TESIS DOCTORAL:

Existence and Multiplicity of Solutions of Functional Differential Equations

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A mi familia,
a mis padres Cristina y Fernando
y a mi hermano Jacobo,
por estar siempre ahí

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Contents

Notation

List of symbols most used throughout the work.

$\mathbb{N}$	Set of natural numbers, that is, $\{1, 2, \dots\}$.
$\mathbb{Z}$	Set of integer numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}^+$	Set of positive real numbers.
$\mathbb{C}$	Set of complex numbers.
$\mathrm{BV}(I)$	Functions of bounded variation defined on the interval I, that is, $\{f : I \to \mathbb{R} \mid V(f) < +\infty\}$ where $V(f) = \sup\limits_{P\in\mathcal{P}_I} \sum_{i=0}^{n_P-1} \lvert f(x_{i+1}) - f(x_i)\rvert$, $P = \{x_0, \dots, x_{n_P}\}$ and $\mathcal{P}_I$ is the set of partitions of I.
$\mathcal{C}(I)$	Space of continuous real functions defined on I.
$\mathcal{C}_c(I)$	Space of compactly supported continuous real functions defined on I.
$\mathcal{C}^k(I),\ k \in \mathbb{N}$	Space of k-times differentiable real functions defined on I such that the j-th derivative is continuous for $j = 0, \dots, k$.
$\mathcal{C}^\infty(I)$	Space of infinitely differentiable real functions defined on I.
D	Derivative operator, in any broad sense.
f_e	Even part of a real function f.
f_o	Odd part of a real function f.
H	Hilbert Transform, that is, $\mathrm{H}f(t) := \frac{1}{\pi} \lim\limits_{\epsilon\to+\infty} \int_{-\epsilon}^{\epsilon} \frac{f(s)}{t-s} \, \mathrm{d}s$.
$\Im$	Imaginary part.
Id	Identity function or operator.
$\mathcal{K}$	Set of compact subsets of $\mathbb{R}$.
$\mathrm{L}^\mathrm{p}(I),\ p \geq 1$	Riesz-Lebesgue p space on the set I, that is, $\mathrm{L}^\mathrm{p}(I) = \left\{u : I \to \mathbb{R} \mid u \text{ Lebesgue measurable },\ \int_I \lvert u\rvert^p < +\infty\right\}$.
$\mathrm{L}^\infty(I)$	The space of essentially bounded functions.
$AC(I)$	Absolutely continuous functions, that is, $AC(I) = \left\{u \in \mathcal{C}(I) \mid \exists f \in \mathrm{L}^1(I),\ u(t) = u(a) + \int_a^t f(s) \, \mathrm{d}s,\ t, a \in I\right\}$.

$R[X]$	Ring of polynomials with coefficients in R and variable X.
$\mathfrak{R}$	Real part.
$\mathrm{S}_{\mathrm{loc}}(\mathrm{I})$	Local version of a function space. If S is a function space which can be defined on any compact set and $I \subset \mathbb{R}$, $\mathrm{S}_{\mathrm{loc}}(\mathrm{I}) := \{u : I \to \mathbb{R}$ such that $u\|_K \in S(K), K \in \mathcal{K}\}$.
S'	The dual of a topological vector space S.
$W^{k,p}(I),\ k,p \in \mathbb{N}$	Sobolev space $k-p$ on the set I, that is, $\left\{u \in L^p(I) \mid u^{(\alpha)} \in L^p(I),\ \alpha = 1, \dots, k\right\}$.
χ_A	Characteristic function on the set $A \subset \mathbb{R}$, that is, $\chi_A(t) = 1, t \in A$, $\chi_A(t) = 0, t \in \mathbb{R} \backslash A$.

Preface

The present Thesis contains most of the work undertaken by the author in the last years. It is indeed a research adventure in the field of solutions of differential equations, therefrom the title *«Existence and multiplicity of solutions of functional differential equations»*. But, how to tackle the study of such broad area? In what solutions are to differential equations, we can take a rather simple approach: there are but two possibilities, either there exist or there exist not, and, in the first case, there can be one or many.

Whether we want to prove if there is one –uniqueness of solution– or many –multiplicity of solution– determines the method to be used. Existence has been traditionally derived in two ways: either through the direct construction of the solution or through topological methods, the later, in most cases, involving global contractions like the Banach contraction theorem. In the first part of the report we will deal with uniqueness in the first of the ways using what is known as the Green's function. Ever since the work of George Green on the subject, it has been clear that one of the most fruitful ways of constructing solutions of different kinds of problems is through the so called Green's function, that is, the obtaining of a solution to a problem of the kind $Lu = h$, $u \in H$, where H is a space of functions, L is a linear operator on H and $h \in L(H)$ by expressing it, if possible, in the form

$$u(t) = \int G(t,s)h(s)\,\mathrm{d}s,$$

with some appropriate boundaries for the integral. It is then understood that this expression provides the so-called maximum and anti-maximum principles, which in lay words convey the simple idea that, if G is positive and h is positive then u is positive (anti-maximum principle) and if G is negative and h is positive then u is negative (maximum principle).

This is just one of the many remarkable properties of Green's functions, but as it usually happens with useful structures, they are hard to obtain. In the case of functional equations this is no exception and throughout the first seven chapters of this Thesis we will explore the construction of these functions and their various applications. We will center our attention in the case of equations with involutions, a particular field of functional differential equations where we can *reduce* –in a specific sense we will detail later– the problem studied to a problem with ordinary differential equations. We will even write a computer program that will allow the automatic calculation of Green's functions in the case of constant coefficients and two-point boundary conditions.

The strength of the Green's functions method relies on them being the kernel of the inverse operator that gives us the unique solution for our problem but, of course, this is not the path to take when we are expecting several solutions. In the second part of this work we explore a particular kind of topological methods which will allow us to prove the multiplicity of solutions and further localize those solutions within a carefully defined cone. The problems to which we will apply this scheme will contain a nonlinearity, that is, a nontrivial, functional, relation between the derivatives of the solution and the solution itself. The key point of this technique relies on a refining of the classical Guo-Krasnosel'skiĭ theorem of cone contraction-expansion. The nonlinearity, which takes real values, will oscillate in some manner, going above and below certain values depending on the variables and these ripples will cause, precisely, the existence of many solutions. This situation is similar to what happens to a bucket of water when we shake it. If we mark a line a little bit above the water level and rock the bucket, ripples start to appear and, when they get high enough, they reach the line we have marked. The more ripples there are, the more times that level is reached.

Simple as it may sound, the conditions that have to be satisfied in order to apply this method can, as we will see, get really convoluted with the increasing generality of the problems studied.

All these discoveries appear in several publications the author has written during the preparation of the Thesis. The reader may consult [34, 35, 39–44, 96, 165, 166].

Part I

Green's functions

Involutions have been an interesting subject of research at least since Rothe first computed the number of different involutions on finite sets in 1800 [152]. After that, Babbage published in 1815 [7] the foundational paper in which functional equations are first considered, in particular those of the form $f(f(t)) = t$ which are called involutions†.

Despite of the progresses on the theory of functional equations, we have to wait for Silberstein who, in 1940 [156], solved the first functional differential equation with an involution. The interest on differential equations with involutions is retaken by Wiener in 1969 [186]. Wiener, together with Watkins, will lead the discoveries in this direction in the following decades [1, 155, 173, 174, 186–189]. Quite a lot of work has been done ever since by several authors. We will make a brief review on this in Chapter 2. In 2013 the first Green's function for a differential equation with an involution was computed [39] and the field rapidly expanded [40, 41, 43, 44].

This first part goes through those discoveries related to Green's functions. In order to do that, first we recall some general results concerning involutions which will help us understand their remarkable analytic and algebraic properties. Chapter 1 will deal about this subject while Chapter 2 will give a brief overview on differential equations with involutions to set the reader in the appropriate research framework. We recommend the reader to go through the monograph [187] which has a whole chapter on the subject and, although it was written more than twenty years ago, it contains most of what is worth knowing on the matter.

In Chapter 3 we start working on the theory of Green's functions for functional differential equations with involutions in the most simple cases: order one problems with constant coefficients and reflection. Here we solve the problem with different boundary conditions, studying the specific characteristics which appear when considering periodic, anti-periodic, initial or arbitrary linear boundary conditions. We also apply some very well known techniques (lower and upper solutions method or Krasnosel'skiĭ's Fixed Point Theorem, for instance) in order to further derive results.

Computing explicitly the Green's function for a problem with nonconstant coefficients is not simple, not even in the case of ordinary differential equations. We face these obstacles in Chapter 4, where we reduce a new, more general problem containing nonconstant coefficients and arbitrary differentiable involutions, to the one studied in Chapter 3. In order to do this we use a double trick. First, we reduce the case of a general involution to the case of the reflection using some of the knowledge gathered in Chapter 1 and then we use a special change of variable (only valid in some cases) that allows the obtaining of the Green's function of problems with nonconstant coefficients from the Green's functions of constant-coefficient analogs.

To end this part of the work, we have Chapter 5, in which we deepen in the algebraic nature of reflections and extrapolate these properties to other algebras. In this way, we do not only generalize the results of Chapter 3 to the case of n-th order problems and general two-point boundary conditions, but also solve functional differential problems in which the Hilbert transform or other adequate operators are involved.

The last chapters of this part are about applying the results we have proved so far to some related problems. First, in Chapter 6, setting again the spotlight on some interesting relation be-

†Babbage, in the preface to his work [7], described very well the importance of involutions: *«Many of the calculations with which we are familiar, consist of two parts, a direct, and an inverse; thus, when we consider an exponent of a quantity: to raise any number to a given power, is the direct operation: to extract a given root of any number, is the inverse method […] In all these cases the inverse method is by far de most difficult, and it might perhaps be added, the most useful».*

tween an equation with reflection and an equation with a φ-Laplacian, we obtain some results concerning the periodicity of solutions of that first problem with reflection. Chapter 7 moves to a more practical setting. It is of the greatest interest to have adequate computer programs in order to derive the Green's functions obtained in Chapter 5 for, in general, the computations involved are very convoluted. Being so, we present in this chapter such an algorithm, implemented in *Mathematica*. We also add some considerations which could lead to simplifying the computations and therefore the time necessary to run the program. The reader can find in the appendix the exact code of the program.

1. Involutions and differential equations

1.1 The straight line problem

Before moving to the study of involutions, we will motivate it with a simple problem derived from some considerations on the straight line.

Let us assume that $x(t) = a\,t + b$, where $a, b \in \mathbb{R}$, is a straight line on the real plane. Then, using the formula of the slope between two points $(-t, x(-t))$ and $(t, x(t))$ we have that

$$x'(t) = \frac{x(t) - x(-t)}{2t}. \tag{1.1.1}$$

Every straight line satisfies this equation. Nevertheless, observe that we are not asking for the slope to be constant and therefore we may ask the following questions in a natural way: Are all of the solutions of equation (1.1.1) straight lines? (see here the spirit of Babbage's words concerning inverse problems), How can we solve differential equations of this kind? How can we guarantee the existence of solution?, How do the solutions of the equation depend on the fact that x' varies depending on both t as well as of the image of t by a symmetry (in this case the reflection), or, more generally of an involution?

In order to answer the first question, we will study the even and odd functions – each one with a different symmetry property– and how does the derivative operator act on them. We do this study in its most basic form, on groups, and then apply it to the real case (a group with the sum).

Definition 1.1.1. Let G and H be groups, $A \subset G$ and define $A^{-1} := \{x^{-1} \mid x \in G\}$. Assume that $A^{-1} \subset A$. We will say that $f : A \to H$ is a *symmetric* or *even function* if $f(x^{-1}) = f(x) \quad \forall x \in A$. We will say that f is an *antisymmetric* or *odd function* if $f(x^{-1}) = f(x)^{-1} \quad \forall x \in A$.

Remark 1.1.2. If f is a homomorphism, f is odd. That is because, first if f is an homeomorphism, A is a subgroup of G and $f(A)$ a subgroup of H. Now if e represents the identity element of G, e' that of H, and $x \in A$, $e' = f(e) = f(xx^{-1}) = f(x)f(x^{-1})$, so $f(x^{-1}) = f(x)^{-1}$. On the other hand, if f is an even homomorphism, all of the elements of $f(A)$ satisfy $y^2 = e'$ for every $y \in f(A)$. For this reason, the only even and odd function with real values, that is, with values in the abelian group $(\mathbb{R}, +)$, is the 0 function.

Remark 1.1.3. The set of even – respectively odd– functions of a subset $A \subset G$ to a commutative group H is a group with the point-wise operation induced by the operation of H, that is, $(fg)(x) := f(x)g(x)$ for every $x \in A, f, g : A \to H$ both even or odd.

Proposition 1.1.4. *Let G be a group, $A \subset G$ such that $A^{-1} \subset A$, V is a vector space on a field $\mathbb{F}$ of characteristic not equal to $2^{\dagger}$. Then there exist two maps $f_e : A \to (V, +)$ and $f_o : A \to (V, +)$, even and odd respectively, such that $f = f_e + f_o$. Furthermore, this decomposition is unique.*

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†This condition is taken in order to be allowed to divide by 2 in the vector space V.

Proof. It is enough to define

$$f_e(x) := \frac{f(x)+f(x^{-1})}{2}, \quad f_o(x) := \frac{f(x)-f(x^{-1})}{2}.$$

It is clear that f_e and f_o are even and odd respectively and that $f = f_e + f_o$.

Assume know that there exist two such decompositions: $f = f_e + f_o = \tilde{f}_e + \tilde{f}_o$. Then, $f_e - \tilde{f}_e = \tilde{f}_o - f_o$, but $f_e - \tilde{f}_e$ is even and $\tilde{f}_o - f_o$ odd, hence $f_e - \tilde{f}_e = \tilde{f}_o - f_o = 0$ and the decomposition is unique. ■

From now on, given a function $f : A \to V$, f_e will stand for its even part and f_o for its odd part.

Corollary 1.1.5. *In the conditions of Proposition 1.1.4, the vector space $\mathcal{F}(G,V) := \{f : G \to V\}$ can be decomposed in the direct sum of vector spaces $\mathcal{F}_e(G,V) := \{f : G \to V \mid f \text{ even}\}$ and $\mathcal{F}_o(G,V) := \{f : G \to V \mid f \text{ odd}\}$, that is, $\mathcal{F}(G,V) = \mathcal{F}_e(G,V) \oplus \mathcal{F}_o(G,V)$.*

For rest of the section, let $A \subset \mathbb{R}$ be such that $-A \subset A$. Given the expression of f_e and f_o in the decomposition we can claim that $\mathcal{D}(A,\mathbb{R}) = \mathcal{D}_e(A,\mathbb{R}) \oplus \mathcal{D}_o(A,\mathbb{R})$ where $\mathcal{D}(A,\mathbb{R})$ are the differentiable functions from A to $\mathbb{R}$ and $\mathcal{D}_e(A,\mathbb{R})$ and $\mathcal{D}_o(A,\mathbb{R})$ the sets of those functions which are, respectively, even differentiable and odd differentiable functions.

The following Proposition is an elemental result in differential calculus.

Proposition 1.1.6. *The derivative operator acts in the following way:*

$$\mathcal{D}_e(A,\mathbb{R}) \oplus \mathcal{D}_o(A,\mathbb{R}) \xrightarrow{D} \mathcal{D}_e(A,\mathbb{R}) \oplus \mathcal{D}_o(A,\mathbb{R})$$

$$(g,h) \longmapsto \left(\begin{smallmatrix} 0 & D \\ D & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} g \\ h \end{smallmatrix}\right) = (h', g')$$

Corollary 1.1.7. *For every $f \in \mathcal{D}(A,\mathbb{R})$ we have that*

(1) $(f')_e = f' \iff f = f_o + c,\ c \in \mathbb{R}$,

(2) $(f')_o = f' \iff f = f_e$.

Now we can solve the *"straight line problem"* as follows: equation (1.1.1) can be written as

$$x'(t) = \frac{x(t) - x(-t)}{2t} = \frac{x_o(t)}{t},$$

and since $\frac{x_o(t)}{t}$ is symmetric, taking into account Proposition 1.1.6, we arrive at the equivalent system of differential equations

$$(x_e)'(t) = 0,$$
$$(x_o)'(t) = \frac{x_o(t)}{t}.$$

Hence, $x_e(t) = c$, $x_o(t) = k\,t$ with $c, k \in \mathbb{R}$, that is, x is the straight line $x(t) = k\,t + c$, which answers the first question we posed.

Further on we will use this decomposition method in order to obtain solutions of more complex differential equations with reflection.

Involutions, as we will see, have very special properties. This is due to their double nature, analytic and algebraic. This chapter is therefore divided in two sections that will explore the two kinds of properties, arriving at last to some parallelism between involutions and complex numbers for its capability to decompose certain polynomials (see Remark 1.3.6). In this chapter we recall results from [39, 41, 46, 132, 187, 189, 196].

1.2 Involutions and their properties

1.2.1 The concept of involution

The concept of involution is fundamental for the theory of groups and algebras, but, at the same time, being an object in mathematical analysis, their analytical properties allow the obtaining of further information concerning this object. In order to be clear in this respect, let us define what we understand by involution in this analytical context. We follow the definitions of [187, 189].

Definition 1.2.1. Let $A \subset \mathbb{R}$ be a set containing more that one point and $f : A \to A$ a function such that f is not the *identity* Id. Then f is an *involution* if

$$f^2 \equiv f \circ f = \mathrm{Id}$$

or, equivalently, if

$$f = f^{-1}.$$

If $A = \mathbb{R}$, we say that f is a *strong involution* [187]. Involutions are also known as *Carleman functions* in the literature [46, 148].

Example 1.2.2. The following involutions are the most common examples:

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x$ is an involution known as *reflection*.

(2) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$, $f(x) = \frac{1}{x}$ known as *inversion*.

(3) Let $a, b, c \in \mathbb{R}$, $c\,b + a^2 \neq 0$, $c \neq 0$,

$$f : \mathbb{R}\left\backslash\left\{\frac{a}{c}\right\}\right. \to \mathbb{R}\left\backslash\left\{\frac{a}{c}\right\}\right., \; f(x) = \frac{ax+b}{cx-a}$$

is a family of functions known as *bilinear involutions*. If $a^2 + b\,c > 0$, the involution is said *hyperbolic* and has two fixed points in its domain.

The definition of involution can be extended in a natural way to arbitrary sets (not necessarily of real numbers) or, in the following way, to order n involutions.

Definition 1.2.3. Let $A \subset \mathbb{R}$, $f : A \to A$, $n \in \mathbb{N}$, $n \geq 2$. We say that f is an *order n involution* if

(1) $f^n \equiv f \circ \overset{n}{\smile}\!\!\cdots \circ f = \mathrm{Id}$,

(2) $f^k \neq \mathrm{Id} \quad \forall k = 1, \dots, n-1.$

Example 1.2.4. The following is an example of an involution defined on a set which is not a subset of $\mathbb{R}$:

$f : \mathbb{C} \to \mathbb{C}, f(z) = e^{\frac{2\pi}{n} i}$ is an order n involution on the complex plane.

Example 1.2.5.

$$f(x) = \begin{cases} x, & x \in (-\infty, 0) \cup (n, +\infty), \\ x+1, & x \in (0,1) \cup (1,2) \cup \cdots \cup (n-2, n-1), \\ x-(n-1), & x \in (n-1, n) \end{cases}$$

is an order n involution in $\mathbb{R}\backslash\{0, 1, \dots, n\}$.

Observe that f is not defined on a connected set of $\mathbb{R}$, neither admits a continuous extension to a connected set. This fact is related to the statement of Theorem 1.2.9.

1.2.2 Properties of involutions

Now we will establish a series of results useful when it comes to study involutions.

Proposition 1.2.6. *Let $A \subset \mathbb{R}$, $f : A \to A$ be an order n involution, then f is invertible.*

Proof. If $h \circ g$ is bijective, then h is surjective, since $(h \circ g)(A) \subset h(A)$, and g injective, since $g(x) = g(y)$ implies $(h \circ g)(x) = (h \circ g)(y)$. Hence, since $f \circ f^{n-1} = f^{n-1} \circ f = \mathrm{Id}$, f is bijective (invertible). ■

The following proposition [189] is a classical result regarding involutions. Here we present it for connected subsets of $\mathbb{R}$.

Proposition 1.2.7. *Let $A \subset \mathbb{R}$ be connected and $f : A \to A$ an order two continuous involution. Then,*

(1) f is strictly decreasing, and

(2) f has a unique fixed point.

Proof. (1). Since f is invertible, it is strictly monotone. $f \neq \mathrm{Id}$, so there exists $x_0 \in A$ such that $f(x_0) \neq x_0$. Let us assume that f is increasing. If $x_0 < f(x_0)$, since A is connected, $f(x_0) < f^2(x_0) = x_0$ (contradiction) and the same occurs if $f(x_0) < x_0$. Thus, f is decreasing.

(2). Since A is connected, A is an interval. Let $a \in A$, then $f(a) \in A$. Let us assume, without lost of generality, that $f(a) > a$. Then, $[a, f(a)] \subset A$ and $f([a, f(a)]) = [a, f(a)]$. Let $g = f - \mathrm{Id}$, g is continuous and $g(a) = f(a) - a > 0$, $g(f(a)) = a - f(a) < 0$, therefore, by Bolzano's Theorem, there exists $\alpha \in (a, f(a))$ such that $g(\alpha) = 0$, i.e. $f(\alpha) = \alpha$.

Since f is strictly decreasing, such point is unique. ■

Remark 1.2.8. If A is not connected, point (2) of Proposition 1.2.7 may not be satisfied. For instance, bilinear involutions have 0 or 2 fixed points.

Now we will prove a theorem that illustrates the importance of order two involutions. Similar proofs can be found in [39, 132, 148].

Theorem 1.2.9. *The only continuous involutions defined in connected sets of $\mathbb{R}$ are of order 2.*

Proof. Let A be a connected subset of $\mathbb{R}$ and $f : A \to A$ a continuous involution of order n. Let us prove in several steps that $n = 2$.

(a) *n is even*. We will prove first that f is decreasing. Since $f \neq \mathrm{Id}$, there exists $x_0 \in A$ such that $f(x_0) \neq x_0$. Let us assume that f is increasing. If $x_0 < f(x_0)$, using that A is connected,

$$x_0 < f(x_0) < f^2(x_0) < \cdots < f^{n-1}(x_0) < f^n(x_0) = x_0,$$

which is a contradiction. The same happens if $f(x_0) < x_0$. Therefore f is decreasing.

The composition of two functions, both increasing or decreasing is increasing. If one is increasing and the other decreasing, then the composition is a decreasing function. Therefore, if n is odd and f is decreasing, f^n is decreasing, which is absurd since $f^n = \mathrm{Id}$.

(b) *$n = 2m$ with m odd*. Otherwise, $n = 4k$ for some $k \in \mathbb{N}$. Then, if $g = f^{2k}$, $g \neq \mathrm{Id}$ and $g^2 = \mathrm{Id}$ and, using Proposition 1.2.7, g is decreasing, but this is a contradiction since $2k$ is even.

(c) $n = 2$. If $n = 2m$ with m odd, $m \geq 3$, take $g = f^2$. Then $g \neq \mathrm{Id}$ and $g^m = \mathrm{Id}$, so g is an involution of order $k \leq m$. But, by part (a), this implies that g is decreasing, which is impossible since $g = f^2$. ■

From now on, if we do not specify the order of the involution, we will assume it is of order two.

The proof of Proposition 1.2.7 suggests a way of constructing an iterative method convergent to the fixed point of the involution. This is illustrated in the following theorem.

Theorem 1.2.10. *Let $A \subset \mathbb{R}$ be a connected set, $f : A \to A$ a continuous involution, α is the unique fixed point of f and f of class two in a neighborhood of α. Then, the iterative method*

$$\begin{cases} x_0 \in A, \\ x_{k+1} = g(x_k), \quad k = 0, 1, \dots, \end{cases}$$

where $g := \frac{f+\mathrm{Id}}{2}$, is globally convergent to α and of order at least 2.

Proof. Let us consider the closed interval of extremal points x_k and $f(x_k)$ that we will denote in this proof by $[x_k, f(x_k)]$. Since x_{k+1} is the middle point of the interval $[x_k, f(x_k)]$, $x_{k+1} \in [x_k, f(x_k)]$ and, furthermore, since $f([x_k, f(x_k)]) = [x_k, f(x_k)]$, we have that $f(x_{k+1}) \in [x_k, f(x_k)]$. Therefore,

$$|f(x_{k+1}) - x_{k+1}| \leq \frac{1}{2}|f(x_k) - x_k| \leq \cdots \leq \frac{1}{2^{k+1}}|f(x_0) - x_0|.$$

Hence,

$$|x_{k+1} - x_k| = \left|\frac{f(x_k) + x_k}{2} - x_k\right| = \frac{1}{2}|f(x_k) - x_k| \leq \frac{1}{2^k}|f(x_0) - x_0|.$$

Thus,

$$
\begin{aligned}
|x_{k+m}-x_k| &\le |x_{k+m}-x_{k+m-1}|+\cdots+|x_{k+1}-x_k| \le \sum_{j=0}^{m-1}\frac{1}{2^{k+j}}|f(x_0)-x_0| \\
&= \frac{1}{2^{k-1}}\left(1-\frac{1}{2^m}\right)|f(x_0)-x_0| \le \frac{1}{2^{k-1}}|f(x_0)-x_0|.
\end{aligned}
$$

As a consequence, $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $[x_0,f(x_0)]$ and therefore convergent. This proves that the method is globally convergent in A.

On the other hand, $f(f(x))=x$ for every $x\in A$, hence $f'(f(x))f'(x)=1$ so

$$1=f'(f(\alpha))f'(\alpha)=(f'(\alpha))^2.$$

Since f is decreasing by Proposition 1.2.7, $f'(\alpha)=-1$. Therefore, $g'(\alpha)=0$ and thus, taking the order 2 Taylor polynomial of g at α, we have that

$$g(x)=\alpha+g'(\alpha)(x-\alpha)+\frac{g''(\xi_x)}{2}(x-\alpha)^2=\alpha+\frac{g''(\xi_x)}{2}(x-\alpha)^2$$

where ξ_x is a point of the interval $[\alpha,x]$.

Hence, if c is an upper bound of g'' in a neighborhood of α,

$$|x_{k+1}-\alpha|=|g(x_k)-\alpha|=\left|\alpha+\frac{g''(\xi_{x_{k+1}})}{2}(x_k-\alpha)^2-\alpha\right|\le\frac{c}{2}|x_k-\alpha|^2,$$

for k sufficiently big, which proves the method is of order at least 2. ■

1.2.3 Characterization of involutions

Involutions can be characterized in a variety of ways. This kind of properties are helpful when it comes to prove some results.

Proposition 1.2.11 ([148, 189]). *Every continuous involution $\varphi:\mathbb{R}\to\mathbb{R}$ with a fixed point $p\in\mathbb{R}$ is of the form $\varphi(t)=\varphi_0(t-p)+p$ where*

$$\varphi_0(t)=\begin{cases} g(t), & t\ge 0,\\ g^{-1}(t), & t<0,\end{cases}$$

and $g:\mathbb{R}\to\mathbb{R}$ is a continuous strictly decreasing function such that $g(0)=0$.

Conversely, every function φ defined in such way is a continuous involution.

Proposition 1.2.12 ([196, Theorem 2.1]). *Let I,J be intervals of $\mathbb{R}$, I symmetric. Every continuous involution $\varphi:J\to J$ is of the form $\varphi(t)=t-h(t)$ where $h=g^{-1}$, $g:I\to J$, $g(t)=(f(t)+t)/2$ and $f:I\to\mathbb{R}$ is a symmetric even function such that $f(0)=0$.*

Conversely, every function φ defined in such way is a continuous involution.

Proposition 1.2.13 ([148, Corollary 1.2, p. 182]). *Let J be an open interval of $\mathbb{R}$. Every continuous involution $\varphi : J \to J$ is of the form $\varphi(t) = h^{-1}(-h(t))$, where $h = h_1 \circ h_2 \circ h_3$, $h_3 : J \to \mathbb{R}$ is a homeomorphism, $h_2(s) = s - a$ where a is the fixed point of the function $h_3 \circ \varphi \circ h$, and $h_1 : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $h(0) = 0$.*

Conversely, every function φ defined in such way is a continuous involution.

Finally, the following Lemma, similar to the previous result, is specially useful when dealing with differential equations (cf. [41, Corollary 2.2]).

Lemma 1.2.14 (Correspondence of Involutions, [41, Lemma 2.1]). *Let φ and ψ be two differentiable involutions*[†] *on the compact intervals J_1 and J_2 respectively. Let t_0 and s_0 be the unique fixed points of φ and ψ respectively. Then, there exists an increasing diffeomorphism $f : J_2 \to J_1$ such that $\psi = f^{-1} \circ \varphi \circ f$.*

Conversely, every function φ defined in such way is a differentiable involution.

Proof. Let $g : [\inf J_2, s_0] \to [\inf J_1, t_0]$ be an increasing diffeomorphism, that is, $g(s_0) = t_0$. Let us define

$$f(s) := \begin{cases} g(s) & \text{if } s \in [\inf J_2, s_0], \\ (\varphi \circ g \circ \psi)(s) & \text{if } s \in (s_0, \sup J_2]. \end{cases}$$

Clearly, $f(\psi(s)) = \varphi(f(s)) \ \ \forall s \in J_2$. Since s_0 is a fixed point for ψ, f is continuous. Furthermore, because φ and ψ are involutions, $\varphi'(t_0) = \psi'(s_0) = -1$, so f is differentiable on J_2. f is invertible with inverse

$$f^{-1}(t) := \begin{cases} g^{-1}(t) & \text{if } t \in [\inf J_1, t_0], \\ (\psi \circ g^{-1} \circ \varphi)(t) & \text{if } t \in (t_0, \sup J_1]. \end{cases}$$

f^{-1} is also differentiable for the same reasons. ∎

We can prove in the same way a continuous version of Lemma 1.2.14.

Corollary 1.2.15. *Let φ and ψ be two continuous involutions on the intervals J_1 and J_2 respectively. Let t_0 and s_0 be the unique fixed points of φ and ψ respectively. Then, there exists an orientation preserving homeomorphism $f : J_2 \to J_1$ such that $\psi = f^{-1} \circ \varphi \circ f$.*

Conversely, every function φ defined in such way is a continuous involution.

Remark 1.2.16. A similar argument could be done in the case of involutions defined on open, possibly not bounded, intervals.

Remark 1.2.17. It is easy to check that if φ is an involution defined on $\mathbb{R}$ with fixed point t_0 then $\psi(t) := \varphi(t + t_0 - s_0) - t_0 + s_0$ is an involution defined on $\mathbb{R}$ with fixed point s_0 (cf. [189, Property 2]). For this particular choice of φ and ψ, we can take $g(s) = s - s_0 + t_0$ in Lemma 1.2.14 and, in such a case, $f(s) = s - s_0 + t_0$ for all $s \in \mathbb{R}$.

Remark 1.2.18. Observe that, if φ and ψ are continuous involutions and $\psi = f^{-1} \circ \varphi \circ f$, then f sends the fixed point of ψ to the fixed point of φ.

[†] Every differentiable involution is a diffeomorphism.

The following Lemma establishes that the involution is defined if we know its behavior up to it's fixed point.

Lemma 1.2.19. *Let φ, ψ be two continuous involutions defined in a compact interval J with a common fixed point $p \in J$. If $\varphi|_{[\inf J,p]} = \psi|_{[\inf J,p]}$ or $\varphi|_{[p,\sup J]} = \psi|_{[p,\sup J]}$, then $\varphi = \psi$.*

Proof. Let $t \in [p,\sup J]$. $\psi(t) \in [\inf J,p]$, so $\varphi(\psi(t)) = \psi(\psi(t)) = t$. Hence, $\psi(t) = \varphi(t)$. The proof for the interval $[p,\sup J]$ is analogous. ∎

The previous results highlight a simple way of obtaining involutions from a given one, just considering the family of homeomorphisms acting on the set of involutions as follows.

Definition 1.2.20. Let

$$\begin{gathered}\mathcal{K} := \{[a,b] \subset \mathbb{R} \,|\, a,b \in \mathbb{R},\ a < b\},\\ \mathcal{H}^J := \{g : J \to J \,|\, g \text{ is a homeomorphism}\},\ \text{for a fixed } J \in \mathcal{K},\\ \mathrm{Inv_C}^J := \{\varphi : J \to J \mid \varphi \text{ is an involution}\},\ \text{for a fixed } J \in \mathcal{K}.\end{gathered}$$

For a fixed $J \in \mathcal{K}$, H^J is a group with the composition and acts transitively on $\mathrm{Inv_C}^J$:

$$\begin{aligned}\mathcal{H}^J \times \mathrm{Inv_C}^J &\longrightarrow \mathrm{Inv_C}^J\\ (f,\varphi) &\longmapsto f^{-1}\circ\varphi\circ f\end{aligned}$$

1.3 Differential Operators with Involutions

1.3.1 Algebraic Theory

Let $A \subset \mathbb{R}$ be a set without isolated points (just so the derivative can be considered in all of A). Let us consider some linear operators in the space of continuous functions $\mathcal{C}^\infty(A,\mathbb{R})$.

To start with, the *differential operator*

$$\begin{aligned}\mathcal{C}^\infty(A,\mathbb{R}) &\xrightarrow{D} \mathcal{C}^\infty(A,\mathbb{R})\\ f &\longmapsto f'\end{aligned}$$

which maps each function to its derivative. Defined in such a way, the linear operator D is surjective.

Let $\varphi \in \mathcal{C}^\infty(A,A)$. Then we can consider the *pullback operator* by φ

$$\begin{aligned}\mathcal{C}^\infty(A,\mathbb{R}) &\xrightarrow{\varphi^*} \mathcal{C}^\infty(A,\mathbb{R})\\ f &\longmapsto f\circ\varphi\end{aligned}$$

Let $a \in \mathcal{C}^\infty(A, \mathbb{R})$. We have also the pointwise *multiplication operator* by a,

$$\begin{array}{ccc} \mathcal{C}^\infty(A, \mathbb{R}) & \xrightarrow{a} & \mathcal{C}^\infty(A, \mathbb{R}) \\ f & \longmapsto & a \cdot f \end{array}$$

Also, if we have a constant $a \in A$ we will define a^* as the operator that acts on $\mathcal{C}^\infty(A, \mathbb{R})$ as $a^* f(t) = f(a)$ for all $t \in A$ (that is, a^* is the *Dirac delta function* at a.

These operators are well defined and present the associative property of the composition, but in general do not commute. To be precise, we have the following equalities:

$$Da = a' + aD, \tag{1.3.1}$$
$$\varphi^* a = \varphi^*(a)\varphi^*, \tag{1.3.2}$$
$$D\varphi^* = \varphi'\varphi^* D, \tag{1.3.3}$$

for each $a \in \mathcal{C}^\infty(A, \mathbb{R})$, $\varphi \in \mathcal{C}^\infty(A, A)$.

From these we derive the following:

$$DaD = a'D + aD^2, \tag{1.3.4}$$
$$Da\varphi^* = a'\varphi^* + a\varphi'\varphi^* D, \tag{1.3.5}$$
$$\varphi^* aD = \varphi^*(a)\varphi^* D, \tag{1.3.6}$$
$$\varphi^* a\varphi^* = \varphi^*(a)(\varphi^*)^2. \tag{1.3.7}$$

These equalities allow to express any composition of such operators (multiplication, pull-back and differential operators) as a composition in a predefined order. In other words, if we fix $\varphi \in \mathcal{C}^\infty(A, A)$ such that $\varphi^k \neq \mathrm{Id}\ \forall k \in \mathbb{N}$ and consider $\mathcal{A}_\varphi$ as the $\mathcal{C}^\infty(A, \mathbb{R})$-free module generated by $\{(\varphi^*)^i D^j\}_{i,j \geq 0}$ (the 0 power is the identity), this is a unitary associative $\mathbb{R}$-algebra with the composition.

Let us assume now that φ is an involution. In this case, the algebra $\mathcal{A}_\varphi$ is generated by $\{(\varphi^*)^i D^j\}_{\substack{i=0,1 \\ j\geq 0}}$.

1.3.2 Differential equations with involutions

We will describe now a method inspired in the annihilator method that will allow us to solve differential equations with involutions. It is in our interest to think of a way of transforming (somehow) expressions in $\mathcal{A}_\varphi$ to expressions in the ring of polynomials $\mathcal{C}^\infty(A)[D]$, since the equations of the form $Lx = 0$ are known for $L \in \mathcal{C}^\infty(A)[D]$ (i.e. L is an ordinary differential operator). In other words, Is there for every $L \in \mathcal{A}_\varphi$ an $R \in \mathcal{A}_\varphi$ such that $RL \in \mathcal{C}^\infty(A)[D]$? Furthermore, it would be convenient that such R is "minimal" in the sense we will detail latter. This is due to the fact that if the difference between the kernel of L and that of RL is minimal, we can obtain the solutions of $Lx = 0$ from those of $RLx = 0$. The ideal case would be the one in which both kernels coincide.

Definition 1.3.1. If $\mathbb{R}[D]$ is the ring of polynomials on the usual differential operator D and $\mathcal{A}$ is any operator algebra containing $\mathbb{R}[D]$, then an equation $Lx = 0$, where $L \in \mathcal{A}$, is said to be a *reducible differential equation* if there exits $R \in \mathcal{A}$ such that $RL \in \mathbb{R}[D]$. A similar definition could be done for nonconstant or complex coefficients.

Proposition 1.3.2. *Let $\varphi \in C^\infty(A, A)$ be an involution and $D + c\varphi^* + d \in \mathcal{A}_\varphi$, $c(t) \neq 0\ \forall t \in A$. Then, there exist $a, b, \alpha, \beta \in C^\infty(A, \mathbb{R})$ such that*

$$(D + a\varphi^* + b)(D + c\varphi^* + d) = D^2 + \alpha D + \beta \in C^\infty(A)[D],$$

and are defined by

$$\begin{cases} a = -c\varphi', \\ b = \varphi'\varphi^*(d) - \dfrac{c'}{c}, \\ \alpha = d + \varphi'\varphi^*(d) - \dfrac{c'}{c}, \\ \beta = d\left(\varphi'\varphi^*(d) - \dfrac{c'}{c}\right) + d' - c\varphi'\varphi^*(c). \end{cases}$$

Proof. Using the identities (1.3.1) – (1.3.7), we have that

$$\begin{aligned}(D + a\varphi^* + b)(D + c\varphi^* + d) = D^2 + (b+d)D + b\,d + d' + a\varphi^*(c) + (a + c\varphi')\varphi^* D \\ + (c' + b\,c + a\varphi^*(d))\varphi^*\end{aligned}$$

Therefore, we have to solve the linear system of four equations and four unknowns

$$\begin{cases} b + d = \alpha, \\ b\,d + d' + a\varphi^*(c) = \beta, \\ a + c\varphi' = 0, \\ c' + b\,c + a\varphi^*(d) = 0, \end{cases}$$

which has as unique solution

$$\begin{cases} a = -c\varphi', \\ b = \varphi'\varphi^*(d) - \dfrac{c'}{c}, \\ \alpha = d + \varphi'\varphi^*(d) - \dfrac{c'}{c}, \\ \beta = d\left(\varphi'\varphi^*(d) - \dfrac{c'}{c}\right) + d' - c\varphi'\varphi^*(c). \end{cases}$$

■

Remark 1.3.3. The previous Proposition can be modified for the case of considering all functions to be just differentiable.

Remark 1.3.4. The condition that φ is an involution is necessary for, otherwise, the term $a\varphi^*(c)(\varphi^*)^2$ would appear and the equation $c' + b\,c + a\varphi^*(d) = 0$ would split in two: $c' + b\,c = 0$ and $a\varphi^*(d) = 0$, forcing $a = 0$, which is incompatible with $a = -c\varphi'$.

Corollary 1.3.5. *Under the conditions of Proposition 1.3.2, if $d = 0$, we have that*

$$\left(D - \varphi' c\varphi^* - \frac{c'}{c}\right)(D + c\varphi^*) = D^2 - \frac{c'}{c}D - \varphi'\varphi^*(c)c.$$

Remark 1.3.6. In this corollary, if c is constant and φ is the reflection we have that

$$\left(D + c\varphi^*\right)(D + c\varphi^*) = D^2 + c^2.$$

Observe the parallelism between this expression and

$$(D + ic)(D - ic) = D^2 + c^2.$$

where i denotes the imaginary unity. We will deepen in this relation in Chapter 5.

Definition 1.3.7. Let $\varphi \in C^\infty(A)$, $L := \sum_{i,j=0}^{m,n} \alpha_{ij}(\varphi^*)^i D^j \in \mathcal{A}_\varphi$ such that $\alpha_{mk}, \alpha_{ln} \neq 0$ for some $k \in \{0, \dots, n\}$ and some $l \in \{0, \dots, m\}$. We call *degree* of L to $\partial L = (m, n)$.

Assume now that φ is an involution of order p. Let $R \in \mathcal{A}_\varphi$. We want to find $L \in \mathcal{A}_\varphi$ and $S \in C^\infty(A)[D]$ such that $LR = S$. Hence, if $\partial R = (m_1, n_1)$, $\partial L = (m_2, n_2)$ and $\partial(S) = (0, n_3)$, we have that $0 \le m_1, m_2 \le p - 1$ and $n_1 + n_2 = n_3$, which means that, in order to find the coefficients of L, we have to solve the linear system $LR = S$, which consists of $(1 + n_1 + n_2)\min\{p, m_1 + m_2 + 1\}$ equations with $(m_2 + 1)(n_2 + 1) + n_3$ unknowns. Assuming $m_1 = m_2 = p - 1$, we have $(1 + n_1 + n_2)p$ equations and $p(n_2 + 1) + n_1 + n_2$ unknowns. Thus, if we pretend to obtain a "minimal" operator as said before, we will try to make the number of equations equal to the number of unknowns, in such a way that the solution of the consequent linear system $LR = S$, if it exists, is unique, which only happens if and only if $n_2 = n_1(p - 1)$.

In the case where φ is an involution, $p = 2$ and hence our condition is $n_2 = n_1$. The case $n_1 = n_2 = 1$ is illustrated by Proposition 1.3.2. Needless to say, the complexity of the equations and its solving increases with the degree of R.

We will use now Proposition 1.3.2 in order to latter study an example.

Example 1.3.8. Sea $T \in \mathbb{R}^+$, $I = [\varphi(T), T] \subset \mathbb{R}$ where φ is a differentiable involution on I, $m, h \in C^1(I)$, $m(T) = m(\varphi(T))$ and $m(t) \neq 0 \ \ \forall t \in I$. Let us consider the operator $L = D + m\varphi^*$ and the boundary value problem

$$Lx(t) = h(t) \ \ \forall t \in I, \quad x(\varphi(T)) = x(T). \tag{1.3.8}$$

Observe that the boundary condition can be expressed, with our notation, as

$$(T^* - (\varphi(T))^*)x = 0,$$

and that $Lx(t) = x'(t) + m(t)x(\varphi(t))$. Following Proposition 1.3.2, if $R = D - m\varphi'\varphi^* - \frac{m'}{m}$, then we have that

$$RL = D^2 - \frac{m'}{m}D - \varphi'\varphi^*(m)m.$$

Remember that $\varphi(\varphi(T)) = T$. Therefore, it is satisfied that

$$x'(T) - x'(\varphi(T)) = (T^* - (\varphi(T))^*)Dx = (T^* - (\varphi(T))^*)(L - m\varphi^*)x$$

$$
\begin{aligned}
&= (T^* - (\varphi(T))^*)Lx - (T^* - (\varphi(T))^*)m\varphi^* x \\
&= h(T) - h(\varphi(T)) - m(T)x(\varphi(T)) + m(\varphi(T))x(\varphi(\varphi(T))) \\
&= h(T) - h(\varphi(T)) - m(T)x(\varphi(T)) + m(T)x(T) \\
&= h(T) - h(\varphi(T)).
\end{aligned}
$$

Hence, any solution of problem (1.3.8) is a solution of problem

$$
\begin{aligned}
RLx &= Rh, \\
x(\varphi(T)) &= x(T), \\
x'(T) - x'(\varphi(T)) &= h(T) - h(\varphi(T)).
\end{aligned}
$$

Rewriting this expression,

$$
\begin{aligned}
x''(t) - \frac{m'(t)}{m(t)}x'(t) - \varphi'(t)m(\varphi(t))m(t)x(t)& \\
=h'(t) - m(t)\varphi'(t)h(\varphi(t)) - \frac{m'(t)}{m(t)}h(t)&, \\
x(\varphi(T)) =x(T), \quad x'(T) - x'(\varphi(T)) = h(T) - h(\varphi(T))&,
\end{aligned}
$$

which is a system of ordinary differential equations with nonhomogeneous boundary conditions.

The reverse problem, determining whether the solution of this system is a solution of (1.3.8), is more difficult and it is not always the case. We will deepen in this fact further on and compute the Green's function in those cases there is a unique solution.

2. General results for differential equations with involutions

As mentioned in the Introduction, this chapter is devoted to those results related to differential equations with involution not directly associated with Green's functions. The proofs of the results can be found in the bibliography cited for each case. We will not deepen into these results, but we summarize their nature for the convenience of the reader. The reader may consult as well the book by Wiener [187] as a good starting point for general results in this direction.

It is interesting to observe the progression and different kinds of results collected in this Chapter with those related to Green's functions that we will show latter on.

2.1 The bases of the study

As was pointed out in the introduction, the study of differential equations with reflection starts with the solving of the Siberstein equation in 1940 [156].

Theorem 2.1.1. *The equation*

$$x'(t)=x\left(\frac{1}{t}\right),\quad t\in\mathbb{R}^+,$$

has exactly the following solutions:

$$x(t)=c\sqrt{t}\cos\left(\frac{\sqrt{3}}{2}\ln t-\frac{\pi}{6}\right),\quad c\in\mathbb{R}.$$

In Silberstein's article it was written $\frac{\pi}{3}$ instead of $\frac{\pi}{6}$, which appears corrected in [186, 187]. Wiener provides a more general result in this line.

Theorem 2.1.2 ([187]). *Let $n\in\mathbb{R}$. The equation*

$$t^n x'(t)=x\left(\frac{1}{t}\right),\quad t\in\mathbb{R}^+$$

has exactly the following solutions:

$$x(t)=\begin{cases} c\,t, & n=-1,\\ c\,t(1-2\ln t), & n=3,\\ c(t^{\lambda_1}+\lambda_1 t^{\lambda_2}), & n<-1 \text{ or } n>3,\\ c\,t^{\frac{1-n}{2}}\left[\cos\left(\alpha\ln t\right)+\sqrt{\dfrac{n+1}{3-n}}\sin\left(\alpha\ln t\right)\right], & n\in(-1,3),\end{cases}$$

where $c \in \mathbb{R}$, λ_1 and λ_2 are the roots of the polynomial $\lambda^2 + (n-1)\lambda + 1$ and

$$\alpha = \frac{\sqrt{(n+1)(3-n)}}{2}.$$

It is also Wiener [186, 187] who formalizes the concept of differential equation with involutions.

Definition 2.1.3 ([186]). An expression of the form

$$f(t, x(\varphi_1(t)), \dots, x(\varphi_k(t)), \dots, x^{n)}(\varphi_1(t)), \dots, x^{n)}(\varphi_k(t))) = 0, \; t \in \mathbb{R}$$

where $\varphi_1, \dots, \varphi_k$ are involutions and f is a real function of $nk + 1$ real variables is called *differential equation with involutions*.

The first objective in the research concerning this kind of equations was to find a way of reducing them to ordinary differential equations of systems of ordinary differential equations. In this sense, we have the following reduction results for the existence of solutions [186, 187].

Theorem 2.1.4. *Consider the equation*

$$x'(t) = f(t, x(t), x(\varphi(t))), \; t \in \mathbb{R} \tag{2.1.1}$$

and assume the following hypotheses are satisfied:

- *The function φ is a continuously differentiable strong involution with fixed point t_0.*
- *The function $f(t, y, z)$ is defined and is continuously differentiable in the space where its arguments take values.*
- *The equation (2.1.1) is uniquely solvable with respect to $x(\varphi(t))$, i.e. there exists a unique function $g(t, x(t), x'(t))$ such that*

$$x(\varphi(t)) = g(t, x(t), x'(t)).$$

Then, the solution of the ordinary differential equation

$$x''(t) = \left[\frac{\partial f}{\partial t} + x'(t)\frac{\partial f}{\partial y} + \varphi'(t) f(\varphi(t), g(t, x(t), x'(t)), x(t))\frac{\partial f}{\partial z}\right](t, x(t), g(t, x(t), x'(t))),$$

with initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = f(t_0, x_0, x_0),$$

is a solution of the equation (2.1.1) with initial conditions $x(t_0) = x_0$.

Corollary 2.1.5 ([186]). *Let us assume that in the equation*

$$x'(t) = f(x(\varphi(t))) \tag{2.1.2}$$

the function φ is a continuously differentiable function with a fixed point t_0 and the function f is monotone and continuously differentiable in $\mathbb{R}$. Then, the solution of the equations

$$\begin{aligned} x''(t) &= f'(f^{-1}(x'(t)))f(x(t))\varphi'(t), \\ x(\varphi(t)) &= f^{-1}(x'(t)), \end{aligned}$$

with initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = f(x_0),$$

is a solution of the equation (2.1.2) with initial condition $x(t_0) = x_0$.

In Lemma 3.1.1 (page 39) we prove a result more general than Corollary 2.1.5. There we show the equivalence of $x'(t) = f(x(\varphi(t)))$ and

$$x''(t) = f'(f^{-1}(x'(t)))f(x(t))\varphi'(t).$$

Lučić has extended these results to more general ones which include higher order derivatives or different involutions. We refer the reader to [128, 129, 187].

On the other hand, Šarkovskiĭ [169] studies the equation $x'(t) = f(x(t), x(-t))$ and, noting $y(t) := x(-t)$, arrives to the conclusion that the solutions of such equation are solutions of the system

$$\begin{aligned} x'(t) &= f(x, y), \\ y'(t) &= -f(y, x), \end{aligned}$$

with the condition $x(0) = y(0)$. Then he applies this expression to the stability of differential-difference equations. We will arrive to this expression by other means in Proposition 3.1.7 (see page 43).

The traditional study of differential equations with involutions has been done for the case of connected domains. Watkins [173] extends these results (in particular Theorem 2.1.4) to the case of nonconnected domains, as it is the case of the inversion $1/t$ in $\mathbb{R}\backslash\{0\}$.

The asymptotic behavior of equations with involutions has also been studied.

Theorem 2.1.6 ([174])**.** *Let $a > 0$. Assume $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuously differentiable involution such that*

$$\varphi(x) - \varphi(b) < \frac{1}{x} - \frac{1}{b}, \text{ for all } x, b \in (a, +\infty),\ x > b.$$

Then the equation $y'(t) = y(\varphi(t))$ has an oscillatory solution.

We will deepen in the fact that such a type of equations oscillate and compute the period later on (see page 124).

Related to this oscillatory behavior is the fact, pointed out by Zampieri [196], that involutions are related to a potential of some second order differential equations.

Definition 2.1.7. An equilibrium point of a planar vector field is called a (local) *center* if all orbits in a neighborhood are periodic and enclose it. The center is called *isochronous* if all periodic orbits have the same period in a neighborhood of the center.

Theorem 2.1.8 ([196]). *Let $\varphi \in C^1(J)$ be an involution, $\omega > 0$, and define*

$$V(x) = \frac{\omega^2}{8}(x - \varphi(x))^2, \; x \in J.$$

Then the origin is an isochronous center for $x''(t) = -V'(x(t))$. Namely, all orbits which intersect J and the interval of the x-axis in the x, x'-plane, are periodic and have the same period $2\pi/\omega$.

On the other hand, if g is a continuous function defined on a neighborhood of $0 \in \mathbb{R}$, such that $g(0) = 0$, there exists $g'(0) > 0$ and the origin is an isochronous center for $x''(t) = g(x(t))$, then there exist an open interval J, $0 \in J$, which is a subset of the domain of g, and an involution $\varphi : J \to J$ such that

$$\int_0^x g(y)\,\mathrm{d}y = \frac{\omega^2}{8}(x - \varphi(x))^2, \; x \in J,$$

where $\omega = \sqrt{g'(0)}$.

2.2 Differential equations with reflection

The particular field of differential equations with reflection has been subject to much study motivated by the simplicity of this particular involution and its good algebraic properties.

O'Regan [136] studies the existence of solutions for problems of the form

$$y^{(k)}(t) = f(t, y(t), y(-t), \dots, y^{(k-1)}(t), y^{(k-1)}(-t)), \; -T \le t \le T, \quad y \in \mathcal{B},$$

where $\mathcal{B}$ represents some initial or boundary value conditions, using a nonlinear alternative result.

On the same line, existence and uniqueness results are proven by Hai [84] for problems of the kind

$$x''(t) + c\,x'(t) + g(t, x(t), x(-t)) = h(t), \; t \in [-1, 1],$$
$$x(-1) = a\,x'(-1), \; x(1) = -b\,x'(1),$$

with $c \in \mathbb{R}$, $a, b \ge 0$.

Wiener and Watkins study in [189] the solution of the equation $x'(t) - a\,x(-t) = 0$ with initial conditions. Equation $x'(t) + a\,x(t) + b\,x(-t) = g(t)$ has been treated by Piao in [141, 142]. For the equation

$$x'(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), \; b \ne 0, \; t \in \mathbb{R},$$

Piao [141] obtains existence results concerning periodic and almost periodic solutions using topological degree techniques (in particular Leray-Schauder Theorem). In [122, 155, 173, 187, 189] some results are introduced to transform this kind of problems with involutions and initial conditions into second order ordinary differential equations with initial conditions or first order two dimensional systems, granting that the solution of the last will be a solution to the first.

Beyond existence, in all its particular forms, the spectral properties of equations with reflection have also been studied. In [117], the focus is set on the eigenvalue problem

$$u'(-t) + \alpha\, u(t) = \lambda\, u(t),\ t \in [-1, 1], \quad u(-1) = \gamma\, u(1).$$

If $\alpha^2 \in (-1, 1)$ and $\gamma \neq \alpha \pm \sqrt{1-\alpha^2}$, the eigenvalues are given by

$$\lambda_k = \sqrt{1-\alpha^2}\left[k\pi + \arctan\left(\frac{1-\gamma}{1+\gamma}\sqrt{\frac{1+\alpha}{1-\alpha}}\right)\right],\ k \in \mathbb{Z},$$

and the related eigenfunctions by

$$\begin{aligned} u_k(t) :=& \sqrt{1+\alpha}\cos\left[k\pi + \arctan\left(\frac{1-\gamma}{1+\gamma}\sqrt{\frac{1+\alpha}{1-\alpha}}\right)\right] t \\ &+ \sqrt{1-\alpha}\sin\left[k\pi + \arctan\left(\frac{1-\gamma}{1+\gamma}\sqrt{\frac{1+\alpha}{1-\alpha}}\right)\right] t,\ k \in \mathbb{Z}. \end{aligned}$$

The study of equations with reflection extends also to partial differential equations. See for instance [23, 187].

Furthermore, asymptotic properties and boundedness of the solutions of initial first order problems are studied in [174] and [1] respectively. Second order boundary value problems have been considered in [82, 83, 137, 187] for Dirichlet and Sturm-Liouville boundary value conditions, higher order equations has been studied in [136]. Other techniques applied to problems with reflection of the argument can be found in [5, 131, 188].

3. Order one problems with constant coefficients

In this chapter we recall some results in [39, 40, 43]. We start studying the first order operator $x'(t) + m\,x(-t)$ coupled with periodic boundary value conditions. We describe the eigenvalues of the operator and obtain the expression of its related Green's function in the nonresonant case. We also obtain the range of the values of the real parameter m for which the integral kernel, which provides the unique solution, has constant sign. In this way, we automatically establish maximum and anti-maximum principles for the equation.

In the last part of the chapter we generalize these results to the case of antiperiodic and general conditions and study the different maximum and anti-maximum principles derived illustrating them with some examples. Also, we put special attention in the case of initial conditions, in which we obtain the Green's function in a particular way and undertake a study of its sign in different circumstances.

3.1 Reduction of differential equations with involutions

Let us consider the problems

$$x'(t) = f(x(\varphi(t))), \quad x(c) = x_c \tag{3.1.1}$$

and

$$x''(t) = f'(f^{-1}(x'(t)))f(x(t))\varphi'(t), \quad x(c) = x_c,\ x'(c) = f(x_c). \tag{3.1.2}$$

Then we have the following Lemma:

Lemma 3.1.1. *Let $(a,b) \subset \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Let $\varphi \in C^1((a,b))$ be an involution. Let c be a fixed point of φ. Then x is a solution of the first order differential equation with involution (3.1.1) if and only if x is a solution of the second order ordinary differential equation (3.1.2).*

We note that this lemma improves Corollary 2.1.5.

Remark 3.1.2. This result is still valid for $f : J_1 \to J_2$, being J_1, J_2 two real intervals as long as the values of the solution x stay in J_2. We will detail more on this subject in Chapter 6.

Proof. That those solutions of (3.1.1) are solutions of (3.1.2) is almost trivial. The boundary conditions are justified by the fact that $\varphi(c) = c$. Differentiating (3.1.1) we get

$$x''(t) = f'(x(\varphi(t)))\,x'(\varphi(t))\,\varphi'(t)$$

and, taking into account that $x'(\varphi(t)) = f(x(t))$ by (3.1.1), we obtain (3.1.2).

Conversely, let x be a solution of (3.1.2). The equation implies that

$$(f^{-1})'(x'(t))x''(t) = f(x(t))\varphi'(t). \tag{3.1.3}$$

Integrating from c to t in (3.1.3),

$$f^{-1}(x'(t)) - x_c = f^{-1}(x'(t)) - f^{-1}(x'(c)) = \int_c^t f(x(s))\varphi'(s)\,\mathrm{d}s \tag{3.1.4}$$

and thus, defining $g(s) := f(x(\varphi(s))) - x'(s)$, we conclude from (3.1.4) that

$$\begin{aligned} x'(t) &= f\left(x_c + \int_c^t f(x(s))\varphi'(s)\,\mathrm{d}s\right) \\ &= f\left(x(\varphi(t)) + \int_c^t (f(x(s)) - x'(\varphi(s)))\varphi'(s)\,\mathrm{d}s\right) \\ &= f\left(x(\varphi(t)) + \int_c^{\varphi(t)} (f(x(\varphi(s))) - x'(s))\,\mathrm{d}s\right) \\ &= f\left(x(\varphi(t)) + \int_c^{\varphi(t)} g(s)\,\mathrm{d}s\right). \end{aligned}$$

Let us fix $t > c$ where $x(t)$ is defined. We will prove that (3.1.1) is satisfied in $[c,t]$ (the proof is done analogously for $t < c$). Recall that φ has to be decreasing, so $\varphi(t) < c$. Also, since f is a diffeomorphism, the derivative of f is bounded on $[c,t]$, so f is Lipschitz on $[c,t]$. Since f, x, x' and φ' are continuous, we can define

$$\begin{aligned} K_1 := \inf\Big\{\alpha \in \mathbb{R}^+ \, : \, &\left|f\left(x(\varphi(r)) + \int_c^{\varphi(r)} g(s)\,\mathrm{d}s\right) - f(x(\varphi(r)))\right| \\ &\leq \alpha \left|\int_c^{\varphi(r)} g(s)\,\mathrm{d}s\right| \ \ \forall r \in [c,t]\Big\}, \end{aligned}$$

and

$$\begin{aligned} K_2 := \inf\Big\{\alpha \in \mathbb{R}^+ \, : \, &\left|f\left(x(r) + \int_c^r g(s)\,\mathrm{d}s\right) - f(x(r))\right| \\ &\leq \alpha \left|\int_c^r g(s)\,\mathrm{d}s\right| \ \ \forall r \in [c,t]\Big\}. \end{aligned}$$

Let $K = \max\{K_1, K_2\}$. Now,

$$\begin{aligned} |g(t)| &= \left|f\left(x(\varphi(t)) + \int_c^{\varphi(t)} g(s)\,\mathrm{d}s\right) - f(x(\varphi(t)))\right| \leq K \left|\int_c^{\varphi(t)} g(s)\,\mathrm{d}s\right| \\ &\leq -K \int_c^{\varphi(t)} |g(s)|\,\mathrm{d}s = -K \int_c^t |g(\varphi(s))||\varphi'(s)\,\mathrm{d}s. \end{aligned}$$

Applying this inequality at $r = \varphi(s)$ inside the integral we deduce that

$$\begin{aligned} |g(t)| &\leq -K \int_c^t K \left|\int_c^s g(r)\,\mathrm{d}r\right| \varphi'(s)\,\mathrm{d}s \leq -K^2 \int_c^t \int_c^t |g(r)|\,\mathrm{d}r \ \ \varphi'(s)\,\mathrm{d}s \\ &= K^2|\varphi(t) - \varphi(c)| \int_c^t |g(r)|\,\mathrm{d}r \leq K^2(c-a) \int_c^t |g(r)|\,\mathrm{d}r. \end{aligned}$$

Thus, by Grönwall's Lemma, $g(t) = 0$ and hence (3.1.1) is satisfied for all $t < b$ where x is defined. ■

Notice that, as an immediate consequence of this result, we have that the unique solution of the equation

$$x''(t) = -\sqrt{1 + (x'(t))^2}\, \sinh x(t), \quad x(0) = x_0,\ x'(0) = \sinh x_0,$$

coincide with the unique solution of

$$x'(t) = \sinh x(-t), \quad x(0) = x_0.$$

Furthermore, Lemma 3.1.1 can be extended, with a very similar proof, to the case with periodic boundary value conditions. Let us consider the equations

$$x'(t) = f(x(\varphi(t))), \quad x(a) = x(b) \tag{3.1.5}$$

and

$$x''(t) = f'(f^{-1}(x'(t)))f(x(t))\varphi'(t), \quad x(a) = x(b) = f^{-1}(x'(a)). \tag{3.1.6}$$

Lemma 3.1.3. *Let $[a,b] \subset \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Let $\varphi \in C^1([a,b])$ be an involution such that $\varphi([a,b]) = [a,b]$. Then x is a solution of the first order differential equation with involution (3.1.5) if and only if x is a solution of the second order ordinary differential equation (3.1.6).*

Proof. Let x be a solution of (3.1.5). Since $\varphi(a) = b$ we trivially get that x is a solution of (3.1.6).

Let x be a solution of (3.1.6). As in the proof of the previous lemma, we have that

$$x'(t) = f\left(x(\varphi(t)) + \int_b^{\varphi(t)} g(s)\,\mathrm{d}s\right),$$

where $g(s) := f(x(\varphi(s))) - x'(s)$.

Let

$$K_1 := \inf\left\{\alpha \in \mathbb{R}^+ : \left|f\left(x(\varphi(r)) + \int_a^{\varphi(r)} g(s)\,\mathrm{d}s\right) - f(x(\varphi(r)))\right| \right.$$
$$\left. \leq \alpha\left|\int_a^{\varphi(r)} g(s)\,\mathrm{d}s\right| \quad \forall r \in [a,b]\right\},$$

$$K_2 := \inf\left\{\alpha \in \mathbb{R}^+ : \left|f\left(x(r) + \int_a^{r} g(s)\,\mathrm{d}s\right) - f(x(r))\right| \right.$$
$$\left. \leq \alpha\left|\int_a^{r} g(s)\,\mathrm{d}s\right| \quad \forall r \in [a,b]\right\}.$$

$$K_1' := \inf\left\{\alpha \in \mathbb{R}^+ : \left|f\left(x(\varphi(r)) + \int_b^{\varphi(r)} g(s)\,\mathrm{d}s\right) - f(x(\varphi(r)))\right| \right.$$
$$\left. \leq \alpha\left|\int_b^{\varphi(r)} g(s)\,\mathrm{d}s\right| \quad \forall r \in [a,b]\right\},$$

$$K_2' := \inf\left\{\alpha \in \mathbb{R}^+ : \left|f\left(x(r) + \int_b^{r} g(s)\,\mathrm{d}s\right) - f(x(r))\right| \right.$$
$$\left. \leq \alpha\left|\int_b^{r} g(s)\,\mathrm{d}s\right| \quad \forall r \in [a,b]\right\}$$

K_1, K_2 be as in the proof of Lemma 3.1.1 but changing c by a and $[c,t]$ by $[a,b]$. Let K_1', K_2' be as K_1, K_2 but changing c by b. Let $K = \max\{K_1, K_2, K_1', K_2'\}$. Then, for t in $[a,b]$,

$$|g(t)| \leq K\left|\int_b^{\varphi(t)} g(s)\,\mathrm{d}s\right| \leq -K\int_a^t |g(\varphi(s))|\varphi'(s)\,\mathrm{d}s$$

$$\leq -K\int_a^t K\left|\int_a^s g(r)\,\mathrm{d}r\right|\varphi'(s)\,\mathrm{d}s \leq K^2|\varphi(t)-\varphi(a)|\int_a^t |g(r)|\,\mathrm{d}r$$
$$\leq K^2(b-a)\int_a^t |g(r)|\,\mathrm{d}r,$$

and we conclude analogously to the other proof. ■

Remark 3.1.4. Condition $x(a) = x(b) = f^{-1}(x'(a))$ in Lemma 3.1.3 can be replaced by $x(a) = x(b) = f^{-1}(x'(b))$. The proof in this case is analogous.

