately bellow, shows that if* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *satisfies at least one of the conditions* (CON1)–(CON3)*, then the previous morphism does define a set of derived actions of* $(\mathfrak{m}, \mathfrak{p}, \eta)$ *on itself.*

Let $(\mathfrak{m}, \mathfrak{p}, \eta)$ and $(\mathfrak{n}, \mathfrak{q}, \mu)$ be Leibniz crossed modules such that (i) and (ii) from Theorem 2.2.18 hold. Therefore, there are Leibniz actions of $\mathfrak{m}$ on $\mathfrak{n}$ and of $\mathfrak{p}$ on $\mathfrak{q}$, so it makes sense to consider the semidirect products of Leibniz algebras $\mathfrak{n} \rtimes \mathfrak{m}$ and $\mathfrak{q} \rtimes \mathfrak{p}$. Furthermore, we have the following result.

Theorem 2.2.21. *There is an action of the Leibniz algebra* $\mathfrak{q} \rtimes \mathfrak{p}$ *on the Leibniz algebra* $\mathfrak{n} \rtimes \mathfrak{m}$*, given by*

$$[(q, p), (n, m)] = ([q, n] + [p, n] + \xi_2(q, m), [p, m]), \tag{2.2.54}$$

$$[(n, m), (q, p)] = ([n, q] + [n, p] + \xi_1(m, q), [m, p]), \tag{2.2.55}$$

for all $(q, p) \in \mathfrak{q} \rtimes \mathfrak{p}$*,* $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$*, with* ξ_1 *and* ξ_2 *as in Theorem 2.2.18. Moreover, the Leibniz homomorphism* $(\mu, \eta) \colon \mathfrak{n} \rtimes \mathfrak{m} \to \mathfrak{q} \rtimes \mathfrak{p}$*, given by*

$$(\mu, \eta)(n, m) = (\mu(n), \eta(m)),$$

for all $(n, m) \in \mathfrak{n} \rtimes \mathfrak{m}$ *is a Leibniz crossed module together with the previous action.*

Proof. In order to prove that (2.2.54) and (2.2.55) define an action of $\mathfrak{q} \rtimes \mathfrak{p}$ on $\mathfrak{n} \rtimes \mathfrak{m}$ it is necessary to confirm that the following identities hold for any $(n, m), (n', m') \in \mathfrak{n} \rtimes \mathfrak{m}$,

$(q,p),(q',p') \in \mathfrak{q} \rtimes \mathfrak{p}$.

$$[(q,p),[(n,m),(n',m')]] = [[(q,p),(n,m)],(n',m')] - [[(q,p),(n',m')],(n,m)], \tag{2.2.56}$$

$$[(n,m),[(q,p),(n',m')]] = [[(n,m),(q,p)],(n',m')] - [[(n,m),(n',m')],(q,p)], \tag{2.2.57}$$

$$[(n,m),[(n',m'),(q,p)]] = [[(n,m),(n',m')],(q,p)] - [[(n,m),(q,p)],(n',m')], \tag{2.2.58}$$

$$[(n,m),[(q,p),(q',p')]] = [[(n,m),(q,p)],(q',p')] - [[(n,m),(q',p')],(q,p)], \tag{2.2.59}$$

$$[(q,p),[(n,m),(q',p')]] = [[(q,p),(n,m)],(q',p')] - [[(q,p),(q',p')],(n,m)], \tag{2.2.60}$$

$$[(q,p),[(q',p'),(n,m)]] = [[(q,p),(q',p')],(n,m)] - [[(q,p),(n,m)],(q',p')]. \tag{2.2.61}$$

Recall that the brackets in $\mathfrak{n} \rtimes \mathfrak{m}$ and $\mathfrak{q} \rtimes \mathfrak{p}$ are given respectively by

$$[(n,m),(n',m')] = ([n,n'] + [m,n'] + [n,m'],[m,m'])$$

and

$$[(q,p),(q',p')] = ([q,q'] + [p,q'] + [q,p'],[p,p'])$$

for all $(n,m),(n',m') \in \mathfrak{n} \rtimes \mathfrak{m}$, $(q,p),(q',p') \in \mathfrak{q} \rtimes \mathfrak{p}$. The procedure in order to check each of the identities is not complicated if one bears in mind the conditions satisfied by $(\mathfrak{m},\mathfrak{p},\eta)$ and $(\mathfrak{n},\mathfrak{q},\mu)$ (see Theorem 2.2.18). Nevertheless, as an example, we show how to prove (2.2.58). Calculations for the rest of the identities are similar. Let $(n,m),(n',m') \in \mathfrak{n} \rtimes \mathfrak{m}$ and $(q,p) \in \mathfrak{q} \rtimes \mathfrak{p}$. By routine calculations we get that

$$\begin{aligned}
[(n,m),[(n',m'),(q,p)]] = (&\underbrace{[n,[n',q]]}_{(1)} \underbrace{+[n,[n',p]]}_{(2)} \underbrace{+[n,\xi_1(m',q)]}_{(3)} \underbrace{+[m,[n',q]]}_{(4)} \\
&\underbrace{+[m,[n',p]]}_{(5)} \underbrace{+[m,\xi_1(m',q)]}_{(6)} \underbrace{+[n,[m',p]]}_{(7)}, \underbrace{[m,[m',p]]}_{(8)}), \\
[[(n,m),(n',m')],(q,p)] = (&\underbrace{[[n,n'],q]}_{(1')} \underbrace{+[[n,n'],p]}_{(2')} \underbrace{+[[n,m'],q]}_{(3')} \underbrace{+[[m,n'],q]}_{(4')} \\
&\underbrace{+[[m,n'],p]}_{(5')} \underbrace{+\xi_1([m,m'],q)}_{(6')} \underbrace{+[[n,m'],p]}_{(7')}, \underbrace{[[m,m'],p]}_{(8')}), \\
[[(n,m),(q,p)],(n',m')] = (&\underbrace{[[n,q],n']}_{(1'')} \underbrace{+[[n,p],n']}_{(2'')} \underbrace{+[[n,q],m']}_{(3'')} \underbrace{+[\xi_1(m,q),n']}_{(4'')} \\
&\underbrace{+[[m,p],n']}_{(5'')} \underbrace{+[\xi_1(m,q),m']}_{(6'')} \underbrace{+[[n,p],m']}_{(7'')}, \underbrace{[[m,p],m']}_{(8'')}).
\end{aligned}$$

Let us show that $(i) = (i') - (i'')$ for $i = 1, \dots, 8$. It is immediate for $i = 1, 2, 8$ due to the action of $\mathfrak{q}$ on $\mathfrak{n}$ and the actions of $\mathfrak{p}$ on $\mathfrak{n}$ and $\mathfrak{m}$. For $i = 5$, the identity follows from the fact that the action $\mathfrak{m}$ on $\mathfrak{n}$ is defined via η together with the equivariance of η. Namely,

$$\begin{aligned}[m,[n',p]] &= [\eta(m),[n',p]] = [[\eta(m),n'],p] - [[\eta(m),p],n'] \\ &= [[m,n'],p] - [\eta([m,p]),n'] = [[m,n'],p] - [[m,p],n'].\end{aligned}$$

The procedure is similar for $i = 7$. For $i = 3$, it is necessary to make use of the Peiffer identity of μ, (LbM1b), the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ and $\mathfrak{q}$ via η and (LbCOM1):

$$\begin{aligned}[n,\xi_1(m',q)] &= [n,\mu\xi_1(m',q)] = [n,[m',q]] = [n,[\eta(m'),q]] \\ &= [[n,\eta(m')],q] - [[n,q],\eta(m')] = [[n,m'],q] - [[n,q],m'].\end{aligned}$$

The conditions required in order to prove the identity for $i = 4$ are the same used for $i = 3$ except (LbCOM1), which is replace by (LbCOM2).

Finally, for $i = 6$, due to (LbM4b) and the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ via η, we know that

$$\xi_1([m,m'],q) = [\xi_1(m,q),m'] - [m,\xi_2(q,m')] = [\xi_1(m,q),m'] - [\eta(m),\xi_2(q,m')],$$

but applying (LbM6b), we get

$$\xi_1([m,m'],q) = [\xi_1(m,q),m'] + [\eta(m),\xi_1(m',q)] = [\xi_1(m,q),m'] + [m,\xi_1(m',q)],$$

so $(6) = (6') - (6'')$ and (2.2.58) holds. Note that (LbM6a) and (LbM6b) are necessary in order to check (2.2.59) and (2.2.60) respectively.

Checking that (μ,η) is indeed a Leibniz homomorphism follows directly from the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ via η together with the conditions (LbEQ1) and (LbEQ2). Regarding the equivariance of (μ,η), given $(n,m) \in \mathfrak{n} \rtimes \mathfrak{m}$ and $(q,p) \in \mathfrak{q} \rtimes \mathfrak{p}$,

$$\begin{aligned}(\mu,\eta)([(q,p),(n,m)]) &= (\mu,\eta)([q,n] + [p,n] + \xi_2(q,m),[p,m]) \\ &= (\mu([q,n]) + \mu([p,n]) + \mu\xi_2(q,m),\eta([p,m])) \\ &= ([q,\mu(n)] + [p,\mu(n)] + [q,m],[p,\eta(m)]) \\ &= ([q,\mu(n)] + [p,\mu(n)] + [q,\eta(m)],[p,\eta(m)]) \\ &= [(q,p),(\mu(n),\eta(m))],\end{aligned}$$

due to the equivariance of μ and η, (LbEQ1), (LbM1a) and the definition of the action of $\mathfrak{m}$ on $\mathfrak{q}$ via η. Similarly, but using (LbEQ2) and (LbM1b) instead of (LbEQ1) and (LbM1a), it can be proved that $(\mu,\eta)([(n,m),(q,p)]) = [(\mu(n),\eta(m)),(q,p)]$.

The Peiffer identity of (μ,η) follows easily from the homonymous property of μ and η, the definition of the action of $\mathfrak{m}$ on $\mathfrak{n}$ via η and the conditions (LbM2a) and (LbM2b). □

Definition 2.2.22. *The Leibniz crossed module $(\mathfrak{n} \rtimes \mathfrak{m}, \mathfrak{q} \rtimes \mathfrak{p}, (\mu,\eta))$ is called the semidirect product of the Leibniz crossed modules $(\mathfrak{n},\mathfrak{q},\mu)$ and $(\mathfrak{m},\mathfrak{p},\eta)$.*

Note that the semidirect product determines an obvious split extension of $(\mathfrak{m},\mathfrak{p},\eta)$ by $(\mathfrak{n},\mathfrak{q},\mu)$

$$(0,0,0) \longrightarrow (\mathfrak{n},\mathfrak{q},\mu) \longrightarrow (\mathfrak{n}\rtimes\mathfrak{m},\mathfrak{q}\rtimes\mathfrak{p},(\mu,\eta)) \rightleftarrows (\mathfrak{m},\mathfrak{p},\eta) \longrightarrow (0,0,0)$$

Conversely, any split extension of $(\mathfrak{m},\mathfrak{p},\eta)$ by $(\mathfrak{n},\mathfrak{q},\mu)$ is isomorphic to their semidirect product, where the action of $(\mathfrak{m},\mathfrak{p},\eta)$ on $(\mathfrak{n},\mathfrak{q},\mu)$ is induced by the splitting morphism.

Now we are in a position to write the following definition.

Definition 2.2.23. *If $(\mathfrak{m},\mathfrak{p},\eta)$ and $(\mathfrak{n},\mathfrak{q},\mu)$ are Leibniz crossed modules and at least one of the following conditions holds,*

1. $\mathrm{Ann}(\mathfrak{n})=0=\mathrm{Ann}(\mathfrak{q})$,
2. $\mathrm{Ann}(\mathfrak{n})=0$ *and* $[\mathfrak{q},\mathfrak{q}]=\mathfrak{q}$,
3. $[\mathfrak{n},\mathfrak{n}]=\mathfrak{n}$ *and* $[\mathfrak{q},\mathfrak{q}]=\mathfrak{q}$,

an action of the crossed module $(\mathfrak{m},\mathfrak{p},\eta)$ on $(\mathfrak{n},\mathfrak{q},\mu)$ is a morphism of Leibniz crossed modules from $(\mathfrak{m},\mathfrak{p},\eta)$ to $\overline{\mathrm{Act}}(\mathfrak{n},\mathfrak{q},\mu)$. In other words, under one of those conditions, $\overline{\mathrm{Act}}(\mathfrak{n},\mathfrak{q},\mu)$ is the actor of $(\mathfrak{n},\mathfrak{q},\mu)$.

Example 2.2.24.
(i) Given a Leibniz algebra $\mathfrak{q}$, it can be regarded as a Leibniz crossed module in two obvious ways, $(\{0\},\mathfrak{q},0)$ and $(\mathfrak{q},\mathfrak{q},\mathrm{id}_{\mathfrak{q}})$. It is easy to check that $\overline{\mathrm{Act}}(\{0\},\mathfrak{q},0)\cong(\{0\},\mathrm{Bider}(\mathfrak{q}),0)$ and $\overline{\mathrm{Act}}(\mathfrak{q},\mathfrak{q},\mathrm{id}_{\mathfrak{q}})\cong(\mathrm{Bider}(\mathfrak{q}),\mathrm{Bider}(\mathfrak{q}),\mathrm{id})$.
(ii) Every Lie crossed module $(\mathbb{n},\mathbb{q},\mu)$ can be regarded as a Leibniz crossed module (see Subsection 3.4.2). Note that in this situation, both the multiplication and the action are antisymmetric. Therefore, given $(d,D)\in\mathrm{Bider}(\mathbb{q},\mathbb{n})$, both d and D are elements in $\mathrm{Der}(\mathbb{q},\mathbb{n})$. Additionally, if we assume that at least one of the conditions from the previous lemma holds, then either $\mathrm{Ann}(\mathbb{n})=0$ or $[\mathbb{q},\mathbb{q}]=\mathbb{q}$. In this situation, one can easily derive from (2.2.18) that $\mathrm{Bider}(\mathbb{q},\mathbb{n})=\{(d,d)\mid d\in\mathrm{Der}(\mathbb{q},\mathbb{n})\}$. Besides, the bracket in $\mathrm{Bider}(\mathbb{q},\mathbb{n})$ becomes antisymmetric and, as a Lie algebra, it is isomorphic to $\mathrm{Der}(\mathbb{q},\mathbb{n})$. Similarly, $\mathrm{Bider}(\mathbb{n},\mathbb{q},\mu)$ is a Lie algebra isomorphic to $\mathrm{Der}(\mathbb{n},\mathbb{q},\mu)$ and $\overline{\mathrm{Act}}(\mathbb{n},\mathbb{q},\mu)$ is a Lie crossed module isomorphic to $\mathrm{Act}(\mathbb{n},\mathbb{q},\mu)$.

2.2.4 On the actor crossed module of associative algebras and dialgebras

A closer look at the procedure followed in the previous subsection in order to construct a general actor for a crossed module of Leibniz algebras, makes us wonder if a similar approach could lead us to the construction of 2-dimensional analogues to the associative algebra of bimultipliers and the associative dialgebra of tetramultipliers. It does not seem reckless to think that, given a crossed module of algebras (respectively dialgebras) (B,A,ρ) (respectively (L,D,μ)), $(\mathrm{Bim}(A,B),\mathrm{Bim}(B,A,\rho),\Delta)$ (respectively

$(\mathrm{Tetra}(D, L), \mathrm{Tetra}(L, D, \mu), \Delta'))$ could be a good candidate for general actor, or even actor under certain conditions. Of course it would be necessary to give a proper definition of all the objects involved. Furthermore, the amount of equations involved in the definition of an action of a crossed module of dialgebras would probably be enormous, around five times what we have for Leibniz algebras.

2.3 Modules over crossed modules

It is a classical fact that the categories of (left or right) modules over a Lie algebra and over its universal enveloping algebra are equivalent. Since we intend to establish the analogous equivalence for the categories of modules over a Lie crossed module and over its universal enveloping crossed module, notion that will be explored in the last chapter, we need a proper definition of left modules over a crossed module of Lie and associative algebras.

Recall that Beck [7] introduced a convenient notion of coefficient module to be used in (co)homology theories. That concept makes sense in a broad context and recovers the usual notions of modules in familiar settings: for groups, commutative algebras and Lie algebras, these are left modules; for associative algebras, the appropriate notion is that of bimodule.

By definition, given an object C of a category $\mathcal{C}$, a Beck module over C is an abelian group object in the slice category $\mathcal{C}/C$. If $\mathcal{C}$ is a category of interest in the sense of Orzech [74], then Beck modules are equivalent to split extensions with singular kernel [74, Theorem 2.7]. This is the case for all the aforementioned familiar categories. Moreover, the description of crossed modules of groups as cat^1-groups makes **XGr** into a category of interest (see for instance [75]). Nevertheless, the same assertion fails in the case of Lie crossed modules. Concretely, the category **XLie** satisfies all the axioms of a category of interest except one (axiom (d) from Definition 1.1.1), which is replaced by a new axiom (see the details in [21] for precrossed modules of Lie algebras). However, since that condition is not used in the proof of [74, Theorem 2.7], we can still apply this general result to **XLie** and identify a module over a Lie crossed module $(\mathbb{m}, \mathbb{p}, \nu)$ with a split extension in **XLie**,

$$(0,0,0) \longrightarrow (\mathbb{n}, \mathbb{q}, \mu) \longrightarrow (\mathbb{m}', \mathbb{p}', \nu') \rightleftarrows (\mathbb{m}, \mathbb{p}, \nu) \longrightarrow (0,0,0)$$

where the kernel $(\mathbb{n}, \mathbb{q}, \mu)$ is an *abelian crossed module* of Lie algebras, that is $\mathbb{n}$, $\mathbb{q}$ are abelian Lie algebras and the action of $\mathbb{q}$ on $\mathbb{n}$ is trivial. Bearing in mind the discussion at the end of Subsection 2.2.2, it makes sense to consider the following definition.

Definition 2.3.1. *Let $(\mathbb{m}, \mathbb{p}, \nu)$ be a Lie crossed module. A left $(\mathbb{m}, \mathbb{p}, \nu)$-module is an abelian Lie crossed module $(\mathbb{n}, \mathbb{q}, \mu)$ together with a morphism of crossed modules (φ, ψ) from $(\mathbb{m}, \mathbb{p}, \nu)$ to* $\mathrm{Act}(\mathbb{n}, \mathbb{q}, \mu)$.

In other words, a left $(\mathbb{m}, \mathbb{p}, \nu)$-module is an abelian Lie crossed module endowed with a $(\mathbb{m}, \mathbb{p}, \nu)$-action.

Let $(\mathbb{n}, \mathbb{q}, \mu)$ and $(\mathbb{n}', \mathbb{q}', \mu')$ be left $(\mathbb{m}, \mathbb{p}, \nu)$-modules with the corresponding morphisms of Lie crossed modules $(\varphi, \psi)\colon (\mathbb{m}, \mathbb{p}, \nu) \to \mathrm{Act}(\mathbb{n}, \mathbb{q}, \mu)$ and $(\varphi', \psi')\colon (\mathbb{m}, \mathbb{p}, \nu) \to \mathrm{Act}(\mathbb{n}', \mathbb{q}', \mu')$. Then a morphism from $(\mathbb{n}, \mathbb{q}, \mu)$ to $(\mathbb{n}', \mathbb{q}', \mu')$ is a pair $(f_{\mathbb{n}}, f_{\mathbb{q}})$ of K-module homomorphisms, $f_{\mathbb{n}}\colon \mathbb{n} \to \mathbb{n}'$ and $f_{\mathbb{p}}\colon \mathbb{p} \to \mathbb{p}'$, such that

$$f_{\mathbb{q}}\mu = \mu' f_{\mathbb{n}}, \tag{2.3.1}$$

$$(f_{\mathbb{n}}, f_{\mathbb{q}})\psi(p) = \psi'(p)(f_{\mathbb{n}}, f_{\mathbb{q}}), \tag{2.3.2}$$

$$f_{\mathbb{n}}\varphi(m) = \varphi'(m) f_{\mathbb{q}}, \tag{2.3.3}$$

for all $m \in \mathbb{m}$, $p \in \mathbb{p}$. There is an obvious composition of such morphisms, which allows us to consider the category of $(\mathbb{m}, \mathbb{p}, \nu)$-modules.

In [75], Paoli also uses the notion of Beck module in order to define modules over crossed modules of groups. In this situation a module over a crossed module of groups (H, G, ∂) would be an abelian crossed module of groups (M, P, η), that is a morphism of abelian groups, endowed with a (H, G, ∂)-action. Nevertheless, in [22], the authors go further and consider a K-module structure for (M, P, η). Namely, they consider $\eta\colon M \to P$ a morphism of K-modules. Moreover, they denote by $\mathrm{Aut}_K(\eta)$ the subgroup of $\mathrm{Aut}(\eta)$ of all K-automorphisms, and by $\mathrm{D}_K(P, M)$ the subgroup of $\mathrm{D}(P, M)$ of all K-linear derivations whose inverse in $\mathrm{D}(P, M)$ is K-linear as well (see Subsection 2.2.1 for the details about the construction of the actor in **XGr**).

It is easy to prove that the homomorphism

$$\Delta_K = \Delta|_{\mathrm{D}_K(P,M)}\colon \mathrm{D}_K(P, M) \to \mathrm{Aut}_K(\eta)$$

is a crossed module of groups, which will be denoted by $\mathrm{Act}_{\mathrm{K}}(M, P, \eta)$, where the action of $\mathrm{Aut}_K(\eta)$ on $\mathrm{D}_K(P, M)$ is induced by the action of $\mathrm{Aut}(\eta)$ on $\mathrm{D}(P, M)$. They give the following definition.

Definition 2.3.2 ([22])**.** *Let (H, G, ∂) be a crossed module of groups. A (H, G, ∂)-module over K is a morphism of K-modules $\eta\colon M \to P$ together with a morphism of crossed modules of groups $(\varphi, \psi)\colon (H, G, \partial) \to \mathrm{Act}_K(M, P, \eta)$.*

Let (M, P, η) and (M', P', η') be a pair of (H, G, ∂)-modules over K with the corresponding morphisms of crossed modules $(\varphi, \psi)\colon (H, G, \partial) \to \mathrm{Act}_K(M, P, \eta)$ and $(\varphi', \psi')\colon (H, G, \partial) \to \mathrm{Act}_K(M', P', \eta')$. A morphism from (M, P, η) to (M', P', η') is a pair (f_M, f_P) of K-module homomorphisms, $f_M\colon M \to M'$ and $f_P\colon P \to P'$ such that

$$f_P\eta = \eta' f_M, \tag{2.3.4}$$

$$(f_M, f_P)\psi(g) = \psi'(g)(f_M, f_P), \tag{2.3.5}$$

$$f_M\varphi(h) = \varphi'(h) f_P, \tag{2.3.6}$$

for all $h \in H$, $g \in G$. Just like in the Lie case, there is an obvious composition of such morphisms and it is possible to consider the category of (H, G, ∂)-modules over K.

In [22], the authors prove that the previous definition agrees with the representation theory of cat^1-groups developed by Forrester-Barker [45], in the sense of the following theorem.

Theorem 2.3.3 ([22])**.** *Let (H, G, ∂) be a crossed module of groups. Then the category of (H, G, ∂)-modules over K is equivalent to the category of linear representations of the corresponding cat^1-group $(H \rtimes G, G, s, t)$.*

Regarding the notion of a left module over a crossed module of associative algebras, Beck's approach is not useful in this situation. The problem is that a Beck module over an algebra is a bimodule but not a left module over it.

Since we know that a left module over an algebra A is a K-module V together with an algebra homomorphism from A to the endomorphism algebra $\mathrm{End}_K(V)$ of V, we were naturally led to search for an adequate construction of an "endomorphism crossed module" such that a left module over a crossed module of algebras would be defined by a morphism of crossed modules to the endomorphism crossed module. A good reason why our construction is relevant is that the Liezation of the endomorphism crossed module is nothing else but the actor crossed module of Lie algebras (see Lemma 3.2.7 at the end of Section 3.2.3). Note that Liezation of crossed modules of algebras is explained in Subsection 3.2.1. At the same time, it is reasonable to expect a module over a Lie crossed module and over its universal enveloping crossed module to be the same.

For this, we take an abelian crossed module of algebras $\delta \colon V \to W$, that is V and W are just K-modules considered as algebras with the trivial multiplication, the action of W on V is trivial and δ is a morphism of K-modules. Then the K-module $\mathrm{Hom}_K(W, V)$ is an algebra with the multiplication given by

$$d_1 \cdot d_2 = d_1 \delta d_2 \tag{2.3.7}$$

for all $d_1, d_2 \in \mathrm{Hom}_K(W, V)$. Let us denote by $\mathrm{End}(V, W, \delta)$ the algebra of all pairs (α, β), with $\alpha \in \mathrm{End}_K(V)$ and $\beta \in \mathrm{End}_K(W)$, such that $\beta\delta = \delta\alpha$. Observe that the multiplication in $\mathrm{End}(V, W, \delta)$ is given by component-wise composition. It is clear that the map

$$\Gamma \colon \mathrm{Hom}_K(W, V) \to \mathrm{End}(V, W, \delta), \quad d \mapsto (d\delta, \delta d),$$

is a morphism of algebras. Moreover, we have the following result.

Lemma 2.3.4. *There is an algebra action of $\mathrm{End}(V, W, \delta)$ on $\mathrm{Hom}_K(W, V)$ given by*

$$(\alpha, \beta) \cdot d = \alpha d \qquad \textit{and} \qquad d \cdot (\alpha, \beta) = d\beta \tag{2.3.8}$$

for all $d \in \mathrm{Hom}_K(W, V)$, $(\alpha, \beta) \in \mathrm{End}(V, W, \delta)$. Moreover, together with this action, $(\mathrm{Hom}_K(W, V), \mathrm{End}(V, W, \delta), \Gamma)$ is a crossed module of algebras.

Proof. Let $d \in \mathrm{Hom}_K(W,V)$ and $(\alpha,\beta) \in \mathrm{End}(V,W,\delta)$. It is obvious that αd and $d\beta$ are elements in $\mathrm{Hom}_K(W,V)$. Besides, it can be readily checked that the identities from Definition 1.2.24 are satisfied simply by using (2.3.7) and (2.3.8). Regarding the equivariance and the Peiffer identity, given $(\alpha,\beta) \in \mathrm{End}(V,W,\delta)$ and $d, d' \in \mathrm{Hom}_K(W,V)$,

$$\begin{aligned}
\Gamma\big((\alpha,\beta)\cdot d\big) &= \Gamma(\alpha d) = (\alpha d\delta, \delta\alpha d) = (\alpha d\delta, \beta\delta d) = (\alpha,\beta)(d\delta,\delta d) = (\alpha,\beta)\Gamma(d),\\
\Gamma\big(d\cdot(\alpha,\beta)\big) &= \Gamma(d\beta) = (d\beta\delta, \delta d\beta) = (d\delta\alpha, \delta d\beta) = (d\delta,\delta d)(\alpha,\beta) = \Gamma(d)(\alpha,\beta),\\
\Gamma(d)\cdot d' &= (d\delta,\delta d)\cdot d' = (d\delta)d' = d\cdot d' = d(\delta d') = d\cdot(d'\delta,\delta d') = d\cdot\Gamma(d'),
\end{aligned}$$

due to the identity $\beta\delta = \delta\alpha$ along with the definition of the action and the multiplications in $\mathrm{End}(V,W,\delta)$ and $\mathrm{Hom}_K(W,V)$. □

Note that $\mathrm{End}(V,W,\delta)$ is a unital algebra and $(\mathrm{Hom}_K(W,V), \mathrm{End}(V,W,\delta), \Gamma)$ is an object of $\mathbf{XAs}^1$, which is called *endomorphism crossed module.*

Definition 2.3.5. *Let (B,A,ρ) be a crossed module of algebras. A left (B,A,ρ)-module is an abelian crossed module of algebras (V,W,δ) together with a morphism of crossed modules of algebras $(\varphi,\psi)\colon (B,A,\rho) \to (\mathrm{Hom}_K(W,V), \mathrm{End}(V,W,\delta), \Gamma)$.*

Let (V,W,δ) and (V',W',δ') be left (B,A,ρ)-modules with the morphisms of crossed modules of algebras $(\varphi,\psi)\colon (B,A,\rho) \to (\mathrm{Hom}_K(W,V), \mathrm{End}(V,W,\delta), \Gamma)$ and $(\varphi',\psi')\colon (B,A,\rho) \to (\mathrm{Hom}_K(W',V'), \mathrm{End}(V',W',\delta'), \Gamma')$. Then a morphism from (V,W,δ) to (V',W',δ') is a pair (f_V, f_W) of morphisms of K-modules $f_V\colon V \to V'$ and $f_W\colon W \to W'$ such that

$$f_W\delta = \delta' f_V, \tag{2.3.9}$$
$$(f_V, f_W)\psi(a) = \psi'(a)(f_V, f_W), \tag{2.3.10}$$
$$f_V\varphi(b) = \varphi'(b) f_W, \tag{2.3.11}$$

for all $b \in B$, $a \in A$. Just like for groups and Lie algebras, there is an obvious composition of such morphisms and it is possible to consider the category of (B,A,ρ)-modules.

Since crossed modules of algebras and cat^1-algebras are equivalent, and we have a definition of left modules over a crossed module of algebras, this may also be considered as a definition of left modules over the corresponding cat^1-algebra. However, a direct definition of a left module over a cat^1-algebra will also be useful so we give it immediately below. First we recall from [45] that a cat^1*-module* (V_1, V_0, s, t) consists of a K-module V_1, a K-submodule V_0 of V_1 and structural morphisms $s, t\colon V_1 \to V_0$ such that $s|_{V_0} = t|_{V_0} = \mathrm{id}_{V_0}$.

Definition 2.3.6. *A left module over a cat^1-algebra (A_1, A_0, σ, τ) is a cat^1-module (V_1, V_0, s, t) together with a left action of A_1 on V_1 and a left action of A_0 on V_0*

such that the structural morphisms commute with the actions, that is $s(a_1 \cdot v_1) = \sigma(a_1) \cdot s(v_1)$, $t(a_1 \cdot v_1) = \tau(a_1) \cdot t(v_1)$ *for all* $a_1 \in A_1$, $v_1 \in V_1$, *and the condition*

$$\operatorname{Ker}\sigma \cdot \operatorname{Ker} t = 0 = \operatorname{Ker}\tau \cdot \operatorname{Ker} s \tag{2.3.12}$$

holds, i.e. $a \cdot v = 0 = a' \cdot v'$ *for all* $a \in \operatorname{Ker}\sigma$, $v \in \operatorname{Ker} s$, $a' \in \operatorname{Ker}\tau$, $v' \in \operatorname{Ker} t$.

Remark 2.3.7. *Note that Forrester-Barker gave in [45] a definition of a left module over a* cat^1*-algebra, but omitted the important condition* (2.3.12)*, which is essential in order to prove that this notion is equivalent to that of a left module over the corresponding crossed module of algebras.*

By complete analogy to [4, Proposition 8] it is possible to prove that cat^1-modules are equivalent to 2-term chain complexes of K-modules. This equivalence is very similar to the one between *cat*^{1}-objects and crossed modules showed in Section 1.2 for the five main categories considered in this thesis. A 2-term chain complex of K-modules is none other than a K-module homomorphism $\delta \colon V \to W$, which can be considered as an abelian crossed module of algebras. The corresponding cat^1-module is $(V \oplus W, W, s, t)$, where $s(v, w) = w$ and $t(v, w) = \delta(v) + w$ for all $v \in V$, $w \in W$.

Theorem 2.3.8. *Let* (B, A, ρ) *be a crossed module of algebras. An abelian crossed module of algebras* (V, W, δ) *is a left module over* (B, A, ρ) *if and only if the corresponding* cat^1*-module* $(V \oplus W, W, s, t)$ *is a left module over the corresponding* cat^1*-algebra* $(B \rtimes A, A, \sigma, \tau)$.

Proof. Let (V, W, δ) be a (B, A, ρ)-module. Then there is a morphism (φ, ψ) of crossed modules of algebras

$$\begin{tikzcd} B \arrow[r, "\rho"] \arrow[d, "\varphi"'] & A \arrow[d, "\psi"] \\ \operatorname{Hom}_K(W, V) \arrow[r, "\Gamma"'] & \operatorname{End}(V, W, \delta). \end{tikzcd} \tag{2.3.13}$$

Let us assume that $\psi(a) = (\alpha_a, \beta_a)$, with $\alpha_a \in \operatorname{End}_K(V)$ and $\beta_a \in \operatorname{End}_K(W)$ for all $a \in A$. It is clear that A acts (to the left) on V and W via ψ, namely $a \cdot v = \alpha_a(v)$ and $a \cdot w = \beta_a(w)$ for all $a \in A$, $v \in V$, $w \in W$. Routine calculations show that the equality

$$(b, a) \cdot (v, w) = (\varphi(b)(w) + (\rho(b) + a) \cdot v, a \cdot w),$$

where $(b, a) \in B \rtimes A$, $(v, w) \in V \oplus W$, defines a left action of the algebra $B \rtimes A$ on the K-module $V \oplus W$ and the structural morphisms commute with these actions. In order to check the condition (2.3.12), note that $\operatorname{Ker}\sigma$ (respectively $\operatorname{Ker}\tau$, $\operatorname{Ker} s$, $\operatorname{Ker} t$) consists of all elements of the form $(b, 0)$ (respectively $(b, -\rho(b))$, $(v, 0)$, $(v, -\delta(v))$), with $b \in B$, $v \in V$. Furthermore, we have that

$$(b, -\rho(b)) \cdot (v, 0) = (\varphi(b)(0) + (\rho(b) - \rho(b)) \cdot v, 0) = (0, 0),$$
$$(b, 0) \cdot (v, -\delta(v)) = (-\varphi(b)\delta(v) + \rho(b) \cdot v, 0) = (0, 0),$$

since $\varphi(b)\delta(v) = \rho(b) \cdot v$ by commutativity of the diagram (2.3.13).

Conversely, given a left $(B \rtimes A, A, \sigma, \tau)$-module structure on the cat^1-module $(V \oplus W, W, s, t)$, we can define the maps φ and ψ in the diagram (2.3.13) by the following equalities:

$$\begin{aligned}(\varphi(b)(w), 0) &= (b, 0) \cdot (0, w),\\ (\alpha_a(v), 0) &= (0, a) \cdot (v, 0),\\ \beta_a(w) &= a \cdot w,\end{aligned}$$

for all $b \in B$, $a \in A$, $v \in V$ and $w \in W$, where $\psi(a) = (\alpha_a, \beta_a)$. The remaining details, which can be checked by straightforward calculations, are left to the reader. $\square$

Chapter 3

Adjunctions between categories of crossed modules

In Section 3.1 we recall the generalizations to crossed modules of the unit group and the group algebra functors and the one-to-one correspondence between module structures over a crossed module of groups and its respective crossed module of associative algebras [22]. Although those 2-dimensional extensions are still adjoint to one another, in Subsection 3.1.2 we prove that the generalization of the group algebra functor does not commute, not even up to isomorphism, with the full embeddings $\mathtt{E}_1$ and $\mathtt{E}'_1$; in contrast to what happens with the universal enveloping crossed module of a Lie or a Leibniz crossed module, as explained in Subsections 3.2.2 and 3.3.2.

In Section 3.2 we construct a pair of adjoint functors between the categories of crossed modules of Lie and associative algebras, which extends the classical one between the categories of Lie and associative algebras. This result is used to establish an isomorphism between the categories of modules over a Lie crossed module and its universal enveloping crossed module.

In Section 3.3 we extend the main results from Section 3.2 to the non-commutative framework, that is we generalize the well-known adjunction between the categories of Leibniz algebras and dialgebras to the categories of Leibniz crossed modules and crossed modules of dialgebras.

Finally, in Section 3.4 we generalize the well-known relations between the categories of Lie, Leibniz, associative algebras and dialgebras to the respective categories of crossed modules and assemble them into four commutative parallelepipeds of categories and functors.

3.1 XGr vs XAs^1

Given a unital algebra A its subset of all invertible elements $\mathtt{U_g}(A)$ forms a group called the group of units of the algebra A. Besides, given a morphism of unital algebras $f\colon A \to B$, it is possible to consider the group homomorphism $\mathtt{U_g}(f)\colon \mathtt{U_g}(A) \to \mathtt{U_g}(B)$, where $\mathtt{U_g}(f) = f|_{\mathtt{U_g}(A)}$. These assignments define a functor $\mathtt{U_g}\colon \mathsf{As}^1 \to \mathsf{Gr}$ called the unit group functor.

The left adjoint to $\mathtt{U_g}$ is the functor $\mathtt{K}\colon \mathsf{Gr} \to \mathsf{As}^1$, which sends every group G to its group algebra $\mathtt{K}(G)$, that is the free module over K on the underlying set of G with the multiplication defined on the basis elements by the group operation in G. Given a group homomorphism $f\colon G \to H$, its corresponding morphism of algebras is $\mathtt{K}(f)\colon \mathtt{K}(G) \to \mathtt{K}(H)$, which is defined by extending f to the elements of $\mathtt{K}(G)$ by linearity.

In this section we recall the construction of the natural generalization of these functors,

$$\mathsf{XGr} \underset{\mathtt{XK}}{\overset{\mathtt{XU_g}}{\rightleftarrows}} \mathsf{XAs}^1$$

which can be found in [22].

3.1.1 The crossed module of units

Let us begin with the definition of $\mathtt{XU_g}$. Given a crossed module of unital algebras (B, A, ρ), consider its corresponding cat^1-algebra as described in the proof of Proposition 1.2.34:

$$B \rtimes A \underset{\tau}{\overset{\sigma}{\rightrightarrows}} A$$

with $\sigma(b, a) = a$ and $\tau(b, a) = \rho(b) + a$ for all $(b, a) \in B \rtimes A$. Note that the unit of $B \rtimes A$ is $(0, 1)$.

Lemma 3.1.1. *Given an object (B, A, ρ) in XAs^1 and its corresponding* cat^1*-algebra $(B \rtimes A, A, \sigma, \tau)$,*

(i) Every element in $\operatorname{Ker} \mathtt{U_g}(\sigma)$ is of the form $(b, 1)$, with $b \in B$, and there is $b' \in B$ such that

$$bb' + b' + b = 0 = b'b + b + b'. \tag{3.1.1}$$

(ii) $(\mathtt{U_g}(B \rtimes A), \mathtt{U_g}(A), \mathtt{U_g}(\sigma), \mathtt{U_g}(\tau))$ is a cat^1*-group.*

Proof. *(i)* Given $(b, a) \in \operatorname{Ker} \mathtt{U_g}(\sigma)$, $1 = \mathtt{U_g}(\sigma)(b, a) = a$. Therefore $(b, a) = (b, 1)$. On the other hand, $(b, 1) \in \mathtt{U_g}(B \rtimes A)$, so there is $(b', a') \in B \rtimes A$ such that $(b, 1)(b', a') = (0, 1) = (b', a')(b, 1)$. Directly from the definition of the multiplication in $B \rtimes A$, we get that $a' = 1$ and $bb' + b' + b = 0 = b'b + b + b'$.

(ii) The identities $\mathtt{U_g}(\sigma)|_{\mathtt{U_g}(A)} = \mathtt{U_g}(\tau)|_{\mathtt{U_g}(A)} = \mathrm{id}_{\mathtt{U_g}(A)}$ follow immediately from $\sigma|_A = \tau|_A = \mathrm{id}_A$ and the definition of the functor $\mathtt{U_g}$.

It only remains to prove that $[\operatorname{Ker}\mathtt{U_g}(\sigma), \operatorname{Ker}\mathtt{U_g}(\tau)] = 1$. Let $(b_1, 1) \in \operatorname{Ker}\mathtt{U_g}(\sigma)$ and $(b_2, a_2) \in \operatorname{Ker}\mathtt{U_g}(\tau)$. Note that $1 = \mathtt{U_g}(\tau)(b_2, a_2) = \rho(b_2) + a_2$, so $a_2 = 1 - \rho(b_2)$. Using the definition of the product in $B \rtimes A$ and (XAs2), we get that:

$$\begin{aligned}(b_1, 1)(b_2, 1-\rho(b_2)) &= (b_1b_2 + b_2 + b_1 - b_1\rho(b_2), 1 - \rho(b_2))\\ &= (b_2 + b_1, 1 - \rho(b_2))\\ &= (b_2b_1 + b_1 - \rho(b_2)b_1 + b_2, 1 - \rho(b_2))\\ &= (b_2, 1 - \rho(b_2))(b_1, 1).\end{aligned}$$

Therefore $[\operatorname{Ker}\mathtt{U_g}(\sigma), \operatorname{Ker}\mathtt{U_g}(\tau)] = 1$. □

Now we can define $\mathtt{XU_g}(B, A, \rho) = (\operatorname{Ker}\mathtt{U_g}(\sigma), \mathtt{U_g}(A), \mathtt{U_g}(\tau)|_{\operatorname{Ker}\mathtt{U_g}(\sigma)})$ for any crossed module of unital algebras (B, A, ρ), where $(\mathtt{U_g}(B \rtimes A), \mathtt{U_g}(A), \mathtt{U_g}(\sigma), \mathtt{U_g}(\tau))$ is the result of applying $\mathtt{U_g}$ to the cat^1-algebra $(B \rtimes A, A, \sigma, \tau)$. Sometimes we will write $\mathtt{U_g}(\tau)$ instead of $\mathtt{U_g}(\tau)|_{\operatorname{Ker}\mathtt{U_g}(\sigma)}$ for ease of notation.

Remark 3.1.2. *Note that $\mathtt{U_g}(A)$ can be regarded as a subgroup of $\mathtt{U_g}(B \times A)$ via the inclusion $a \mapsto (0, a)$ and the action of $\mathtt{U_g}(A)$ on $\operatorname{Ker}\mathtt{U_g}(\sigma)$ is defined in the proof of Proposition 1.2.12 as conjugation. Namely, given $a \in \mathtt{U_g}(A)$ and $(b, 1) \in \operatorname{Ker}\mathtt{U_g}(\sigma)$,*

$${}^{a}(b, 1) = {}^{(0,a)}(b, 1) = (0, a)(b, 1)(0, a^{-1}) = (ab, a)(0, a^{-1}) = (aba^{-1}, 1).$$

For any morphism of crossed modules of algebras $(\varphi, \psi)\colon (B, A, \rho) \to (B', A', \rho')$, the corresponding morphism of crossed modules of groups $\mathtt{XU_g}(\varphi, \psi)$ is given by:

$$\begin{tikzcd}
\operatorname{Ker}\mathtt{U_g}(\sigma) \arrow[r, "\mathtt{U_g}(\tau)"] \arrow[d, "{(\varphi,1)}"'] & \mathtt{U_g}(A) \arrow[d, "\psi"] \\
\operatorname{Ker}\mathtt{U_g}(\sigma') \arrow[r, "\mathtt{U_g}(\tau')"'] & \mathtt{U_g}(A').
\end{tikzcd}$$

Recall that a group G (resp. a unital algebra A) can be regarded as a crossed module of groups (resp. a crossed module of unital algebras) in two obvious ways: via the trivial map $1\colon \{1\} \to G$ (resp. $0\colon \{0\} \to A$) or via the identity map $\mathrm{id}_G\colon G \to G$ (resp. $\mathrm{id}_A\colon A \to A$) with the action of G (resp. of A) on itself defined by conjugation (resp. by multiplication). See Example 1.2.5 (i) for groups and Example 1.2.28 (i) for algebras. We have the functors:

$$\mathtt{E}_0, \mathtt{E}_1\colon \mathbf{Gr} \longrightarrow \mathbf{XGr} \qquad\qquad \mathtt{E}'_0, \mathtt{E}'_1\colon \mathbf{As}^1 \longrightarrow \mathbf{XAs}^1$$

where $\mathtt{E}_0(G) = (\{1\}, G, 1)$, $\mathtt{E}_1(G) = (G, G, \mathrm{id}_G)$, $\mathtt{E}'_0(A) = (\{0\}, A, 0)$ and $\mathtt{E}'_1(A) = (A, A, \mathrm{id}_A)$, given any group G and any unital algebra A.

It is easy to check that the following diagram is commutative.

$$
\begin{tikzcd}
\mathbf{As}^1 \arrow[r, "\mathtt{E}_0'"] \arrow[d, "\mathtt{U}_\mathtt{g}"'] & \mathbf{XAs}^1 \arrow[d, "\mathtt{XU}_\mathtt{g}"] \\
\mathbf{Gr} \arrow[r, "\mathtt{E}_0"'] & \mathbf{XGr}\,.
\end{tikzcd}
$$

Note that $\operatorname{Ker}\mathtt{U}_\mathtt{g}(\sigma) = \{1\}$ for $\mathtt{U}_\mathtt{g}(\sigma)\colon \mathtt{U}_\mathtt{g}(\{0\} \rtimes A) \to \mathtt{U}_\mathtt{g}(A)$, with $\mathtt{U}_\mathtt{g}(\sigma)(0,a) = a$, for all $a \in A$. Concerning the embeddings $\mathtt{E}_1$ and $\mathtt{E}_1'$, we have the following result:

Proposition 3.1.3 ([22]). *There is a natural isomorphism of functors*

$$\mathtt{XU}_\mathtt{g} \circ \mathtt{E}_1' \cong \mathtt{E}_1 \circ \mathtt{U}_\mathtt{g}\,.$$

Proof. Let $A \in \mathbf{As}^1$. We need to prove that $\mathtt{XU}_\mathtt{g}(A, A, \mathrm{id}_A)$ is naturally isomorphic to $(\mathtt{U}_\mathtt{g}(A), \mathtt{U}_\mathtt{g}(A), \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$. According to the foregoing definition of $\mathtt{XU}_\mathtt{g}$, we first need to consider the cat^1-algebra $(A \rtimes A, A, \sigma, \tau)$, with $\sigma(a, a') = a'$ and $\tau(a, a') = a + a'$ for all $(a, a') \in A \rtimes A$. Then we apply $\mathtt{U}_\mathtt{g}$ to $(A \rtimes A, A, \sigma, \tau)$ and $\mathtt{XU}_\mathtt{g}(A, A, \mathrm{id}_A) = (\operatorname{Ker}\mathtt{U}_\mathtt{g}(\sigma), \mathtt{U}_\mathtt{g}(A), \mathtt{U}_\mathtt{g}(\tau))$. It is clear that $(\mathtt{U}_\mathtt{g}(\tau), \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$ is a morphism of crossed modules of groups from $\mathtt{XU}_\mathtt{g}(A, A, \mathrm{id}_A)$ to $(\mathtt{U}_\mathtt{g}(A), \mathtt{U}_\mathtt{g}(A), \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$ (see Example 1.2.8 (iii)). Note that every element in $\operatorname{Ker}\mathtt{U}_\mathtt{g}(\sigma)$ is of the form $(a, 1)$, with $a \in A$ (see Lemma 3.1.1 (i)) and $\mathtt{U}_\mathtt{g}(\tau)(a, 1) = a + 1$. It is easy to check that $\phi\colon A \to A \rtimes A$, $a \mapsto (a-1, 1)$ is a morphism of algebras. Therefore $\mathtt{U}_\mathtt{g}(\phi) = \phi|_{\mathtt{U}_\mathtt{g}(A)}$ is a group homomorphism from $\mathtt{U}_\mathtt{g}(A)$ to $\mathtt{U}_\mathtt{g}(A \rtimes A)$. Furthermore, $\phi(\mathtt{U}_\mathtt{g}(A)) \subset \operatorname{Ker}\mathtt{U}_\mathtt{g}(\sigma)$. The diagram

$$
\begin{tikzcd}
\mathtt{U}_\mathtt{g}(A) \arrow[r, "\mathrm{id}_{\mathtt{U}_\mathtt{g}(A)}"] \arrow[d, "\phi"'] & \mathtt{U}_\mathtt{g}(A) \arrow[d, "\mathrm{id}_{\mathtt{U}_\mathtt{g}(A)}"] \\
\operatorname{Ker}\mathtt{U}_\mathtt{g}(\sigma) \arrow[r, "\mathtt{U}_\mathtt{g}(\tau)"'] & \mathtt{U}_\mathtt{g}(A)
\end{tikzcd}
$$

is clearly commutative. Additionally, given $a_1, a_2 \in \mathtt{U}_\mathtt{g}(A)$,

$$
\begin{aligned}
\phi({}^{a_1}a_2) = \phi(a_1 a_2 a_1^{-1}) &= (a_1 a_2 a_1^{-1} - 1, 1) = (a_1 a_2 a_1^{-1} - a_1 a_1^{-1}, 1) \\
&= (a_1(a_2 - 1)a_1^{-1}, 1) = {}^{a_1}(a_2 - 1, 1) = {}^{a_1}\phi(a_2),
\end{aligned}
$$

so $(\phi, \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$ is a morphism of crossed modules of groups. It is easy to check that $(\phi, \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$ is the inverse of $(\mathtt{U}_\mathtt{g}(\tau), \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$ and the naturality of $(\mathtt{U}_\mathtt{g}(\tau), \mathrm{id}_{\mathtt{U}_\mathtt{g}(A)})$ is obvious. $\square$

3.1.2 The group algebra crossed module

Let us now recall the construction of the left adjoint to $\mathtt{XK}$. Let (H, G, ∂) be a crossed module of groups and consider its corresponding cat^1-group as in the proof of Proposition 1.2.12, that is

$$\begin{tikzcd} H \rtimes G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G \end{tikzcd}$$

with $s(h, g) = g$ and $t(h, g) = \partial(h)g$ for all $(h, g) \in H \rtimes G$. If we apply $\mathtt{K}$ to the previous diagram we get

$$\begin{tikzcd} \mathtt{K}(H \rtimes G) \arrow[r, shift left, "\mathtt{K}(s)"] \arrow[r, shift right, "\mathtt{K}(t)"'] & \mathtt{K}(G) \end{tikzcd}$$

Although it is true that $\mathtt{K}(s)|_{\mathtt{K}(G)} = \mathtt{K}(t)|_{\mathtt{K}(G)} = \mathrm{id}_{\mathtt{K}(G)}$, in general, the second condition for *cat*^{1}-algebras (CAs2) is not satisfied. For instance, let us take $h \in H \setminus \{1\}$ and $g \in G$ such that ${}^g h \neq h^{-1}$. It is clear that $v = (h, g) - (1, g) \in \operatorname{Ker} \mathtt{K}(s)$ and $w = (h, g) - (1, \partial(h)g) \in \operatorname{Ker} \mathtt{K}(t)$. Besides,

$$vw = (h^g h, gg) - ({}^g h, gg) - (h, g\partial(h)g) + (1, g\partial(h)g),$$

is a linear combination of elements from the basis of $\mathtt{K}(H \rtimes G)$ with non-zero coefficients due to the condition ${}^g h \neq h^{-1}$. Hence, $\operatorname{Ker} \mathtt{K}(s) \operatorname{Ker} \mathtt{K}(t) \neq 0$ in general.