Remark 3.1.5. It is important to notice that the proofs of Lemmas 3.1.1 and 3.1.3 are still valid if we weaken the regularity hypothesis on f and f^{-1} to f and f^{-1} absolutely continuous and f locally Lipschitz. It is enough to check that we have sufficient regularity for using the chain rule (cf. [37, Lemma 1 and Remark 3]).

Let $I := [-T,T] \subset \mathbb{R}$ and consider a problem of the kind

$$x'(t) = f(t,x(-t),x(t)), \quad x(-T) = x(T). \tag{3.1.7}$$

If we consider now the endomorphism $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\xi(t,z,w) = (t,z-w,z+w) \quad \forall z,w \in \mathbb{R},$$

with inverse

$$\xi^{-1}(t,y,x) = \left(t,\frac{x+y}{2},\frac{x-y}{2}\right) \quad \forall\, x,y \in \mathbb{R}.$$

It is clear that

$$f(t,x(-t),x(t)) = (f\circ\xi)(t,x_e(t),x_o(t)),$$

and

$$f(-t,x(t),x(-t)) = (f\circ\xi)(-t,x_e(t),-x_o(t)).$$

where x_e and x_o denote the even and odd parts of x respectively.

On the other hand, we define

$$\begin{aligned} g_e(t) := f_e(t,x(-t),x(t)) &= \frac{f(t,x(-t),x(t)) + f(-t,x(t),x(-t))}{2} \\ &= \frac{(f\circ\xi)(t,x_e(t),x_o(t)) + (f\circ\xi)(-t,x_e(t),-x_o(t))}{2} \end{aligned}$$

and

$$g_o(t) := f_o(t,x(-t),x(t)) = \frac{(f\circ\xi)(t,x_e(t),x_o(t)) - (f\circ\xi)(-t,x_e(t),-x_o(t))}{2},$$

which are an even and an odd function respectively. Furthermore, since x_e is even, $x_e(-T) = x_e(T)$ and since x_o is odd, $x_o(-T) = -x_o(T)$. Taking into account Proposition 1.1.6, we can state the following theorem.

Theorem 3.1.6. *If x is a solution of problem (3.1.7) and $y(t) = x(-t)$, then $(z,w) : I \to \mathbb{R}^2$ satisfying $(t,z,w) = \xi^{-1}(t,y,x)$ is a solution of the system of boundary value ordinary differential equations*

$$\begin{aligned} z'(t) &= \frac{(f\circ\xi)(t,z(t),w(t)) - (f\circ\xi)(-t,z(t),-w(t))}{2}, \quad t\in I,\\ w'(t) &= \frac{(f\circ\xi)(t,z(t),w(t)) + (f\circ\xi)(-t,z(t),-w(t))}{2}, \quad t\in I,\\ (z,w)(-T) &= (z,-w)(T). \end{aligned} \tag{3.1.8}$$

We can take this one step further trying to "undo" what we did:

$$\begin{aligned} x'(t) &= (z+w)'(t) = (f\circ\xi)(t,z(t),w(t)) = f(t,y(t),x(t)),\\ y'(t) &= (z-w)'(t) = -(f\circ\xi)(-t,z(t),-w(t)) = -f(-t,x(t),y(t)),\\ (y,x)(-T) &= ((z-w)(-T),(z+w)(-T)) = ((z+w)(T),(z-w)(T)) = (x,y)(T). \end{aligned}$$

We get then the following result.

Proposition 3.1.7. *(z,w) is a solution of problem (3.1.8) if and only if (y,x) such that $\xi(t,z,w) = (t,y,x)$ is a solution of the system of boundary value ordinary differential equations*

$$\begin{aligned} x'(t) &= f(t,y(t),x(t)),\\ y'(t) &= -f(-t,x(t),y(t)),\\ (y,x)(-T) &= (x,y)(T). \end{aligned} \tag{3.1.9}$$

The next corollary can also be obtained in a straightforward way without going trough problem (3.1.8).

Corollary 3.1.8. *If x is a solution of problem (3.1.7) and $y(t) = x(-t)$, then $(y,x) : I \to \mathbb{R}^2$ is a solution of the problem (3.1.9).*

Solving problems (3.1.8) or (3.1.9) we can check whether x, obtained from the relation $(t,y,x) = \xi(t,z,w)$ is a solution to problem (3.1.7). Unfortunately, not every solution of (3.1.8) – or (3.1.9)– is a solution of (3.1.7), as we show in the following example.

Example 3.1.9. Consider the problem

$$x'(t) = x(t)\,x(-t),\ t\in I;\quad x(-T) = x(T). \tag{3.1.10}$$

Using Proposition 3.1.7 and Theorem 3.1.6, we know that the solutions of problem (3.1.10) are those of problem

$$\begin{aligned} x'(t) &= x(t)\,y(t), \quad t\in I;\\ y'(t) &= -x(t)\,y(t), \quad t\in I;\\ (y,x)(-T) &= (x,y)(T). \end{aligned} \tag{3.1.11}$$

To solve the problem, observe that, adding the two equations, we get $x'(t) + y'(t) = 0$, so $y(t) = c - x(t)$ for some constant $c \in \mathbb{R}$. Substituting y in the first equation we get

$x'(t)=x(t)(c-x(t))$. It is easy to check that the only solutions of problem (3.1.11) defined on I are of the kind

$$(x,y)=\left(\frac{c\,k\,e^{ct}}{k\,e^{ct}+1},\frac{c}{k\,e^{c\,t}+1}\right),$$

with $c,k\in\mathbb{R}$. However, in order to have $x(T)=x(-T)$, a condition necessary for x to be a solution of problem (3.1.10), the only possibility is to have $ck=0$, and so $x(t)=0$ is the only solution of problem (3.1.11) which is a solution of problem (3.1.10). Hence, using Corollary 3.1.8, we conclude that $x\equiv 0$ is the only solution of problem (3.1.10).

In a completely analogous way, we can study the initial value problem

$$x'(t)=f(t,x(-t),x(t)),\quad x(0)=x_0. \tag{3.1.12}$$

In such a case we would have the following versions of the previous results.

Theorem 3.1.10. *If $x:(-\epsilon,\epsilon)\to\mathbb{R}$ is a solution of problem (3.1.12) and $y(t)=x(-t)$, then $(z,w):(-\epsilon,\epsilon)\to\mathbb{R}^2$ satisfying $(t,z,w)=\xi^{-1}(t,y,x)$ is a solution of the system of boundary value ordinary differential equations*

$$\begin{aligned} z'(t)&=\frac{(f\circ\xi)(t,z(t),w(t))-(f\circ\xi)(-t,z(t),-w(t))}{2},\quad t\in I,\\ w'(t)&=\frac{(f\circ\xi)(t,z(t),w(t))+(f\circ\xi)(-t,z(t),-w(t))}{2},\quad t\in I,\\ (z,w)(0)&=(x_0,0). \end{aligned} \tag{3.1.13}$$

Proposition 3.1.11. *(z,w) is a solution of problem (3.1.13) if and only if (y,x), such that $\xi(t,z,w)=(t,y,x)$, is a solution of the system of ordinary differential equations with initial conditions*

$$\begin{aligned} x'(t)&=f(t,y(t),x(t)),\\ y'(t)&=-f(-t,x(t),y(t)),\\ (y,x)(0)&=(x_0,x_0). \end{aligned} \tag{3.1.14}$$

Corollary 3.1.12. *If $x:(-\epsilon,\epsilon)\to\mathbb{R}$ is a solution of problem (3.1.12) and $y(t)=x(-t)$, then $(y,x):(-\epsilon,\epsilon)\to\mathbb{R}^2$ is a solution of problem (3.1.14).*

Remark 3.1.13. The relation $y(t)=x(-t)$ is used in [187] to study conditions under which the problem

$$x'(t)=f(t,x(t),x(-t)),\quad t\in\mathbb{R}$$

has a unique bounded solution.

3.2 Solution of the equation $x'(t)+m\,x(-t)=h(t)$

In this section we will solve a first order linear equation with reflection coupled with periodic boundary value conditions using its Green's function. More concisely, we consider the following differential functional equation:

$$x'(t)+m\,x(-t)=h(t),\ \ t\in I, \tag{3.2.1a}$$

$$x(T) - x(-T) = 0, \tag{3.2.1b}$$

where m is a real nonzero constant, $T \in \mathbb{R}^+$ and $h \in \mathrm{L}^1(I)$.

Applying the result obtained in Example 1.3.8 to this particular case arrive to a problem of the kind

$$\begin{aligned} x''(t) + m^2 x(t) &= f(t), \quad t \in I, \\ x(T) - x(-T) &= 0, \\ x'(T) - x'(-T) &= 0, \end{aligned} \tag{3.2.2}$$

where $f \in \mathrm{L}^1(I)$. Observe that there is some abuse in this reduction of the problem. First, observe that f, if taken as in Example 1.3.8, should be $h'(t) + m\,h(-t)$ but, here, $h \in \mathrm{L}^1(I)$ so we cannot guarantee it is differentiable. This paradox is solved by developing a density argument. $C^\infty(I)$ functions are dense in $\mathrm{L}^1(I)$ so, in general, we may assume the independent term h is differentiable as necessary and then argue that, since $C^\infty(I)$ is dense in $\mathrm{L}^1(I)$, the expression of the Green's function obtained for the original problem should hold for $h \in \mathrm{L}^1(I)$ as well (as will always be the case).

Also, the second boundary condition is, following Example 1.3.8,

$$x'(T) - x'(-T) = h(T) - h(-T),$$

but, since $h \in \mathrm{L}^1(I)$, we may as well assume that $h(T) = h(-T)$. We will use this density argument several times throughout the work, so the reader should pay careful attention when it appears.

There is much literature on how to solve this problem and the properties of the solution (see for instance [2, 30, 31]). It is very well known that for all $m^2 \neq (k\pi/T)^2$, $k = 0, 1, \dots$, problem (3.2.2) has a unique solution given by the expression

$$u(t) = \int_{-T}^{T} G(t,s) f(s)\,\mathrm{d}s,$$

where G is the so-called Green's function.

This function is unique insofar as it satisfies the following properties [28]:

(I) $\quad G \in C(I^2, \mathbb{R})$,

(II) $\quad \frac{\partial G}{\partial t}$ and $\frac{\partial^2 G}{\partial t^2}$ exist and are continuous in $\{(t,s) \in I^2 \mid s \neq t\}$,

(III) $\quad \frac{\partial G}{\partial t}(t,t^-)$ and $\frac{\partial G}{\partial t}(t,t^+)$ exist for all $t \in I$ and satisfy

$$\frac{\partial G}{\partial t}(t,t^-) - \frac{\partial G}{\partial t}(t,t^+) = 1 \quad \forall t \in I,$$

(IV) $\quad \frac{\partial^2 G}{\partial t^2} + m^2 G = 0$ in $\{(t,s) \in I^2 \mid s \neq t\}$,

(V) $\quad (a) \quad G(T,s) = G(-T,s) \quad \forall s \in I$,

$\quad (b) \quad \frac{\partial G}{\partial t}(T,s) = \frac{\partial G}{\partial t}(-T,s) \quad \forall s \in (-T,T)$.

The solution to problem (3.2.2) is unique whenever $T \in \mathbb{R}^+ \backslash \{k\pi/|m|\}_{k\in\mathbb{N}}$, so the solution to (3.2.1) is unique in such a case. We will assume uniqueness conditions from now on.

The following proposition gives us some more properties of the Green's function for (3.2.2).

Proposition 3.2.1. *For all $t, s \in I$, the Green's function associated to problem (3.2.2) satisfies the following properties as well:*

$$
\begin{aligned}
(VI) &\quad G(t,s) = G(s,t),\\
(VII) &\quad G(t,s) = G(-t,-s),\\
(VIII) &\quad \tfrac{\partial G}{\partial t}(t,s) = \tfrac{\partial G}{\partial s}(s,t),\\
(IX) &\quad \tfrac{\partial G}{\partial t}(t,s) = -\tfrac{\partial G}{\partial t}(-t,-s),\\
(X) &\quad \tfrac{\partial G}{\partial t}(t,s) = -\tfrac{\partial G}{\partial s}(t,s).
\end{aligned}
$$

Proof. (VI). The differential operator $L = \frac{\mathrm{d}^2}{\mathrm{d}\,t^2} + m^2$ associated to equation (3.2.2) is self-adjoint, so in an analogous way to [2, Chapter 33] or [28, Section 1.3], we deduce that function G is symmetric.

(VII). Let u be a solution to (3.2.2) and define $v(t) := u(-t)$, then v is a solution of problem (3.2.2) with $f(-t)$ instead of $f(t)$. This way

$$v(t) = \int_{-T}^{T} G(t,s) f(-s)\,\mathrm{d}s = \int_{-T}^{T} G(t,-s) f(s)\,\mathrm{d}s,$$

but we have also

$$v(t) = u(-t) = \int_{-T}^{T} G(-t,s) f(s)\,\mathrm{d}s,$$

therefore

$$\int_{-T}^{T} [G(t,-s) - G(-t,s)] f(s) = 0$$

and, since continuous functions are dense in $L^2(I)$, $G(t,-s) = G(-t,s)$ on I^2, this is,

$$G(t,s) = G(-t,-s) \quad \forall t, s \in I.$$

To prove $(VIII)$ and (IX) it is enough to differentiate (VI) and (VII) with respect to t.

(X) Assume f is differentiable. Let u be a solution to (3.2.2), then $u \in C^1(I)$ and $v \equiv u'$ is a solution of

$$
\begin{aligned}
x''(t) + m^2 x(t) &= f'(t), \quad t \in I,\\
x(T) - x(-T) &= 0,\\
x'(T) - x'(-T) &= f(T) - f(-T).
\end{aligned}
$$

Therefore,

$$v(t) = \int_{-T}^{T} G(t,s) f'(s)\,\mathrm{d}s - G(t,-T)[f(T) - f(-T)],$$

where the second term in the right hand side stands for the nonhomogeneity of the boundary conditions and properties (III), (IV) and (V) (a).

Hence, from $(V)\,(a)$ and (VI), we have that

$$\begin{aligned} v(t) &= G(t,s)f(s)|_{s=-T}^{s=T} - \int_{-T}^{t} \frac{\partial G}{\partial s}(t,s)f(s)\,\mathrm{d}s - \int_{t}^{T} \frac{\partial G}{\partial s}(t,s)f(s)\,\mathrm{d}s \\ & -G(t,-T)\left[f(T)-f(-T)\right] = -\int_{-T}^{T} \frac{\partial G}{\partial s}(t,s)f(s)\,\mathrm{d}s. \end{aligned}$$

On the other hand,

$$v(t)=u'(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-T}^{t} G(t,s)f(s)\,\mathrm{d}s+\frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{T} G(t,s)f(s)\,\mathrm{d}s=\int_{-T}^{T}\frac{\partial G}{\partial t}(t,s)f(s)\,\mathrm{d}s.$$

Since differentiable functions are dense in $L^2(I)$, we conclude that

$$\frac{\partial G}{\partial t}(t,s) = -\frac{\partial G}{\partial s}(t,s).$$

■

Now we are in a position to prove the main result of this section, in which we deduce the expression of the Green's function related to problem (3.2.1).

Proposition 3.2.2. *Suppose that $m \neq k\pi/T$, $k \in \mathbb{Z}$. Then problem (3.2.1) has a unique solution given by the expression*

$$u(t) := \int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s, \tag{3.2.3}$$

where

$$\overline{G}(t,s) := m\,G(t,-s) - \frac{\partial G}{\partial s}(t,s)$$

is called the Green's function *related to problem (3.2.1).*

Proof. As we have previously remarked, problem (3.2.1) has at most one solution for all $m \neq k\pi/T$, $k \in \mathbb{Z}$. Let us see that function u defined in (3.2.3) fulfills (3.2.1) (we assume $t>0$, the other case is analogous):

$$\begin{aligned} & u'(t)+m\,u(-t) \\ = \; & \frac{\mathrm{d}}{\mathrm{d}t}\int_{-T}^{-t}\overline{G}(t,s)h(s)\,\mathrm{d}s+\frac{\mathrm{d}}{\mathrm{d}t}\int_{-t}^{t}\overline{G}(t,s)h(s)\,\mathrm{d}s \\ & +\frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{T}\overline{G}(t,s)h(s)\,\mathrm{d}s+m\int_{-T}^{T}\overline{G}(-t,s)h(s)\,\mathrm{d}s \\ = \; & (\overline{G}(t,t^-)-\overline{G}(t,t^+))h(t)+\int_{-T}^{T}\left[m\frac{\partial G}{\partial t}(t,-s)-\frac{\partial^2 G}{\partial t\partial s}(t,s)\right]h(s)\,\mathrm{d}s \\ & +m\int_{-T}^{T}\left[mG(-t,-s)-\frac{\partial G}{\partial s}(-t,s)\right]h(s)\,\mathrm{d}s. \end{aligned}$$

Using (III), we deduce that this last expression is equal to

$$h(t) + \int_{-T}^{T} \left[m\frac{\partial G}{\partial t}(t,-s) - \frac{\partial^2 G}{\partial t \partial s}(t,s) + m^2 G(-t,-s) - m\frac{\partial G}{\partial s}(-t,s) \right] h(s)\,\mathrm{d}s.$$

which is, by (IV), (VII), (IX) and (X), equal to

$$h(t) + \int_{-T}^{T} \left(m\left[\frac{\partial G}{\partial t}(t,-s) - \frac{\partial G}{\partial s}(-t,s) \right] + \frac{\partial^2 G}{\partial t^2}(t,s) + m^2 G(t,s) \right) h(s)\,\mathrm{d}s = h(t).$$

Therefore, (3.2.1a) is satisfied.

Condition (V) allows us to verify the boundary condition:

$$\begin{aligned} & u(T) - u(-T) \\ & = \int_{-T}^{T} \left[mG(T,-s) - \frac{\partial G}{\partial s}(T,s) - mG(-T,-s) + \frac{\partial G}{\partial s}(-T,s) \right] h(s) = 0. \end{aligned}$$

■

As the original Green's function, $\overline{G}$ satisfies several properties.

Proposition 3.2.3. *$\overline{G}$ satisfies the following properties:*

(I') $\frac{\partial \overline{G}}{\partial t}$ *exists and is continuous in* $\{(t,s) \in I^2 \mid s \neq t\}$,

(II') $\overline{G}(t,t^-)$ *and* $\overline{G}(t,t^+)$ *exist for all* $t \in I$ *and satisfy* $\overline{G}(t,t^-) - \overline{G}(t,t^+) = 1 \;\; \forall t \in I$,

(III') $\frac{\partial \overline{G}}{\partial t}(t,s) + m\overline{G}(-t,s) = 0$ *for a. e.* $t,s \in I,\ s \neq t$,

(IV') $\overline{G}(T,s) = \overline{G}(-T,s) \;\; \forall s \in (-T,T)$,

(V') $\overline{G}(t,s) = \overline{G}(-s,-t) \;\; \forall t,s \in I$.

Proof. Properties (I'), (II') and (IV') are straightforward from the analogous properties for function G.

(III'). In the proof of Proposition 3.2.2 we implicitely showed that function u defined in (3.2.3), and thus the unique solution of (3.2.1), satisfies

$$u'(t) = h(t) + \int_{-T}^{T} \frac{\partial \overline{G}}{\partial t}(t,s)h(s)\,\mathrm{d}s.$$

Hence, since $u'(t) - h(t) + m\,u(-t) = 0$,

$$\int_{-T}^{T} \frac{\partial \overline{G}}{\partial t}(t,s)h(s)\,\mathrm{d}s + m\int_{-T}^{T} \overline{G}(-t,s)h(s)\,\mathrm{d}s = 0,$$

this is,

$$\int_{-T}^{T}\left[\frac{\partial\overline{G}}{\partial t}(t,s)+m\overline{G}(-t,s)\right]h(s)\,\mathrm{d}s=0 \text{ for all } h\in \mathrm{L}^1(I),$$

and thus

$$\frac{\partial\overline{G}}{\partial t}(t,s)+m\overline{G}(-t,s)=0 \text{ for a. e. } t,s\in I,\ s\neq t.$$

(V'). This result is proven using properties $(VI)-(X)$:

$$\begin{aligned}\overline{G}(-s,-t)&=mG(-s,t)-\frac{\partial G}{\partial s}(-s,-t)=mG(t,-s)+\frac{\partial G}{\partial t}(-s,-t)\\&=mG(t,-s)-\frac{\partial G}{\partial t}(s,t)=mG(t,-s)-\frac{\partial G}{\partial s}(t,s)=\overline{G}(t,s).\end{aligned}$$

■

Remark 3.2.4. Due to the expression of G given in next section, properties (II) and (I') can be improved by adding that G and $\overline{G}$ are analytic on $\{(t,s)\in I^2 \mid s\neq t\}$ and $\{(t,s)\in I^2 \mid |s|\neq|t|\}$ respectively.

Using properties $(II')-(V')$ we obtain the following corollary of Proposition 3.2.2.

Corollary 3.2.5. *Suppose that $m\neq k\pi/T$, $k\in\mathbb{Z}$. Then the problem*

$$\begin{aligned}&x'(t)+m\,x(-t)=h(t),\quad t\in I:=[-T,T],\\&x(-T)-x(T)=\lambda,\end{aligned}$$

with $\lambda\in\mathbb{R}$ has a unique solution given by the expression

$$u(t):=\int_{-T}^{T}\overline{G}(t,s)h(s)\,\mathrm{d}s+\lambda\overline{G}(t,-T).$$

3.2.1 Constant sign of function $\overline{G}$

We will now give a result on the positivity or negativity of the Green's function for problem (3.2.1). In order to achieve this, we need a new lemma and the explicit expression of the function $\overline{G}$.

Let $\alpha:=mT$ and $\overline{G}_\alpha$ be the Green's function for problem (3.2.1) for a particular value of the parameter α. Note that $\operatorname{sign}(\alpha)=\operatorname{sign}(m)$ because T is always positive.

Lemma 3.2.6. $\overline{G}_\alpha(t,s)=-\overline{G}_{-\alpha}(-t,-s)\quad\forall t,s\in I.$

Proof. Let $u(t):=\int_{-T}^{T}\overline{G}_\alpha(t,s)h(s)\,\mathrm{d}s$ be a solution to (3.2.1). Let $v(t):=-u(-t)$. Then $v'(t)-m\,v(-t)=u'(-t)+m\,u(t)=h(-t)$, and therefore

$$v(t)=\int_{-T}^{T}\overline{G}_{-\alpha}(t,s)h(-s)\,\mathrm{d}s.$$

On the other hand, by definition of v,

$$v(t)=-\int_{-T}^{T}\overline{G}_\alpha(-t,s)h(s)\,\mathrm{d}s=-\int_{-T}^{T}\overline{G}_\alpha(-t,-s)h(-s)\,\mathrm{d}s,$$

therefore we can conclude that $\overline{G}_\alpha(t,s)=-\overline{G}_{-\alpha}(-t,-s)$ for all $t,\ s\in I$. ■

Corollary 3.2.7. *$\overline{G}_\alpha$ is positive if and only if $\overline{G}_{-\alpha}$ is negative on I^2.*

With this corollary, to make a complete study of the positivity and negativity of the Green's function, it is enough to find out for what values $\alpha = m\,T \in \mathbb{R}^+$ function $\overline{G}$ is positive and for which is not. This will be very useful to state maximum and anti-maximum principles for (3.2.1) due to the way we express its solution as an integral operator with kernel $\overline{G}$.

Using the algorithm described in [31] we can obtain the explicit expression of G:

$$2m\sin(mT)G(t,s) = \begin{cases} \cos m(T+s-t) & \text{if} \quad s \le t, \\ \cos m(T-s+t) & \text{if} \quad s > t. \end{cases}$$

Therefore,

$$2\sin(mT)\overline{G}(t,s) = \begin{cases} \cos m(T-s-t) + \sin m(T+s-t) & \text{if} \quad -t \le s < t, \\ \cos m(T-s-t) - \sin m(T-s+t) & \text{if} \quad -s \le t < s, \\ \cos m(T+s+t) + \sin m(T+s-t) & \text{if} \quad -|t| > s, \\ \cos m(T+s+t) - \sin m(T-s+t) & \text{if} \quad t < -|s|. \end{cases}$$

Realize that $\overline{G}$ is continuous in $\{(t,s) \in I^2 \mid t \ne s\}$. Making the change of variables $t = Tz$, $s = Ty$, we can simplify this expression to

$$2\sin(\alpha)\overline{G}(z,y) = \begin{cases} \cos\alpha(1-y-z) + \sin\alpha(1+y-z) & \text{if} \quad -z \le y < z, \\ \cos\alpha(1-y-z) - \sin\alpha(1-y+z) & \text{if} \quad -y \le z < y, \\ \cos\alpha(1+y+z) + \sin\alpha(1+y-z) & \text{if} \quad -|z| > y, \\ \cos\alpha(1+y+z) - \sin\alpha(1-y+z) & \text{if} \quad z < -|y|. \end{cases}$$

Using the trigonometric identity

$$\cos(a-b) \pm \sin(a+b) = (\cos a \pm \sin a)(\cos b \pm \sin b),$$

we can factorise this expression as follows:

$$2\sin(\alpha)\overline{G}(z,y) = \begin{cases} [\cos\alpha(1-z) + \sin\alpha(1-z)][\sin\alpha y + \cos\alpha y] & \text{if} \quad -z \le y < z, \\ [\cos\alpha z - \sin\alpha z][\sin\alpha(y-1) + \cos\alpha(y-1)] & \text{if} \quad -y \le z < y, \\ [\cos\alpha(1+y) + \sin\alpha(1+y)][\cos\alpha z - \sin\alpha z] & \text{if} \quad -|z| > y, \\ [\cos\alpha y + \sin\alpha y][\cos\alpha(z+1) - \sin\alpha(z+1)] & \text{if} \quad z < -|y|. \end{cases} \tag{3.2.5}$$

Note that

$$\begin{aligned} &\cos\xi + \sin\xi > 0 \;\; \forall \xi \in \left(2k\pi - \frac{\pi}{4}, 2k\pi + \frac{3\pi}{4}\right), \, k \in \mathbb{Z} \\ &\cos\xi + \sin\xi < 0 \;\; \forall \xi \in \left(2k\pi + \frac{3\pi}{4}, 2k\pi + \frac{7\pi}{4}\right), \, k \in \mathbb{Z} \\ &\cos\xi - \sin\xi > 0 \;\; \forall \xi \in \left(2k\pi - \frac{3\pi}{4}, 2k\pi + \frac{\pi}{4}\right), \, k \in \mathbb{Z} \\ &\cos\xi - \sin\xi < 0 \;\; \forall \xi \in \left(2k\pi + \frac{\pi}{4}, 2k\pi + \frac{5\pi}{4}\right), \, k \in \mathbb{Z} \end{aligned} \tag{3.2.6}$$

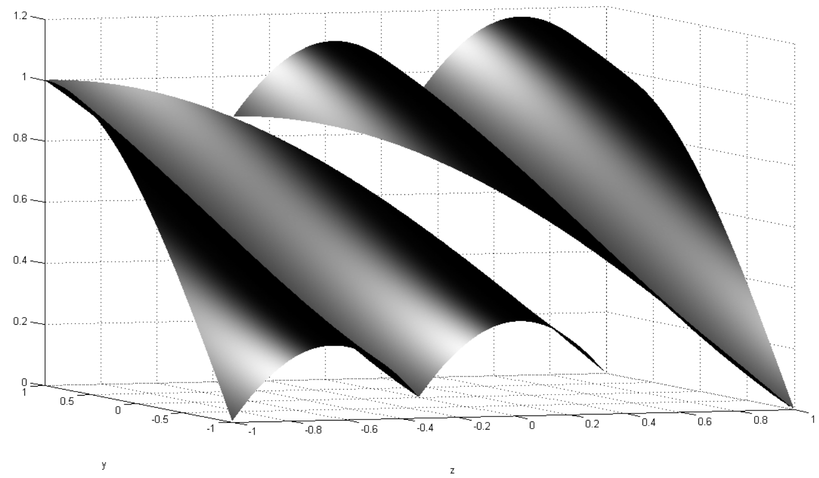


Figure 3.2.1: Plot of the function $\overline{G}(z,y)$ for $\alpha = \frac{\pi}{4}$.

As we have seen, the Green's function $\overline{G}$ is not defined on the diagonal of I^2. For easier manipulation, we will define it in the diagonal as follows:

$$G(t,t) = \begin{cases} \lim\limits_{s\to t^+} G(t,s) & \text{if} \quad m>0 \\ \lim\limits_{s\to t^-} G(t,s) & \text{if} \quad m<0 \end{cases} \quad \text{for } t \in (-T,T);$$

$$G(T,T) = \lim_{s\to T^-} G(s,s), \quad G(-T,-T) = \lim_{s\to -T^+} G(s,s)$$

Using expression (3.2.5) and formulae (3.2.6) we can prove the following theorem.

Theorem 3.2.8.

(1) If $\alpha \in (0,\frac{\pi}{4})$ then $\overline{G}$ is strictly positive on I^2.

(2) If $\alpha \in (-\frac{\pi}{4},0)$ then $\overline{G}$ is strictly negative on I^2.

(3) If $\alpha = \frac{\pi}{4}$ then $\overline{G}$ vanishes on $P := \{(-T,-T),(0,0),(T,T),(T,-T)\}$ and is strictly positive on $(I^2)\backslash P$.

(4) If $\alpha = -\frac{\pi}{4}$ then $\overline{G}$ vanishes on P and is strictly negative on $(I^2)\backslash P$.

(5) If $\alpha \in \mathbb{R}\backslash[-\frac{\pi}{4},\frac{\pi}{4}]$ then $\overline{G}$ is not positive nor negative on I^2.

Proof. Lemma 3.2.6 allows us to restrict the proof to the positive values of α.

We study here the positive values of $\overline{G}(z,y)$ in $A := \{(z,y) \in [-1,1]^2 \mid z \geq |y|\}$. The rest of cases are done in an analogous fashion. Let

$$B_1 := \bigcup_{k_1\in\mathbb{Z}} \left(1-\frac{\pi}{\alpha}\left(2k_1+\frac{3}{4}\right), 1-\frac{\pi}{\alpha}\left(2k_1-\frac{1}{4}\right)\right),$$

$$
\begin{aligned}
B_2 &:= \bigcup_{k_2\in\mathbb{Z}} \frac{\pi}{\alpha}\left(2k_2-\frac{1}{4}, 2k_2+\frac{3}{4}\right),\\
C_1 &:= \bigcup_{k_1\in\mathbb{Z}} \left(1-\frac{\pi}{\alpha}\left(2k_1+\frac{7}{4}\right), 1-\frac{\pi}{\alpha}\left(2k_1+\frac{3}{4}\right)\right),\\
C_2 &:= \bigcup_{k_2\in\mathbb{Z}} \frac{\pi}{\alpha}\left(2k_2+\frac{3}{4}, 2k_2+\frac{7}{4}\right),\\
B &:= \{(z,y)\in B_1\times B_2 \,|\, z>|y|\}, \quad \text{and} \quad C := \{(z,y)\in C_1\times C_2 \,|\, z>|y|\}.
\end{aligned}
$$

Realize that $B\cap C=\emptyset$. Moreover, we have that $\overline{G}(z,y)>0$ on A if and only if $A\subset B\cup C$.

To prove the case $A\subset B$, it is a necessary and sufficient condition that $[-1,1]\subset B_2$ and $[0,1]\subset B_1$.

$[-1,1]\subset B_2$ if and only if $k_2\in\frac{1}{2}(\frac{\alpha}{\pi}-\frac{3}{4},\frac{1}{4}-\frac{\alpha}{\pi})$ for some $k_2\in\mathbb{Z}$, but, since $\alpha>0$, this only happens if $k_2=0$. In such a case $[-1,1]\subset\frac{\pi}{4\alpha}(-1,3)$, which implies $\alpha<\frac{\pi}{4}$. Hence, $\frac{\pi}{\alpha}>4$, so $[0,1]\subset(1-\frac{3}{4}\frac{\pi}{\alpha},1+\frac{1}{4}\frac{\pi}{\alpha})=\left(1-\frac{\pi}{\alpha}\left(2k_1+\frac{3}{4}\right),1-\frac{\pi}{\alpha}\left(2k_1-\frac{1}{4}\right)\right)$ for $k_1=0$. Therefore $A\subset B$.

We repeat this study for the case $A\subset C$ and all the other subdivisions of the domain of $\overline{G}$, proving the statement. ■

The following definitions [25] lead to a direct corollary of Theorem 3.2.8.

Definition 3.2.9. Let $\mathcal{F}_\lambda(I)$ be the set of real differentiable functions f defined on I such that $f(-T)-f(T)=\lambda$. A linear operator $R:\mathcal{F}_\lambda(I)\to\mathrm{L}^1(I)$ is said to be

(1) *strongly inverse positive on* $\mathcal{F}_\lambda(I)$ if $Rx\succ 0$ on I $\Rightarrow x>0$ on I $\forall x\in\mathcal{F}_\lambda(I)$,

(2) *strongly inverse negative on* $\mathcal{F}_\lambda(I)$ if $Rx\succ 0$ on I $\Rightarrow x<0$ on I $\forall x\in\mathcal{F}_\lambda(I)$,

where $x\succ 0$ stands for $x\geq 0$ and $\int_{-T}^{T}x(t)\,\mathrm{d}t>0$. Respectively, $x\prec 0$ stand for stands for $x\leq 0$ and $\int_{-T}^{T}x(t)\,\mathrm{d}t<0$.

Corollary 3.2.10. *The operator* $R_m:\mathcal{F}_\lambda(I)\to\mathrm{L}^1(I)$ *defined as* $R_m(x(t))=x'(t)+m\,x(-t)$*, with* $m\in\mathbb{R}\backslash\{0\}$*, satisfies*

(1) R_m *is strongly inverse positive on* $\mathcal{F}_\lambda(I)$ *if and only if* $m\in(0,\frac{\pi}{4T}]$ *and* $\lambda\geq 0$*,*

(2) R_m *is strongly inverse negative on* $\mathcal{F}_\lambda(I)$ *if and only if* $m\in[-\frac{\pi}{4T},0)$ *and* $\lambda\geq 0$*.*

This last corollary establishes a maximum and anti-maximum principle (cf. [25, Lemma 2.5, Remark 2.3]).

The function $\overline{G}$ has a fairly convoluted expression which does not allow us to see in a straightforward way its dependence on m (see Figure 3.2.1). This dependency can be analyzed, without computing and evaluating the derivative with respect to m, just using the properties of equation (3.2.1a) in those regions where the operator R_m is inverse positive or inverse negative. A different method to the one used here but pursuing a similar purpose can be found in [30, Lemma 2.8] for the Green's function related to the second order *Hill's equation*. In [28, Section 1.8] the reader can find a weaker result for n-th order equations.

Proposition 3.2.11. *Let $G_{m_i} : I \to \mathbb{R}$ be the Green's function and u_i the solution to the problem (3.2.1) with constant $m = m_i$, $i = 1, 2$ respectively. Then the following assertions hold.*

(1) If $0 < m_1 < m_2 \leq \frac{\pi}{4T}$ then $u_1 > u_2 > 0$ on I for every $h \succ 0$ on I and $G_{m_1} > G_{m_2} > 0$ on I^2.

(2) If $-\frac{\pi}{4T} \leq m_1 < m_2 < 0$ then $0 > u_1 > u_2 > 0$ on I for every $h \succ 0$ on I and $0 > G_{m_1} > G_{m_2}$ on I^2.

Proof. (1). Let $h \succ 0$ in equation (3.2.1a). Then, by Corollary 3.2.10, $u_i > 0$ on I, $i = 1, 2$. We have that

$$u_i'(t) + m_i u_i(-t) = h(t) \quad i = 1, 2.$$

Therefore, for a. e. $t \in I$,

$$0 = (u_2 - u_1)'(t) + m_2 u_2(-t) - m_1 u_1(-t) > (u_2 - u_1)'(t) + m_1 (u_2 - u_1)(-t),$$

and $0 = (u_2 - u_1)(T) - (u_2 - u_1)(-T)$. Hence, from Corollary 3.2.10, $u_2 < u_1$ on I.

On the other hand, for all $t \in I$, it is satisfied that

$$0 > (u_2 - u_1)(t) = \int_{-T}^{T} (G_{m_2}(t,s) - G_{m_1}(t,s)) h(s)\, \mathrm{d}s \quad \forall h \succ 0. \tag{3.2.7}$$

This makes clear that $0 \prec G_{m_2} \prec G_{m_1}$ a. e. on I^2.

To prove that $G_{m_2} < G_{m_1}$ on I^2, let $s \in I$ be fixed, and define $v_i : \mathbb{R} \to \mathbb{R}$ as the $2T$–periodic extension to the whole real line of $G_{m_i}(\cdot, s)$.

Using $(I') - (IV')$, we have that $v_2 - v_1$ is a continuosly differentiable function on $I_s \equiv (s, s + 2T)$. Futhermore, it is clear that $(v_2 - v_1)'$ is absolutely continuous on I_s. Using (III'), we have that

$$(v_2 - v_1)'(t) + m_2 v_2(-t) - m_1 v_1(-t) = 0 \quad \text{on } I_s.$$

As consequence, $v_i''(t) + m_i^2 v_i(t) = 0$ a. e. on I_s. Moreover, using (II') and (IV') we know that

$$(v_2 - v_1)(s) = (v_2 - v_1)(s + 2T), \quad (v_2 - v_1)'(s) = (v_2 - v_1)'(s + 2T).$$

Hence, for all $t \in I_s$, we have that

$$0 = (v_2 - v_1)''(t) + m_2^2 v_2(t) - m_1^2 v_1(t) > (v_2 - v_1)''(t) + m_1^2 (v_2 - v_1)(t).$$

The periodic boundary value conditions, together the fact that for this range of values of m_1, operator $v'' + m_1^2 v$ is strongly inverse positive (see Corollary 3.2.10), we conclude that $v_2 < v_1$ on I_s, this is, $G_{m_2}(t,s) < G_{m_1}(t,s)$ for all t, $s \in I$.

(2). This is straightforward using part (1), Lemma 3.2.6 and Theorem 3.2.8:

$$G_{m_2}(t,s) = -G_{-m_2}(-t,-s) < -G_{-m_1}(-t,-s) = G_{m_1}(t,s) < 0 \quad \forall t, s \in I.$$

By equation (3.2.7), $u_2 < u_1$ on I.

■

Remark 3.2.12. In (1) and (2) we could have added that $u_1 < u_2 \ \forall h \prec 0$. These are straightforward consequences of the rest of the proposition.

The next subsection is devoted to point out some applications of the given results to the existence of solutions of nonlinear periodic boundary value problems. Due to the fact that the proofs follow similar steps to the ones given in some previous papers (see [25, 167]), we omit them.

3.2.2 Lower and upper solutions method

Lower and upper solutions methods are a variety of widespread techniques that supply information about the existence –and sometimes construction– of solutions of differential equations. Depending on the particular type of differential equation and the involved boundary value conditions, it is subject to these techniques change but are in general suitable –with proper modifications– to other cases.

For this application we will follow the steps in [25] and use Corollary 3.2.10 to establish conditions under which the more general problem

$$x'(t) = f(t, x(-t)) \ \ \forall t \in I, \quad x(-T) = x(T), \tag{3.2.8}$$

has a solution. Here $f : I \times \mathbb{R} \to \mathbb{R}$ is an L^{p}-*Carathéodory function*, that is, $f(\cdot, x)$ is measurable for all $x \in \mathbb{R}$, $f(t, \cdot)$ is continuous for a. e. $t \in I$, and for every $R > 0$, there exists $h_R \in \mathrm{L}^{\mathrm{p}}(I)$ such that, if with $|x| < R$ then

$$|f(t,x)| \le h_R(t) \quad \text{for a. e. } t \in I.$$

Definition 3.2.13. We say $u \in C(I)$ is an absolutely continuous function in I if there exists $f \in \mathrm{L}^1(I)$ such that for all $a \in I$,

$$u(t) = u(a) + \int_a^t f(s)\,\mathrm{d}s, \ t \in I.$$

We denote by $AC(I)$ the set of absolutely continuous functions defined on I.

Definition 3.2.14. We say that $\alpha \in AC(I)$ is a *lower solution* of (3.2.8) if α satisfies

$$\alpha'(t) \ge f(t, \alpha(-t)) \quad \text{for a. e. } t \in I, \quad \alpha(-T) - \alpha(T) \ge 0.$$

Definition 3.2.15. We say that $\beta \in AC(I)$ is an *upper solution* of (3.2.8) if β satisfies

$$\beta'(t) \le f(t, \beta(-t)) \quad \text{for a. e. } t \in I, \quad \beta(-T) - \beta(T) \le 0.$$

We establish now a theorem that proves the existence of solutions of (3.2.8) under some conditions. The proof follows the same steps of [25, Theorem 3.1] and we omit it here.

Theorem 3.2.16. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function. If there exist $\alpha \ge \beta$ lower and upper solutions of (3.2.8) respectively and $m \in (0, \frac{\pi}{4T}]$ such that*

$$f(t,x) - f(t,y) \ge -m(x-y) \quad \textit{for a. e. } t \in I \textit{ with } \beta(t) \le y \le x \le \alpha(t),$$

then there exist two monotone sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, nonincreasing and nondecreasing respectively, with $\alpha_0 = \alpha$, $\beta_0 = \beta$, which converge uniformly to the extremal solutions in $[\beta, \alpha]$ of (3.2.8).

Furthermore, the estimate $m = \frac{\pi}{4T}$ is best possible in the sense that, for every fixed $m > \frac{\pi}{4T}$, there are problems with its unique solution outside of the interval $[\beta, \alpha]$.

In an analogous way we can prove the following theorem.

Theorem 3.2.17. *Let $f : I\times\mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function. If there exist $\alpha \le \beta$ lower and upper solutions of (3.2.8) respectively and $m \in [-\frac{\pi}{4T}, 0)$ such that*

$$f(t,x) - f(t,y) \le -m(x-y) \quad \text{for a. e. } t \in I \text{ with } \alpha(t) \le y \le x \le \beta(t),$$

then there exist two monotone sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, nonincreasing and nondecreasing respectively, with $\alpha_0 = \alpha$, $\beta_0 = \beta$, which converge uniformly to the extremal solutions in $[\alpha, \beta]$ of (3.2.8).

Furthermore, the estimate $m = -\frac{\pi}{4T}$ is best possible in the sense that, for every fixed $m < -\frac{\pi}{4T}$, there are problems with its unique solution outside of the interval $[\alpha, \beta]$.

3.2.3 Existence of solutions via Krasnosel'skiĭ's Fixed Point Theorem

In this section we implement the methods used in [120] for the existence of solutions of second order differential equations to prove new existence results for problem

$$x'(t) = f(t,x(-t),x(t)) \;\; \forall t \in I, \quad x(-T) = x(T), \tag{3.2.9}$$

where $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $2T$-periodic on t and an L^1-Carathéodory function, that is, $f(\cdot,u,v)$ is measurable for each fixed u and v and $f(t,\cdot,\cdot)$ is continuous for a. e. $t \in [-T,T]$, and for each $r > 0$, there exists $\varphi_r \in \mathrm{L}^1([-T,T])$ such that

$$f(t,u,v) \le \varphi_r(t) \;\text{ for all }\; (u,v) \in [-r,r]\times[-r,r], \;\text{ and a. e. }\; t \in [-T,T].$$

.

Let us first establish the fixed point theorem we are going to use [120].

Definition 3.2.18. Let $\mathcal{T}$ be a real topological vector space. A cone K in $\mathcal{T}$ is closed set such that is closed under the sum (that is, $x+y \in K$ for all $x,y \in K$), closed under the multiplication by nonnegative scalars (that is $\lambda x \in K$ for all $\lambda \in [0,+\infty), x \in K$) and such that $K \cap (-K) = \{0\}$ (that is, if $x, -x \in K$, then $x = 0$).

Theorem 3.2.19 (Krasnosel'skiĭ). *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let $A : \mathcal{P} \cap (\overline{\Omega_2}\backslash\Omega_1) \to \mathcal{P}$ be a compact and continuous operator such that one of the following conditions is satisfied:*

(1) $\|Au\| \le \|u\|$ if $u \in \mathcal{P} \cap \partial\Omega_1$ and $\|Au\| \ge \|u\|$ if $u \in \mathcal{P} \cap \partial\Omega_2$,

(2) $\|Au\| \ge \|u\|$ if $u \in \mathcal{P} \cap \partial\Omega_1$ and $\|Au\| \le \|u\|$ if $u \in \mathcal{P} \cap \partial\Omega_2$.

Then, A has at least one fixed point in $\mathcal{P} \cap (\overline{\Omega_2} \backslash \Omega_1)$.

In the following, let $m \in \mathbb{R}\backslash\{0\}$ and G be the Green function for problem

$$x'(t) + m\,x(-t) = h(t), \quad x(-T) = x(T).$$

Let $M = \sup\{G(t,s) \ : \ t,s \in I\}, \quad L = \inf\{G(t,s) \ : \ t,s \in I\}$.

Theorem 3.2.20. *Let $m \in (0, \frac{\pi}{4T})$. Assume there exist r, $R \in \mathbb{R}^+$, $r < R$ such that*

$$f(t,x,y) + m\,x \geq 0 \ \ \forall x,y \in \left[\frac{L}{M}r, \frac{M}{L}R\right], \text{ a. e. } t \in I.$$

Then, if one of the following conditions holds,

(1)

$$f(t,x,y) + m\,x \geq \frac{M}{2TL^2}x \ \ \forall x,y \in \left[\frac{L}{M}r, r\right], \text{ a. e. } t \in I,$$
$$f(t,x,y) + m\,x \leq \frac{1}{2TM}x \ \ \forall x,y \in \left[R, \frac{M}{L}R\right], \text{ a. e. } t \in I;$$

(2)

$$f(t,x,y) + m\,x \leq \frac{1}{2TM}x \ \ \forall x,y \in \left[\frac{L}{M}r, r\right], \text{ a. e. } t \in I,$$
$$f(t,x,y) + m\,x \geq \frac{M}{2TL^2}x \ \ \forall x,y \in \left[R, \frac{M}{L}R\right], \text{ a. e. } t \in I;$$

problem (3.2.9) has a positive solution.

If $\mathcal{B} = (\mathcal{C}(I), \|\cdot\|_\infty)$, by defining the absolutely continuous operator $A : \mathcal{B} \to \mathcal{B}$ such that

$$(A\,x)(t) := \int_{-T}^{T} G(t,s)\,[f(s,x(-s),x(s)) + m\,x(-s)], \mathrm{d}s$$

we deduce the result following the same steps as in [167].

We present now two corollaries (analogous to the ones in [167]). The first one is obtained by strengthening the hypothesis and making them easier to check.

Corollary 3.2.21. *Let $m \in (0, \frac{\pi}{4T})$, $f(t,x,y) \geq 0$ for all $x,y \in \mathbb{R}^+$ and a. e. $t \in I$. Then, if one of the following condition holds:*

(1)

$$\lim_{x,y \to 0^+} \frac{f(t,x,y)}{x} = +\infty, \qquad \lim_{x,y \to +\infty} \frac{f(t,x,y)}{x} = 0,$$

(2)

$$\lim_{x,y \to 0^+} \frac{f(t,x,y)}{x} = 0, \qquad \lim_{x,y \to +\infty} \frac{f(t,x,y)}{x} = +\infty$$

uniformly for a. e. $t \in I$, then problem (3.2.9) has a positive solution.

Corollary 3.2.22. *Let $m \in (0, \frac{\pi}{4T})$. Assume there exist $r, R \in \mathbb{R}^+$, $r < R$ such that*

$$f(t,x,y) + m\,x \leq 0 \ \ \forall x,y \in \left[-\frac{M}{L}R, -\frac{L}{M}r\right], \text{ a. e. } t \in I.$$

Then, if one of the following conditions holds,

(1)

$$f(t,x,y) + m\,x \leq \frac{M}{2TL^2}x \ \ \forall x,y \in \left[-r, -\frac{L}{M}r\right], \text{ a. e. } t \in I,$$
$$f(t,x,y) + m\,x \geq \frac{1}{2TM}x \ \ \forall x,y \in \left[-\frac{M}{L}R, -R\right], \text{ a. e. } t \in I;$$

(2)

$$f(t,x,y) + m\,x \geq \frac{1}{2TM}x \ \ \forall x,y \in \left[-r, -\frac{L}{M}r\right], \text{ a. e. } t \in I,$$
$$f(t,x,y) + m\,x \leq \frac{M}{2TL^2}x \ \ \forall x,y \in \left[-\frac{M}{L}R, -R\right], \text{ a. e. } t \in I;$$

problem (3.2.9) has a negative solution.

Similar results to these –with analogous proofs– can be given when the Green's function is negative.

Theorem 3.2.23. *Let $m \in (-\frac{\pi}{4T}, 0)$. Assume there exist $r, R \in \mathbb{R}^+$, $r < R$ such that*

$$f(t,x,y) + m\,x \leq 0 \ \ \forall x,y \in \left[\frac{M}{L}r, \frac{L}{M}R\right], \text{ a. e. } t \in I.$$

Then, if one of the following conditions holds,

(1)

$$f(t,x,y) + m\,x \leq \frac{L}{2TM^2}x \ \ \forall x,y \in \left[\frac{M}{L}r, r\right], \text{ a. e. } t \in I,$$
$$f(t,x,y) + m\,x \geq \frac{1}{2TL}x \ \ \forall x,y \in \left[R, \frac{L}{M}R\right], \text{ a. e. } t \in I;$$

(2)

$$f(t,x,y) + m\,x \geq \frac{1}{2TL}x \ \ \forall x,y \in \left[\frac{M}{L}r, r\right], \text{ a. e. } t \in I,$$
$$f(t,x,y) + m\,x \leq \frac{L}{2TM^2}x \ \ \forall x,y \in \left[R, \frac{L}{M}R\right], \text{ a. e. } t \in I;$$

problem (3.2.9) has a positive solution.

Corollary 3.2.24. *Let $m \in (-\frac{\pi}{4T}, 0)$. Assume there exist $r, R \in \mathbb{R}^+$, $r < R$ such that*

$$f(t,x,y) + m\,x \geq 0 \ \ \forall x,y \in \left[-\frac{L}{M}R, -\frac{M}{L}r\right], \text{ a. e. } t \in I.$$

Then, if one of the following conditions holds,

(1)

$$f(t,x,y)+m\,x \geq \frac{L}{2TM^2}x \;\; \forall x,y\in\left[-r,-\frac{M}{L}r\right], \textit{a. e. } t\in I,$$
$$f(t,x,y)+m\,x \leq \frac{1}{2TL}x \;\; \forall x,y\in\left[-\frac{L}{M}R,-R\right], \textit{a. e. } t\in I;$$

(2)

$$f(t,x,y)+m\,x \leq \frac{1}{2TL}x \;\; \forall x,y\in\left[-r,-\frac{M}{L}r\right], \textit{a. e. } t\in I,$$
$$f(t,x,y)+m\,x \geq \frac{L}{2TM^2}x \;\; \forall x,y\in\left[-\frac{L}{M}R,-R\right], \textit{a. e. } t\in I;$$

problem (3.2.9) has a negative solution.

We could also state analogous corollaries to Corollary 3.2.21 for Theorem 3.2.23 and Corollaries 3.2.22 and 3.2.24.

3.2.4 Examples

We will now analyze two examples to which we can apply the previous results. Observe that both examples do not lie under the hypothesis of the existence results for bounded solutions for differential equations with reflection of the argument in [187] nor in those of the more general results found in [1, 155, 173, 174, 189] or any other existence results known to the authors.

Example 3.2.25. Consider the problem

$$x'(t)=\lambda\,\sinh\,(t-x(-t)),\quad \forall\, t\in I,\quad x(-T)=x(T). \tag{3.2.10}$$

It is easy to check that $\alpha\equiv T$ and $\beta\equiv -T$ are lower and upper solutions for problem (3.2.10) for all $\lambda\geq 0$. Since $f(t,y):=\lambda\,\sinh\,(t-y)$ satisfies that $|\frac{\partial f}{\partial y}(t,y)|\leq\lambda\cosh\,(2\,T)$, for all $(t,y)\in I^2$, we know, from Theorem 3.2.16, that problem (3.2.10) has extremal solutions on $[-T,T]$ for all

$$0\leq\lambda\leq\frac{\pi}{4\,T\,\cosh\,(2\,T)}.$$

Example 3.2.26. Consider the problem

$$x'(t)=t^2x^2(t)\,[\cos^2(x^2(-t))+1]\;\;\forall t\in I,\quad x(-T)=x(T). \tag{3.2.11}$$

By defining $f(t,x,y)$ as the $2T$-periodic extension on t of the function

$$t^2x^2[\cos^2(y^2)+1],$$

we may to apply Corollary 3.2.21 to deduce that problem (3.2.11) has a positive solution. Using the analogous corollary for Corollary 3.2.24, we know that it also has a negative solution.

3.3 The antiperiodic case

As we will see in this section, the antiperiodic case satisfies properties which are analogous to the periodic one.

We consider the antiperiodic problem

$$x'(t) + m\,x(-t) = h(t), \quad x(-T) + x(T) = 0, \tag{3.3.1}$$

we have that the reduced problem for $h \equiv 0$ corresponds with the harmonic oscillator with antiperiodic boundary value conditions

$$x''(t) + m^2 x(t) = 0, \quad x(-T) + x(T) = 0, \quad x'(-T) + x'(T) = 0$$

of which the Green's function, H, is given by the expression

$$2m\cos(mT)H(t,s) = \begin{cases} \sin m(t-s-T) & \text{if } -T \le s \le t \le T, \\ \sin m(s-t-T) & \text{if } -T \le t < s \le T. \end{cases}$$

It is straight forward to check that the following properties are fulfilled.

(A_1) $H \in \mathcal{C}(I^2, \mathbb{R})$.

(A_2) $\frac{\partial H}{\partial t}$ y $\frac{\partial^2 H}{\partial t^2}$ exist and are continuous on $I^2 \backslash D$ where $D := \{(t,s) \in I^2 \, : \, t = s\}$.

Also,

$$2m\cos(mT)\frac{\partial H}{\partial t}(t,s) = \begin{cases} m\cos m(t-s-T) & \text{if } -T \le s \le t \le T, \\ -m\cos m(s-t-T) & \text{if } -T \le t < s \le T, \end{cases}$$

$$\lim_{s\to t^-} 2m\cos(mT)\frac{\partial H}{\partial t}(t,s) = m\cos mT,$$

$$\lim_{s\to t^+} 2m\cos(mT)\frac{\partial H}{\partial t}(t,s) = -m\cos mT,$$

hence

(A_3) $\frac{\partial H}{\partial t}(t,t^-) - \frac{\partial H}{\partial t}(t,t^+) = 1 \;\; \forall t \in I$.

Furthermore, we have the following

(A_4) $\frac{\partial^2 H}{\partial t^2}(t,s) + m^2 H(t,s) = 0 \;\; \forall (t,s) \in I^2 \backslash D$.

(A_5) $\quad a)$ $H(T,s) + H(-T,s) = 0 \;\; \forall s \in I$,

$\quad b)$ $\frac{\partial H}{\partial t}(T,s) + \frac{\partial H}{\partial t}(-T,s) = 0 \;\; \forall s \in I$.

For every $t,\, s \in I$, we have that

(A_6) $H(t,s) = H(s,t)$.

(A_7) $H(t,s) = H(-t,-s)$.

(A_8) $\frac{\partial H}{\partial t}(t,s) = \frac{\partial H}{\partial s}(s,t)$.

(A_9) $\frac{\partial H}{\partial t}(t,s) = -\frac{\partial H}{\partial t}(-t,-s)$.

(A_{10}) $\frac{\partial H}{\partial t}(t,s) = -\frac{\partial H}{\partial s}(t,s)$.

The properties $(A_1) - (A_{10})$ are equivalent to the properties $(I) - (X)$ in the previous section. This allows us to prove the following proposition in an analogous fashion to Proposition 3.2.2.

Proposition 3.3.1. *Assume $m \neq (k+\frac{1}{2})\frac{\pi}{T}$, $k \in \mathbb{Z}$. Then problem (3.3.1) has a unique solution*

$$u(t) := \int_{-T}^{T} \overline{H}(t,s)h(s)\,\mathrm{d}s,$$

where

$$\overline{H}(t,s) := mH(t,-s) - \frac{\partial H}{\partial s}(t,s)$$

is the Green's function relative to problem (3.3.1).

The Green's function $\overline{H}$ has the following explicit expression:

$$2\cos(mT)\overline{H}(t,s) = \begin{cases} \sin m(-T+s+t) + \cos m(-T-s+t) & \text{si} \quad t > |s|, \\ \sin m(-T+s+t) - \cos m(-T+s-t) & \text{si} \quad |t| < s, \\ \sin m(-T-s-t) + \cos m(-T-s+t) & \text{si} \quad -|t| > s, \\ \sin m(-T-s-t) - \cos m(-T+s-t) & \text{si} \quad t < -|s|. \end{cases}$$

The following properties of $\overline{H}$ hold and are equivalent to properties $(I') - (V')$ in the previous section.

(A_1') $\frac{\partial \overline{H}}{\partial t}$ exists and is continuous on $I^2 \backslash D$,

(A_2') $\overline{H}(t,t^-)$ y $\overline{H}(t,t^+)$ exist for all $t \in I$ and satisfy

$$\overline{H}(t,t^-) - \overline{H}(t,t^+) = 1 \;\; \forall t \in I,$$

(A_3') $\frac{\partial \overline{H}}{\partial t}(t,s) + m\overline{H}(-t,s) = 0$ a. e. $t,s \in I$, $s \neq t$,

(A_4') $\overline{H}(T,s) + \overline{H}(-T,s) = 0 \;\; \forall s \in (-T,T)$,

(A_5') $\overline{H}(t,s) = \overline{H}(-s,-t) \;\; \forall t,s \in I$.

Despite the parallelism with the periodic problem, we cannot generalize the maximum and anti-maximum results of [39] because property (A_4') guarantees that $\overline{H}(\cdot,s)$ changes sign for a. e. s and, by property (A_5'), that $\overline{H}(t,\cdot)$ changes sign for a. e. t fixed.

3.3.1 The general case

In this section we study equation $x'(t) + mx(t) = h(t)$ under the conditions imposed by a linear functional F, this is, we study the problem

$$x'(t) + mx(-t) = h(t), \quad F(x) = c, \tag{3.3.2}$$

where $c \in \mathbb{R}$ and $F \in W^{1,1}(I)'$.

Remember that that $W^{1,1}(I) := \{f : I \to \mathbb{R} \,:\, f' \in \mathrm{L}^1(I)\}$ and we denote by $W^{1,1}(I)'$ its dual. Also, we will denote by $C_c(I)$ the space of compactly supported functions on I.

Recall that the solutions of equation $x''(t) + m^2 x(t) = 0$ are parametrized by two real numbers a and b in the following way: $u(t) = a\cos mt + b\sin mt$. Since every solution of equation $x'(t) + mx(-t) = 0$ has to be of this form, if we impose the equation to be satisfied, we obtain a relationship between the parameters: $b = -a$, and hence the solutions of $x'(t) + mx(-t) = 0$ are given by $u(t) = a(\cos mt - \sin mt)$, $a \in \mathbb{R}$.

Observe that $2\sin(mT)\,\overline{G}(t,-T) = \cos mt - \sin mt$, and $\overline{G}(t,-T)$ is the unique solution of the problem

$$x'(t) + mx(-t) = 0, \quad x(-T) - x(T) = 1.$$

Hence, if we look for a solution of the form

$$x(t) = \int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s + \lambda\overline{G}(t,-T), \tag{3.3.3}$$

and impose the condition $F(x) = c$, we have that

$$c = F\left(\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s\right) + \lambda F(\overline{G}(t,-T))$$

and hence, for

$$\lambda = \frac{c - F\left(\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s\right)}{F(\overline{G}(t,-T))},$$

expression (3.3.3) is a solution of problem (3.3.2) as long as $F(\overline{G}(t,-T)) \neq 0$ or, which is the same,

$$F(\cos mt) \neq F(\sin mt).$$

We summarize this argument in the following result.

Corollary 3.3.2. *Assume $m \neq k\pi/T$, $k \in \mathbb{Z}$, $F \in W^{1,1}(I)'$ such that $F(\cos mt) \neq F(\sin mt)$. Then problem* (3.3.2) *has a unique solution given by*

$$u(t) := \int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s + \frac{c - F\left(\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s\right)}{F(\overline{G}(t,-T))}\overline{G}(t,-T), \quad t \in I. \tag{3.3.4}$$

Remark 3.3.3. The condition $m \neq k\pi/T$, $k \in \mathbb{Z}$ together with the rest of the hypothesis of the corollary is sufficient for the existence of a unique solution of problem (3.3.2) but is not necessary, as it has been illustrated in Proposition 3.3.1, because such a condition is only necessary for the existence of $\overline{G}$.

3.4 Examples

We now apply the previous results in order to get some specific applications.

Application 3.4.1. Let $F \in W^{1,1}(I)' \cap C_c(I)'$ and assume $F(\cos m t) \neq F(\sin m t)$. The Riesz Representation Theorem guarantees the existence of a –probably signed– regular Borel measure of bounded variation μ on I such that $F(x) := \int_{-T}^{T} x \,\mathrm{d}\mu$ and $\|F\|_{C_c(I)'} = |\mu|(I)$, where $|\mu|(I)$ is the total variation of the measure μ on I.

Let us compute now an estimate for the value of the solution u at t.

$$\begin{aligned}
|u(t)| &= \left| \int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s + \frac{c - F\left(\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s\right)}{F(\overline{G}(t,-T))} \overline{G}(t,-T) \right| \\
&\leq \sup_{s\in I} |\overline{G}(t,s)| \|h\|_1 + \frac{|c - \int_{-T}^{T}\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s\,\mathrm{d}\mu(t)|}{|F(\overline{G}(t,-T))|} |\overline{G}(t,-T)| \\
&\leq \sup_{s\in I} |\overline{G}(t,s)| \|h\|_1 + \frac{|c| + \sup_{t,s\in I} |\overline{G}(t,s)| |\mu|(I) \|h\|_1}{|F(\overline{G}(t,-T))|} |\overline{G}(t,-T)| \\
&= \left| \frac{c\,\overline{G}(t,-T)}{F(\overline{G}(t,-T))} \right| + \left[\sup_{s\in I} |\overline{G}(t,s)| + \left| \frac{\overline{G}(t,-T)}{F(\overline{G}(t,-T))} \right| \sup_{t,s\in I} |\overline{G}(t,s)| |\mu|(I) \right] \|h\|_1.
\end{aligned}$$

Define operator Ξ as $\Xi(f)(t) := \int_{-T}^{T} \overline{G}(t,s)f(s)\,\mathrm{d}s$. And let us consider, for notational purposes, $\Xi(\delta_{-T})(t) := \overline{G}(t,-T)$. Hence, equation (3.3.4) can be rewritten as

$$u(t) = \Xi(h)(t) + \frac{c - F(\Xi(h))}{F(\Xi(\delta_{-T}))} \Xi(\delta_{-T})(t), \quad t \in I. \tag{3.4.1}$$

Consider now the following lemma.