Nevertheless, we can consider the quotient $\overline{\mathtt{K}}(H \rtimes G) = \mathtt{K}(H \rtimes G)/X$, where $X = \operatorname{Ker} \mathtt{K}(s) \operatorname{Ker} \mathtt{K}(t) + \operatorname{Ker} \mathtt{K}(t) \operatorname{Ker} \mathtt{K}(s)$, together with the induced morphisms $\overline{\mathtt{K}}(s)$ and $\overline{\mathtt{K}}(t)$. In this way, the diagram

$$\begin{tikzcd} \overline{\mathtt{K}}(H \rtimes G) \arrow[r, shift left, "\overline{\mathtt{K}}(s)"] \arrow[r, shift right, "\overline{\mathtt{K}}(t)"'] & \mathtt{K}(G) \end{tikzcd}$$

is clearly a cat^1-algebra. Note that $X \subset \operatorname{Ker} \mathtt{K}(t)$ and $\mathtt{K}(t)|_{\mathtt{K}(G)} = \mathrm{id}_{\mathtt{K}(G)}$, since $t|_G = \mathrm{id}_G$. Given $v,\ w \in K(\{1\} \rtimes G) \simeq \mathtt{K}(G)$, if $v - w \in X$, then $0 = \mathtt{K}(t)(v - w) = v - w$. Therefore $\mathtt{K}(G)$ can be regarded as a subalgebra of $\overline{\mathtt{K}}(H \rtimes G)$.

We can now define $\mathtt{XK}(H, G, \partial)$ as the crossed module of algebras corresponding to the previous cat^1-algebra, that is $(\operatorname{Ker} \overline{\mathtt{K}}(s), \mathtt{K}(G), \overline{\mathtt{K}}(t)|_{\operatorname{Ker} \overline{\mathtt{K}}(s)})$. Sometimes we will write $\overline{\mathtt{K}}(t)$ instead of $\overline{\mathtt{K}}(t)|_{\operatorname{Ker} \overline{\mathtt{K}}(s)}$ to ease notation.

For any morphism of crossed modules $(\varphi, \psi)\colon (H, G, \partial) \to (H', G', \partial')$, $\mathtt{XK}(\varphi, \psi)$ is given by

$$\begin{tikzcd} \operatorname{Ker} \overline{\mathtt{K}}(s) \arrow[r, "\overline{\mathtt{K}}(t)"] \arrow[d, "\overline{\mathtt{K}}(\varphi,\psi)|_{\operatorname{Ker} \overline{\mathtt{K}}(s)}"'] & \mathtt{K}(G) \arrow[d, "\mathtt{K}(\psi)"] \\ \operatorname{Ker} \overline{\mathtt{K}}(s') \arrow[r, "\overline{\mathtt{K}}(t')"'] & \mathtt{K}(G') \end{tikzcd}$$

where $\overline{\mathtt{K}}(\varphi,\psi)$ is the algebra homomorphism induced by $\mathtt{K}(\varphi,\psi)$, which is itself the K-linear extension of $(\varphi,\psi)\colon H \rtimes G \to H' \rtimes G'$, given by $(\varphi,\psi)(h,g) = (\varphi(h),\psi(g))$ for all $(h,g) \in H \rtimes G$.

The functor $\mathtt{XK}\colon \mathbf{XGr} \to \mathbf{XAs}^1$ is a natural generalization of the functor $\mathtt{K}$, in the sense that the following diagram commutes,

$$\begin{tikzcd} \mathbf{Gr} \arrow[r, "\mathtt{E}_0"] \arrow[d, "\mathtt{K}"'] & \mathbf{XGr} \arrow[d, "\mathtt{XK}"] \\ \mathbf{As}^1 \arrow[r, "\mathtt{E}'_0"'] & \mathbf{XAs}^1 . \end{tikzcd}$$

In fact, given G a group, it is clear that $\mathtt{E}_0(\mathtt{K}(G)) = (\{0\},\mathtt{K}(G),0)$. Now, let us follow the steps in the construction of $\mathtt{XK}(\{1\},G,1)$. First we consider the cat^1-group

$$\{1\} \rtimes G \overset{s}{\underset{t}{\rightrightarrows}} G$$

with $s(1,g) = t(1,g) = g$. Then we apply the functor $\mathtt{K}$,

$$\mathtt{K}(\{1\} \rtimes G) \overset{\mathtt{K}(s)}{\underset{\mathtt{K}(t)}{\rightrightarrows}} \mathtt{K}(G).$$

It is clear that $\mathtt{K}(s) = \mathtt{K}(t)$ is an isomorphism between the algebras $\mathtt{K}(\{1\} \rtimes G)$ and $\mathtt{K}(G)$. Hence $\operatorname{Ker}\mathtt{K}(s) = \operatorname{Ker}\mathtt{K}(t) = \{0\}$ and $(\mathtt{K}(\{1\} \rtimes G),\mathtt{K}(G),\mathtt{K}(s),\mathtt{K}(t))$ is a cat^1-algebra. Moreover $\mathtt{XK}(\{1\},G,1) = (\operatorname{Ker}\mathtt{K}(t),\mathtt{K}(G),\mathtt{K}(t)|_{\operatorname{Ker}\mathtt{K}(t)}) = (\{0\},\mathtt{K}(G),0)$.

Nevertheless, for the functor $\mathtt{XK}$ there is no analogue to Proposition 3.1.3, that is we claim that $\mathtt{XK} \circ \mathtt{E}_1 \not\cong \mathtt{E}'_1 \circ \mathtt{K}$. Let us first illustrate it with an example.

Let $G = \{e,x\}$ be the group with two elements and $K = \mathbb{R}$. $\mathtt{E}'_1(\mathtt{K}(G)) = (\mathtt{K}(G),\mathtt{K}(G),\mathrm{id}_{\mathtt{K}(G)})$, where $\mathtt{K}(G)$ is a vector space over $\mathbb{R}$ of dimension 2, endowed with the product resulting from extending by bilinearity the group operation in G. A base of $\mathtt{K}(G)$ is $\{e,x\}$. We will denote $e = (1,0)$ and $x = (0,1)$.

Let us now construct $\mathtt{XK}(G,G,\mathrm{id}_G)$. First, we need to consider the cat^1-group

$$G \rtimes G \overset{s}{\underset{t}{\rightrightarrows}} G,$$

with

$$\begin{aligned} s(e,e) &= e, & t(e,e) &= e, \\ s(e,x) &= x, & t(e,x) &= x, \\ s(x,e) &= e, & t(x,e) &= x, \\ s(x,x) &= x, & t(x,x) &= e. \end{aligned}$$

The next step is to apply $\mathtt{K}$ on that diagram. In this way, we get:

$$\mathtt{K}(G \rtimes G) \underset{\mathtt{K}(t)}{\overset{\mathtt{K}(s)}{\rightrightarrows}} \mathtt{K}(G),$$

with $\mathtt{K}(s)$ and $\mathtt{K}(t)$ as a result of extending by linearity s and t respectively. A basis for $\mathtt{K}(G \rtimes G)$ is given by $\{(e,e),(e,x),(x,e),(x,x)\}$. We will denote

$$\begin{aligned}(e,e) &= (1,0,0,0),\\ (e,x) &= (0,1,0,0),\\ (x,e) &= (0,0,1,0),\\ (x,x) &= (0,0,0,1).\end{aligned}$$

Bearing in mind the previous notation,

$$\begin{aligned}\mathtt{K}(s)(1,0,0,0) &= (1,0), & \mathtt{K}(t)(1,0,0,0) &= (1,0),\\ \mathtt{K}(s)(0,1,0,0) &= (0,1), & \mathtt{K}(t)(0,1,0,0) &= (0,1),\\ \mathtt{K}(s)(0,0,1,0) &= (1,0), & \mathtt{K}(t)(0,0,1,0) &= (0,1),\\ \mathtt{K}(s)(0,0,0,1) &= (0,1), & \mathtt{K}(t)(0,0,0,1) &= (1,0).\end{aligned}$$

Hence,

$$\mathtt{K}(s) = \begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix} \quad \text{and} \quad \mathtt{K}(t) = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\end{pmatrix}.$$

Now we can easily calculate $\operatorname{Ker}\mathtt{K}(s)$ and $\operatorname{Ker}\mathtt{K}(t)$:

$$\begin{aligned}\operatorname{Ker}\mathtt{K}(s) &= \{(a_1,a_2,a_3,a_4) \mid a_1 = -a_3, a_2 = -a_4, a_i \in \mathbb{R},\ \text{for } i = 1,2,3,4\}\\ &= \{(a_1,a_2,-a_1,-a_2) \mid a_1,a_2 \in \mathbb{R}\}\\ &= \langle (1,0,-1,0),(0,1,0,-1)\rangle\,,\\ \operatorname{Ker}\mathtt{K}(t) &= \{(a_1,a_2,a_3,a_4) \mid a_1 = -a_4, a_2 = -a_3, a_i \in \mathbb{R},\ \text{for } i = 1,2,3,4\}\\ &= \{(a_1,a_2,-a_2,-a_1) \mid a_1,a_2 \in \mathbb{R}\}\\ &= \langle (1,0,0,-1),(0,1,-1,0)\rangle\,.\end{aligned}$$

Note that Forrester-Barker gives in [45] an explicit description of one basis for $\operatorname{Ker}\mathtt{K}(s)$ and for $\operatorname{Ker}\mathtt{K}(t)$. Those bases agree with the ones described above. Namely, according to Forrester-Barker,

$$\begin{aligned}
\operatorname{Ker}\mathtt{K}(s) &= \langle\{(g,g')-(e,g') \mid g,g'\in G, g\neq e\}\rangle\\
&= \langle\{(x,e)-(e,e),(x,x)-(e,x)\}\rangle\\
&= \langle(0,0,1,0)-(1,0,0,0),(0,0,0,1)-(0,1,0,0)\rangle\\
&= \langle(-1,0,1,0),(0,-1,0,1)\rangle\,,\\
\operatorname{Ker}\mathtt{K}(t) &= \langle\{(g,g')-(e,gg') \mid g,g'\in G, g\neq e\}\rangle\\
&= \langle\{(x,e)-(e,x),(x,x)-(e,e)\}\rangle\\
&= \langle(0,0,1,0)-(0,1,0,0),(0,0,0,1)-(1,0,0,0)\rangle\\
&= \langle(0,-1,1,0),(-1,0,0,1)\rangle\,.
\end{aligned}$$

Note that $\operatorname{Ker}\mathtt{K}(s)+\operatorname{Ker}\mathtt{K}(t)$ has dimension 3 and $\operatorname{Ker}\mathtt{K}(s)\cap\operatorname{Ker}\mathtt{K}(t)=\langle(1,1,-1,-1)\rangle$.

Observe that the product in $\mathtt{K}(G\rtimes G)$ and the product in $\mathtt{K}(G)$ are both commutative, since G is an abelian group. Therefore $X=\operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t)+\operatorname{Ker}\mathtt{K}(t)\operatorname{Ker}\mathtt{K}(s)=\operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t)$. Fairly straightforward calculations show that $\operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t)=\langle(1,1,-1-1)\rangle$ which agrees with what we would get if we followed the description given by Forrester-Barker.

In order to get a cat^1-algebra, we need to consider the diagram

$$\mathtt{K}(G\rtimes G)/X \underset{\overline{\mathtt{K}}(t)}{\overset{\overline{\mathtt{K}}(s)}{\rightrightarrows}} \mathtt{K}(G),$$

with $\overline{\mathtt{K}}(s)$ and $\overline{\mathtt{K}}(t)$ induced by $\mathtt{K}(s)$ and $\mathtt{K}(t)$. Note that $\mathtt{K}(G\rtimes G)/X$ has dimension 3. We can consider the basis $B=\{(0,1,0,0)+X,(0,0,1,0)+X,(0,0,0,1)+X\}$.

Let us give an explicit basis for $\operatorname{Ker}\overline{\mathtt{K}}(s)$:

$$\operatorname{Ker}\overline{\mathtt{K}}(s)=\{v+X\in\mathtt{K}(G\rtimes G)/X \mid \overline{\mathtt{K}}(s)(v+X)=(0,0)\},$$

but $X=\operatorname{Ker}\mathtt{K}(s)\cap\operatorname{Ker}\mathtt{K}(t)\subset\operatorname{Ker}\mathtt{K}(s)$, so

$$\operatorname{Ker}\overline{\mathtt{K}}(s)=\{v+X\in\mathtt{K}(G\rtimes G)/X \mid \mathtt{K}(s)(v)=(0,0)\}=\operatorname{Ker}\mathtt{K}(s)/X.$$

Moreover,

$$(-1,0,1,0)+X=(0,a,b,c)+X \Leftrightarrow (-1,-a,1-b,-c)\in X \Leftrightarrow a=1,b=0,c=-1,$$

$$\begin{aligned}(0,-1,0,1)+X=(0,a,b,c)+X &\Leftrightarrow (0,-1-a,-b,1-c)\in X\\ &\Leftrightarrow a=-1,b=0,c=1.\end{aligned}$$

Hence, $\operatorname{Ker}\overline{\mathtt{K}}(s)=\langle(1,0,-1)_B\rangle$ and it has dimension 1.

In the following diagram,

$$\begin{tikzcd}
\operatorname{Ker}\overline{\mathtt{K}}(s) \arrow[r, "\overline{\mathtt{K}}(t)"] \arrow[d, "\overline{\mathtt{K}}(t)"'] & \mathtt{K}(G) \arrow[d, "\mathrm{id}_{\mathtt{K}(G)}"] \\
\mathtt{K}(G) \arrow[r, "\mathrm{id}_{\mathtt{K}(G)}"'] & \mathtt{K}(G)
\end{tikzcd}$$

$\operatorname{Ker}\overline{\mathtt{K}}(s)$ is a $\mathbb{R}$-vector space of dimension 1, while $\mathtt{K}(G)$ has dimension 2. Therefore $\mathtt{XK}(G, G, \mathrm{id}_G) \ncong (\mathtt{K}(G), \mathtt{K}(G), \mathrm{id}_{\mathtt{K}(G)})$.

The problem shown in the previous example persists in a general situation. Let us now assume that G is any group, not necessarily the one with to elements. The first step in order to construct $\mathtt{XK}(G, G, \mathrm{id}_G)$ is to take the cat^1-group

$$G \rtimes G \underset{t}{\overset{s}{\rightrightarrows}} G,$$

with $s(g, g') = g'$ and $t(g, g') = gg'$. Consider the morphism of groups $\epsilon\colon G \to G \rtimes G$, $\epsilon(g) = (g, 1)$. It is clear that $s(\epsilon(g)) = 1$ and $t(\epsilon(g)) = g$ for every $g \in G$.

Now, if we use the functor $\mathtt{K}$, we get:

$$\mathtt{K}(G) \xrightarrow{\mathtt{K}(\epsilon)} \mathtt{K}(G \rtimes G) \underset{\mathtt{K}(t)}{\overset{\mathtt{K}(s)}{\rightrightarrows}} \mathtt{K}(G).$$

It is clear that $\mathtt{K}(t)\,\mathtt{K}(\epsilon) = \mathrm{id}_{\mathtt{K}(G)}$, while $\mathtt{K}(s)(\mathtt{K}(\epsilon)(\sum \lambda g)) = \mathtt{K}(s\epsilon)(\sum \lambda g) = \mathtt{K}(1)(\sum \lambda g)$ $= \sum \lambda 1$ for any formal linear combination of elements in G. Note that $\mathtt{K}(s)\,\mathtt{K}(\epsilon)$ is not the trivial map unless $K = \{0\}$.

Now, if we consider $X = \operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t) + \operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t)$, the diagram

$$\mathtt{K}(G \rtimes G)/X \underset{\overline{\mathtt{K}}(t)}{\overset{\overline{\mathtt{K}}(s)}{\rightrightarrows}} \mathtt{K}(G),$$

is a cat^1-algebra, with $\overline{\mathtt{K}}(s)$ and $\overline{\mathtt{K}}(t)$ induced by $\mathtt{K}(s)$ and $\mathtt{K}(t)$ respectively. The crossed module of algebras $\mathtt{XK}(G, G, \mathrm{id}_G)$ is defined as $(\operatorname{Ker}\overline{\mathtt{K}}(s), \mathtt{K}(G), \overline{\mathtt{K}}(t)|_{\operatorname{Ker}\overline{\mathtt{K}}(s)})$.

As we stated previously, in Forrester-Barker's PhD thesis [45] there is and explicit description of a basis for $\operatorname{Ker}\mathtt{K}(s) \subset \mathtt{K}(G \rtimes G)$, namely $\{(g, g') - (1, g') \mid g' \in G,\ g \in G \setminus \{1\}\}$. He also proves that $\{(g, g') - (1, g') + X \mid g \in G \setminus \{1\},\ g' \in G\}$ is a set of generators of $\operatorname{Ker}\overline{\mathtt{K}}(s)$.

Let $(g, g') - (1, g') + X \in \operatorname{Ker}\overline{\mathtt{K}}(s)$. Then $\overline{\mathtt{K}}(t)((g, g') - (1, g') + X) = gg' - g'$. Now, if we apply $\pi\,\mathtt{K}(\epsilon)$ to $gg' - g'$, with π the projection $\pi\colon \mathtt{K}(G \rtimes G) \to \mathtt{K}(G \rtimes G)/X$, we get $\pi\,\mathtt{K}(\epsilon)(gg' - g') = (gg', 1) - (g', 1) + X$, but $(gg', 1) - (g', 1) \sim (g, g') - (1, g')$. Actually $(gg', 1) - (g', 1) - (g, g') + (1, g') = ((g, 1) - (1, 1))((g', 1) - (1, g'))$, with $(g, 1) - (1, 1) \in \operatorname{Ker}\mathtt{K}(s)$ and $(g', 1) - (1, g') \in \operatorname{Ker}\mathtt{K}(t)$. Hence $(gg', 1) - (g', 1) - (g, g') + (1, g') \in X = \operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t) + \operatorname{Ker}\mathtt{K}(t)\operatorname{Ker}\mathtt{K}(s)$.

The previous calculations show that $\pi\,\mathtt{K}(\epsilon)\overline{\mathtt{K}}(t)|_{\mathrm{Ker}\,\overline{\mathtt{K}}(s)} = \mathrm{id}_{\mathrm{Ker}\,\overline{\mathtt{K}}(s)}$. Observe that for $\overline{\mathtt{K}}(t)|_{\mathrm{Ker}\,\overline{\mathtt{K}}(s)}$ to be an isomorphism between $\mathrm{Ker}\,\overline{\mathtt{K}}(s)$ and $\mathtt{K}(G)$, it should be a surjective morphism. Therefore $\pi\,\mathtt{K}(\epsilon)$ should take values in $\mathrm{Ker}\,\overline{\mathtt{K}}(s)$ for every element in $\mathtt{K}(G)$. In that case, $0 = \overline{\mathtt{K}}(s)\pi\,\mathtt{K}(\epsilon) = \mathtt{K}(s)\,\mathtt{K}(\epsilon)$, which is not true unless $K = \{0\}$.

Therefore $\mathtt{XK} \circ \mathtt{E}_1 \ncong \mathtt{E}'_1 \circ \mathtt{K}$.

3.1.3 Adjunction between $\mathbf{XGr}$ and $\mathbf{XAs}^1$

The following result is a natural generalization of the well-known classical adjunction between the categories $\mathbf{Gr}$ and $\mathbf{As}^1$.

Theorem 3.1.4 ([22]). *The functor* $\mathtt{XK}$ *is left adjoint to the functor* $\mathtt{XU_g}$*.*

Proof. Given (H, G, ∂) in $\mathbf{XGr}$ and (B, A, ρ) in $\mathbf{XAs}^1$, we have to construct a natural bijection

$$\mathrm{Hom}_{\mathbf{XGr}}\big((H,G,\partial), \mathtt{XU_g}(B,A,\rho)\big) \cong \mathrm{Hom}_{\mathbf{XAs}^1}\big(\mathtt{XK}(H,G,\partial),(B,A,\rho)\big).$$

Let $(\varphi,\psi) \in \mathrm{Hom}_{\mathbf{XGr}}\big((H,G,\partial), \mathtt{XU_g}(B,A,\rho)\big)$, that is

$$\begin{tikzcd}
H \arrow[r, "\partial"] \arrow[d, "\varphi"'] & G \arrow[d, "\psi"] \\
\mathrm{Ker}\,\mathtt{U_g}(\sigma) \arrow[r, "\mathtt{U_g}(\tau)"'] & \mathtt{U_g}(A)
\end{tikzcd}$$

such that $\mathtt{U_g}(\tau)\varphi = \psi\partial$ and $\varphi({}^{g}h) = {}^{\psi(g)}\varphi(h)$ for all $h \in H$, $g \in G$.

We can consider the corresponding morphism of cat^1-groups by using the functor $\mathtt{cat_{Gr}}$ as defined in the proof of Proposition 1.2.12:

$$\begin{tikzcd}
H \rtimes G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "{(\varphi,\psi)}"'] \arrow[dd, bend right=60, "\varphi'"'] & G \arrow[d, "\psi"] \\
\mathrm{Ker}\,\mathtt{U_g}(\sigma) \rtimes \mathtt{U_g}(A) \arrow[r, shift left, "\tilde{s}"] \arrow[r, shift right, "\tilde{t}"'] \arrow[d, "\simeq"'] & \mathtt{U_g}(A) \arrow[d, "\mathrm{id}_{\mathtt{U_g}(A)}"] \\
\mathtt{U_g}(B \rtimes A) \arrow[r, shift left, "\mathtt{U_g}(\sigma)"] \arrow[r, shift right, "\mathtt{U_g}(\tau)"'] & \mathtt{U_g}(A).
\end{tikzcd}$$

Note that the isomorphism in the diagram above is explicitly described at the end of the proof of Proposition 1.2.12, as well as $\tilde{s}$ and $\tilde{t}$.

Since the functor $\mathtt{K}$ is left adjoint to the functor $\mathtt{U_g}$, we have the induced commutative diagrams of algebras

$$\begin{tikzcd} \mathtt{K}(H \rtimes G) \arrow[r, "\mathtt{K}(s)"] \arrow[d, "\varphi'^*"'] & \mathtt{K}(G) \arrow[d, "\psi^*"] \\ B \rtimes A \arrow[r, "\sigma"'] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathtt{K}(H \rtimes G) \arrow[r, "\mathtt{K}(t)"] \arrow[d, "\varphi'^*"'] & \mathtt{K}(G) \arrow[d, "\psi^*"] \\ B \rtimes A \arrow[r, "\tau"'] & A. \end{tikzcd}$$

Due to the identity $\operatorname{Ker}\sigma \operatorname{Ker}\tau + \operatorname{Ker}\tau \operatorname{Ker}\sigma = 0$, we have a uniquely defined morphism of cat^1-algebras

$$\begin{tikzcd} \mathtt{K}(H \rtimes G)/X \arrow[r, shift left, "\overline{\mathtt{K}}(s)"] \arrow[r, shift right, "\overline{\mathtt{K}}(t)"'] \arrow[d, "\overline{\varphi}'^*"'] & \mathtt{K}(G) \arrow[d, "\psi^*"] \\ B \rtimes A \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & A \end{tikzcd}$$

where $X = \operatorname{Ker}\mathtt{K}(s)\operatorname{Ker}\mathtt{K}(t) + \operatorname{Ker}\mathtt{K}(t)\operatorname{Ker}\mathtt{K}(s)$.

Finally, we can make use the functor $\mathtt{Xm}_{\mathbf{As}}$, which is described in the proof of Proposition 1.2.34, in order to get a uniquely defined homomorphism (φ^{**}, ψ^*) in $\operatorname{Hom}_{\mathbf{XAs}^1}\big(\mathtt{XK}(H, G, \partial), (B, A, \rho)\big)$:

$$\begin{tikzcd} \operatorname{Ker}\overline{\mathtt{K}}(s) \arrow[r, "\overline{\mathtt{K}}(t)"] \arrow[d, "\overline{\varphi}'^*"'] \arrow[dd, bend right=60, "\varphi^{**}"'] & \mathtt{K}(G) \arrow[d, "\psi^*"] \\ \operatorname{Ker}\sigma \arrow[r, "\tau"] \arrow[d, "\simeq"'] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r, "\rho"] & A. \end{tikzcd}$$

Note that in the diagram above $\overline{\varphi}'^* = \overline{\varphi}'^*|_{\operatorname{Ker}\overline{\mathtt{K}}(s)}$, $\operatorname{Ker}\sigma = B \rtimes \{0\}$ and $\tau = \tau|_{\operatorname{Ker}\sigma} = \rho$.

Now, let $(\phi, \chi) \in \operatorname{Hom}_{\mathbf{XAs}^1}\big(\mathtt{XK}(H, G, \partial), (B, A, \rho)\big)$, that is

$$\begin{tikzcd} \operatorname{Ker}\overline{\mathtt{K}}(s) \arrow[r, "\overline{\mathtt{K}}(t)"] \arrow[d, "\phi"'] & \mathtt{K}(G) \arrow[d, "\chi"] \\ B \arrow[r, "\rho"'] & A \end{tikzcd}$$

such that $\rho\phi = \chi\overline{\mathtt{K}}(t)$ and ϕ preserves the action of $\mathtt{K}(G)$ on $\operatorname{Ker}\overline{\mathtt{K}}(s)$ via χ.

We can consider the corresponding morphism of cat^1-algebras by using the functor

$\mathtt{cat}_{\mathtt{As}}$ as defined in the proof of Proposition 1.2.34.

$$\begin{tikzcd}
\mathtt{K}(H \rtimes G)/X \arrow[r, shift left, "\overline{\mathtt{K}}(s)"] \arrow[r, shift right, "\overline{\mathtt{K}}(t)"'] \arrow[d, "\simeq"'] \arrow[dd, bend right=60, "\phi'"'] & \mathtt{K}(G) \arrow[d, "\mathrm{id}_{\mathtt{K}(G)}"] \\
\operatorname{Ker} \overline{\mathtt{K}}(s) \rtimes \mathtt{K}(G) \arrow[r, shift left, "\tilde{\sigma}"] \arrow[r, shift right, "\tilde{\tau}"'] \arrow[d, "(\phi{,}\chi)"'] & \mathtt{K}(G) \arrow[d, "\chi"] \\
B \rtimes A \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & A.
\end{tikzcd}$$

Note that the isomorphism in the diagram above is explicitly described at the end of the proof of Proposition 1.2.34, as well as $\tilde{\sigma}$ and $\tilde{\tau}$.

Let $\overline{\phi} = \phi'\pi$, with $\pi\colon \mathtt{K}(H \rtimes G) \to \mathtt{K}(H \rtimes G)/X$ the canonical projection. Hence, we have the commutative diagrams of algebras

$$\begin{tikzcd}
\mathtt{K}(H \rtimes G) \arrow[r, "\mathtt{K}(s)"] \arrow[d, "\overline{\phi}"'] & \mathtt{K}(G) \arrow[d, "\chi"] \\
B \rtimes A \arrow[r, "\sigma"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathtt{K}(H \rtimes G) \arrow[r, "\mathtt{K}(t)"] \arrow[d, "\overline{\phi}"'] & \mathtt{K}(G) \arrow[d, "\chi"] \\
B \rtimes A \arrow[r, "\tau"'] & A.
\end{tikzcd}$$

Since the functor $\mathtt{K}$ is left adjoint to the functor $\mathtt{U}_{\mathtt{g}}$, we have the morphism of cat^1-groups

$$\begin{tikzcd}
H \rtimes G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\phi^*"'] & G \arrow[d, "\chi^*"] \\
\mathtt{U}_{\mathtt{g}}(B \rtimes A) \arrow[r, shift left, "\mathtt{U}_{\mathtt{g}}(\sigma)"] \arrow[r, shift right, "\mathtt{U}_{\mathtt{g}}(\tau)"'] & \mathtt{U}_{\mathtt{g}}(A).
\end{tikzcd}$$

Finally, we can use the functor $\mathtt{Xm}_{\mathtt{Gr}}$, described in the proof of Proposition 1.2.12, to get a uniquely defined morphism (ϕ^{**}, χ^*) in $\mathrm{Hom}_{\mathbf{XGr}}\big((H, G, \partial), \mathtt{XU}_{\mathtt{g}}(B, A, \rho)\big)$:

$$\begin{tikzcd}
H \arrow[r, "\partial"] \arrow[d, "\simeq"'] \arrow[dd, bend right=60, "\phi^{**}"'] & G \arrow[d, "\mathrm{id}_G"] \\
\operatorname{Ker} s \arrow[r, "t"] \arrow[d, "\phi^*|_{\operatorname{Ker} s}"'] & G \arrow[d, "\chi^*"] \\
\operatorname{Ker} \mathtt{U}_{\mathtt{g}}(\sigma) \arrow[r, "\mathtt{U}_{\mathtt{g}}(\tau)"'] & \mathtt{U}_{\mathtt{g}}(A).
\end{tikzcd}$$

Note that $\operatorname{Ker} s = H \rtimes \{1\}$ and $t = t|_{\operatorname{Ker} s} = \partial$ in the diagram above. □

Recall that in Section 2.3 there is an explicit definition of left modules over crossed modules of groups and crossed modules of algebras. In [22], the authors prove that there is a one-to-one correspondence between left modules over a crossed module of groups and its respective crossed module of associative algebras:

Theorem 3.1.5 ([22])**.** *Let (H, G, ∂) be a crossed module of groups. Then the category of (H, G, ∂)-modules over K is isomorphic to the category of (left) $\mathtt{XK}(H, G, \partial)$-modules.*

At the end of Subsection 3.2.3 we will prove an analogue to the previous theorem for modules over a Lie crossed module and its corresponding crossed module of algebras.

3.2 XLie vs XAs

Any algebra A becomes a Lie algebra with the bracket $[a, b] = ab - ba$, $a, b \in A$. Moreover, any algebra homomorphism $f\colon A \to B$ is a Lie algebra homomorphism if we consider the bracket previously defined both in A and B. Those assignments define a functor that will be denoted by $\mathtt{Lie}_{\mathtt{As}}$.

Remark 3.2.1. *The reason to denote the previous functor by $\mathtt{Lie}_{\mathtt{As}}$ instead of simply $\mathtt{Lie}$ is that we will use another Liezation functor in the next section, and we want them to be easily distinguishable, even though the context itself will probably make the difference clear.*

The functor $\mathtt{Lie}_{\mathtt{As}}$ has a left adjoint, $\mathtt{U}\colon$ **Lie** $\to$ **As**, which sends a Lie algebra $\mathbb{p}$ to its universal enveloping algebra $\mathtt{U}(\mathbb{p})$. We recall that $\mathtt{U}(\mathbb{p}) = \mathrm{T}(\mathbb{p})/J$, with $\mathrm{T}(\mathbb{p})$ the tensor algebra of $\mathbb{p}$, that is

$$\mathrm{T}(\mathbb{p}) = K \oplus \mathbb{p} \oplus (\mathbb{p} \otimes \mathbb{p}) \oplus (\mathbb{p} \otimes \mathbb{p} \otimes \mathbb{p}) \cdots$$

and J the ideal generated by $p_1 \otimes p_2 - p_2 \otimes p_1 - [p_1, p_2]$, for all $p_1, p_2 \in \mathbb{p}$. Given a Lie homomorphism $f\colon \mathbb{p} \to \mathbb{m}$, we can define $\mathtt{U}(f)\colon \mathtt{U}(\mathbb{p}) \to \mathtt{U}(\mathbb{m})$ by extending the definition of f to the elements of $\mathtt{U}(\mathbb{p})$ by linearity.

In this section we generalize the functors $\mathtt{Lie}_{\mathtt{As}}$ and $\mathtt{U}$ to the categories **XAs** and **XLie** in such a way that those generalizations are still adjoint to one another.

3.2.1 Liezation of crossed modules of algebras

Let us first show that the functor $\mathtt{Lie}_{\mathtt{As}}$ preserves the semidirect product.

Lemma 3.2.2. *Given A and B two associative algebras together with an action of A on B, there is an action of $\mathtt{Lie}_{\mathtt{As}}(A)$ on $\mathtt{Lie}_{\mathtt{As}}(B)$ given by*

$$[a, b] = ab - ba$$

for all $a \in A$, $b \in B$.

Proof. We have to prove that

$$[[a,a'],b] = [a,[a',b]] - [a',[a,b]],$$
$$[a,[b,b']] = [[a,b],b'] + [b,[a,b']],$$

for every $a, a' \in \mathtt{Lie}_{\mathtt{As}}(A)$ and $b, b' \in \mathtt{Lie}_{\mathtt{As}}(B)$. Both equalities can be proved by straightforward calculations, the first one due to the equalities from Definition 1.2.24 involving two elements from A and one from B, and the second one due to the other three equalities from the same definition. □

Lemma 3.2.3. *Let A and B be two associative algebras together with an action of A on B. Then $\mathtt{Lie}_{\mathtt{As}}(B \rtimes A) = \mathtt{Lie}_{\mathtt{As}}(B) \rtimes \mathtt{Lie}_{\mathtt{As}}(A)$.*

Proof. Bearing in mind the previous lemma, it makes sense to consider the semidirect product $\mathtt{Lie}_{\mathtt{As}}(B) \rtimes \mathtt{Lie}_{\mathtt{As}}(A)$, since $\mathtt{Lie}_{\mathtt{As}}(A)$ acts on $\mathtt{Lie}_{\mathtt{As}}(B)$. It is clear that $\mathtt{Lie}_{\mathtt{As}}(B \rtimes A)$ and $\mathtt{Lie}_{\mathtt{As}}(B) \rtimes \mathtt{Lie}_{\mathtt{As}}(A)$ are equal as K-modules, so we only need to check that they share the same bracket. Let $(b,a),(b',a') \in B \times A$. If we use the bracket in $\mathtt{Lie}_{\mathtt{As}}(B) \rtimes \mathtt{Lie}_{\mathtt{As}}(A)$, we get

$$\begin{aligned}[(b,a),(b',a')] &= ([b,b'] + [a,b'] - [a',b], [a,a']) \\ &= (bb' - b'b + ab' - b'a + ba' - a'b, aa' - a'a).\end{aligned}$$

On the other hand, if we use the bracket in $\mathtt{Lie}_{\mathtt{As}}(B \rtimes A)$, we get

$$\begin{aligned}[(b,a),(b',a')] &= (b,a)(b',a') - (b',a')(b,a) \\ &= (bb' + ab' + ba', aa') - (b'b + a'b + b'a - ab', a'a),\end{aligned}$$

so the brackets are equal. □

As noted in [34], we can associate to a crossed module of algebras (B, A, ρ) the Lie crossed module $\big(\mathtt{Lie}_{\mathtt{As}}(B), \mathtt{Lie}_{\mathtt{As}}(A), \mathtt{Lie}_{\mathtt{As}}(\rho)\big)$ with the action of $\mathtt{Lie}_{\mathtt{As}}(A)$ on $\mathtt{Lie}_{\mathtt{As}}(B)$ described in Lemma 3.2.2. Note that $\mathtt{Lie}_{\mathtt{As}}(\rho) = \rho$. Directly from (XAs1), (XAs2) and the definition of the Lie action, we get that $\big(\mathtt{Lie}_{\mathtt{As}}(B), \mathtt{Lie}_{\mathtt{As}}(A), \mathtt{Lie}_{\mathtt{As}}(\rho)\big)$ satisfies (XLie1) and (XLie2).

Furthermore, any morphism of crossed modules of algebras $(\varphi, \psi)\colon (B, A, \rho) \to (B', A', \rho')$ is a morphism of Lie crossed modules from $\big(\mathtt{Lie}_{\mathtt{As}}(B), \mathtt{Lie}_{\mathtt{As}}(A), \rho\big)$ to $\big(\mathtt{Lie}_{\mathtt{As}}(B'), \mathtt{Lie}_{\mathtt{As}}(A'), \rho'\big)$, since

$$\varphi([a,b]) = \varphi(ab - ba) = \psi(a)\varphi(b) - \varphi(b)\psi(a) = [\psi(a), \varphi(b)],$$

for all $a \in A$, $b \in B$. The previous assignments define a functor $\mathtt{XLie}_{\mathtt{As}}\colon \mathsf{\textbf{XAs}} \to \mathsf{\textbf{XLie}}$, which is a natural generalization of the functor $\mathtt{Lie}_{\mathtt{As}}\colon \mathsf{\textbf{As}} \to \mathsf{\textbf{Lie}}$ in the following sense. Recall that there are full embeddings

$$\mathtt{I}_0, \mathtt{I}_1\colon \mathsf{\textbf{Lie}} \longrightarrow \mathsf{\textbf{XLie}} \qquad\qquad \mathtt{I}'_0, \mathtt{I}'_1\colon \mathsf{\textbf{As}} \longrightarrow \mathsf{\textbf{XAs}}$$

where $\mathtt{I}_0(\mathbb{p}) = (\{0\}, \mathbb{p}, 0)$, $\mathtt{I}_1(\mathbb{p}) = (\mathbb{p}, \mathbb{p}, \mathrm{id}_{\mathbb{p}})$, $\mathtt{I}'_0(A) = (\{0\}, A, 0)$ and $\mathtt{I}'_1(A) = (A, A, \mathrm{id}_A)$ for any Lie algebra $\mathbb{p}$ and any associative algebra A. It is immediate that the following diagram is commutative for $i = 0, 1$.

$$\begin{tikzcd} \mathbf{As} \arrow[r, "\mathtt{I}'_i"] \arrow[d, "\mathtt{Lie}_{\mathbf{As}}"'] & \mathbf{XAs} \arrow[d, "\mathtt{XLie}_{\mathbf{As}}"] \\ \mathbf{Lie} \arrow[r, "\mathtt{I}_i"'] & \mathbf{XLie}. \end{tikzcd}$$

3.2.2 Universal enveloping crossed module of a Lie crossed module

Let us now construct the left adjoint to the functor $\mathtt{XU}$, which generalizes the universal enveloping algebra functor $\mathtt{U}\colon \mathbf{Lie} \to \mathbf{As}$ to crossed modules.

Let $(\mathbb{m}, \mathbb{p}, \nu)$ be a Lie crossed module and consider its corresponding cat^1-Lie algebra as in the proof of Proposition 1.2.23, that is

$$\mathbb{m} \rtimes \mathbb{p} \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{p}$$

with $s(m,p) = p$ and $t(m,p) = \nu(m) + p$ for all $(m,p) \in \mathbb{m} \rtimes \mathbb{p}$. If we apply $\mathtt{U}$ to the previous diagram, we get

$$\mathtt{U}(\mathbb{m} \rtimes \mathbb{p}) \underset{\mathtt{U}(t)}{\overset{\mathtt{U}(s)}{\rightrightarrows}} \mathtt{U}(\mathbb{p}).$$

Although it is true that $\mathtt{U}(s)|_{\mathtt{U}(\mathbb{p})} = \mathtt{U}(t)|_{\mathtt{U}(\mathbb{p})} = \mathrm{id}_{\mathtt{U}(\mathbb{p})}$, in general, the second condition for cat^1-algebras (CAs2) is not satisfied. For instance, if we take $m \in \mathbb{m}\backslash\{0\}$, it is clear that $(m, 0) \in \operatorname{Ker}\mathtt{U}(s)$ and $(m, -\nu(m)) \in \operatorname{Ker}\mathtt{U}(t)$. However $(m, 0) \otimes (m, -\nu(m)) \neq 0$, so $\operatorname{Ker}\mathtt{U}(s)\operatorname{Ker}\mathtt{U}(t) \neq 0$.

Nevertheless, we can consider the quotient $\overline{\mathtt{U}}(\mathbb{m} \rtimes \mathbb{p}) = \mathtt{U}(\mathbb{m} \rtimes \mathbb{p})/X$, where $X = \operatorname{Ker}\mathtt{U}(s)\operatorname{Ker}\mathtt{U}(t) + \operatorname{Ker}\mathtt{U}(t)\operatorname{Ker}\mathtt{U}(s)$, and the induced morphisms $\overline{\mathtt{U}}(s)$ and $\overline{\mathtt{U}}(t)$. In this way, the diagram

$$\overline{\mathtt{U}}(\mathbb{m} \rtimes \mathbb{p}) \underset{\overline{\mathtt{U}}(t)}{\overset{\overline{\mathtt{U}}(s)}{\rightrightarrows}} \mathtt{U}(\mathbb{p})$$

is clearly a cat^1-algebra. Note that $X \subset \operatorname{Ker}\mathtt{U}(t)$ and $\mathtt{U}(t)|_{\mathtt{U}(\mathbb{p})} = \mathrm{id}_{\mathtt{U}(\mathbb{p})}$, since $t|_{\mathbb{p}} = \mathrm{id}_{\mathbb{p}}$. Given $v,\ w \in \mathtt{U}(\{0\} \rtimes \mathbb{p}) \simeq \mathtt{U}(\mathbb{p})$, if $v - w \in X$, then $0 = \mathtt{U}(t)(v-w) = v - w$. Therefore $\mathtt{U}(\mathbb{p})$ can be regarded as a subalgebra of $\overline{\mathtt{U}}(\mathbb{m} \rtimes \mathbb{p})$.

We can now define $\mathtt{XU}(\mathbb{m}, \mathbb{p}, \nu)$ as the crossed module of associative algebras given by $(\operatorname{Ker}\overline{\mathtt{U}}(s), \mathtt{U}(\mathbb{p}), \overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)})$. We will write $\overline{\mathtt{U}}(t)$ instead of $\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}$ to ease notation.

For any morphism of Lie crossed modules $(\varphi,\psi)\colon (\mathfrak{m},\mathfrak{p},\nu)\to(\mathfrak{m}',\mathfrak{p}',\nu')$, $\mathtt{XU}(\varphi,\psi)$ is given by

$$\begin{tikzcd} \operatorname{Ker}\overline{\mathtt{U}}(s) \arrow[r, "\overline{\mathtt{U}}(t)"] \arrow[d, "\overline{\mathtt{U}}(\varphi,\psi)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}"'] & \mathtt{U}(\mathfrak{p}) \arrow[d, "\mathtt{U}(\psi)"] \\ \operatorname{Ker}\overline{\mathtt{U}}(s') \arrow[r, "\overline{\mathtt{U}}(t')"'] & \mathtt{U}(\mathfrak{p}') \end{tikzcd}$$

where $\overline{\mathtt{U}}(\varphi,\psi)$ is the algebra homomorphism induced by $\mathtt{U}(\varphi,\psi)$, which is itself the linear extension of $(\varphi,\psi)\colon \mathfrak{m}\rtimes\mathfrak{p}\to\mathfrak{m}'\rtimes\mathfrak{p}'$, given by $(\varphi,\psi)(m,p)=(\varphi(m),\psi(p))$ for all $(m,p)\in\mathfrak{m}\rtimes\mathfrak{p}$.

The functor $\mathtt{XU}\colon \textbf{XLie}\to\textbf{XAs}$ is a natural generalization of the functor $\mathtt{U}$, in the sense that it makes the following diagram commute,

$$\begin{tikzcd} \textbf{Lie} \arrow[r, "\mathtt{I}_0"] \arrow[d, "\mathtt{U}"'] & \textbf{XLie} \arrow[d, "\mathtt{XU}"] \\ \textbf{As} \arrow[r, "\mathtt{I}'_0"'] & \textbf{XAs}\,. \end{tikzcd}$$

Regarding the embeddings $\mathtt{I}_1$ and $\mathtt{I}'_1$, we have the following result.

Proposition 3.2.4. *There is a natural isomorphism of functors*

$$\mathtt{XU}\circ\mathtt{I}_1\cong\mathtt{I}'_1\circ\mathtt{U}\,.$$

Proof. Let $\mathfrak{p}\in\textbf{Lie}$. We need to show that $\mathtt{XU}(\mathfrak{p},\mathfrak{p},\mathrm{id}_{\mathfrak{p}})$ is naturally isomorphic to $(\mathtt{U}(\mathfrak{p}),\mathtt{U}(\mathfrak{p}),\mathrm{id}_{\mathtt{U}(\mathfrak{p})})$. In order to do so, we will prove that $(\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)},\mathrm{id}_{\mathtt{U}(\mathfrak{p})})$ is an isomorphism of crossed modules of algebras from $(\operatorname{Ker}\overline{\mathtt{U}}(s),\mathtt{U}(\mathfrak{p}),\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)})$ to $(\mathtt{U}(\mathfrak{p}),\mathtt{U}(\mathfrak{p}),\mathrm{id}_{\mathtt{U}(\mathfrak{p})})$.

Note that $(\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)},\mathrm{id}_{\mathtt{U}(\mathfrak{p})})$ is indeed a morphism of crossed modules (see Example 1.2.30 (iii)). Recall that the first step in the construction of $\mathtt{XU}(\mathfrak{p},\mathfrak{p},\mathrm{id}_{\mathfrak{p}})$ requires us to consider the cat^1-Lie algebra

$$\mathfrak{p}\rtimes\mathfrak{p} \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{p}$$

with $s(p,p')=p'$ and $t(p,p')=p+p'$ for all $p,p'\in\mathfrak{p}$. Let us define the Lie homomorphism $\epsilon\colon\mathfrak{p}\to\mathfrak{p}\rtimes\mathfrak{p}$, $\epsilon(p)=(p,0)$. It is clear that $s\epsilon=0$ and $t\epsilon=\mathrm{id}_{\mathfrak{p}}$.