Lemma 3.4.2 ([34, Lemma 5.5]). *Let $f : [p-c, p+c] \to \mathbb{R}$ be a symmetric function with respect to p, decreasing in $[p, p+c]$. Let $g : [a,b] \to \mathbb{R}$ be a straight line such that $g([a,b]) \subset [p-c, p+c]$. Under these hypothesis, the following hold.*

(1) *If $g(a) < g(b) < p$ or $p < g(b) < g(a)$ then $f(g(a)) < f(g(b))$,*

(2) *if $g(b) < g(a) < p$ or $p < g(a) < g(b)$ then $f(g(a)) > f(g(b))$,*

(3) *if $g(a) < p < g(b)$ then $f(g(a)) < f(g(b))$ if and only if $g(\frac{a+b}{2}) < p$,*

(4) *if $g(b) < p < g(a)$ then $f(g(a)) < f(g(b))$ if and only if $g(\frac{a+b}{2}) > p$.*

Remark 3.4.3. An analogous result can be established, with the proper changes in the inequalities, if f is increasing in $[p, p+c]$.

Proof. It is clear that $f(g(a)) < f(g(b))$ if and only if $|g(a) - p| > |g(b) - p|$, so (1) and (2) are straightforward. Also, realize that, since g is affine, we have that $g\left(\frac{a+b}{2}\right) = \frac{g(a)+g(b)}{2}$.

Let us prove (3) as (4) is analogous:

$$|g(b)-p| - |g(a)-p| = g(b) - p - (p - g(a)) = g(a) + g(b) - 2p = 2\left[g\left(\frac{a+b}{2}\right) - p\right].$$

Therefore $|g(a) - p| > |g(b) - p|$ if and only if $g\left(\frac{a+b}{2}\right) < p$. ■

With this Lemma, we can prove the following proposition.

Proposition 3.4.4. *Assume $\alpha = mT \in (0, \pi/4)$, $F \in W^{1,1}(I)' \cap C_c(I)'$ such that μ is its associated Borel measure and $F(\cos mt) > F(\sin mt)$. Then the solution to problem* (3.3.2) *is positive if*

$$c > \frac{2M|\mu|(I)\|h\|_1}{1-\tan\alpha}. \tag{3.4.2}$$

Proof. Observe that $\Xi(\delta_{-T})(t) > 0 \ \forall t \in I$ for every $\alpha \in (0, \frac{\pi}{4})$ because $F(\cos mt) > F(\sin mt)$. Hence, if we assume that u is positive, solving for c in (3.4.1), we have that

$$c > F(\Xi(h)) - F(\Xi(\delta_{-T}))\frac{\Xi(h)(t)}{\Xi(\delta_{-T})(t)} \quad \forall t \in I.$$

Reciprocally, if this inequality is satisfied, u is positive.

It is easy to check using Lemma 3.4.2, that

$$\min_{t\in I} \overline{G}(t,-T) = \frac{1}{2}(\cot\alpha - 1) \quad \text{and} \quad \max_{t\in I} \overline{G}(t,-T) = \frac{1}{2}(\cot\alpha + 1).$$

Let $M := \max_{t,s\in I} \overline{G}(t,s)$.

Then

$$\begin{aligned} F(\Xi(h)) - F(\Xi(\delta_{-T}))\frac{\Xi(h)(t)}{\Xi(\delta_{-T})(t)} &\leq |F(\Xi(h))| + \left|2F(\Xi(\delta_{-T}))\frac{\Xi(h)(t)}{\cot\alpha - 1}\right| \\ \leq & M|\mu|(I)\|h\|_1 + (\cot\alpha + 1)|\mu|(I)\frac{M\|h\|_1}{\cot\alpha - 1} = \frac{2M|\mu|(I)\|h\|_1}{1-\tan\alpha}. \end{aligned}$$

Thus, a sufficient condition for u to be positive is

$$c > \frac{2M|\mu|(I)\|h\|_1}{1-\tan\alpha} =: k_1.$$

■

Condition (3.4.2) can be excessively strong in some cases, which can be illustrated with the following example.

Example 3.4.5. Let us assume that $F(x) = \int_{-T}^{T} x(t)\,\mathrm{d}t$. For this functional,

$$\frac{2M|\mu|(I)\|h\|_1}{1-\tan\alpha} = \frac{4MT\|h\|_1}{1-\tan\alpha}.$$

In [34, Lemma 5.11], it is proven that $\int_{-T}^{T} \overline{G}(t,s)\,\mathrm{d}t = \frac{1}{m}$. Hence, we have the following

$$\begin{aligned} & F(\Xi(h)) - F(\Xi(\delta_{-T}))\frac{\Xi(h)(t)}{\Xi(\delta_{-T})(t)} \\ =& \int_{-T}^{T}\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s\,\mathrm{d}t - \int_{-T}^{T} \overline{G}(t,-T)\,\mathrm{d}t \frac{\int_{-T}^{T} \overline{G}(t,s)h(s)\,\mathrm{d}s}{\overline{G}(t,-T)} \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{m}\int_{-T}^{T}h(s)\,\mathrm{d}s-\frac{1}{m}\frac{\int_{-T}^{T}\overline{G}(t,s)h(s)\,\mathrm{d}s}{\overline{G}(t,-T)}\\
&\leq\frac{1}{m}\int_{-T}^{T}|h(s)|\,\mathrm{d}s+\frac{1}{m}\frac{\int_{-T}^{T}\overline{G}(t,s)|h(s)|\,\mathrm{d}s}{\overline{G}(t,-T)}\\
&\leq\left(1+\max_{t\in I}\frac{\max_{s\in I}\overline{G}(t,s)}{G(t,-T)}\right)\frac{\|h\|_1}{m}\leq\left(1+\frac{M}{\min_{t\in I}G(t,-T)}\right)\frac{\|h\|_1}{m}\\
&=\left(1+\frac{2M}{\cot\alpha-1}\right)\frac{\|h\|_1}{m}.
\end{aligned}
$$

This provides a new sufficient condition to ensure that $u>0$.

$$
c>\left(1+\frac{2M}{\cot\alpha-1}\right)\frac{\|h\|_1}{m}=:k_2.
$$

Observe that

$$
\frac{k_2}{k_1}=\frac{1+(2M-1)\tan\alpha}{4M\alpha}.
$$

In order to quantify the improvement of the estimate, we have to know the value of M.

Lemma 3.4.6. $M=\frac{1}{2}(1+\csc\alpha)$.

Proof. By [34, Lemma 5.9] we know that, after the change of variable $t=Tz, y=Ts$,

$$
(\sin\alpha)\Phi(y)=\max_{z\in[-1,1]}\overline{G}(z,y)=\begin{cases}\cos\left[\alpha(y-1)+\frac{\pi}{4}\right]\cos\left(\alpha y-\frac{\pi}{4}\right) & \text{if} \quad y\in[0,1],\\ \cos\left(\alpha y+\frac{\pi}{4}\right)\cos\left[\alpha(y+1)-\frac{\pi}{4}\right] & \text{if} \quad y\in[-1,0).\end{cases}
$$

Observe that Φ is symmetric, hence, it is enough to study it on $[0,1]$. Differentiating and equalizing to zero it is easy to check that the maximum is reached at $z=\frac{1}{2}$. ∎

Thus,

$$
f(\alpha):=\frac{k_2}{k_1}=\frac{1}{2\alpha}\cdot\frac{1+\sec\alpha}{1+\csc\alpha}.
$$

f is strictly decreasing on $\left(0,\frac{\pi}{4}\right)$, $f(0^+)=1$ and $f\left(\frac{\pi}{4}^-\right)=\frac{2}{\pi}$.

Example 3.4.7. We give now an example for which we compute the optimal constant c that ensures the solution is positive and compare it to the aforementioned estimate. Consider the problem

$$
x'(t)+x(-t)=e^t,\ t\in\left[-\frac{1}{2},\frac{1}{2}\right],\quad \int_{-\frac{1}{2}}^{\frac{1}{2}}x(s)\,\mathrm{d}s=c. \tag{3.4.3}
$$

For this specific case,

$$
k_2=\frac{\cos\alpha+1}{\cos\alpha-\sin\alpha}\frac{\|h\|_1}{m}=\frac{2\cot\frac{1}{4}\sinh\frac{1}{2}}{\cot\frac{1}{4}-1}=4.91464\ldots
$$

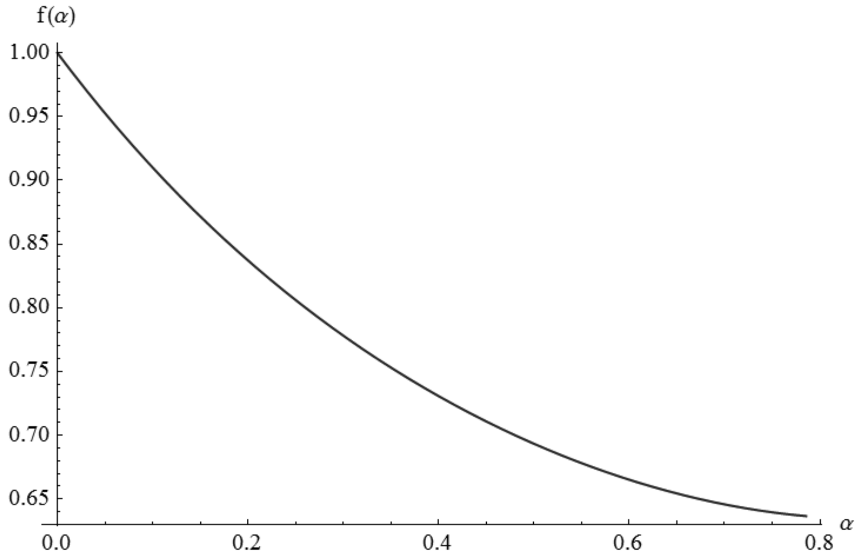


Figure 3.4.1: $\frac{k_2}{k_1}$ as a function of α.

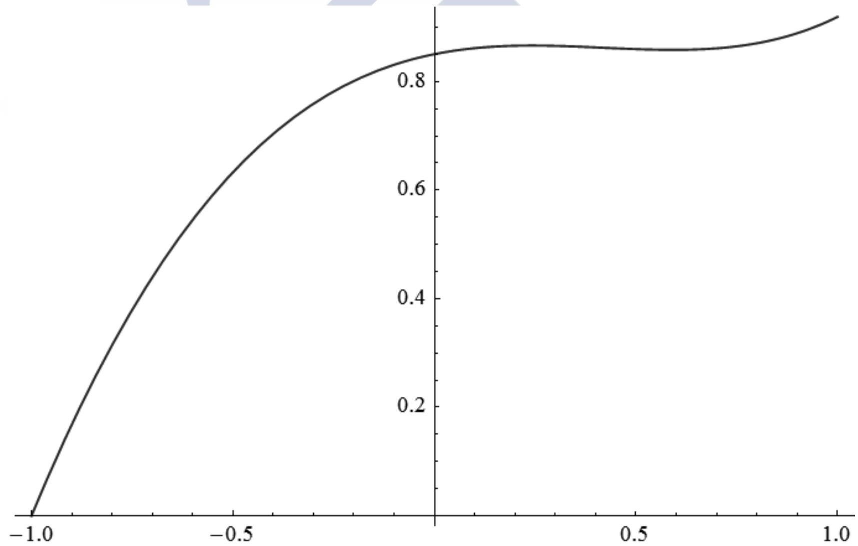


Figure 3.4.2: Solution of problem (3.4.3) for $c = 0.850502\ldots$

Now, using the expression of $\overline{G}$, it is clear that

$$u(t) = \sinh t + \frac{c}{2\sin\frac{1}{2}}(\cos t - \sin t)$$

is the unique solution of problem (3.4.3). It is easy to check that the minimum of the solution is reached at -1 for $c \in [0,1]$. Also that the solution is positive for $c > 2\sin\frac{1}{2}\sinh 1/(\cos 1 + \sin 1) = 0.850502\ldots$, which illustrates that the estimate is far from being optimal.

3.5 Solutions of the initial value problem

In this section we analyze a particular case for the boundary conditions in the previous section: the initial – or, better said, middle point– problem. We will show that this specific case admits an interesting way of constructing the Green's function. The results of the Section follow [43].

3.5.1 The n-th order problem

Consider the following n-th order differential equation with involution with involution

$$Lu := \sum_{k=0}^{n}\left[a_k u^{(k)}(-t) + b_k u^{(k)}(t)\right] = h(t),\ t\in\mathbb{R};\quad u(t_0)=c, \tag{3.5.1}$$

where $h \in L^1_{\text{loc}}(\mathbb{R})$, t_0, c, a_k, $b_k \in \mathbb{R}$ for $k=0,\dots n-1$; $a_n=0$; $b_n=1$. A solution to this problem will be a function $u\in W^{n,1}_{\text{loc}}(\mathbb{R})$, that is, u is k times differentiable in the sense of distributions and each of the derivatives satisfies $u^{k)}|_K \in \mathrm{L}^1(K)$ for every compact set $K\subset\mathbb{R}$ and $k=0,\dots,n$.

Theorem 3.5.1. *Assume that there exist $\tilde u$ and $\tilde v$, functions such that satisfy*

$$\sum_{i=0}^{n-j}\binom{i+j}{j}\left[(-1)^{n+i-1}a_{i+j}\tilde u^{(i)}(-t)+b_{i+j}\tilde u^{(i)}(t)\right]=0,\ t\in\mathbb{R};\ j=0,\dots,n-1, \tag{3.5.2}$$

$$\sum_{i=0}^{n-j}\binom{i+j}{j}\left[(-1)^{n+i}a_{i+j}\tilde v^{(i)}(-t)+b_{i+j}\tilde v^{(i)}(t)\right]=0,\ t\in\mathbb{R};\ j=0,\dots,n-1, \tag{3.5.3}$$

$$(\tilde u_e\tilde v_e-\tilde u_o\tilde v_o)(t)\neq 0,\ t\in\mathbb{R}. \tag{3.5.4}$$

and also one of the following

- $(h1)$ $L\tilde u = 0$ *and* $\tilde u(t_0)\neq 0$,
- $(h2)$ $L\tilde v = 0$ *and* $\tilde v(t_0)\neq 0$,
- $(h3)$ $a_0+b_0\neq 0$ *and* $(a_0+b_0)\int_0^{t_0}(t_0-s)^{n-1}\dfrac{\tilde v(t_0)\tilde u_e(s)-\tilde u(t_0)\tilde v_o(s)}{(\tilde u_e\tilde v_e-\tilde u_o\tilde v_o)(s)}\,\mathrm{d}s\neq 1.$

Then problem (3.5.1) *has a solution.*

Proof. Define

$$\varphi := \frac{h_o\tilde v_e - h_e\tilde v_o}{\tilde u_e\tilde v_e-\tilde u_o\tilde v_o},\quad\text{and}\quad \psi := \frac{h_e\tilde u_e - h_o\tilde u_0}{\tilde u_e\tilde v_e-\tilde u_o\tilde v_o}.$$

Observe that φ is odd, ψ is even and $h=\varphi\tilde u+\psi\tilde v$. So, in order to ensure the existence of solution of problem (3.5.1) it is enough to find y and z such that $Ly=\varphi\tilde u$ and $Lz=\psi\tilde v$ for, in that case, defining $u=y+z$, we can conclude that $Lu=h$. We will deal with the initial condition later on.

Take $y=\tilde\varphi\,\tilde u$, where

$$\tilde\varphi(t) := \int_0^t\int_0^{s_n}\cdots\int_0^{s_2}\varphi(s_1)\,\mathrm{d}s_1\cdots\mathrm{d}s_n = \frac{1}{(n-1)!}\int_0^t(t-s)^{n-1}\varphi(s)\,\mathrm{d}s.$$

Observe that $\tilde{\varphi}$ is even if n is odd and vice-versa. In particular, we have that

$$\tilde{\varphi}^{(j)}(t) = (-1)^{j+n-1}\,\tilde{\varphi}^{(j)}(-t)\,, \quad j = 0, \dots, n.$$

Thus,

$$\begin{aligned}
Ly(t) &= \sum_{k=0}^{n}\left[a_k(\tilde{\varphi}\tilde{u})^{(k)}(-t) + b_k(\tilde{\varphi}\tilde{u})^{(k)}(t)\right] \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\left[(-1)^k a_k\tilde{\varphi}^{(j)}(-t)\tilde{u}^{(k-j)}(-t) + b_k\tilde{\varphi}^{(j)}(t)\tilde{u}^{(k-j)}(t)\right] \\
&= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}\tilde{\varphi}^{(j)}(t)\left[(-1)^{k+j+n-1}a_k\tilde{u}^{(k-j)}(-t) + b_k\tilde{u}^{(k-j)}(t)\right] \\
&= \sum_{j=0}^{n}\tilde{\varphi}^{(j)}(t)\sum_{k=j}^{n}\binom{k}{j}\left[(-1)^{k+j+n-1}a_k\tilde{u}^{(k-j)}(-t) + b_k\tilde{u}^{(k-j)}(t)\right] \\
&= \sum_{j=0}^{n}\tilde{\varphi}^{(j)}(t)\sum_{i=0}^{n-j}\binom{i+j}{j}\left[(-1)^{i+n-1}a_{i+j}\tilde{u}^{(i)}(-t) + b_{i+j}\tilde{u}^{(i)}(t)\right] = \tilde{\varphi}^{(n)}(t)\tilde{u}(t) \\
&= \varphi(t)\tilde{u}(t).
\end{aligned}$$

Hence, $Ly = \varphi\tilde{u}$.

All the same, by taking $z = \tilde{\psi}\tilde{v}$ with $\tilde{\psi}(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\psi(s)\,\mathrm{d}s$, we have that $Lz = \psi\tilde{v}$.

Hence, defining $\bar{u} := y + z = \tilde{\varphi}\,\tilde{u} + \tilde{\psi}\tilde{v}$ we have that $\bar{u}$ satisfies $L\,\bar{u} = h$ and $\bar{u}(0) = 0$.

If we assume $(h1)$,

$$w = \bar{u} + \frac{c - \bar{u}(t_0)}{\tilde{u}(t_0)}\tilde{u}$$

is clearly a solution of problem (3.5.1).

When $(h2)$ is fulfilled a solution of problem (3.5.1) is given by

$$w = \bar{u} + \frac{c - \bar{u}(t_0)}{\tilde{v}(t_0)}\tilde{v}.$$

If $(h3)$ holds, using the aforementioned construction we can find w_1 such that $L\,w_1 = 1$ and $w_1(0) = 0$. Now, $w_2 := w_1 - 1/(a_0 + b_0)$ satisfies $L\,w_2 = 0$. Observe that the second part of condition $(h3)$ is precisely $w_2(t_0) \neq 0$, and hence, defining

$$w = \bar{u} + \frac{c - \bar{u}(t_0)}{w_2(t_0)}w_2$$

we have that w is a solution of problem (3.5.1). ■

Remark 3.5.2. Having in mind condition $(h1)$ in Theorem 3.5.1, it is immediate to verify that $L\,\tilde{u} = 0$ provided that

$$a_i = 0 \text{ for all } i \in \{0, \ldots, n-1\} \text{ such that } n+i \text{ is even.}$$

In an analogous way, for $(h2)$, one can show that $L\tilde{v} = 0$ when

$$a_i = 0 \text{ for all } i \in \{0, \ldots, n-1\} \text{ such that } n+i \text{ is odd.}$$

3.5.2 The first order problem

After proving the general result for the n-th order case, we concentrate our work in the first order problem

$$u'(t) + a\,u(-t) + b\,u(t) = h(t), \text{ for a. e. } t \in \mathbb{R}; \quad u(t_0) = c, \tag{3.5.5}$$

with $h \in \mathrm{L}^1{}_{\mathrm{loc}}(\mathbb{R})$ and $t_0, a, b, c \in \mathbb{R}$. A solution of this problem will be $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$.

In order to do so, we first study the homogeneous equation

$$u'(t) + a\,u(-t) + b\,u(t) = 0,\ t \in \mathbb{R}. \tag{3.5.6}$$

By differentiating and making the proper substitutions we arrive to the equation

$$u''(t) + (a^2 - b^2)u(t) = 0,\ t \in \mathbb{R}. \tag{3.5.7}$$

Let $\omega := \sqrt{|a^2 - b^2|}$. Equation (3.5.7) presents three different cases:

(C1). $a^2 > b^2$. In such a case, $u(t) = \alpha\cos\omega t + \beta\sin\omega t$ is a solution of (3.5.7) for every $\alpha, \beta \in \mathbb{R}$. If we impose equation (3.5.6) to this expression we arrive to the general solution

$$u(t) = \alpha\left(\cos\omega t - \frac{a+b}{\omega}\sin\omega t\right)$$

of equation (3.5.6) with $\alpha \in \mathbb{R}$.

(C2). $a^2 < b^2$. Now, $u(t) = \alpha\cosh\omega t + \beta\sinh\omega t$ is a solution of (3.5.7) for every $\alpha, \beta \in \mathbb{R}$. To get equation (3.5.6) we arrive to the general solution

$$u(t) = \alpha\left(\cosh\omega t - \frac{a+b}{\omega}\sinh\omega t\right)$$

of equation (3.5.6) with $\alpha \in \mathbb{R}$.

(C3). $a^2 = b^2$. In this a case, $u(t) = \alpha t + \beta$ is a solution of (3.5.7) for every $\alpha, \beta \in \mathbb{R}$. So, equation (3.5.6) holds provided that one of the two following cases is fulfilled:

(C3.1). $a = b$, where

$$u(t) = \alpha(1 - 2a\,t)$$

is the general solution of equation (3.5.6) with $\alpha \in \mathbb{R}$, and

(C3.2). $a = -b$, where

$$u(t) = \alpha$$

is the general solution of equation (3.5.6) with $\alpha \in \mathbb{R}$.

Now, according to Theorem 3.5.1, we denote $\tilde{u}, \tilde{v}$ satisfying

$$\tilde{u}'(t) + a\tilde{u}(-t) + b\tilde{u}(t) = 0, \quad \tilde{u}(0) = 1, \tag{3.5.8}$$
$$\tilde{v}'(t) - a\tilde{v}(-t) + b\tilde{v}(t) = 0, \quad \tilde{v}(0) = 1. \tag{3.5.9}$$

Observe that $\tilde{u}$ and $\tilde{v}$ can be obtained from the explicit expressions of the cases (C1)–(C3) by taking $\alpha = 1$.

Remark 3.5.3. Note that if u is in the case (C3.1), v is in the case (C3.2) and vice-versa.

We have now the following properties of functions $\tilde{u}$ and $\tilde{v}$.

Lemma 3.5.4. *For every $t, s \in \mathbb{R}$, the following properties hold.*

(1) $\tilde{u}_e \equiv \tilde{v}_e$, $\tilde{u}_o \equiv k\,\tilde{v}_o$ *for some real constant k almost everywhere,*

(2) $\tilde{u}_e(s)\tilde{v}_e(t) = \tilde{u}_e(t)\tilde{v}_e(s)$, $\tilde{u}_o(s)\tilde{v}_o(t) = \tilde{u}_o(t)\tilde{v}_o(s)$,

(3) $\tilde{u}_e\tilde{v}_e - \tilde{u}_o\tilde{v}_o \equiv 1$.

(4) $\tilde{u}(s)\tilde{v}(-s) + \tilde{u}(-s)\tilde{v}(s) = 2[\tilde{u}_e(s)\tilde{v}_e(s) - \tilde{u}_o(s)\tilde{v}_o(s)] = 2$.

Proof. (1) and (3) can be checked by inspection of the different cases. (2) is a direct consequence of (1). (4) is obtained from the definition of even and odd parts and (3). ■

Now, Theorem 3.5.1 has the following corollary.

Corollary 3.5.5. *Problem* (3.5.5) *has a unique solution if and only if* $\tilde{u}(t_0) \neq 0$.

Proof. Considering Lemma 3.5.4 (3), $\tilde{u}$ and $\tilde{v}$, defined as in (3.5.8) and (3.5.9) respectively, satisfy the hypothesis of Theorem 3.5.1, $(h1)$, therefore a solution exists.

Now, assume w_1 and w_2 are two solutions of (3.5.5). Then $w_2 - w_1$ is a solution of (3.5.6). Hence, $w_2 - w_1$ is of one of the forms covered in the cases (C1)–(C3) and, in any case, a multiple of $\tilde{u}$, that is $w_2 - w_1 = \lambda\,\tilde{u}$ for some $\lambda \in \mathbb{R}$. Also, it is clear that $(w_2 - w_1)(t_0) = 0$, but we have $\tilde{u}(t_0) \neq 0$ as a hypothesis, therefore $\lambda = 0$ and $w_1 = w_2$. This is, problem (3.5.5) has a unique solution.

Assume now that w is a solution of (3.5.5) and $\tilde{u}(t_0) = 0$. Then $w + \lambda\,\tilde{u}$ is also a solution of (3.5.5) for every $\lambda \in \mathbb{R}$, which proves the result. ■

This last Theorem raises an obvious question: In which circumstances $\tilde{u}(t_0) \neq 0$? In order to answer this question, it is enough to study the cases (C1)–(C3). We summarize this study in the following Lemma which can be checked easily.

Lemma 3.5.6. $\tilde{u}(t_0) = 0$ *only in the following cases,*

- *if* $a^2 > b^2$ *and* $t_0 = \frac{1}{\omega}\left(\arctan\frac{\omega}{a+b} + k\pi\right)$ *for some* $k \in \mathbb{Z}$,
- *if* $a^2 < b^2$, $a\,b > 0$[†] *and* $t_0 = \frac{1}{\omega}\operatorname{arctanh}\frac{\omega}{a+b}$,

[†] $a\,b > 0$ is equivalent to $|b - a| < |b + a|$.

- *if $a = b$ and $t_0 = \frac{1}{2a}$.*

Definition 3.5.7. Let $t_1, t_2 \in \mathbb{R}$. We define the *oriented characteristic function* of the pair (t_1, t_2) as

$$\chi_{t_1}^{t_2}(t) := \begin{cases} 1, & t_1 \le t \le t_2, \\ -1, & t_2 \le t < t_1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3.5.8. The previous definition implies that, for any given integrable function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{t_1}^{t_2} f(s)\,\mathrm{d}s = \int_{-\infty}^{\infty} \chi_{t_1}^{t_2}(s) f(s)\,\mathrm{d}s.$$

Also, $\chi_{t_1}^{t_2} = -\chi_{t_2}^{t_1}$.

The following corollary gives us the expression of the Green's function for problem (3.5.5).

Corollary 3.5.9. *Suppose $\tilde{u}(t_0) \neq 0$. Then the unique solution of problem* (3.5.5) *is given by*

$$u(t) := \int_{-\infty}^{\infty} G(t,s)h(s)\,\mathrm{d}s + \frac{c - \bar{u}(t_0)}{\tilde{u}(t_0)}\tilde{u}(t), \quad t \in \mathbb{R},$$

where

$$G(t,s) := \frac{1}{2}\left([\tilde{u}(-s)\tilde{v}(t) + \tilde{v}(-s)\tilde{u}(t)]\chi_0^t(s) + [\tilde{u}(-s)\tilde{v}(t) - \tilde{v}(-s)\tilde{u}(t)]\chi_{-t}^0(s) \right),$$
(3.5.10)

for every $t, s \in \mathbb{R}$.

Proof. First observe that $G(t,\cdot)$ is bounded and of compact support for every fixed $t \in \mathbb{R}$, so the integral $\int_{-\infty}^{\infty} G(t,s)h(s)\,\mathrm{d}s$ is well defined. It is not difficult to verify, for any $t \in \mathbb{R}$, the following equalities:

$$\begin{aligned} u'(t) - \frac{c - \bar{u}(t_0)}{\tilde{u}(t_0)}\tilde{u}'(t) =& \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t [\tilde{u}(-s)\tilde{v}(t) + \tilde{v}(-s)\tilde{u}(t)]\, h(s)\,\mathrm{d}s \right. \\ &\left. + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-t}^0 [\tilde{u}(-s)\tilde{v}(t) - \tilde{v}(-s)\tilde{u}(t)]\, h(s)\,\mathrm{d}s \right) \\ =& \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t [\tilde{u}(-s)\tilde{v}(t) + \tilde{v}(-s)\tilde{u}(t)]\, h(s)\,\mathrm{d}s \right. \\ &\left. + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t [\tilde{u}(s)\tilde{v}(t) - \tilde{v}(s)\tilde{u}(t)]\, h(-s)\,\mathrm{d}s \right) \\ =& h(t) + \frac{1}{2}\left(\int_0^t [\tilde{u}(-s)\tilde{v}'(t) + \tilde{v}(-s)\tilde{u}'(t)]\, h(s)\,\mathrm{d}s \right. \\ &\left. + \int_0^t [\tilde{u}(s)\tilde{v}'(t) - \tilde{v}(s)\tilde{u}'(t)]\, h(-s)\,\mathrm{d}s \right). \end{aligned} \tag{3.5.11}$$

On the other hand,

$$\begin{aligned} &a\left[u(-t) - \frac{c - \bar{u}(t_0)}{\tilde{u}(t_0)}\tilde{u}(-t)\right] + b\left[u(t) - \frac{c - \bar{u}(t_0)}{\tilde{u}(t_0)}\tilde{u}(t)\right] \\ =& \frac{1}{2}a\int_0^{-t} ([\tilde{u}(-s)\tilde{v}(-t) + \tilde{v}(-s)\tilde{u}(-t)]h(s) \end{aligned}$$

$$
\begin{aligned}
&+[\tilde u(s)\tilde v(-t)-\tilde v(s)\tilde u(-t)]h(-s))\,\mathrm{d}s\\
&+\frac{1}{2}b\int_0^t([\tilde u(-s)\tilde v(t)+\tilde v(-s)\tilde u(t)]h(s)+[\tilde u(s)\tilde v(t)-\tilde v(s)\tilde u(t)]h(-s))\,\mathrm{d}s\\
=&-\frac{1}{2}a\int_0^t([\tilde u(s)\tilde v(-t)+\tilde v(s)\tilde u(-t)]h(-s)\\
&+[\tilde u(-s)\tilde v(-t)-\tilde v(-s)\tilde u(-t)]h(s))\,\mathrm{d}s\\
&+\frac{1}{2}b\int_0^t([\tilde u(-s)\tilde v(t)+\tilde v(-s)\tilde u(t)]h(s)+[\tilde u(s)\tilde v(t)-\tilde v(s)\tilde u(t)]h(-s))\,\mathrm{d}s\\
=&\frac{1}{2}\int_0^t(-a[\tilde u(-s)\tilde v(-t)-\tilde v(-s)\tilde u(-t)]+b[\tilde u(-s)\tilde v(t)+\tilde v(-s)\tilde u(t)])h(s)\,\mathrm{d}s\\
&+\frac{1}{2}\int_0^t(-a[\tilde u(s)\tilde v(-t)+\tilde v(s)\tilde u(-t)]+b[\tilde u(s)\tilde v(t)-\tilde v(s)\tilde u(t)])h(-s)\,\mathrm{d}s\\
=&\frac{1}{2}\int_0^t(\tilde u(-s)[-a\tilde v(-t)+b\tilde v(t)]+\tilde v(-s)[a\tilde u(-t)+b\tilde u(t)]h(s)\,\mathrm{d}s\\
&+\frac{1}{2}\int_0^t(\tilde u(s)[-a\tilde v(-t)+b\tilde v(t)]-\tilde v(s)[a\tilde u(-t)+b\tilde u(t)])h(-s)\,\mathrm{d}s\\
=&-\frac{1}{2}\left(\int_0^t(\tilde u(-s)\tilde v'(t)+\tilde v(-s)\tilde u'(t))h(s)\,\mathrm{d}s\right.\\
&\left.+\int_0^t(\tilde u(s)\tilde v'(t)-\tilde v(s)\tilde u'(t))h(-s)\,\mathrm{d}s\right). \qquad (3.5.12)
\end{aligned}
$$

Thus, adding (3.5.11) and (3.5.12), it is clear that $u'(t)+a\,u(-t)+b\,u(t)=h(t)$.

We now check the initial condition.

$$
\begin{aligned}
&u(t_0)=c-\bar u(t_0)+\\
&\frac{1}{2}\int_0^{t_0}([\tilde u(-s)\tilde v(t_0)+\tilde v(-s)\tilde u(t_0)]h(s)+[\tilde u(s)\tilde v(t_0)-\tilde v(s)\tilde u(t_0)]h(-s))\,\mathrm{d}s.
\end{aligned}
$$

It can be directly checked that, for all $t\in\mathbb{R}$,

$$\bar u(t)=\frac{1}{2}\int_0^t([\tilde u(-s)\tilde v(t)+\tilde v(-s)\tilde u(t)]h(s)+[\tilde u(s)\tilde v(t)-\tilde v(s)\tilde u(t)]h(-s))\,\mathrm{d}s,$$

is a solution of problem (3.5.5), which proves the result. ■

Denote now $G_{a,b}$ the Green's function for problem (3.5.5) with constant coefficients a and b. The following Lemma is analogous to Lemma 3.2.6.

Lemma 3.5.10. $G_{a,b}(t,s)=-G_{-a,-b}(-t,-s),\quad$ *for all* $t,s\in I$.

Proof. Let $u(t):=\int_{-\infty}^{\infty}G_{a,b}(t,s)h(s)\,\mathrm{d}s$ be the solution to

$$u'(t)+a\,u(-t)+b\,u(t)=h(t),\quad u(0)=0.$$

Let $v(t):=-u(-t)$. Then $v'(t)-a\,v(-t)-b\,v(t)=h(-t)$, and therefore $v(t)=\int_{-\infty}^{\infty}G_{-a,-b}(t,s)h(-s)\,\mathrm{d}s$.

On the other hand, by definition of v,

$$v(t)=-\int_{-\infty}^{\infty}G_{a,b}(-t,s)h(s)\,\mathrm{d}s=-\int_{-\infty}^{\infty}G_{a,b}(-t,-s)h(-s)\,\mathrm{d}s,$$

therefore we can conclude that $G_{a,b}(t,s)=-G_{-a,-b}(-t,-s)$ for all $t,\ s\in I$. ■

As a consequence of the previous result, we arrive at the following immediate conclusion.

Corollary 3.5.11. *$G_{a,b}$ is positive in I^2 if and only if $G_{-a,-b}$ is negative on I^2.*

3.6 Sign of the Green's Function

In this section we use the above obtained expressions to obtain the explicit expression of the Green's function, depending on the values of the constants a and b. Moreover we study the sign of the function and deduce suitable comparison results.

We separate the study in three cases, taking into consideration the expression of the general solution of equation (3.5.6).

3.6.1 The case (C1)

Now, assume the case $(C1)$, i.e., $a^2 > b^2$. Using equation (3.5.10), we get the following expression of G for this situation:

$$G(t,s) = \left[\cos(\omega(s-t)) + \frac{b}{\omega}\sin(\omega(s-t))\right]\chi_0^t(s) + \frac{a}{\omega}\sin(\omega(s+t))\chi_{-t}^0(s),$$

which we can rewrite as

$$G(t,s) = \begin{cases} \cos\omega(s-t) + \dfrac{b}{\omega}\sin\omega(s-t), & 0 \le s \le t, & \text{(3.6.1a)} \\ -\cos\omega(s-t) - \dfrac{b}{\omega}\sin\omega(s-t), & t \le s \le 0, & \text{(3.6.1b)} \\ \dfrac{a}{\omega}\sin\omega(s+t), & -t \le s < 0, & \text{(3.6.1c)} \\ -\dfrac{a}{\omega}\sin\omega(s+t), & 0 < s \le -t, & \text{(3.6.1d)} \\ 0, & \text{otherwise.} & \text{(3.6.1e)} \end{cases}$$

Studying the expression of G we can obtain maximum and antimaximum principles. In order to do this, we will be interested in those maximal strips (in the sense of inclusion) of the kind $[\alpha,\beta] \times \mathbb{R}$ where G does not change sign depending on the parameters.

So, we are in a position to study the sign of the Green's function in the different triangles of definition. The result is the following:

Lemma 3.6.1. *Assume $a^2 > b^2$ and define*

$$\eta(a,b) := \begin{cases} \dfrac{1}{\sqrt{a^2-b^2}}\arctan\dfrac{\sqrt{a^2-b^2}}{b}, & b > 0, \\ \dfrac{\pi}{2|a|}, & b = 0, \\ \dfrac{1}{\sqrt{a^2-b^2}}\left(\arctan\dfrac{\sqrt{a^2-b^2}}{b} + \pi\right), & b < 0. \end{cases}$$

Then, the Green's function of problem (3.5.5) *is*

- *positive on* $\{(t,s),\ 0<s<t\}$ *if and only if* $t \in (0,\eta(a,b))$,
- *negative on* $\{(t,s),\ t<s<0\}$ *if and only if* $t \in (-\eta(a,-b),0)$.

If $a>0$*, the Green's function of problem* (3.5.5) *is*

- *positive on* $\{(t,s),\ -t<s<0\}$ *if and only if* $t \in (0,\pi/\sqrt{a^2-b^2})$,
- *positive on* $\{(t,s),\ 0<s<-t\}$ *if and only if* $t \in (-\pi/\sqrt{a^2-b^2},0)$,

and, if $a<0$*, the Green's function of problem* (3.5.5) *is*

- *negative on* $\{(t,s),\ -t<s<0\}$ *if and only if* $t \in (0,\pi/\sqrt{a^2-b^2})$,
- *negative on* $\{(t,s),\ 0<s<-t\}$ *if and only if* $t \in (-\pi/\sqrt{a^2-b^2},0)$.

Proof. For $0<b<a$, the argument of the $\sin$ in (3.6.1c) is positive, so (3.6.1c) is positive for $t<\pi/\omega$. On the other hand, it is easy to check that (3.6.1a) is positive as long as $t<\eta(a,b)$.

The rest of the proof continues similarly. ■

As a corollary of the previous result we obtain the following one:

Lemma 3.6.2. *Assume* $a^2>b^2$*. Then,*

- *if* $a>0$*, the Green's function of problem* (3.5.5) *is nonnegative on* $[0,\eta(a,b)]\times\mathbb{R}$,
- *if* $a<0$*, the Green's function of problem* (3.5.5) *is nonpositive on* $[-\eta(a,-b),0]\times\mathbb{R}$,
- *the Green's function of problem* (3.5.5) *changes sign in any other strip not a subset of the aforementioned.*

Proof. The proof follows from the previous result together with the fact that

$$\eta(a,b) \le \frac{\pi}{2\omega} < \frac{\pi}{\omega}.$$

■

Remark 3.6.3. Realize that the strips defined in the previous Lemma are optimal in the sense that G changes sign in a bigger rectangle. The same observation applies to the similar results we will prove for the other cases. This fact implies that we cannot have maximum or anti-maximum principles on bigger intervals for the solution, something that is widely known and which the following results, together with Example 3.6.12, illustrate.

Since $G(t,0)$ changes sign at $t=\eta(a,b)$. It is immediate to verify that, defining function $h_\epsilon(s)=1$ for all $s\in(-\epsilon,\epsilon)$ and $h(s)=0$ otherwise, we have a solution $u(t)$ of problem (3.5.5) that takes the value c for $t=\eta(a,b)+\delta(\epsilon)$ with $\delta(\epsilon)>0$ such that $\lim\limits_{\epsilon\to 0}\delta(\epsilon)=0$. Hence, the estimates are optimal for this case.

However, one can study problems with particular non homogeneous part h for which the solution is positive for a bigger interval. This is shown in the following example.

Example 3.6.4. Consider the problem $x'(t)-5x(-t)+4x(t)=\cos^2 3t, x(0)=0$.

Clearly, we are in the case (C1). For this problem,

$$\begin{aligned}\bar{u}(t) :=& \int_0^t \left[\cos(3(s-t))+\frac{4}{3}\sin(3(s-t))\right]\cos^2 3s\,\mathrm{d}s-\frac{5}{3}\int_{-t}^0\sin(3(s+t))\,\mathrm{d}s\\ =&\frac{1}{18}\left(6\cos 3t+3\cos 6t+2\sin 3t+2\sin 6t-9\right).\end{aligned}$$

$\bar{u}(0)=0$, so $\bar{u}$ is the solution of our problem.

Studying $\bar{u}$, we can arrive to the conclusion that $\bar{u}$ is nonnegative in the interval $[0,\gamma]$, being zero at both ends of the interval and

$$\begin{aligned}\gamma =&\frac{1}{3}\arccos\left(\frac{1}{39}\left[\sqrt[3]{47215-5265\sqrt{41}}+\sqrt[3]{5\left(9443+1053\sqrt{41}\right)}-35\right]\right)\\ =&0.201824\ldots\end{aligned}$$

Also, $\bar{u}(t)<0$ for $t=\gamma+\epsilon$ with $\epsilon\in\mathbb{R}^+$ sufficiently small. Furthermore, as Figure 3.6.1 shows, the solution is periodic of period $2\pi/3$.

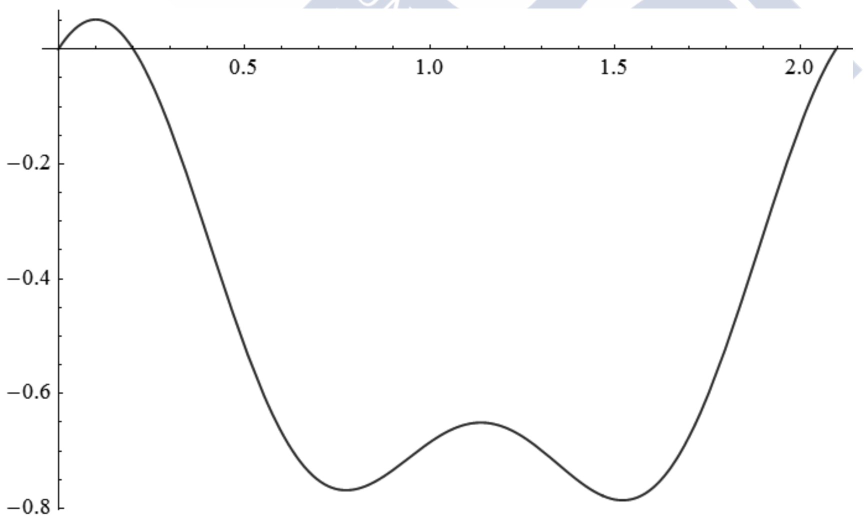


Figure 3.6.1: Graph of the function $\bar{u}$ on the interval $[0,2\pi/3]$. Observe that $\bar{u}$ is positive on $(0,\gamma)$ and negative on $(\gamma,2\pi/3)$.

If we use Lemma 3.6.2, we have that, a priori, $\bar{u}$ is nonpositive on $[-4/15,0]$ which we know is true by the study we have done of $\bar{u}$, but this estimate is, as expected, far from the interval $[\gamma-1,0]$ in which $\bar{u}$ is nonpositive. This does not contradict the optimality of the a priori estimates, as we have shown before, some other examples could be found for which the interval where the solution has constant sign is arbitrarily close to the one given by the *a priori* estimate.

3.6.2 The case (C2)

We study here the case (C2). In this case, it is clear that

$$G(t,s) = \left[\cosh(\omega(s-t)) + \frac{b}{\omega}\sinh(\omega(s-t))\right]\chi_0^t(s) + \frac{a}{\omega}\sinh(\omega(s+t))\chi_{-t}^0(s),$$

which we can rewrite as

$$G(t,s) = \begin{cases} \cosh\omega(s-t) + \frac{b}{\omega}\sinh\omega(s-t), & 0 \le s \le t, \quad \text{(3.6.2a)} \\ -\cosh\omega(s-t) - \frac{b}{\omega}\sinh\omega(s-t), & t \le s \le 0, \quad \text{(3.6.2b)} \\ \frac{a}{\omega}\sinh\omega(s+t), & -t \le s \le 0, \quad \text{(3.6.2c)} \\ -\frac{a}{\omega}\sinh\omega(s+t), & 0 \le s \le -t, \quad \text{(3.6.2d)} \\ 0, & \text{otherwise.} \quad \text{(3.6.2e)} \end{cases}$$

Studying the expression of G we can obtain maximum and antimaximum principles. With this information, we can state the following lemma.

Lemma 3.6.5. *Assume $a^2 < b^2$ and define*

$$\sigma(a,b) := \frac{1}{\sqrt{b^2-a^2}}\operatorname{arctanh}\frac{\sqrt{b^2-a^2}}{b}.$$

Then,

- *if $a > 0$, the Green's function of problem* (3.5.5) *is positive on $\{(t,s),\ -t < s < 0\}$ and $\{(t,s),\ 0 < s < -t\}$,*
- *if $a < 0$, the Green's function of problem* (3.5.5) *is negative on $\{(t,s),\ -t < s < 0\}$ and $\{(t,s),\ 0 < s < -t\}$,*
- *if $b > 0$, the Green's function of problem* (3.5.5) *is negative on $\{(t,s),\ t < s < 0\}$,*
- *if $b > 0$, the Green's function of problem* (3.5.5) *is positive on $\{(t,s),\ 0 < s < t\}$ if and only if $t \in (0, \sigma(a,b))$,*
- *if $b < 0$, the Green's function of problem* (3.5.5) *is positive on $\{(t,s),\ 0 < s < t\}$,*
- *if $b < 0$, the Green's function of problem* (3.5.5) *is negative on $\{(t,s),\ t < s < 0\}$ if and only if $t \in (\sigma(a,b), 0)$.*

Proof. For $0 < a < b$, he argument of the $\sinh$ in (3.6.2d) is negative, so (3.6.2d) is positive. The argument of the $\sinh$ in (3.6.2c) is positive, so (3.6.2c) is positive. It is easy to check that (3.6.2a) is positive as long as $t < \sigma(a,b)$.

On the other hand, (3.6.2b) is always negative.

The rest of the proof continues similarly. ■

As a corollary of the previous result we obtain the following one:

Lemma 3.6.6. *Assume $a^2 < b^2$. Then,*

- *if $0 < a < b$, the Green's function of problem* (3.5.5) *is nonnegative on* $[0, \sigma(a,b)] \times \mathbb{R}$,
- *if $b < -a < 0$, the Green's function of problem* (3.5.5) *is nonnegative on* $[0, +\infty) \times \mathbb{R}$,
- *if $b < a < 0$, the Green's function of problem* (3.5.5) *is nonpositive on* $[\sigma(a,b), 0] \times \mathbb{R}$,
- *if $b > -a > 0$, the Green's function of problem* (3.5.5) *is nonpositive on* $(-\infty, 0] \times \mathbb{R}$,
- *the Green's function of problem* (3.5.5) *changes sign in any other strip not a subset of the aforementioned.*

Example 3.6.7. Consider the problem

$$x'(t) + \lambda x(-t) + 2\lambda x(t) = e^t, \quad x(1) = c \tag{3.6.3}$$

with $\lambda > 0$.

Clearly, we are in the case (C2).

$$\sigma(\lambda, 2\lambda) = \frac{1}{\lambda\sqrt{3}} \ln[7 + 4\sqrt{3}] = \frac{1}{\lambda} \cdot 1.52069\ldots$$

If $\lambda \neq 1/\sqrt{3}$, then

$$\begin{aligned}
\bar{u}(t) := & \int_0^t \left[\cosh(\lambda\sqrt{3}(s-t)) + \frac{2}{\sqrt{3}}\sinh(\lambda\sqrt{3}(s-t))\right] e^s \,\mathrm{d}s \\
& + \frac{1}{\sqrt{3}} \int_{-t}^0 \sinh(\omega(s+t)) e^s \,\mathrm{d}s \\
= & \frac{1}{3\lambda^2 - 1}\left[(\lambda - 1)(\sqrt{3}\sinh(\sqrt{3}\lambda t) - \cosh(\sqrt{3}\lambda t)) + (2\lambda - 1)e^t - \lambda e^{-t}\right],
\end{aligned}$$

$$\tilde{u}(t) = \cosh(\lambda\sqrt{3}t) - \sqrt{3}\sinh(\lambda\sqrt{3}t).$$

With these equalities, it is straightforward to construct the unique solution w of problem (3.6.3). For instance, in the case $\lambda = c = 1$,

$$\bar{u}(t) = \sinh(t),$$

and

$$w(t) = \sinh t + \frac{1 - \sinh 1}{\cosh(\lambda\sqrt{3}) - \sqrt{3}\sinh(\lambda\sqrt{3})}\left(\cosh(\lambda\sqrt{3}t) - \sqrt{3}\sinh(\lambda\sqrt{3}t)\right).$$

Observe that for $\lambda = 1$, $c = \sinh 1$, $w(t) = \sinh t$. Lemma 3.6.6 guarantees the nonnegativity of w on $[0, 1.52069\ldots]$, but it is clear that the solution $w(t) = \sinh t$ is positive on the whole positive real line.

3.6.3 The case (C3)

We study here the case (C3) for $a = b$. In this case, it is clear that

$$G(t,s) = [1 + a(s-t)]\chi_0^t(s) + a(s+t)\chi_{-t}^0(s),$$

which we can rewrite as

$$G(t,s) = \begin{cases} 1 + a(s-t), & 0 \le s \le t, \\ -1 - a(s-t), & t \le s \le 0, \\ a(s+t), & -t \le s \le 0, \\ -a(s+t), & 0 \le s \le -t, \\ 0, & \text{otherwise.} \end{cases}$$

Studying the expression of G we can obtain maximum and antimaximum principles. With this information, we can prove the following Lemma as we did with the analogous ones for cases (C1) and (C2).

Lemma 3.6.8. *Assume $a = b$. Then, if $a > 0$, the Green's function of problem* (3.5.5) *is*

- *positive on $\{(t,s),\ -t < s < 0\}$ and $\{(t,s),\ 0 < s < -t\}$,*
- *negative on $\{(t,s),\ t < s < 0\}$,*
- *positive on $\{(t,s),\ 0 < s < t\}$ if and only if $t \in (0, 1/a)$,*

and, if $a < 0$, the Green's function of problem (3.5.5) *is*

- *negative on $\{(t,s),\ -t < s < 0\}$ and $\{(t,s),\ 0 < s < -t\}$,*
- *positive on $\{(t,s),\ 0 < s < t\}$.*
- *negative on $\{(t,s),\ t < s < 0\}$ if and only if $t \in (1/a, 0)$.*

As a corollary of the previous result we obtain the following one:

Lemma 3.6.9. *Assume $a = b$. Then,*

- *if $0 < a$, the Green's function of problem* (3.5.5) *is nonnegative on $[0, 1/a] \times \mathbb{R}$,*
- *if $a < 0$, the Green's function of problem* (3.5.5) *is nonpositive on $[1/a, 0] \times \mathbb{R}$,*
- *the Green's function of problem* (3.5.5) *changes sign in any other strip not a subset of the aforementioned.*

For this particular case we have another way of computing the solution to the problem.

Proposition 3.6.10. *Let $a = b$ and assume $2at_0 \neq 1$. Let $H(t) := \int_{t_0}^t h(s)\,\mathrm{d}s$ and $\mathcal{H}(t) := \int_{t_0}^t H(s)\,\mathrm{d}s$. Then problem* (3.5.5) *has a unique solution given by*

$$u(t) = H(t) - 2a\mathcal{H}_o(t) + \frac{2at - 1}{2at_0 - 1}c.$$

Proof. The equation is satisfied, since

$$\begin{aligned}&u'(t)+a(u(t)+u(-t))=u'(t)+2au_e(t)\\=&h(t)-2aH_e(t)+\frac{2ac}{2at_0-1}+2aH_e(t)-\frac{2ac}{2at_0-1}=h(t).\end{aligned}$$

The initial condition is also satisfied for, clearly, $u(t_0)=c$. The uniqueness of solution is derived from the fact that $2at_0\neq 1$ and Lemma 3.5.6. ■

Example 3.6.11. Consider the problem $x'(t)+\lambda(x(t)-x(-t))=|t|^p$, $x(0)=1$ for $\lambda,p\in\mathbb{R}$, $p>-1$. For $p\in(-1,0)$ we have a singularity at 0. We can arrive to the solution

$$u(t)=\frac{1}{p+1}t|t|^p+1-2\lambda t,$$

where $\bar{u}(t)=\frac{1}{p+1}t|t|^p$ and $\tilde{u}(t)=1-2\lambda t$.

$\bar{u}$ is positive in $(0,+\infty)$ and negative in $(-\infty,0)$ independently of λ, so the solution has better properties than the ones guaranteed by Lemma 3.6.9.

The next example shows that the estimate is sharp.

Example 3.6.12. Consider the problem

$$u_\epsilon'(t)+u_\epsilon(t)+u_\epsilon(-t)=h_\epsilon(t),\ t\in\mathbb{R};\quad u_\epsilon(0)=0, \tag{3.6.4}$$

where $\epsilon\in\mathbb{R}$, $h_\epsilon(t)=12t(\epsilon-t)\chi_{[0,\epsilon]}(t)$ and $\chi_{[0,\epsilon]}$ is the characteristic function of the interval $[0,\epsilon]$. Observe that h_ϵ is continuous. By means of the expression of the Green's function for problem (3.6.4), we have that its unique solution is given by

$$u_\epsilon(t)=\begin{cases}-2\epsilon^3t-\epsilon^4, & \text{if}\quad t<-\epsilon,\\ -t^4-2\epsilon t^3, & \text{if}\quad -\epsilon<t<0,\\ t^4-(4+2\epsilon)t^3+6\epsilon t^2, & \text{if}\quad 0<t<\epsilon,\\ -2\epsilon^3t+2\epsilon^3+\epsilon^4, & \text{if}\quad t>\epsilon.\end{cases}$$

The a priory estimate on the solution tells us that u_ϵ is nonnegative at least in $[0,1]$. Studying the function u_ϵ, it is easy to check that u_ϵ is zero at 0 and $1+\epsilon/2$, positive in $(-\infty,1+\epsilon/2)\backslash\{0\}$ and negative in $(1+\epsilon/2,+\infty)$.

The case (C3.2) is very similar,

$$G(t,s)=\begin{cases}1+a(t-s), & 0\le s\le t,\\ -1-a(t-s), & t\le s\le 0,\\ a(s+t), & -t\le s\le 0,\\ -a(s+t), & 0\le s\le -t,\\ 0, & \text{otherwise}.\end{cases}$$

Lemma 3.6.13. *Assume $a=-b$. Then, if $a>0$, the Green's function of problem* (3.5.5) *is*

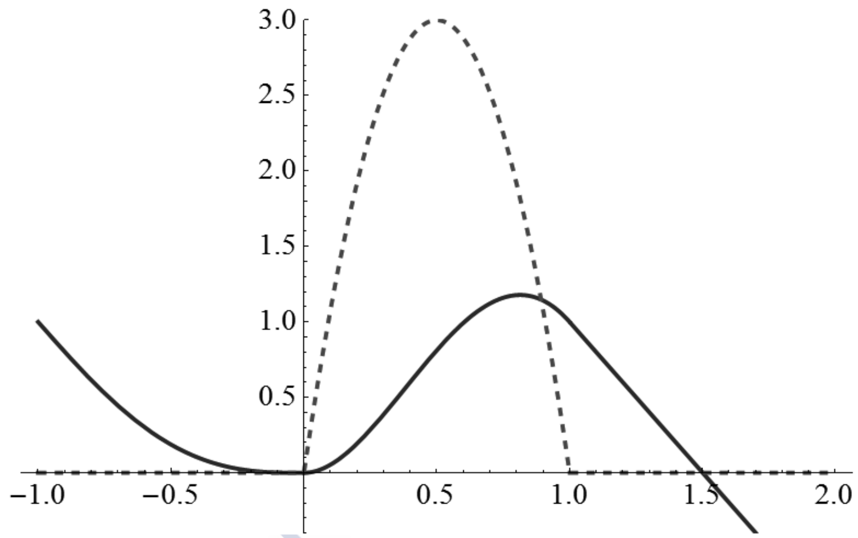


Figure 3.6.2: Graph of the function u_1 and h_1 (dashed). Observe that u becomes zero at $t = 1 + \epsilon/2 = 3/2$.

- *positive on* $\{(t,s),\ -t < s < 0\}$*,* $\{(t,s),\ 0 < s < t\}$ *and* $\{(t,s),\ 0 < s < -t\}$*,*
- *negative on* $\{(t,s),\ t < s < 0\}$ *if and only if* $t \in (-1/a, 0)$*,*

and, if $a > 0$*, the Green's function of problem* (3.5.5) *is*

- *negative on* $\{(t,s),\ -t < s < 0\}$*,* $\{(t,s),\ t < s < 0\}$ *and* $\{(t,s),\ 0 < s < -t\}$*,*
- *positive on* $\{(t,s),\ 0 < s < t\}$ *if and only if* $t \in (0, -1/a)$*.*

As a corollary of the previous result we obtain the following one:

Lemma 3.6.14. *Assume* $a = -b$*. Then,*

- *if* $a > 0$*,the Green's function of problem* (3.5.5) *is nonnegative on* $[0, +\infty) \times \mathbb{R}$*,*
- *if* $a < 0$ *the Green's function of problem* (3.5.5) *is nonpositive on* $(-\infty, 0] \times \mathbb{R}$*,*
- *the Green's function of problem* (3.5.5) *changes sign in any other strip not a subset of the aforementioned.*

Again, for this particular case we have another way of computing the solution to the problem.

Proposition 3.6.15. *Let* $a = -b$*,* $H(t) := \int_0^t h(s)\,\mathrm{d}s$ *and* $\mathcal{H}(t) := \int_0^t H(s)\,\mathrm{d}s$*. Then problem* (3.5.5) *has a unique solution given by*

$$u(t) = H(t) - H(t_0) - 2a(\mathcal{H}_e(t) - \mathcal{H}_e(t_0)) + c.$$

Proof. The equation is satisfied, since

$$u'(t) + a(u(t) - u(-t)) = u'(t) + 2au_o(t) = h(t) - 2aH_o(t) + 2aH_o(t) = h(t).$$

The initial condition is also satisfied for, clearly, $u(t_0) = c$. ■

Example 3.6.16. Consider the problem

$$x'(t) + \lambda(x(-t) - x(t)) = \frac{\lambda t^2 - 2t + \lambda}{(1+t^2)^2},\ x(0) = \lambda$$

for $\lambda \in \mathbb{R}$. We can apply the theory in order to get the solution

$$u(t) = \frac{1}{1+t^2} + \lambda(1+2\lambda t)\arctan t - \lambda^2 \ln(1+t^2) + \lambda - 1$$

where $\bar{u}(t) = \frac{1}{1+t^2} + \lambda(1+2\lambda t)\arctan t - \lambda^2 \ln(1+t^2) - 1.$

Observe that the real function

$$h(t) := \frac{\lambda t^2 - 2t + \lambda}{(1+t^2)^2}$$

is positive on $\mathbb{R}$ if $\lambda > 1$ and negative on $\mathbb{R}$ for all $\lambda < -1$. Therefore, Lemma 3.6.14 guarantees that $\bar{u}$ will be positive on $(0, \infty)$ for $\lambda > 1$ and in $(-\infty, 0)$ when $\lambda < -1$.

4. The nonconstant case

In the previous chapter we dealt with order one differential equations with reflection, constant coefficients and different boundary conditions. Now, following [41] we reduce a new, more general problem containing nonconstant coefficients and arbitrary differentiable involutions, to the one studied in Chapter 3. As we will see, we will do this in three steps. First we add a term depending on $x(t)$ which does not change much with respect to the previous situations. Then, moving from the reflection to a general involution is fairly simple using some of the knowledge gathered in Chapter 1.

The last step, changing from constant to nonconstant coefficients, is another matter. In the nonconstant case computing the Green's function gets trickier and it is only possible in some situations. We use a special change of variable (only valid in some cases) that allows the obtaining the Green's function of problems with nonconstant coefficients from the Green's functions of constant-coefficient analogs.

4.1 Order one linear problems with involutions

Assume φ is a differentiable involution on a compact interval $J_1 \subset \mathbb{R}$. Let $a, b, c, d \in \mathrm{L}^1(J_1)$ and consider the following problem

$$d(t)x'(t) + c(t)x'(\varphi(t)) + b(t)x(t) + a(t)x(\varphi(t)) = h(t),\ x(\inf J_1) = x(\sup J_1). \tag{4.1.1}$$

It would be interesting to know under what circumstances problem (4.1.1) is equivalent to another problem of the same kind but with a different involution, in particular the reflection. The following corollary of Lemma 1.2.14 will help us to clarify this situation.

Corollary 4.1.1 (CHANGE OF INVOLUTION). *Under the hypothesis of Lemma 1.2.14, problem* (4.1.1) *is equivalent to*

$$\frac{d(f(s))}{f'(s)}y'(s) + \frac{c(f(s))}{f'(\psi(s))}y'(\psi(s)) + b(f(s))y(s) + a(f(s))y(\psi(s)) = h(f(s)),$$
$$y(\inf J_2) = y(\sup J_2). \tag{4.1.2}$$

Proof. Consider the change of variable $t = f(s)$ and $y(s) := x(t) = x(f(s))$. Then, using Lemma 1.2.14, it is clear that

$$\frac{\mathrm{d}y}{\mathrm{d}s}(s) = \frac{\mathrm{d}x}{\mathrm{d}t}(f(s))\frac{\mathrm{d}f}{\mathrm{d}s}(s) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}s}(\psi(s)) = \frac{\mathrm{d}x}{\mathrm{d}t}(\varphi(f(s)))\frac{\mathrm{d}f}{\mathrm{d}s}(\psi(s)).$$

Making the proper substitutions in problem (4.1.1) we get problem (4.1.2) and vice-versa. ■

This last results allows us to restrict our study of problem (4.1.1) to the case where φ is the reflection $\varphi(t) = -t$.

Now, take $T \in \mathbb{R}^+, I := [-T, T]$. Equation (4.1.1), for the case $\varphi(t) = -t$, can be reduced to the following system

$$\Lambda \begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix} \begin{pmatrix} x_o \\ x_e \end{pmatrix} + \begin{pmatrix} h_e \\ h_o \end{pmatrix},$$

where

$$\Lambda = \begin{pmatrix} c_e + d_e & d_o - c_o \\ c_o + d_o & d_e - c_e \end{pmatrix}.$$

To see this, just compute the even and odd parts of both sides of the equation taking into account Corollary 1.1.7.

Now, if $\det(\Lambda(t)) = d(t)d(-t) - c(t)c(-t) \neq 0$ for a. e. $t \in I$, $\Lambda(t)$ is invertible a. e. and

$$\begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix} \begin{pmatrix} x_o \\ x_e \end{pmatrix} + \Lambda^{-1} \begin{pmatrix} h_e \\ h_o \end{pmatrix}.$$

So the general case where $c \not\equiv 0$ is reduced to the case $c = 0$, taking

$$\Lambda^{-1} \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix}$$

as coefficient matrix.

Hence, in the following section we will further restrict our assumptions to the case where $c \equiv 0$ in problem (4.1.1).

4.2 Study of the homogeneous equation

In this section we will study some different cases for the homogeneous equation

$$x'(t) + a(t)x(-t) + b(t)x(t) = 0, \ t \in I, \tag{4.2.1}$$

where $a, b \in \mathrm{L}^1(I)$. The solutions of equation (4.2.1) satisfy

$$\begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix} \begin{pmatrix} x_o \\ x_e \end{pmatrix}. \tag{4.2.2}$$

Realize that, a priori, solutions of system (4.2.2) need not to be pairs of even and odd functions, nor provide solutions of (4.2.1).

In order to solve this system, we will restrict problem (4.2.2) to those cases where the matrix

$$M(t) = \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix}(t)$$

satisfies that $[M(t), M(s)] := M(t)M(s) - M(s)M(t) = 0 \ \ \forall t, s \in I$, for in that case, the solution of the system (4.2.2) is given by the exponential of the integral of M. To see this, we have to present a definition and a related result [119].

Definition 4.2.1. Let $S \subset \mathbb{R}$ be an interval. Define $\mathcal{M} \subset \mathcal{C}^1(\mathbb{R}, \mathcal{M}_{n\times n}(\mathbb{R}))$ such that for every $M \in \mathcal{M}$,

- there exists $P \in \mathcal{C}^1(\mathbb{R}, \mathcal{M}_{n\times n}(\mathbb{R}))$ such that $M(t) = P^{-1}(t)J(t)P(t)$ for every $t \in S$ where $P^{-1}(t)J(t)P(t)$ is a Jordan decomposition of $M(t)$;
- the superdiagonal elements of J are independent of t, as well as the dimensions of the Jordan boxes associated to the different eigenvalues of M;
- two different Jordan boxes of J correspond to different eigenvalues;
- if two eigenvalues of M are ever equal, they are identical in the whole interval S.