The next step is to apply the functor $\mathtt{U}$ on the previous cat^1-Lie algebra and take the quotient of $\mathtt{U}(\mathfrak{p}\rtimes\mathfrak{p})$ by $X=\operatorname{Ker}\mathtt{U}(s)\operatorname{Ker}\mathtt{U}(t)+\operatorname{Ker}\mathtt{U}(t)\operatorname{Ker}\mathtt{U}(s)$ in order to guarantee that we have a cat^1-algebra. In the next diagram of algebras,

$$\begin{tikzcd}
\mathtt{U}(\mathbb{p}) \arrow[r, "\mathtt{U}(\epsilon)"] & \mathtt{U}(\mathbb{p} \rtimes \mathbb{p}) \arrow[r, shift left, "\mathtt{U}(s)"] \arrow[r, shift right, "\mathtt{U}(t)"'] \arrow[d, "\pi"'] & \mathtt{U}(\mathbb{p}) \\
& \mathtt{U}(\mathbb{p} \rtimes \mathbb{p})/X \arrow[ur, shift left, "\overline{\mathtt{U}}(s)"] \arrow[ur, shift right, "\overline{\mathtt{U}}(t)"'] &
\end{tikzcd}$$

where π is the canonical projection, it is easy to see that $\overline{\mathtt{U}}(s)\pi\,\mathtt{U}(\epsilon) = \mathtt{U}(s)\,\mathtt{U}(\epsilon) = \mathtt{U}(s\epsilon) = 0$ and $\overline{\mathtt{U}}(t)\pi\,\mathtt{U}(\epsilon) = \mathtt{U}(t)\,\mathtt{U}(\epsilon) = \mathtt{U}(t\epsilon) = \mathrm{id}_{\mathtt{U}(\mathbb{p})}$. Hence $\pi\,\mathtt{U}(\epsilon)$ takes values in $\operatorname{Ker}\overline{\mathtt{U}}(s)$ and it is a right inverse for $\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}$.

Now we need to show that $\pi\,\mathtt{U}(\epsilon)\overline{\mathtt{U}}(t) = \mathrm{id}_{\operatorname{Ker}\overline{\mathtt{U}}(s)}$. Note that $X \subset \operatorname{Ker}\mathtt{U}(s)$, so $\operatorname{Ker}\overline{\mathtt{U}}(s) = \operatorname{Ker}\mathtt{U}(s)/X$ and, as a K-module, $\operatorname{Ker}\mathtt{U}(s)$ is generated by all the elements of the form

$$(p_1, p'_1) \otimes \cdots \otimes (p_{i-1}, p'_{i-1}) \otimes (p_i, 0) \otimes (p_{i+1}, p'_{i+1}) \otimes \cdots \otimes (p_n, p'_n), \qquad (3.2.1)$$

with $n \in \mathbb{N}$, $n \geq 1$, $p_i, p'_i \in \mathbb{p}$, $1 \leq i \leq n$. By the definition of $\mathtt{U}(t)$ and $\mathtt{U}(\epsilon)$, the value of $\mathtt{U}(\epsilon)\,\mathtt{U}(t)$ on (3.2.1) is

$$(p_1+p'_1, 0)\otimes\cdots\otimes(p_{i-1}+p'_{i-1}, 0)\otimes(p_i, 0)\otimes(p_{i+1}+p'_{i+1}, 0)\otimes\cdots\otimes(p_n+p'_n, 0). \quad (3.2.2)$$

In $\operatorname{Ker}\mathtt{U}(s)/X$,

$$\begin{aligned}
&(p_1 + p'_1, 0) \otimes \cdots \otimes (p_{i-1} + p'_{i-1}, 0) \otimes (p_i, 0) \otimes (p_{i+1} + p'_{i+1}, 0) \otimes \cdots \otimes (p_n + p'_n, 0) \\
&= (p_1, p'_1) \otimes \cdots \otimes (p_{i-1} + p'_{i-1}, 0) \otimes (p_i, 0) \otimes (p_{i+1} + p'_{i+1}, 0) \otimes \cdots \otimes (p_n + p'_n, 0),
\end{aligned}$$

since $(p'_1, -p'_1)\otimes\cdots\otimes(p_{i-1}+p'_{i-1}, 0)\otimes(p_i, 0)\otimes(p_{i+1}+p'_{i+1}, 0)\otimes\cdots\otimes(p_n+p'_n, 0) \in X$. By repeating this process we can easily derive that

$$\begin{aligned}
&(p_1 + p'_1, 0) \otimes \cdots \otimes (p_{i-1} + p'_{i-1}, 0) \otimes (p_i, 0) \otimes (p_{i+1} + p'_{i+1}, 0) \otimes \cdots \otimes (p_n + p'_n, 0) \\
&= (p_1, p'_1) \otimes \cdots \otimes (p_{i-1} + p'_{i-1}, 0) \otimes (p_i, 0) \otimes (p_{i+1} + p'_{i+1}, 0) \otimes \cdots \otimes (p_n + p'_n, 0) \\
&= \cdots \\
&= (p_1, p'_1) \otimes \cdots \otimes (p_{i-1}, p'_{i-1}) \otimes (p_i, 0) \otimes (p_{i+1}, p'_{i+1}) \otimes \cdots \otimes (p_n, p'_n).
\end{aligned}$$

Thus, the elements (3.2.1) and (3.2.2) are equal in $\operatorname{Ker}\mathtt{U}(s)/X$ and it follows that

$$\pi\,\mathtt{U}(\epsilon)\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)} = \mathrm{id}_{\operatorname{Ker}\overline{\mathtt{U}}(s)}\,.$$

Therefore we have found an inverse for the morphism of crossed modules of algebras

$(\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}, \mathrm{id}_{\mathtt{U}(\mathbb{p})})$, which is given by

$$\begin{tikzcd}
\mathtt{U}(\mathbb{p}) \arrow[r, "\mathrm{id}_{\mathtt{U}(\mathbb{p})}"] \arrow[d, "\pi\,\mathtt{U}(\epsilon)"'] & \mathtt{U}(\mathbb{p}) \arrow[d, "\mathrm{id}_{\mathtt{U}(\mathbb{p})}"] \\
\operatorname{Ker}\overline{\mathtt{U}}(s) \arrow[r, "\overline{\mathtt{U}}(t)"'] & \mathtt{U}(\mathbb{p}).
\end{tikzcd}$$

Note that we have already proved that the above diagram commutes. Besides, given $p, p' \in \mathbb{p}$, if we consider them as elements in $\mathtt{U}(\mathbb{p})$, $\pi\,\mathtt{U}(\epsilon)(p \otimes p') = (p,0) \otimes (p',0) + X = (0,p) \otimes (p',0) + X = (p,0) \otimes (0,p') + X$, so $\pi\,\mathtt{U}(\epsilon)$ clearly preserves the action of $\mathtt{U}(\mathbb{p})$ on itself via $\mathrm{id}_{\mathtt{U}(\mathbb{p})}$ when extended to all the elements in $\mathtt{U}(\mathbb{p})$.

Finally, $(\overline{\mathtt{U}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}, \mathrm{id}_{\mathtt{U}(\mathbb{p})})$ is natural, as shown in the diagram below, where $\alpha \colon \mathbb{p} \to \mathbb{p}'$ is a Lie homomorphism. Note that $\overline{\mathtt{U}}(\alpha,\alpha)$ is actually $\overline{\mathtt{U}}(\alpha,\alpha)|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}$.

$$\begin{tikzcd}
& \operatorname{Ker}\overline{\mathtt{U}}(s) \arrow[rr, "\overline{\mathtt{U}}(t)"] \arrow[dl, "\overline{\mathtt{U}}(\alpha,\alpha)"'] \arrow[dd, dashed, "\overline{\mathtt{U}}(t)"] & & \mathtt{U}(\mathbb{p}) \arrow[dl, "\mathtt{U}(\alpha)"] \arrow[dd, "\mathrm{id}_{\mathtt{U}(\mathbb{p})}"] \\
\operatorname{Ker}\overline{\mathtt{U}}(s') \arrow[rr, "\overline{\mathtt{U}}(t')"] \arrow[dd, "\overline{\mathtt{U}}(t')"'] & & \mathtt{U}(\mathbb{p}') \arrow[dd, "\mathrm{id}_{\mathtt{U}(\mathbb{p}')}"] & \\
& \mathtt{U}(\mathbb{p}) \arrow[rr, dashed, "\mathrm{id}_{\mathtt{U}(\mathbb{p})}"] \arrow[dl, dashed, "\mathtt{U}(\alpha)"'] & & \mathtt{U}(\mathbb{p}) \arrow[dl, "\mathtt{U}(\alpha)"] \\
\mathtt{U}(\mathbb{p}') \arrow[rr, "\mathrm{id}_{\mathtt{U}(\mathbb{p}')}"'] & & \mathtt{U}(\mathbb{p}') &
\end{tikzcd}$$

□

3.2.3 Adjunction between **XLie** and **XAs**

The following result is a natural generalization of the well-known classical adjunction between the categories **Lie** and **As**.

Theorem 3.2.5. *The functor* $\mathtt{XU}$ *is left adjoint to the Liezation functor* $\mathtt{XLie}_{\mathsf{As}}$.

Proof. Given $(\mathbb{m}, \mathbb{p}, \nu)$ in **XLie** and (B, A, ρ) in **XAs**, we have to construct a natural bijection

$$\operatorname{Hom}_{\mathsf{XLie}}\big((\mathbb{m}, \mathbb{p}, \nu), \mathtt{XLie}_{\mathsf{As}}(B, A, \rho)\big) \cong \operatorname{Hom}_{\mathsf{XAs}}\big(\mathtt{XU}(\mathbb{m}, \mathbb{p}, \nu), (B, A, \rho)\big).$$

Let $(\varphi,\psi) \in \operatorname{Hom}_{\mathbf{XLie}}\big((\mathfrak{m},\mathfrak{p},\nu), \mathtt{XLie}_{\mathtt{As}}(B,A,\rho)\big)$, that is

$$\begin{tikzcd} \mathfrak{m} \arrow[r, "\nu"] \arrow[d, "\varphi"'] & \mathfrak{p} \arrow[d, "\psi"] \\ \mathtt{Lie}_{\mathtt{As}}(B) \arrow[r, "\rho"'] & \mathtt{Lie}_{\mathtt{As}}(A) \end{tikzcd}$$

such that $\rho\varphi = \psi\nu$ and φ preserves the action of $\mathfrak{p}$ on $\mathfrak{m}$ via ψ.

We can consider the following morphism of cat^1-Lie algebras by using the functor $\mathtt{cat}_{\mathtt{Lie}}$ as defined in the proof of Proposition 1.2.23, as well as Lemma 3.2.3:

$$\begin{tikzcd} \mathfrak{m} \rtimes \mathfrak{p} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\varphi'"'] & \mathfrak{p} \arrow[d, "\psi"] \\ \mathtt{Lie}_{\mathtt{As}}(B \rtimes A) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathtt{Lie}_{\mathtt{As}}(A) \end{tikzcd}$$

where $\varphi'(m,p) = (\varphi(m),\psi(p))$ for all $(m,p) \in \mathfrak{m} \rtimes \mathfrak{p}$. Since the functor $\mathtt{U}$ is left adjoint to the functor $\mathtt{Lie}_{\mathtt{As}}$, we have the induced commutative diagrams of algebras

$$\begin{tikzcd} \mathtt{U}(\mathfrak{m} \rtimes \mathfrak{p}) \arrow[r, "\mathtt{U}(s)"] \arrow[d, "\varphi'^*"'] & \mathtt{U}(\mathfrak{p}) \arrow[d, "\psi^*"] \\ B \rtimes A \arrow[r, "\sigma"'] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathtt{U}(\mathfrak{m} \rtimes \mathfrak{p}) \arrow[r, "\mathtt{U}(t)"] \arrow[d, "\varphi'^*"'] & \mathtt{U}(\mathfrak{p}) \arrow[d, "\psi^*"] \\ B \rtimes A \arrow[r, "\tau"'] & A. \end{tikzcd}$$

Besides, we have a uniquely defined morphism of cat^1-algebras:

$$\begin{tikzcd} \mathtt{U}(\mathfrak{m} \rtimes \mathfrak{p})/X \arrow[r, shift left, "\overline{\mathtt{U}}(s)"] \arrow[r, shift right, "\overline{\mathtt{U}}(t)"'] \arrow[d, "\overline{\varphi}'^*"'] & \mathtt{U}(\mathfrak{p}) \arrow[d, "\psi^*"] \\ B \rtimes A \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & A \end{tikzcd}$$

where $X = \operatorname{Ker}\mathtt{U}(s)\operatorname{Ker}\mathtt{U}(t) + \operatorname{Ker}\mathtt{U}(t)\operatorname{Ker}\mathtt{U}(s)$.

Finally, we can use the functor $\mathtt{Xm}_{\mathtt{As}}$, described in the proof of Proposition 1.2.34, to get a uniquely defined morphism (φ^{**},ψ^*) in $\operatorname{Hom}_{\mathbf{XAs}}\big(\mathtt{XU}(\mathfrak{m},\mathfrak{p},\nu),(B,A,\rho)\big)$:

$$\begin{tikzcd} \operatorname{Ker}\overline{\mathtt{U}}(s) \arrow[r, "\overline{\mathtt{U}}(t)"] \arrow[d, "\overline{\varphi}'^*"'] \arrow[dd, bend right=60, "\varphi^{**}"'] & \mathtt{U}(\mathfrak{p}) \arrow[d, "\psi^*"] \\ \operatorname{Ker}\sigma \arrow[r, "\tau"] \arrow[d, "\simeq"'] & A \arrow[d, "\mathrm{id}_A"] \\ B \arrow[r, "\rho"'] & A. \end{tikzcd}$$

Note that in the diagram above $\overline{\varphi}'^* = \overline{\varphi}'^*|_{\operatorname{Ker}\overline{\mathtt{U}}(s)}$, $\operatorname{Ker}\sigma = B \rtimes \{0\}$ and $\tau = \tau|_{\operatorname{Ker}\sigma} = \rho$.

Now, let $(\phi, \chi) \in \operatorname{Hom}_{\mathbf{XAs}}\big(\mathtt{XU}(\mathbb{m}, \mathbb{p}, \nu), (B, A, \rho)\big)$, that is

$$\begin{tikzcd}
\operatorname{Ker}\overline{\mathtt{U}}(s) \arrow[r, "\overline{\mathtt{U}}(t)"] \arrow[d, "\phi"'] & \mathtt{U}(\mathbb{p}) \arrow[d, "\chi"] \\
B \arrow[r, "\rho"'] & A
\end{tikzcd}$$

such that $\rho\phi = \chi\overline{\mathtt{U}}(t)$ and ϕ preserves the action of $\mathtt{U}(\mathbb{p})$ on $\operatorname{Ker}\overline{\mathtt{U}}(s)$ via χ.

We can consider the corresponding morphism of cat^1-algebras by using the functor $\mathtt{cat}_{\mathtt{As}}$ as defined in the proof of Proposition 1.2.34.

$$\begin{tikzcd}
\mathtt{U}(\mathbb{m} \rtimes \mathbb{p})/X \arrow[r, shift left, "\overline{\mathtt{U}}(s)"] \arrow[r, shift right, "\overline{\mathtt{U}}(t)"'] \arrow[d, "\simeq"'] \arrow[dd, bend right=60, "\phi'"'] & \mathtt{U}(\mathbb{p}) \arrow[d, "\mathrm{id}_{\mathtt{U}(\mathbb{p})}"] \\
\operatorname{Ker}\overline{\mathtt{U}}(s) \rtimes \mathtt{U}(\mathbb{p}) \arrow[r, shift left, "\tilde{\sigma}"] \arrow[r, shift right, "\tilde{\tau}"'] \arrow[d, "(\phi,\chi)"'] & \mathtt{U}(\mathbb{p}) \arrow[d, "\chi"] \\
B \rtimes A \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & A.
\end{tikzcd}$$

Observe that the isomorphism in the diagram above is explicitly described at the end of the proof of Proposition 1.2.34, as well as $\tilde{\sigma}$ and $\tilde{\tau}$.

Let $\overline{\phi} = \phi'\pi$, with $\pi\colon \mathtt{U}(\mathbb{m} \rtimes \mathbb{p}) \to \mathtt{U}(\mathbb{m} \rtimes \mathbb{p})/X$ the canonical projection. Hence, we have the commutative diagrams of algebras

$$\begin{tikzcd}
\mathtt{U}(\mathbb{m} \rtimes \mathbb{p}) \arrow[r, "\mathtt{U}(s)"] \arrow[d, "\overline{\phi}"'] & \mathtt{U}(\mathbb{p}) \arrow[d, "\chi"] \\
B \rtimes A \arrow[r, "\sigma"'] & A
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathtt{U}(\mathbb{m} \rtimes \mathbb{p}) \arrow[r, "\mathtt{U}(t)"] \arrow[d, "\overline{\phi}"'] & \mathtt{U}(\mathbb{p}) \arrow[d, "\chi"] \\
B \rtimes A \arrow[r, "\tau"'] & A.
\end{tikzcd}$$

Since the functor $\mathtt{U}$ is left adjoint to the functor $\mathtt{Lie}_{\mathtt{As}}$, we have the morphism of cat^1-Lie algebras

$$\begin{tikzcd}
\mathbb{m} \rtimes \mathbb{p} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\phi^*"'] & \mathbb{p} \arrow[d, "\chi^*"] \\
\mathtt{Lie}_{\mathtt{As}}(B \rtimes A) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathtt{Lie}_{\mathtt{As}}(A).
\end{tikzcd}$$

Finally, we can use the functor $\mathtt{Xm}_{\mathtt{Lie}}$, described in the proof of Proposition 1.2.23, to get a uniquely defined morphism (ϕ^{**}, χ^*) in $\mathrm{Hom}_{\mathbf{XLie}}\big((\mathbb{m}, \mathbb{p}, \nu), \mathtt{XLie}_{\mathtt{As}}(B, A, \rho)\big)$:

$$
\begin{tikzcd}
\mathbb{m} \arrow[r, "\nu"] \arrow[d, "\simeq"'] \arrow[ddd, bend right=60, "\phi^{**}"'] & \mathbb{p} \arrow[d, "\mathrm{id}_{\mathbb{p}}"] \\
\operatorname{Ker} s \arrow[r, "t"] \arrow[d, "\phi^*|_{\operatorname{Ker} s}"'] & \mathbb{p} \arrow[d, "\chi^*"] \\
\operatorname{Ker} \sigma \arrow[r, "\tau"] \arrow[d, "\simeq"'] & \mathtt{Lie}_{\mathtt{As}}(A) \arrow[d, "\mathrm{id}_{\mathtt{Lie}_{\mathtt{As}}(A)}"] \\
\mathtt{Lie}_{\mathtt{As}}(B) \arrow[r, "\rho"'] & \mathtt{Lie}_{\mathtt{As}}(A).
\end{tikzcd}
$$

Note that in the diagram above $\operatorname{Ker} s = \mathbb{m} \rtimes \{0\}$, $t = t|_{\operatorname{Ker} s} = \nu$, $\operatorname{Ker} \sigma = \mathtt{Lie}_{\mathtt{As}}(B) \rtimes \{0\}$ and $\tau = \tau|_{\operatorname{Ker} \sigma} = \rho$. □

Bearing in mind the previous theorem along with Propositions 1.2.20 and 1.2.31, one can easily deduce the following result.

Theorem 3.2.6. *The inner and outer squares in the following diagrams are commutative or commute up to isomorphism for $i = 0, 1$.*

$$
\begin{tikzcd}
\mathbf{As} \arrow[r, shift right, "\mathtt{Lie}_{\mathtt{As}}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathtt{I}'_i"] \arrow[d, phantom, "\dashv"] & \mathbf{Lie} \arrow[l, shift right, "\mathtt{U}"'] \arrow[d, shift right, "\mathtt{I}_i"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{XAs} \arrow[u, shift left, "\Phi'_i"] \arrow[r, shift left, "\mathtt{XLie}_{\mathtt{As}}"] \arrow[r, phantom, "\top"] & \mathbf{XLie} \arrow[u, shift right, "\Phi_i"'] \arrow[l, shift left, "\mathtt{XU}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{As} \arrow[r, shift right, "\mathtt{Lie}_{\mathtt{As}}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\mathtt{I}'_i"'] \arrow[d, phantom, "\dashv"] & \mathbf{Lie} \arrow[l, shift right, "\mathtt{U}"'] \arrow[d, shift left, "\mathtt{I}_i"] \arrow[d, phantom, "\vdash"] \\
\mathbf{XAs} \arrow[u, shift right, "\Phi'_{i+1}"'] \arrow[r, shift left, "\mathtt{XLie}_{\mathtt{As}}"] \arrow[r, phantom, "\top"] & \mathbf{XLie} \arrow[u, shift left, "\Phi_{i+1}"] \arrow[l, shift left, "\mathtt{XU}"]
\end{tikzcd}
\tag{3.2.3}
$$

Proof. Let us begin with the first diagram. Directly from the definition of the functors involved, $\mathtt{XLie}_{\mathtt{As}} \circ \mathtt{I}'_i = \mathtt{I}_i \circ \mathtt{Lie}_{\mathtt{As}}$ for i=0,1. Besides, from the adjunctions described in the first diagram, we get that $\mathtt{U} \circ \Phi_i$ is left adjoint to $\mathtt{I}_i \circ \mathtt{Lie}_{\mathtt{As}}$, while $\Phi'_i \circ \mathtt{XU}$ is left adjoint to $\mathtt{XLie}_{\mathtt{As}} \circ \mathtt{I}'_i$, for $i = 0, 1$. Hence, $\mathtt{U} \circ \Phi_i \cong \Phi'_i \circ \mathtt{XU}$ for $i = 0, 1$.

Regarding the second diagram, the commutativity of the outer square is obvious for $i = 0$, while in Proposition 3.2.4 we proved that there is a natural isomorphism $\mathtt{XU} \circ \mathtt{I}_1 \cong \mathtt{I}'_1 \circ \mathtt{U}$. Therefore, by a similar reasoning to the one used for the first diagram, we get that $\mathtt{Lie}_{\mathtt{As}} \circ \Phi'_{i+1} \cong \Phi_{i+1} \circ \mathtt{XLie}_{\mathtt{As}}$ for $i = 0, 1$. □

Recall that in Section 2.3 we gave an explicit definition of left modules over crossed modules of Lie and associative algebras.

Lemma 3.2.7. *Let $\delta\colon V \to W$ be a K-module homomorphism, regarded as an abelian crossed module of algebras (Lie algebras). Then, using the same notations as in Lemma 2.3.4, the Lie crossed module* $\mathtt{XLie}_{\mathtt{As}}\big(\mathrm{Hom}_K(W,V), \mathrm{End}(V,W,\delta), \Gamma\big)$ *coincides with the actor crossed module* $\mathrm{Act}(V,W,\delta) = \big(\mathrm{Der}(W,V), \mathrm{Der}(V,W,\delta), \Delta\big)$.

Proof. Since V and W are considered as abelian Lie algebras together with the trivial action of W on V, it is clear that $\mathrm{Der}(W,V) = \mathtt{Lie}\big(\mathrm{Hom}_K(W,V)\big)$, $\mathrm{Der}(V,W,\delta) = \mathtt{Lie}_{\mathtt{As}}\big(\mathrm{End}(V,W,\delta)\big)$ and $\Delta = \mathtt{Lie}_{\mathtt{As}}(\Gamma)$. Moreover, the Lie action of $\mathrm{Der}(V,W,\delta)$ on $\mathrm{Der}(W,V)$ is induced by the algebra action of $\mathrm{End}(V,W,\delta)$ on $\mathrm{Hom}_K(W,V)$. □

Theorem 3.2.8. *Let $(\mathbb{m},\mathbb{p},\nu)$ be a Lie crossed module. Then the categories of left $(\mathbb{m},\mathbb{p},\nu)$-modules and left $\mathtt{XU}(\mathbb{m},\mathbb{p},\nu)$-modules are isomorphic.*

Proof. By using Theorem 3.2.5 and Lemma 3.2.7, left $(\mathbb{m},\mathbb{p},\nu)$-module structures on a K-module homomorphism $\delta\colon V \to W$ are in bijective correspondence with left $\mathtt{XU}(\mathbb{m},\mathbb{p},\nu)$-module structures on it:

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{XLie}}\big((\mathbb{m},\mathbb{p},\nu), \mathrm{Act}(V,W,\delta)\big)\\
&= \mathrm{Hom}_{\mathsf{XLie}}\big((\mathbb{m},\mathbb{p},\nu), \mathtt{XLie}_{\mathtt{As}}\big(\mathrm{Hom}_K(W,V), \mathrm{End}(V,W,\delta), \Gamma\big)\big)\\
&\approx \mathrm{Hom}_{\mathsf{XAs}}\big(\mathtt{XU}(\mathbb{m},\mathbb{p},\nu), \big(\mathrm{Hom}_K(W,V), \mathrm{End}(V,W,\delta), \Gamma\big)\big).
\end{aligned}$$

Due to the conditions satisfied by a morphism between modules over a crossed module of Lie and associative algebras (see (2.3.1)–(2.3.3) and (2.3.9)–(2.3.11)), it is easy to check that this correspondence is functorial. □

Finally, let us remark that right modules (over crossed modules of Lie and associative algebras) could be defined similarly and could equally be used everywhere instead of left modules.

3.3 XLb vs XDias

As explained by Loday in [65], a Leibniz algebra is a non-commutative version of a Lie algebra. When we replace Lie algebras by Leibniz algebras, the role of associative algebras is played by associative dialgebras.

Any dialgebra D becomes a Leibniz algebra with the bracket given by $[x,y] = x \dashv y - y \vdash x$ for all $x,y \in D$. Moreover, any dialgebra homomorphism $f\colon D \to L$ is a Leibniz algebra homomorphism if we consider the bracket previously defined both in D and L. Thus we have a functor $\mathtt{Lb}\colon \mathsf{Dias} \to \mathsf{Lb}$.

The functor $\mathtt{Lb}$ admits a left adjoint, $\mathtt{U_d}\colon \mathsf{Lb} \to \mathsf{Dias}$, which assigns to a Leibniz algebra $\mathfrak{p}$ its *universal enveloping dialgebra* $\mathtt{U_d}(\mathfrak{p})$ (see [65]), which is defined as the quotient $\mathrm{T}(\mathfrak{p}) \otimes \mathfrak{p} \otimes \mathrm{T}(\mathfrak{p})/J$ of the free dialgebra over the underlying K-module of $\mathfrak{p}$

by the ideal J generated by $p_1 \dashv p_2 - p_2 \vdash p_1 - [p_1, p_2]$, for all $p_1, p_2 \in \mathfrak{p}$. Note that the two products in $\mathrm{T}(\mathfrak{p}) \otimes \mathfrak{p} \otimes \mathrm{T}(\mathfrak{p})$ are induced by

$$\begin{aligned}(p_{-n} \cdots p_{-1} \otimes p_0 \otimes p_1 \cdots p_m) &\dashv (q_{-s} \cdots q_{-1} \otimes q_0 \otimes q_1 \cdots q_t)\\ &= p_{-n} \cdots p_{-1} \otimes p_0 \otimes p_1 \cdots p_m q_{-s} \cdots q_t,\\ (p_{-n} \cdots p_{-1} \otimes p_0 \otimes p_1 \cdots p_m) &\vdash (q_{-s} \cdots q_{-1} \otimes q_0 \otimes q_1 \cdots q_t)\\ &= p_{-n} \cdots p_m q_{-s} \cdots q_{-1} \otimes q_0 \otimes q_1 \cdots q_t,\end{aligned}$$

where $p_i,\ q_i \in \mathfrak{p}$. Note that central dots in the previous expressions represent tensor products, but they are omitted in order to indicate clearly the middle entry, which is p_0 in the first case and q_0 in the second one.

Given a Leibniz homomorphism $f \colon \mathfrak{p} \to \mathfrak{m}$, we can define $\mathtt{U_d}(f) \colon \mathtt{U_d}(\mathfrak{p}) \to \mathtt{U_d}(\mathfrak{m})$ by extending the definition of f to the elements of $\mathtt{U_d}(\mathfrak{p})$ by linearity.

In this section we extend the functors $\mathtt{Lb}$ and $\mathtt{U_d}$ to the categories **XDias** and **XLb** in such a way that those extensions are still adjoint to one another.

3.3.1 From **XDias** to **XLb**

Let us first show that the functor $\mathtt{Lb}$ preserves the semidirect product.

Lemma 3.3.1. *Given D and L two associative dialgebras together with an action of D on L, there is an action of $\mathtt{Lb}(D)$ on $\mathtt{Lb}(L)$ given by*

$$\begin{aligned}[x, a] &= x \dashv a - a \vdash x,\\ [a, x] &= a \dashv x - x \vdash a,\end{aligned}$$

for all $x \in \mathtt{Lb}(D)$, $a \in \mathtt{Lb}(L)$.

Proof. The six conditions from the Definition 1.2.37 can be checked by straightforward calculations, using the 30 equalities from the Definition 1.2.50. We show here how to check the first condition; the rest of them can be done analogously. Let $a \in \mathtt{Lb}(L)$, $x, x' \in \mathtt{Lb}(D)$. We will prove that $[a, [x, x']] = [[a, x], x'] - [[a, x'], x]$.

$$\begin{aligned}[a, [x, x']] &= [a, x \dashv x' - x' \vdash x] = a \dashv (x \dashv x' - x' \vdash x) - (x \dashv x' - x' \vdash x) \vdash a\\ &= \underbrace{a \dashv (x \dashv x')}_{(2)} \underbrace{- a \dashv (x' \vdash x)}_{(1)} \underbrace{- (x \dashv x') \vdash a}_{(4)} \underbrace{+ (x' \vdash x) \vdash a}_{(5)},\\ [[a, x], x'] &= [a \dashv x - x \vdash a, x'] = (a \dashv x - x \vdash a) \dashv x' - x' \vdash (a \dashv x - x \vdash a)\\ &= \underbrace{(a \dashv x) \dashv x'}_{(2)} \underbrace{- (x \vdash a) \dashv x'}_{(3)} \underbrace{- x' \vdash (a \dashv x)}_{(3')} \underbrace{+ x' \vdash (x \vdash a)}_{(5)},\\ -[[a, x'], x] &= \underbrace{-(a \dashv x') \dashv x}_{(1)} \underbrace{+ (x' \vdash a) \dashv x}_{(3')} \underbrace{+ x \vdash (a \dashv x')}_{(3)} \underbrace{- x \vdash (x' \vdash a)}_{(4)}.\end{aligned}$$

The addends labelled by (1) are equal because of the equality in Definition 1.2.50 constructed from (Di1), with the first element in L and the other two in D. The same happens with the parts labelled by (2), (4) and (5), using the corresponding equalities from (Di2), (Di4) and (Di5). The parts labelled by (3) cancel each other due to the equality in Definition 1.2.50 constructed from (Di3) with the first and the third elements in D and the second in L. The same applies to $(3')$. $\square$

Lemma 3.3.2. *Let D and L be two associative dialgebras together with an action of D on L. Then $\mathtt{Lb}(L \rtimes D) = \mathtt{Lb}(L) \rtimes \mathtt{Lb}(D)$.*

Proof. Due to the previous lemma, $\mathtt{Lb}(D)$ acts on $\mathtt{Lb}(L)$, so it makes sense to consider the semidirect product Leibniz algebra $\mathtt{Lb}(L) \rtimes \mathtt{Lb}(D)$. It is clear that $\mathtt{Lb}(L \rtimes D)$ and $\mathtt{Lb}(L) \rtimes \mathtt{Lb}(D)$ are equal as K-modules, so we only need to check that they share the same bracket. Let $(a, x), (a', x') \in L \times D$. If we use the bracket in $\mathtt{Lb}(L) \rtimes \mathtt{Lb}(D)$, we get:

$$\begin{aligned}
[(a, x), (a', x')] &= ([a, a'] + [x, a'] + [a, x'], [x, x']) \\
&= (a \dashv a' - a' \vdash a + x \dashv a' - a' \vdash x + a \dashv x' - x' \vdash a, x \dashv x' - x' \vdash x).
\end{aligned}$$

On the other hand, if we use the bracket in $\mathtt{Lie}_{\mathtt{As}}(B \rtimes A)$, we get

$$\begin{aligned}
[(a, x), (a', x')] &= (a, x) \dashv (a', x') - (a', x') \vdash (a, x) \\
&= (a \dashv a' + x \dashv a' + a \dashv x', x \dashv x') - (a' \vdash a + x' \vdash a + a' \vdash x, x' \vdash x),
\end{aligned}$$

so the brackets are equal. $\square$

Given a crossed module of dialgebras (L, D, μ), we can now define $\mathtt{XLb}(L, D, \mu)$ as the Leibniz crossed module $(\mathtt{Lb}(L), \mathtt{Lb}(D), \mathtt{Lb}(\mu))$, with the action of $\mathtt{Lb}(D)$ on $\mathtt{Lb}(L)$ described in Lemma 3.3.1. Note that $\mathtt{Lb}(\mu) = \mu$. Directly from (XDi1), (XDi2) and the definition of the Leibniz action, we get that $(\mathtt{Lb}(L), \mathtt{Lb}(D), \mathtt{Lb}(\mu))$ satisfies (XLb1) and (XLb2).

Moreover, any morphism of crossed modules of dialgebras $(\varphi, \psi)\colon (L, D, \mu) \to (L', D', \mu')$ is morphism of crossed modules of Leibniz algebras from $(\mathtt{Lb}(L), \mathtt{Lb}(D), \mu)$ to $(\mathtt{Lb}(L'), \mathtt{Lb}(D'), \mu')$, since

$$\begin{aligned}
\varphi([x, a]) &= \varphi(x \dashv a - a \vdash x) = \psi(x) \dashv \varphi(a) - \varphi(a) \vdash \psi(x) = [\psi(x), \varphi(a)], \\
\varphi([a, x]) &= \varphi(a \dashv x - x \vdash a) = \varphi(a) \dashv \psi(x) - \psi(x) \vdash \varphi(a) = [\varphi(a), \psi(x)],
\end{aligned}$$

for all $x \in \mathtt{Lb}(D)$, $a \in \mathtt{Lb}(L)$.

The previous assignments define a functor $\mathtt{XLb}\colon \mathbf{XDias} \to \mathbf{XLb}$, which is a natural generalization of the functor $\mathbf{Lb}\colon \mathbf{Dias} \to \mathbf{Lb}$ in the following sense. Recall that we have the full embeddings

$$\mathsf{J}_0, \mathsf{J}_1\colon \mathbf{Lb} \longrightarrow \mathbf{XLb} \qquad\qquad \mathsf{J}'_0, \mathsf{J}'_1\colon \mathbf{Dias} \longrightarrow \mathbf{XDias}$$

where $\mathtt{J}_0(\mathfrak{p}) = (\{0\}, \mathfrak{p}, 0)$, $\mathtt{J}_1(\mathfrak{p}) = (\mathfrak{p}, \mathfrak{p}, \mathrm{id}_\mathfrak{p})$, $\mathtt{J}'_0(D) = (\{0\}, D, 0)$ and $\mathtt{J}'_1(D) = (D, D, \mathrm{id}_D)$. It is obvious that the following diagram is commutative for $i = 0, 1$.

$$\begin{tikzcd}
\mathbf{Dias} \arrow[r, "\mathtt{J}'_i"] \arrow[d, "\mathtt{Lb}"'] & \mathbf{XDias} \arrow[d, "\mathtt{XLb}"] \\
\mathbf{Lb} \arrow[r, "\mathtt{J}_i"'] & \mathbf{XLb}\,.
\end{tikzcd}$$

3.3.2 Universal enveloping crossed module of a Leibniz crossed module

Let us now construct the left adjoint to the functor $\mathtt{XU_d}$, which generalizes the universal enveloping dialgebra functor $\mathtt{U_d}\colon \mathbf{Lb} \to \mathbf{Dias}$ to crossed modules.

Let $(\mathfrak{m}, \mathfrak{p}, \eta)$ be a Leibniz crossed module and consider its corresponding cat^1-Leibniz algebra as in the proof of Proposition 1.2.46, that is

$$\begin{tikzcd}
\mathfrak{m} \rtimes \mathfrak{p} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathfrak{p}
\end{tikzcd}$$

with $s(m,p) = p$ and $t(m,p) = \eta(m) + p$ for all $(m,p) \in \mathfrak{m} \rtimes \mathfrak{p}$. Now, if we apply $\mathtt{U_d}$ to the previous diagram, we get

$$\begin{tikzcd}
\mathtt{U_d}(\mathfrak{m} \rtimes \mathfrak{p}) \arrow[r, shift left, "\mathtt{U_d}(s)"] \arrow[r, shift right, "\mathtt{U_d}(t)"'] & \mathtt{U_d}(\mathfrak{p}).
\end{tikzcd}$$

Although it is true that $\mathtt{U_d}(s)|_{\mathtt{U_d}(\mathfrak{p})} = \mathtt{U_d}(t)|_{\mathtt{U_d}(\mathfrak{p})} = \mathrm{id}_{\mathtt{U_d}(\mathfrak{p})}$, in general, the second condition for *cat*^{1}-dialgebras (CDi2) is not satisfied. For instance, if we take $m \in \mathfrak{m} \setminus \{0\}$, it is clear that $(m, 0) \in \operatorname{Ker} \mathtt{U_d}(s)$ and $(m, -\eta(m)) \in \operatorname{Ker} \mathtt{U_d}(t)$. However, $(m, 0) \otimes (m, -\nu(m)) \neq 0$, so $\operatorname{Ker} \mathtt{U_d}(s) * \operatorname{Ker} \mathtt{U_d}(t) \neq 0$ for $* = \dashv$ and $* = \vdash$.

Nevertheless, we can consider the quotient $\overline{\mathtt{U_d}}(\mathfrak{m} \rtimes \mathfrak{p}) = \mathtt{U_d}(\mathfrak{m} \rtimes \mathfrak{p})/X$, where $X = \operatorname{Ker} \mathtt{U_d}(s) \dashv \operatorname{Ker} \mathtt{U_d}(t) + \operatorname{Ker} \mathtt{U_d}(t) \dashv \operatorname{Ker} \mathtt{U_d}(s) + \operatorname{Ker} \mathtt{U_d}(s) \vdash \operatorname{Ker} \mathtt{U_d}(t) + \operatorname{Ker} \mathtt{U_d}(t) \vdash \operatorname{Ker} \mathtt{U_d}(s)$, and the induced morphisms $\overline{\mathtt{U_d}}(s)$ and $\overline{\mathtt{U_d}}(t)$. In this way, the diagram

$$\begin{tikzcd}
\overline{\mathtt{U_d}}(\mathfrak{m} \rtimes \mathfrak{p}) \arrow[r, shift left, "\overline{\mathtt{U_d}}(s)"] \arrow[r, shift right, "\overline{\mathtt{U_d}}(t)"'] & \mathtt{U_d}(\mathfrak{p})
\end{tikzcd}$$

is clearly a cat^1-dialgebra. Note that $X \subset \operatorname{Ker} \mathtt{U_d}(t)$ and $\mathtt{U_d}(t)|_{\mathtt{U_d}(\mathfrak{p})} = \mathrm{id}_{\mathtt{U_d}(\mathfrak{p})}$, since $t|_\mathfrak{p} = \mathrm{id}_\mathfrak{p}$. Given $v,\ w \in \mathtt{U_d}(\{0\} \rtimes \mathfrak{p}) \simeq \mathtt{U_d}(\mathfrak{p})$, if $v - w \in X$, then $0 = \mathtt{U_d}(t)(v - w) = v - w$. Therefore $\mathtt{U_d}(\mathfrak{p})$ can be regarded as a subalgebra of $\overline{\mathtt{U_d}}(\mathfrak{m} \rtimes \mathfrak{p})$.

We can now define $\mathtt{XU_d}(\mathfrak{m}, \mathfrak{p}, \eta)$ as the crossed module of dialgebras given by $(\operatorname{Ker} \overline{\mathtt{U_d}}(s), \mathtt{U_d}(\mathfrak{p}), \overline{\mathtt{U_d}}(t)|_{\operatorname{Ker} \overline{\mathtt{U_d}}(s)})$. Sometimes we will write $\overline{\mathtt{U_d}}(t)$ instead of $\overline{\mathtt{U_d}}(t)|_{\operatorname{Ker} \overline{\mathtt{U_d}}(s)}$ to ease notation.

For any morphism of Leibniz crossed modules $(\varphi,\psi)\colon (\mathfrak{m},\mathfrak{p},\eta) \to (\mathfrak{m}',\mathfrak{p}',\eta')$, $\mathtt{XU_d}(\varphi,\psi)$ is given by

$$\begin{tikzcd}
\operatorname{Ker}\overline{\mathtt{U_d}}(s) \arrow[r, "\overline{\mathtt{U_d}}(t)"] \arrow[d, "\overline{\mathtt{U_d}}(\varphi,\psi)|_{\operatorname{Ker}\overline{\mathtt{U_d}}(s)}"'] & \mathtt{U_d}(\mathfrak{p}) \arrow[d, "\mathtt{U_d}(\psi)"] \\
\operatorname{Ker}\overline{\mathtt{U_d}}(s') \arrow[r, "\overline{\mathtt{U_d}}(t')"'] & \mathtt{U_d}(\mathfrak{p}')
\end{tikzcd}$$

where $\overline{\mathtt{U_d}}(\varphi,\psi)$ is the algebra homomorphism induced by $\mathtt{U_d}(\varphi,\psi)$, which is itself the linear extension of $(\varphi,\psi)\colon \mathfrak{m}\rtimes\mathfrak{p} \to \mathfrak{m}'\rtimes\mathfrak{p}'$, given by $(\varphi,\psi)(m,p) = (\varphi(m),\psi(p))$ for all $(m,p)\in\mathfrak{m}\rtimes\mathfrak{p}$.

The functor $\mathtt{XU_d}\colon \mathbf{XLb} \to \mathbf{XDias}$ is a natural generalization of the functor $\mathtt{U_d}$, in the sense that it makes the following diagram commute,

$$\begin{tikzcd}
\mathbf{Lb} \arrow[r, "\mathtt{J}_0"] \arrow[d, "\mathtt{U_d}"'] & \mathbf{XLb} \arrow[d, "\mathtt{XU_d}"] \\
\mathbf{Dias} \arrow[r, "\mathtt{J}_0'"'] & \mathbf{XDias}\,.
\end{tikzcd}$$

Regarding the embeddings $\mathtt{J}_1$ and $\mathtt{J}_1'$, we have the following result.

Proposition 3.3.3. *There is a natural isomorphism of functors*

$$\mathtt{XU_d}\circ\mathtt{J}_1 \cong \mathtt{J}_1'\circ\mathtt{U_d}\,.$$

Proof. Let $\mathfrak{p}\in\mathbf{Lb}$. We need to show that $\mathtt{XU_d}(\mathfrak{p},\mathfrak{p},\mathrm{id}_{\mathfrak{p}})$ is naturally isomorphic to $(\mathtt{U_d}(\mathfrak{p}),\mathtt{U_d}(\mathfrak{p}),\mathrm{id}_{\mathtt{U_d}(\mathfrak{p})})$. In order to do so, we will prove that $(\overline{\mathtt{U_d}}(t)|_{\operatorname{Ker}\overline{\mathtt{U_d}}(s)},\mathrm{id}_{\mathtt{U_d}(\mathfrak{p})})$ is an isomorphism of crossed modules of dialgebras between $(\operatorname{Ker}\overline{\mathtt{U_d}}(s),\mathtt{U_d}(\mathfrak{p}),\overline{\mathtt{U_d}}(t)|_{\operatorname{Ker}\overline{\mathtt{U_d}}(s)})$ and $(\mathtt{U_d}(\mathfrak{p}),\mathtt{U_d}(\mathfrak{p}),\mathrm{id}_{\mathtt{U_d}(\mathfrak{p})})$.

Note that $(\overline{\mathtt{U_d}}(t)|_{\operatorname{Ker}\overline{\mathtt{U_d}}(s)},\mathrm{id}_{\mathtt{U_d}(\mathfrak{p})})$ is indeed a morphism of crossed modules of dialgebras (see Example 1.2.56 (iii)). Recall that the first step in the construction of $\mathtt{XU_d}(\mathfrak{p},\mathfrak{p},\mathrm{id}_{\mathfrak{p}})$ requires us to consider the cat^1-Leibniz algebra

$$\begin{tikzcd}
\mathfrak{p}\rtimes\mathfrak{p} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathfrak{p}
\end{tikzcd}$$

with $s(p,p') = p'$ and $t(p,p') = p+p'$ for all $p,p'\in\mathfrak{p}$. Let us define the Leibniz homomorphism $\epsilon\colon \mathfrak{p}\to\mathfrak{p}\rtimes\mathfrak{p}$, $\epsilon(p) = (p,0)$. It is clear that $s\epsilon = 0$ and $t\epsilon = \mathrm{id}_{\mathfrak{p}}$.

The next step is to apply the functor $\mathtt{U_d}$ on the previous cat^1-Leibniz algebra and take the quotient of $\mathtt{U_d}(\mathfrak{p}\rtimes\mathfrak{p})$ by $X = \operatorname{Ker}\mathtt{U_d}(s)\dashv\operatorname{Ker}\mathtt{U_d}(t) + \operatorname{Ker}\mathtt{U_d}(t)\dashv\operatorname{Ker}\mathtt{U_d}(s) +$

$\operatorname{Ker}\mathtt{U}_\mathtt{d}(s) \vdash \operatorname{Ker}\mathtt{U}_\mathtt{d}(t) + \operatorname{Ker}\mathtt{U}_\mathtt{d}(t) \vdash \operatorname{Ker}\mathtt{U}_\mathtt{d}(s)$ in order to guarantee that we have a cat^1-dialgebra. In the next diagram of dialgebras,

$$\begin{tikzcd}
\mathtt{U}_\mathtt{d}(\mathfrak{p}) \arrow[r, "\mathtt{U}_\mathtt{d}(\epsilon)"] & \mathtt{U}_\mathtt{d}(\mathfrak{p} \rtimes \mathfrak{p}) \arrow[r, shift left, "\mathtt{U}_\mathtt{d}(s)"] \arrow[r, shift right, "\mathtt{U}_\mathtt{d}(t)"'] \arrow[d, "\pi"'] & \mathtt{U}_\mathtt{d}(\mathfrak{p}) \\
& \mathtt{U}_\mathtt{d}(\mathfrak{p} \rtimes \mathfrak{p})/X \arrow[ur, shift left, "\overline{\mathtt{U}_\mathtt{d}}(s)"] \arrow[ur, shift right, "\overline{\mathtt{U}_\mathtt{d}}(t)"'] &
\end{tikzcd}$$

where π is the canonical projection, it is easy to see that $\overline{\mathtt{U}_\mathtt{d}}(s)\pi\,\mathtt{U}(\epsilon) = \mathtt{U}_\mathtt{d}(s)\,\mathtt{U}_\mathtt{d}(\epsilon) = \mathtt{U}_\mathtt{d}(s\epsilon) = 0$ and $\overline{\mathtt{U}_\mathtt{d}}(t)\pi\,\mathtt{U}_\mathtt{d}(\epsilon) = \mathtt{U}_\mathtt{d}(t)\,\mathtt{U}_\mathtt{d}(\epsilon) = \mathtt{U}_\mathtt{d}(t\epsilon) = \mathrm{id}_{\mathtt{U}_\mathtt{d}(\mathfrak{p})}$. Hence $\pi\,\mathtt{U}_\mathtt{d}(\epsilon)$ takes values in $\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s)$ and it is a right inverse for $\overline{\mathtt{U}_\mathtt{d}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s)}$.