Theorem 4.2.2 ([119]). *Let $M \in \mathcal{M}$. Then, the following statements are equivalent.*

- *M commutes with its derivative.*
- *M commutes with its integral.*
- *M commutes functionally, that is $M(t)M(s) = M(s)M(t)$ for all $t, s \in S$.*
- *$M = \sum_{k=0}^{r} \gamma_k(t)C^k$ for some $C \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $\gamma_k \in \mathcal{C}^1(S, \mathbb{R})$, $k = 1, \dots, r$.*

Furthermore, any of the last properties imply that $M(t)$ has a set of constant eigenvectors, i.e. a Jordan decomposition $P^{-1}J(t)P$ where P is constant.

Even though the coefficients a and b may in general not have enough regularity to apply Theorem 4.2.2, we will see that we can obtain a basis of constant eigenvectors whenever the matrix M functionally commutes. That, as we will see, is enough for the solution of the system (4.2.2) to be given by the exponential of the integral of M.

Observe that,

$$[M(t), M(s)] = 2\begin{pmatrix} a_e(t)b_e(s)-a_e(s)b_e(t) & a_o(s)[a_e(t)+b_e(t)]-a_o(t)[a_e(s)+b_e(s)] \\ a_o(t)[a_e(s)+b_e(s)]-a_o(s)[a_e(t)+b_e(t)] & a_e(s)b_e(t)-a_e(t)b_e(s) \end{pmatrix}.$$

Let $A(t) := \int_0^t a(s)\,\mathrm{d}s$, $B(t) := \int_0^t b(s)\,\mathrm{d}s$. Let $\overline{M}$ be a primitive (save possibly a constant matrix) of M,that is, the matrix,

$$M = \begin{pmatrix} A_e - B_e & -A_o - B_o \\ A_o - B_o & -A_e - B_e \end{pmatrix}.$$

We study now the different cases where $[M(t), M(s)] = 0 \;\; \forall t, s \in I$. We will always assume $a \not\equiv 0$, since the case $a \equiv 0$ is the well-known case of an ordinary differential equation. Let us see the different possible cases.

(D1). $b_e = k\,a,\ k \in \mathbb{R},\ |k| < 1$. In this case, $a_o = 0$ and $\overline{M}$ has the form

$$\overline{M} = \begin{pmatrix} B_e & -(1+k)A_o \\ (1-k)A_o & -B_e \end{pmatrix}.$$

$\overline{M}$ has two complex conjugate eigenvalues. What is more, both M and $\overline{M}$ functionally commute, and they have a basis of constant eigenvectors given by the constant matrix

$$Y := \begin{pmatrix} i\sqrt{1-k^2} & -i\sqrt{1-k^2} \\ k-1 & k-1 \end{pmatrix}.$$

We have that

$$Y^{-1}\overline{M}(t)Y = Z(t) := \begin{pmatrix} -B_e - i\sqrt{1-k^2}A_o & 0 \\ 0 & -B_e + i\sqrt{1-k^2}A_o \end{pmatrix}.$$

Hence,

$$\begin{aligned} e^{\overline{M}(t)} =& e^{YZ(t)Y^{-1}} = Ye^{Z(t)}Y^{-1} \\ =& e^{-B_e(t)} \begin{pmatrix} \cos\left(\sqrt{1-k^2}A(t)\right) & -\frac{1+k}{\sqrt{1-k^2}} \sin\left(\sqrt{1-k^2}A(t)\right) \\ \frac{\sqrt{1-k^2}}{1+k} \sin\left(\sqrt{1-k^2}A(t)\right) & \cos\left(\sqrt{1-k^2}A(t)\right) \end{pmatrix}. \end{aligned}$$

Therefore, if a solution to equation (4.2.1) exists, it has to be of the form

$$u(t) = \alpha e^{-B_e(t)} \cos\left(\sqrt{1-k^2}A(t)\right) + \beta e^{-B_e(t)} \frac{1+k}{\sqrt{1-k^2}} \sin\left(\sqrt{1-k^2}A(t)\right).$$

with $\alpha, \beta \in \mathbb{R}$. It is easy to check that all the solutions of equation (4.2.1) are of this form with $\beta = -\alpha$.

(D2). $b_e = k\,a$, $k \in \mathbb{R}$, $|k| > 1$. This case is much similar to (D1) In this case $\overline{M}$ has again the form

$$\overline{M} = \begin{pmatrix} B_e & -(1+k)A_o \\ (1-k)A_o & -B_e \end{pmatrix}.$$

$\overline{M}$ has two real eigenvalues and a basis of constant eigenvectors given by the constant matrix

$$Y := \begin{pmatrix} \sqrt{k^2-1} & -\sqrt{k^2-1} \\ k-1 & k-1 \end{pmatrix}.$$

We have that

$$Y^{-1}\overline{M}(t)Y = Z(t) := \begin{pmatrix} -B_e - \sqrt{k^2-1}A_o & 0 \\ 0 & -B_e + \sqrt{k^2-1}A_o \end{pmatrix}.$$

And so,

$$\begin{aligned} e^{\overline{M}(t)} =& e^{YZ(t)Y^{-1}} = Ye^{Z(t)}Y^{-1} \\ =& e^{-B_e(t)} \begin{pmatrix} \cosh\left(\sqrt{1-k^2}A(t)\right) & -\frac{1+k}{\sqrt{k^2-1}} \sinh\left(\sqrt{1-k^2}A(t)\right) \\ \frac{\sqrt{1-k^2}}{1+k} \sinh\left(\sqrt{k^2-1}A(t)\right) & \cosh\left(\sqrt{1-k^2}A(t)\right) \end{pmatrix}. \end{aligned}$$

Therefore, it yields solutions of system (4.2.2) of the form

$$u(t) = \alpha e^{-B_e(t)} \cosh\left(\sqrt{k^2-1}A(t)\right) + \beta e^{-B_e(t)} \frac{1+k}{\sqrt{k^2-1}} \sinh\left(\sqrt{k^2-1}A(t)\right),$$

which are solutions of equation (4.2.1) when $\beta = -\alpha$.

(D3). $b_e = a$.

$$\overline{M} = \begin{pmatrix} B_e & -(1+k)A_o \\ 0 & -B_e \end{pmatrix}.$$

Since the matrix is triangular, we can solve sequentially for x_e and x_o. In this case the solutions of system (4.2.2) are of the form

$$u(t) = \alpha e^{-B_e(t)} + 2\beta e^{-B_e(t)} A(t) \tag{4.2.3}$$

which are solutions of equation (4.2.1) when $\beta = -\alpha$.

(D4). $b_e = -a$.

$$\overline{M} = \begin{pmatrix} B_e & 0 \\ (1-k)A_o & -B_e \end{pmatrix}.$$

We can solve sequentially for x_o and x_e and the solutions of system (4.2.2) are the same as in case (D3), but they are solutions of equation (4.2.1) when $\beta = 0$.

(D5). $b_e = a_e = 0$.

$$M = \begin{pmatrix} A_e - B_e & 0 \\ 0 & -A_e - B_e \end{pmatrix}$$

In this case the solutions of system (4.2.2) are of the form

$$u(t) = \alpha e^{A(t)-B(t)} + \beta e^{-A(t)-B(t)},$$

which are solutions of equation (4.2.1) when $\alpha = 0$.

Remark 4.2.3. Observe that functional matrices appearing in cases (D1)–(D5) belong to $\mathcal{M}$.

4.3 The cases (D1)–(D3) for the complete problem

In the more complicated setting of the following nonhomogeneous problem

$$x'(t) + a(t)\,x(-t) + b(t)\,x(t) = h(t), \; a.e.\, t \in I, \quad x(-T) = x(T), \tag{4.3.1}$$

we have still that, in the cases (D1)–(D3), it can be sorted out very easily. In fact, we get the expression of the Green's function for the operator. We remark that in the three considered cases along this section the function a must be even on I. We note also that a is allowed to change its sign on I.

First, we are going to prove a generalization of Proposition 3.2.2.

Consider problem (4.3.1) with a and b constants.

$$x'(t) + a\,x(-t) + b\,x(t) = h(t), \; t \in I, \quad x(-T) = x(T). \tag{4.3.2}$$

Considering the homogeneous case ($h = 0$), differentiating and making proper substitutions, we arrive to the problem.

$$x''(t) + (a^2 - b^2)x(t) = 0, \; t \in I, \quad x(-T) = x(T), \quad x'(-T) = x'(T). \tag{4.3.3}$$

Which, for $b^2 < a^2$, is the problem of the harmonic oscillator.

It was shown in Section 3.2 that, under uniqueness conditions, the Green's function G for problem (4.3.3) (that is, problem (3.2.2) satisfies the following properties in the case $b^2 < a^2$), but they can be extended almost automatically to the case $b^2 > a^2$.

Lemma 4.3.1. *The Green's function G related to problem* (4.3.3)*, satisfies the following properties.*

(I) $\quad G \in \mathcal{C}(I^2, \mathbb{R})$,

(II) $\quad \frac{\partial G}{\partial t}$ *and* $\frac{\partial^2 G}{\partial t^2}$ *exist and are continuous in* $\{(t,s) \in I^2 \mid s \neq t\}$,

(III) $\quad \frac{\partial G}{\partial t}(t,t^-)$ *and* $\frac{\partial G}{\partial t}(t,t^+)$ *exist for all* $t \in I$ *and satisfy*

$$\frac{\partial G}{\partial t}(t,t^-) - \frac{\partial G}{\partial t}(t,t^+) = 1 \ \ \forall t \in I,$$

(IV) $\quad \frac{\partial^2 G}{\partial t^2} + (a^2 - b^2)G = 0$ *in* $\{(t,s) \in I^2 \mid s \neq t\}$,

(V) $\quad (a)\quad G(T,s) = G(-T,s) \ \ \forall s \in I$,

$\quad\quad (b)\quad \frac{\partial G}{\partial t}(T,s) = \frac{\partial G}{\partial t}(-T,s) \ \ \forall s \in (-T,T)$.

(VI) $\quad G(t,s) = G(s,t)$,

(VII) $\quad G(t,s) = G(-t,-s)$,

$(VIII)$ $\quad \frac{\partial G}{\partial t}(t,s) = \frac{\partial G}{\partial s}(s,t)$,

(IX) $\quad \frac{\partial G}{\partial t}(t,s) = -\frac{\partial G}{\partial t}(-t,-s)$,

(X) $\quad \frac{\partial G}{\partial t}(t,s) = -\frac{\partial G}{\partial s}(t,s)$.

With these properties, we can prove the following Theorem in the same way we proved Theorem 3.2.2.

Theorem 4.3.2. *Suppose that $a^2 - b^2 \neq n^2\,(\pi/T)^2$, $n = 0, 1, \dots$ Then problem* (4.3.2) *has a unique solution given by the expression*

$$u(t) := \int_{-T}^{T} \overline{G}(t,s) h(s)\, \mathrm{d}s,$$

where

$$\overline{G}(t,s) := a\,G(t,-s) - b\,G(t,s) + \frac{\partial G}{\partial t}(t,s)$$

is called the Green's function *related to problem* (4.3.2)*.*

This last theorem leads us to the question “Which is the Green's function for the case (D3) with a, b constants?”. The following Lemma answers that question.

Lemma 4.3.3. *Let $a \neq 0$ be a constant and let G_{D3} be a real function defined as*

$$G_{D3}(t,s) := \frac{t-s}{2} - a\,s\,t + \begin{cases} -\frac{1}{2} + a\,s & \text{if } |s| < t, \\ \frac{1}{2} - a\,s & \text{if } |s| < -t, \\ \frac{1}{2} + a\,t & \text{if } |t| < s, \\ -\frac{1}{2} - a\,t & \text{if } |t| < -s. \end{cases}$$

Then the following properties hold.

- $\frac{\partial G_{D3}}{\partial t}(t,s) + a(G_{D3}(t,s) + G_{D3}(-t,s)) = 0$ *for a. e.* $t,s \in (-1,1)$.
- $\frac{\partial G_{D3}}{\partial t}(t,t^+) - \frac{\partial G_{D3}}{\partial t}(t,t^-) = 1 \;\; \forall t \in (-1,1)$.
- $G_{D3}(-1,s) = G_{D3}(1,s) \;\; \forall s \in (-1,1)$.

These properties are straightforward to check. Consider the following problem

$$x'(t) + a[x(t) + x(-t)] = h(t), t \in [-1,1]; \quad x(1) = x(-1). \tag{4.3.4}$$

In case of having a solution, it is unique, for if u, v are solutions, $u - v$ is in the case $(D3)$ for equation (4.2.1), that is, $(u-v)(t) = \alpha(1-2at)$. Since $(u-v)(-T) = (u-v)(T), u = v$. With this and Lemma 4.3.3 in mind, G_{D3} is the Green's function for the problem (4.3.4), that is, the Green's function for the case (D3) with a, b constants and $T = 1$. For other values of T, it is enough to make a change of variables $\tilde{t} = Tt, \tilde{s} = Ts$.

Remark 4.3.4. The function G_{D3} can be obtained from the Green's functions for the case (D1) with a constant, $b_o \equiv 0$ and $T = 1$ taking the limit $k \to 1^-$ for $T = 1$.

The following theorem shows how to obtain a Green's function for non constant coefficients of the equation using the Green's function for constant coefficients. We can find the same principle, that is, to compose a Green's function with some other function in order to obtain a new Green's function, in [29, Theorem 5.1, Remark 5.1] and also in [74, Section 2].

But first, we need to know how the Green's function should be defined in such a case. Theorem 4.3.2 gives us the expression of the Green's function for problem (4.3.2), $\overline{G}(t,s) := a\,G(t,-s) - b\,G(t,s) + \frac{\partial G}{\partial t}(t,s)$. For instance, in the case (D1), if $\omega = \sqrt{|a^2 - b^2|}$, we have that

$$\begin{aligned} &2\omega\sin(\omega T)\overline{G}(t,s) \\ &:= \begin{cases} a\cos[\omega(s+t-T)] + b\cos[\omega(s-t+T)] + \omega\sin[\omega(s-t+T)], & t > |s|, \\ a\cos[\omega(s+t-T)] + b\cos[\omega(-s+t+T)] - \omega\sin[\omega(-s+t+T)], & s > |t|, \\ a\cos[\omega(s+t+T)] + b\cos[\omega(-s+t+T)] - \omega\sin[\omega(-s+t+T)], & -t > |s|, \\ a\cos[\omega(s+t+T)] + b\cos[\omega(s-t+T)] + \omega\sin[\omega(s-t+T)], & -s > |t|. \end{cases} \end{aligned}$$

Also, observe that $\overline{G}$ is continuous except at the diagonal, where

$$\overline{G}(t,t^-) - \overline{G}(t,t^+) = 1.$$

Similarly, we can obtain the explicit expression of the Green's function $\overline{G}$ for the case (D2). Taking again $\omega = \sqrt{|a^2 - b^2|}$,

$$\omega^2 \left(e^{2T\omega^2} - 1\right) \overline{G}(t,s)$$
$$:= \begin{cases} e^{\omega^2(t-s)} \left(e^{\omega^2(s+T)} - 1\right) \left[b\left(e^{\omega^2(T-t)} - 1\right) + \omega^2\right] \\ -4ae^{\frac{1}{2}\omega^2(s+t+2T)} \sinh\left(\frac{1}{2}\omega^2[s-T]\right) \sinh\left(\frac{1}{2}\omega^2[t-T]\right), & |s| < t, \\ e^{-s\omega^2} \left(e^{\omega^2(s+T)} - 1\right) \\ \cdot\left[a\left(e^{s\omega^2} - e^{\omega^2(s+t+T)}\right) + \left(\omega^2 - b\right) e^{t\omega^2} + be^{T\omega^2}\right], & -s > |t|, \\ e^{-s\omega^2} \left(e^{s\omega^2} - e^{T\omega^2}\right) \\ \cdot\left[a\left(-e^{\omega^2(s+t)}\right) + ae^{\omega^2(s+T)} + \left(\omega^2 - b\right) e^{\omega^2(t+T)} + b\right], & s > |t|, \\ -a\left(e^{\omega^2(s+T)} - 1\right)\left(e^{\omega^2(t+T)} - 1\right) \\ + \left(\omega^2 - b\right)\left(e^{\omega^2(t+T)} - e^{\omega^2(-s+t+2T)}\right) + b\left(-e^{\omega^2(T-s)}\right) + b, & |s| < -t. \end{cases}$$

In any case, we have that the Green's function for problem (4.3.2) can be expressed as

$$\overline{G}(t,s) := \begin{cases} k_1(t,s), & t > |s|, \\ k_2(t,s), & s > |t|, \\ k_3(t,s), & -t > |s|, \\ k_4(t,s), & -s > |t|, \end{cases}$$

were the $k_j, j = 1,\dots,4$ are analytic functions defined on $\mathbb{R}^2$.

In order to simplify the statement of the following Theorem, consider the following conditions.

$(\mathbf{D1}^*)$. (D1) is satisfied, $(1-k^2)A(T)^2 \neq (n\,\pi)^2$ for all $n = 0, 1, \dots$

$(\mathbf{D2}^*)$. (D2) is satisfied and $A(T) \neq 0$.

$(\mathbf{D3}^*)$. (D3) is satisfied and $A(T) \neq 0$.

Assume one of $(D1^*)$–$(D3^*)$. In that case, by Theorem 4.3.2 and Lemma 4.3.3, we are under uniqueness conditions for the solution for the following problem [39].

$$x'(t) + x(-t) + k\,x(t) = h(t), \quad t \in [-|A(T)|, |A(T)|], \quad x(A(T)) = x(-A(T)). \tag{4.3.5}$$

The Green's function $\overline{G_2}$ for problem (4.3.5) is just an specific case of $\overline{G}$ and can be expressed as

$$\overline{G}_2(t,s) := \begin{cases} k_1(t,s), & t > |s|, \\ k_2(t,s), & s > |t|, \\ k_3(t,s), & -t > |s|, \\ k_4(t,s), & -s > |t|. \end{cases}$$

Define now

$$G_1(t,s) := e^{B_e(s)-B_e(t)} H(t,s) = e^{B_e(s)-B_e(t)} \begin{cases} k_1(A(t),A(s)), & t > |s|, \\ k_2(A(t),A(s)), & s > |t|, \\ k_3(A(t),A(s)), & -t > |s|, \\ k_4(A(t),A(s)), & -s > |t|. \end{cases} \tag{4.3.6}$$

Defined this way, G_1 is continuous except at the diagonal, where $G_1(t,t^-) - \overline{G}_1(t,t^+) = 1$. Now we can state the following Theorem.

Theorem 4.3.5. *Assume one of* $(D1^*)$–$(D3^*)$. *Let* G_1 *be defined as in* (4.3.6). *Assume* $G_1(t,\cdot)h(\cdot) \in \mathrm{L}^1(I)$ *for every* $t \in I$. *Then problem* (4.3.1) *has a unique solution given by*

$$u(t) = \int_{-T}^{T} G_1(t,s)h(s)\,\mathrm{d}s.$$

Proof. First realize that, since a is even, A is odd, so $A(-t) = -A(t)$. It is important to note that if a has not constant sign in I, then A may be not injective on I.

From the properties of $\overline{G_2}$ as a Green's function, it is clear that

$$\frac{\partial \overline{G_2}}{\partial t}(t,s) + \overline{G_2}(-t,s) + k\,\overline{G_2}(t,s) = 0 \quad \text{for a. e. } t,s \in A(I),$$

and so,

$$\frac{\partial H}{\partial t}(t,s) + a(t)H(-t,s) + ka(t)\,H(t,s) = 0 \quad \text{for a. e. } t,s \in I,$$

Hence

$$\begin{aligned}
& u'(t) + a(t)\,u(-t) + (b_o(t) + k\,a(t))\,u(t) \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int_{-T}^{T} G_1(t,s)h(s)\,\mathrm{d}s + a(t)\int_{-T}^{T} G_1(-t,s)h(s)\,\mathrm{d}s + (b_o(t) \\
& + k\,a(t))\int_{-T}^{T} G_1(t,s)h(s)\,\mathrm{d}s \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int_{-T}^{t} e^{B_e(s)-B_e(t)}H(t,s)h(s)\,\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{T} e^{B_e(s)-B_e(t)}H(t,s)h(s)\,\mathrm{d}s \\
& + a(t)\int_{-T}^{T} e^{B_e(s)-B_e(t)}H(-t,s)h(s)\,\mathrm{d}s \\
& + (b_o(t) + k\,a(t))\int_{-T}^{T} e^{B_e(s)-B_e(t)}H(t,s)h(s)\,\mathrm{d}s \\
=& [H(t,t^-) - H(t,t^+)]h(t) + a(t)\,e^{-B_e(t)}\int_{-T}^{T} e^{B_e(s)}\frac{\partial H}{\partial t}(t,s)h(s)\,\mathrm{d}s \\
& - b_o(t)e^{-B_e(t)}\int_{-T}^{T} e^{B_e(s)}H(t,s)h(s)\,\mathrm{d}s + a(t)e^{-B_e(t)}\int_{-T}^{T} e^{B_e(s)}H(-t,s)h(s)\,\mathrm{d}s \\
& + (b_o(t) + k\,a(t))e^{-B_e(t)}\int_{-T}^{T} e^{B_e(s)}H(t,s)h(s)\,\mathrm{d}s \\
=& h(t) + a(t)e^{-B_e(t)}\int_{-T}^{T} e^{B_e(s)}\left[\frac{\partial H}{\partial t}(t,s) + a(t)H(-t,s) + ka(t)\,H(t,s)\right]h(s)\,\mathrm{d}s \\
=& h(t).
\end{aligned}$$

The boundary conditions are also satisfied.

$$u(T) - u(-T) = e^{-B_e(T)}\int_{-T}^{T} e^{B_e(s)}[H(T,s) - H(-T,s)]h(s)\,\mathrm{d}s = 0.$$

In order to check the uniqueness of solution, let u and v be solutions of problem (4.3.5). Then $u - v$ satisfies equation (4.2.1) and so is of the form given in Section 4.2. Also, $(u-v)(T) - (u-v)(-T) = 2(u-v)_o(T) = 0$, but this can only happen, by what has been imposed by conditions $(D1^*)$–$(D3^*)$, if $u - v \equiv 0$, thus proving the uniqueness of solution. ■

Example 4.3.6. Consider the problem

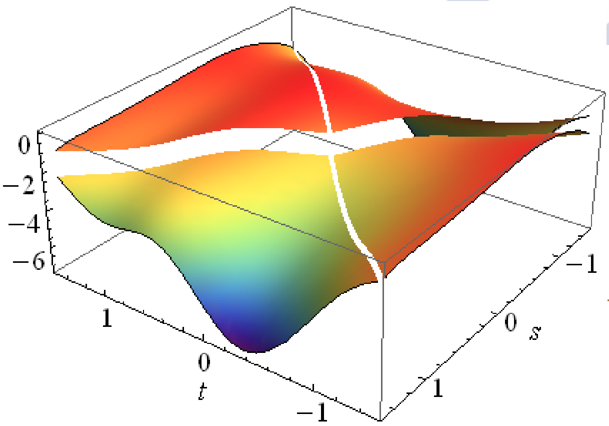
$$x'(t) = \cos(\pi t)x(-t) + \sinh(t)x(t) = \cos(\pi t) + \sinh(t),\ x(3/2) = x(-3/2).$$

Clearly we are in the case (D1) with $k = 0$. If we compute the Green's function according to Theorem 4.3.5 we obtain

$$2\sin(\sin(\pi T))G_1(t,s) = e^{\cosh(s)-\cosh(t)}\hat{G}_1(t,s),$$

where

$$\hat{G}_1(t,s)$$
$$= \begin{cases} \sin\left(\frac{\sin(\pi s)}{\pi} - \frac{\sin(\pi t)}{\pi} - \frac{\sin(\pi T)}{\pi}\right) + \cos\left(\frac{\sin(\pi s)}{\pi} + \frac{\sin(\pi t)}{\pi} - \frac{\sin(\pi T)}{\pi}\right), |t| < s, \\ \sin\left(\frac{\sin(\pi s)}{\pi} - \frac{\sin(\pi t)}{\pi} + \frac{\sin(\pi T)}{\pi}\right) + \cos\left(\frac{\sin(\pi s)}{\pi} + \frac{\sin(\pi t)}{\pi} + \frac{\sin(\pi T)}{\pi}\right), |t| < -s, \\ \sin\left(\frac{\sin(\pi s)}{\pi} - \frac{\sin(\pi t)}{\pi} + \frac{\sin(\pi T)}{\pi}\right) + \cos\left(\frac{\sin(\pi s)}{\pi} + \frac{\sin(\pi t)}{\pi} - \frac{\sin(\pi T)}{\pi}\right), |s| < t, \\ \sin\left(\frac{\sin(\pi s)}{\pi} - \frac{\sin(\pi t)}{\pi} - \frac{\sin(\pi T)}{\pi}\right) + \cos\left(\frac{\sin(\pi s)}{\pi} + \frac{\sin(\pi t)}{\pi} + \frac{\sin(\pi T)}{\pi}\right), |s| < -t. \end{cases}$$



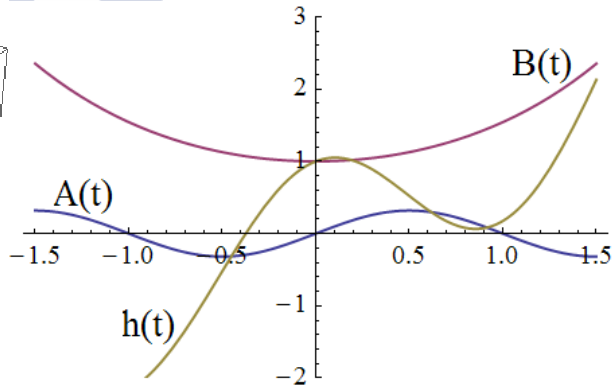


Figure 4.3.1: Graphs of the kernel *(left)* and of the functions involved in the problem *(right)*.

One of the most important direct consequences of Theorem 4.3.5 is the existence of maximum and antimaximum principles in the case $b \equiv 0$†.

Corollary 4.3.7. *Under the conditions of Theorem 4.3.5, if a is nonnegative on I, we have the following properties:*

- *If $A(T) \in (0, \frac{\pi}{4})$ then G_1 is strictly positive on I^2.*
- *If $A(T) \in (-\frac{\pi}{4}, 0)$ then G_1 is strictly negative on I^2.*
- *If $A(T) = \frac{\pi}{4}$ then G_1 vanishes on*

$$P := \{(-A(T), -A(T)), (0,0), (A(T),A(T)), (A(T), -A(T))\}$$

and is strictly positive on $(I^2)\backslash P$.

†Note that this discards the case (D3), for which $b \equiv 0$ implies $a \equiv 0$, because we are assuming $a \not\equiv 0$.

- *If* $A(T) = -\frac{\pi}{4}$ *then* G_1 *vanishes on* P *and is strictly negative on* $(I^2)\backslash P$.
- *If* $A(T) \in \mathbb{R}\backslash[-\frac{\pi}{4}, \frac{\pi}{4}]$ *then* G_1 *is not positive nor negative on* I^2.

Furthermore, the operator $R_a : \mathcal{F}_\lambda(I) \to \mathrm{L}^1(I)$ *defined as*

$$R_a(x(t)) = x'(t) + a(t)\,x(-t)$$

satisfies

- R_a *is strongly inverse positive if and only if* $A(T) \in (0, \frac{\pi}{4T}]$ *and* $\lambda \geq 0$,
- R_a *is strongly inverse negative if and only if* $A(T) \in [-\frac{\pi}{4T}, 0)$ *and* $\lambda \geq 0$.

The second part of this last corollary, drawn from positivity (or negativity) of the Green's function could have been obtained, as we show below, without having so much knowledge about the Green's function. In order to show this, consider the following proposition in the line of the work of Torres [167, Theorem 2.1].

Proposition 4.3.8. *Consider the homogeneous initial value problem*

$$x'(t) + a(t)\,x(-t) + b(t)\,x(t) = 0, \text{ a. e. } t \in I;\; x(t_0) = 0. \tag{4.3.7}$$

If problem (4.3.7) *has a unique solution (*$x \equiv 0$*) on* I *for all* $t_0 \in I$ *then, if the Green's function for* (4.3.1) *exists, it has constant sign.*

What is more, if we further assume $a + b$ *has constant sign, the Green's function has the same sign as* $a + b$.

Proof. Without lost of generality, consider a to be a $2T$-periodic L^1-function defined on $\mathbb{R}$ (the solution of (4.3.1) will be considered in I). Let G_1 be the Green's function for problem (4.3.1). Since $G_1(T, s) = G_1(-T, s)$ for all $s \in I$, and G_1 is continuous except at the diagonal, it is enough to prove that $G_1(t, s) \neq 0 \;\; \forall t, s \in I$.

Assume, on the contrary, that there exists $t_1, s_1 \in I$ such that $G_1(t_1, s_1) = 0$. Let g be the $2T$-periodic extension of $G_1(\cdot, s_1)$. Let us assume $t_1 > s_1$ (the other case would be analogous). Let f be the restriction of g to $(s_1, s_1 + 2T)$. f is absolutely continuous and satisfies (4.3.7) a. e. in I for $t_0 = t_1$, hence, $f \equiv 0$. This contradicts the fact of G_1 being a Green's function, therefore G_1 has constant sign.

Realize now that $x \equiv 1$ satisfies

$$x'(t) + a(t)x(-t) + b(t)x(t) = a(t) + b(t),\; x(-T) = x(T).$$

Hence, $\int_{-T}^{T} G_1(t, s)\,(a(s) + b(s))\,\mathrm{d}s = 1$ for all $t \in I$. Since both G_1 and $a + b$ have constant sign, they have the same sign. ■

The following corollaries are an straightforward application of this result to the cases $(D1)$ – $(D3)$ respectively.

Corollary 4.3.9. *Assume a has constant sign in I. Under the assumptions of (D1) and Theorem 4.3.5, G_1 has constant sign if*

$$|A(T)| < \frac{\arccos(k)}{2\sqrt{1-k^2}}.$$

Furthermore, $\operatorname{sign}(G_1) = \operatorname{sign}(a)$.

Proof. The solutions of (4.2.1) for the case (D1), as seen before, are given by

$$u(t) = \alpha e^{-B_e(t)}\left[\cos\left(\sqrt{1-k^2}A(t)\right) - \frac{1+k}{\sqrt{1-k^2}}\sin\left(\sqrt{1-k^2}A(t)\right)\right].$$

Using a particular case of the phasor addition formula[†] –see Appendix A,–

$$u(t) = \alpha e^{-B_e(t)}\sqrt{\frac{2}{1-k}}\sin\left(\sqrt{1-k^2}A(t)+\theta\right),$$

where $\theta \in [-\pi, \pi)$ is the angle such that

$$\sin\theta = \sqrt{\frac{1-k}{2}} \quad \text{and} \quad \cos\theta = -\sqrt{\frac{1+k}{2}}. \tag{4.3.8}$$

Observe that this implies that $\theta \in \left(\frac{\pi}{2}, \pi\right)$.

In order for the hypothesis of Proposition 4.3.8 to be satisfied, it is enough and sufficient to ask for $0 \notin u(I)$ for some $\alpha \neq 0$. Equivalently, that

$$\sqrt{1-k^2}A(t)+\theta \neq \pi n \;\; \forall n \in \mathbb{Z} \;\; \forall t \in I,$$

That is,

$$A(t) \neq \frac{\pi n - \theta}{\sqrt{1-k^2}} \;\; \forall n \in \mathbb{Z} \;\; \forall t \in I.$$

Since A is odd and injective and $\theta \in \left(\frac{\pi}{2}, \pi\right)$, this is equivalent to

$$|A(T)| < \frac{\pi - \theta}{\sqrt{1-k^2}}. \tag{4.3.9}$$

Now, using the double angle formula for the sine and (4.3.8),

$$\frac{1-k}{2} = \sin^2\theta = \frac{1-\cos(2\theta)}{2}, \quad \text{this is,} \quad k = \cos(2\theta),$$

which implies, since $2\theta \in (\pi, 2\pi)$,

$$\theta = \pi - \frac{\arccos(k)}{2},$$

[†] $\alpha\cos\gamma + \beta\sin\gamma = \sqrt{\alpha^2+\beta^2}\sin(\gamma+\theta)$, where $\theta \in [-\pi, \pi)$ is the angle such that $\cos\theta = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$, $\sin\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$.

where $\arccos$ is defined such that it's image is $[0, \pi)$. Plugging this into inequality (4.3.9) yields

$$|A(T)| < \sigma(k) := \frac{\arccos(k)}{2\sqrt{1-k^2}}, \quad k \in (-1, 1). \tag{4.3.10}$$

The sign of the Green's function is given by Proposition 4.3.8 and $\operatorname{sign}(G_1) = \operatorname{sign}(a+b)$. Now, we have that $|k| < 1$ and $a + b = (k+1)a + b_o$. Because of the continuity of G_1 with respect to the parameters a and b, G_1 has to have the same sign in the case $b_0 \equiv 0$ – observe that b_0 does not affect inequality (4.3.10)– so, actually, $\operatorname{sign}(G_1) = \operatorname{sign}((k+1)a) = \operatorname{sign}(a)$. ■

Remark 4.3.10. In the case a is a constant ω and $k = 0$, $A(I) = [-|\omega|T, |\omega|T]$, and the condition can be written as $|\omega|T < \frac{\pi}{4}$, which is consistent with Theorem 3.2.8.

Remark 4.3.11. Observe that σ is strictly decreasing on $(-1, 1)$ and

$$\lim_{k\to -1^+} \sigma(k) = +\infty, \quad \lim_{k\to 1^-} \sigma(k) = \frac{1}{2}.$$

Corollary 4.3.12. *Under the conditions of (D3) and Theorem 4.3.5, G_1 has constant sign in I if $|A(T)| < \frac{1}{2}$.*

Proof. This corollary is a direct consequence of equation (4.2.3), Proposition 4.3.8 and Theorem 4.3.5. Observe that the result is consistent with $\sigma(1^-) = \frac{1}{2}$. ■

In order to prove the next corollary, we need the following «hyperbolic version» of the phasor addition formula. It's proof can be done without difficulty.

Lemma 4.3.13. *Let $\alpha, \beta, \gamma \in \mathbb{R}$, then*

$$\alpha\cosh\gamma + \beta\sinh\gamma = \sqrt{|\alpha^2-\beta^2|}\begin{cases} \cosh\left(\frac{1}{2}\ln\left|\frac{\alpha+\beta}{\alpha-\beta}\right| + \gamma\right) & \text{if} \quad \alpha > |\beta|, \\ -\cosh\left(\frac{1}{2}\ln\left|\frac{\alpha+\beta}{\alpha-\beta}\right| + \gamma\right) & \text{if} \quad -\alpha > |\beta|, \\ \sinh\left(\frac{1}{2}\ln\left|\frac{\alpha+\beta}{\alpha-\beta}\right| + \gamma\right) & \text{if} \quad \beta > |\alpha|, \\ -\sinh\left(\frac{1}{2}\ln\left|\frac{\alpha+\beta}{\alpha-\beta}\right| + \gamma\right) & \text{if} \quad -\beta > |\alpha|, \\ \alpha\, e^{\gamma} & \text{if} \quad \alpha = \beta, \\ \alpha\, e^{-\gamma} & \text{if} \quad \alpha = -\beta. \end{cases}$$

Corollary 4.3.14. *Assume a has constant sign in I. Under the assumptions of (D2) and Theorem 4.3.5, G_1 has constant sign if $k < -1$ or*

$$|A(T)| < -\frac{\ln(k-\sqrt{k^2-1})}{2\sqrt{k^2-1}}.$$

Furthermore, $\operatorname{sign}(G_1) = \operatorname{sign}(k\,a)$.

Proof. The solutions of (4.2.1) for the case (D2), as seen before, are given by

$$u(t) = \alpha e^{-B_e(t)}\left[\cosh\left(\sqrt{k^2-1}A(t)\right) - \frac{1+k}{\sqrt{k^2-1}}\sinh\left(\sqrt{k^2-1}A(t)\right)\right].$$

If $k > 1$, then $1 < \frac{1+k}{\sqrt{k^2-1}}$, so, using Lemma 4.3.13,

$$u(t) = -\alpha e^{-B_e(t)}\sqrt{\frac{2k}{k-1}}\sinh\left(\frac{1}{2}\ln\left|k-\sqrt{k^2-1}\right| + \sqrt{k^2-1}A(t)\right),$$

In order for the hypothesis of Proposition 4.3.8 to be satisfied, it is enough and sufficient to ask that $0 \not\in u(I)$ for some $\alpha \neq 0$. Equivalently, that

$$\frac{1}{2}\ln(k-\sqrt{k^2-1}) + \sqrt{k^2-1}A(t) \neq 0 \;\; \forall t \in I,$$

That is,

$$A(t) \neq -\frac{\ln(k-\sqrt{k^2-1})}{2\sqrt{k^2-1}} \;\; \forall t \in I.$$

Since A is odd and injective, this is equivalent to

$$|A(T)| < \sigma(k) := -\frac{\ln(k-\sqrt{k^2-1})}{2\sqrt{k^2-1}}, \quad k > 1.$$

Now, if $k < -1$, then $\left|\frac{1+k}{\sqrt{k^2-1}}\right| < 1$, so using Lemma 4.3.13,

$$u(t) = \alpha e^{-B_e(t)}\sqrt{\frac{2k}{k-1}}\cosh\left(\frac{1}{2}\ln\left|k-\sqrt{k^2-1}\right| + \sqrt{k^2-1}A(t)\right) \neq 0,$$

for all $t \in I$, $\alpha \neq 0$, so the hypothesis of Proposition 4.3.8 are satisfied.

The sign of the Green's function is given by Proposition 4.3.8 and $\operatorname{sign}(G_1) = \operatorname{sign}(a+b)$. Now, we have that $|k| > 1$ and $a + b = (k^{-1}+1)b_e + b_o$. Because of the continuity of G_1 with respect to the parameters a and b, G_1 has to have the same sign in the case $b_0 \equiv 0$ so, actually, $\operatorname{sign}(G_1) = \operatorname{sign}((k^{-1}+1)b_e) = \operatorname{sign}(b_e) = \operatorname{sign}(k\,a)$. ∎

Remark 4.3.15. If we consider σ defined piecewise as in Corollaries 4.3.9 and 4.3.14 and continuously continued through $1/2$, we get

$$\sigma(k) := \begin{cases} \frac{\arccos(k)}{2\sqrt{1-k^2}} & \text{if } k \in (-1,1), \\ \frac{1}{2} & \text{if } k = 1, \\ -\frac{\ln(k-\sqrt{k^2-1})}{2\sqrt{k^2-1}} & \text{if } k > 1. \end{cases}$$

This function is not only continuous (it is defined thus), but also analytic. In order to see this it is enough to consider the extended definition of the logarithm and the square root to the complex numbers. Remember that $\sqrt{-1} := i$ and that the principal branch of the logarithm is defined as $\ln_0(z) = \ln|z| + i\theta$ where $\theta \in [-\pi,\pi)$ and $z = |z|e^{i\theta}$ for all $z \in \mathbb{C}\backslash\{0\}$. Clearly, $\ln_0|_{(0,+\infty)} = \ln$.

Now, for $|k| < 1$, $\ln_0(k-\sqrt{1-k^2}i) = i\theta$ with $\theta \in [-\pi,\pi)$ such that $\cos\theta = k$, $\sin\theta = -\sqrt{1-k^2}$, that is, $\theta \in [-\pi, 0]$. Hence, $i\ln_0(k-\sqrt{1-k^2}i) = -\theta \in [0,\pi]$. Since $\cos(-\theta) = k$, $\sin(-\theta) = \sqrt{1-k^2}$, it is clear that

$$\arccos(k) = -\theta = i\ln_0(k-\sqrt{1-k^2}i).$$

We thus extend arccos to $\mathbb{C}$ by

$$\arccos(z) := i\ln_0(z - \sqrt{1-z^2}i),$$

which is clearly an analytic function. So, if $k > 1$,

$$\begin{aligned}\sigma(k) &= -\frac{\ln(k-\sqrt{k^2-1})}{2\sqrt{k^2-1}} = -\frac{\ln_0(k-i\sqrt{1-k^2})}{2i\sqrt{1-k^2}} \\ &= \frac{i\ln_0(k-i\sqrt{1-k^2})}{2\sqrt{1-k^2}} = \frac{\arccos(k)}{2\sqrt{1-k^2}}.\end{aligned}$$

σ is positive, strictly decreasing and

$$\lim_{k\to-1^+}\sigma(k) = +\infty, \quad \lim_{k\to+\infty}\sigma(k) = 0.$$

In a similar way to Corollaries 4.3.9, 4.3.12 and 4.3.14, we can prove results not assuming a to be a constant sign function. The result is the following.

Corollary 4.3.16. *Under the assumptions of Theorem 4.3.5 and conditions (D1), (D2) or (D3) (let k be the constant involved in such conditions), G_1 has constant sign if $\max A(I) < \sigma(k)$.*

4.4 The cases (D4) and (D5)

Consider the following problem derived from the nonhomogeneous problem (4.3.1).

$$\begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \begin{pmatrix} a_o - b_o & -a_e - b_e \\ a_e - b_e & -a_o - b_o \end{pmatrix}\begin{pmatrix} x_o \\ x_e \end{pmatrix} + \begin{pmatrix} h_e \\ h_o \end{pmatrix}. \tag{4.4.1}$$

The following theorems tell us what happens when we impose the boundary conditions.

Theorem 4.4.1. *If condition (D4) holds, then problem* (4.3.1) *has solution if and only if*

$$\int_0^T e^{B_e(s)}h_e(s)\,\mathrm{d}s = 0,$$

and in that case the solutions of (4.3.1) *are given by*

$$u_c(t) = e^{-B_e(t)}\left[c + \int_0^t\left(e^{B_e(s)}h(s) + 2a_e(s)\int_0^s e^{B_e(r)}h_e(r)\,\mathrm{d}r\right)\mathrm{d}s\right] \quad \text{for } c\in\mathbb{R}. \tag{4.4.2}$$

Proof. We know that any solution of problem (4.3.1) has to satisfy (4.4.1). In the case (D4), the matrix in (4.4.1) is lower triangular

$$\begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \begin{pmatrix} -b_o & 0 \\ 2a_e & -b_o \end{pmatrix}\begin{pmatrix} x_o \\ x_e \end{pmatrix} + \begin{pmatrix} h_e \\ h_o \end{pmatrix}, \tag{4.4.3}$$

so, the solutions of (4.4.3) are given by

$$x_o(t) = e^{-B_e(t)}\left[\tilde{c} + \int_0^t e^{B_e(s)}h_e(s)\,\mathrm{d}s\right],$$

$$x_e(t) = e^{-B_e(t)}\left[c + \int_0^t \left(e^{B_e(s)} h_o(s) + 2a_e(s)\left[\tilde{c} + \int_0^s e^{B_e(r)} h_e(r)\,\mathrm{d}r\right]\right)\mathrm{d}s\right],$$

where $c, \tilde{c} \in \mathbb{R}$.

x_e is even independently of the value of c. Nevertheless, x_o is odd only when $\tilde{c} = 0$. Hence, a solution of (4.3.1), if it exists, it has the form (4.4.2).

To show the other implication it is enough to check that u_c is a solution of the problem (4.3.1).

$$\begin{aligned} u_c'(t) =& - b_o(t)e^{-B_e(t)}\left[c + \int_0^t \left(e^{B_e(s)} h(s) + 2a_e(s)\int_0^s e^{B_e(r)} h_e(r)\,\mathrm{d}r\right)\mathrm{d}s\right] \\ & + e^{-B_e(t)}\left(e^{B_e(t)} h(t) + 2a_e(t)\int_0^t e^{B_e(r)} h_e(r)\,\mathrm{d}r\right) \\ =& h(t) - b_o(t)u(t) + 2a_e(t)e^{-B_e(t)}\int_0^t e^{B_e(r)} h_e(r)\,\mathrm{d}r. \end{aligned}$$

Now,

$$\begin{aligned} & a_e(t)\left(u_c(-t) - u_c(t)\right) + 2a_e(t)e^{-B_e(t)}\int_0^t e^{B_e(r)} h_e(r)\,\mathrm{d}r \\ =& a_e(t)e^{-B_e(t)}\left[c - \int_0^t \left(e^{B_e(s)} h(-s) - 2a_e(s)\int_0^s e^{B_e(r)} h_e(r)\,\mathrm{d}r\right)\mathrm{d}s\right] \\ & - a_e(t)e^{-B_e(t)}\left[c + \int_0^t \left(e^{B_e(s)} h(s) + 2a_e(s)\int_0^s e^{B_e(r)} h_e(r)\,\mathrm{d}r\right)\mathrm{d}s\right] \\ & + 2a_e(t)e^{-B_e(t)}\int_0^t e^{B_e(r)} h_e(r)\,\mathrm{d}r \\ =& - 2a_e(t)e^{-B_e(t)}\int_0^t e^{B_e(r)} h_e(r)\,\mathrm{d}s + 2a_e(t)e^{-B_e(t)}\int_0^t e^{B_e(r)} h_e(r)\,\mathrm{d}r = 0. \end{aligned}$$

Hence,

$$u_c'(t) + a_e(t)u_c(-t) + (-a_e(t) + b_o(t))u_c(t) = h(t),\ a.e.t \in I.$$

The boundary condition $u_c(-T) - u_c(T) = 0$ is equivalent to $(u_c)_o(T) = 0$, this is,

$$\int_0^T e^{B_e(s)} h_e(s)\,\mathrm{d}s = 0$$

and the result is concluded. ■

Theorem 4.4.2. *If condition (D5) holds, then problem* (4.3.1) *has solution if and only if*

$$\int_0^T e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s = 0, \tag{4.4.4}$$

and in that case the solutions of (4.3.1) *are given by*

$$u_c(t) = e^{A(t)-B(t)}\int_0^t e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s + e^{-A(t)-B(t)}\left[c + \int_0^t e^{A(s)+B(s)} h_o(s)\,\mathrm{d}s\right] \tag{4.4.5}$$

for every $c \in \mathbb{R}$.

Proof. In the case (D5), $b_o = b$ and $a_o = a$. Also, the matrix in (4.4.1) is diagonal

$$\begin{pmatrix} x_o' \\ x_e' \end{pmatrix} = \begin{pmatrix} a_o - b_o & 0 \\ 0 & -a_o - b_o \end{pmatrix}\begin{pmatrix} x_o \\ x_e \end{pmatrix} + \begin{pmatrix} h_e \\ h_o \end{pmatrix}. \tag{4.4.6}$$

and the solutions of (4.4.6) are given by

$$\begin{aligned} x_o(t) &= e^{A(t)-B(t)}\left[\tilde{c} + \int_0^t e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s\right], \\ x_e(t) &= e^{-A(t)-B(t)}\left[c + \int_0^t e^{A(s)+B(s)} h_o(s)\,\mathrm{d}s\right], \end{aligned}$$

where c, $\tilde{c} \in \mathbb{R}$. Since a and b are odd, A and B are even. So, x_e is even independently of the value of c. Nevertheless, x_o is odd only when $\tilde{c} = 0$. In such a case, since we need, as in Theorem 4.4.1, that $x_o(T) = 0$, we get condition (4.4.4), which allows us to deduce the first implication of the Theorem.

Any solution u_c of (4.3.1) has the expression (4.4.5).

To show the second implication, it is enough to check that u is a solution of the problem (4.3.1).

$$\begin{aligned} u_c'(t) =&(a(t) - b(t))e^{A(t)-B(t)} \int_0^t e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s \\ &- (a(t) + b(t))e^{-A(t)-B(t)}\left[c + \int_0^t e^{A(s)+B(s)} h_o(s)\,\mathrm{d}s\right] + h(t). \end{aligned}$$

Now,

$$\begin{aligned} &a(t)u_c(-t) + b(t)u_c(t) \\ =&a(t)\left(-e^{A(t)-B(t)} \int_0^t e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s + e^{-A(t)-B(t)}\left[c + \int_0^t e^{A(s)+B(s)} h_o(s)\,\mathrm{d}s\right]\right) \\ &+ b(t)\left(e^{A(t)-B(t)} \int_0^t e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s + e^{-A(t)-B(t)}\left[c + \int_0^t e^{A(s)+B(s)} h_o(s)\,\mathrm{d}s\right]\right) \\ =&- (a(t) - b(t))e^{A(t)-B(t)} \int_0^t e^{B(s)-A(s)} h_e(s)\,\mathrm{d}s \\ &+ (a(t) + b(t))e^{-A(t)-B(t)}\left[c + \int_0^t e^{A(s)+B(s)} h_o(s)\,\mathrm{d}s\right]. \end{aligned}$$

So clearly,

$$u_c'(t) + a(t)u_c(-t) + b(t)u_c(t) = h(t) \quad \text{for a.e. } t \in I.$$

which ends the proof. ■

4.5 The other cases

When we are not on the cases (D1)-(D5), since the fundamental matrix of M is not given by its exponential matrix, it is more difficult to precise when problem (4.3.1) has a solution. Here we present some partial results.

Consider the following ordinary differential equation

$$x'(t) + [a(t) + b(t)]x(t) = h(t), \quad x(-T) = x(T). \tag{4.5.1}$$

The following lemma gives us the explicit Green's function for this problem. Let $v = a + b$.

Lemma 4.5.1. *Let h, a, b in problem* (4.5.1) *be in* $\mathrm{L}^1(I)$ *and assume* $\int_{-T}^T v(t)\,\mathrm{d}t \neq 0$. *Then problem* (4.5.1) *has a unique solution given by*

$$u(t) = \int_{-T}^T G_3(t,s)h(t)\,\mathrm{d}s,$$

where

$$G_3(t,s) = \begin{cases} \tau\, e^{\int_t^s v(r)\,\mathrm{d}r}, & s \le t, \\ (\tau-1)e^{\int_t^s v(r)\,\mathrm{d}r}, & s > t, \end{cases} \quad \textit{and} \quad \tau = \frac{1}{1-e^{-\int_{-T}^{T} v(r)\,\mathrm{d}r}}. \tag{4.5.2}$$

Proof.

$$\frac{\partial G_3}{\partial t}(t,s) = \begin{cases} -\tau\, v(t)\, e^{\int_t^s v(r)\,\mathrm{d}r}, & s \le t, \\ -(\tau-1)\, v(t) e^{\int_t^s v(r)\,\mathrm{d}r}, & s > t, \end{cases} = -v(t)G_3(t,s).$$

Therefore,

$$\frac{\partial G_3}{\partial t}(t,s) + v(t)G_3(t,s) = 0,\ s \ne t.$$

Hence,

$$\begin{aligned} & u'(t)+v(t)u(t) \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\int_{-T}^{t} G_3(t,s)h(s)\,\mathrm{d}s + \frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{T} G_3(t,s)h(s)\,\mathrm{d}s + v(t)\int_{-T}^{T} G_3(t,s)h(s)\,\mathrm{d}s \\ =& [G_3(t,t^-)-G_3(t,t^+)]h(t) + \int_{-T}^{T}\left[\frac{\partial G_3}{\partial t}(t,s)+v(t)G_3(t,s)\right]h(t)\,\mathrm{d}s \\ =& h(t) \ \text{ a. e. } t \in I. \end{aligned}$$

The boundary conditions are also satisfied.

$$\begin{aligned} u(T)-u(-T) &= \int_{-T}^{T}\left[\tau\, e^{\int_T^s v(r)\,\mathrm{d}r} - (\tau-1)e^{\int_{-T}^s v(r)\,\mathrm{d}r}\right]h(s)\,\mathrm{d}s \\ &= \int_{-T}^{T}\left[\frac{e^{\int_T^s v(r)\,\mathrm{d}r}}{1-e^{-\int_{-T}^{T} v(r)\,\mathrm{d}r}} - \frac{e^{-\int_{-T}^{T} v(r)\,\mathrm{d}r}\, e^{\int_{-T}^s v(r)\,\mathrm{d}r}}{1-e^{-\int_{-T}^{T} v(r)\,\mathrm{d}r}}\right]h(s)\,\mathrm{d}s \\ &= \int_{-T}^{T}\left[\frac{e^{\int_T^s v(r)\,\mathrm{d}r}}{1-e^{-\int_{-T}^{T} v(r)\,\mathrm{d}r}} - \frac{e^{\int_T^s v(r)\,\mathrm{d}r}}{1-e^{-\int_{-T}^{T} v(r)\,\mathrm{d}r}}\right]h(s)\,\mathrm{d}s = 0. \end{aligned}$$

■

Lemma 4.5.2.

$$|G_3(t,s)| \le F(v) := \frac{e^{\|v\|_1}}{|e^{\|v^+\|_1} - e^{\|v^-\|_1}|}. \tag{4.5.3}$$

Proof. Observe that

$$\tau = \frac{1}{1-e^{\|v^-\|_1-\|v^+\|_1}} = \frac{e^{\|v^+\|_1}}{e^{\|v^+\|_1}-e^{\|v^-\|_1}}.$$

Hence,

$$\tau - 1 = \frac{e^{\|v^-\|_1}}{e^{\|v^+\|_1}-e^{\|v^-\|_1}}.$$

On the other hand,

$$e^{\int_t^s v(r)\,\mathrm{d}r} \le \begin{cases} e^{\|v^-\|_1}, & s \le t, \\ e^{\|v^+\|_1}, & s > t, \end{cases}$$

which ends the proof. ■

The next result proves the existence and uniqueness of solution of (4.3.1) when υ is 'sufficiently small'.

Theorem 4.5.3. *Let h, a, b in problem* (4.3.1) *be in* $\mathrm{L}^1(I)$ *and assume* $\int_{-T}^{T}\upsilon(t)\,\mathrm{d}t\neq 0$. *Let* $W:=\{(2T)^{\frac{1}{p}}(\|a\|_{p^*}+\|b\|_{p^*})\}_{p\in[1,+\infty]}$ *where* $p^{-1}+(p^*)^{-1}=1$. *If*

$$F(\upsilon)\|a\|_1(\inf W)<1,$$

where $F(\upsilon)$ is defined as in (4.5.3)*, then problem* (4.3.1) *has a unique solution.*

Proof. With some manipulation we get

$$\begin{aligned}h(t)=&x'(t)+a(t)\left(\int_t^{-t}x'(s)\,\mathrm{d}s+x(t)\right)+b(t)x(t)\\=&x'(t)+\upsilon(t)x(t)+a(t)\int_t^{-t}(h(s)-a(s)x(-s)-b(s)x(s))\,\mathrm{d}s.\end{aligned}$$

Hence,

$$x'(t)+\upsilon(t)x(t)=a(t)\int_t^{-t}(a(s)x(-s)+b(s)x(s))\,\mathrm{d}s+a(t)\int_{-t}^{t}h(s)\,\mathrm{d}s+h(t).$$

Using G_3 defined as in (4.5.2) and Lemma 4.5.1, it is clear that

$$\begin{aligned}x(t)=&\int_{-T}^{T}G_3(t,s)a(s)\int_s^{-s}(a(r)x(-r)+b(r)x(r))\,\mathrm{d}r\,\mathrm{d}s\\&+\int_{-T}^{T}G_3(t,s)\left[a(s)\int_{-s}^{s}h(r)\,\mathrm{d}r+h(s)\right]\mathrm{d}s,\end{aligned}$$

this is, x is a fixed point of an operator of the form $Hx(t)+\beta(t)$, so, by Banach contraction Theorem, it is enough to prove that $\|H\|<1$ for some compatible norm of H.

Using Fubini's Theorem,

$$Hx(t)=-\int_{-T}^{T}\rho(t,r)(a(r)x(-r)+b(r)x(r))\,\mathrm{d}r,$$

where $\rho(t,r)=\left[\int_{|r|}^{T}-\int_{-T}^{-|r|}\right]G_3(t,s)a(s)\,\mathrm{d}s$.

If $\int_{-T}^{T}\upsilon(t)\,\mathrm{d}t=\|\upsilon^+\|_1-\|\upsilon^-\|_1>0$ then G_3 is positive and

$$\rho(t,r)\le\int_{-T}^{T}G_3(t,s)|a(s)|\,\mathrm{d}s\le F(\upsilon)\|a\|_1.$$

We have the same estimate for $-\rho(t,r)$.

If $\int_{-T}^{T}\upsilon(t)\,\mathrm{d}t<0$ we proceed with an analogous argument and arrive to the conclusion that G_3 is negative and $|\rho(t,s)|<F(\upsilon)\|a\|_1$.

Hence,

$$\begin{aligned}|Hx(t)|\le&F(\upsilon)\|a\|_1\int_{-T}^{T}|a(r)x(-r)+b(r)x(r)|\,\mathrm{d}r\\=&F(\upsilon)\|a\|_1\|a(r)x(-r)+b(r)x(r)\|_1.\end{aligned}$$

Thus, it is clear that

$$\|Hx\|_p\le(2T)^{\frac{1}{p}}F(\upsilon)\|a\|_1(\|a\|_{p^*}+\|b\|_{p^*})\|x\|_p,\ p\in[1,\infty],$$

which ends the proof. ■

Remark 4.5.4. In the hypothesis of Theorem 4.5.3, realize that $F(v) \geq 1$.

The following result will let us obtain some information on the sign of the solution of problem (4.3.1). In order to prove it, we will use a Theorem from Chapter 8 –Theorem 8.4.11– which is demonstrated independently.

Consider an interval $[w,d] \subset I$, the cone

$$K = \{u \in C(I) : \min_{t\in[w,d]} u(t) \geq c\|u\|\},$$

and the following problem

$$x'(t) = h(t,x(t),x(-t)),\ t \in I, \quad x(-T) = x(T), \tag{4.5.4}$$

where h is an L^1-Carathéodory function. Consider the following conditions.

$(\mathrm{I}^1_{\rho,\omega})$ There exist $\rho > 0$ and $\omega \in \left(0, \frac{\pi}{4T}\right]$ such that $f_\omega^{-\rho,\rho} < \omega$ where

$$f_\omega^{-\rho,\rho} := \sup\left\{\frac{h(t,u,v)+\omega v}{\rho} : (t,u,v) \in [-T,T]\times[-\rho,\rho]\times[-\rho,\rho]\right\}.$$

$(\mathrm{I}^0_{\rho,\omega})$ There exists $\rho > 0$ such that

$$f^\omega_{(\rho,\rho/c)} \cdot \inf_{t\in[w,d]} \int_w^d \overline{G}(t,s)\,ds > 1,$$

where

$$f^\omega_{(\rho,\rho/c)} = \inf\left\{\frac{h(t,u,v)+\omega v}{\rho} : (t,u,v) \in [w,d]\times[\rho,\rho/c]\times[-\rho/c,\rho/c]\right\}.$$

Theorem 4.5.5 (Part of Theorem 8.4.11). *Let $\omega \in \left(0, \frac{\pi}{2}T\right]$. Let $[w,d] \subset I$ such that $w = T - d \in (\max\{0, T-\frac{\pi}{4\omega}\}, \frac{T}{2})$. Let*

$$c = \frac{[1-\tan(\omega d)][1-\tan(\omega w)]}{[1+\tan(\omega d)][1+\tan(\omega w)]}. \tag{4.5.5}$$

Problem (4.5.4) *has at least one nonzero solution in K if either of the following conditions hold.*

(S_1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1/c < \rho_2$ such that $(\mathrm{I}^0_{\rho_1,\omega})$ and $(\mathrm{I}^1_{\rho_2,\omega})$ hold.*

(S_2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $(\mathrm{I}^1_{\rho_1,\omega})$ and $(\mathrm{I}^0_{\rho_2,\omega})$ hold.*

Theorem 4.5.6. *Let $h \in \mathrm{L}^\infty(I)$, $a, b \in \mathrm{L}^1(I)$ be such that $0 < |b(t)| < a(t) < \omega < \frac{\pi}{2}T$ for a. e. $t \in I$ and $\inf h > 0$. Then there exists a solution u of* (4.3.1) *such that, $u > 0$ in $\left(\max\{0, T-\frac{\pi}{4\omega}\}, \min\{T, \frac{\pi}{4\omega}\}\right)$.*

Proof. Problem (4.3.1) can be rewritten as

$$x'(t) = h(t) - b(t)\,x(t) - a(t)\,x(-t),\ \ t \in I, \quad x(-T) = x(T).$$

With this formulation, we can apply Theorem 4.5.5. Since $0 < a(t) - |b(t)| < \omega$ for a. e. $t \in I$, take $\rho_2 \in \mathbb{R}^+$ large enough such that $h(t) < (a(t) - |b(t)|)\rho_2$ for a. e. $t \in I$. Hence, $h(t) < (a(t) - \omega)\rho_2 - |b(t)|\rho_2 + \rho_2\omega$ for a. e. $t \in I$, in particular,

$$h(t) < (a(t) - \omega)v - |b(t)|u + \rho_2\omega \leq (a(t) - \omega)v + b(t)\,u + \rho_2\omega$$

for a. e. $t \in I$; $u, v \in [-\rho_2, \rho_2]$. Therefore,

$$\sup\left\{\frac{h(t) - b(t)u - a(t)v + \omega v}{\rho_2}\ :\ (t,v) \in [-T,T] \times [-\rho_2, \rho_2]\right\} < \omega,$$

and thus, $(\mathrm{I}^1_{\rho_2,\omega})$ is satisfied.

Let $[w,d] \subset I$ be such that $[w,d] \subset \left(T - \frac{\pi}{4\omega}, \frac{\pi}{4\omega}\right)$. Let c be defined as in (4.5.5) and $\epsilon = \omega\int_w^d \overline{G}(t,s)\,\mathrm{d}s$.

Choose $\delta \in (0,1)$ such that $h(t) > \left[\left(1 + \frac{c}{\epsilon}\right)\omega - (a(t) - |b(t)|)\right]\rho_2\delta$ for a. e. $t \in I$ and define $\rho_1 := \delta c\rho_2$. Therefore, $h > [(a(t) - \omega)v + b(t)\,u(t)]\,\frac{\omega}{\epsilon}\rho_1$ for a. e. $t \in I$, $u \in [\rho_1, \frac{\rho_1}{c}]$ and $v \in [-\frac{\rho_1}{c}, \frac{\rho_1}{c}]$. Thus,

$$\inf\left\{\frac{h(t) - b(t)u - a(t)v + \omega v}{\rho_1}\ :\ (t,v) \in [w,d] \times [-\rho_1/c, \rho_1/c]\right\} > \frac{\omega}{\epsilon},$$

and hence, $(\mathrm{I}^0_{\rho_1,\omega})$ is satisfied.

Finally, (S_1) in Theorem 4.5.5 is satisfied and we get the desired result. ∎

Remark 4.5.7. In the hypothesis of Theorem 4.5.6, if $\omega < \frac{\pi}{4}T$, we can take $[w,d] = [-T,T]$ and continue with the proof of Theorem 4.5.6 as done above. This guarantees that u is positive in $[-T,T]$.

5. General linear equations

In this chapter we study differential problems in which the reflection operator and the Hilbert transform are involved. We reduce these problems to ordinary differential equations in order to solve them. Also, we describe a general method for obtaining the Green's function of reducible functional differential equations and illustrate it with the case of homogeneous boundary value problems with reflection and several specific examples.

It is important to point out that these transformations, necessary to reduce the problem to an ordinary one, are of a purely algebraic nature. It is, in this sense, similar to the *algebraic analysis* theory which, through the study of Ore algebras and modules, obtains important information about some functional problems, including explicit solutions [21, 50]. Nevertheless, the algebraic structures we deal with here are somewhat different, e. g., they are not in general Ore algebras [†]

Among the reducible functional differential equations, those with reflection have gathered great interest, some of it due to their applications to supersymmetric quantum mechanics [73, 147, 153] or to other areas of analysis like topological methods [34].

In this chapter, following [44] we put special emphasis in two operators appearing in the equations: the reflection operator and the Hilbert transform. Both of them have exceptional algebraic properties which make them fit for our approach.