Now we need to show that $\pi\,\mathtt{U}_\mathtt{d}(\epsilon)\overline{\mathtt{U}_\mathtt{d}}(t) = \mathrm{id}_{\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s)}$. Note that $X \subset \operatorname{Ker}\mathtt{U}_\mathtt{d}(s)$, so $\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s) = \operatorname{Ker}\mathtt{U}_\mathtt{d}(s)/X$ and, as a K-module, $\operatorname{Ker}\mathtt{U}_\mathtt{d}(s)$ is generated by all the elements of the form

$$(p_{-n}, p'_{-n}) \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_m, p'_m) \tag{3.3.1}$$

with $n, m \in \mathbb{N}$, $p_i, p'_i \in \mathfrak{p}$, $-n \le i \le m$. By the definition of $\mathtt{U}_\mathtt{d}(t)$ and $\mathtt{U}_\mathtt{d}(\epsilon)$, the value of $\mathtt{U}_\mathtt{d}(\epsilon)\,\mathtt{U}_\mathtt{d}(t)$ on (3.3.1) is

$$(p_{-n} + p'_{-n}, 0) \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_m + p'_m, 0). \tag{3.3.2}$$

Furthermore, one can easily derive the following equalities in $\operatorname{Ker}\mathtt{U}_\mathtt{d}(s)/X$:

$$\begin{aligned}
&(p_{-n} + p'_{-n}, 0) \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_m + p'_m, 0) \\
&= (p_{-n}, p'_{-n}) \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_m + p'_m, 0) \\
&= \cdots \\
&= (p_{-n}, p'_{-n}) \otimes \cdots \otimes (p_i, 0) \otimes \cdots \otimes (p_m, p'_m).
\end{aligned}$$

Thus, the elements (3.3.1) and (3.3.2) are equal in $\operatorname{Ker}\mathtt{U}_\mathtt{d}(s)/X$ and it follows that

$$\pi\,\mathtt{U}_\mathtt{d}(\epsilon)\overline{\mathtt{U}_\mathtt{d}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s)} = \mathrm{id}_{\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s)}\,.$$

Therefore we have found an inverse for the morphism of crossed modules of dialgebras $(\overline{\mathtt{U}_\mathtt{d}}(t)|_{\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s)}, \mathrm{id}_{\mathtt{U}_\mathtt{d}(\mathfrak{p})})$, which is given by

$$\begin{tikzcd}
\mathtt{U}_\mathtt{d}(\mathfrak{p}) \arrow[r, "\mathrm{id}_{\mathtt{U}_\mathtt{d}(\mathfrak{p})}"] \arrow[d, "\pi\,\mathtt{U}_\mathtt{d}(\epsilon)"'] & \mathtt{U}_\mathtt{d}(\mathfrak{p}) \arrow[d, "\mathrm{id}_{\mathtt{U}_\mathtt{d}(\mathfrak{p})}"] \\
\operatorname{Ker}\overline{\mathtt{U}_\mathtt{d}}(s) \arrow[r, "\overline{\mathtt{U}_\mathtt{d}}(t)"'] & \mathtt{U}_\mathtt{d}(\mathfrak{p}).
\end{tikzcd}$$

Note that we have already proved that the above diagram commutes. Moreover, given $p_i, p'_i \in \mathfrak{p}$ for $i = -1, 0, 1$, let $v = p_{-1} \otimes p_0 \otimes p_1$ and $w = p'_{-1} \otimes p'_0 \otimes p'_1$. Then,

$$\begin{aligned}
\pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(v \dashv w) &= (p_{-1}, 0) \otimes (p_0, 0) \otimes (p_1, 0)(p'_{-1}, 0)(p'_0, 0)(p'_1, 0) + X \\
&= (0, p_{-1}) \otimes (0, p_0) \otimes (0, p_1)(p'_{-1}, 0)(p'_0, 0)(p'_1, 0) + X \\
&= v \dashv \pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(w). \\
\pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(v \dashv w) &= (p_{-1}, 0) \otimes (p_0, 0) \otimes (p_1, 0)(p'_{-1}, 0)(p'_0, 0)(p'_1, 0) + X \\
&= (p_{-1}, 0) \otimes (p_0, 0) \otimes (p_1, 0)(0, p'_{-1})(0, p'_0)(0, p'_1) + X \\
&= \pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(v) \dashv w.
\end{aligned}$$

Similarly, $\pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(v \vdash w) = v \vdash \pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(w) = \pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)(v) \vdash w$, so $\pi\, \mathtt{U}_{\mathtt{d}}(\epsilon)$ preserves the action of $\mathtt{U}_{\mathtt{d}}(\mathfrak{p})$ on itself via $\mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p})}$ when extended to all the elements in $\mathtt{U}_{\mathtt{d}}(\mathfrak{p})$.

Finally, $(\overline{\mathtt{U}_{\mathtt{d}}}(t)|_{\operatorname{Ker} \overline{\mathtt{U}_{\mathtt{d}}}(s)}, \mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p})})$ is natural, as shown in the diagram below, where $\alpha\colon \mathfrak{p} \to \mathfrak{p}'$ is a Leibniz homomorphism. Note that $\overline{\mathtt{U}_{\mathtt{d}}}(\alpha, \alpha)$ is actually $\overline{\mathtt{U}_{\mathtt{d}}}(\alpha, \alpha)|_{\operatorname{Ker} \overline{\mathtt{U}_{\mathtt{d}}}(s)}$.

$$\begin{tikzcd}
& \operatorname{Ker} \overline{\mathtt{U}_{\mathtt{d}}}(s) \arrow[rr, "\overline{\mathtt{U}_{\mathtt{d}}}(t)"] \arrow[dl, "\overline{\mathtt{U}_{\mathtt{d}}}(\alpha,\alpha)"'] \arrow[dd, dashed, "\overline{\mathtt{U}_{\mathtt{d}}}(t)"] & & \mathtt{U}(\mathfrak{p}) \arrow[dl, "\mathtt{U}_{\mathtt{d}}(\alpha)"] \arrow[dd, "\mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p})}"] \\
\operatorname{Ker} \overline{\mathtt{U}_{\mathtt{d}}}(s') \arrow[rr, "\overline{\mathtt{U}_{\mathtt{d}}}(t')"' near end] \arrow[dd, "\overline{\mathtt{U}_{\mathtt{d}}}(t')"'] & & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}') \arrow[dd, "\mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p}')}" near start] & \\
& \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[rr, dashed, "\mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p})}" near start] \arrow[dl, dashed, "\mathtt{U}_{\mathtt{d}}(\alpha)"] & & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[dl, "\mathtt{U}_{\mathtt{d}}(\alpha)"] \\
\mathtt{U}_{\mathtt{d}}(\mathfrak{p}') \arrow[rr, "\mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p}')}"'] & & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}') &
\end{tikzcd}$$

$\square$

3.3.3 Adjunction between **XLb** and **XDias**

The following result is a natural generalization of the well-known adjunction between the categories **Lb** and **Dias**.

Theorem 3.3.4. *The functor* $\mathtt{XU}_{\mathtt{d}}$ *is left adjoint to the functor* $\mathtt{XLb}$*.*

Proof. Given $(\mathfrak{m}, \mathfrak{p}, \eta)$ in **XLb** and (L, D, μ) in **XDias**, we have to construct a natural bijection

$$\operatorname{Hom}_{\mathbf{XLb}}\big((\mathfrak{m}, \mathfrak{p}, \eta), \mathtt{XLb}(L, D, \mu)\big) \cong \operatorname{Hom}_{\mathbf{XDias}}\big(\mathtt{XU}_{\mathtt{d}}(\mathfrak{m}, \mathfrak{p}, \eta), (L, D, \mu)\big),$$

Let $(\varphi, \psi) \in \mathrm{Hom}_{\mathbf{XLb}}\big((\mathfrak{m}, \mathfrak{p}, \eta), \mathtt{XLb}(L, D, \mu)\big)$, that is

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, "\eta"] \arrow[d, "\varphi"'] & \mathfrak{p} \arrow[d, "\psi"] \\
\mathtt{Lb}(L) \arrow[r, "\mu"'] & \mathtt{Lb}(D)
\end{tikzcd}$$

such that $\mu\varphi = \psi\eta$ and φ preserves the action of $\mathfrak{p}$ on $\mathfrak{m}$ via ψ.

We can consider the following morphism of cat^1-Leibniz algebras by using the functor $\mathtt{cat}_{\mathtt{Lb}}$ as defined in the proof of Proposition 1.2.46, as well as Lemma 3.3.2.

$$\begin{tikzcd}
\mathfrak{m} \rtimes \mathfrak{p} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\varphi'"'] & \mathfrak{p} \arrow[d, "\psi"] \\
\mathtt{Lb}(L \rtimes D) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathtt{Lb}(D)
\end{tikzcd}$$

where $\varphi'(m, p) = (\varphi(m), \psi(p))$ for all $(m, p) \in \mathfrak{m} \rtimes \mathfrak{p}$. Since the functor $\mathtt{U_d}$ is left adjoint to the functor $\mathtt{Lb}$, we have the induced commutative diagrams of dialgebras

$$\begin{tikzcd}
\mathtt{U_d}(\mathfrak{m} \rtimes \mathfrak{p}) \arrow[r, "\mathtt{U_d}(s)"] \arrow[d, "\varphi'^*"'] & \mathtt{U_d}(\mathfrak{p}) \arrow[d, "\psi^*"] \\
L \rtimes D \arrow[r, "\sigma"'] & D
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathtt{U_d}(\mathfrak{m} \rtimes \mathfrak{p}) \arrow[r, "\mathtt{U_d}(t)"] \arrow[d, "\varphi'^*"'] & \mathtt{U_d}(\mathfrak{p}) \arrow[d, "\psi^*"] \\
L \rtimes D \arrow[r, "\tau"'] & D
\end{tikzcd}$$

Besides, we have a uniquely defined morphism of cat^1-dialgebras:

$$\begin{tikzcd}
\mathtt{U_d}(\mathfrak{m} \rtimes \mathfrak{p})/X \arrow[r, shift left, "\overline{\mathtt{U_d}}(s)"] \arrow[r, shift right, "\overline{\mathtt{U_d}}(t)"'] \arrow[d, "\overline{\varphi}'^*"'] & \mathtt{U_d}(\mathfrak{p}) \arrow[d, "\psi^*"] \\
L \rtimes D \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & D
\end{tikzcd}$$

where $X = \mathrm{Ker}\,\mathtt{U_d}(s) \dashv \mathrm{Ker}\,\mathtt{U_d}(t) + \mathrm{Ker}\,\mathtt{U_d}(t) \dashv \mathrm{Ker}\,\mathtt{U_d}(s) + \mathrm{Ker}\,\mathtt{U_d}(s) \vdash \mathrm{Ker}\,\mathtt{U_d}(t) + \mathrm{Ker}\,\mathtt{U_d}(t) \vdash \mathrm{Ker}\,\mathtt{U_d}(s)$.

Finally, we can make use of the functor $\mathtt{Xm}_{\mathtt{Dias}}$, which is described in the proof of Proposition 1.2.61, in order to get a uniquely defined morphism (φ^{**}, ψ^*) in

$\operatorname{Hom}_{\mathsf{XDias}}\big(\mathtt{XU}_{\mathtt{d}}(\mathfrak{m},\mathfrak{p},\eta),(L,D,\mu)\big)$:

$$\begin{tikzcd}
\operatorname{Ker}\overline{\mathtt{U}_{\mathtt{d}}}(s) \arrow[r, "\overline{\mathtt{U}_{\mathtt{d}}}(t)"] \arrow[d, "\overline{\varphi}'^{*}"'] \arrow[dd, bend right=60, "\varphi^{**}"'] & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[d, "\psi^{*}"] \\
\operatorname{Ker}\sigma \arrow[r, "\tau"] \arrow[d, "\simeq"'] & D \arrow[d, "\mathrm{id}_D"] \\
L \arrow[r, "\mu"'] & D.
\end{tikzcd}$$

Note that, in the diagram above, $\overline{\varphi}'^{*}=\overline{\varphi}'^{*}|_{\operatorname{Ker}\overline{\mathtt{U}_{\mathtt{d}}}(s)}$, $\operatorname{Ker}\sigma=L\rtimes\{0\}$, and $\tau=\tau|_{\operatorname{Ker}\sigma}=\mu$.

Now, let $(\phi,\chi)\in\operatorname{Hom}_{\mathsf{XDias}}\big(\mathtt{XU}_{\mathtt{d}}(\mathfrak{m},\mathfrak{p},\eta),(L,D,\mu)\big)$, that is

$$\begin{tikzcd}
\operatorname{Ker}\overline{\mathtt{U}_{\mathtt{d}}}(s) \arrow[r, "\overline{\mathtt{U}_{\mathtt{d}}}(t)"] \arrow[d, "\phi"'] & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[d, "\chi"] \\
L \arrow[r, "\mu"'] & D
\end{tikzcd}$$

such that $\mu\phi=\chi\overline{\mathtt{U}_{\mathtt{d}}}(t)$ and ϕ preserves the action of $\mathtt{U}_{\mathtt{d}}(\mathfrak{p})$ on $\operatorname{Ker}\overline{\mathtt{U}_{\mathtt{d}}}(s)$ via χ.

We can consider the corresponding morphism of cat^1-dialgebras by using the functor $\mathtt{cat}_{\mathtt{Dias}}$ as defined in the proof of Proposition 1.2.61.

$$\begin{tikzcd}
\mathtt{U}_{\mathtt{d}}(\mathfrak{m}\rtimes\mathfrak{p})/X \arrow[r, shift left, "\overline{\mathtt{U}_{\mathtt{d}}}(s)"] \arrow[r, shift right, "\overline{\mathtt{U}_{\mathtt{d}}}(t)"'] \arrow[d, "\simeq"'] \arrow[dd, bend right=60, "\phi'"'] & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[d, "\mathrm{id}_{\mathtt{U}_{\mathtt{d}}(\mathfrak{p})}"] \\
\operatorname{Ker}\overline{\mathtt{U}_{\mathtt{d}}}(s)\rtimes\mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[r, shift left, "\tilde{\sigma}"] \arrow[r, shift right, "\tilde{\tau}"'] \arrow[d, "(\phi,\chi)"'] & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[d, "\chi"] \\
L\rtimes D \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & D.
\end{tikzcd}$$

Observe that the isomorphism in the diagram above is explicitly described at the end of the proof of Proposition 1.2.61, as well as $\tilde{\sigma}$ and $\tilde{\tau}$.

Let $\overline{\phi}=\phi'\pi$, with $\pi\colon\mathtt{U}_{\mathtt{d}}(\mathfrak{m}\rtimes\mathfrak{p})\to\mathtt{U}_{\mathtt{d}}(\mathfrak{m}\rtimes\mathfrak{p})/X$ the canonical projection. Hence, we have the commutative diagrams of dialgebras

$$\begin{tikzcd}
\mathtt{U}_{\mathtt{d}}(\mathfrak{m}\rtimes\mathfrak{p}) \arrow[r, "\mathtt{U}_{\mathtt{d}}(s)"] \arrow[d, "\overline{\phi}"'] & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[d, "\chi"] \\
L\rtimes D \arrow[r, "\sigma"'] & D
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathtt{U}_{\mathtt{d}}(\mathfrak{m}\rtimes\mathfrak{p}) \arrow[r, "\mathtt{U}_{\mathtt{d}}(t)"] \arrow[d, "\overline{\phi}"'] & \mathtt{U}_{\mathtt{d}}(\mathfrak{p}) \arrow[d, "\chi"] \\
L\rtimes D \arrow[r, "\tau"'] & D.
\end{tikzcd}$$

Since the functor $\mathtt{U_d}$ is left adjoint to the functor $\mathtt{Lb}$, we have the morphism of cat^1-Leibniz algebras

$$\begin{tikzcd}
\mathfrak{m} \rtimes \mathfrak{p} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\phi^*"'] & \mathfrak{p} \arrow[d, "\chi^*"] \\
\mathtt{Lb}(L \rtimes D) \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & \mathtt{Lb}(D).
\end{tikzcd}$$

Finally, we can use the functor $\mathtt{Xm_{Lb}}$, described in the proof of Proposition 1.2.46, to get a uniquely defined morphism (ϕ^{**}, χ^*) in $\mathrm{Hom}_{\mathbf{XLb}}\big((\mathfrak{m}, \mathfrak{p}, \eta), \mathtt{XLb}(L, D, \mu)\big)$:

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, "\eta"] \arrow[d, "\simeq"'] \arrow[ddd, bend right=60, "\phi^{**}"'] & \mathfrak{p} \arrow[d, "\mathrm{id}_{\mathfrak{p}}"] \\
\mathrm{Ker}\, s \arrow[r, "t"] \arrow[d, "\phi^*|_{\mathrm{Ker}\, s}"'] & \mathfrak{p} \arrow[d, "\chi^*"] \\
\mathrm{Ker}\, \sigma \arrow[r, "\tau"] \arrow[d, "\simeq"'] & \mathtt{Lb}(D) \arrow[d, "\mathrm{id}_{\mathtt{Lb}(D)}"] \\
\mathtt{Lb}(L) \arrow[r, "\mu"'] & \mathtt{Lb}(D).
\end{tikzcd}$$

Note that in the diagram above $\mathrm{Ker}\, s = \mathfrak{m} \rtimes \{0\}$, $t = t|_{\mathrm{Ker}\, s} = \eta$, $\mathrm{Ker}\, \sigma = \mathtt{Lb}(L) \rtimes \{0\}$ and $\tau = \tau|_{\mathrm{Ker}\, \sigma} = \mu$. □

On account of the previous theorem along with Propositions 1.2.43 and 1.2.57, one can easily deduce the following result.

Theorem 3.3.5. *The inner and outer squares in the following diagrams are commutative or commute up to isomorphism for $i = 0, 1$.*

$$\begin{tikzcd}
\mathbf{Dias} \arrow[r, shift right, "\mathtt{Lb}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathtt{J}'_i"] \arrow[d, phantom, "\dashv"] & \mathbf{Lb} \arrow[l, shift right, "\mathtt{U_d}"'] \arrow[d, shift right, "\mathtt{J}_i"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{XDias} \arrow[r, shift left, "\mathtt{XLb}"] \arrow[r, phantom, "\top"] \arrow[u, shift left, "\mathtt{\Psi}'_i"] & \mathbf{XLb} \arrow[l, shift left, "\mathtt{XU_d}"] \arrow[u, shift right, "\mathtt{\Psi}_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Dias} \arrow[r, shift right, "\mathtt{Lb}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\mathtt{J}'_i"'] \arrow[d, phantom, "\dashv"] & \mathbf{Lb} \arrow[l, shift right, "\mathtt{U_d}"'] \arrow[d, shift left, "\mathtt{J}_i"] \arrow[d, phantom, "\vdash"] \\
\mathbf{XDias} \arrow[r, shift left, "\mathtt{XLb}"] \arrow[r, phantom, "\top"] \arrow[u, shift right, "\mathtt{\Psi}'_{i+1}"'] & \mathbf{XLb} \arrow[l, shift left, "\mathtt{XU_d}"] \arrow[u, shift left, "\mathtt{\Psi}_{i+1}"]
\end{tikzcd}
\tag{3.3.3}$$

Proof. Let us begin with the first diagram. Directly from the definition of the functors involved, $\mathtt{XLb} \circ \mathtt{J}'_i = \mathtt{J}_i \circ \mathtt{Lb}$, for i=0,1. Besides, from the adjunctions described in the first diagram, we get that $\mathtt{U_d} \circ \mathtt{\Psi}_i$ is left adjoint to $\mathtt{J}_i \circ \mathtt{Lb}$, while $\mathtt{\Psi}'_i \circ \mathtt{XU_d}$ is left adjoint to $\mathtt{XLb} \circ \mathtt{J}'_i$, for $i = 0, 1$. Hence $\mathtt{U_d} \circ \mathtt{\Psi}_i \cong \mathtt{\Psi}'_i \circ \mathtt{XU_d}$ for $i = 0, 1$.

Regarding the second diagram, the commutativity of the outer square is obvious for $i = 0$, while in Proposition 3.3.3 we proved that there is a natural isomorphism $\mathtt{XU_d} \circ \mathtt{J}_1 \cong \mathtt{J}_1' \circ \mathtt{U_d}$. Therefore, by a similar reasoning to the one used for the first diagram, we get that $\mathtt{Lb} \circ \Psi_{i+1}' \cong \Psi_{i+1} \circ \mathtt{XLb}$ for $i = 0, 1$. □

3.4 Relations between crossed modules of Lie, Leibniz, associative algebras and dialgebras

Any associative algebra A can be regarded as a dialgebra with $a_1 \dashv a_2 = a_1 a_2 = a_1 \vdash a_2$ for all $a_1, a_2 \in A$. Also, any morphism of algebras $f\colon A \to B$ is a morphism of dialgebras when we consider the previous dialgebra structure both in A and B. Therefore there is an inclusion functor $\mathbf{As} \xrightarrow{\subset} \mathbf{Dias}$. In [65], it is described a left adjoint $\mathtt{As}\colon \mathbf{Dias} \to \mathbf{As}$ as follows. Given any dialgebra D, $\mathtt{As}(D)$ is the quotient of D by the ideal generated by the elements $x_1 \dashv x_2 - x_1 \vdash x_2$, with $x_1, x_2 \in D$. It is clear that $\dashv = \vdash$ in $\mathtt{As}(D)$, so it is an associative algebra (not necessarily unital). For any morphism of dialgebras $g\colon D \to L$, we define $\mathtt{As}(g) = \overline{g}\colon \mathtt{As}(D) \to \mathtt{As}(L)$, with $\overline{g}(\overline{x}) = \overline{g(x)}$, for all $\overline{x} \in \mathtt{As}(D)$. It is obvious that $\overline{g}$ is well defined and it is a morphism of algebras.

Additionally, every Lie algebra is a Leibniz algebra and every Lie homomorphism can be regarded as a Leibniz homomorphism. There is a left adjoint $\mathtt{Lie_{Lb}}\colon \mathbf{Lb} \to \mathbf{Lie}$ to the inclusion functor $\mathbf{Lie} \xrightarrow{\subset} \mathbf{Lb}$ (see, for instance, [64]) defined as follows. Given any Leibniz algebra $\mathfrak{p}$, $\mathtt{Lie_{Lb}}(\mathfrak{p})$ is the quotient of $\mathfrak{p}$ by the ideal generated by the elements $[p, p]$, with $p \in \mathfrak{p}$. Analogously, given a Leibniz homomorphism $g\colon \mathfrak{m} \to \mathfrak{p}$, we will denote $\mathtt{Lie_{Lb}}(g)$ by $\overline{g}$.

We have the following diagram of adjunctions:

$$\begin{tikzcd}
\mathbf{As} \arrow[rr, shift right, "\mathtt{Lie_{As}}"'] \arrow[rr, phantom, "\perp"] \arrow[dd, shift left, "\subset"] \arrow[dd, phantom, "\dashv"] & & \mathbf{Lie} \arrow[ll, shift right, "\mathtt{U}"'] \arrow[dd, shift right, "\subset"'] \arrow[dd, phantom, "\vdash"] \\
& & \\
\mathbf{Dias} \arrow[uu, shift left, "\mathtt{As}"] \arrow[rr, shift left, "\mathtt{Lb}"] \arrow[rr, phantom, "\top"] & & \mathbf{Lb} \arrow[ll, shift left, "\mathtt{U_d}"] \arrow[uu, shift right, "\mathtt{Lie_{Lb}}"']
\end{tikzcd} \tag{3.4.1}$$

in which $\subset \circ \mathtt{Lie_{As}} = \mathtt{Lb} \circ \subset$. Regarding the outer square, since $\mathtt{U}$ is left adjoint to $\mathtt{Lie_{As}}$ and $\mathtt{Lie_{Lb}}$ is left adjoint to $\subset$, $\mathtt{U} \circ \mathtt{Lie_{Lb}}$ is left adjoint to $\subset \circ \mathtt{Lie_{As}}$. Likewise, $\mathtt{As} \circ \mathtt{U_d}$ is left adjoint to $\mathtt{Lb} \circ \subset$. Therefore $\mathtt{U} \circ \mathtt{Lie_{Lb}}$ is naturally isomorphic to $\mathtt{As} \circ \mathtt{U_d}$.

In the two previous sections we generalized the adjunctions $\mathtt{U} \dashv \mathtt{Lie_{As}}$ and $\mathtt{U_d} \dashv \mathtt{Lb}$. Now we will do the same with $\mathtt{As} \dashv \subset$ and $\mathtt{Lie_{Lb}} \dashv \subset$ in order to get a diagram for crossed modules analogous to the one above.

3.4.1 Adjunction between **XAs** and **XDias**

Let (B, A, ρ) be a crossed module of algebras. Both A and B can be regarded as dialgebras and the identities $a * b = ab$ and $b * a = ba$ for all $a \in A$, $b \in B$, define an action of dialgebras of A on B. Moreover, since $\dashv$ and $\vdash$ are equal both as products and actions, (XDi1) and (XDi2) follow from (XAs1) and (XAs2) respectively. Besides, every morphism of crossed modules of algebras can be regarded as a morphism of crossed modules of dialgebras.