5.1 Differential operators with reflection

In this Section we will study a particular family of operators, those that are combinations of the differential operator D, the pullback operator of the reflection $\varphi(t) = -t$, denoted by $\varphi^*(f)(t) = f(-t)$, and the identity operator, Id. In order to freely apply the operator D without worrying too much about it's domain of definition, we will consider that D acts on the set of functions locally of bounded variation on $\mathbb{R}$, $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$[‡]. Given a compact interval K, the space $\mathrm{BV}(K)$ is defined as the set $\{f : I \to \mathbb{R} \mid V(f) < +\infty\}$ where $V(f) = \sup\limits_{P\in\mathcal{P}_K} \sum_{i=0}^{n_P-1} |f(x_{i+1}) - f(x_i)|$, $P = \{x_0, \dots, x_{n_P}\}$, $\min K = x_0 < x_1 < \dots < x_{n_P-1} < x_{n_P} = \max K$ and $\mathcal{P}_K$ is the set of partitions of K. $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ is the set

$$\{f : \mathbb{R} \to \mathbb{R} \mid f|_K \in \mathrm{BV}(K), \text{ for all } K \subset \mathbb{R} \text{ compact}\}.$$

It is well known that any function locally of bounded variation $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ can be expressed as

$$f(x) = f(x_0) + \int_{x_0}^{x} g(y)\,\mathrm{d}y + h(x),$$

[†] We refer the reader to [118, 149–151] for an algebraic approach to the abstract theory of boundary value problems and its applications to symbolic computation.

[‡] Since we will be working with $\mathbb{R}$ as a domain throughout this chapter, it will be in our interest to take the local versions of the classical function spaces. By local version we mean that, if we restrict the function to a compact set, the restriction belongs to the classical space defined with that compact set as domain for its functions.

for any $x_0 \in \mathbb{R}$, where $g \in \mathrm{L}^1(\mathbb{R})$, and h is the function which is constant except for a countable number of discontinuities (cf. [37, 116]). This implies that the distributional derivative (we will call it *weak derivative* as shorthand) of f is

$$f' = g + \sum_{n\in\mathbb{N}} h_n \delta_{x_n}, \tag{5.1.1}$$

where δ_x is the Dirac distribution at x, the x_n are the points at which h has discontinuities and h_n is the magnitude of the discontinuity at x_n. In this way, we will define $Df := g$ (we will restate this definition in a more general way further on).

As we did in Section 2.2, we now consider the real abelian group $\mathbb{R}[D,\varphi^*]$ of generators $\{D^k, \varphi^* D^k\}_{k=0}^{\infty}$. If we take the usual composition for operators in $\mathbb{R}[D,\varphi^*]$, we observe that $D\varphi^* = -\varphi^* D$, so composition is closed in $\mathbb{R}[D,\varphi^*]$, which makes it a *non commutative algebra*. In general, $D^k\varphi^* = (-1)^k\varphi^* D^k$ for $k = 0, 1, \dots$

The elements of $\mathbb{R}[D,\varphi^*]$ are of the form

$$L = \sum_i a_i \varphi^* D^i + \sum_j b_j D^j \in \mathbb{R}[D,\varphi^*]. \tag{5.1.2}$$

For convenience, we consider the sums on i and j such that $i, j \in \{0, 1, \dots\}$, but taking into account that the real coefficients a_i, b_j are zero for big enough indices – that is, we are dealing with finite sums.

Despite the non commutativity of the composition in $\mathbb{R}[D,\varphi^*]$ there are interesting relations in this algebra.

First, notice that $\mathbb{R}[D,\varphi^*]$ is not a unique factorization domain. Take a polynomial $P = D^2 + \beta D + \alpha$ where $\alpha, \beta \in \mathbb{R}$, and define the following operators.

If $\beta^2 - 4\alpha \geq 0$,

$$
\begin{aligned}
L_1 &:= D + \frac{1}{2}\left(\beta - \sqrt{\beta^2 - 4\alpha}\right),\\
R_1 &:= D + \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\alpha}\right),\\
L_2 &:= \varphi^* D - \sqrt{2}D + \frac{1}{2}\left(\beta - \sqrt{\beta^2 - 4\alpha}\right)\varphi^* + \frac{\left(-\beta + \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},\\
R_2 &:= \varphi^* D - \sqrt{2}D - \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)\varphi^* - \frac{\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},\\
L_3 &:= \varphi^* D - \sqrt{2}D + \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)\varphi^* - \frac{\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},\\
R_3 &:= \varphi^* D - \sqrt{2}D + \frac{1}{2}\left(-\beta + \sqrt{\beta^2 - 4\alpha}\right)\varphi^* + \frac{\left(-\beta + \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},\\
L_4 &:= \varphi^* D + \sqrt{2}D + \frac{1}{2}\left(\beta - \sqrt{\beta^2 - 4\alpha}\right)\varphi^* + \frac{\left(\beta - \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},
\end{aligned}
$$

$$R_4 := \varphi^* D + \sqrt{2} D - \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)\varphi^* + \frac{\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},$$
$$L_5 := \varphi^* D + \sqrt{2} D + \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)\varphi^* + \frac{\left(\beta + \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}},$$
$$R_5 := \varphi^* D + \sqrt{2} D + \frac{1}{2}\left(-\beta + \sqrt{\beta^2 - 4\alpha}\right)\varphi^* + \frac{\left(\beta - \sqrt{\beta^2 - 4\alpha}\right)}{\sqrt{2}}.$$

If $\beta = 0$ and $\alpha \le 0$,

$$L_6 := \varphi^* D + \sqrt{-\alpha}\varphi^*,$$
$$L_7 := \varphi^* D - \sqrt{-\alpha}\varphi^*.$$

If $\beta = 0$ and $\alpha \ge 0$,

$$L_8 := D + \sqrt{\alpha}\varphi^*,$$
$$L_9 := D - \sqrt{\alpha}\varphi^*.$$

If $\beta = 0$ and $\alpha \le 1$,

$$L_{10} := \varphi^* D - \sqrt{1-\alpha}\varphi^* + 1,$$
$$R_{10} := -\varphi^* D + \sqrt{1-\alpha}\varphi^* + 1,$$
$$L_{11} := \varphi^* D + \sqrt{1-\alpha}\varphi^* + 1,$$
$$R_{11} := -\varphi^* D - \sqrt{1-\alpha}\varphi^* + 1.$$

If $\beta = 0$, $\alpha \neq 0$ and $\alpha \le 1$,

$$L_{12} := \varphi^* D - \sqrt{1-\alpha} D + \alpha,$$
$$R_{12} := -\frac{1}{\alpha}\varphi^* D + \frac{\sqrt{1-\alpha}}{\alpha} D + 1,$$
$$L_{13} := \varphi^* D + \sqrt{1-\alpha} D + \alpha,$$
$$R_{13} := -\frac{1}{\alpha}\varphi^* D - \frac{\sqrt{1-\alpha}}{\alpha} D + 1.$$

Then,

$$P = L_1 R_1 = R_1 L_1 = R_2 L_2 = R_3 L_3 = R_4 L_4 = R_5 L_5,$$

and, when $\beta = 0$,

$$P = -L_6^2 = -L_7^2 = L_8^2 = L_9^2 = R_{10} L_{10} = L_{10} R_{10} = R_{11} L_{11}$$

$$= L_{11}R_{11} = R_{12}L_{12} = L_{12}R_{12} = R_{13}L_{13} = L_{13}R_{13}.$$

Observe that only L_1 and R_1 commute in the case of $\beta \neq 0$.

This rises the question on whether we can decompose every differential polynomial P in the composition of two 'order one' (or degree (1,1), cf. page 31) elements of $\mathbb{R}[D, \varphi^*]$, but this is not the case in general. Just take $Q = D^2 + D + 1$ (observe that Q is not in any of the aforementioned cases). Consider a decomposition of the kind

$$(a\varphi^* D + bD + c\varphi^* + d)\,(e\varphi^* D + gD + h\varphi^* + j) = Q,$$

where a, b, c, d, e, g, h and j are real coefficients to be determined. The resulting system

$$\begin{cases} dh + cj = 0, \\ de - cg + bh + aj = 0, \\ be - ag = 0, \\ -ae + bg = 1, \\ ch + dj = 1, \\ -ce + dg + ah + bj = 1, \end{cases}$$

has no solution for real coefficients.

Let $\mathbb{R}[D]$ be the ring of polynomials with real coefficients on the variable D. The following result states a very useful property of the algebra $\mathbb{R}[D, \varphi^*]$.

Theorem 5.1.1. *Take L as defined in* (5.1.2) *and take*

$$R = \sum_k a_k \varphi^* D^k + \sum_l (-1)^{l+1} b_l D^l \in \mathbb{R}[D, \varphi^*]. \tag{5.1.3}$$

Then $RL = LR \in \mathbb{R}[D]$.

Proof.

$$\begin{aligned} RL = & \sum_{i,k} (-1)^k a_i a_k D^{i+k} + \sum_{j,k} b_j a_k \varphi^* D^{j+k} + \sum_{i,l} (-1)^l (-1)^{l+1} a_i b_l \varphi^* D^{i+l} \\ & + \sum_{j,l} (-1)^{l+1} b_j b_l D^{j+l} \\ = & \sum_{i,k} (-1)^k a_i a_k D^{i+k} + \sum_{j,l} (-1)^{l+1} b_j b_l D^{j+l}. \end{aligned} \tag{5.1.4}$$

Hence, $RL \in \mathbb{R}[D]$.

Observe that, if we take R in the place of L in the hypothesis of the Theorem, we obtain L in the place of R and so, by expression (5.1.4) $LR \in \mathbb{R}[D]$. ■

Remark 5.1.2. Some interesting remarks on the coefficients of the operator $S = RL$ defined in Theorem 5.1.1 can be made.

If we have

$$S = \sum_k c_k D^k = RL = \sum_{i,k} (-1)^k a_i a_k D^{i+k} + \sum_{j,l} (-1)^{l+1} b_j b_l D^{j+l},$$

then

$$c_k = \sum_{i=0}^{k} (-1)^i (a_i a_{k-i} - b_i b_{k-i}).$$

A closer inspection reveals that

$$c_k = \begin{cases} 0, & k \text{ odd}, \\ 2\sum_{i=0}^{\frac{k}{2}-1} (-1)^i \left(a_i a_{k-i} - b_i b_{k-i}\right) + (-1)^{\frac{k}{2}} \left(a_{\frac{k}{2}}^2 - b_{\frac{k}{2}}^2\right), & k \text{ even}. \end{cases}$$

This has some important consequences. If $L = \sum_{i=0}^{n} a_i \varphi^* D^i + \sum_{j=0}^{n} b_j D^j$ with $a_n \neq 0$ or $b_n \neq 0$, we have that $c_{2n} = (-1)^n (a_n^2 - b_n^2)^\dagger$ and so, if $a_i = \pm b_i$, then $c_{2i} = 0$. This shows that composing two elements of $\mathbb{R}[D, \varphi^*]$ we can get another element which has simpler terms in the sense of derivatives of less order. We illustrate this with two examples.

Example 5.1.3. Take $n \geq 3$, $L = \varphi^* D^n + D^n + D - \mathrm{Id}$ and $R = -\varphi^* D^n + (-1)^n D^n - D - \mathrm{Id}$. Then, $RL = -2D^{\alpha(n)} - D^2 + \mathrm{Id}$ where $\alpha(n) = n$ if n is even and $\alpha(n) = n+1$ if n is odd.

If we take $n \geq 0$, $L = \varphi^* D^{2n+1} + D^{2n+1} + \mathrm{Id}$ and $R = \varphi^* D^{2n+1} + D^{2n+1} - \mathrm{Id}$. Then, $RL = -\mathrm{Id}$.

Example 5.1.4. Consider the equation

$$x^{(3)}(t) + x^{(3)}(-t) + x(t) = \sin t.$$

Applying the operator $\varphi^* D^3 + D^3 - \mathrm{Id}$ to both sides of the equation we obtain $x(t) = \sin t + 2\cos t$. This is the unique solution of the equation, to which we had not imposed any extra conditions.

5.2 Boundary Value Problems

In this section we obtain the Green's function of n-th order boundary value problems with reflection and constant coefficients. We point out that the same approach used in this section is also valid for initial problems among other types of conditions.

Let $I = [a, b] \subset \mathbb{R}$ be an interval and $f \in \mathrm{L}^1(I)$. Consider now the following problem with the usual derivative.

$$\begin{aligned} Su(t) &:= \sum_{k=0}^{n} a_k u^{(k)}(t) = f(t),\ t \in I, \\ B_i u &:= \sum_{j=0}^{n-1} \alpha_{ij} u^{(j)}(a) + \beta_{ij} u^{(j)}(b) = 0,\ i = 1, \dots, n. \end{aligned} \tag{5.2.1}$$

The following Theorem from [31] states the cases where we can find a unique solution for problem (5.2.1)[‡].

[†] This is so because if $i \in \{0, \dots, n-1\}$, then $2i - i \in \{n+1, \dots, 2n\}$ and a_i (respectively b_n) are nonzero only for $n \in \{0, \dots, n\}$.

[‡] In [31], this result is actually stated for nonconstant coefficients, but the case of constant coefficients is enough for our purposes.

Theorem 5.2.1. *Assume the following homogeneous problem has a unique solution*

$$Su(t) = 0,\ t \in I,\ B_i u = 0,\ i = 1, \dots n.$$

Then there exists a unique function, called Green's function*, such that*

(G1) G *is defined on the square* I^2[†].

(G2) The partial derivatives $\frac{\partial^k G}{\partial t^k}$ *exist and are continuous on* I^2 *for* $k = 0, \dots, n-2$.

(G3) $\frac{\partial^{n-1} G}{\partial t^{n-1}}$ *and* $\frac{\partial^n G}{\partial t^n}$ *exist and are continuous on* $I^2 \backslash \{(t,t)\ :\ t \in I\}$.

(G4) The lateral limits $\frac{\partial^{n-1} G}{\partial t^{n-1}}(t,t^+)$ *and* $\frac{\partial^{n-1} G}{\partial t^{n-1}}(t,t^-)$ *exist for every* $t \in (a,b)$ *and*

$$\frac{\partial^{n-1} G}{\partial t^{n-1}}(t,t^-) - \frac{\partial^{n-1} G}{\partial t^{n-1}}(t,t^+) = \frac{1}{a_n}.$$

(G5) For each $s \in (a,b)$ *the function* $G(\cdot,s)$ *is a solution of the differential equation* $Su = 0$ *on* $I\backslash\{s\}$.

(G6) For each $s \in (a,b)$ *the function* $G(\cdot,s)$ *satisfies the boundary conditions*

$$B_i u = 0\, i = 1, \dots, n.$$

Furthemore, the function $u(t) := \int_a^b G(t,s) f(s)\,\mathrm{d}s$ *is the unique solution of the problem* (5.2.1).

Using the properties (G1)–(G6) and Theorem 5.1.1 one can prove Theorem 5.2.3. The proof of this result will be a direct consequence of Theorem 5.3.8.

Definition 5.2.2. Given an operator $\mathcal{L}$ for functions of one variable, define the operator $\mathcal{L}_\vdash$ as $\mathcal{L}_\vdash G(t,s) := \mathcal{L}(G(\cdot,s))|_t$ for every s and any suitable function G.

Theorem 5.2.3. *Let* $I = [-T,T]$. *Consider the problem*

$$Lu(t) = h(t),\ t \in I,\ B_i u = 0,\ i = 1, \dots, n, \tag{5.2.2}$$

where L *is defined as in* (5.1.2), $h \in \mathrm{L}^1(I)$ *and*

$$B_i u := \sum_{j=0}^{n-1} \alpha_{ij} u^{(j)}(-T) + \beta_{ij} u^{(j)}(T).$$

Then, there exists $R \in \mathbb{R}[D, \varphi^*]$ *– defined as in* (5.1.3)*– such that* $S := RL \in \mathbb{R}[D]$ *and the unique solution of problem* (5.2.2) *is given by* $\int_a^b R_\vdash G(t,s) h(s)\,\mathrm{d}s$ *where* G *is the Green's function associated to the problem* $Su = 0$, $B_i Ru = 0$, $B_i u = 0$, $i = 1, \dots, n$, *assuming that the homogeneous problem* $Su = 0$, $B_i Ru = 0$, $B_i u = 0$, $i = 1, \dots, n$, *has a unique solution.*

[†] In most applications it is not necessary to define the Green's function on the diagonal for we will be integrating the expression $\int_a^b G(t,s) f(s)\,\mathrm{d}s$. Hence, the uniqueness mentioned in Theorem 5.2.1 has to be understood 'save for the values on the diagonal'.

For the following example, let us explain some notations. Let $k,p \in \mathbb{N}$. We denote by $W^{k,p}(I)$ the Sobolev Space defined by

$$W^{k,p}(I) = \{u \in \mathrm{L}^{\mathrm{p}}(I) : D^\alpha u \in \mathrm{L}^{\mathrm{p}}(I) \ \forall \alpha \le k\}.$$

Given a constant $a \in \mathbb{R}$ we can consider the pullback by this constant as a functional $a^* : \mathcal{C}(I) \to \mathbb{R}$ such that $a^* f = f(a)$ in the same way we defined it for functions.

Example 5.2.4. Consider the following problem.

$$u''(t) + a\,u(-t) + b\,u(t) = h(t),\ t \in I, \quad u(-T) = u(T),\ u'(-T) = u'(T). \tag{5.2.3}$$

where $h \in W^{2,1}(I)$. Then, the operator we are considering is $L = D^2 + a\,\varphi^* + b$. If we take $R := D^2 - a\,\varphi^* + b$, we have that $RL = D^4 + 2b\,D^2 + b^2 - a^2$.

The boundary conditions are $((T^*) - (-T)^*)u = 0$ and $((T^*) - (-T)^*)Du = 0$. Taking this into account, we add the conditions

$$0 = ((T^*) - (-T)^*)Ru = ((T^*) - (-T)^*)(D^2 - a\,\varphi^* + b)u = ((T^*) - (-T)^*)D^2 u,$$

$$0 = ((T^*) - (-T)^*)RDu = ((T^*) - (-T)^*)(D^2 - a\,\varphi^* + b)Du = ((T^*) - (-T)^*)D^3 u.$$

That is, our new *reduced* problem is

$$u^{(4)}(t) + 2b\,u''(t) + (b^2 - a^2)u(t) = f(t),\ t \in I, \quad u^{(k)}(-T) = u^{(k)}(T),\ k = 0, \dots, 3. \tag{5.2.4}$$

where $f(t) = R\,h(t) = h''(t) - a\,h(-t) + b\,h(t)$.

Observe that this problem is equivalent to the system of equations (a chain of two order two problems)

$$u''(t) + (b + a)u(t) = v(t),\ t \in I, \quad u(-T) = u(T),\ u'(-T) = u'(T),$$
$$v''(t) + (b - a)v(t) = f(t),\ t \in I, \quad v(-T) = v(T),\ v'(-T) = v'(T).$$

Thus, it is clear that

$$u(t) = \int_{-T}^{T} G_1(t,s)v(s)\,\mathrm{d}s,\ v(t) = \int_{-T}^{T} G_2(t,s)f(s)\,\mathrm{d}s,$$

where, G_1 and G_2 are the Green's functions related to the previous second order problems. Explicitly, in the case $b > |a|$ (the study for other cases would be analogous cf. page 85),

$$2\sqrt{b+a}\sin(\sqrt{b+a}\,T)G_1(t,s) = \begin{cases} \cos\sqrt{b+a}(T + s - t) & \text{if} \quad s \le t, \\ \cos\sqrt{b+a}(T - s + t) & \text{if} \quad s > t. \end{cases}$$

and

$$2\sqrt{b-a}\sin(\sqrt{b-a}\,T)G_2(t,s) = \begin{cases} \cos\sqrt{b-a}(T + s - t) & \text{if} \quad s \le t, \\ \cos\sqrt{b-a}(T - s + t) & \text{if} \quad s > t. \end{cases}$$

Hence, the Green's function G for problem (5.2.4) is given by

$$G(t,s) = \int_{-T}^{T} G_1(t,r)G_2(r,s)\,\mathrm{d}r.$$

Therefore, using Theorem 5.2.3, the Green's function for problem (5.2.3) is

$$\overline{G}(t,s) = R_{\vdash} G(t,s) = \frac{\partial^2 G}{\partial t^2}(t,s) - a\,G(-t,s) + b\,G(t,s).$$

Remark 5.2.5. We can reduce the assumptions on the regularity of h to $h \in \mathrm{L}^1(I)$ just taking into account the density of $W^{2,1}(I)$ in $\mathrm{L}^1(I)$.

Remark 5.2.6. Example 5.1.4 illustrates the importance of the existence and uniqueness of solution of the problem $Su = 0$, $B_iRu = 0$, $B_iu = 0$ in the hypothesis of Theorem 5.2.3. In general, when we compose two linear ordinary differential equations, respectively of orders m and n and a number m and n of conditions, we obtain a new problem of order $m+n$ and $m+n$ conditions. As we see this is not the case in the reduction provided by Theorem 5.2.3. In the case the order of the reduced problem is less than $2n$ anything is possible: we may have an infinite number of solutions, no solution or uniqueness of solution being the problem nonhomogeneous. The following example illustrates this last case.

Example 5.2.7. Consider the problem

$$Lu(t) := u^{(4)}(t) + u^{(4)}(-t) + u''(-t) = h(t),\ t \in [-1,1],\ u(1) = u(-1) = 0,$$

where $h \in W^{4,1}([-1,1])$.

For this case, $Ru(t) := -u^{(4)}(t) + u^{(4)}(-t) + u''(-t)$ and the reduced equation is $RLu = 2u^{(6)} + u^{(4)} = Rh$, which has order $6 < 2\cdot 4 = 8$, so there is a reduction of the order. Now we have to be careful with the new *reduced* boundary conditions.

$$\begin{aligned}
B_1u(t) &= u(1) = 0,\\
B_2u(t) &= u(-1) = 0,\\
B_1Ru(t) &= -u^{(4)}(1) + u^{(4)}(-1) + u''(-1) = 0,\\
B_2Ru(t) &= -u^{(4)}(-1) + u^{(4)}(1) + u''(1) = 0,\\
B_1Lu(t) &= u^{(4)}(1) + u^{(4)}(-1) + u''(-1) = h(1),\\
B_2Lu(t) &= u^{(4)}(-1) + u^{(4)}(1) + u''(1) = h(-1).
\end{aligned} \tag{5.2.5}$$

Being the two last conditions the obtained from applying the original boundary conditions to the original equation.

(5.2.5) is a system of linear equations which can be solved for u and its derivatives as

$$u(1) = u(-1) = 0, -u''(1) = u''(-1) = \frac{1}{2}(h(1) - h(-1)),\ u^{(4)}(\pm 1) = \frac{h(\pm 1)}{2}. \tag{5.2.6}$$

Consider now the reduced problem

$$\begin{aligned}
&2u^{(6)}(t) + u^{(4)}(t) = Rh(t) =: f(t),\ t \in [-1,1],\\
&u(1) = u(-1) = 0,\ -u''(1) = u''(-1) = \frac{1}{2}(h(1) - h(-1)),\ u^{(4)}(\pm 1) = \frac{h(\pm 1)}{2},
\end{aligned}$$

and the change of variables $v(t) := u^{(4)}(t)$. Now we look the solution of

$$2v''(t) + v(t) = f(t),\ t \in [-1,1],\ v(\pm 1) = \frac{h(\pm 1)}{2},$$

Which is given by

$$v(t) = \int_{-1}^{1} G(t,s)f(s)\,\mathrm{d}s - \frac{h(1)\csc\sqrt{2}}{2}\sin\left(\frac{t-1}{\sqrt{2}}\right) + \frac{h(-1)\csc\sqrt{2}}{2}\sin\left(\frac{t+1}{\sqrt{2}}\right),$$

where

$$G(t,s):=\frac{\csc\sqrt{2}}{\sqrt{2}}\begin{cases}\sin\left(\frac{s+1}{\sqrt{2}}\right)\sin\left(\frac{t-1}{\sqrt{2}}\right), & -1\le s\le t\le 1,\\ \sin\left(\frac{s-1}{\sqrt{2}}\right)\sin\left(\frac{t+1}{\sqrt{2}}\right), & -1\le t<s\le 1.\end{cases}$$

Now, it is left to solve the problem

$$u^{(4)}(t)=v(t),\ u(1)=u(-1)=0,\ -u''(1)=u''(-1)=\frac{1}{2}(h(1)-h(-1)).$$

The solution is given by

$$u(t)=\int_{-1}^{1}K(t,s)v(s)\,\mathrm{d}s-\frac{h(1)-h(-1)}{12}t(t-1)(t+1).$$

where

$$K(t,s)=\frac{1}{12}\begin{cases}(s+1)(t-1)\left(s^2+2s+t^2-2t-2\right), & -1\le s\le t\le 1,\\ (s-1)(t+1)\left(s^2-2s+t^2+2t-2\right), & -1\le t<s\le 1.\end{cases}$$

Hence, taking $J(t,s)=\int_{-1}^{1}H(t,r)G(r,s)\,\mathrm{d}s$,

$$J(t,s):=$$

$$\frac{\csc\sqrt{2}}{12\sqrt{2}}\begin{cases}\sqrt{2}\sin(\sqrt{2})(s+1)(t-1)[s(s+2)+(t-2)t-14] & \\ +24\cos\left(\frac{s-t+2}{\sqrt{2}}\right)-24\cos\left(\frac{s+t}{\sqrt{2}}\right), & -1\le s\le t\le 1,\\ \sqrt{2}\sin(\sqrt{2})(s-1)(t+1)[(s-2)s+t(t+2)-14] & \\ +24\cos\left(\frac{s-t-2}{\sqrt{2}}\right)-24\cos\left(\frac{s+t}{\sqrt{2}}\right), & -1\le t<s\le 1.\end{cases}$$

Therefore,

$$\begin{aligned}u(t)=&\int_{-1}^{1}J(t,s)f(s)\,\mathrm{d}s\\ &-\frac{h(1)\csc\sqrt{2}}{2}\left[\frac{1}{6}(t-5)(t-1)(t+3)\sin\left(\sqrt{2}\right)+4\sin\left(\frac{t-1}{\sqrt{2}}\right)\right]\\ &+\frac{h(-1)\csc\sqrt{2}}{2}\left[\frac{1}{6}(t-3)(t+1)(t+5)\sin\left(\sqrt{2}\right)+4\sin\left(\frac{t+1}{\sqrt{2}}\right)\right]\\ &-\frac{h(1)-h(-1)}{12}t(t-1)(t+1).\end{aligned}$$

5.3 The reduced problem

The usefulness of a theorem of the kind of Theorem 5.2.3 is clear, for it allows the obtaining of the Green's function of any problem of differential equations with constant coefficients and involutions. The proof of this Theorem relies heavily on the properties $(G1)-(G6)$, so our main goal now is to consider abstractly these properties in order to apply them in a more general context with different kinds of operators.

Let X be a vector subspace of $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, and $(\mathbb{R},\tau)$ the real line with its usual topology. Define $X_U := \{f|_U \ : \ f \in X\}$ for every $U \in \tau$ (observe that X_U is a vector space as well). Assume that X satisfies the following property.

(P) *For every partition of $\mathbb{R}$, $\{S_j\}_{j\in J} \cup \{N\}$, consisting of measurable sets where N has no accumulation points and the S_j are open, if $f_j \in X_{S_j}$ for every $j \in J$, then there exists $f \in X$ such that $f|_{S_j} = f_j$ for every $j \in J$.*

Example 5.3.1. The set of locally absolutely continuous functions $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ does not satisfy **(P)**. To see this just take the following partition of $\mathbb{R}$: $S_1 = (-\infty,0)$, $S_2 = (0,+\infty)$, $N = \{0\}$ and consider $f_1 \equiv 0, f_2 \equiv 1$. $f_j \in \mathrm{AC}(\mathbb{R})_{S_j}$ for $j = 1,2$, but any function f such that $f|_{S_j} = f_j$, $j = 1,2$ has a discontinuity at 0, so it cannot be absolutely continuous. That is, **(P)** is not satisfied.

Example 5.3.2. $X = \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ satisfies **(P)**. Take a partition of $\mathbb{R}$, $\{S_j\}_{j\in J} \cup \{N\}$, with the properties of **(P)** and a family of functions $(f_j)_{j\in J}$ such that $f_j \in X_{S_j}$ for every $j \in J$. We can further assume, without lost of generality, that the S_j are connected. Define a function f such that $f|_{S_j} := f_j$ and $f|_N = 0$. Take a compact set $K \subset \mathbb{R}$. Then, by Bolzano-Weierstrass' and Heine-Borel's Theorems, $K \cap N$ is finite for N has no accumulation points. Therefore, $J_K := \{j \in J : S_j \cap K \neq \emptyset\}$ is finite as well. To see this denote by ∂S the boundary of a set S and observe that $N \cap K = \cup_{j\in J} \partial(S_j \cup K)$ and that the sets $\partial(S_j \cap K) \cap \partial(S_k \cap K)$ are finite for every $j,k \in J$.

Thus, the variation of f in K is $V_K(f) \leq \sum_{j\in J_K} V_{S_j}(f) < \infty$ since f is of bounded variation on each S_j. Hence, X satisfies **(P)**.

Throughout this section we will consider a function space X satisfying **(P)** and two families of linear operators $L = \{L_U\}_{U\in\tau}$ and $R = \{R_U\}_{U\in\tau}$ that satisfy

Locality: $L_U \in \mathcal{L}(X_U, \mathrm{L}^1_{\mathrm{loc}}(U)), R_U \in \mathcal{L}(\mathrm{im}(L_U), \mathrm{L}^1_{\mathrm{loc}}(U))$,

Restriction: $L_V(f|_V) = L_U(f)|_V$, $R_V(f|_V) = R_U(f)|_V$ for every $U, V \in \tau$ such that $V \subset U$†.

The following definition allows us to give an example of an space that satisfies the properties of locality and restriction.

Definition 5.3.3. Let $f : \mathbb{R} \to \mathbb{R}$ and assume there exists a partition $\{S_j\}_{j\in J} \cup \{N\}$ of $\mathbb{R}$ consisting of measurable sets where N is of zero Lebesgue measure satisfying that the weak derivative g_i exists for every $f|_{S_j}$, then a function g such that $g|_{S_j} = g_j$ is called the *very weak derivative* (vw-derivative) of f.

Remark 5.3.4. The vw-derivative is uniquely defined save for a zero measure set and is equivalent to the weak derivative for absolutely continuous functions.

†The definitions here presented of L and R are deeply related to Sheaf Theory. Since the authors want to make this work as self-contained as possible, we will not deepen into that fact.

Nevertheless, the vw-derivative is different from the derivative of distributions. For instance, the derivative of the Heavyside function in the distributional sense is de Dirac delta at 0, whereas its vw-derivative is zero. What is more, the kernel of the vw-derivative is the set of functions which are constant on a family of open sets $\{S_j\}_{j\in J}$ such that $\mathbb{R}\backslash(\cup_{j\in J}S_j)$ has Lebesgue measure zero.

Example 5.3.5. Take $X = \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ and $L = D$ to be the very weak derivative. Then L satisfies the locality and restriction hypotheses.

Remark 5.3.6. The vw-derivative, as defined here, is the D operator defined in (5.1.1) for functions of bounded variation. In other words, the vw-derivative ignores the jumps and considers only those parts with enough regularity.

Remark 5.3.7. The locality property allows us to treat the maps L and R as if they were just linear operators in $\mathcal{L}(X,\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and $\mathcal{L}(\mathrm{im}(L),\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ respectively, although we must not forget their more complex structure.

Assume $X_U\subset\mathrm{im}(L_U)\subset\mathrm{im}(R_U)$ for every $U\in\tau$. $B_i\in\mathcal{L}(\mathrm{im}(R_{\mathbb{R}}),\mathbb{R})$, $i=1,\dots,m$ and $h\in\mathrm{im}(L_{\mathbb{R}})$. Consider now the following problem

$$Lu=h,\ B_iu=0,\ i=1,\dots,m. \tag{5.3.1}$$

Let

$$Z:=\{G:\mathbb{R}^2\to\mathbb{R}\,|\,G(t,\cdot)\in X\cap\ \ (\mathbb{R})\text{ and }\operatorname{supp}\{G(t,\cdot)\}\text{ is compact, }s\in\mathbb{R}\}.$$

Z is a vector space.

Let $f\in\mathrm{im}(L_{\mathbb{R}})$ and consider the problem

$$RLv=f,\ B_iv=0,\ B_iRv=0,\ i=1,\dots,m. \tag{5.3.2}$$

Let $G\in Z$ and define the operator H_G such that $H_G(h)|_t:=\int_{\mathbb{R}}G(t,s)h(s)\,\mathrm{d}s$. We have now the following theorem relating problems (5.3.1) and (5.3.2). Recall that, by definition, $\mathcal{L}_{\vdash}G(t,s):=\mathcal{L}(G(\cdot,s))|_t$.

Theorem 5.3.8. *Assume L and R are the aforementioned operators with the locality and restriction properties and let $h\in\mathrm{Dom}(R_{\mathbb{R}})$. Assume L commutes with R and that there exists $G\in Z$ such that*

$$\begin{aligned}
(I)&\quad (RL)_{\vdash}G=0,\\
(II)&\quad B_{i\,\vdash}G=0,\ i=1,\dots,m,\\
(III)&\quad (B_iR)_{\vdash}G=0,\ i=1,\dots,m,\\
(IV)&\quad RLH_Gh=H_{(RL)_{\vdash}G}h+h,\\
(V)&\quad LH_{R_{\vdash}G}h=H_{L_{\vdash}R_{\vdash}G}h+h.\\
(VI)&\quad B_iH_G=H_{B_{i\,\vdash}G},\ i=1,\dots,m,\\
(VII)&\quad B_iRH_G=B_iH_{R_{\vdash}G}=H_{(B_iR)_{\vdash}G},\ i=1,\dots,m,
\end{aligned}$$

Then, $v:=H_G(h)$ is a solution of problem (5.3.2) *and $u:=H_{R_{\vdash}G}(h)$ is a solution of problem* (5.3.1).

Proof. (I) and (IV) imply that

$$RLv = RLH_G h = H_{(RL)_\vdash G} h + h = H_0 h + h = h.$$

On the other hand, (III) and (VII) imply that, for every $i = 1, \dots, m$,

$$B_i Rv = B_i RH_G h = H_{(B_i R)_\vdash G} h = 0.$$

All the same, by (II) and (VI),

$$B_i v = B_i H_G h = H_{B_{i\vdash} G} h = 0.$$

Therefore, v is a solution to problem (5.3.2).

Now, using (I) and (V) and the fact that $LR = RL$, we have that

$$Lu = LH_{R_\vdash G} h = H_{L_\vdash R_\vdash G} h + h = H_{(LR)_\vdash G} h + h = H_{(RL)_\vdash G} h + h = h.$$

Taking into account (III) and (VII),

$$B_i u = B_i H_{R_\vdash G}(h) = H_{(B_i R)_\vdash G} h = 0,\ i = 1, \dots, m.$$

Hence, u is a solution of problem (5.3.1). ■

The following Corollary is proved in the same way as the previous Theorem.

Corollary 5.3.9. *Assume $G \in Z$ satisfies*

(1) $L_\vdash G = 0$,
(2) $B_{i\vdash} G = 0,\ i = 1, \dots, m$,
(3) $LH_G h = H_{L_\vdash G} h + h$,
(4) $B_i H_G h = H_{B_{i\vdash} G} h$.

Then $u = H_G h$ is a solution of problem (5.3.1).

Proof of Theorem 5.2.3. Originally, we would need to take $h \in \operatorname{Dom}(R)$, but by a simple density argument –$\mathcal{C}^\infty(I)$ is dense in $\mathrm{L}^1(I)$– we can take $h \in \mathrm{L}^1(I)$. If we prove that the hypothesis of Theorem 5.3.8 are satisfied, then the existence of solution will be proved.

First, Theorem 5.1.1 guarantees the commutativity of L and R. Now, Theorem 5.2.1 implies hypothesis $(I) - (VII)$ of Theorem 5.3.8 in terms of the vw-derivative. Indeed, (I) is straightforward from $(G5)$. (II) and (III) are satisfied because $(G1) - -(G6)$ hold and $B_i u, B_i Ru = 0$. $(G2)$ and $(G4)$ imply (IV) and (V). (VI) and (VII) hold because of $(G2)$, $(G5)$ and the fact that the boundary conditions commute with the integral.

On the other hand, the solution to problem (5.2.2) must be unique for, otherwise, the reduced problem $Su = 0, B_i Ru = 0, B_i u = 0, i = 1, \dots, n$ would have several solutions, contradicting the hypotheses. ■

The following Lemma, in the line of Theorem 4.3.5, extends the application of Theorem 5.2.3 to the case of nonconstant coefficients with some restrictions for problems similar to the one in Example 5.2.4.

Lemma 5.3.10. *Consider the problem*

$$u''(t)+a(t)\,u(-t)+b(t)\,u(t)=h(t), u(-T)=u(T), \tag{5.3.3}$$

where $a\in W^{2,1}_{\text{loc}}(\mathbb{R})$ is nonnegative and even,

$$b=k\,a+\frac{a''}{4a}-\frac{5}{16}\left(\frac{a'}{a}\right)^2,$$

for some constant $k\in\mathbb{R}$, $k^2\neq 1$ and b is integrable.

Define $A(t):=\int_0^t\sqrt{a(s)}\,\mathrm{d}s$, consider

$$u''(t)+u(-t)+k\,u(t)=h(t),\ u(-A(T))=u(A(T))$$

and assume it has a Green's function G.

Then

$$u(t)=\int_{-T}^{T}H(t,s)h(s)\,\mathrm{d}s$$

is a solution of problem (5.3.3) *where*

$$H(t,s):=\sqrt[4]{\frac{a(s)}{a(t)}}G(A(t),A(s)),$$

And $H(t,\cdot)h(\cdot)$ is assumed to be integrable in $[-T,T]$.

Proof. Let G be the Green's function of the problem

$$u''(t)+u(-t)+k\,u(t)=h(t),\ u(-A(T))=u(A(T)),\ u\in \mathrm{W}^{2,1}_{\text{loc}}(\mathbb{R}).$$

Observe that, since $|k|\neq 1$, we are in the cases $(D1)-(D2)$ in Chapter 4. Now, we show that H satisfies the equation, that is,

$$\frac{\partial^2 H}{\partial t^2}(t,s)+a(t)H(-t,s)+b(t)H(t,s)=0 \text{ for a. e. } t,s\in\mathbb{R}.$$

$$\begin{aligned}\frac{\partial^2 H}{\partial t^2}(t,s)=&\frac{\partial^2}{\partial t^2}\left[\sqrt[4]{\frac{a(s)}{a(t)}}G(A(t),A(s))\right]\\ =&\frac{\partial}{\partial t}\left[-\frac{a'(t)}{4}\sqrt[4]{\frac{a(s)}{a^5(t)}}G(A(t),A(s))+\sqrt[4]{a(s)a(t)}\frac{\partial G}{\partial t}(A(t),A(s))\right]\\ =&-\frac{a''(t)}{4}\sqrt[4]{\frac{a(s)}{a^5(t)}}G(A(t),A(s))+\frac{5}{16}(a'(t))^2\sqrt[4]{\frac{a(s)}{a^9(t)}}G(A(t),A(s))\\ &+\sqrt[4]{a(s)a^3(t)}\frac{\partial^2 G}{\partial t^2}(A(t),A(s)).\end{aligned}$$

Therefore,

$$\frac{\partial^2 H}{\partial t^2}(t,s)+a(t)H(-t,s)+b(t)H(t,s)$$

$$
\begin{aligned}
&= \sqrt[4]{a(s)a^3(t)}\,\frac{\partial^2 G}{\partial t^2}(A(t),A(s)) + a(t)\sqrt[4]{\frac{a(s)}{a(t)}}G(-A(t),A(s)) \\
&\quad + k\,a(t)\sqrt[4]{\frac{a(s)}{a(t)}}G(A(t),A(s)) \\
&= \sqrt[4]{a(s)a^3(t)}\left(\frac{\partial^2 G}{\partial t^2}(A(t),A(s)) + G(-A(t),A(s)) + k\,G(A(t),A(s))\right) = 0.
\end{aligned}
$$

The boundary conditions are satisfied as well. ■

The same construction of Lemma 5.3.10 is valid for the case of the initial value problem. We illustrate this in the following example.

Example 5.3.11. Let $a(t) = |t|^p$, $k > 1$. Taking b as in Lemma 5.3.10,

$$b(t) = k|t|^p - \frac{p(p+4)}{16t^2},$$

consider problems

$$u''(t) + a(t)\,u(-t) + b(t)\,u(t) = h(t),\ u(0) = u'(0) = 0 \tag{5.3.4}$$

and

$$u''(t) + u(-t) + k\,u(t) = h(t),\ u(0) = u'(0) = 0. \tag{5.3.5}$$

Using an argument similar as the one in Example 5.2.4 and considering $R = D^2 - \varphi^* + k$, we can reduce problem (5.3.5) to

$$u^{(4)}(t) + 2ku''(t) + (k^2-1)u(t) = f(t),\ u^{(j)}(0) = 0,\ j = 0,\dots,3, \tag{5.3.6}$$

which can be decomposed in

$$
\begin{aligned}
u''(t) + (k+1)u(t) &= v(t),\ t \in I, \quad u(0) = u'(0) = 0, \\
v''(t) + (k-1)v(t) &= f(t),\ t \in I, \quad v(0) = v'(0) = 0,
\end{aligned}
$$

which have as Green's functions, respectively,

$$\tilde{G}_1(t,s) = \frac{\sin\left(\sqrt{k+1}\,(t-s)\right)}{\sqrt{k+1}}\chi_0^t(s),\ t \in \mathbb{R},$$

$$\tilde{G}_2(t,s) = \frac{\sin\left(\sqrt{k-1}\,(t-s)\right)}{\sqrt{k-1}}\chi_0^t(s),\ t \in \mathbb{R}.$$

Then, the Green's function for problem (5.3.6) is

$$
\begin{aligned}
G(t,s) &= \int_s^t \tilde{G}_1(t,r)\tilde{G}_2(r,s)\,\mathrm{d}r \\
&= \frac{1}{2\sqrt{k^2-1}}\left[\sqrt{k-1}\sin\left(\sqrt{k+1}(s-t)\right) - \sqrt{k+1}\sin\left(\sqrt{k-1}(s-t)\right)\right]\chi_0^t(s).
\end{aligned}
$$

Observe that

$$R_{\vdash}G(t,s) = -\left[\frac{\sin\left(\sqrt{k-1}(s-t)\right)}{2\sqrt{k-1}} + \frac{\sin\left(\sqrt{k+1}(s-t)\right)}{2\sqrt{k+1}}\right]\chi_0^t(s).$$

Hence, considering

$$A(t) := \frac{2}{p+2}|t|^{\frac{p}{2}}t,$$

the Green's function of problem (5.3.4) follows the expression

$$H(t,s) := \sqrt[4]{\frac{a(s)}{a(t)}}G(A(t),A(s)),$$

This is,

$$H(t,s) = -\left|\frac{s}{t}\right|^{\frac{p}{4}}\left[\frac{\sin\left(\frac{2\sqrt{k-1}\left(s|s|^{p/2}-t|t|^{p/2}\right)}{p+2}\right)}{2\sqrt{k-1}} + \frac{\sin\left(\frac{2\sqrt{k+1}\left(s|s|^{p/2}-t|t|^{p/2}\right)}{p+2}\right)}{2\sqrt{k+1}}\right]\chi_0^t(s).$$

5.4 The Hilbert transform and other algebras

In this section we devote our attention to new algebras to which we can apply the previous results. To achieve this goal we recall the definition and remarkable properties of the Hilbert transform [114].

We define the *Hilbert transform* H of a function f as

$$\mathrm{H}f(t) := \frac{1}{\pi}\lim_{\epsilon\to\infty}\int_{-\epsilon}^{\epsilon}\frac{f(s)}{t-s}\,\mathrm{d}s \equiv \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{t-s}\,\mathrm{d}s,$$

where the last integral is to be understood as the Cauchy principal value.

Among its properties, we would like to point out the following.

- $\mathrm{H}:\mathrm{L}^{\mathrm{p}}(\mathbb{R})\to\mathrm{L}^{\mathrm{p}}(\mathbb{R})$ is a linear bounded operator for every $p\in(1,+\infty)$ and

$$\|\mathrm{H}\|_p = \begin{cases}\tan\frac{\pi}{2p}, & p\in(1,2],\\ \cot\frac{\pi}{2p}, & p\in[2,+\infty),\end{cases}$$

  in particular $\|\mathrm{H}\|_2 = 1$.

- H *is an anti-involution:* $\mathrm{H}^2 = -\mathrm{Id}$.

- Let $\sigma(t) = at+b$ for $a,b\in\mathbb{R}$. Then $\mathrm{H}\sigma^* = \operatorname{sign}(a)\sigma^*\mathrm{H}$ (in particular, $\mathrm{H}\varphi^* = -\varphi^*\mathrm{H}$). Furthermore, if a linear bounded operator $\mathrm{O}:\mathrm{L}^{\mathrm{p}}(\mathbb{R})\to\mathrm{L}^{\mathrm{p}}(\mathbb{R})$ satisfies this property, $\mathrm{O} = \beta H$ where $\beta\in\mathbb{R}$.

- H *commutes with the derivative:* $\mathrm{H}D = D\mathrm{H}$.

- $\mathsf{H}(f*g) = f*\mathsf{H}g = \mathsf{H}f*g$ where $*$ denotes the convolution.
- H *is an isometry in* $\mathrm{L}^2(\mathbb{R})$*:* $\langle \mathsf{H}f, \mathsf{H}g\rangle = \langle f, g\rangle$ where $\langle\ ,\ \rangle$ is the scalar product in $\mathrm{L}^2(\mathbb{R})$. In particular $\|\mathsf{H}f\|_2 = \|f\|_2$.

Consider now the same construction we did for $\mathbb{R}[D,\varphi^*]$ changing φ^* by H and denote this algebra as $\mathbb{R}[D,\mathsf{H}]$. In this case we are dealing with a commutative algebra. Actually, this algebra is isomorphic to the complex polynomials $\mathbb{C}[D]$. Just consider the isomorphism

$$\mathbb{R}[D,\mathsf{H}] \xrightarrow{\quad\Xi\quad} \mathbb{C}[D]$$

$$\sum_j (a_j\mathsf{H}+b_j)D^j \longmapsto \sum_j (a_j\, i+b_j)D^j$$

Observe that $\Xi|_{\mathbb{R}[D]} = \mathrm{Id}|_{\mathbb{R}[D]}$.

We now state a result analogous to Theorem 5.1.1.

Theorem 5.4.1. *Take*

$$L = \sum_j (a_j\mathsf{H}+b_j)D^j \in \mathbb{R}[D,\mathsf{H}]$$

and define

$$R = \sum_j (a_j\mathsf{H}-b_j)D^j.$$

Then $LR = RL \in \mathbb{R}[D]$.

Remark 5.4.2. Theorem 5.4.1 is clear from the point of view of $\mathbb{C}[D]$. Since $\Xi(R) = -\overline{\Xi(L)}$,

$$RL = \Xi^{-1}(-\Xi(L)\overline{\Xi(L)}) = \Xi^{-1}(-|\Xi(L)|^2).$$

Therefore, $|\Xi(L)|^2 \in \mathbb{R}[D]$, implies $RL \in \mathbb{R}[D]$.

Remark 5.4.3. Since $\mathbb{R}[D,\mathsf{H}]$ is isomorphic to $\mathbb{C}[D]$, the Fundamental Theorem of Algebra also applies to $\mathbb{R}[D,\mathsf{H}]$, which shows a clear classification of the decompositions of an element of $\mathbb{R}[D,\mathsf{H}]$ in contrast with those of $\mathbb{R}[D,\varphi^*]$ which, in page 89, was shown not to be a unique factorization domain.

In the following example we will use some properties of the Hilbert transform [114]:

$$\begin{aligned}\mathsf{H}\cos &= \sin,\\ \mathsf{H}\sin &= -\cos,\\ \mathsf{H}(tf(t))(t) &= t\,\mathsf{H}f(t) - \frac{1}{\pi}\int_{-\infty}^{\infty} f(s)\,\mathrm{d}s,\end{aligned}$$

where, as we have noted before, the integral is considered as the principal value.

Example 5.4.4. Consider the problem

$$Lu(t) \equiv u'(t) + a\mathsf{H}u(t) = h(t) := \sin at,\ u(0) = 0, \tag{5.4.1}$$

where $a > 0$. Composing the operator $L = D + a\mathsf{H}$ with the operator $R = D - a\mathsf{H}$ we obtain $S = RL = D^2 + a^2$, the harmonic oscillator operator. The extra boundary conditions obtained applying R are $u'(0) - a\mathsf{H}u(0) = 0$. The general solution to the problem $u''(t) + a^2u(t) = Rh(t) = 2a\cos at, u(0) = 0$ is given by

$$v(t) = \int_0^t \frac{\sin(a[t-s])}{a} Rh(s)\,\mathrm{d}s + \alpha \sin at = (t+\alpha)\sin at,$$

where α is a real constant. Hence,

$$\mathsf{H}v(t) = -(t+\alpha)\cos at.$$

If we impose the boundary conditions $v'(0) - a\mathsf{H}v(0) = 0$ then we get $\alpha = 0$. Hence, the unique solution of problem (5.4.1) is

$$u(t) = t\sin at.$$

Remark 5.4.5. It can be checked that the kernel of $D + a\mathsf{H}$ $(a > 0)$ is spanned by $\sin t$ and $\cos t$ and, also, the kernel of $D - a\mathsf{H}$ is just 0. This defies, in the line of Remark 5.2.6, the usual relation between the degree of the operator and the dimension of the kernel which is held for ordinary differential equations, that is, the operator of a linear ordinary differential equation of order n has a kernel of dimension n. In this case we have the order one operator $D + a\mathsf{H}$ with a dimension two kernel and the injective order one operator $D - a\mathsf{H}$.

Now, we consider operators with reflection and Hilbert transforms, and denote the algebra as $\mathbb{R}[D, \mathsf{H}, \varphi^*]$. We can again state a reduction Theorem.

Theorem 5.4.6. *Take*

$$L = \sum_i a_i\varphi^*\mathsf{H}D^i + \sum_i b_i\mathsf{H}D^i + \sum_i c_i\varphi^*D^i + \sum_i d_iD^i \in \mathbb{R}[D, \mathsf{H}, \varphi^*]$$

and define

$$R = \sum_j a_j\varphi^*\mathsf{H}D^j + \sum_j (-1)^j b_j\mathsf{H}D^j + \sum_j c_j\varphi^*D^j - \sum_j (-1)^j d_jD^j.$$

Then $LR = RL \in \mathbb{R}[D]$.

5.4.1 Hyperbolic numbers as operators

Finally, we use the same idea behind the isomorphism Ξ to construct an operator algebra isomorphic to the algebra of polynomials on the hyperbolic numbers.

The hyperbolic numbers[†] are defined, in a similar way to the complex numbers, as follows,

$$\mathbb{D} = \{x + jy \ :\ x, y \in \mathbb{R},\ j \notin \mathbb{R},\ j^2 = 1\}.$$

[†]See [6, 166] for an introduction to hyperbolic numbers and some of their properties and applications.

The arithmetic in $\mathbb{D}$ is that obtained assuming the commutative, associative and distributive properties for the sum and product. In a parallel fashion to the complex numbers, if $w \in \mathbb{D}$, with $w = x + jy$, we can define

$$\overline{w} := x - jy, \quad \Re(w) := x, \quad \Im(w) := y,$$

and, since $w\overline{w} = x^2 - y^2 \in \mathbb{R}$, we set

$$|w| := \sqrt{|w\overline{w}|},$$

which is called the *Minkowski norm*. It is clear that $|w_1 w_2| = |w_1||w_2|$ for every $w_1, w_2 \in \mathbb{D}$ and, if $|w| \neq 0$, then $w^{-1} = \overline{w}/|w|^2$. If we add the norm

$$\|w\| = \sqrt{2(x^2 + y^2)},$$

we have that $(\mathbb{D}, \|\cdot\|)$ is a Banach algebra, so the exponential and the hyperbolic trigonometric functions are well defined. Although, unlike $\mathbb{C}$, $\mathbb{D}$ is not a division algebra (not every nonzero element has an inverse), we can derive calculus (differentiation, integration, holomorphic functions...) for $\mathbb{D}$ as well [6].

In this setting, we want to derive an operator J defined on a suitable space of functions such that satisfies the same algebraic properties as the hyperbolic imaginary unity j. In other words, we want the map

$$\begin{array}{ccc} \mathbb{R}[D, J] & \xrightarrow{\quad\Theta\quad} & \mathbb{D}[D] \\ & & \\ \sum_k (a_k J + b_k) D^j & \hookrightarrow & \sum_k (a_k j + b_k) D^k \end{array}$$

to be an algebra isomorphism. This implies:

- J is a linear operator,
- $J \notin \mathbb{R}[D]$.
- $J^2 = \mathrm{Id}$, that is, J is an involution,
- $JD = DJ$.

There is a simple characterization of linear involutions on a vector space: every linear involution J is of the form

$$J = \pm(2P - \mathrm{Id})$$

where P is a projection operator, that is, $P^2 = P$. It is clear that $\pm(2P - \mathrm{Id})$ is, indeed a linear operator and an involution. On the other hand, it is simple to check that, if J is a linear involution, $P := (\pm J + \mathrm{Id})/2$ is a projection, so $J = \pm(2P - \mathrm{Id})$.

Hence, it is sufficient to look for a projection P commuting with de derivative.

Example 5.4.7. Consider the space $W = \mathrm{L}^2([-\pi,\pi])$ and define

$$Pf(t) := \sum_{n\in\mathbb{N}} \int_{-\pi}^{\pi} f(s)\cos(2\,n\,s)\,\mathrm{d}s\,\cos(2\,n\,t) \text{ for every} f \in W,$$

that is, take only the sum over the even coefficients of the Fourier series of f. Clearly $PD = DP$. $J := 2P - \mathrm{Id}$ satisfies the aforementioned properties.

The algebra $\mathbb{R}[D,J]$, being isomorphic to $\mathbb{D}[D]$, satisfies also very good algebraic properties (see, for instance, [146]). In order to get an analogous theorem to Theorem 5.1.1 for the algebra $\mathbb{R}[D,J]$ it is enough to take, as in the case of $\mathbb{R}[D,J]$, $R = \Theta^{-1}(\overline{\Theta(L)})$.

6. An application to the φ-Laplacian

This chapter is devoted to the study of the existence and periodicity of solutions of initial differential problems, paying special attention to the explicit computation of the period. These problems are also connected with some particular initial and boundary value problems with reflection, which allows us to prove existence of solutions of the latter using the existence of the first.

Let us consider the problems (3.1.1) and (3.1.2) again for a differentiable involution φ. Observe that, from problem (3.1.6), we have that

$$\begin{aligned}0 =&\frac{x''(t)}{f'(f^{-1}(x'(t)))} - f(x(t))\varphi'(t) = (f^{-1})'(x'(t))x''(t) - f(x(t))\varphi'(t)\\ =&(f^{-1}\circ x')'(t) - f(x(t))\varphi'(t).\end{aligned}$$

So, clearly, problem (3.1.6) is equivalent to the problem

$$(f^{-1}\circ x')'(t) - \varphi'(t)f(x(t)) = 0, \quad x(a) = x(b), \quad x'(a) = f(x(a)). \qquad (6.0.1)$$

Which involves the f^{-1}-Laplacian $(f^{-1}\circ x')'$, although, contrary to most literature, the other term in the equation does not involve f^{-1} but f. As we will see, this is not more than a further generalization in the line of the p-q-Laplacian.

Problems concerning the φ-Laplacian (or, particularly, the p-Laplacian) have been studied extensively in recent literature. Drábek, Manásevich and others study the eigenvalues of problems with the p-Laplacian in [15, 61, 63, 64, 145] using variational methods. The existence of positive solutions is treated in [62], the existence of an exact number of solutions in [154] and topological existence results can be found in [55]. Anti-maximum principles and sign properties of the solutions are studied in [32, 36]. In [49] the authors study a variant of the p-Laplacian equation with an approach based on variational methods, in [16] they study the eigenvalues of the Dirichlet problem and in [60] they find some oscillation criteria for equations with the p-Laplacian.

The φ-Laplacian is studied from different points of view in several papers, e. g. [2, 9–13, 33, 38, 48, 53, 54, 86, 110, 127, 136]. Actually, if we consider the problem with the f^{-1}-Laplacian

$$(f^{-1}\circ x_c')'(t) + f(x_c(t)) = 0, \qquad x_c(a) = c, \quad x_c'(a) = f(c), \qquad (6.0.2)$$

and we assume there exist $\bar{c}_1, \bar{c}_2 \in \mathbb{R}$, $\bar{c}_1 < \bar{c}_2$, such that a unique solution of problem (6.0.2) exists for every $c \in [\bar{c}_1, \bar{c}_2]$ and $(x_{\bar{c}_1}(b) - \bar{c}_1)(x_{\bar{c}_2}(b) - \bar{c}_2) < 0$, then problem (3.1.6) must have at least a solution due to the continuity of x_c on c and Bolzano's Theorem. For this reason we will be interested in studying the properties of problem (6.0.2) and its solutions in this chapter. In the sections to come we study this problem and more general versions of it.

In the following section we will study the existence, uniqueness and periodicity of solutions of problem (6.1.1) and in Section 6.2 we will apply these results to the case of problems with reflection. The results of this chapter can be found in [42].

6.1 General solutions

First, we write in a general way the solutions of equations involving the g-f-Laplacian.

Let $\tau_i, \sigma_i \in [-\infty, \infty]$, $i = 1, \dots, 4$, $\tau_1 < \tau_2$, $\sigma_1 < \sigma_2$, $\tau_3 < \tau_4$, $\sigma_3 < \sigma_4$. Let $f : (\tau_1, \tau_2) \to (\sigma_1, \sigma_2)$ and $g : (\tau_3, \tau_4) \to (\sigma_3, \sigma_4)$ be invertible functions such that f and g^{-1} are continuous. Assume there is $s_0 \in (\tau_1, \tau_2)$ such that $f(s_0) = 0$ and define $F(t) := \int_{s_0}^{t} f(s)\,\mathrm{d}s$. Observe that F is 0 at s_0 and of constant sign everywhere else. The following Lemma is an straightforward application of the properties of the integral.

Lemma 6.1.1. *If f is continuous, invertible and increasing (decreasing) then $F_- \equiv F|_{(-\infty, s_0]}$ is strictly decreasing (increasing) and $F_+ \equiv F|_{[s_0, +\infty)}$ is strictly increasing (decreasing). Furthermore, if $\tau_1 = -\infty$, $F(-\infty) = +\infty$ $(-\infty)$ and if $\tau_2 = +\infty$, $F(+\infty) = +\infty$ $(-\infty)$.*

All the same, define $G(t) := \int_{g^{-1}(\{0\})}^{t} g^{-1}(s)\,\mathrm{d}s$ and consider the problem

$$(g \circ x')'(t) + f(x(t)) = 0, \quad \text{a. e. } t \in \mathbb{R}, \quad x(a) = c_1, \quad x'(a) = c_2, \tag{6.1.1}$$

for some fixed $c_1, c_2 \in \mathbb{R}$.

Definition 6.1.2. A *solution* x of problem (6.1.1) will be $x \in \mathcal{C}^1(I)$, such that $g \circ x'$ is absolutely continuous on I, where I is an open interval with $a \in I$. The solution must further satisfy that the equation in problem (6.1.1) holds a. e. and the initial conditions are satisfied as well.

Theorem 6.1.3. *Let $f : (\tau_1, \tau_2) \to (\sigma_1, \sigma_2)$ and $g : (\tau_3, \tau_4) \to (\sigma_3, \sigma_4)$ be invertible functions such that f and g^{-1} are continuous and assume $0 \in (\tau_1, \tau_2) \cap (\tau_3, \tau_4)$, $f(0) = 0$, $g(0) = 0$, f and g increasing, $F(c_1) + G(g(c_2)) < \min\{G(\sigma_3), G(\sigma_4)\}$. Then there exists a unique local solution of problem* (6.1.1).

Furthermore, if $F(c_1) + G(g(c_2)) < \min\{F(\tau_1), F(\tau_2)\}$, then such solution is defined on the whole real line and is periodic of smallest period

$$\begin{aligned} T := \int_{F_-^{-1}(G(g(c_2))+F(c_1))}^{F_+^{-1}(G(g(c_2))+F(c_1))} \Bigg[& \frac{1}{g^{-1} \circ G_+^{-1}(G(g(c_2)) + F(c_1) - F(r))} \\ & - \frac{1}{g^{-1} \circ G_-^{-1}(G(g(c_2)) + F(c_1) - F(r))} \Bigg] \mathrm{d}r. \end{aligned} \tag{6.1.2}$$

Proof. For the first part of the Theorem and without loss of generality, we will prove the existence of solution in an interval of the kind $[a, a + \delta)$, $\delta \in \mathbb{R}^+$. The proof would be analogous for an interval of the kind $(a - \delta, a]$.

Let $y(t) = g(x'(t))$. Then problem (6.1.1) is equivalent to

$$x'(t) = g^{-1}(y(t)), \quad y'(t) = -f(x(t)), \ t \in \mathbb{R} \quad x(a) = c_1, \ y(a) = g(c_2).$$

Hence,

$$f(x(t))x'(t) + g^{-1}(y(t))y'(t) = 0, \ t \in \mathbb{R},$$

so, integrating both sides from a to t,

$$F(x(t)) + G(y(t)) = k, \quad t \in \mathbb{R},$$

where $k = F(c_1) + G(g(c_2))$. That is, undoing the change of variables,

$$G(g(x'(t))) = G(g(c_2)) + F(c_1) - F(x(t)),\ t \in \mathbb{R}. \tag{6.1.3}$$

If $c_1 = c_2 = 0$ it is clear that the only possible solution is $x \equiv 0$ for, in that case, $G(g(x'(t))) + F(x(t)) = 0$ and, since G and F are nonnegative and increasing, $x'(t) = x(t) = 0$ for $t \in \mathbb{R}$. Assume, without loss of generality, that c_2 is nonnegative and c_1 negative (the other cases are similar). If $c_2 = 0$ then, integrating (6.1.1),

$$g \circ x'(t) = -\int_a^t f(x(s))\,\mathrm{d}s,$$

which implies x' is positive in some interval $[a, a+\delta)$.

If c_2 is positive, then x' has to be positive at least in some neighborhood of a, so, in a right neighborhood of a, we can solve for $g \circ x'$ in (6.1.3) as

$$g \circ x'(t) = G_+^{-1}(F(c_1) - F(x(t)) + G(g(c_2))). \tag{6.1.4}$$

In order to solve for x' in (6.1.4), we need $F(c_1) + G(g(c_2)) < G(\sigma_4)$. Then,

$$x'(t) = g^{-1} \circ G_+^{-1}(F(c_1) - F(x(t)) + G(g(c_2))). \tag{6.1.5}$$

Integrating between a and t,

$$t = \int_a^t \frac{x'(s)}{g^{-1} \circ G_+^{-1}(F(c_1) - F(x(s)) + G(g(c_2)))}\,\mathrm{d}s + a = H_+(x(t)),$$

where

$$H_+(r) := \int_{c_1}^r \frac{1}{g^{-1} \circ G_+^{-1}(F(c_1) - F(s) + G(g(c_2)))}\,\mathrm{d}s + a.$$

H_+ is strictly increasing in its domain due to the positivity of the denominator in the integrand. Hence, for t sufficiently close to a,

$$x(t) = H_+^{-1}(t).$$

Therefore, a solution of problem (6.1.1) exists and is unique (by construction) on an interval $[a, a+\delta)$.

If we assume $F(c_1) + G(g(c_2)) < \min\{F(\tau_1), F(\tau_2)\}$, $c_2 > 0$ (the case $c_2 = 0$ is similar), H_+ is well defined on

$$I := \left(F_-^{-1}(F(c_1) + G(g(c_2))), F_+^{-1}(F(c_1) + G(g(c_2)))\right).$$

Now, we study the range of H_+.

$g(x'(t))$ is positive as long as $x'(t)$ is positive. Hence, consider

$$t_0 := \sup\{t \in [a, +\infty) \ :\ x'(s) > 0 \text{ for a. e. } s \in [a, t)\} \in [a, +\infty].$$

G is positive on nonzero values, so equation (6.1.3) implies that

$$F(x(t)) < G(g(c_2)) + F(c_1)$$

for all $t \in (a, t_0)$.

Assume $t_0 = +\infty$. Now, $x'(t) > 0$ a. e. in $[a, +\infty)$ so there exists

$$x(+\infty) \in \left(c_1, F_+^{-1}(G(g(c_2)) + F(c_1))\right].$$

On the other hand, since x is increasing in $[a, +\infty)$ and $c_1 < 0$, by equation (6.1.5) we have that x' is increasing as long as x is negative. This means that, eventually (in finite time), x will be positive and therefore, x' is decreasing in $[\tilde{a}, +\infty)$ for $\tilde{a}$ big enough, so there exists $x'(+\infty) \geq 0$. If we assume $x'(+\infty) = \epsilon > 0$, this implies that $x(+\infty) = +\infty$, for there would exist $M \in \mathbb{R}$ such that $x'(t) > \epsilon/2$ for every $t \geq M$, so $x'(+\infty) = 0$. Taking the limit $t \to +\infty$ in equation (6.1.3), $x(+\infty) = F_+^{-1}(G(g(c_2)) + F(c_1))$.

Now, take $\epsilon \in (0, f(x(+\infty)))$. Since $g \circ x'(+\infty) = 0$ and $g \circ x'$ is continuous and decreasing in $[\tilde{a}, +\infty)$, there exists $M \in \mathbb{R}^+$ such that $|g(x'(M_2)) - g(x'(M_1))| < \epsilon$ for every $M_1, M_2 > M$. Since f is continuous, there exits $\tilde{M} > M$ such that $f(x(M_3)) > \epsilon$ for every $M_3 > \tilde{M}$. Take M_3 in such a way. Then, integrating equation (6.1.1) between M_3 and $M_3 + 1$,

$$(g \circ x')(M_3 + 1) - (g \circ x')(M_3) = \int_{M_3}^{M_3+1} f(x(s))\, \mathrm{d}s > \epsilon,$$

a contradiction. Therefore, $t_0 \in \mathbb{R}$.

Observe that $x'(t_0) = 0$, so x attains its maximum at t_0 and $x(t_0) = F_+^{-1}(G(g(c_2)) + F(c_1))$ by equation (6.1.3), that is, $x(t_0) = \sup I$. In order for this value to be well defined it is necessary that $G(g(c_2)) + F(c_1) \leq F(\tau_2)$.

Now, we have that H_+ is well defined at $\sup I$ (assuming it is defined continuous at that point). Indeed,

$$t_0 = \lim_{t \to t_0} H_+(x(t)) = H_+(F_+^{-1}(G(g(c_2)) + F(c_1))).$$

We prove now that there is a neighborhood $(t_0, t_0 + \epsilon)$ where x' is negative, which means that we can take

$$t_1 := \sup\{t \in [t_0, +\infty) \ : \ x'(s) < 0 \text{ for a. e. } s \in [t_0, t)\}.$$

Fix ξ such that $0 < \xi < f(x(t_0))$ and take ϵ such that $f(x(t)) > \xi$ in $(t_0, t_0 + \epsilon)$. Take $t \in (t_0, t_0 + \epsilon)$, then, integrating equation (6.1.1) between t_0 and t,

$$g(x'(t)) = -\int_{t_0}^{t} f(x(s))\, \mathrm{d}s < -\xi(t - t_0) < 0.$$

We deduce that $t_1 < +\infty$ by the same kind of reasoning we used to prove $t_0 < +\infty$. Observe that $x'(t_1) = 0$ and $x(t_1) = F_-^{-1}\left(G(g(c_2)) + F(c_1)\right)$. This last equality comes from evaluating equation (6.1.3) at t_1 and Rolle's Theorem as we show now: the other possibility would be $x(t_1) = F_+^{-1}\left(G(g(c_2)) + F(c_1)\right)$. Observe that, by equation (6.1.5), x' is continuous, so $x \in \mathcal{C}^1([a, t_1))$. Since $x(t_0) = x(t_1)$, there would exist $\tilde{t} \in (t_0, t_1)$ such that $x'(\tilde{t}) = 0$, a contradiction.

Now, we have that $x'(t) = g^{-1} \circ G_-^{-1}(G(g(c_2)) + F(c_1) - F(x(t)))$, that is,

$$1 = x'(t)/(g^{-1} \circ G_-^{-1})(G(g(c_2)) + F(c_1) - F(x(t))).$$

Thus,

$$\begin{aligned} t_1 - t_0 &= \int_{t_0}^{t_1} \frac{x'(s)\,\mathrm{d}s}{g^{-1}\circ G_-^{-1}(G(g(c_2)) + F(c_1) - F(x(s)))} \\ &= \int_{F_+^{-1}(G(g(c_2))+F(c_1))}^{F_-^{-1}(G(g(c_2))+F(c_1))} \frac{\mathrm{d}r}{g^{-1}\circ G_-^{-1}(G(g(c_2)) + F(c_1) - F(r))}. \end{aligned}$$

If we define

$$H_-(s) := \int_{F_+^{-1}(G(g(c_2))+F(c_1))}^{s} \frac{\mathrm{d}r}{g^{-1}\circ G_-^{-1}(G(g(c_2)) + F(c_1) - F(r))} + t_0,$$

H_- is strictly decreasing in its domain and $x(t) = H_-^{-1}(t)$ for $t \in [t_0, t_1]$.

We can again deduce that

$$t_2 := \sup\{t \in [t_1, +\infty) \,:\, x'(s) > 0 \text{ for a. e. } s \in [t_1, t)\} < +\infty.$$

Using the positivity and growth conditions of the functions involved, it is easy to check that $x(t_1) = F_-^{-1}(G(g(c_2)) + F(c_1)) < c_1 < F_+^{-1}(G(g(c_2)) + F(c_1)) = x(t_2)$, so there exists a unique $b \in (t_1, t_2)$ such that $x(b) = c_1$. Now,

$$\begin{aligned} b - t_1 &= \int_{t_1}^{b} \frac{x'(s)\,\mathrm{d}s}{g^{-1}\circ G_+^{-1}(G(g(c_2)) + F(c_1) - F(x(s)))} \\ &= \int_{F_-^{-1}(G(g(c_2))+F(c_1))}^{c_1} \frac{\mathrm{d}r}{g^{-1}\circ G_+^{-1}(G(g(c_2)) + F(c_1) - F(r))}. \end{aligned}$$

Defining $T := b - a$ and extending x periodically in the following way (we have x already defined in $[a, a+T]$),

$$x(t) = x\left(t - \left\lfloor \frac{t-a}{T} \right\rfloor T\right),$$

where $\lfloor t \rfloor := \sup\{k \in \mathbb{Z} \,:\, k \le t\}$, it is easy to check that x, extended in such a way, is a global periodic solution of problem (6.1.1).

Take $z(t) := x(t-T)$, $t \in \mathbb{R}$, we show that z is a solution of the problem in $[a+T, a+2T]$.

$$0 = (g\circ x')'(t) + f(x(t)) = (g\circ z')'(t+T) + f(z(t+T)) \text{ for a. e. } t \in \mathbb{R}$$

This is equivalent to

$$(g\circ z')'(t) + f(z(t)) = 0 \quad \text{for a. e } t \in \mathbb{R}.$$

Also,

$$\begin{aligned} z(a+T) &= x(a) = c_1, \\ z'(a+T) &= x'(a) = c_2. \end{aligned}$$

■

Remark 6.1.4. A similar argument can be done for the case f and g have different growth type (e. g. f increasing and g decreasing), but taking the negative branch of the inverse function G^{-1} in (6.1.5).

Remark 6.1.5. In the hypotheses of Theorem 6.1.3, if instead of $g(0) = f(0) = 0$ we have that $g(s_0) = f(s_0) = 0$, define $\tilde{f}(x) := f(x+s_0), \tilde{g}(x) := g(x+s_0)$. Then $\tilde{f}(0) = \tilde{g}(0) = 0$ and problem (6.1.1) is equivalent to

$$(\tilde{g} \circ v')'(t) + \tilde{f}(v(t)) = 0, \quad v(a) = c_1 - s_0, \quad v(a) = c_2,$$

with $v(t) = x(t) - s_0$. Hence, we can apply Theorem 6.1.3 to this case.

Remark 6.1.6. Using the notation of Theorem 6.1.3, the explicit form of the solution of problem (6.1.1) is given by

$$x(t) = \begin{cases} H_+^{-1}\left(t - \left\lfloor \dfrac{t-a}{T} \right\rfloor T\right), & t \in [a + 2Tk, a + (2k+1)T],\ k \in \mathbb{Z}, \\ H_-^{-1}\left(t - \left\lfloor \dfrac{t-a}{T} \right\rfloor T\right), & t \in [a + (2k-1)T, a + 2kT],\ k \in \mathbb{Z}, \end{cases}$$

Remark 6.1.7. Consider the following particular case of problem (6.1.1) with $f(0) = 0, g(0) = 0$, f and g increasing and the hypothesis for a unique global solution of the following problem are satisfied in Theorem 6.1.3.

$$(g \circ x')'(t) + f(x(t)) = 0, \qquad x(0) = 0, \quad x'(0) = 1. \tag{6.1.6}$$

It is clear that, in the case $g(x) = f(x) = x$, the unique solution of problem (6.1.6) is $\sin(t)$, which suggests the definition of the $\sin_{g,f}$ function as the unique solution of problem (6.1.6) for general g and f. Correspondingly,

$$\arcsin^+_{g,f}(r) := H_+(r).$$

This function, defined as such, coincides with the $\arcsin_p$ function defined in [24, 115] for the p-Laplacian $f(x) = g(x) = |x|^{p-2}x$, the function $\arcsin_{p,q}$ defined in [14, 65, 108] for the p-q-Laplacian $f(x) = |x|^{q-2}x, g(x) = |x|^{p-2}x$, which first appeared with a slightly different definition in [64], and the hyperbolic version of this function, also in [14, 108], which corresponds to the case $f(x) = |x|^{q-2}x, g(x) = -|x|^{p-2}x$. [164] derives generalized Jacobian functions in a similar way, defining

$$\operatorname{arcsn}_{p,q}(t,k) := \int_0^t \frac{1}{\sqrt[p]{(1-s^q)(1-k^q s^q)}}\,\mathrm{d}s,$$

of which the inverse (see [164, Proposition 3.2]) is precisely a solution of

$$(f_p \circ x'(t))' + \frac{q}{p^*} f_q(x(t))(1 + k^q - 2k^q|x(t)|^q) = 0,$$

where f_r is the r-Laplacian for $r = p, q$ and $p^* p = p^* + p$. Observe that this case is also covered by our definition.

In all of the aforementioned works they are interested on the inverse of the $\arcsin_{g,f}$ function, the $\sin_{g,f}$ function, which they extend to the whole real line by symmetry and periodicity. Observe that in our case f and g need not to be odd functions, contrary to the above examples, but we can still give the definition of the $\sin_{g,f}$ function in the whole real line. Also, this lack of symmetry gives rise to a richer set of right inverses of $\sin_{g,f}$, for instance,

$$\arcsin^{-}_{g,f}(r) := H_-(r).$$

In general, if we have a problem of the kind

$$\Phi((g\circ x')', x(t)) = 0; \quad x(0) = 0,\, x'(0) = 1,$$

and we know it has a unique solution in a neighborhood of 0, then we can define $\sin_{g,\Phi}$ as such unique solution and its inverse, in a neighborhood of 0, $\arcsin_{g,\Phi}$.

6.1.1 A particular case

Having in mind problem (6.0.2), we now consider a particular case of problem (6.1.1) for the rest of this section. Assume f is invertible and both f and f^{-1} are continuous. For convenience, assume also that f is increasing and $f(0) = 0$. Consider the following problem.

$$(f^{-1}\circ x')'(t) + \lambda f(x(t)) = 0, \quad x(a) = c,\, x'(a) = f(c), \tag{6.1.7}$$

where $\lambda \in \mathbb{R}^+$.

The following corollary is just the restatement of Theorem 6.1.3 for this particular case.

Corollary 6.1.8. *Let $f : (\tau_1, \tau_2) \to (\sigma_1, \sigma_2)$ be an invertible function such that f is continuous and assume $0 \in (\tau_1, \tau_2)$, $f(0) = 0$ and f increasing, $\lambda > 0$, $(1+\lambda)F(c) < \min\{F(\tau_1), F(\tau_2)\}$. Then there exists a unique local solution of problem* (6.1.7).

Furthermore, if $(1+\lambda^{-1})F(c) < \min\{F(\tau_1), F(\tau_2)\}$, then such solution is defined on $\mathbb{R}$ and is periodic of first period

$$\begin{aligned} T := \int_{F_-^{-1}((1+\lambda^{-1})F(c))}^{F_+^{-1}\left((1+\lambda^{-1})F(c)\right)} &\left[\frac{1}{f(F_+^{-1}((1+\lambda)F(c) - \lambda F(r)))}\right. \\ &\left. -\frac{1}{f(F_-^{-1}((1+\lambda)F(c) - \lambda F(r)))}\right] \mathrm{d}r. \end{aligned} \tag{6.1.8}$$

There are some particular cases where the formula (6.1.8) can be simplified.

If f is odd then F is even and, with the change of variables $r = |c|\, s$, we have that expression (6.1.8) becomes

$$T = \int_0^{\frac{F_+^{-1}((1+\lambda^{-1})F(c))}{|c|}} \frac{4\,|c|\,\mathrm{d}r}{f(F_+^{-1}((1+\lambda)F(c) - \lambda F(|c|r)))}.$$

Also, if we further assume that f is defined in $\mathbb{R}$ and that $f(rt) = h(r)f(t)$ for every $r, t \in \mathbb{R}$ (see Remark 6.1.10 for a classification of such functions) and some function h, then

$$F(rt) = \int_0^{rt} f(s)\,\mathrm{d}s = \int_0^t f(rs)r\,\mathrm{d}s = r\,h(r)\int_0^t f(s)\,\mathrm{d}s = rh(r)F(t),$$

so F satisfies the same kind of property for $\tilde h(r) = r\,h(r)$.

Clearly, for $t > 0$,

$$F_-^{-1}(\tilde h(r)t) = r\,F_-^{-1}(t), \quad F_+^{-1}(\tilde h(r)t) = r\,F_+^{-1}(t).$$

Observe that $\tilde{h}(r) = F(r)/F(1)$, and therefore $\tilde{h}|_{(-\infty,0]}, \tilde{h}|_{[0,+\infty)}$ are invertible. Also, $\tilde{h}_+^{-1}(t) = F_+^{-1}(tF(1))$ for $t > 0$. Hence,

$$\frac{F_+^{-1}((1+\lambda^{-1})F(c))}{|c|} = \frac{F_+^{-1}(\tilde{h}(\tilde{h}_+^{-1}(1+\lambda^{-1}))F(c))}{|c|} = \frac{\tilde{h}_+^{-1}(1+\lambda^{-1})F_+^{-1}(F(c))}{|c|}$$
$$= \tilde{h}_+^{-1}(1+\lambda^{-1}) = F_+^{-1}((1+\lambda^{-1})F(1)).$$

All the same, $F_-^{-1}((1+\lambda^{-1})F(c))/|c| = -F_-^{-1}((1+\lambda^{-1})F(1))$.

Also,

$$\begin{aligned} &f(F_+^{-1}((1+\lambda)F(c) - \lambda F(|c|r))) = f(F_+^{-1}((1+\lambda)\tilde{h}(|c|)F(1) - \lambda\tilde{h}(|c|)F(r))) \\ =&f(F_+^{-1}(\tilde{h}(|c|)[(1+\lambda)F(1) - \lambda F(r)])) = f(|c|F_+^{-1}((1+\lambda)F(1) - \lambda F(r))) \\ =&h(|c|)f(F_+^{-1}((1+\lambda)F(1) - \lambda F(r))) \\ =&(f(|c|)/f(1))f(F_+^{-1}((1+\lambda)F(1) - \lambda F(r))). \end{aligned}$$

With these considerations in mind, we have that we can further reduce expression (6.1.8) to

$$T(c,\lambda) = \frac{4|c|f(1)}{f(|c|)}\int_0^{F_+^{-1}((1+\lambda^{-1})F(1))} \frac{\mathrm{d}r}{f(F_+^{-1}((1+\lambda)F(1) - \lambda F(r)))}.$$

Example 6.1.9. Let $f(t) := |t|^{p-2}t$, $p > 1$. Then

$$T(c,\lambda,p) = 4|c|^{2-p}\int_0^{(1+\lambda^{-1})^{\frac{1}{p}}} [1+\lambda-\lambda r^p]^{\frac{1-p}{p}}\,\mathrm{d}r.$$

Observe that with the change of variable $r = (1+\lambda^{-1})^{\frac{1}{p}}s$ we have that

$$\begin{aligned} T(c,\lambda,p) &= 4|c|^{2-p}\int_0^1 (1+\lambda^{-1})^{\frac{1}{p}}[(1+\lambda)(1-s^p)]^{\frac{1-p}{p}}\,\mathrm{d}s \\ &= 4|c|^{2-p}\lambda^{-\frac{1}{p}}(1+\lambda)^{\frac{2}{p}-1}\int_0^1 (1-s^p)^{\frac{1-p}{p}}\,\mathrm{d}s \\ &= 4|c|^{2-p}\lambda^{-\frac{1}{p}}(1+\lambda)^{\frac{2}{p}-1}\frac{\Gamma\left(\frac{1}{p}\right)^2}{p\,\Gamma\left(\frac{2}{p}\right)}. \end{aligned}$$

T is increasing on $|c|$ if $p \in (1,2)$ and decreasing on $|c|$ if $p > 2$ and independent of $|c|$ if $p = 2$.

If we take $\lambda = 1$,

$$T(c,1,p) = 2^{\frac{2}{p}+1}|c|^{2-p}\frac{\Gamma\left(\frac{1}{p}\right)^2}{p\,\Gamma\left(\frac{2}{p}\right)}.$$

In particular, $T(c,1,2) = 2\pi$ (independently of c).

We can also consider the dependence of T on λ. We do this study for this particular example and in the following section we develop a general theory.

$$\frac{\partial T}{\partial\lambda}(c,\lambda,p) = -\frac{4|c|^{2-p}}{p\lambda}\left(1+\frac{1}{\lambda}\right)^{\frac{1}{p}}(1+\lambda)^{\frac{1-2p}{p}}(1+(p-1)\lambda)\int_0^1 (1-s^p)^{\frac{1-p}{p}}\,\mathrm{d}s < 0.$$

Therefore the period T is decreasing on λ.

Remark 6.1.10. If a continuous function f satisfies that $f(rt) = h(r)f(t)$, we can obtain the explicit expression of f. Let $c = f(1)$, $g(t) := f(t)/f(1)$ and $\alpha = \ln g(e)$. Then $g(ts) = g(t)g(s)$. Also, for $t \neq 0$, $1 = g(1) = g(t/t) = g(t)g(1/t)$ and therefore $g(t^{-1}) = g(t)^{-1}$. If $n \in \mathbb{N}$, $g(t^n) = g(t)^n$, so, for $t \geq 0$, $g(t) = g(t^{\frac{n}{n}}) = g(t^{\frac{1}{n}})^n$ and $g(t^{\frac{1}{n}}) = g(t)^{\frac{1}{n}}$. Hence, $g(t^{\frac{p}{q}}) = g(t)^{\frac{p}{q}}$ for every $p, q \in \mathbb{N}$, $q \neq 0$ and, by the density of $\mathbb{Q}$ in $\mathbb{R}$ and the continuity of f, $g(t^r) = g(t)^r$ for all $t \geq 0, r \in \mathbb{R}^+$.

Now, for $t > 0$, $g(t) = g(e^{\ln t}) = g(e)^{\ln t} = e^{\ln g(e) \ln t} = t^{\ln g(e)} = t^\alpha$. Hence, $f(t) = \beta t^\alpha$ for $t \geq 0$. On the other hand, $1 = g(1) = (g(-1))^2$, so $g(-1) = \pm 1$. Also, $f(-t) = g(-1)f(t)$ and thus, $f(-t) = \pm\beta t^\alpha$ for $t > 0$. In summary,

$$f(t) = \begin{cases} \beta t^\alpha & \text{if } t \geq 0, \\ \pm\beta(-t)^\alpha & \text{if } t < 0. \end{cases}$$

If we further ask for f to be injective, $f(t) = \beta |t|^{\alpha-1} t$, that is, f is an α-laplacian.

6.1.2 Dependence of T on λ and c

Based on the approach used in Example 6.1.9, we study now the dependence of T on λ and c in a general way. For simplicity, we will assume $c > 0$. For the case $c < 0$, just do the change of variable $y(t) = -x(t)$.

We continue to assume the hypotheses for (6.1.7) and further assume that f is a differentiable function. Let us divide the interval of integration in equation (6.1.2) in $[F_-^{-1}((1+\lambda^{-1})F(c)), 0]$ and $[0, F_+^{-1}((1+\lambda^{-1})F(c))]$. Observe that F is injective restricted to any of the two intervals. For the nonnegative interval, taking the change of variables

$$r = F_+^{-1}\left((1+\lambda^{-1})F(cs)\right),$$

we have that

$$\begin{aligned}
&\int_0^{F_+^{-1}((1+\lambda^{-1})F(c))} \left[\frac{1}{f(F_+^{-1}((1+\lambda)F(c) - \lambda F(r)))}\right. \\
&\left. - \frac{1}{f(F_-^{-1}((1+\lambda)F(c) - \lambda F(r)))}\right] \mathrm{d}r \\
= &\int_0^1 \left[\frac{1}{f(F_+^{-1}((1+\lambda)[F(c) - F(cs)])} - \frac{1}{f(F_-^{-1}((1+\lambda)[F(c) - F(cs)])}\right] \\
&\cdot \frac{[1+\lambda^{-1}]cf(cs)}{f\left(F_+^{-1}\left((1+\lambda^{-1})F(cs)\right)\right)} \mathrm{d}s.
\end{aligned}$$

All the same, with the change of variables

$$r = F_-^{-1}\left((1+\lambda^{-1})F(cs)\right),$$

$$\int_{F_-^{-1}((1+\lambda^{-1})F(c))}^0 \left[\frac{1}{f(F_+^{-1}((1+\lambda)F(c) - \lambda F(r)))}\right.$$

$$
\begin{aligned}
&\left.-\frac{1}{f(F_-^{-1}((1+\lambda)F(c)-\lambda F(r)))}\right]\mathrm{d}r\\
&=\int_1^0\left[\frac{1}{f(F_+^{-1}((1+\lambda)[F(c)-F(cs)])}-\frac{1}{f(F_-^{-1}((1+\lambda)[F(c)-F(cs)])}\right]\\
&\cdot\frac{[1+\lambda^{-1}]cf(cs)}{f\left(F_-^{-1}\left((1+\lambda^{-1})F(cs)\right)\right)}\mathrm{d}s.
\end{aligned}
$$

Now let, for $\lambda\in\mathbb{R}^+$ and $s\in[0,1]$,

$$
\begin{aligned}
&\alpha(\lambda,s,c):=(1+\lambda^{-1})cf(cs),\quad \frac{\partial\alpha}{\partial\lambda}(\lambda,s,c)=-\lambda^{-2}cf(cs),\\
&\beta_\pm(\lambda,s,c):=f\left(F_\pm^{-1}\left((1+\lambda^{-1})F(cs)\right)\right),\\
&\frac{\partial\beta_\pm}{\partial\lambda}(\lambda,s,c)=-\lambda^{-2}F(cs)\frac{f'\left(F_\pm^{-1}\left((1+\lambda^{-1})F(cs)\right)\right)}{f\left(F_\pm^{-1}\left((1+\lambda^{-1})F(cs)\right)\right)},\\
&\gamma_\pm(\lambda,s,c):=f(F_\pm^{-1}((1+\lambda)[F(c)-F(cs)])),\\
&\frac{\partial\gamma_\pm}{\partial\lambda}(\lambda,s,c)=[F(c)-F(cs)]\frac{f'\left(F_\pm^{-1}((1+\lambda)[F(c)-F(cs)])\right)}{f\left(F_\pm^{-1}((1+\lambda)[F(c)-F(cs)])\right)}.
\end{aligned}
$$

Then

$$
T(\lambda,c)=\int_0^1\alpha(\lambda,s,c)\left[\frac{1}{\beta_+(\lambda,s,c)}-\frac{1}{\beta_-(\lambda,s,c)}\right]\left[\frac{1}{\gamma_+(\lambda,s,c)}-\frac{1}{\gamma_-(\lambda,s,c)}\right]\mathrm{d}s.
$$

(6.1.9)

Therefore,

$$
\begin{aligned}
\frac{\partial T}{\partial\lambda}(\lambda,c)=\int_0^1\Bigg\{&\frac{\partial\alpha}{\partial\lambda}(\lambda,s,c)\left[\frac{1}{\beta_+(\lambda,s,c)}-\frac{1}{\beta_-(\lambda,s,c)}\right]\left[\frac{1}{\gamma_+(\lambda,s,c)}-\frac{1}{\gamma_-(\lambda,s,c)}\right]\\
&+\alpha(\lambda,s,c)\left[\frac{\frac{\partial\beta_-}{\partial\lambda}(\lambda,s,c)}{\beta_-(\lambda,s,c)^2}-\frac{\frac{\partial\beta_+}{\partial\lambda}(\lambda,s,c)}{\beta_+(\lambda,s,c)^2}\right]\left[\frac{1}{\gamma_+(\lambda,s,c)}-\frac{1}{\gamma_-(\lambda,s,c)}\right]\\
&+\alpha(\lambda,s,c)\left[\frac{1}{\beta_+(\lambda,s,c)}-\frac{1}{\beta_-(\lambda,s,c)}\right]\left[\frac{\frac{\partial\gamma_-}{\partial\lambda}(\lambda,s,c)}{\gamma_-(\lambda,s,c)^2}-\frac{\frac{\partial\gamma_+}{\partial\lambda}(\lambda,s,c)}{\gamma_+(\lambda,s,c)^2}\right]\Bigg\}\mathrm{d}s.
\end{aligned}
$$

Observe that $\alpha,f|_{[0,1]},f',F,F_+^{-1},\beta_+,\frac{\partial\beta_-}{\partial\lambda},\gamma_+,\frac{\partial\gamma_+}{\partial\lambda}$ are nonnegative, while $\frac{\partial\alpha}{\partial\lambda},F_-^{-1},\beta_-,\frac{\partial\beta_+}{\partial\lambda}$, γ_-, $\frac{\partial\gamma_-}{\partial\lambda}$ are nonpositive. In general we cannot tell the sign of $T(\lambda,c)$ from this expression, but making certain assumptions we can simplify it to derive information.

Assume now f is and odd function. Then $F_-^{-1}=-F_+^{-1}$, $\beta_-=-\beta_+$ and $\gamma_-=-\gamma_+$, so

$$
\begin{aligned}
\frac{\partial T}{\partial\lambda}(\lambda,c)=&4\int_0^1\frac{1}{\beta_+(\lambda,s,c)\gamma_+(\lambda,s,c)}\Bigg[\frac{\partial\alpha}{\partial\lambda}(\lambda,s,c)\\
&-\alpha(\lambda,s,c)\left(\frac{\frac{\partial\beta_+}{\partial\lambda}(\lambda,s,c)}{\beta_+(\lambda,s,c)}+\frac{\frac{\partial\gamma_+}{\partial\lambda}(\lambda,s,c)}{\gamma_+(\lambda,s,c)}\right)\Bigg]\mathrm{d}s.
\end{aligned}
$$

Now, if we differentiate equation (6.1.9) with respect to c,

$$\begin{aligned}\frac{\partial T}{\partial c}(\lambda,c) = \int_0^1 \Bigg\{ & \frac{\partial \alpha}{\partial c}(\lambda,s,c)\left[\frac{1}{\beta_+(\lambda,s,c)}-\frac{1}{\beta_-(\lambda,s,c)}\right]\left[\frac{1}{\gamma_+(\lambda,s,c)}-\frac{1}{\gamma_-(\lambda,s,c)}\right] \\ & + \alpha(\lambda,s,c)\left[\frac{\frac{\partial \beta_-}{\partial c}(\lambda,s,c)}{\beta_-(\lambda,s,c)^2}-\frac{\frac{\partial \beta_+}{\partial c}(\lambda,s,c)}{\beta_+(\lambda,s,c)^2}\right]\left[\frac{1}{\gamma_+(\lambda,s,c)}-\frac{1}{\gamma_-(\lambda,s,c)}\right] \\ & + \alpha(\lambda,s,c)\left[\frac{1}{\beta_+(\lambda,s,c)}-\frac{1}{\beta_-(\lambda,s,c)}\right]\left[\frac{\frac{\partial \gamma_-}{\partial c}(\lambda,s,c)}{\gamma_-(\lambda,s,c)^2}-\frac{\frac{\partial \gamma_+}{\partial c}(\lambda,s,c)}{\gamma_+(\lambda,s,c)^2}\right]\Bigg\}\,\mathrm{d}s.\end{aligned}$$

Observe that

$$\begin{aligned}\frac{\partial \alpha}{\partial c}(\lambda,s,c) &= (1+\lambda^{-1})\left[f(cs)+csf'(cs)\right], \\ \frac{\partial \beta_\pm}{\partial c}(\lambda,s,c) &= (1+\lambda^{-1})\,sf(cs)\frac{f'\left(F_\pm^{-1}\left((1+\lambda^{-1})F(cs)\right)\right)}{f\left(F_\pm^{-1}\left((1+\lambda^{-1})F(cs)\right)\right)}, \\ \frac{\partial \gamma_\pm}{\partial c}(\lambda,s,c) &= (1+\lambda)\left[f(c)-sf(cs)\right]\frac{f'\left(F_\pm^{-1}((1+\lambda)[F(c)-F(cs)])\right)}{f\left(F_\pm^{-1}((1+\lambda)[F(c)-F(cs)])\right)}.\end{aligned}$$

Hence, $\frac{\partial \alpha}{\partial c}$, $\frac{\partial \beta_+}{\partial c}$ is positive and $\frac{\partial \beta_-}{\partial c}$ negative for $c\geq 0$. Assume now f is an odd function.

$$\begin{aligned}\frac{\partial T}{\partial c}(\lambda,c) = & 4\int_0^1 \frac{1}{\beta_+(\lambda,s,c)\gamma_+(\lambda,s,c)}\left[\frac{\partial \alpha}{\partial c}(\lambda,s,c)\right. \\ & \left.-\alpha(\lambda,s,c)\left(\frac{\frac{\partial \beta_+}{\partial c}(\lambda,s,c)}{\beta_+(\lambda,s,c)}+\frac{\frac{\partial \gamma_+}{\partial c}(\lambda,s,c)}{\gamma_+(\lambda,s,c)}\right)\right]\mathrm{d}s.\end{aligned}$$

Example 6.1.11. Let $f:(-1,1)\to\mathbb{R}$, $f(x):=x/\sqrt{1-x^2}$, $x\in\mathbb{R}$ and consider problem (6.1.7)†. Then

$$F(x)=1-\sqrt{1-x^2},\ F_+^{-1}(x)=\sqrt{2x-x^2}.$$

In order for the conditions in Corollary 6.1.8 to be satisfied we need

$$(1+\lambda)F(c)<1,\quad (1+\lambda^{-1})F(c)<1,$$

that is

$$|c|<\min\left\{\frac{\sqrt{\lambda(\lambda+2)}}{\lambda+1},\frac{\sqrt{2\lambda+1}}{\lambda+1}\right\}.$$

In Figure 6.1.1 we plot how the period varies as a function of c and λ. Observe how the period is decreasing in both parameters and $\lim_{c,\lambda\to 0}T(\lambda,c)=+\infty$.

†The diffeomorphisms f in this example has been widely studied by Bereanu and Mawhin (see, for instance, [8, 10]) and is a type of singular φ-Laplacian known as the mean curvature operator of the Minkowski space. Its inverse, the mean curvature operator of the Euclidean space, also studied in [8], appears in Example 6.1.12.

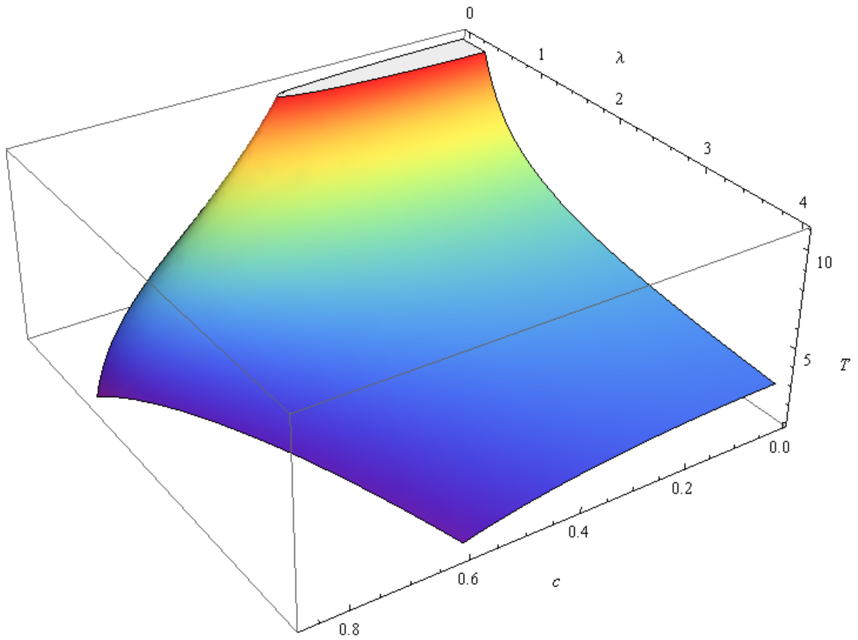


Figure 6.1.1: Graph of the period T function of c and λ.

Example 6.1.12. Let f be the bounded φ-Laplacian [8] given by $f : \mathbb{R} \to (-1, 1), f(x) := x/\sqrt{1+x^2}$, $x \in \mathbb{R}$ and consider problem (6.1.7). f is effectively the inverse function of the one in the previous example. Then

$$F(x) = \sqrt{1+x^2} - 1,\ F_+^{-1}(x) = \sqrt{2x+x^2}.$$

The conditions in Corollary 6.1.8 are satisfied without any further restrictions. In Figure 6.1.2 we plot how the period varies as a function of c and λ. Observe in this plot how the period is decreasing in λ, increasing in c and $\lim_{\lambda\to 0} T(c,\lambda) = \lim_{c\to+\infty} T(c,\lambda) = +\infty$.

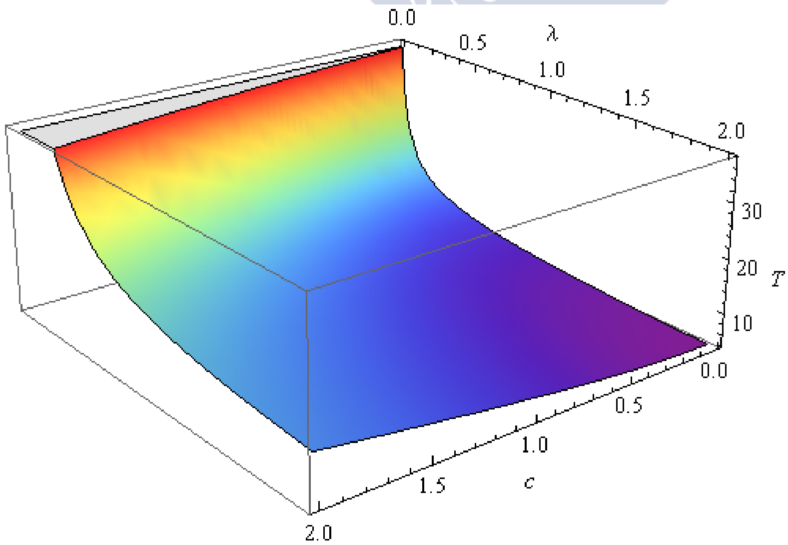


Figure 6.1.2: Graph of the period T function of c and λ.

6.2 Problems with reflection

Let us consider again the problem that motivated this chapter, the obtaining of solutions of problem (3.1.5) in the case $\varphi(t) = -t$. Hence, consider again the problems (3.1.1) and (3.1.2) in the case $\varphi(t) = -t$.

Observe that Lemma 3.1.1 (following Remark 3.1.5) can be trivially extended to the following lemma.

Lemma 6.2.1. *Let $f : (\tau_1, \tau_2) \to (\sigma_1, \sigma_2)$ an locally Lipschitz a. c. function with a. c. inverse. Then x is a solution of the first order differential equation with involution (3.1.5) if and only if x is a solution of the second order ordinary differential equation (3.1.6).*

As was previously shown, problem (3.1.6) is equivalent to problem (6.0.1). We can now state the following corollary of Theorem 6.1.3 regarding the periodicity of problem (3.1.5) as foreseen at the beginning of the chapter.

Corollary 6.2.2. *Let $f : (\tau_1, \tau_2) \to (\sigma_1, \sigma_2)$ an increasing locally Lipschitz a. c. function with a. c. inverse such that $0 \in (\tau_1, \tau_2)$, $f(0) = 0$ and $c > 0$. Assume $2F(c) < \min\{F(\tau_1), F(\tau_2)\}$. Then, if $x_c(t)$ is a solution of problem* (6.0.2) *and we assume there exist $\bar{c}_1, \bar{c}_2 \in \mathbb{R}$, $\bar{c}_1 < \bar{c}_2$, such that $2\max\{F(\bar{c}_1), F(\bar{c}_2)\} < \min\{F(\tau_1), F(\tau_2)\}$ and $(x_{\bar{c}_1}(b) - \bar{c}_1)(x_{\bar{c}_2}(b) - \bar{c}_2) < 0$, then problem* (3.1.5) *must have at least a solution.*

We now give an example in which there is no need to find $\bar{c}_1, \bar{c}_2 \in \mathbb{R}$ in the conditions of Corollary 6.2.2 because the function determining the period has a simple inverse.

Example 6.2.3. Take again $f(t) := |t|^{p-2}t$, $p > 1$, $c > 0$ and consider the problem

$$x'(t) = |x(-t)|^{p-2}x(-t),\ t \in \mathbb{R},\ x(0) = c. \tag{6.2.1}$$

By Corollaries 6.1.8 and 6.2.2 and Example 6.1.9, we have that the solutions of are periodic for every $c \neq 0$ and

$$T(c,1,p) = 2^{\frac{2}{p}+1} c^{2-p} \frac{\Gamma\left(\frac{1}{p}\right)^2}{p\,\Gamma\left(\frac{2}{p}\right)}.$$

Consider now the problem

$$x'(t) = |x(-t)|^{p-2}x(-t),\ t \in \mathbb{R},\ x(a) = x(b). \tag{6.2.2}$$

There is a unique solution for problem (6.2.2) for $p \in (2, +\infty)$. Just take the unique solution of problem (6.2.1) with

$$c = \left(\frac{b-a}{2^{\frac{2}{p}+1}} p \frac{\Gamma\left(\frac{2}{p}\right)}{\Gamma\left(\frac{1}{p}\right)^2}\right)^{\frac{1}{2-p}}.$$

Observe that for $p \in (0, 2)$ the function f is not locally Lipschitz, and therefore we cannot apply Lemma 6.2.1.

7. A *Mathematica* implementation

In this chapter we develop an algorithm implemented in *Mathematica* which allows the obtaining of the Green's function associated to a differential equation with constant coefficients, reflection and boundary conditions. We also point out possible ways to improve the computational time of the algorithm based on particular decompositions of the problem. The results in this chapter were sent for publication [165].

In order to establish a useful framework to work with these equations, we go back to the notation in Chapter 5. We consider the differential operator D, the pullback operator of the reflection $\varphi(t) = -t$, denoted by $\varphi^*(u)(t) = u(-t)$, and the identity operator, Id.

Let $T \in \mathbb{R}^+$ and $I := [-T, T]$. We consider again the algebra $\mathbb{R}[D, \varphi^*]$.

7.1 The algorithm

Theorem 5.2.3 gives a way of computing the Green's function of a problem with reflection via reduction of the problem. The possibility of computing the Green's function relies entirely on whether the reduced problem has a unique solution or not.

Once we have reduced the problem, we check whether it has a unique solution and, in that case, we use part of the algorithm described in Chapter 6 to derive its Green's function. Then it is left to compute the function $R_\vdash G$ as expressed in Theorem 5.2.3 which will be the Green's function to our problem. Figure 7.1.1 shows the flow diagram of the algorithm.

7.1.1 Characteristics of the *Mathematica* notebook

We work with the following input variables:

- *Coefficients* $\mathbf{a_k}$: The coefficients associated to the terms $u^{k)}(t)$.
- *Coefficients* $\mathbf{b_k}$: The coefficients associated to the terms $u^{k)}(-t)$.
- $\mathbf{T}$: A positive number, half of the length of the interval on which the solution is defined.
- *Boundary conditions*: A vector in *Mathematica* notation which specifies the boundary conditions.

The input variables may be numbers or abstract symbols. The vectors of coefficents must be introduced in *Mathematica* notation (there is a default example when the program starts so to get an idea, see Figure 7.1.2). Furthermore, there is a checkbox which allows *Mathematica* to consider the numbers in the input variables as numerical approximations, which greatly reduces the computation time.

While running, the steps of the computation will be shown in the 'Progress' frame. These messages will be, in order, 'Processing data...', 'Solving homogeneous equation...', 'Computing

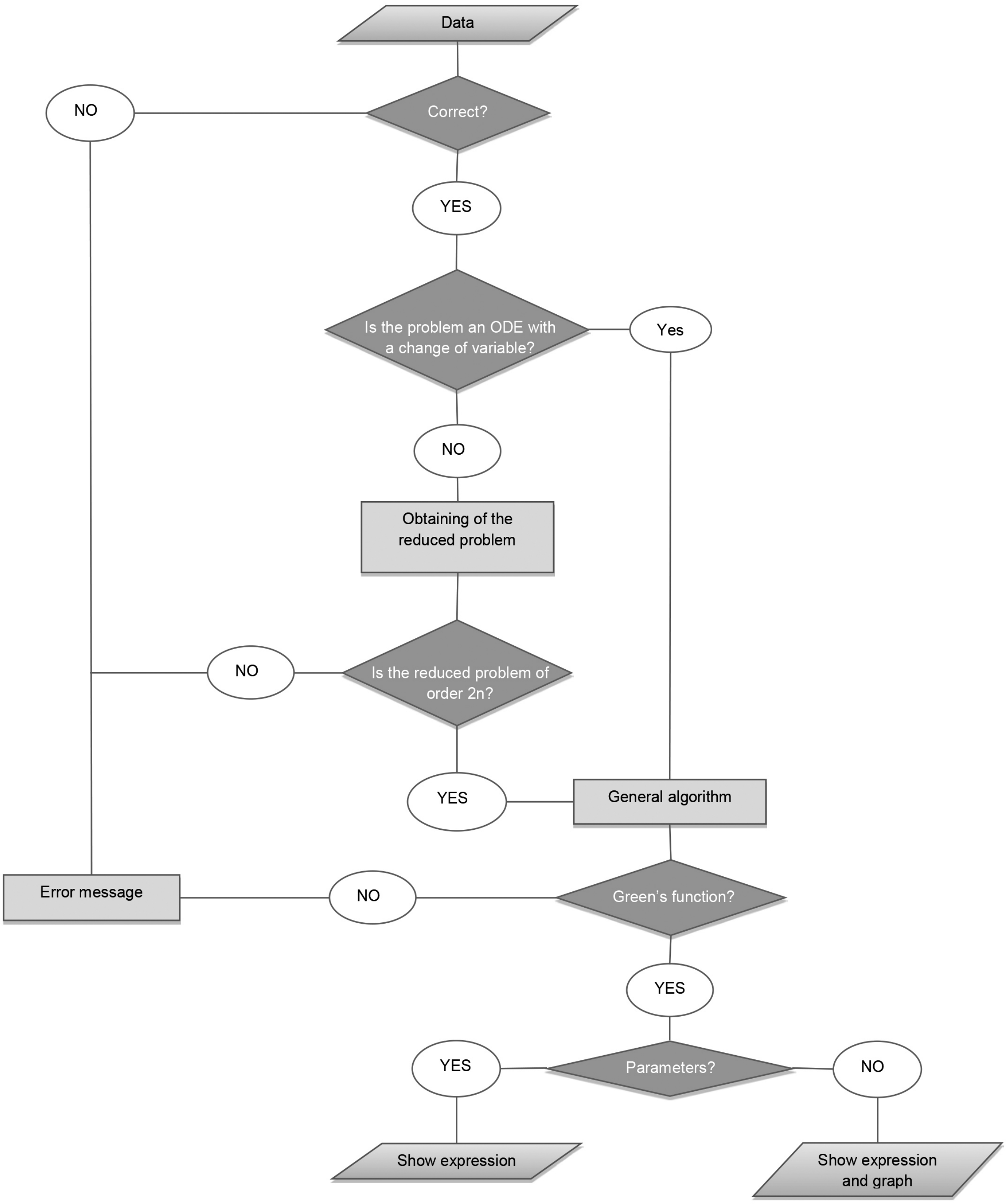


Figure 7.1.1: Flow diagram of the algorithm.

fundamental matrix...', 'Constructing Green's function... (100 s max)' and, finally 'done', right before the graphical output appears (see Figure 7.1.3). Usually, the step that takes the longest is the construction of the Green's function. The '100 s max' comment makes reference to the total time limit set for those *Mathematica* commands during this process which can be aborted after some time giving a valid result, like, for instance Simplify or FullSimplify . This does not mean that other operations on which no time limit can be placed cannot make the whole process take longer.

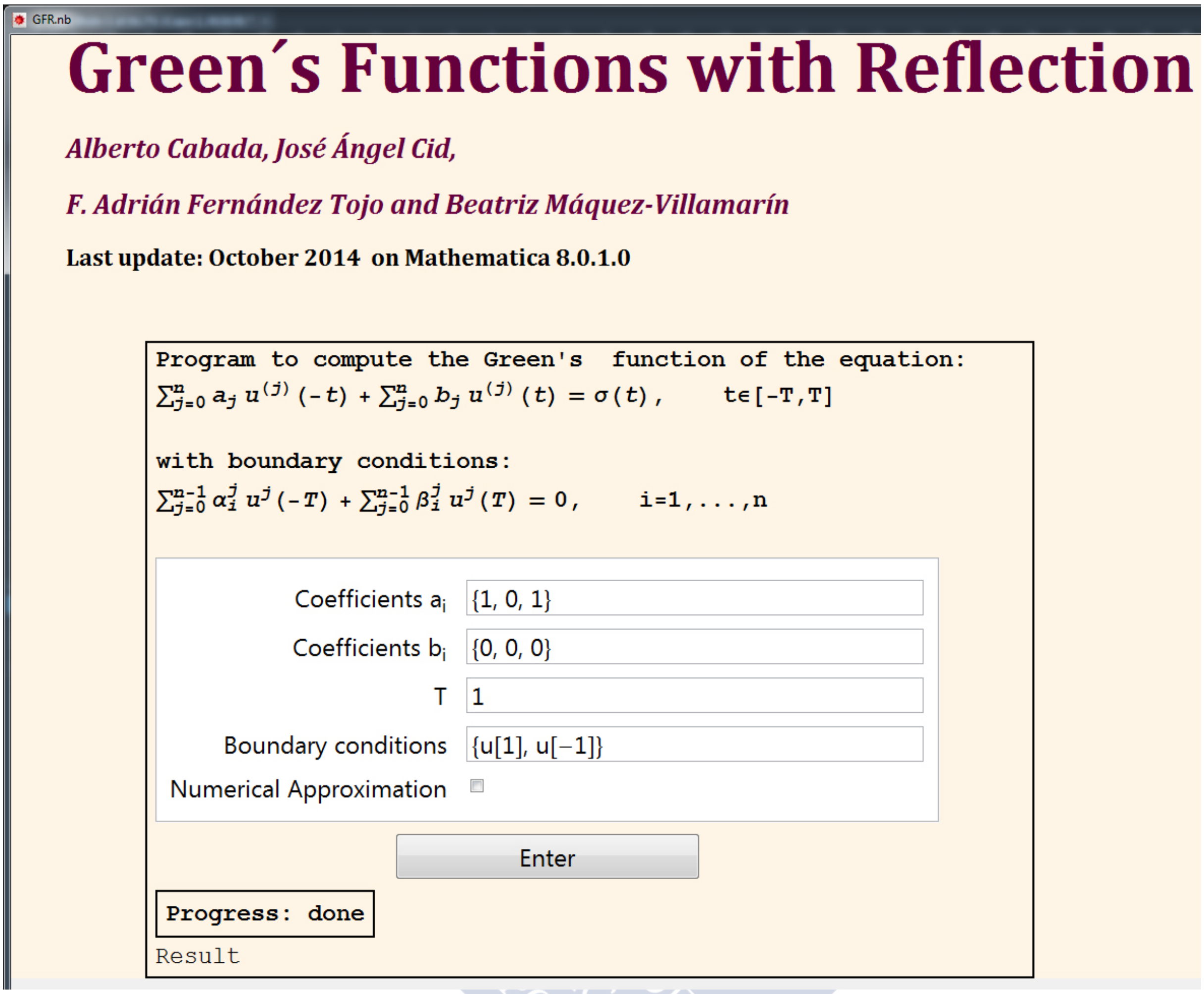


Figure 7.1.2: The Mathematica Notebook after initialization.

7.1.2 Validation of the input variables and error messages

The fist step in the algorithm is to check whether the input data is correct. The order of the equation will be computed automatically as the index of the highest nonzero coefficient in the vectors (a_k) and (b_k). If the order is zero ($(a_k) = (b_k) = 0$), then an error message will appear. The program will check as well whether the length of the vectors (a_k) and (b_k) is consistent, if the boundary conditions are valid, if T is a positive real number (in the case it is a number) and so on. Most important, it will check as well if the condition $a_n = \pm b_n$ is satisfied for in that case we cannot use the algorithm to derive a Green's function.

7.1.3 Computing the reduced problem

The program reads the input values in the variables and vectors `c1, c2, T, cc1` and `Nap`, which correspond, respectively, to (a_k), (b_k), T, the boundary conditions and whether the 'Numer-

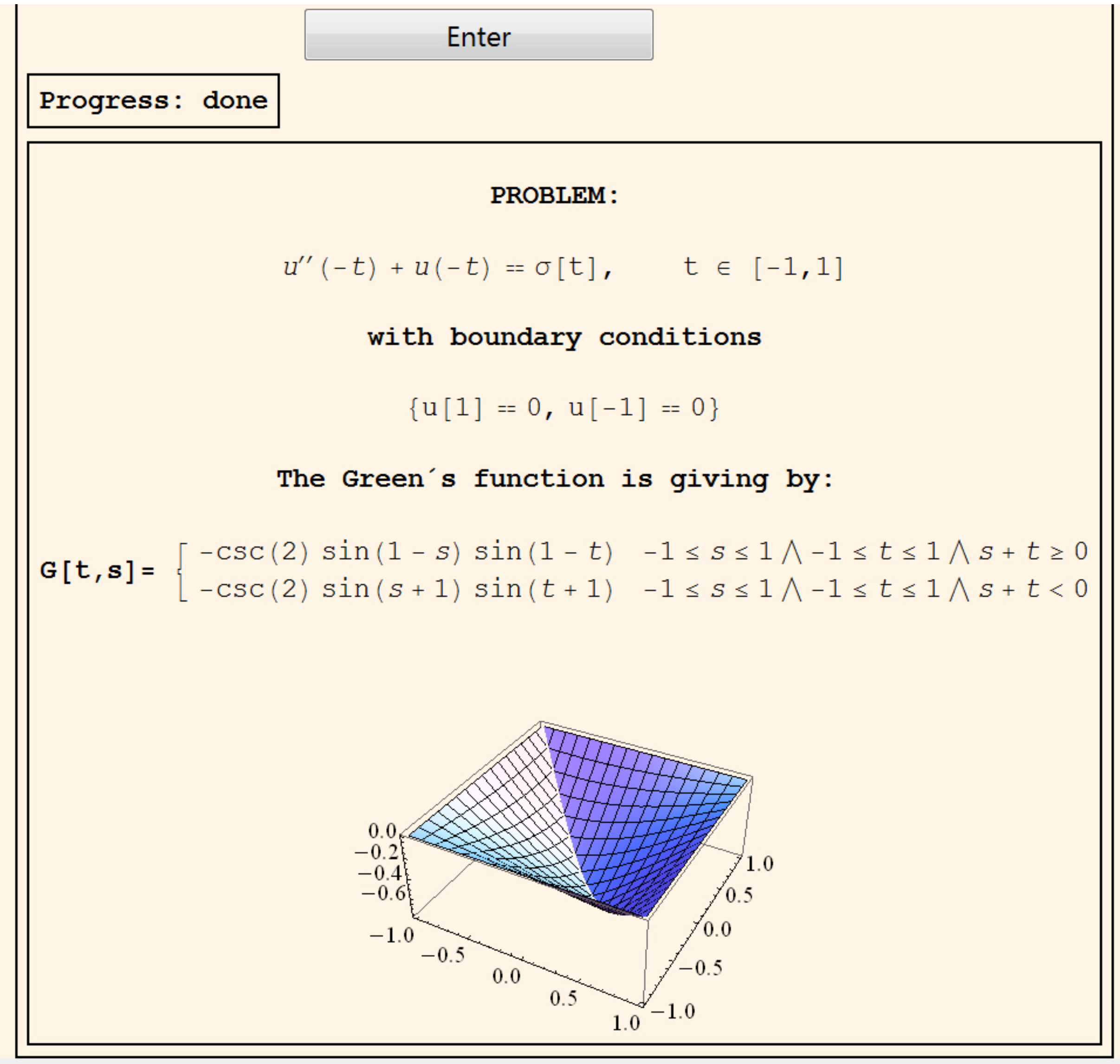


Figure 7.1.3: Result of the default problem.

ical approximation' checkbox is activated. If the 'Numerical approximation' checkbox is activated, the program will automatically transform the values of c1 and, c2 to numerical values if possible:

```
If [Nap,
        If [Element[c1, Reals ], c1=  N[c1 ]];
        If [Element[c2, Reals ], c2=N[c2 ]];
    ]
```

The program now separates the problem in three different cases. First, if there is no reflection $((a_i) = 0,$ If [TrueQ[Norm[c1]==0]) the Green's function will be obtained by the algorithm described in [31] for the nonhomogeneous case. If all of the terms depend on the reflection, that is, $((b_i) = 0,$ If [TrueQ[Norm[c2]==0]), we can apply the change of variable $s = -t$ and turn it into a problem with an ordinary differential equation and use the mentioned algorithm. Then it is left to undo the change of variable for the Green's function and so obtain it for our problem (here Gb is a variable where the Green's function is stored before the change of variable):

```
G[t_,s_]=Chop[PiecewiseExpand[Gb[−s][t]/c[[m+1]], TimeConstraint −> 15]];
```

Finally, there is the case where no shortcut is possible (If [Not[TrueQ[Norm[c1]*Norm[c2]==0]]). In these circumstances, we define the operator L related to de equation as

```
L[f_][x_] := Sum[c1b[[k + 1]] Derivative [k][f][−x] + c2b[[k + 1]] Derivative [k][f
    ][x], {k, 0, n}];
```

and the associated operator R as

```
R[f_][x_] := Sum[c1b[[k + 1]] Derivative [k][f][−x] −(−1)^k c2b[[k + 1]] Derivative
    [k][f][x], {k, 0, n}];
```

Now we obtain the coefficients of the reduced equation:

```
Do[c[[j + 1]] = Sum[(−1)^i*(c1b[[i + 1]]*c1b[[j − i + 1]] − c2b[[i + 1]]*c2b[[j −
    i + 1]]) , {i, 0, j}], {j, 0, m}];
```

and the new boundary conditions:

```
aux2[u_]:= Join [aux[u], Expand[aux[R[u ]]]];
```

which are the original conditions (stored in the vector aux) together with the ones obtained composing such conditions with the operator R. Now we proceed as usual with the classical algorithm and obtain the Green's function composing with the operator R:

```
Gb1[t_,s_]=PiecewiseExpand[R[Gb[s]][t], TimeConstraint −> 15];
```

7.1.4 Final remarks

Although the algorithm allows the obtaining of the Green's function for any order of the equation, the implementation in *Mathematica* suffers severe limitations in this regard. Often, for big orders or several parameters, the computations are too long and convoluted for Mathematica to obtain the result in a reasonable time and, when it succeeds, the output is frequently gargantuan.

We can think of various possibilities in order to palliate the computational time problem. One of them could be computing the Green's function for the reduced problem using matrix exponentiation. Another one could be the one we sketch next.

First observe that, from Remark 5.1.2, we know that the reduced equation has no derivatives in odd indices. This allows to use the following Lemma. For convenience, if p is a real (complex) polynomial, we will denote by p_- the polynomial with the same principal coefficient and opposite eigenvalues.

Lemma 7.1.1. *Let $n \in \mathbb{N}$ and $p(x) = \sum_{k=0}^{n} \alpha_{2k}x^{2k}$ a real polynomial of order $2n$. Then there is a complex polynomial q of order n such that $p = \alpha_{2n}qq_-$. Furthermore, if $\tilde{p}(x) = \sum_{k=0}^{n} \alpha_{2k}x^{k}$ has no negative eigenvalues, q is a real polynomial.*

Proof. First observe that p is a polynomial on x^2, and therefore, if λ is an eigenvalue of p, so has to be $-\lambda$. Hence, using the Fundamental Theorem of Algebra the first part of the result can be derived by separating the monomials that compose p in two different polynomials with opposite eigenvalues.

Let us do that explicitly to show how in the case $\tilde{p}$ has no negative eigenvalues, q is a real polynomial.

Take the change of variables $y = x^2$. Then, $p(x) = \tilde{p}(y)$ and, by the Fundamental Theorem of Algebra,

$$\begin{aligned}\tilde{p}(y) = \sum_{k=0}^{n} \alpha_{2k} y^k =& \alpha_{2n} y^{\sigma} (y - \lambda_1^2) \cdots (y - \lambda_m^2)(y + \lambda_{m+1}^2) \\ & \cdots (y + \lambda_{\overline{m}}^2)(y^2 + \mu_1 y + \nu_1^2) \cdots (y^2 + \mu_l y + \nu_l^2),\end{aligned}$$

for some integers $\sigma, m, \overline{m}, l$ and real numbers $\lambda_1, \ldots, \lambda_{\overline{m}}, \nu_1, \ldots, \nu_l, \mu_1, \ldots, \mu_l$ such that $\lambda_k > 0$ and $\nu_k > |\mu_k|/2$ for every k in the appropriate set of indices[†]. Hence,

$$\begin{aligned}p(x) =& \alpha_{2n} x^{2\sigma} (x^2 - \lambda_1^2) \cdots (x^2 - \lambda_m^2)(x^2 + \lambda_{m+1}^2) \\ & \cdots (x^2 + \lambda_{\overline{m}} r)(x^4 + \mu_1 x^2 + \nu_1^2) \cdots (x^4 + \mu_l x^2 + \nu_l^2).\end{aligned}$$

Now we have that

$$(x^2 - \lambda_k^2) = (x + \lambda_k)(x - \lambda_k), \quad (x^2 + \lambda_k^2) = (x + \lambda_k i)(x - \lambda_k i)$$

$$\text{and} \quad (x^4 + \mu_k x^2 + \nu_k^2) = (x^2 - x\sqrt{2\nu_k - \mu_k} + \nu_k)(x^2 + x\sqrt{2\nu_k - \mu_k} + \nu_k),$$

for any k in the appropriate set of indices. Define

$$\begin{aligned}q(x) =& x^{\sigma}(x - \lambda_1) \cdots (x - \lambda_m)(x - \lambda_{m+1} i) \cdots (x - \lambda_{\overline{m}} i)(x^2 - x\sqrt{2\nu_1 - \mu_1} + \nu_1) \\ & \cdots (x^2 - x\sqrt{2\nu_l - \mu_l} + \nu_l)\end{aligned}$$

and

$$\begin{aligned}q_-(x) =& x^{\sigma}(x + \lambda_1) \cdots (x + \lambda_m)(x + \lambda_{m+1} i) \cdots (x + \lambda_{\overline{m}} i)(x^2 + x\sqrt{2\nu_1 - \mu_1} + \nu_1) \\ & \cdots (x^2 + x\sqrt{2\nu_l - \mu_l} + \nu_l).\end{aligned}$$

We have that $p = \alpha_{2n} q q_-$. The nonzero eigenvalues of q are

$$\begin{aligned}& \lambda_1, \ldots, \lambda_m, \lambda_{m+1} i, \ldots, \lambda_{\overline{m}} i, \frac{1}{2}\left(\sqrt{2\nu_1 - \mu_1} \pm i\sqrt{2\nu_1 + \mu_1}\right), \\ & \ldots, \frac{1}{2}\left(\sqrt{2\nu_l - \mu_l} \pm i\sqrt{2\nu_l + \mu_l}\right)\end{aligned}$$

and those of q_- are precisely

$$\begin{aligned}& -\lambda_1, \ldots, -\lambda_m, -\lambda_{m+1} i, \ldots, -\lambda_{\overline{m}} i, -\frac{1}{2}\left(\sqrt{2\nu_1 - \mu_1} \pm i\sqrt{2\nu_1 + \mu_1}\right), \\ & \ldots, -\frac{1}{2}\left(\sqrt{2\nu_l - \mu_l} \pm i\sqrt{2\nu_l + \mu_l}\right).\end{aligned}$$

Clearly, if $\tilde{p}$ has no negative real eigenvalues, q and q_- are real polynomials. ■

[†] The $y^2 + \mu_k y + \nu_k^2$ correspond to the pairs of complex roots of the polynomial. This means that the discriminant $\Delta = \mu_k^2 - 4\nu_k < 0$, that is, $\nu_k > |\mu_k|/2$.

Remark 7.1.2. Descartes' rule of signs establishes that the number of positive roots (with multiple roots of the same value counted separately) of a real polynomial on one variable is either equal to the number of sign differences between consecutive nonzero coefficients, or less than it by an even number, considering the case the terms of the polynomial are ordered by descending variable exponent. This implies that a sufficient criterion for a polynomial $p(x)$ to have no negative roots is for $p(-x)$ to have all coefficients with positive sign, that is, for $p(x)$ to have positive even coefficients and negative odd coefficients.

There exist algorithmic ways of determining the exact number of positive (or real) roots of a polynomial. For more information on this issue see, for instance, [126, 190, 191].

The following Lemma establishes a relation between the coefficients of q and q_-.

Lemma 7.1.3. *Let $n \in \mathbb{N}$ and $q(x) = \sum_{k=0}^{n} \alpha_k x^k$ be a complex polynomial. Then $q_-(x) = \sum_{k=0}^{n} (-1)^{k+n} \alpha_k x^k$.*

Proof. We proceed by induction[†]. For $n = 1$, $q(x) = \alpha(x - \lambda_1)$. Clearly, q has the eigenvalue λ_1 and $q_-(x) = \alpha(x + \lambda_1) = (-1)^{1+1}\alpha x + (-1)^1 \alpha \lambda_1$ the eigenvalue $-\lambda_1$.

Assume the result is true for some $n \geq 1$. Then, for $n + 1$, q is of the form $q(x) = (x - \lambda_{n+1}) r(x)$ where $r(x) = \sum_{k=0}^{n} \alpha_k x^k$ is a polynomial of order n, that is,

$$q(x) = (x - \lambda_{n+1}) \sum_{k=0}^{n} \alpha_k x^k = x^{n+1} + \sum_{k=1}^{n} [\alpha_{k-1} - \lambda_{n+1}\alpha_k]\, x^k - \lambda_{n+1}\alpha_0.$$

Now, $q_-(x) = (x + \lambda_{n+1}) r_-(x)$. Since the formula is valid for n,

$$\begin{aligned} q_-(x) = (x + \lambda_{n+1}) r_-(x) &= (x + \lambda_{n+1}) \sum_{k=0}^{n} (-1)^{k+n} \alpha_k x^k \\ &= x^{n+1} + \sum_{k=1}^{n} (-1)^{k+n+1} [\alpha_{k-1} - \lambda_{n+1}\alpha_k]\, x^k - (-1)^{n+1} \lambda_{n+1} \alpha_0. \end{aligned}$$

So the formula is valid for $n + 1$ as well. ■

This last Lemma allows the computation of the polynomials q and q_- related to the polynomial RL on the variable D using the formula given in Remark 5.1.2. We will assume that RL is of order $2n$, that is, $a_n^2 - b_n^2 \neq 0$. Otherwise the problem of computing q and q_- would be the same but these polynomials would be of less order. Also, assume RL, considered as a polynomial on D^2, has no negative roots in order for q to be a real polynomial. If $L = \sum_{k=0}^{n} (a_k \varphi^* + b_k) D^k$ and $q(D) = D^n + \sum_{k=0}^{n-1} \alpha_k D^k$ then $RL = \sum_{k=0}^{2n} c_k D^k = (-1)^n (a_n^2 - b_n^2) q(D) q_-(D)$. This relation establishes the following system of quadratic equations:

$$c_{2k} = 2 \sum_{l=0}^{k-1} (-1)^l \left(a_l a_{2k-l} - b_l b_{2k-l}\right) + (-1)^k \left(a_k^2 - b_k^2\right)$$

[†] The result can be directly proven by considering the last statement in Remark 7.1.2. If we take a polynomial $p(x) = a(x - \lambda_1) \cdots (x - \lambda_n)$, the polynomial $p(-x)$ has exactly opposite eigenvalues. Actually, $p(-x) = a(-x - \lambda_1) \cdots (-x - \lambda_n) = (-1)^n a(x + \lambda_1) \cdots (x + \lambda_n)$. It is easy to check that the coefficients of $p(-x)$ are precisely as described in the statement of Lemma 7.1.3 save for the factor $(-1)^n$.

$$=\left(a_n^2-b_n^2\right)\left[2\sum_{l=0}^{k-1}(-1)^l\left(\alpha_l\alpha_{2k-l}\right)+(-1)^k\alpha_k^2\right],$$

for $k=0,\dots,n$ where $a_k, b_k, \alpha_k = 0$ if $k\notin\{0,\dots,n\}$ and $\alpha_n = 1$. These are n equations with n unknowns: $\alpha_0,\dots,\alpha_n$. We present here the case of $n=2$ to illustrate the solution of these equations.