Conversely, given a crossed module of dialgebras (L, D, μ), we define $\mathtt{XAs}(L, D, \mu)$ as the crossed module of algebras $(\overline{\mathtt{As}}(L), \mathtt{As}(D), \overline{\mu})$, with $\overline{\mathtt{As}}(L)$ the quotient of L by the ideal generated by the elements $a \dashv a' - a \vdash a'$, $x \dashv a - x \vdash a$, $a \dashv x - a \vdash x$, for all $x \in D$, $a,\ a' \in L$. From the action of D on L we get an action of $\mathtt{As}(D)$ on $\overline{\mathtt{As}}(L)$, given by $\overline{x}\,\overline{a} = \overline{x \dashv a} = \overline{x \vdash a}$ and $\overline{a}\,\overline{x} = \overline{a \dashv x} = \overline{a \vdash x}$ for all $\overline{x} \in \mathtt{As}(D)$, $\overline{a} \in \mathtt{As}(L)$. Since $\dashv$ and $\vdash$ are equal as products in $\mathtt{As}(D)$ and $\overline{\mathtt{As}}(L)$, and as an action of $\mathtt{As}(D)$ on $\overline{\mathtt{As}}(L)$, (XAs1) and (XAs2) follow from (XDi1) and (XDi2) respectively. Besides, given a morphism of crossed modules of dialgebras $(\varphi, \psi)\colon (L, D, \mu) \to (L', D', \mu')$, $(\overline{\varphi}, \overline{\psi})\colon (\overline{\mathtt{As}}(L), \mathtt{As}(D), \overline{\mu}) \to (\overline{\mathtt{As}}(L'), \mathtt{As}(D'), \overline{\mu'})$ is a morphism of crossed modules of algebras.

Proposition 3.4.1. *The functor* $\mathtt{XAs}\colon$ **XDias** $\to$ **XAs** *is left adjoint to the inclusion functor* **XAs** $\hookrightarrow$ **XDias**.

Proof. Let (L, D, μ) be a crossed module of dialgebras, (B, A, ρ) a crossed module of algebras and $(\varphi, \psi)\colon (L, D, \mu) \to (B, A, \rho)$ a morphism of crossed modules of dialgebras. Consider the quotient maps $\pi_L\colon L \to \overline{\mathtt{As}}(L)$ and $\pi_D\colon D \to \mathtt{As}(D)$, which form morphism of crossed modules of dialgebras. There is a unique morphism of crossed modules of algebras $(\tilde{\varphi}, \tilde{\psi})\colon (\overline{\mathtt{As}}(L), \mathtt{As}(D), \overline{\mu}) \to (B, A, \rho)$, such that $\tilde{\varphi}\pi_L = \varphi$ and $\tilde{\psi}\pi_D = \psi$, where $\tilde{\varphi}(\overline{a}) = \varphi(a)$ and $\tilde{\psi}(\overline{x}) = \psi(x)$, for all $\overline{a} \in \overline{\mathtt{As}}(L)$, $\overline{x} \in \mathtt{As}(D)$. Note that $\tilde{\varphi}$ is well defined due to the fact that φ is a morphism of dialgebras together with φ preserving the action of D on L via ψ and the identity $\dashv = \vdash$ in the dialgebra structure of B, while the correctness of the definition of $\tilde{\psi}$ follows from ψ being a morphism of dialgebras and the identity $\dashv = \vdash$ in A.

$$
\begin{tikzcd}
L \arrow[r, "\mu"] \arrow[rrd, dashed, "\pi_L"] \arrow[dd, "\varphi"'] & D \arrow[rrd, "\pi_D"] \arrow[dd, "\psi"] & & \\
& & \overline{\mathtt{As}}(L) \arrow[r, dashed, "\overline{\mu}"] \arrow[lld, dashed, "\tilde{\varphi}"] & \mathtt{As}(D) \arrow[lld, "\tilde{\psi}"] \\
B \arrow[r, "\rho"'] & A & &
\end{tikzcd}
$$

Hence $(\pi_L, \pi_D)\colon (L, D, \mu) \to (\overline{\mathtt{As}}(L), \mathtt{As}(D), \overline{\mu})$ is universal among the morphisms from (L, D, μ) to (B, A, ρ), that is $\mathtt{XAs} \dashv \subset$. $\square$

The functor $\mathtt{XAs}\colon \textsf{\textbf{XDias}} \to \textsf{\textbf{XAs}}$ is a natural generalization of $\mathtt{As}\colon \textsf{\textbf{Dias}} \to \textsf{\textbf{As}}$ in the sense that the following inner and outer diagrams are commutative or commute up to isomorphism for $i = 0, 1$.

$$
\begin{tikzcd}
\textsf{\textbf{As}} \arrow[r, shift left, hook, "\subset"'] \arrow[d, shift left, "\mathtt{I}'_i"] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \textsf{\textbf{Dias}} \arrow[l, shift left=2, "\mathtt{As}"'] \arrow[d, shift right, "\mathtt{J}'_i"'] \arrow[d, phantom, "\vdash"] \\
\textsf{\textbf{XAs}} \arrow[u, shift left=2, "\Phi'_i"] \arrow[r, shift left, hook, "\subset"] \arrow[r, phantom, "\top"] & \textsf{\textbf{XDias}} \arrow[l, shift left=2, "\mathtt{XAs}"] \arrow[u, shift right=2, "\Psi'_i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\textsf{\textbf{As}} \arrow[r, shift left, hook, "\subset"'] \arrow[d, shift right, "\mathtt{I}'_i"'] \arrow[r, phantom, "\perp"] \arrow[d, phantom, "\dashv"] & \textsf{\textbf{Dias}} \arrow[l, shift left=2, "\mathtt{As}"'] \arrow[d, shift left, "\mathtt{J}'_i"] \arrow[d, phantom, "\vdash"] \\
\textsf{\textbf{XAs}} \arrow[u, shift right=2, "\Phi'_{i+1}"'] \arrow[r, shift left, hook, "\subset"] \arrow[r, phantom, "\top"] & \textsf{\textbf{XDias}} \arrow[l, shift left=2, "\mathtt{XAs}"] \arrow[u, shift left=2, "\Psi'_{i+1}"]
\end{tikzcd}
\tag{3.4.2}
$$

Note that $\mathtt{I}'_1 \circ \mathtt{As} = \mathtt{XAs} \circ \mathtt{J}'_1$ holds for any given dialgebra D, because $\overline{\mathtt{As}}(D) = \mathtt{As}(D)$ for the crossed module (D, D, id_D), since the action of D on itself is given by the left and the right products in D.

3.4.2 Adjunction between XLie and XLb

Let $(\mathbb{m}, \mathbb{p}, \nu)$ be a Lie crossed module. Both $\mathbb{m}$ and $\mathbb{p}$ can be considered as Leibniz algebras and the Lie action of $\mathbb{p}$ on $\mathbb{m}$ is itself a Leibniz action if we assume that $[m, p] = -[p, m]$ for all $m \in \mathbb{m}$, $p \in \mathbb{p}$. Moreover, due to the anti-commutativity of the Lie brackets in $\mathbb{p}$ and $\mathbb{m}$ along with the anti-commutativity of the action, (XLb1) and (XLb2) follow from (XLie1) and (XLie2) respectively. Besides, every morphism of Lie crossed modules can be regarded as a morphism of Leibniz crossed modules.

Conversely, given a Leibniz crossed module $(\mathfrak{m}, \mathfrak{p}, \eta)$, we define $\mathtt{XLie}_{\mathtt{Lb}}(\mathfrak{m}, \mathfrak{p}, \eta) = (\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m}), \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p}), \overline{\eta})$, with $\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m})$ the quotient of $\mathfrak{m}$ by the ideal generated by the elements $[m, m]$ and $[p, m] + [m, p]$ for all $m \in \mathfrak{m}$, $p \in \mathfrak{p}$. From the action of $\mathfrak{p}$ on $\mathfrak{m}$ we get an induced action of $\mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p})$ on $\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m})$. (XLie1) and (XLie2) follow from (XLb1) and (XLb2). Besides, given a morphism of Leibniz crossed modules $(\varphi, \psi)\colon (\mathfrak{m}, \mathfrak{p}, \eta) \to (\mathfrak{m}', \mathfrak{p}', \eta')$, $(\overline{\varphi}, \overline{\psi})$ is a morphism of Lie crossed modules from $(\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m}), \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p}), \overline{\eta})$ to $(\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m}'), \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p}'), \overline{\eta'})$.

Proposition 3.4.2. *The functor* $\mathtt{XLie}_{\mathtt{Lb}}\colon \textsf{\textbf{XLb}} \to \textsf{\textbf{XLie}}$ *is left adjoint to the inclusion functor* $\textsf{\textbf{XLie}} \hookrightarrow \textsf{\textbf{XLb}}$.

Proof. Let $(\mathfrak{m}, \mathfrak{p}, \eta)$ be a Leibniz crossed module, $(\mathbb{m}, \mathbb{p}, \nu)$ a Lie crossed module and $(\varphi, \psi)\colon (\mathfrak{m}, \mathfrak{p}, \eta) \to (\mathbb{m}, \mathbb{p}, \nu)$ a morphism of Leibniz crossed modules. Consider the quotient maps $\pi_{\mathfrak{m}}\colon \mathfrak{m} \to \overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m})$ and $\pi_{\mathfrak{p}}\colon \mathfrak{p} \to \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p})$, which form a morphism of Leibniz crossed modules. There is a unique morphism of Lie crossed modules $(\tilde{\varphi}, \tilde{\psi})\colon (\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m}), \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p}), \overline{\eta}) \to (\mathbb{m}, \mathbb{p}, \nu)$, such that $\tilde{\varphi}\pi_{\mathfrak{m}} = \varphi$ and $\tilde{\psi}\pi_{\mathfrak{p}} = \psi$, where $\tilde{\varphi}(\overline{m}) = \varphi(m)$ and $\tilde{\psi}(\overline{p}) = \psi(p)$, for all $\overline{m} \in \overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m})$, $\overline{p} \in \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p})$. Note that $\tilde{\varphi}$ is well defined due to the fact that φ is a morphism of Leibniz algebras together with φ preserving the action of $\mathfrak{p}$ on $\mathfrak{m}$ via ψ and the antisymmetry of the bracket in $\mathbb{m}$, while the correctness of the definition of $\tilde{\psi}$ follows from ψ being a morphism of Leibniz algebras and the antisymmetry of the bracket in $\mathbb{p}$.

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, "\eta"] \arrow[dd, "\varphi"'] \arrow[drr, dashed, "\pi_{\mathfrak{m}}"] & \mathfrak{p} \arrow[dd, "\psi"] \arrow[drr, "\pi_{\mathfrak{p}}"] & & \\
& & \overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m}) \arrow[r, dashed, "\overline{\eta}"] \arrow[dll, dashed, "\tilde{\varphi}"] & \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p}) \arrow[dll, "\tilde{\psi}"] \\
\mathbb{m} \arrow[r, "\nu"'] & \mathbb{p} & &
\end{tikzcd}$$

Hence $(\pi_{\mathfrak{m}}, \pi_{\mathfrak{p}}) \colon (\mathfrak{m}, \mathfrak{p}, \eta) \to (\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{m}), \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p}), \overline{\eta})$ is universal among the morphisms from $(\mathfrak{m}, \mathfrak{p}, \eta)$ to $(\mathbb{m}, \mathbb{p}, \nu)$, that is $\mathtt{XLie}_{\mathtt{Lb}} \dashv \subset$. □

The functor $\mathtt{XLie}_{\mathtt{Lb}} \colon \mathbf{XLb} \to \mathbf{XLie}$ is a natural generalization of $\mathtt{Lie}_{\mathtt{Lb}} \colon \mathbf{Lb} \to \mathbf{Lie}$ in the sense that the following inner and outer diagrams are commutative or commute up to isomorphism for $i = 0, 1$.

$$\begin{tikzcd}
\mathbf{Lie} \arrow[r, shift left, "\subset"'] \arrow[d, shift left, "\mathtt{I}_i"] \arrow[d, phantom, "\dashv"] \arrow[r, phantom, "\perp"] & \mathbf{Lb} \arrow[l, shift left=2, "\mathtt{Lie}_{\mathtt{Lb}}"'] \arrow[d, shift right, "\mathtt{J}_i"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{XLie} \arrow[u, shift left, "\Phi_i"] \arrow[r, shift left, "\subset"] \arrow[r, phantom, "\top"] & \mathbf{XLb} \arrow[u, shift right, "\Psi_i"'] \arrow[l, shift left=2, "\mathtt{XLie}_{\mathtt{Lb}}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Lie} \arrow[r, shift left, "\subset"'] \arrow[d, shift right, "\mathtt{I}_i"'] \arrow[d, phantom, "\dashv"] \arrow[r, phantom, "\perp"] & \mathbf{Lb} \arrow[l, shift left=2, "\mathtt{Lie}_{\mathtt{Lb}}"'] \arrow[d, shift left, "\mathtt{J}_i"] \arrow[d, phantom, "\vdash"] \\
\mathbf{XLie} \arrow[u, shift right, "\Phi_{i+1}"'] \arrow[r, shift left, "\subset"] \arrow[r, phantom, "\top"] & \mathbf{XLb} \arrow[u, shift left, "\Psi_{i+1}"] \arrow[l, shift left=2, "\mathtt{XLie}_{\mathtt{Lb}}"]
\end{tikzcd}
\tag{3.4.3}$$

Note that $\mathtt{I}_1 \circ \mathtt{Lie}_{\mathtt{Lb}} = \mathtt{XLie}_{\mathtt{Lb}} \circ \mathtt{J}_1$ holds for any given Leibniz algebra $\mathfrak{p}$ because $\overline{\mathtt{Lie}_{\mathtt{Lb}}}(\mathfrak{p}) = \mathtt{Lie}_{\mathtt{Lb}}(\mathfrak{p})$ for the crossed module $(\mathfrak{p}, \mathfrak{p}, \mathrm{id}_{\mathfrak{p}})$, since the action of $\mathfrak{p}$ on itself is given by the bracket in $\mathfrak{p}$.

3.4.3 Extended diagram for categories of crossed modules

We have the following diagram of adjunctions

$$\begin{tikzcd}
\mathbf{XAs} \arrow[r, shift right, "\mathtt{XLie}_{\mathtt{As}}"'] \arrow[d, shift left, "\subset"] \arrow[d, phantom, "\dashv"] \arrow[r, phantom, "\perp"] & \mathbf{XLie} \arrow[l, shift right=2, "\mathtt{XU}"'] \arrow[d, shift right, "\subset"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{XDias} \arrow[u, shift left, "\mathtt{XAs}"] \arrow[r, shift left, "\mathtt{XLb}"] \arrow[r, phantom, "\top"] & \mathbf{XLb} \arrow[u, shift right, "\mathtt{XLie}_{\mathtt{Lb}}"'] \arrow[l, shift left=2, "\mathtt{XU}_{\mathtt{d}}"]
\end{tikzcd}
\tag{3.4.4}$$

in which $\subset \circ\, \mathtt{XLie}_{\mathtt{As}} = \mathtt{XLb} \circ \subset$. Regarding the outer square, since $\mathtt{XU}$ is left adjoint to $\mathtt{XLie}_{\mathtt{As}}$ and $\mathtt{XLie}_{\mathtt{Lb}}$ is left adjoint to $\subset$, $\mathtt{XU} \circ \mathtt{XLie}_{\mathtt{Lb}}$ is left adjoint to $\subset \circ\, \mathtt{XLie}_{\mathtt{As}}$. Likewise, $\mathtt{XAs} \circ \mathtt{XU}_{\mathtt{d}}$ is left adjoint to $\mathtt{XLb} \circ \subset$. Therefore $\mathtt{XU} \circ \mathtt{XLie}_{\mathtt{Lb}}$ is naturally isomorphic to $\mathtt{XAs} \circ \mathtt{XU}_{\mathtt{d}}$.

Finally, by assembling the commutative squares (3.2.3), (3.3.3), (3.4.1), (3.4.2), (3.4.3) and (3.4.4), we get the following result.

Theorem 3.4.3. *In the following parallelepipeds of categories and functors*

$$\begin{tikzcd}
 & \mathsf{As} \arrow[rr, shift right, "\mathtt{Lie}_{\mathtt{As}}"'] \arrow[rr, phantom, "\perp"] \arrow[dl, shift left, "\supset"] \arrow[dd, dashed, shift left, "\mathtt{I}'_i"] \arrow[dd, phantom, "\dashv"] & & \mathsf{Lie} \arrow[ll, shift right, "\mathtt{U}"'] \arrow[dl, shift right, "\supset"'] \arrow[dd, shift right, "\mathtt{I}_i"'] \arrow[dd, phantom, "\vdash"] \\
\mathsf{Dias} \arrow[ur, shift left, "\mathtt{As}"] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left, "\mathtt{Lb}"] \arrow[rr, phantom, "\top"] \arrow[dd, shift left, "\mathtt{J}'_i"] \arrow[dd, phantom, "\dashv"] & & \mathsf{Lb} \arrow[ll, shift left, "\mathtt{U}_{\mathtt{d}}"] \arrow[ur, shift right, "\mathtt{Lie}_{\mathtt{Lb}}"'] \arrow[ur, phantom, "\top"] \arrow[dd, shift right, "\mathtt{J}_i"'] \arrow[dd, phantom, "\vdash"] & \\
 & \mathsf{XAs} \arrow[uu, dashed, shift left, "\Phi'_i"] \arrow[rr, dashed, shift right, "\mathtt{XLie}_{\mathtt{As}}"'] \arrow[rr, phantom, "\perp"] \arrow[dl, dashed, shift left, "\supset"] & & \mathsf{XLie} \arrow[uu, shift right, "\Phi_i"'] \arrow[ll, dashed, shift right, "\mathtt{XU}"'] \arrow[dl, shift right, "\supset"'] \\
\mathsf{XDias} \arrow[uu, shift left, "\Psi'_i"] \arrow[ur, dashed, shift left, "\mathtt{XAs}"] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left, "\mathtt{XLb}"] \arrow[rr, phantom, "\top"] & & \mathsf{XLb} \arrow[uu, shift right, "\Psi_i"'] \arrow[ll, shift left, "\mathtt{XU}_{\mathtt{d}}"] \arrow[ur, shift right, "\mathtt{XLie}_{\mathtt{Lb}}"'] \arrow[ur, phantom, "\top"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathsf{As} \arrow[rr, shift right, "\mathtt{Lie}_{\mathtt{As}}"'] \arrow[rr, phantom, "\perp"] \arrow[dl, shift left, "\supset"] \arrow[dd, dashed, shift right, "\mathtt{I}'_i"'] \arrow[dd, phantom, "\dashv"] & & \mathsf{Lie} \arrow[ll, shift right, "\mathtt{U}"'] \arrow[dl, shift right, "\supset"'] \arrow[dd, shift left, "\mathtt{I}_i"] \arrow[dd, phantom, "\vdash"] \\
\mathsf{Dias} \arrow[ur, shift left, "\mathtt{As}"] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left, "\mathtt{Lb}"] \arrow[rr, phantom, "\top"] \arrow[dd, shift right, "\mathtt{J}'_i"'] \arrow[dd, phantom, "\dashv"] & & \mathsf{Lb} \arrow[ll, shift left, "\mathtt{U}_{\mathtt{d}}"] \arrow[ur, shift right, "\mathtt{Lie}_{\mathtt{Lb}}"'] \arrow[ur, phantom, "\top"] \arrow[dd, shift left, "\mathtt{J}_i"] \arrow[dd, phantom, "\vdash"] & \\
 & \mathsf{XAs} \arrow[uu, dashed, shift right, "\Phi'_{i+1}"'] \arrow[rr, dashed, shift right, "\mathtt{XLie}_{\mathtt{As}}"'] \arrow[rr, phantom, "\perp"] \arrow[dl, dashed, shift left, "\supset"] & & \mathsf{XLie} \arrow[uu, shift left, "\Phi_{i+1}"] \arrow[ll, dashed, shift right, "\mathtt{XU}"'] \arrow[dl, shift right, "\supset"'] \\
\mathsf{XDias} \arrow[uu, shift right, "\Psi'_{i+1}"'] \arrow[ur, dashed, shift left, "\mathtt{XAs}"] \arrow[ur, phantom, "\perp"] \arrow[rr, shift left, "\mathtt{XLb}"] \arrow[rr, phantom, "\top"] & & \mathsf{XLb} \arrow[uu, shift left, "\Psi_{i+1}"] \arrow[ll, shift left, "\mathtt{XU}_{\mathtt{d}}"] \arrow[ur, shift right, "\mathtt{XLie}_{\mathtt{Lb}}"'] \arrow[ur, phantom, "\top"] &
\end{tikzcd}$$

all the inner and outer squares of adjoint functors are commutative or commute up to *isomorphism for* $i = 0, 1$*. Note that, for each face in the diagrams above, left adjoints form the outer square, while right adjoints form the inner square.*

Chapter 4

Conclusions and further research

Our initial goal was to achieve a generalization to crossed modules of the adjunction between the Liezation functor $\mathtt{Lie}_{\mathtt{As}}\colon \mathsf{As} \to \mathsf{Lie}$ and the universal enveloping algebra functor $\mathtt{U}\colon \mathsf{Lie} \to \mathsf{As}$, together with an isomorphism between the categories of modules over a Lie crossed module and its universal enveloping crossed module. These results can be found in Section 3.2 and [17].

The next logical step was to construct the corresponding 2-dimensional analogue to the adjunction between the functors $\mathtt{Lb}\colon \mathsf{Dias} \to \mathsf{Lb}$ and $\mathtt{U}_{\mathtt{d}}\colon \mathsf{Lb} \to \mathsf{Dias}$. Additionally, we assembled all the resulting commutative (or commutative up to isomorphism) squares of categories and functors into four parallelepipeds. These constructions can be found in Sections 3.3 and 3.4 as well as in [18].

As a consequence of the aforementioned results we noticed that the known generalization to crossed modules of the group algebra functor does not behave as naturally as one could expect, which is discussed in Subsection 3.1.2.

Some of our main results involve crossed modules of dialgebras, so it was mandatory to give a proper definition and study some of their basic properties (see Subsection 1.2.5). Furthermore, it seemed natural to give an equivalent description in terms of strict 2-dialgebras, which is presented in Section 1.3.

It is a well-known fact that the group of automorphisms and the Lie algebra of derivations are the actors in the categories of groups and Lie algebras, respectively. Additionally, it is also known that the Leibniz algebra of biderivations and the algebra of bimultipliers, under certain hypotheses, play the role of actor in the categories of Leibniz algebras and associative algebras, respectively. Bearing these facts in mind, we constructed the dialgebra of tetramultipliers, which is the actor of a given dialgebra in some particular cases (see Subsection 2.1.1).

The actor in the category of groups was extended to crossed modules by Norrie

[73], while the 2-dimensional analogue to the actor in the category of Lie algebras was described by Casas and Ladra [27]. On account of those generalizations we considered the possibility of extending the Leibniz algebra of biderivations to crossed modules, what led us to the construction of a Leibniz crossed module that works as the actor under certain conditions. Our approach and the subsequent results are presented in Subsection 2.2.3.

In many occasions, research leads to a few answers and a lot of new questions and our work is not an exception. In this way, several resulting problems remain open for further investigations. We would like to mention here a few of them. For instance, it would be interesting to give a proper definition of semistrict 2-dialgebras by replacing the identities from the axioms in the definition of their strict version by trilinear natural isomorphisms, along with their corresponding coherence diagrams.

Additionally, the steps in the construction of the (sometimes) actor of a Leibniz crossed module suggest two candidates for the 2-dimensional analogues of the algebra of bimultipliers and the dialgebra of tetramultipliers, which are, under certain conditions, the actors in the categories of algebras and dialgebras, respectively.

It is a well-known fact that the group algebra $\mathtt{K}(G)$ of a group G and the universal enveloping algebra $\mathtt{U}(\mathbb{p})$ of a Lie algebra $\mathbb{p}$ are Hopf algebras. Therefore, it might be natural to consider the possibility of $\mathtt{XK}(H, G, \partial)$ and $\mathtt{XU}(\mathbb{m}, \mathbb{p}, \nu)$ being crossed modules of Hopf algebras. Nevertheless, we should first find an appropriate definition of crossed module of Hopf algebras. One could consider the definition presented in [43], although Faria Martins explored another option in the recent article [42].

The universal enveloping dialgebra functor $\mathtt{U}_\mathtt{d}\colon \mathsf{Lb} \to \mathsf{Dias}$ is left adjoint to $\mathtt{Lb}\colon \mathsf{Dias} \to \mathsf{Lb}$ (see [65]). However, in contrast to the Lie case, this functor is not adequate to study representations. Loday and Pirashvili proved in [66] that, given a Leibniz algebra $\mathfrak{p}$, the category of right (respectively left) $\mathtt{UL}(\mathfrak{p})$-modules is equivalent to the category of $\mathfrak{p}$-representations (respectively $\mathfrak{p}$-co-representations), where $\mathtt{UL}(\mathfrak{p})$ is the universal enveloping algebra of $\mathfrak{p}$. It might be interesting to try to extend this equivalence to crossed modules.

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