Example 7.1.4. For $n=2$, we have that

$$\begin{aligned} RL &= \left(a_2^2-b_2^2\right)D^4+\left(-a_1^2+2a_0a_2+b_1^2-2b_0b_2\right)D^2+a_0^2-b_0^2, \\ \left(a_2^2-b_2^2\right)q(D)q_-(D) &= \left(a_2^2-b_2^2\right)D^4+\left(2\alpha_0-\alpha_1^2\right)\left(a_2^2-b_2^2\right)D^2+\alpha_0^2\left(a_2^2-b_2^2\right), \end{aligned}$$

and the system of equations is

$$\begin{aligned} a_0^2-b_0^2 &= \left(a_2^2-b_2^2\right)\alpha_0^2, \\ -a_1^2+2a_0a_2+b_1^2-2b_0b_2 &= \left(a_2^2-b_2^2\right)\left(2\alpha_0-\alpha_1^2\right). \end{aligned} \tag{7.1.1}$$

Before computing the solutions let us state explicitly de limitations that the fact that RL, considered as an order 2 polynomial on D^2, that is $RL(x)=ax^2+bx+c$, has no negative roots implies. There are two options:

(1) There are two complex roots, that is, $\Delta=b^2-4ac<0$. This is equivalent to $ac>0\wedge|b|<2\sqrt{ac}$. Expressed in terms of the coefficients of RL:

$(b_0^2-a_0^2)(b_2^2-a_2^2)>0$, and $|-a_1^2+2a_0a_2+b_1^2-2b_0b_2|<2\sqrt{(b_0^2-a_0^2)(b_2^2-a_2^2)}$.

(2) There are two nonnegative roots, that is $\Delta=b^2-4ac\geq 0$ and

$$\left(-b+\sqrt{b^2-4ac}\right)/(2a)\leq 0.$$

This is equivalent to $(a,c\geq 0\wedge -b\geq 2\sqrt{ac})\vee(a,c\leq 0\wedge b\geq 2\sqrt{ac})$. Expressed in terms of the coefficients of RL:

$$\left[(b_0^2-a_0^2),(b_2^2-a_2^2)\geq 0\wedge-(-a_1^2+2a_0a_2+b_1^2-2b_0b_2)\geq 2\sqrt{(b_0^2-a_0^2)(b_2^2-a_2^2)}\right]$$

OR

$$\left[(b_0^2-a_0^2),(b_2^2-a_2^2)\leq 0\wedge-(-a_1^2+2a_0a_2+b_1^2-2b_0b_2)\geq 2\sqrt{(b_0^2-a_0^2)(b_2^2-a_2^2)}\right].$$

Now, with these conditions, the solutions the system of equations (7.1.1) are:

Case (I). We have two solutions:

$$\alpha_0=\sqrt{\frac{b_0^2-a_0^2}{b_2^2-a_2^2}},$$

$$\alpha_1 = \pm\sqrt{\frac{2\,\mathrm{sign}(a_2^2-b_2^2)\sqrt{(b_0^2-a_0^2)(b_2^2-a_2^2)}-(-a_1^2+2a_0a_2+b_1^2-2b_0b_2)}{a_2^2-b_2^2}}.$$

Case (II). We have four solutions depending on whether we choose $\xi = 1$ or $\xi = -1$:

$$\alpha_0 = \xi\sqrt{\frac{b_0^2-a_0^2}{b_2^2-a_2^2}},$$

$$\alpha_1 = \pm\sqrt{\frac{2\xi\,\mathrm{sign}(a_2^2-b_2^2)\sqrt{(b_0^2-a_0^2)(b_2^2-a_2^2)}-(-a_1^2+2a_0a_2+b_1^2-2b_0b_2)}{a_2^2-b_2^2}}.$$

These solution provide well defined real numbers by conditions (I) and (II).

Now we could consider those cases where the problem can be decomposed easily. Consider that the reduced problem given by Theorem 5.2.3, $Su = Rh, B_jRu = 0, B_ju = 0, j = 1,\dots,n$ can be expressed as an equivalent factored problem

$$\begin{aligned} L_1u = y, &\quad V_ju = 0, j = 1,\dots,n,\\ L_2y = Rh, &\quad \widetilde{V_j}y = 0, j = 1,\dots,n,\end{aligned}$$

where the conditions $V_ju = 0$, $\widetilde{V_j}L_1u = 0$, $j = 1,\dots,n$ are equivalent to the conditions $B_jRu = 0, B_ju = 0, j = 1,\dots,n$. Then the Green's function of problem $Su = Rh, B_jRu = 0$, $B_ju = 0, j = 1,\dots,n$ can be expressed as

$$G(t,s) = \int_{-T}^{T} G_1(t,r)G_2(r,s)\,\mathrm{d}r,$$

where G_1 is the Green's function associated to the problem $L_1u = y$, $V_ju = 0, j = 1,\dots,n$, and G_2 the one associated to the problem $L_2y = Rh$, $\widetilde{V_j}y = 0, j = 1,\dots,n$, in the case both Green's functions exist. This procedure was already illustrated in Example 5.2.4.

Computationally, this procedure poses a big advantage: it is always easier to obtain the Green's function two order n problems than to do so for one order $2n$ problem. Furthermore, if the hypothesis of Lemma 7.1.1 are satisfied and we are able to obtain a factorization of the aforementioned kind using q and q_- in the place of L_1 and L_2, we have an extra advantage: the differential equation given by q_- is the adjoint equation of the one given by q multiplied by the factor $(-1)^n$. This fact, together with the following result (which can be found, although not stated as in this work, in [28]), illustrates that in this case it may be possible to solve problem (5.2.2) just computing the Green's function of one order n problem.

Theorem 7.1.5. *Consider an interval $J = [a,b] \subset \mathbb{R}$, functions $\sigma, a_i \in \mathrm{L}^1(J)$, $i = 1,\dots,n$, real numbers $\alpha_{ij}, \beta_{ij}, h_i$, $i = 1,\dots,n, j = 0,\dots,n-1$, $D(L_n) \subset W^{n,1}(J)$ a vector subspace, the operator*

$$L_nu(t) = a_0u^{(n)}(t) + a_1(t)u^{(n-1)}(t) + \dots + a_{n-1}(t)u'(t) + a_n(t)u(t),\ t \in J,\ u \in D(L_n),$$

with $a_0 = 1$ and the problem

$$L_nu(t) = \sigma(t),\ t \in J, \quad U_i(u) = h_i,\ i = 1,\dots,n, \tag{7.1.2}$$

where

$$U_i(u) := \sum_{j=0}^{n-1}\left(\alpha_{ij}u^{(j)}(a)+\beta_{ij}u^{(j)}(b)\right), \quad i=1,\dots,n.$$

Then, the associated adjoint problem is

$$L_n^\dagger v(t) = \sum_{j=0}^{n}(-1)^j a_{n-j}(t)u^{(j)}(t),\ t\in J,\ v\in D(L_n^\dagger), \tag{7.1.3}$$

where $D(L_n^\dagger) =$

$$\left\{v\in W^{n,2}(J)\ :\ (b^*-a^*)\left(\sum_{j=1}^{n}\sum_{i=0}^{j-1}(-1)^{(j-i-1)}(a_{n-j}v)^{j-i-1}u^{(i)}\right)=0,\ u\in D(L_n)\right\}$$

Furthermore, if $G(t,s)$ is the Green's function of problem (7.1.2)*, then the one associated to problem* (7.1.3) *is $G(s,t)$.*

Hence, if we can decompose problem (5.2.2) in two adjoint problems, its Green's function will be

$$G(t,s) = \int_{-T}^{T}G_1(t,r)G_2(r,s)\,\mathrm{d}r = \int_{-T}^{T}G_1(t,r)G_1(s,r)\,\mathrm{d}r.$$

We note though, that unless the operator q_- is the adjoint equation times $(-1)^n$, the boundary conditions may be not the adjoint ones.

Part II

Topological Methods

We have so far studied differential equations with reflection finding, when possible, the Green's function in order to derive the solution in the case of uniqueness. Still, many situations, in which nonlinearities are involved, escape the direct construction of solutions and different methods become necessary.

Topological methods come handy in these situations, in particular those related to the fixed point index. These tools permit to guarantee the existence and multiplicity of fixed points of continuous maps through an index which counts them with sign. We have already used in Subsection 3.2.3 the celebrated cone contraction-expansion fixed point theorem of Krasnosel'skiĭ. Here we avoid its limitations using an approach developed by Infante and Webb [97] and used in several publications [34, 35, 87–95, 98–100, 175–184].

In the following four chapters we will use this method to solve four different kinds of problems increasing in complexity: a problem with reflection, a problem with deviated arguments (applied to a thermostat model), a problem with nonlinear Neumann boundary conditions and a problem with functional nonlinearities in both the equation and the boundary conditions.

The structure of the method is fairly consistent and is developed as follows.

(1) State the *nature of the problem* being studied and its specific characteristics.

(2) Elaborate a list of *properties*, of the elements involved in the problem, which is necessary to ask for so we can grant that the existence / multiplicity / nonexistence results can be applied. For instance, the operator F of which its fixed points will be solutions for our problem has to be continuous.

(3) Define an *appropriate cone* K in which we will localize the solutions of our problem. Here we have to take an important decision: large cones allow the finding of more solutions but, at the same time, they do not provide good localization results.

(4) Prove that the operator F is *compact*, *continuous* and *maps* K to K.

(5) Find *sufficient conditions* for which the fixed point index of the operator F is 0 and ± 1 respectively in (at least) two nested subsets of the cone. If we find n nested subsets for which the index alternates from 0 to ± 1 we can guarantee the existence of at least $n-1$ different nontrivial solutions (cf. [123]).

    By making the cone smaller, we trade solutions for simpler conditions. Also, we may use conditions for the index related to the eigenvalues of the operators involved (see Chapters 10 and 11).

(6) Finally, we can apply the results derived to a vast variety of problems and illustrate its usefulness with some examples.

As we will see, the particularities of each problem make it impossible to take a common approach to all of the problems studied. Still, there will be important similarities in the different cases which will lead to comparable results. The results in Chapters 8, 9 and 10 have been published in [34], [34] and [96] respectively. Those in Chapter 11 are ready to be sent for publication soon.

Due to the bast amount of notation necessary to develop this theory, we will consider it only valid for the chapter in question, so we can use the same symbols for similar (but different) purposes.

8. A cone approximation to a problem with reflection

We have studied previously (see Chapter 3), the first order operator $u'(t) + \omega u(-t)$ coupled with periodic boundary value conditions, describing the eigenvalues of the operator and providing the expression of the associated Green's function in the nonresonant case. We provide the range of values of the real parameter ω for which the Green's function has constant sign and apply these results to prove the existence of constant sign solutions for the nonlinear periodic problem with reflection of the argument (see page 55)

$$u'(t) = h(t, u(t), u(-t)),\ t \in [-T, T], \quad u(-T) = u(T). \tag{8.0.1}$$

The methodology, analogous to the one used by Torres [167] in the case of ordinary differential equations, is to rewrite the problem (8.0.1) as an Hammerstein integral equation with reflections of the type

$$u(t) = \int_{-T}^{T} k(t,s)\,[h(s, u(s), u(-s)) + \omega u(-s)]\,\mathrm{d}s, \quad t \in [-T, T],$$

where the kernel k has constant sign, and to make use of the well-known Guo-Krasnosel'skiĭ theorem on cone compression-expansion (see Theorem 3.2.19).

In this chapter we continue this study and we prove new results regarding the existence of nontrivial solutions of Hammerstein integral equations with reflections of the form

$$u(t) = \int_{-T}^{T} k(t,s) g(s) f(s, u(s), u(-s))\,\mathrm{d}s, \quad t \in [-T, T],$$

where the kernel k is allowed to be not of constant sign. In order to do this, we extend the results of [98], valid for *Hammerstein integral equations* without reflections, to the new context. We make use of a cone of functions that are allowed to change sign combined with the classical fixed point index for compact maps (we refer to [4] or [81] for further information). As an application of our theory we prove the existence of nontrivial solutions of the periodic problem with reflections (8.0.1). The results of this chapter were published in [34]

8.1 The case of kernels that change sign

We begin with the case of kernels that are allowed to change sign. We impose the following conditions on k, f, g that occur in the integral equation

$$u(t) = \int_{-T}^{T} k(t,s) g(s) f(s, u(s), u(-s))\,\mathrm{d}s =: Fu(t), \tag{8.1.1}$$

where T is fixed in $(0, \infty)$.

(C_1) The kernel k is measurable, and for every $\tau \in [-T, T]$ we have

$$\lim_{t \to \tau} |k(t,s) - k(\tau,s)| = 0 \text{ for almost every (a. e.) } s \in [-T, T].$$

(C_2) There exist a subinterval $[a,b] \subseteq [-T,T]$, a measurable function Φ with $\Phi \geq 0$ a. e. and a constant $c = c(a,b) \in (0,1]$ such that

$$|k(t,s)| \leq \Phi(s) \text{ for all } t \in [-T,T] \text{ and a. e. } s \in [-T,T],$$
$$k(t,s) \geq c\,\Phi(s) \text{ for all } t \in [a,b] \text{ and a. e. } s \in [-T,T].$$

(C_3) The function g is measurable and satisfies that $g\,\Phi \in \mathrm{L}^1([-T,T])$, $g(t) \geq 0$ a. e. $t \in [-T,T]$ and $\int_a^b \Phi(s)g(s)\,\mathrm{d}s > 0$.

(C_4) The nonlinearity $f : [-T,T] \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ satisfies the L^∞-Carathéodory conditions, that is, $f(\cdot,u,v)$ is measurable for each fixed u and v and $f(t,\cdot,\cdot)$ is continuous for a. e. $t \in [-T,T]$, and for each $r > 0$, there exists $\phi_r \in \mathrm{L}^\infty([-T,T])$ such that

$$f(t,u,v) \leq \phi_r(t) \text{ for all } (u,v) \in [-r,r] \times [-r,r], \text{ and a. e. } t \in [-T,T].$$

We recall the following definition.

Definition 8.1.1. Let X be a Banach Space. A *cone* on X is a closed, convex subset of X such that $\lambda\, x \in K$ for $x \in K$ and $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

Here we work in the space $C[-T,T]$, endowed with the usual supremum norm, and we use the cone

$$K = \{u \in C[-T,T] : \min_{t\in[a,b]} u(t) \geq c\|u\|\}, \tag{8.1.2}$$

where c and $[a,b]$ are defined in (C_2). Note that $K \neq \{0\}$.

The cone K has been essentially introduced by Infante and Webb in [98] and later used in [34,66,69,70,87,93,94,97,99,100,134]. K is similar to a type of *cone of nonnegative functions* first used by Krasnosel'skiĭ, see e.g. [121], and D. Guo, see e.g. [81]. Note that functions in K are positive on the subset $[a,b]$ but are allowed to change sign in $[-T,T]$.

We require some knowledge of the classical fixed point index for compact maps, see for example [4] or [81] for further information. If Ω is a bounded open subset of K (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to K. When D is an open bounded subset of X we write $D_K = D \cap K$, an open subset of K.

Next Lemma is a direct consequence of classical results from degree theory [81].

Lemma 8.1.2. *Let Ω be an open bounded set with $0 \in \Omega_K$ and $\overline{\Omega}_K \neq K$. Assume that $F : \overline{\Omega}_K \to K$ is a continuous compact map such that $x \neq Fx$ for all $x \in \partial\Omega_K$. Then the fixed point index $i_K(F,\Omega_K)$ has the following properties.*

(1) If there exists $e \in K\backslash\{0\}$ such that $x \neq Fx + \lambda e$ for all $x \in \partial\Omega_K$ and all $\lambda > 0$, then $i_K(F,\Omega_K) = 0$.

(2) If $\mu x \neq Fx$ for all $x \in \partial\Omega_K$ and for every $\mu \geq 1$, then $i_K(F,\Omega_K) = 1$.

(3) If $i_K(F,\Omega_K) \neq 0$, then F has a fixed point in Ω_K.

(4) Let Ω^1 be open in X with $\overline{\Omega^1} \subset \Omega_K$. If $i_K(F,\Omega_K) = 1$ and $i_K(F,\Omega_K^1) = 0$, then F has a fixed point in $\Omega_K\backslash\overline{\Omega_K^1}$. The same result holds if $i_K(F,\Omega_K) = 0$ and $i_K(F,\Omega_K^1) = 1$.

Definition 8.1.3. We use the following sets:

$$K_\rho = \{u \in K : \|u\| < \rho\},\ V_\rho = \{u \in K : \min_{t\in[a,b]} u(t) < \rho\}.$$

The set V_ρ was introduced in [100] and is equal to the set called $\Omega_{\rho/c}$ in [97]. The notation V_ρ makes shows that choosing c as large as possible yields a weaker condition to be satisfied by f in Lemma 8.1.10. A key feature of these sets is that they can be nested, that is

$$K_\rho \subset V_\rho \subset K_{\rho/c}.$$

Lemma 8.1.4. *The operator* $N_f(u,v)(t) = \int_0^1 k(t,s)g(s)f(s,u(s),v(s))\,\mathrm{d}s$ *maps* $C(I)\times \mathrm{L}^\infty(I)$ *to* $C(I)$ *and is compact and continuous.*

Proof. Fix $(u,v) \in C(I)\times \mathrm{L}^\infty(I)$ and let $(t_n)_{n\in\mathbb{N}} \subset I$ be such that $\lim\limits_{n\to\infty}(t_n) = t \in I$. Take $r = \|(u,v)\| := \|u\| + \|v\|$ and consider

$$h_n(s) := k(t_n,s)\,g(s)\,f(s,u(s),v(s)),\ \text{for a.e. } s \in I.$$

We have, by (C_1), that

$$\lim_{n\to\infty} h_n(s) = h(s) := k(t,s)\,g(s)\,f(s,u(s),v(s)),\ \text{for a.e. } s \in I.$$

On the other hand, $|h_n| \le \Phi g\,\|\phi_r\| \in \mathrm{L}^1(I)$ so, by the Dominated Convergence Theorem, we have $\lim\limits_{n\to\infty} N_f(u,v)(t_n) = N_f(u,v)(t)$ and therefore $N_f(u,v) \in C(I)$.

Now let's see that N_f is compact, indeed, let $(u_n,v_n)_{n\in\mathbb{N}} \subset C(I)\times \mathrm{L}^\infty(I)$ be such that $\|(u_n,v_n)\| \le R \in \mathbb{R}^+$ for all $n \in \mathbb{N}$.

Define $y_n(s) = f(s,u_n(s),v_n(s))$. By Condition (C_4) we know that $\|y_n\| \le \|\phi_R\| \in \mathrm{L}^\infty(I)$, therefore $(y_n(s))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ and by the Bolzano-Weierstrass Theorem it has a convergent subsequence $(y_{n_k}(s))_{k\in\mathbb{N}}$. Take $y(s) := \lim\limits_{k\to\infty} y_{n_k}(s)$.

Now, since

$$\|k(t,\cdot)g(\cdot)y_{n_k}(\cdot)\| \le \Phi(\cdot)g(\cdot)\|\phi_R\|,\quad \text{for all } t \in I,$$

we can apply the Dominated Convergence Theorem and therefore

$$\lim_{k\to\infty} N_f(u_{n_k},v_{n_k})(t) = \int_0^1 k(t,s)g(s)y(s)\,\mathrm{d}s,\ \text{for all } t \in I.$$

So we have proved that there exists the point-wise limit on I. To conclude the assertion of compactness we verify that such convergence is uniform in I. To this end, we take into account that for all $t \in I$ it is verified that

$$\begin{aligned}|N_f(u_{n_k},v_{n_k})(t) - N_f(u,v)(t)| &\le \int_0^1 |k(t,s)|g(s)|y_{n_k}(s) - y(s)|\,\mathrm{d}s\\ &\le \int_0^1 \Phi(s)g(s)|y_{n_k}(s) - y(s)|\,\mathrm{d}s.\end{aligned}$$

Since the last expression on the right is independent of t we have that such convergence is uniform in I, and the assertion holds.

The continuity is proved in a similar manner. ■

Remark 8.1.5. If N_f maps $C(I) \times C(I)$ to $C(I)$ the proof works exactly the same.

Theorem 8.1.6. *Assume that hypotheses* (C_1)–(C_4) *hold. Then* F *maps* $\overline{K_r}$ *into* K *and is compact and continuous. In particular* F *maps* K *into* K.

Proof. For $u \in \overline{K_r}$ and $t \in [-T, T]$ we have,

$$\begin{aligned} |Fu(t)| &\le \int_{-T}^{T} |k(t,s)| g(s) f(s, u(s), u(-s))\, \mathrm{d}s \\ &\le \int_{-T}^{T} \Phi(s) g(s) f(s, u(s), u(-s))\, \mathrm{d}s, \end{aligned}$$

and

$$\min_{t\in[a,b]} Fu(t) \ge +c \int_{-T}^{T} \Phi(s) g(s) f(s, u(s), u(-s))\, \mathrm{d}s \ge c\|Fu\|.$$

Therefore we have that $Fu \in K$ for every $u \in \overline{K_r}$.

The compactness of F follows from Lemma 8.1.4. ■

In the sequel, we give a condition that ensures that, for a suitable $\rho > 0$, the index is 1 on K_ρ.

Lemma 8.1.7. *Assume that*

(I_ρ^1) *there exists* $\rho > 0$ *such that*

$$f^{-\rho,\rho} \cdot \sup_{t\in[-T,T]} \int_{-T}^{T} |k(t,s)| g(s)\, \mathrm{d}s < 1$$

where

$$f^{-\rho,\rho} = \sup\left\{ \frac{f(t,u,v)}{\rho} : \ (t,u,v) \in [-T,T] \times [-\rho,\rho] \times [-\rho,\rho] \right\}.$$

Then the fixed point index, $i_K(F, K_\rho)$, *is equal to 1.*

Proof. We show that $\mu u \ne Fu$ for every $u \in \partial K_\rho$ and for every $\mu \ge 1$. In fact, if this does not happen, there exist $\mu \ge 1$ and $u \in \partial K_\rho$ such that $\mu u = Fu$, that is

$$\mu u(t) = \int_{-T}^{T} k(t,s) g(s) f(s, u(s), u(-s))\, \mathrm{d}s,$$

Taking the absolute value and then the supremum for $t \in [-T, T]$ gives

$$\begin{aligned} \mu\rho &\le \sup_{t\in[-T,T]} \int_{-T}^{T} |k(t,s)| g(s) f(s, u(s), u(-s))\, \mathrm{d}s \\ &\le \rho f^{-\rho,\rho} \cdot \sup_{t\in[-T,T]} \int_{-T}^{T} |k(t,s)| g(s)\, \mathrm{d}s < \rho. \end{aligned}$$

This contradicts the fact that $\mu \ge 1$ and proves the result. ■

For the next remark consider the following lemma.

Lemma 8.1.8. *Let* $\omega \in \mathrm{L}^1([0,1])$ *and denote*

$$\omega^+(s) = \max\{\omega(s), 0\}, \quad \omega^-(s) = \max\{-\omega(s), 0\}.$$

Then we have

$$\left|\int_0^1 \omega(s)\,\mathrm{d}s\right| \le \max\left\{\int_0^1 \omega^+(s)\,\mathrm{d}s, \int_0^1 \omega^-(s)\,\mathrm{d}s\right\} \le \int_0^1 |\omega(s)|\,\mathrm{d}s.$$

Proof. Observing that, since $\omega = \omega^+ - \omega^-$,

$$\begin{aligned} \int_0^1 \omega(s)\,\mathrm{d}s &= \int_0^1 \omega^+(s)\,\mathrm{d}s - \int_0^1 \omega^-(s)\,\mathrm{d}s \le \int_0^1 \omega^+(s)\,\mathrm{d}s, \\ -\int_0^1 \omega(s)\,\mathrm{d}s &= \int_0^1 \omega^-(s)\,\mathrm{d}s - \int_0^1 \omega^+(s)\,\mathrm{d}s \le \int_0^1 \omega^-(s)\,\mathrm{d}s, \end{aligned}$$

we get the first inequality, the second comes from the fact that $|\omega| = \omega^+ + \omega^-$. ■

Remark 8.1.9. We point out that, as in [181], a stronger (but easier to check) condition than (I_ρ^1) is given by the following.

$$\frac{f^{-\rho,\rho}}{m} < 1,$$

where

$$\frac{1}{m} := \sup_{t\in[0,1]} \left\{\max\left\{\int_0^1 k^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k^-(t,s)g(s)\,\mathrm{d}s\right\}\right\},$$

which is finite since $k^+g \le \Phi g \in \mathrm{L}^1([-T,T])$.

Let us see now a condition that guarantees the index is equal to zero on $V_\rho := \{u \in K : \min_{t\in[\hat a,\hat b]} u(t) < \rho\}$ for some appropriate $\rho > 0$.

Lemma 8.1.10. *Assume that*

(I_ρ^0) *there exists* $\rho > 0$ *such that*

$$f_{(\rho,\rho/c)}/M(a,b) > 1,$$

where

$$\begin{aligned} f_{(\rho,\rho/c)} &:= \inf\left\{\frac{f(t,u,v)}{\rho} : (t,u,v) \in [a,b]\times[\rho,\rho/c]\times[-\rho/c,\rho/c]\right\}, \\ \frac{1}{M(a,b)} &:= \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,\mathrm{d}s. \end{aligned}$$

Then $i_K(F, V_\rho) = 0$.

Proof. Let $e(t) \equiv 1$, then $e \in K$. We prove that

$$u \ne Fu + \lambda e \quad \text{for all } u \in \partial V_\rho \text{ and } \lambda \ge 0.$$

In fact, if not, there exist $u \in \partial V_\rho$ and $\lambda \ge 0$ such that $u = Fu + \lambda e$. Then we have

$$u(t) = \int_{-T}^{T} k(t,s)g(s)f(s,u(s),u(-s))\,\mathrm{d}s + \lambda.$$

Thus, taking into account that $k, g, f \geq 0$ in $[a,b] \times [-T,T]$, we get, for $t \in [a,b]$,

$$u(t) = \int_{-T}^{T} k(t,s)g(s)f(s,u(s),u(-s))\,\mathrm{d}s + \lambda \geq \int_{a}^{b} k(t,s)g(s)f(s,u(s),u(-s))\,\mathrm{d}s$$
$$\geq \rho f_{(\rho,\rho/c)} \left(\int_{a}^{b} k(t,s)g(s)\,\mathrm{d}s\right).$$

Taking the minimum over $[a,b]$ gives $\rho > \rho$ a contradiction. ■

The above Lemmas can be combined to prove the following theorem. Here we deal with the existence of at least one, two or three solutions. We stress that, by expanding the lists in conditions $(S_5), (S_6)$ below, it is possible to state results for four or more solutions in K, see for example the paper by Lan [123] for the type of results that might be stated.

We omit the proof which follows directly from the properties of the fixed point index stated in Lemma 8.1.2, (3). In it we would basically construct, using the K_ρ and V_ρ, an strictly increasing –in the subset order sense– sequence of subsets of the cone K, $A^1 \subset A^2 \subset \dots$ satisfying $\overline{A^j_K} \subset \mathring{A}^{j+1}_K$, $j \in \mathbb{N}$, and such the index alternates its value throughout the sequence, thus guaranteeing the existence of solution in the intersection of every two consecutive sets in the sequence. Since the sequence is strictly increasing, all the solutions found are different.

Theorem 8.1.11. *The integral equation* (8.1.1) *has at least one nonzero solution in K if either of the following conditions hold.*

(S_1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1/c < \rho_2$ such that $(\mathrm{I}^0_{\rho_1})$ and $(\mathrm{I}^1_{\rho_2})$ hold.*

(S_2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $(\mathrm{I}^1_{\rho_1})$ and $(\mathrm{I}^0_{\rho_2})$ hold.*

The integral equation (8.1.1) *has at least two nonzero solutions in K if one of the following conditions hold.*

(S_3) *There exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1/c < \rho_2 < \rho_3$ such that $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$ and $(\mathrm{I}^0_{\rho_3})$ hold.*

(S_4) *There exist $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ with $\rho_1 < \rho_2$ and $\rho_2/c < \rho_3$ such that $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$ and $(\mathrm{I}^1_{\rho_3})$ hold.*

The integral equation (8.1.1) *has at least three nonzero solutions in K if one of the following conditions hold.*

(S_5) *There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ with $\rho_1/c < \rho_2 < \rho_3$ and $\rho_3/c < \rho_4$ such that $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$, $(\mathrm{I}^0_{\rho_3})$ and $(\mathrm{I}^1_{\rho_4})$ hold.*

(S_6) *There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ with $\rho_1 < \rho_2$ and $\rho_2/c < \rho_3 < \rho_4$ such that $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$, $(\mathrm{I}^1_{\rho_3})$ and $(\mathrm{I}^0_{\rho_4})$ hold.*

8.2 The case of nonnegative kernels

We now assume the functions k, f, g that occur in (8.1.1) satisfy the conditions $(C_1) - (C_4)$ in the previous section, where (C_2) and (C_4) are replaced with the following.

(C_2') The kernel k is nonnegative for $t \in [-T,T]$ and a. e. $s \in [-T,T]$ and there exist a subinterval $[a,b] \subseteq [-T,T]$, a measurable function Φ, and a constant $c = c(a,b) \in (0,1]$ such that

$$k(t,s) \le \Phi(s) \text{ for } t \in [-T,T] \text{ and a. e. } s \in [-T,T],$$
$$k(t,s) \ge c\Phi(s) \text{ for } t \in [a,b] \text{ and a. e. } s \in [-T,T].$$

(C_4') The nonlinearity $f : [-T,T] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfies L^∞-Carathéodory conditions, that is, $f(\cdot,u,v)$ is measurable for each fixed u and v and $f(t,\cdot,\cdot)$ is continuous for a. e. $t \in [-T,T]$, and for each $r > 0$, there exists $\phi_r \in \mathrm{L}^\infty([-T,T])$ such that

$$f(t,u,v) \le \phi_r(t) \text{ for all } (u,v) \in [0,r] \times [0,r], \text{ and a. e. } t \in [-T,T].$$

These hypotheses enable us to work in the cone of nonnegative functions

$$K' = \{u \in C[-T,T] : u \ge 0 \text{ on } [-T,T], \min_{t \in [a,b]} u(t) \ge c\|u\|\}, \tag{8.2.1}$$

that is smaller than the cone (8.1.2). It is possible to show that F is compact and leaves the cone K' invariant. The conditions on the index are given by the following Lemmas, the proofs are omitted as they are similar to the ones in the previous section.

Lemma 8.2.1. *Assume that*

$\left(\overline{\mathrm{I}_\rho^1}\right)$ *there exists $\rho > 0$ such that $f^{0,\rho} < m$, where*

$$f^{0,\rho} = \sup\left\{\frac{f(t,u,v)}{\rho} : (t,u,v) \in [-T,T] \times [0,\rho] \times [0,\rho]\right\}.$$

Then $i_K(F, K_\rho) = 1$.

Lemma 8.2.2. *Assume that*

$\left(\overline{\mathrm{I}_\rho^0}\right)$ *there exist $\rho > 0$ such that $f_{(\rho,\rho/c)'} > M$, where*

$$f_{(\rho,\rho/c)'} = \inf\left\{\frac{f(t,u,v)}{\rho} : (t,u,v) \in [a,b] \times [\rho,\rho/c] \times [0,\rho/c]\right\}.$$

Then $i_K(F, V_\rho) = 0$.

A result equivalent to Theorem 8.1.11 is valid in this case, with nontrivial solutions belonging to the cone (8.2.1).

8.3 The case of kernels with extra positivity

We now assume the functions k, f, g that occur in (8.1.1) satisfy the conditions (C_1), (C_2'), (C_3) and (C_4') with $[a,b]=[-T,T]$; in particular note that the kernel satisfies the stronger positivity requirement

$$c\Phi(s) \le k(t,s) \le \Phi(s) \text{ for } t\in[-T,T] \text{ and a. e. } s\in[-T,T].$$

These hypotheses enable us to work in the cone

$$K'' = \{u\in C[-T,T] : \min_{t\in[-T,-T]} u(t) \ge c\|u\|\}.$$

Remark 8.3.1. Note that a function in K'' that possesses a nontrivial norm, has the useful property that is strictly positive on $[-T,T]$.

Once again F is compact and leaves the cone K'' invariant. The assumptions on the index are as follows.

Lemma 8.3.2. *Assume that*

$\left(\widetilde{\mathrm{I}}^1_\rho\right)$ *there exists* $\rho>0$ *such that* $f^{c\rho,\rho}<m$, *where*

$$f^{c\rho,\rho} = \sup\left\{\frac{f(t,u,v)}{\rho} : (t,u,v)\in[-T,T]\times[c\rho,\rho]\times[c\rho,\rho]\right\}.$$

Then $i_K(F,K_\rho)=1$.

Lemma 8.3.3. *Assume that*

$\left(\widetilde{\mathrm{I}}^1_\rho\right)$ *there exist* $\rho>0$ *such that* $f_{(\rho,\rho/c)''}>M$, *where*

$$f_{(\rho,\rho/c)''} = \inf\left\{\frac{f(t,u,v)}{\rho} : (t,u,v)\in[a,b]\times[\rho,\rho/c]\times[\rho,\rho/c]\right\}.$$

Then $i_K(F,V_\rho)=0$.

A result similar to Theorem 8.1.11 holds in this case.

Remark 8.3.4. If f is defined only on $[-T,T]\times[u_1,u_2]\times[v_1,v_2]$ we can extend it, with continuity, to $[-T,T]\times\mathbb{R}\times\mathbb{R}$ considering firstly

$$\bar f(t,u,v) := \begin{cases} f(t,u_1,v), & u\le u_1,\\ f(t,u,v), & u_1\le u\le u_2,\\ f(t,u_2,v), & u_2\le u,\end{cases}$$

and secondly

$$\tilde f(t,u,v) := \begin{cases} \bar f(t,u,v_1), & \le v_1,\\ \bar f(t,u,v), & v_1\le v\le v_2,\\ \bar f(t,u,v_2), & v_2\le v.\end{cases}$$

Remark 8.3.5. Note that results similar to those presented so far in the chapter hold when the kernel k is negative on a strip, negative and strictly negative. This gives nontrivial solutions that are negative on an interval, negative and strictly negative respectively.

8.4 An application

We now turn our attention to the first order functional periodic boundary value problem

$$u'(t) = h(t, u(t), u(-t)),\ t \in [-T, T], \tag{8.4.1}$$

$$u(-T) = u(T), \tag{8.4.2}$$

We apply the shift argument of Subsection 3.2.3 –a similar idea has been used in [167,184],– by fixing $\omega \in \mathbb{R}^+$ and considering the equivalent expression

$$u'(t) + \omega u(-t) = h(t, u(t), u(-t)) + \omega u(-t) := f(t, u(t), u(-t)),\ t \in [-T, T], \tag{8.4.3}$$

$$u(-T) = u(T). \tag{8.4.4}$$

Following the ideas developed in Subsection 3.2.3, we can conclude that the functional boundary value problem (8.4.3)-(8.4.4) can be rewritten into a Hammerstein integral equation of the type

$$u(t) = \int_{-T}^{T} k(t,s) f(s, u(s), u(-s))\, \mathrm{d}s, \tag{8.4.5}$$

Also, $k(t,s)$ can be expressed in the following way (see page 50):

$$2\sin(\omega T) k(t,s) = \begin{cases} \cos\omega(T-s-t) + \sin\omega(T+s-t), & t > |s|, \\ \cos\omega(T-s-t) - \sin\omega(T-s+t), & |t| < s, \\ \cos\omega(T+s+t) + \sin\omega(T+s-t), & |t| < -s, \\ \cos\omega(T+s+t) - \sin\omega(T-s+t), & t < -|s|. \end{cases} \tag{8.4.6}$$

The results that follow are meant to prove that we are under the hypothesis of Theorem 8.1.6.

Apart from Theorem 3.2.3, Lemma 3.2.6 and Theorem 3.2.8, there are some things to be said about the kernel k when $\zeta = \omega T \in \mathbb{R}\backslash[-\frac{\pi}{4}, \frac{\pi}{4}]$. First, realize that, using the trigonometric identities $\cos(a-b) \pm \sin(a+b) = (\cos a \pm \sin a)(\cos b \pm \sin b)$ and $\cos(a) + \sin(a) = \sqrt{2}\cos(a - \frac{\pi}{4})$ and making the change of variables $t = Tz$, $s = Ty$, we can express k as

$$\sin(\zeta) k(z,y) = \begin{cases} \cos[\zeta(1-z) - \frac{\pi}{4}]\cos(\zeta y - \frac{\pi}{4}), & z > |y|, \\ \cos(\zeta z + \frac{\pi}{4})\cos[\zeta(y-1) - \frac{\pi}{4}], & |z| < y, \\ \cos(\zeta z + \frac{\pi}{4})\cos[\zeta(1+y) - \frac{\pi}{4}], & -|z| > y, \\ \cos[\zeta(z+1) + \frac{\pi}{4}]\cos(\zeta y - \frac{\pi}{4}), & z < -|y|. \end{cases} \tag{8.4.7}$$

Lemma 8.4.1. *The following properties hold:*

(1) If $\zeta \in (\frac{\pi}{4}, \frac{\pi}{2})$, then k is strictly positive in

$$S := \left[\left(-\frac{\pi}{4|\zeta|}, \frac{\pi}{4|\zeta|} - 1\right) \cup \left(1 - \frac{\pi}{4|\zeta|}, \frac{\pi}{4|\zeta|}\right)\right] \times [-1, 1].$$

(2) If $\zeta \in (-\frac{\pi}{2}, -\frac{\pi}{4})$, k is strictly negative in S.

Proof. By Lemma 3.2.6, it is enough to prove that k is strictly positive in S for $\zeta \in (\frac{\pi}{4}, \frac{\pi}{2})$. We do here the proof for the connected component $\left(1-\frac{\pi}{4\zeta}, \frac{\pi}{4\zeta}\right) \times [-1,1]$ of S. For the other one the proof is analogous.

If $z \in \left(1-\frac{\pi}{4\zeta}, \frac{\pi}{4\zeta}\right)$, then $\zeta z + \frac{\pi}{4} \in \left(\zeta, \frac{\pi}{2}\right) \subset \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, and hence $\cos\left(\zeta z + \frac{\pi}{4}\right) > 0$.

Also, if $z \in \left(1-\frac{\pi}{4\zeta}, \frac{\pi}{4\zeta}\right)$, then $\zeta(1-z) - \frac{\pi}{4} \in \left(\zeta - \frac{\pi}{2}, 0\right) \subset \left(-\frac{\pi}{4}, 0\right)$ and therefore $\cos\left(\zeta(1-z) - \frac{\pi}{4}\right) > 0$.

If $y \in \left(-\frac{\pi}{4\zeta}, \frac{\pi}{4\zeta}\right)$, then $\zeta y - \frac{\pi}{4} \in \left(-\frac{\pi}{2}, 0\right)$ so $\cos\left(\zeta y - \frac{\pi}{4}\right) > 0$.

If $y \in \left(1-\frac{\pi}{4\zeta}, 1\right)$, then $\zeta(y-1) - \frac{\pi}{4} \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ so $\cos\left(\zeta(y-1) - \frac{\pi}{4}\right) > 0$.

If $y \in \left(-1, \frac{\pi}{4\zeta} - 1\right)$, then $\zeta(y+1) + \frac{\pi}{4} \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ so $\cos\left(\zeta(y+1) + \frac{\pi}{4}\right) > 0$.

With these inequalities the result is straightforward from equation (8.4.7). ■

Lemma 8.4.2. *If $\zeta \in (\frac{\pi}{4}, \frac{\pi}{2})$ then $\sin(\zeta)|k(z,y)| \leq \Phi(y) := \sin(\zeta)\ \max_{r\in[-1,1]} k(r,y)$ where Φ admits the following expression:*

$$\Phi(y) = \begin{cases} \cos\left[\zeta(y-1) - \frac{\pi}{4}\right], & y \in [\beta, 1], \\ \cos\left[\zeta(y-1) + \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right), & y \in \left[1-\frac{\pi}{4\zeta}, \beta\right), \\ \cos\left(\zeta y - \frac{\pi}{4}\right), & y \in \left[\beta - 1, 1 - \frac{\pi}{4\zeta}\right), \\ \cos\left(\zeta y + \frac{\pi}{4}\right)\cos\left[\zeta(y+1) - \frac{\pi}{4}\right], & y \in [-\frac{\pi}{4\zeta}, \beta - 1), \\ \cos\left[\zeta(y+1) - \frac{\pi}{4}\right], & y \in [-1, -\frac{\pi}{4\zeta}), \end{cases}$$

and β is the only solution of the equation

$$\cos\left[\zeta(y-1) + \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right) - \cos\left[\zeta(y-1) - \frac{\pi}{4}\right] = 0 \tag{8.4.8}$$

in the interval $\left[\frac{1}{2}, 1\right]$.

Proof. First observe that, for convenience, we are redefining Φ multiplying it by $\sin(\zeta)$. Let

$$v(y) := \cos\left[\zeta(y-1) + \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right) - \cos\left[\zeta(y-1) - \frac{\pi}{4}\right],$$

then

$$v'(y) = \zeta\left[\sin\left(\zeta(y-1) - \frac{\pi}{4}\right) - \sin\left(\zeta(2y-1)\right)\right].$$

Observe that $y \in \left[\frac{1}{2}, 1\right]$ implies

$$\zeta(y-1) - \frac{\pi}{4} \in \left[-\frac{\zeta}{2} - \frac{\pi}{4}, -\frac{\pi}{4}\right] \subset \left[-\frac{3\pi}{4}, -\frac{\pi}{4}\right]$$

and

$$\zeta(2y-1) \in (0,\zeta) \subset \left(0,\frac{\pi}{2}\right),$$

therefore $v'(y) < 0 \ \ \forall y \in \left(\frac{1}{2},1\right)$. Furthermore, since $\zeta \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$,

$$\begin{aligned} v\left(\frac{1}{2}\right) &= \cos^2\left(\frac{\zeta}{2}-\frac{\pi}{4}\right) - \cos\left(\frac{\zeta}{2}+\frac{\pi}{4}\right) \\ &= 1-\left[\cos\left(-\frac{\zeta}{2}\right)+\frac{\sqrt{2}}{2}\right]\left[\sin\left(-\frac{\zeta}{2}\right)+\frac{\sqrt{2}}{2}\right] \geq \frac{\sqrt{4-2\sqrt{2}}}{2} > 0, \\ v(1) &= \frac{\sqrt{2}}{2}\left[1-\cos\left(\zeta-\frac{\pi}{4}\right)\right] \leq 0. \end{aligned}$$

Hence, equation (8.4.8) has a unique solution β in $\left[\frac{1}{2},1\right]$. Besides, since $v(\frac{\pi}{4\zeta}) = \sqrt{2}\sin(\zeta - \frac{\pi}{4}) > 0$, we have that $\beta > \frac{\pi}{4\zeta}$. Furthermore, it can be checked that

$$-1 < -\frac{\pi}{4\zeta} < \beta - 1 < \frac{\pi}{4\zeta} - 1 < 0 < 1 - \frac{\pi}{4\zeta} < \frac{\pi}{4\zeta} < \beta < 1.$$

Now, realize that

$$\begin{aligned} &\sin(\zeta)k(z,y) \leq \xi(z,y) \\ &:= \begin{cases} \cos[\zeta(1-\max\{1-\frac{\pi}{4\zeta},|y|\}) - \frac{\pi}{4}]\cos(\zeta y - \frac{\pi}{4}), & z > |y|, \\ \cos(\zeta\min\{\frac{\pi}{4\zeta},y\} - \frac{\pi}{4})\cos[\zeta(y-1) - \frac{\pi}{4}], & |z| < y, \\ \cos(\zeta\max\{-\frac{\pi}{4\zeta},y\} + \frac{\pi}{4})\cos[\zeta(1+y) - \frac{\pi}{4}], & -|z| > y, \\ \frac{\sqrt{2}}{2}\cos(\zeta y - \frac{\pi}{4}), & z < -|y|, \end{cases} \end{aligned} \tag{8.4.9}$$

while $\xi(z,y) \leq \Phi(y)$.

We study now the different cases for the value of y.

- If $y \in [\beta,1]$, then

$$\xi(z,y) = \begin{cases} \cos\left[\zeta(y-1)+\frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right), & z > y, & \text{(8.4.10a)} \\ \cos\left[\zeta(y-1)-\frac{\pi}{4}\right], & |z| < y, & \text{(8.4.10b)} \\ \frac{\sqrt{2}}{2}\cos\left(\zeta y - \frac{\pi}{4}\right), & z < -y. & \text{(8.4.10c)} \end{cases}$$

It is straightforward that $\cos[\zeta(y-1)+\frac{\pi}{4}] > \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, so (8.4.10a)>(8.4.10c). By our study of equation (8.4.8), we have that that

$$\cos\left[\zeta(y-1)+\frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right) \leq \cos\left[\zeta(y-1)-\frac{\pi}{4}\right].$$

Therefore (8.4.10a)≥(8.4.10b) and $\Phi(y) = \cos\left[\zeta(y-1) - \frac{\pi}{4}\right]$.

- If $y \in \left[\frac{\pi}{4\zeta}, \beta\right)$, then ξ is as in (8.4.10) and (8.4.10a)>(8.4.10c), but in this case

$$\cos\left[\zeta(y-1) + \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right) \geq \cos\left[\zeta(y-1) - \frac{\pi}{4}\right],$$

so (8.4.10a)≤(8.4.10b) and $\Phi(y) = \cos\left[\zeta(y-1) + \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right)$.

- If $y \in \left[1 - \frac{\pi}{4\zeta}, \frac{\pi}{4\zeta}\right)$, then

$$\xi(z,y) = \begin{cases} \cos\left[\zeta(y-1) + \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right), & z > y, & \text{(8.4.11a)} \\ \cos\left[\zeta(y-1) - \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right), & |z| < y, & \text{(8.4.11b)} \\ \frac{\sqrt{2}}{2}\cos\left(\zeta y - \frac{\pi}{4}\right), & z < -y. & \text{(8.4.11c)} \end{cases}$$

We have that

$$\cos\left[\zeta(y-1) + \frac{\pi}{4}\right] - \cos\left[\zeta(y-1) - \frac{\pi}{4}\right] = \sqrt{2}\sin[\zeta(1-y)] > 0,$$

therefore (8.4.11a)≥(8.4.11b) and $\Phi(y) = \cos[\zeta(y-1) + \frac{\pi}{4}]\cos(\zeta y - \frac{\pi}{4})$.

- If $y \in \left[0, 1 - \frac{\pi}{4\zeta}\right)$, then

$$\xi(z,y) = \begin{cases} \cos\left(\zeta y - \frac{\pi}{4}\right), & z > y, & \text{(8.4.12a)} \\ \cos\left[\zeta(y-1) - \frac{\pi}{4}\right]\cos\left(\zeta y - \frac{\pi}{4}\right), & |z| < y, & \text{(8.4.12b)} \\ \frac{\sqrt{2}}{2}\cos\left(\zeta y - \frac{\pi}{4}\right), & z < -y. & \text{(8.4.12c)} \end{cases}$$

$\cos\left[\zeta(y-1) - \frac{\pi}{4}\right] < \frac{\sqrt{2}}{2}$, so (8.4.12b)≤(8.4.12c)≤(8.4.12a) and $\Phi(y) = \cos\left(\zeta y - \frac{\pi}{4}\right)$.

- If $y \in [\beta - 1, 0)$, then

$$\xi(z,y) = \begin{cases} \cos\left(\zeta y - \frac{\pi}{4}\right), & z > -y, \\ \cos\left(\zeta y + \frac{\pi}{4}\right)\cos\left[\zeta(1+y) - \frac{\pi}{4}\right], & -|z| > y, \\ \frac{\sqrt{2}}{2}\cos\left(\zeta y - \frac{\pi}{4}\right), & z < y. \end{cases} \tag{8.4.13}$$

Let $y = \overline{y} - 1$, then

$$\cos\left(\zeta y + \frac{\pi}{4}\right)\cos\left[\zeta(1+y) - \frac{\pi}{4}\right] \leq \cos\left(\zeta y - \frac{\pi}{4}\right)$$

if and only if

$$\cos\left[\zeta(\bar{y}-1)+\frac{\pi}{4}\right]\cos\left(\zeta\bar{y}-\frac{\pi}{4}\right)\leq\cos\left[\zeta(\bar{y}-1)-\frac{\pi}{4}\right]$$

which is true as $\bar{y}\in[\beta,1)$ and our study of equation (8.4.8). Hence, $\Phi(y)=\cos\left(\zeta y-\frac{\pi}{4}\right)$.

- If $y\in\left[\frac{\pi}{4\zeta}-1,\beta-1\right)$, then

  ξ is the same as in (8.4.13) but in this case

$$\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]\geq\cos\left(\zeta y-\frac{\pi}{4}\right)$$

so $\Phi(y)=\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]$.

- If $y\in\left[-\frac{\pi}{4\zeta},\frac{\pi}{4\zeta}-1\right)$, then

$$\xi(z,y)=\begin{cases}\cos\left[\zeta(1-y)-\frac{\pi}{4}\right]\cos\left(\zeta y-\frac{\pi}{4}\right), & z>-y,\\ \cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right], & -|z|>y,\\ \frac{\sqrt{2}}{2}\cos\left(\zeta y-\frac{\pi}{4}\right), & z<y.\end{cases}$$

$$\begin{aligned}&\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]-\cos\left[\zeta(1-y)-\frac{\pi}{4}\right]\cos\left(\zeta y-\frac{\pi}{4}\right)\\&=-\sin\zeta\sin(2\zeta y)>0,\end{aligned}$$

then $\Phi(y)=\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]$.

- If $y\in\left[-1,-\frac{\pi}{4\zeta}\right)$, then

$$\xi(z,y)=\begin{cases}\cos\left[\zeta(1-y)-\frac{\pi}{4}\right]\cos\left(\zeta y-\frac{\pi}{4}\right), & z>-y,\\ \cos\left[\zeta(1+y)-\frac{\pi}{4}\right], & -|z|>y,\\ \frac{\sqrt{2}}{2}\cos\left(\zeta y-\frac{\pi}{4}\right), & z<y.\end{cases}$$

Since

$$\begin{aligned}\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]&\geq\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]\\&>\cos\left[\zeta(1-y)-\frac{\pi}{4}\right]\cos\left(\zeta y-\frac{\pi}{4}\right),\end{aligned}$$

we have that $\Phi(y)=\cos\left[\zeta(1+y)-\frac{\pi}{4}\right]$.

It can be checked that, just studying the arguments of the cosines involved, that

$$-\sin(\zeta)k(z,y)\leq\frac{1}{2}\leq\Phi(y),$$

therefore $\sin(\zeta)|k(z,y)|\leq\Phi(y)$ for all $z,y\in[-1,1]$. ■

Lemma 8.4.3. *Let $\zeta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $b \geq a \geq 0$ such that $a + b = 1$. Then*

$$\sin(\zeta) k(z,y) \geq c(a)\Phi(y) \text{ for } z \in [a,b]\,,\; y \in [-1,1]\,,$$

where

$$c(a) := \inf_{y\in[-1,1]} \left\{ \frac{\sin(\zeta) \inf\limits_{z\in[a,b]} k(z,y)}{\Phi(y)} \right\} = \frac{[1-\tan(\zeta a)][1-\tan(\zeta b)]}{[1+\tan(\zeta a)][1+\tan(\zeta b)]}.$$

Proof. We know by Lemma 8.4.1 that k is positive in $S_r := [a,b] \times [-1,1]$. Furthermore, it is proved in Proposition 3.2.3 that

$$\frac{\partial k}{\partial t}(t,s) + \omega\, k(-t,s) = 0 \quad \forall t,s \in [-T,T],$$

so, differentiating and doing the proper substitutions we get that

$$\frac{\partial^2 k}{\partial t^2}(t,s) + \omega^2 k(t,s) = 0 \quad \forall t,s \in [-T,T].$$

Therefore, $\frac{\partial^2 k}{\partial t^2} < 0$ in S_r, which means that any minimum of k with respect to t has to be in the boundary of the differentiable regions of S_r. Thus, in S_r,

$$\sin(\zeta)k(z,y) \geq \eta(z,y)$$
$$:= \begin{cases} \cos([\max\{|\zeta a + \frac{\pi}{4}|, |\zeta b + \frac{\pi}{4}|\}) \cos[\zeta(y-1) - \frac{\pi}{4}], & |z| < y,\ y \in [b,1], \\ \cos([\max\{|\zeta a + \frac{\pi}{4}|, |\zeta y + \frac{\pi}{4}|\}) \cos[\zeta(y-1) - \frac{\pi}{4}], & |z| < y,\ y \in [a,b), \\ \cos[\max\{|\zeta(1-y) - \frac{\pi}{4}|, |\zeta(1-b) - \frac{\pi}{4}|] \cos(\zeta y - \frac{\pi}{4}), & z > |y|,\ y \in [a,b), \\ \cos[\max\{|\zeta(1-a) - \frac{\pi}{4}|, |\zeta(1-b) - \frac{\pi}{4}|] \cos(\zeta y - \frac{\pi}{4}), & z > |y|,\ y \in [-a,a), \\ \cos[\max\{|\zeta(1-y) - \frac{\pi}{4}|, |\zeta(1-b) - \frac{\pi}{4}|] \cos(\zeta y - \frac{\pi}{4}), & z > |y|,\ y \in [-b,-a), \\ \cos([\max\{|\zeta a + \frac{\pi}{4}|, |\zeta y + \frac{\pi}{4}|\}) \cos[\zeta(1+y) - \frac{\pi}{4}], & -|z| > y,\ y \in [-b,-a), \\ \cos([\max\{|\zeta a + \frac{\pi}{4}|, |\zeta b + \frac{\pi}{4}|\}) \cos[\zeta(1+y) - \frac{\pi}{4}], & -|z| > y,\ y \in [-1,-b). \end{cases}$$

By definition, $\eta(z,y) \geq \Psi(y) := \sin(\zeta) \inf_{r\in[a,b]} k(r,y)$. Also, realize that the arguments of the cosine in (8.4.7) are affine functions and that the cosine function is strictly decreasing in $[0,\pi]$ and symmetric with respect to zero. We can apply Lemma 3.4.2 to get

$$\eta(z,y) = \begin{cases} \cos\left(\zeta b + \dfrac{\pi}{4}\right) \cos\left[\zeta\,(y-1) - \dfrac{\pi}{4}\right], & |z| < y,\ y \in [b,1], & \text{(8.4.14a)} \\ \cos\left(\zeta y + \dfrac{\pi}{4}\right) \cos\left[\zeta\,(y-1) - \dfrac{\pi}{4}\right], & |z| < y,\ y \in [a,b), & \text{(8.4.14b)} \\ \cos\left(\zeta\,(1-b) - \dfrac{\pi}{4}\right) \cos\left(\zeta y - \dfrac{\pi}{4}\right), & z > |y|,\ y \in [-b,b), & \text{(8.4.14c)} \\ \cos\left(\zeta y + \dfrac{\pi}{4}\right) \cos\left[\zeta\,(1+y) - \dfrac{\pi}{4}\right], & -|z| > y,\ y \in [-b,-a), & \text{(8.4.14d)} \\ \cos\left(\zeta b + \dfrac{\pi}{4}\right) \cos\left[\zeta\,(1+y) - \dfrac{\pi}{4}\right], & -|z| > y,\ y \in [-1,-b). & \text{(8.4.14e)} \end{cases}$$

Finally, we have to compare the cases (8.4.14b) with (8.4.14c) for $y \in [a,b)$ and (8.4.14d) with (8.4.14c) for $y \in [-b,-a)$. Using again Lemma 3.4.2, we obtain the following inequality.

$$\begin{aligned}
&\cos\left(\zeta(1-b)-\frac{\pi}{4}\right)\cos\left(\zeta y-\frac{\pi}{4}\right)-\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(y-1)-\frac{\pi}{4}\right]\\
&\geq \cos\left(\zeta(1-b)-\frac{\pi}{4}\right)\cos\left(\zeta b-\frac{\pi}{4}\right)-\cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left[\zeta(b-1)-\frac{\pi}{4}\right]\\
&= \sin\zeta > 0.
\end{aligned}$$

Thus, (8.4.14c)>(8.4.14b) for $y \in [a,b)$.

To compare (8.4.14d) with (8.4.14c) for $y \in [-b,b)$ realize that k is continuous in the diagonal $z=-y$ (see Theorem 3.2.3). Hence, since the expressions of (8.4.14d) and (8.4.14c) are already locally minimyzing (in their differentiable components) for the variable z, we have that (8.4.14d)≥(8.4.14c) for $y \in [-b,-a)$. Therefore,

$$\Psi(y)=\begin{cases}\cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left[\zeta(y-1)-\frac{\pi}{4}\right], & y\in[b,1],\\ \cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(y-1)-\frac{\pi}{4}\right], & y\in[a,b),\\ \cos\left(\zeta(1-b)-\frac{\pi}{4}\right)\cos\left(\zeta y-\frac{\pi}{4}\right), & y\in[-b,a),\\ \cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left[\zeta(1+y)-\frac{\pi}{4}\right], & y\in[-1,-b).\end{cases} \tag{8.4.15}$$

It can be checked that the following order holds:

$$-1<-\frac{\pi}{4\zeta}<-b<\beta-1<1-\frac{\pi}{4\zeta}<a<b<\beta<1.$$

Thus, we get the following expression $\Psi(y)/\Phi(y)=$

$$\begin{cases}
\cos\left(\zeta b+\frac{\pi}{4}\right), & y\in[\beta,1], & \text{(8.4.16a)}\\
\dfrac{\cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left(\zeta(y-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta y-\frac{\pi}{4}\right)\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}, & y\in[b,\beta), & \text{(8.4.16b)}\\
\dfrac{\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left(\zeta(y-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta y-\frac{\pi}{4}\right)\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}, & y\in[a,b), & \text{(8.4.16c)}\\
\dfrac{\cos\left(\zeta(1-b)-\frac{\pi}{4}\right)}{\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}, & y\in\left[1-\frac{\pi}{4\zeta},a\right), & \text{(8.4.16d)}\\
\cos\left(\zeta(1-b)-\frac{\pi}{4}\right), & y\in\left[\beta-1,1-\frac{\pi}{4\zeta}\right), & \text{(8.4.16e)}\\
\dfrac{\cos\left(\zeta(1-b)-\frac{\pi}{4}\right)\cos\left(\zeta y-\frac{\pi}{4}\right)}{\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left(\zeta(1+y)-\frac{\pi}{4}\right)}, & y\in[-b,\beta-1), & \text{(8.4.16f)}\\
\dfrac{\cos\left(\zeta b+\frac{\pi}{4}\right)}{\cos\left(\zeta y+\frac{\pi}{4}\right)}, & y\in\left[-\frac{\pi}{4\zeta},-b\right), & \text{(8.4.16g)}\\
\cos\left(\zeta b+\frac{\pi}{4}\right), & y\in\left[-1,-\frac{\pi}{4\zeta}\right). & \text{(8.4.16h)}
\end{cases}$$

To find the infimum of this function we will go through several steps in which we discard different cases. First, it can be checked that the inequalities (8.4.16g)≥(8.4.16h)= (8.4.16a) and (8.4.16d)≥(8.4.16e), so we need not to think about (8.4.16d), (8.4.16g) and (8.4.16h) anymore.

Now, realize that $|\zeta(1-b)-\frac{\pi}{4}|\le|\zeta b+\frac{\pi}{4}|\le\pi$. Since the cosine is decreasing in $[0,\pi]$ and symmetric with respect to zero this implies that (8.4.16e)≥(8.4.16a).

Note that (8.4.16c) can be written as

$$g_1(y):=\frac{[1-\tan(\zeta y)](1-\tan[\zeta(1-y)])}{[1+\tan(\zeta y)](1+\tan[\zeta(1-y)])}.$$

Its derivative is

$$g_1'(y)=-\frac{4\zeta[\tan^2(\zeta y)-\tan^2\zeta(y-1)]}{(\tan\zeta y+1)^2[\tan\zeta(y-1)]^2},$$

which only vanishes at $y=\frac{1}{2}$ for $y\in[a,b]$.

$$g_1''\left(\frac{1}{2}\right)=-\frac{16\zeta^2\tan\left(\frac{\zeta}{2}\right)\left(\tan^2\frac{\zeta}{2}+1\right)}{\left(\tan\frac{\zeta}{2}+1\right)^4}<0,$$

Therefore $y=\frac{1}{2}$ is a maximum of the function. Since g_1 is symmetric with respect to $\frac{1}{2}$ and a is the symmetric point of b with respect to $\frac{1}{2}$, $g(a)=g(b)$ is the infimum of (8.4.16c) which is contemplated in (8.4.16b) for $y=b$.

Making the change of variables $y=\overline{y}-1$ we have that (8.4.16f) can be written as

$$\frac{\cos\left(\zeta(1-b)-\frac{\pi}{4}\right)\cos\left(\zeta(\overline{y}-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta\overline{y}-\frac{\pi}{4}\right)\cos\left(\zeta(\overline{y}-1)+\frac{\pi}{4}\right)},\overline{y}\in[a,\beta). \tag{8.4.16f'}$$

Since (8.4.16e)≥(8.4.16a), now we have that (8.4.16f')≥(8.4.16b) in $[b,\beta)$.

Let

$$g_2(y):=\frac{\cos\left(\zeta(y-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta y-\frac{\pi}{4}\right)\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}.$$

Then

$$g_2'(y)=\frac{\zeta}{4}\cdot\frac{\sin\left[\zeta(2-y)-\frac{\pi}{4}\right]+\sin\left[\zeta(3y-2)-\frac{\pi}{4}\right]+4\cos\left[\zeta y-\frac{\pi}{4}\right]}{\sin^2\left[\zeta y+\frac{\pi}{4}\right]\cos^2\left[\zeta(1-y)-\frac{\pi}{4}\right]^2}.$$

Since the argument in the cosine of the numerator is in the interval $[-\frac{\pi}{4},\frac{\pi}{4}]$ for $y\in[a,1]$, we have that $g_2'(y)>0$ for $y\in[a,1]$, which implies that g_2 is increasing in that interval and (8.4.16b) and (8.4.16f) reach their infimum in the left extreme point of their intervals of definition.

We have then that

$$c(a)=\inf_{y\in[-1,1]}\frac{\Psi(y)}{\Phi(y)}$$

$$= \min\left\{\cos\left(\zeta b+\frac{\pi}{4}\right), \frac{\cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left(\zeta(b-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta b-\frac{\pi}{4}\right)\cos\left(\zeta(b-1)+\frac{\pi}{4}\right)}, \frac{\cos\left(-\zeta b-\frac{\pi}{4}\right)}{\cos\left(-\zeta b+\frac{\pi}{4}\right)}\right\}.$$

The third element of the set is greater or equal than the first. The second element can be simplified to $\cos\left(\zeta b+\frac{\pi}{4}\right)g_2(b)$. Since g_2 is increasing in $[a,1]$,

$$\cos\left(\zeta b+\frac{\pi}{4}\right)g_2(b) \leq \cos\left(\zeta b+\frac{\pi}{4}\right)g_2(1) = \cos\left(\zeta b+\frac{\pi}{4}\right)\frac{\cos(\zeta)}{\sin(\zeta)}$$
$$\leq \cos\left(\zeta b+\frac{\pi}{4}\right).$$

Therefore,

$$c(a) = \frac{\cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left(\zeta(b-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta b-\frac{\pi}{4}\right)\cos\left(\zeta(b-1)+\frac{\pi}{4}\right)} = \frac{[1-\tan(\zeta a)][1-\tan(\zeta b)]}{[1+\tan(\zeta a)][1+\tan(\zeta b)]}.$$

■

Remark 8.4.4. Let us find an upper estimate of $c(a)$. Just assume $a=b=\frac{1}{2}$.

$$c(a) \leq c(1/2) = \left(\frac{1-\tan\frac{\zeta}{2}}{1+\tan\frac{\zeta}{2}}\right)^2 \leq \left(\frac{1-\tan\frac{\pi}{8}}{1+\tan\frac{\pi}{8}}\right)^2 = \frac{(2-\sqrt{2})^2}{2} = 0.17157\ldots$$

We can do the same study for $\zeta \in (0,\frac{\pi}{4}]$. The proofs are almost the same, but in this case the calculations are much easier.

Lemma 8.4.5. *If $\zeta \in (0,\frac{\pi}{4}]$ then $\sin(\zeta)|k(z,y)| \leq \Phi(y) := \max_{r\in[-1,1]} k(r,y)$ where Φ admits the following expression:*

$$\Phi(y) = \begin{cases} \cos\left[\zeta(y-1)+\frac{\pi}{4}\right]\cos\left(\zeta y-\frac{\pi}{4}\right), & y\in[0,1], \\ \cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(y+1)-\frac{\pi}{4}\right], & y\in[-1,0). \end{cases}$$

Proof. This time, a simplified version of inequality (8.4.9) holds,

$$\sin(\zeta)k(z,y) \leq \xi(z,y) := \begin{cases} \cos[\zeta(1-|y|)-\frac{\pi}{4}]\cos(\zeta y-\frac{\pi}{4}), & z>|y|, \\ \cos(\zeta y-\frac{\pi}{4})\cos[\zeta(y-1)-\frac{\pi}{4}], & |z|<y, \\ \cos(\zeta y+\frac{\pi}{4})\cos[\zeta(1+y)-\frac{\pi}{4}], & -|z|>y, \\ \frac{\sqrt{2}}{2}\cos(\zeta y-\frac{\pi}{4}), & z<-|y|, \end{cases}$$

so we only need to study two cases. If $y>0$, we are in the same situation as in the case $y\in[1-\frac{\pi}{4\zeta},\frac{\pi}{4\zeta})$ studied in Lemma 8.4.2. Hence, $\Phi(y) = \cos\left[\zeta(y-1)+\frac{\pi}{4}\right]\cos\left(\zeta y-\frac{\pi}{4}\right)$. If $y<0$ we are in the same situation as in the case $y\in[-\frac{\pi}{4\zeta},\frac{\pi}{4\zeta}-1)$. Therefore, $\Phi(y) = \cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left[\zeta(y+1)-\frac{\pi}{4}\right]$. ■

Lemma 8.4.6. *Let $\zeta \in (0, \frac{\pi}{4}]$ and $b \geq a \geq 0$ such that $a + b = 1$. Then*

$$\sin\left(\zeta\right) k\left(z,y\right) \geq c(a)\Phi\left(y\right) \text{ for } z \in \left[a,b\right], y \in \left[-1,1\right],$$

where

$$c(a) := \inf_{y\in[-1,1]}\left\{\frac{\sin(\zeta) \inf_{z\in[a,b]} k(z,y)}{\Phi(y)}\right\} = \frac{[1-\tan(\zeta a)][1-\tan(\zeta b)]}{[1+\tan(\zeta a)][1+\tan(\zeta b)]}.$$

Proof. Let Ψ be as in (8.4.15). In this case we get the simpler expression

$$\frac{\Psi\left(y\right)}{\Phi(y)} = \begin{cases} \dfrac{\cos\left(\zeta b+\frac{\pi}{4}\right)\cos\left(\zeta(y-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta y-\frac{\pi}{4}\right)\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}, & y \in \left[b,1\right], \\ \dfrac{\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left(\zeta(y-1)-\frac{\pi}{4}\right)}{\cos\left(\zeta y-\frac{\pi}{4}\right)\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}, & y \in \left[a,b\right), \\ \dfrac{\cos\left(\zeta(1-b)-\frac{\pi}{4}\right)}{\cos\left(\zeta(y-1)+\frac{\pi}{4}\right)}, & y \in \left[0,a\right), \\ \dfrac{\cos\left(\zeta(1-b)-\frac{\pi}{4}\right)\cos\left(\zeta y-\frac{\pi}{4}\right)}{\cos\left(\zeta y+\frac{\pi}{4}\right)\cos\left(\zeta(1+y)-\frac{\pi}{4}\right)}, & y \in \left[-b,0\right), \\ \cos\left(\zeta b+\dfrac{\pi}{4}\right), & y \in \left[-1,-b\right). \end{cases}$$

By the same kind of arguments used in the proof of Lemma 8.4.3, we get the desired result. ■

Lemma 8.4.7.

$$\sup_{t\in[-T,T]} \int_{-T}^{T} |k(t,s)|\,\mathrm{d}s = \begin{cases} \dfrac{1}{\omega}, & \zeta \in \left(0,\dfrac{\pi}{4}\right], \\ \dfrac{1}{\omega}\left[1+\dfrac{\sqrt{2}\cos\frac{2\zeta+\pi}{3}\sin\frac{\pi-4\zeta}{12}+\cos\frac{\pi-\zeta}{3}\left(1-\sin\frac{2\zeta+\pi}{3}\right)}{\sin\zeta}\right], & \zeta \in \left[\dfrac{\pi}{4},\dfrac{\pi}{2}\right]. \end{cases}$$

Proof. First of all, if $\zeta \in \left[0,\frac{\pi}{4}\right]$, then $|k(t,s)| = k(t,s)$. The solution of the problem

$$x'\left(t\right) + \omega x\left(-t\right) = 1,\ x\left(-T\right) = x\left(T\right)$$

is $u\left(t\right) \equiv \frac{1}{\omega}$, but at the same time it has to be of the kind in equation (8.4.5), so $u\left(t\right) = \int_{-T}^{T} k(t,s)\,\mathrm{d}s$. This proves the first part.

If $\zeta \in \left[\frac{\pi}{4},\frac{\pi}{2}\right]$, then

$$\int_{-T}^{T} |k(t,s)|\,\mathrm{d}s = \int_{-T}^{T} k^{+}(t,s)\,\mathrm{d}s + \int_{-T}^{T} k^{-}(t,s)\,\mathrm{d}s = \frac{1}{\omega} + 2\int_{-T}^{T} k^{-}(t,s)\,\mathrm{d}s.$$

We make two observations here.

From equation (8.4.6), it can be checked that $k\,(t+T,s+T)=k(t,s)$ and $k\,(t+T,s)=k\,(t,s+T)$ for a.e. $t,s\in[-T,0]$. Hence, for $t\in[-T,0]$ and a function $\xi:\mathbb{R}\to\mathbb{R}$, using the change of variables $r=s+T$, $\tau=s-T$, we have that

$$\begin{aligned}\int_{-T}^{T}\xi\,(k\,(t+T,s))\,\mathrm{d}s &= \int_{-T}^{0}\xi\,(k\,(t+T,s))\,ds+\int_{0}^{T}\xi\,(k\,(t+T,s))\,\mathrm{d}s\\ &= \int_{-T}^{0}\xi\,(k\,(t,s+T))\,\mathrm{d}s+\int_{-T}^{0}\xi\,(k\,(t+T,\tau+T))\,\mathrm{d}\tau\\ &= \int_{0}^{T}\xi\,(k\,(t,r))\,\mathrm{d}r+\int_{-T}^{0}\xi\,(k\,(t,\tau))\,\mathrm{d}\tau=\int_{-T}^{T}\xi\,(k(t,s))\,\mathrm{d}s.\end{aligned}$$

Therefore,

$$\sup_{t\in[-T,T]}\int_{-T}^{T}|k(t,s)|\,\mathrm{d}s=\sup_{t\in[-T,0]}\int_{-T}^{T}|k(t,s)|\,\mathrm{d}s.$$

The second observation is that, taking into account Lemma 8.4.1, $k(t,s)$ is positive in $\left(-\frac{\pi}{4\omega},T-\frac{\pi}{4\omega}\right)\times[-T,T]$, so

$$\sup_{t\in[-T,0]}\int_{-T}^{T}|k(t,s)|\,\mathrm{d}s=\sup_{t\in[-T,0]\setminus\left(-\frac{\pi}{4\omega},1-\frac{\pi}{4\omega}\right)}\int_{-T}^{T}|k(t,s)|\,\mathrm{d}s.$$

Using the same kind of arguments as in Lemma 8.4.1, it can be checked that $k(t,s)$ is negative in

$$\left(-T,-\frac{\pi}{4\omega}\right)\times\left(t,-\frac{\pi}{4\omega}\right)\text{ if }t\in\left(-T,-\frac{\pi}{4\omega}\right)$$

and in

$$\left(\frac{\pi}{4\omega}-1,0\right)\times\left(t,1-\frac{\pi}{4\omega}\right)\text{ if }t\in\left(\frac{\pi}{4\omega}-1,0\right),$$

so it is enough to compute $\eta\,(t):=\int_{B(t)}k^{-}(t,s)\,\mathrm{d}s$ where

$$B\,(t)=\{s\in[-T,T]\,:\,(t,s)\in\operatorname{supp}\,(k^{-})\}.$$

We have that $2\omega\sin(\zeta)\eta(t)=$

$$\begin{cases}\cos\left(\omega t+\zeta+\frac{\pi}{4}\right)\left[1+\sin\left(\omega t-\frac{\pi}{4}\right)\right], & t\in\left(-T,-\frac{\pi}{4\omega}\right),\\ \sqrt{2}\cos\left(\omega t+\zeta+\frac{\pi}{4}\right)\sin\omega t+\cos\left(\omega t+\frac{\pi}{4}\right)\left[1-\sin\left(\omega t+\zeta+\frac{\pi}{4}\right)\right], & t\in\left(\frac{\pi}{4\omega}-1,0\right).\end{cases}$$

With the change of variable $t=zT$,

$$2\omega\sin\left(\zeta\right)\eta\,(z)=\begin{cases}\eta_1\,(z) & \text{if }z\in\left(-1,-\frac{\pi}{4\zeta}\right),\\ \eta_2\,(z) & \text{if }z\in\left(\frac{\pi}{4\zeta}-1,0\right),\end{cases}$$

where

$$\eta_1\,(z)=\cos\left[\zeta\,(z+1)+\frac{\pi}{4}\right]\left[1+\sin\left(\zeta z-\frac{\pi}{4}\right)\right]$$

and

$$\eta_2\,(z)=\sqrt{2}\cos\left[\zeta\,(z+1)+\frac{\pi}{4}\right]\sin\zeta z+\cos\left(\zeta z+\frac{\pi}{4}\right)\left[1-\sin\left(\zeta\,(z+1)+\frac{\pi}{4}\right)\right].$$

It can be checked that

$$\eta_1'(-1) \le 0,\ \eta_1'\left(-\frac{\pi}{4\zeta}\right) = 0,\ \eta_1''(z) \ge 0 \text{ for } z \in \left[-1, -\frac{\pi}{4\zeta}\right],$$

$$\eta_1'(-1) = \eta_2(0),$$

$$\eta_2'\left(\frac{\pi}{4\omega} - 1\right) > 0,\ \eta_2'(0) < 0,\ \eta_2''(z) \ge 0 \text{ for } z \in \left[\frac{\pi}{4\zeta} - 1, 0\right].$$

With these facts we conclude that there is a unique maximum of the function $\eta(z)$ in the interval $\left(\frac{\pi}{4\zeta} - 1, 0\right)$, precisely where $\eta_2'(z) = \zeta\left(\cos\left[\zeta(1+2z)\right] - \sin\left(\frac{\pi}{4} + z\zeta\right)\right) = 0$, this is, for $z = \frac{1}{3}(\frac{\pi}{4} - 1)$, and therefore the statement of the theorem holds. ∎

Lemma 8.4.8. *Let $\omega \in \left[\frac{\pi}{4}T, \frac{\pi}{2}T\right]$ and $T - \frac{\pi}{4\omega} < a < b = T - a < \frac{\pi}{4\omega}$. Then*

$$2\omega\sin(\zeta) \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s = \sin\omega(T-2a) + \cos\zeta - \cos 2\omega a.$$

Proof. We can check that

$$2\omega\sin(\zeta)\int_{-T}^{s} k(t,r)\,\mathrm{d}r$$
$$= \begin{cases} \sin\omega(T+s+t) - \cos\omega(T+s-t) - \sin\omega t + \cos\omega t, & |t| \le -s, \\ \sin\omega(T+s+t) - \cos\omega(T-s+t) - \sin\omega t + \cos\omega t, & |s| \le -t, \\ -\sin\omega(T-s-t) - \cos\omega(T+s-t) - \sin\omega t + \cos\omega t + 2\sin\omega t, & |s| \le t, \\ -\sin\omega(T-s-t) - \cos\omega(T-s+t) - \sin\omega t + \cos\omega t + 2\sin\omega t, & |t| \le s. \end{cases}$$

Therefore $\int_a^b k(t,s)\,\mathrm{d}s = \int_{-T}^b k(t,s)\,\mathrm{d}s - \int_{-T}^a k(t,s)\,\mathrm{d}s$, this is,

$$2\omega\sin(\zeta)\int_a^b k(t,s)\,\mathrm{d}s$$
$$= \sin\omega(T-a-t) - \sin\omega(a-t) + \cos\omega(T+a-t) - \cos\omega(a+t), t \in [a,b].$$

Using similar arguments to the ones used in the proof of Lemma 8.4.3 we can show that

$$2\omega\sin(\zeta) \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s = \sin\omega(T-2a) + \cos\zeta - \cos 2\omega a.$$

∎

With the same method, we can prove the following corollary.

Corollary 8.4.9. *Let $\omega \in \left(0, \frac{\pi}{4}T\right]$ and $0 < a < b = T - a < 1$. Then*

$$2\omega\sin(\zeta) \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s = \sin\omega(T-2a) + \cos\zeta - \cos 2\omega a.$$

Remark 8.4.10. If $\omega \in \left(0, \frac{\pi}{4}T\right]$, then

$$\inf_{t\in[-T,T]} \int_{-T}^{T} k(t,s)\,\mathrm{d}s = \frac{1}{\omega},$$

just because of the observation in the proof of Lemma 8.4.7.

Now we can state conditions (I^0_ρ) and (I^1_ρ) for the special case of problem (8.4.1)–(8.4.2):

$(\mathrm{I}^1_{\rho,\omega})$ Let

$$f_\omega^{-\rho,\rho} := \sup\left\{\frac{h(t,u,v)+\omega v}{\rho}:\ (t,u,v)\in[-T,T]\times[-\rho,\rho]\times[-\rho,\rho]\right\}.$$

There exist $\rho>0$ and $\omega\in\left(0,\frac{\pi}{4}\right]$ such that $f_\omega^{-\rho,\rho}<\omega$,

or

there exist $\rho>0$ and $\omega\in\left(\frac{\pi}{4},\frac{\pi}{2}\right]$ such that

$$f_\omega^{-\rho,\rho}\cdot\left[1+\frac{\sqrt{2}\cos\frac{2\zeta+\pi}{3}\sin\frac{\pi-4\zeta}{12}+\cos\frac{\pi-\zeta}{3}\left(1-\sin\frac{2\zeta+\pi}{3}\right)}{\sin\zeta}\right]<\omega.$$

$(\mathrm{I}^0_{\rho,\omega})$ there exist $\rho>0$ such that such that

$$f^\omega_{(\rho,\rho/c)}\cdot\inf_{t\in[a,b]}\int_a^b k(t,s)\,\mathrm{d}s>1,$$

where

$$f^\omega_{(\rho,\rho/c)}=\inf\left\{\frac{h(t,u,v)+\omega v}{\rho}:\ (t,u,v)\in[a,b]\times[\rho,\rho/c]\times[-\rho/c,\rho/c]\right\}.$$

Theorem 8.4.11. *Assume (C_1)–(C_4) hold. Let $\omega\in\left(0,\frac{\pi}{2}T\right]$. Let $[a,b]\subset[-T,T]$ such that $a=1-b\in(\max\{0,T-\frac{\pi}{4\omega}\},\frac{T}{2})$. Let*

$$c=\frac{[1-\tan(\omega a)][1-\tan(\omega b)]}{[1+\tan(\omega a)][1+\tan(\omega b)]}.$$

Problem (8.4.1)–(8.4.2) *has at least one nonzero solution in K if either of the following conditions hold.*

(S_1) *There exist $\rho_1,\rho_2\in(0,\infty)$ with $\rho_1/c<\rho_2$ such that $(\mathrm{I}^0_{\rho_1,\omega})$ and $(\mathrm{I}^1_{\rho_2,\omega})$ hold.*

(S_2) *There exist $\rho_1,\rho_2\in(0,\infty)$ with $\rho_1<\rho_2$ such that $(\mathrm{I}^1_{\rho_1,\omega})$ and $(\mathrm{I}^0_{\rho_2,\omega})$ hold.*

Problem (8.4.1)–(8.4.2) *has at least two nonzero solutions in K if one of the following conditions hold.*

(S_3) *There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$ with $\rho_1/c<\rho_2<\rho_3$ such that $(\mathrm{I}^0_{\rho_1,\omega})$, $(\mathrm{I}^1_{\rho_2,\omega})$ and $(\mathrm{I}^0_{\rho_3,\omega})$ hold.*

(S_4) *There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$ with $\rho_1<\rho_2$ and $\rho_2/c<\rho_3$ such that $(\mathrm{I}^1_{\rho_1,\omega})$, $(\mathrm{I}^0_{\rho_2,\omega})$ and $(\mathrm{I}^1_{\rho_3,\omega})$ hold.*

Problem (8.4.1)–(8.4.2) *has at least three nonzero solutions in K if one of the following conditions hold.*

(S_5) *There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ with $\rho_1/c < \rho_2 < \rho_3$ and $\rho_3/c < \rho_4$ such that $(\mathrm{I}^0_{\rho_1,\omega})$, $(\mathrm{I}^1_{\rho_2,\omega})$, $(\mathrm{I}^0_{\rho_3,\omega})$ and $(\mathrm{I}^1_{\rho_4,\omega})$ hold.*

(S_6) *There exist $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ with $\rho_1 < \rho_2$ and $\rho_2/c < \rho_3 < \rho_4$ such that $(\mathrm{I}^1_{\rho_1,\omega})$, $(\mathrm{I}^0_{\rho_2,\omega})$, $(\mathrm{I}^1_{\rho_3,\omega})$ and $(\mathrm{I}^0_{\rho_4,\omega})$ hold.*

Example 8.4.12. Consider problem (8.4.1)–(8.4.2) with

$$h(t,u,v) = \frac{1}{4}\left(\frac{1}{5+(t-1)^2} + \frac{u^2}{5} + 2|u| + \frac{1}{3+7v^2}\right) - \frac{3}{2}v.$$

Then, for $\omega = 3/2$,

$$f(t,u,v) = \frac{1}{4}\left(\frac{1}{5+(t-1)^2} + \frac{u^2}{5} + 2|u| + \frac{1}{3+7v^2}\right).$$

Let $T = 1$, $\zeta = 3/2$, $a = 12/25$, $b = 13/25$, $\rho_1 = 1/4$, $\rho_2 = 2.2\cdot 10^5$. Conditions (C_1)–(C_3) are satisfied by the results proved before (in this case $g \equiv 1$). (C_1) is satisfied by equation (8.4.6) and (C_2) and (C_3) by Lemmas 8.4.2 and 8.4.3. (C_4) is implied in a straightforward way from the expression of h, so we are in the hypothesis of Theorem 8.4.11. Also,

$$\begin{aligned}
c &= 0.000353538\ldots,\\
r_1 &:= \omega\left[1 + \frac{\sqrt{2}\cos\frac{2\zeta+\pi}{3}\sin\frac{\pi-4\zeta}{12} + \cos\frac{\pi-\zeta}{3}\left(1-\sin\frac{2\zeta+\pi}{3}\right)}{\sin\zeta}\right]^{-1} = 1.2021\ldots,\\
r_2 &:= \left(\inf_{t\in[a,b]}\int_a^b k(t,s)\,\mathrm{d}s\right)^{-1} = \left(\frac{\sin\omega(T-2a)+\cos\zeta-\cos 2\omega a}{2\omega\sin\zeta}\right)^{-1}\\
&= 10783.8\ldots,\\
f_\omega^{-\rho_1,\rho_1} &= \frac{f(T,\rho_1,0)}{\rho_1} = 1.04583,\\
f^\omega_{(\rho_2,\rho_2/c)} &= \frac{f(a,\rho_2,\rho_2/c)}{\rho_2} = 11000.5\ldots
\end{aligned}$$

We have that $f_\omega^{-\rho_1,\rho_1} < r_1$ and $f^\omega_{(\rho_2,\rho_2/c)} > r_2$, so condition (S_2) in the previous theorem is satisfied, and therefore problem (8.4.1)-(8.4.2) has at least one solution.

9. A thermostat model with deviated arguments

The existence of solutions of boundary value problems with deviated arguments has been investigated recently by a number of authors using the upper and lower solutions method [68], monotone iterative methods [101, 106, 162, 163]†, the classic Avery-Peterson Theorem [102–105] or, in the special case of reflections, the classical fixed point index as in Chapter 8. One motivation for studying these problems is that they often arise when dealing with real world problems, for example when modeling the stationary distribution of the temperature of a wire of length one which is bent, see the recent paper by Figueroa and Pouso [68] for details. Most of the works mentioned above are devoted to the study of *positive* solutions, while in this chapter we focus our attention on the existence of *nontrivial* solutions. In particular we show how the fixed point index theory can be used to develop a theory for the existence of multiple nonzero solutions for a class of perturbed Hammerstein integral equations with deviated arguments of the form

$$u(t) = \gamma(t)\alpha[u] + \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s, \quad t \in [a,b],$$

where $\alpha[u]$ is a linear functional on $C([a,b])$ given by

$$\alpha[u] = \int_a^b u(s)\,\mathrm{d}A(s), \tag{9.0.1}$$

involving a Stieltjes integral with a *signed* measure, that is, A has bounded variation.

Here σ is a continuous function such that $\sigma([a,b]) \subseteq [a,b]$. We point out that when $\sigma(t) = a+b-t$ this type of perturbed Hammerstein integral equation is well-suited to treat problems with reflections. We apply our theory to prove the existence of nontrivial solutions of the first order functional periodic boundary value problem

$$u'(t) = h(t,u(t),u(-t)),\, t \in [-T,T];\, u(-T) - u(T) = \alpha[u],$$

which generalises the boundary conditions in Chapter 8 by adding a nonlocal term. The formulation of the nonlocal boundary conditions in terms of linear functionals is fairly general and includes, as special cases, multi-point and integral conditions, namely

$$\alpha[u] = \sum_{j=1}^{m} \alpha_j u(\eta_j) \quad \text{or} \quad \alpha[u] = \int_0^1 \phi(s)u(s)\,\mathrm{d}s.$$

where the α_j and ϕ might change sign. The study of multi-point problems has been initiated by 1908 by Picone [143] and continued by a number of authors. For an introduction to nonlocal problems we refer to the reviews of Whyburn [185], Conti [52], Ma [130], Ntouyas [135] and Štikonas [158] and to the papers [109, 112, 180].

† The tight relationship between the monotone iterative method and the upper and lower solutions method has been highlighted in [26]. Therefore, to make a difference between them is mostly a convention.

We study as well the existence of nontrivial solutions of the boundary value problem

$$u''(t) + g(t)f(t, u(t), u(\sigma(t))) = 0,\ t \in (0,1), \tag{9.0.2}$$

$$u'(0) + \alpha[u] = 0,\ \beta u'(1) + u(\eta) = 0,\ \eta \in [0,1]. \tag{9.0.3}$$

This type of problems arises when modeling the problem of a cooling or heating system controlled by a thermostat, something that has been studied in several papers, for instance [20, 45, 72]. Nonlocal heat flow problems of the type (9.0.2)-(9.0.3) were studied, without the presence of deviated arguments, by Infante and Webb in [99], who were motivated by the previous work of Guidotti and Merino [80]. This study continued in a series of papers, see [66, 88, 90, 100, 113, 139, 175, 176, 179] and references therein. The case of deviating arguments has been the subject of a recent paper by Figueroa and Pouso, see [68]. In Section 9.3 we describe with more details the physical interpretation of the boundary value problem (9.0.2)–(9.0.3).

We stress that the existence of nontrivial solutions of perturbed Hammerstein integral equations, without the presence of deviated arguments, namely

$$u(t) = \gamma(t)\hat{\alpha}[u] + \int_a^b k(t,s)f(s,u(s))\,\mathrm{d}s,$$

where $\hat{\alpha}[\cdot]$ is an *affine* functional given by a *positive* measure, have been investigated by Infante and Webb in [100], also by means of fixed point index. We make use of ideas from [100] paper, but our results are somewhat different and complementary in the case of undeviated arguments.

We work in the space $C([a,b])$ of continuous functions endowed with the usual supremum norm, and use the well-known classical fixed point index for compact maps, we refer to the review of Amann [4] and to the book of Guo and Lakshmikantham [81] for further information. The results in this chapter where published in [34]-

9.1 On a class of perturbed Hammerstein integral equations

We impose the following conditions on $k, f, g, \gamma, \alpha, \sigma$ that occur in the integral equation

$$u(t) = \gamma(t)\alpha[u] + \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s =: Fu(t). \tag{9.1.1}$$

(C_1) The kernel k is measurable, and for every $\tau \in [a,b]$ we have

$$\lim_{t\to\tau} |k(t,s) - k(\tau,s)| = 0 \text{ for a.e. } s \in [a,b].$$

(C_2) There exist a subinterval $[\hat{a},\hat{b}] \subseteq [a,b]$, a measurable function Φ with $\Phi \geq 0$ a. e. in $[a,b]$ and a constant $c_1 = c_1(\hat{a},\hat{b}) \in (0,1]$ such that

$$|k(t,s)| \leq \Phi(s) \text{ for all } t \in [a,b] \text{ and a. e. } s \in [a,b],$$
$$k(t,s) \geq c_1\,\Phi(s) \text{ for all } t \in [\hat{a},\hat{b}] \text{ and a. e. } s \in [a,b].$$

(C_3) A is of bounded variation, $\mathcal{K}_A(s) := \int_a^b k(t,s)\,\mathrm{d}A(t) \geq 0$ for a. e. $s \in [a,b]$ and $\mathcal{K}_A \in \mathrm{L}^1([a,b])$.

(C_4) The function g is measurable and satisfies that

$$g\,\Phi, g\,\mathcal{K}_A \in \mathrm{L}^1([a,b]),\ g(t) \geq 0 \text{ a. e. } t \in [a,b] \text{ and } \int_{\hat{a}}^{\hat{b}} \Phi(s)g(s)\,\mathrm{d}s > 0.$$

(C_5) $0 \not\equiv \gamma \in \mathcal{C}([a,b])$, $0 \leq \alpha[\gamma] < 1$ and there exists $c_2 \in (0,1]$ such that $\gamma(t) \geq c_2\|\gamma\|$ for all $t \in [\hat{a},\hat{b}]$.

(C_6) The nonlinearity $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ satisfies L^∞-Carathéodory conditions.

$$f(t,u,v) \leq \phi_r(t) \ \text{ for all } \ (u,v) \in [-r,r] \times [-r,r], \ \text{ and a. e. } \ t \in [a,b].$$

(C_7) The function $\sigma : [a,b] \to [a,b]$ is continuous.

Here we work in the cone

$$K = \{u \in \mathcal{C}([a,b]) : \min_{t\in[\hat{a},\hat{b}]} u(t) \geq c\|u\|,\ \alpha[u] \geq 0\},$$

where $c = \min\{c_1, c_2\}$ and c_1 and c_2 are given in (C2) and (C5) respectively. Note that, from (C_5), $K \neq \{0\}$ since $0 \neq \gamma \in K$.

The cone K is a modification of a cone of positive functions introduced in [181], that allows the use of signed measures.

Theorem 9.1.1. *Assume that hypotheses (C_1)–(C_7) hold. Then F maps K into K and is compact and continuous.*

Proof. Let $u \in K$, $t \in [a,b]$ we have,

$$\begin{aligned} |Fu(t)| &\leq |\gamma(t)|\alpha[u] + \int_a^b |k(t,s)|g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \\ &\leq \alpha[u]\|\gamma\| + \int_a^b \Phi(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s. \end{aligned}$$

Taking the supremum on $t \in [a,b]$ we get

$$\|Fu\| \leq \alpha[u]\|\gamma\| + \int_a^b \Phi(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s$$

and, combining this fact with (C_2) and (C_5),

$$\min_{t\in[\hat{a},\hat{b}]} Fu(t) \geq c_2\alpha[u]\|\gamma\| + c_1\int_a^b \Phi(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \geq c\|Fu\|.$$

Furthermore, by (C_3), (C_5) and (9.0.1),

$$\alpha[Fu] = \alpha[\gamma]\alpha[u] + \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \geq 0.$$

Therefore we have $Fu \in K$ for every $u \in K$.

The continuity and compactness of F follows from Lemma 8.1.4. ■

In the sequel, we give a condition that ensures that, for a suitable $\rho > 0$, the index is 1 on $K_\rho := \{u \in K : \|u\| < \rho\}$.

Lemma 9.1.2. *Assume that*

(I_{ρ}^{1}) *there exists $\rho > 0$ such that*

$$f^{-\rho,\rho} \cdot \sup_{t\in[a,b]} \left\{ \frac{|\gamma(t)|}{1-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \int_a^b |k(t,s)|g(s)\,\mathrm{d}s \right\} < 1$$

where

$$f^{-\rho,\rho} := \sup\left\{ \frac{f(t,u,v)}{\rho} : \ (t,u,v) \in [a,b]\times[-\rho,\rho]\times[-\rho,\rho] \right\}.$$

Then the fixed point index, $i_K(F, K_\rho)$, is equal to 1.

Proof. We show that $\mu u \neq Fu$ for every $u \in \partial K_\rho$ and for every $\mu \geq 1$. In fact, if this does not happen, there exist $\mu \geq 1$ and $u \in \partial K_\rho$ such that $\mu u = Fu$, that is

$$\mu u(t) = \gamma(t)\alpha[u] + \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s.$$

Furthermore, applying α to both sides of the equation,

$$\mu\alpha[u] = \alpha[\gamma]\alpha[u] + \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s,$$

thus, from (C_5), $\mu - \alpha[\gamma] \geq 1 - \alpha[\gamma] > 0$, and we deduce that

$$\alpha[u] = \frac{1}{\mu-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s$$

and we get, substituting,

$$\begin{aligned}\mu u(t) =& \frac{\gamma(t)}{\mu-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \\ &+ \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s.\end{aligned}$$

Taking the absolute value, and then the supremum for $t \in [a,b]$, gives

$$\begin{aligned}\mu\rho \leq& \sup_{t\in[a,b]} \left\{ \frac{|\gamma(t)|}{1-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \right. \\ &\left. + \int_a^b |k(t,s)|g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \right\} \\ \leq& \rho f^{-\rho,\rho} \cdot \sup_{t\in[a,b]} \left\{ \frac{|\gamma(t)|}{1-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \int_a^b |k(t,s)|g(s)\,\mathrm{d}s \right\} < \rho.\end{aligned}$$

This contradicts the fact that $\mu \geq 1$ and proves the result. ■

Remark 9.1.3. We point out, in similar way as in [181], that a stronger (but easier to check) condition than (I_{ρ}^{1}) is given by the following.

$$f^{-\rho,\rho} \left(\frac{\|\gamma\|}{1-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \frac{1}{m} \right) < 1, \tag{9.1.2}$$

where

$$\frac{1}{m} := \sup_{t\in[a,b]} \int_a^b |k(t,s)|g(s)\,\mathrm{d}s.$$

Let's see now a condition that guarantees that the index is equal to zero on

$$V_\rho := \{u \in K : \min_{t\in[\hat{a},\hat{b}]} u(t) < \rho\},$$

for some appropriate $\rho > 0$.

Lemma 9.1.4. *Assume that*

(I^0_ρ) *there exists $\rho > 0$ such that*

$$f_{\rho,\rho/c} \cdot \inf_{t\in[\hat{a},\hat{b}]} \left\{ \frac{\gamma(t)}{1-\alpha[\gamma]} \int_{\hat{a}}^{\hat{b}} \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \int_{\hat{a}}^{\hat{b}} k(t,s)g(s)\,\mathrm{d}s \right\} > 1,$$

where

$$f_{\rho,\rho/c} := \inf\left\{ \frac{f(t,u,v)}{\rho} : \ (t,u,v) \in [\hat{a},\hat{b}] \times [\rho,\rho/c] \times [\theta,\rho/c] \right\},$$

and

$$\theta := \begin{cases} \rho, & \text{if } \sigma([\hat{a},\hat{b}]) \subseteq [\hat{a},\hat{b}], \\ -\rho/c, & \text{otherwise.} \end{cases}$$

Then $i_K(F,V_\rho) = 0$.

Proof. Since $0 \not\equiv \gamma \in K$ we can choose $e = \gamma$ in Lemma 8.1.2, so we now prove that

$$u \neq Fu + \mu\gamma \quad \text{for all } u \in \partial V_\rho \quad \text{and every } \mu > 0.$$

In fact, if not, there exist $u \in \partial V_\rho$ and $\mu > 0$ such that $u = Fu + \mu\gamma$. Then we have

$$u(t) = \gamma(t)\alpha[u] + \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s + \mu\gamma(t)$$

and

$$\alpha[u] = \alpha[\gamma]\alpha[u] + \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s + \mu\alpha[\gamma],$$

and therefore

$$\alpha[u] = \frac{1}{1-\alpha[\gamma]} \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s + \frac{\mu\alpha[\gamma]}{1-\alpha[\gamma]}.$$

Thus we get, for $t \in [\hat{a},\hat{b}]$,

$$\begin{aligned} u(t) =& \frac{\gamma(t)}{1-\alpha[\gamma]} \left(\int_a^b \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s + \mu\alpha[\gamma] \right) \\ &+ \int_a^b k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s + \mu\gamma(t) \\ \geq& \frac{\gamma(t)}{1-\alpha[\gamma]} \int_{\hat{a}}^{\hat{b}} \mathcal{K}_A(s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \\ &+ \int_{\hat{a}}^{\hat{b}} k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s \\ \geq& \rho f_{\rho,\rho/c} \left(\frac{\gamma(t)}{1-\alpha[\gamma]} \int_{\hat{a}}^{\hat{b}} \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \int_{\hat{a}}^{\hat{b}} k(t,s)g(s)\,\mathrm{d}s \right). \end{aligned}$$

Taking the minimum over $[\hat{a},\hat{b}]$ gives $\rho > \rho$, a contradiction. ■

Remark 9.1.5. We point out, that a stronger condition than (I_{ρ}^{0}) is given by the following.

$$f_{\rho,\rho/c}\left(\frac{c_2\|\gamma\|}{1-\alpha[\gamma]}\int_{\hat{a}}^{\hat{b}}\mathcal{K}_A(s)g(s)\,\mathrm{d}s+\frac{1}{M(\hat{a},\hat{b})}\right)>1, \tag{9.1.3}$$

where

$$\frac{1}{M(\hat{a},\hat{b})}:=\inf_{t\in[\hat{a},\hat{b}]}\int_{\hat{a}}^{\hat{b}}k(t,s)g(s)\,\mathrm{d}s. \tag{9.1.4}$$

Remark 9.1.6. Depending on the nature of the nonlinearity f and due to the way θ is defined, sometimes it could be useful to take a smaller $[\hat{a},\hat{b}]$ such that $\sigma([\hat{a},\hat{b}])\subseteq[\hat{a},\hat{b}]$. This fact is illustrated in Section 9.3.

The above Lemmas can be combined to prove the following Theorem. Here we deal with the existence of at least one, two or three solutions. We stress that, by expanding the lists in conditions (S_5), (S_6) below, it is possible to state results for four or more positive solutions.

Theorem 9.1.7. *Assume (C_1)-(C_7) are satisfied. The integral equation* (9.1.1) *has at least one nonzero solution in K if any of the following conditions hold.*

(S_1) *There exist $\rho_1,\rho_2\in(0,\infty)$ with $\rho_1/c<\rho_2$ such that $(\mathrm{I}_{\rho_1}^{0})$ and $(\mathrm{I}_{\rho_2}^{1})$ hold.*

(S_2) *There exist $\rho_1,\rho_2\in(0,\infty)$ with $\rho_1<\rho_2$ such that $(\mathrm{I}_{\rho_1}^{1})$ and $(\mathrm{I}_{\rho_2}^{0})$ hold.*

The integral equation (9.1.1) *has at least two nonzero solutions in K if one of the following conditions hold.*

(S_3) *There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$ with $\rho_1/c<\rho_2<\rho_3$ such that $(\mathrm{I}_{\rho_1}^{0})$, $(\mathrm{I}_{\rho_2}^{1})$ and $(\mathrm{I}_{\rho_3}^{0})$ hold.*

(S_4) *There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$ with $\rho_1<\rho_2$ and $\rho_2/c<\rho_3$ such that $(\mathrm{I}_{\rho_1}^{1})$, $(\mathrm{I}_{\rho_2}^{0})$ and $(\mathrm{I}_{\rho_3}^{1})$ hold.*

The integral equation (9.1.1) *has at least three nonzero solutions in K if one of the following conditions hold.*

(S_5) *There exist $\rho_1,\rho_2,\rho_3,\rho_4\in(0,\infty)$ with $\rho_1/c<\rho_2<\rho_3$ and $\rho_3/c<\rho_4$ such that $(\mathrm{I}_{\rho_1}^{0})$, $(\mathrm{I}_{\rho_2}^{1})$, $(\mathrm{I}_{\rho_3}^{0})$ and $(\mathrm{I}_{\rho_4}^{1})$ hold.*

(S_6) *There exist $\rho_1,\rho_2,\rho_3,\rho_4\in(0,\infty)$ with $\rho_1<\rho_2$ and $\rho_2/c<\rho_3<\rho_4$ such that $(\mathrm{I}_{\rho_1}^{1})$, $(\mathrm{I}_{\rho_2}^{0})$, $(\mathrm{I}_{\rho_3}^{1})$ and $(\mathrm{I}_{\rho_4}^{0})$ hold.*

Remark 9.1.8. A similar approach can be used, depending on the signs of k and γ, to prove the existence of solutions that are negative on sub-interval, nonpositive, strictly negative, nonnegative and strictly positive the same way we did in the previous chapter.

9.2 An application to a problem with reflection

We now turn our attention to the first order functional periodic boundary value problem

$$u'(t) = h(t, u(t), u(-t)),\ t \in I := [-T, T], \tag{9.2.1}$$

$$u(-T) - u(T) = \alpha[u], \tag{9.2.2}$$

where α is a linear functional on $C(I)$ given by

$$\alpha[u] = \int_{-T}^{T} u(s)\,\mathrm{d}A(s),$$

involving a Stieltjes integral with a *signed* measure.

We use again the shift argument of the Chapter 8, by fixing $\omega \in \mathbb{R}\backslash\{0\}$ and considering the equivalent expression

$$u'(t) + \omega u(-t) = h(t, u(t), u(-t)) + \omega u(-t) =: f(t, u(t), u(-t)),\ t \in I, \tag{9.2.3}$$

with the boundary conditions

$$u(-T) - u(T) = \alpha[u]. \tag{9.2.4}$$

Note that the Green's function k of the periodic problem only exists when $\omega T \neq l\pi$ for every $l \in \mathbb{Z}$. Hence, Corollary 3.2.5 guarantees that problem (9.2.3)–(9.2.4) is equivalent to the perturbed Hammerstein integral equation

$$u(t) = k(t, -T)\alpha[u] + \int_{-T}^{T} k(t,s) f(s, u(s), u(-s))\,\mathrm{d}s,$$

where k is the associated Green's function given by equation (8.4.6). Thus, we are working with an equation of the type (9.1.1) where

$$\gamma(t) = k(t, -T) = \cos \omega t - \sin \omega t = \sqrt{2} \sin\left(\frac{\pi}{4} - \omega t\right).$$

In order to apply Theorem 9.1.7, we must verify conditions (C_1)–(C_7) and study when (I^0_ρ) and (I^1_ρ) are fulfilled.

Let $\zeta := \omega T$. Then we have

$$\|\gamma\| = \begin{cases} \sqrt{2}\sin\left(\frac{\pi}{4} + \zeta\right), & \zeta \in \left(0, \frac{\pi}{4}\right), \\ \sqrt{2}, & \zeta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right). \end{cases}$$

Also, using Lemma 3.4.2, the constant c_2 is given by

$$\|\gamma\| c_2 = \inf_{t\in[\hat{a},\hat{b}]} \gamma(t) = \begin{cases} \gamma(\hat{b}), & \zeta \in \left(0, \frac{\pi}{4}\right] \quad \text{or} \quad \left|\hat{a} + \frac{\pi}{4\zeta}\right| < \left|\hat{b} + \frac{\pi}{4\zeta}\right|, \\ \gamma(\hat{a}), & \zeta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right] \text{ and } \left|\hat{a} + \frac{\pi}{4\zeta}\right| \geq \left|\hat{b} + \frac{\pi}{4\zeta}\right|. \end{cases}$$

The constant c_1 was given in Chapter 8, Theorem 8.4.11, for the case $\hat{a} + \hat{b} = 1$ and has the following expression

$$c_1 = \frac{(1 - \tan \omega\hat{a})(1 - \tan \omega\hat{b})}{(1 + \tan \omega\hat{a})(1 + \tan \omega\hat{b})}. \tag{9.2.5}$$

Observe that using the fact that $k(t,s) = k(t+T, s+T)$, $k(t+T,s) = k(t,s+T)$ for $t,s \in [-T,0]$ and formula (9.2.5) for $[\hat{a},\hat{b}] = [0,T]$ we get that

$$c_1 = \frac{1 - \tan \zeta}{1 + \tan \zeta} = \cot\left(\frac{\pi}{4} + \zeta\right).$$

Consider now the set $\hat{S} := \{(\hat{a},\hat{b}) \in \mathbb{R}^2 \, : \, \hat{a} < \hat{b}, \ (C_2) \text{ is satisfied for } [\hat{a},\hat{b}]\}$ and $M(\hat{a},\hat{b})$ defined as in (9.1.4) (with $g \equiv 1$). Since a smaller constant $M(\hat{a},\hat{b})$ relaxes the growth conditions imposed on the nonlinearity f by the inequality (9.1.3), we turn our attention to the quantity

$$\frac{1}{M_{opt}} := \sup_{(\hat{a},\hat{b}) \in \hat{S}} \frac{1}{M(\hat{a},\hat{b})}.$$

A similar study has been done, in the case of second-order boundary value problems in [94, 175, 176] and for fourth order boundary value problems in [92, 144, 182].

Before computing this value, we need some relevant information about the kernel k.

First, observe that with the change of variables $t = \bar{x}T$, $s = \bar{y}T$, $\bar{k}(\bar{x},\bar{y}) = k(t,s)$, $a = \bar{a}T$, $b = \bar{b}T$ we have

$$\frac{1}{M_{opt}} = T \sup_{(\bar{a},\bar{b}) \in \tilde{S}} \min_{x \in [\bar{a},\bar{b}]} \int_{\bar{a}}^{\bar{b}} \bar{k}(\bar{x},\bar{y}) \, \mathrm{d}\bar{y},$$

where $\tilde{S} := \{(\bar{a},\bar{b}) \in \mathbb{R}^2 \, : \, (\bar{a}T, \bar{b}T) \in \hat{S}\}$.

Recall (see Lemma 3.2.6) that there is a symmetry between the cases ω and $-\omega$ given by the fact that $\bar{k}_\omega(x,y) = -\bar{k}_{-\omega}(-x,-y)$, so we can restrict our problem to the case $\omega > 0$.

We proved in the previous Chapter that $\bar{k}$ satisfies the equation $\frac{\partial \bar{k}}{\partial x}(x,y) + \omega\bar{k}(-x,y) = 0$. Also, the strip S, defined in Lemma 8.4.1, satisfies that, if $(x,y) \in S$, then $(-x,y) \in S$, so, wherever $\bar{k} \geq 0$, $\frac{\partial \bar{k}}{\partial t} \leq 0$. Hence, we have

$$\frac{1}{M(\omega)} = T \sup_{(\bar{a},\bar{b}) \in \tilde{S}} \int_{\bar{a}}^{\bar{b}} \bar{k}(\bar{b},y) \, \mathrm{d}y.$$

Notice that, fixed $\bar{b}$, it is of our interest to take $\bar{a}$ as small as possible (as long as (C_2) is satisfied) for we are integrating a positive function on the interval $[\bar{a},\bar{b}]$.

With these considerations in mind, we will prove that

$$M_{opt} = \begin{cases} \omega, & \text{if} \quad \zeta \in (0, \frac{\pi}{4}), \\ \frac{\omega}{\cos \zeta}, & \text{if} \quad \zeta \in [\frac{\pi}{4}, \frac{\pi}{2}), \end{cases}$$

by studying two cases: (A) and (B).

(A) If $\zeta \in (0, \frac{\pi}{4})$, $\overline{k}$ is positive and

$$\frac{1}{M_{opt}} = T \sup_{\overline{b} \in [-1,1]} \int_{-1}^{\overline{b}} \overline{k}(\overline{b}, y)\, \mathrm{d}y.$$

(A1) If $\overline{b} \leq 0$, let

$$\begin{aligned} g_1(\overline{b}) :=& 2\sin\zeta \int_{-1}^{\overline{b}} \overline{k}(\overline{b}, y)\, \mathrm{d}y = \int_{-1}^{\overline{b}} [\cos\zeta(1+y+\overline{b}) + \sin\zeta(1+y-\overline{b})]\, \mathrm{d}s \\ =& \frac{1}{\zeta}\left[\sin\zeta(1+2\overline{b}) - \sin\zeta\overline{b} + \cos\zeta\overline{b} - \cos\zeta\right]. \end{aligned}$$

Then, taking into account that $\overline{b} \in [-1, 0]$ and $\zeta \in \left(0, \frac{\pi}{4}\right)$ and studying the range of the arguments of the sines and cosines involved, we get

$$g_1'(\overline{b}) = 2\cos\zeta(1+2\overline{b}) - \sqrt{2}\sin\left(\zeta\overline{b} + \frac{\pi}{4}\right) \geq 2\frac{\sqrt{2}}{2} - \sqrt{2}\frac{\sqrt{2}}{2} = \sqrt{2} - 1 > 0.$$

Therefore, the maximum of g_1 in $[-1, 0]$ is reached at 0.

(A2) If $\overline{b} \geq 0$,

$$\begin{aligned} g_1(\overline{b}) =& \int_{-1}^{-\overline{b}} [\cos\zeta(1+y+\overline{b}) + \sin\zeta(1+y-\overline{b})]\, \mathrm{d}s \\ &+ \int_{-\overline{b}}^{\overline{b}} [\cos\zeta(1-y-\overline{b}) + \sin\zeta(1+y-\overline{b})]\, \mathrm{d}s \\ =& -\frac{1}{\zeta}\left[\cos\zeta - \cos\zeta b - 2\sin\zeta + \sin\zeta b + \sin\zeta(1-2b)\right]. \end{aligned}$$

Now, we have

$$g_1'''(\overline{b}) = -\zeta^2\left[8\cos\zeta(1-2\overline{b}) - \sqrt{2}\sin\left(\zeta\overline{b} + \frac{\pi}{4}\right)\right] < 0.$$

Therefore, g_1' reaches its minimum in $[0, 1]$ at 0 or 1.

$$g_1'(0) = 2\cos\zeta - 1,\ g_1'(1) = \cos\zeta - \sin\zeta > 0.$$

Thus, $g_1' > 0$ in $[0, 1]$, this is, the maximum of g_1 in $[0, 1]$ is reached at 1. In conclusion, by the continuity of g_1, the maximum of g_1 in $[-1, 1]$ is reached at 1 and so

$$\frac{1}{M_{opt}} = T\int_{-1}^{1} \overline{k}(1, y)\, \mathrm{d}y = T\frac{g_1(1)}{2\sin\zeta} = \frac{T}{\zeta} = \frac{1}{\omega}.$$

Observe now that, since $[\overline{a}, \overline{b}] = [-1, 1]$, $c = c_1 = c_2 = \cot\left(\frac{\pi}{4} + \zeta\right)$.

(B) Now assume $\zeta \in [\frac{\pi}{4}, \frac{\pi}{2})$. $\overline{k}$ is positive on S.

Assume $\overline{b} > 0$. Also, since $\overline{k}(x, y) = \overline{k}(-y, -x)$, fixed $b \in S$, the smallest $\overline{a}$ that can be taken is $\overline{a} = 1 - \frac{\pi}{4\zeta}$, so

$$g_2(\overline{b}) := 2\sin\zeta \int_{1-\frac{\pi}{4\zeta}}^{\overline{b}} \overline{k}(\overline{b}, y)\, \mathrm{d}y$$

$$=\frac{1}{\zeta}\left[\cos\left(\frac{\pi}{4}+(\overline{b}-2)\,\zeta\right)+\cos\left(\frac{\pi}{4}+\overline{b}\zeta\right)-\cos\zeta+\sin\left((2\overline{b}-1)\,\zeta\right)\right].$$

Thus, we have

$$\begin{aligned} g_2'''(\overline{b}) =&\zeta^2\left[\sin\left(\frac{\pi}{4}+(b-2)\,\zeta\right)+\sin\left(\frac{\pi}{4}+b\zeta\right)-8\cos\left((1-2b)\,\zeta\right)\right]\\ >&\zeta^2\left(2-8\frac{\sqrt{2}}{2}\right)<0. \end{aligned}$$

Therefore, g_2' reaches its minimum in $Y:=\left[1-\frac{\pi}{4\zeta},\frac{\pi}{4\zeta}\right]$ at $1-\frac{\pi}{4\zeta}$ or $\frac{\pi}{4\zeta}$.

$$g_2'\left(1-\frac{\pi}{4\zeta}\right)=2\sin\zeta,\; g_2'\left(\frac{\pi}{4\zeta}\right)=2(\sin\zeta-\cos^2\zeta)>0.$$

Thus, $g_2'>0$ in Y, this is, the maximum of g_2 in Y is reached at $\frac{\pi}{4\zeta}$ and so

$$T\int_{1-\frac{\pi}{4\zeta}}^{\frac{\pi}{4\zeta}}\overline{k}\left(\frac{\pi}{4\zeta},y\right)\mathrm{d}y=T\frac{g_2\left(\frac{\pi}{4\zeta}\right)}{2\sin\zeta}=\frac{T\cos\zeta}{\zeta}=\frac{\cos\zeta}{\omega}.$$

Now, the case $\overline{b}\le 0$ can be reduced to the case $\overline{b}\ge 0$ just taking into account that $\overline{k}(z,y)=\overline{k}(z+1,y+1)$ for $z,y\in[-1,0]$ (cf. Chapter 7) and making the change of variables $\overline{y}=y-1$, so

$$\int_{1-\frac{\pi}{4\zeta}}^{\frac{\pi}{4\zeta}}\overline{k}\left(\frac{\pi}{4\zeta},y\right)\mathrm{d}y=\int_{-\frac{\pi}{4\zeta}}^{\frac{\pi}{4\zeta}-1}k\left(\frac{\pi}{4\zeta},\overline{y}+1\right)\mathrm{d}\overline{y}=\int_{-\frac{\pi}{4\zeta}}^{\frac{\pi}{4\zeta}-1}k\left(\frac{\pi}{4\zeta}-1,\overline{y}\right)\mathrm{d}\overline{y}.$$

Hence we have

$$\frac{1}{M_{opt}}=\frac{\cos\zeta}{\omega}.$$

Consider again the case $\zeta\in\left(0,\frac{\pi}{4}\right)$ and $\hat{a}_{opt},\hat{b}_{opt},c(\hat{a}_{opt},\hat{b}_{opt})$, the values for which M_{opt} is reached. In the following table we summarize these findings.

ζ	$\hat{a}_{opt}$	$\hat{b}_{opt}$	M_{opt}	$c(\hat{a}_{opt},\hat{b}_{opt})$	$\Vert\gamma\Vert$
$\left(0,\frac{\pi}{4}\right)$	-1	1	ω	$\cot\left(\frac{\pi}{4}+\zeta\right)$	$\sqrt{2}\sin\left(\frac{\pi}{4}+\zeta\right)$

When $\zeta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)$ we have the following.

ζ	$\hat{a}_{opt}$	$\hat{b}_{opt}$	M_{opt}	$\Vert\gamma\Vert$
$\left[\frac{\pi}{4},\frac{\pi}{2}\right)$	$1-\frac{\pi}{4\zeta}$ $-\frac{\pi}{4\zeta}$	$\frac{\pi}{4\zeta}$ $\frac{\pi}{4\zeta}-1$	$\frac{\omega}{\cos\zeta}$	$\sqrt{2}$

We point out that in this second case we cannot take an interval $[\hat{a}, \hat{b}]$ at which M_{opt} is reached because c_1 and c_2 tend to zero as we approach that interval, but we may take $[\hat{a}, \hat{b}]$ as close as possible to these values, in order to approximate M_{opt}.

With all these ingredients we can apply Theorem 9.1.7 in order to solve (9.2.1)-(9.2.2) for some given f and α.

9.3 An application to a thermostat problem

9.3.1 The model

We work here with the model of a light bulb with a temperature regulating system (thermostat). The model includes a bulb in which a metal filament, bended on itself, is inserted with only its two extremes outside of the bulb. There is a sensor that allows to measure the temperature of the filament at a point η (see Figure 9.3.1). The bulb is sealed with some gas in its interior.

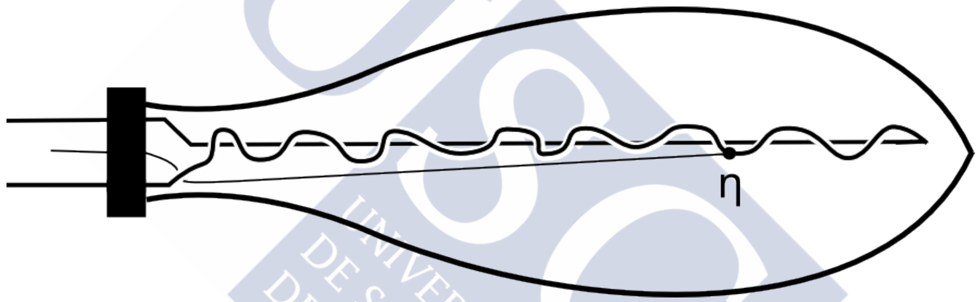


Figure 9.3.1: Sketch of the light bulb model with a sensor at the point η.

As variables, we take u for the temperature, $t \in [0,1]$ for a point in the filament and x for the time[†].

We control the light bulb via two thermopairs connected to the extremes of the filament. This allows us to measure (and hence modify via a resistance or with some other heating or cooling system) the variation of the temperature with respect to x. Also, we will be able to measure the total light ouput of the light bulb.

The problem can then be stated as

$$\begin{aligned}\frac{du}{dx}(t,x) =& d_1\frac{d^2u}{dt^2}(t,x) + \int_0^1 u^4(y,x)v(s,t,u(t,x))\,\mathrm{d}s - d_2u^4(t,x)\\ &+ j(t,u(t,x)) + (d_3 + d_4u(t,x))\hat{I}^2 + d_5(u_0 - u(t,x)),\end{aligned} \tag{9.3.1}$$

$$\frac{du}{dt}(0,x) + d_6\int_0^1 u^4(s,x)\,\mathrm{d}s = 0, \quad \beta\frac{du}{dt}(1,x) + u(\eta,x) = 0, \tag{9.3.2}$$

where $d_1, \ldots, d_5$ and u_0 are physical (real) constants that can be determined either theoretically or experimentally; d_6, $\hat{I}$ and β are real constants to be chosen; $\eta \in [0,1]$ is the position

[†]We use this unusual notation in order to be consistent with the rest of the section. Since we are looking for stationary solutions of the model, the temporal variable will no longer appear after the model is set.

of the sensor at the filament and v is some real continuous function. We explain now each component of the equation.

The term $d_1 \frac{d^2u}{dt^2}(t,x)$ comes from the traditional heat equation, $\frac{du}{dx} = d_1 \frac{d^2u}{dt^2}$. The integral in the equation stands for the temperature (that is, power per space unit squared), in form of blackbody radiation, absorbed by the point t and emitted from every other point s of the filament. The function v gives the rate of this absorption depending on t, s and also on u, since the reflectivity of metals changes with temperature (see [168]). The equation behind the fourth power in the integral comes from the Stefan-Boltzmann equation for blackbody power emission, $j^\star = \tilde{k}u^4(t,x)$, where $j^\star$ is the irradiance and $\tilde{k}$ a constant. Observe that considering the power emission from the rest of the filament is important, since, as early as 1914 (see [51]), it has been observed that an interior and much brighter (90 to 100 percent) helix appears in helical filaments of tungsten. Although a $200\,^\circ$C difference would be necessary to account for the extra brightness, experiments show that most of it is due to reflection, being the difference in the temperature less than $5\,^\circ$C.

The term $-d_2u^4(t,x)$ accounts again for the Stefan-Boltzmann equation, this time for the irradiance of the point, $j(t,u(t,x))$ for the energy absorbed from the bulb (via reflection and/or blackbody emission) and $(d_3 + d_4u(t,x))\hat{I}^2$ is the heat produced by the intensity of the electrical current, $\hat{I}$, going through the filament via Ohm's law taking into account a first order approximation of the variation of the resistivity of the metal with temperature. Finally, $d_5(u_0 - u(t,x))$ is the heat transfer from the filament to the gas due to Newton's law of cooling, where u_0 is the temperature at the interior of the bulb which we may assume constant.

The first boundary condition controls the variation of the temperature at the left extreme depending on the total irradiance of the bulb, while the second boundary condition controls the variation of the temperature at the right end of the filament depending on the temperature at η.

Consider now the term

$$\Gamma[u](t,x) := \int_0^1 u^4(s,x)v(s,t,u(t,x))\,\mathrm{d}s.$$

For a fixed x, Γ is an operator on $C[0,1]$. If we consider the wire to be bended on itself, in such a way that every point of the filament touches one and only one other point of the filament, by the continuity of the temperature on the filament, we may take the approximation $\Gamma[u](t,x) = d_7u^4(\sigma(t,x))$ for some constant d_7 and a function σ which maps every point in the filament to the other point it is affected by. Now, σ is an involution.

With these ingredients, and looking for stationary solutions of problem (9.3.1)-(9.3.2), we arrive to a boundary value problem of the form

$$u''(t) + g(t)f(t,u(t),u(\sigma(t))) = 0,\ t \in (0,1), \tag{9.3.3}$$

$$u'(0) + \alpha[u] = 0,\ \beta u'(1) + u(\eta) = 0,\ \eta \in [0,1]. \tag{9.3.4}$$

Remark 9.3.1. In some other light bulb model it could happen that every point of the filament is 'within reach' of more than one other point, which would mean we could have a multivalued function σ or just two functions σ_1 and σ_2 in the equation (9.3.3). Our theory can be extended to the case of having more than one function σ. A possible approach to the multivalued case would require to extend the theory in [94].

9.3.2 The associated perturbed integral equation

We now turn our attention to the second order boundary value problem (9.3.3)-(9.3.4).

In a similar way as in Chapter 7, the solution of the boundary value problem (9.3.3)-(9.3.4) can be expressed as

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s),u(\sigma(s)))\,\mathrm{d}s,$$

where $\gamma(t) = \beta + \eta - t$, and

$$k(t,s) = \beta + \begin{cases} \eta - s, & s \le \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t - s, & s \le t \\ 0, & s > t. \end{cases}$$

Here we focus on the case $\beta \ge 0$ and $0 < \beta + \eta < 1$, that leads (in similar way to [100]) to the existence of solutions that are positive on a subinterval. The constant c for this problem (see for example [94]) is

$$c = \begin{cases} \beta/(\beta+\eta), & \hat{b} \le \eta,\ \beta+\eta \ge \frac{1}{2}, \\ \beta/(1-(\beta+\eta)), & \hat{b} \le \eta,\ \beta+\eta < \frac{1}{2}, \\ (\beta+\eta-\hat{b})/(\beta+\eta), & \hat{b} > \eta,\ \beta+\eta \ge \frac{1}{2}, \\ (\beta+\eta-\hat{b})/(1-(\beta+\eta)), & \hat{b} > \eta,\ \beta+\eta < \frac{1}{2}. \end{cases}$$

Also, we have

$$\Phi(s) = \|\gamma\| = \begin{cases} \beta+\eta, & \beta+\eta \ge \frac{1}{2}, \\ 1-(\beta+\eta), & \beta+\eta < \frac{1}{2}, \end{cases}$$

and

$$c_2\|\gamma\| = \beta + \eta - \hat{b}.$$

Theorem 9.1.7 can be applied to this problem for given f, α and g. We now set $g \equiv 1$ and recall (see [100]) that

$$\sup_{t\in[0,1]} \int_0^1 |k(t,s)|\,\mathrm{d}s = \max\Big\{\beta + \frac{1}{2}\eta^2, \beta^2 - \beta + \frac{1}{2}(1-\eta^2)\Big\}.$$

Furthermore, note that the solution of the problem

$$w''(t) = -1, \quad w'(0) = 0, \quad \beta w'(1) + w(\eta) = 0,$$

is given by $w(t) = \beta + \frac{1}{2}(\eta^2 - t^2)$, which implies that

$$w(t) = \int_0^1 k(t,s)\,\mathrm{d}s = \beta + \frac{1}{2}(\eta^2 - t^2).$$

Using this fact, equation (9.0.1) and Fubini's Theorem we have

$$\begin{aligned} \int_0^1 \mathcal{K}_A(s)\,\mathrm{d}s &= \int_0^1 \int_0^1 k(t,s)\,\mathrm{d}A(t)\,\mathrm{d}s \\ &= \int_0^1 \int_0^1 k(t,s)\,\mathrm{d}s\,\mathrm{d}A(t) = \alpha\left[\beta + \frac{1}{2}(\eta^2 - t^2)\right]. \end{aligned}$$

With all these facts, the conditions (9.1.2) and (9.1.3) can be rewritten, respectively, for problem (9.0.2)–(9.0.3) as

$$f^{-\rho,\rho} < m_\alpha, \tag{$\tilde{\mathrm{I}}^1_\rho$}$$

where

$$\frac{1}{m_\alpha} := \frac{(\beta+\eta)\chi_{[\frac{1}{2},+\infty)}(\beta+\eta) + (1-\beta-\eta)\chi_{(-\infty,\frac{1}{2})}(\beta+\eta)}{1-\alpha[\beta+\eta-t]} \cdot \alpha\left[\beta + \frac{1}{2}(\eta^2 - t^2)\right] + \max\left\{\beta + \frac{1}{2}\eta^2, \beta^2 - \beta + \frac{1}{2}(1-\eta^2)\right\},$$

χ_B is the characteristic function of the set B; and

$$f_{\rho,\rho/c} > M_\alpha, \tag{$\tilde{\mathrm{I}}^0_\rho$}$$

where

$$\frac{1}{M_\alpha} := \frac{\beta+\eta-\hat{b}}{1-\alpha[\beta+\eta-t]} \cdot \alpha\left[\int_{\hat{a}}^{\hat{b}} k(t,s)\,\mathrm{d}s\right] + \frac{1}{M(\hat{a},\hat{b})}.$$

Therefore, we can restate Theorem 9.1.7 as follows.

Theorem 9.3.2. *Theorem 9.1.7 is satisfied if we change the conditions* (I^0_ρ) *and* (I^1_ρ) *by* $(\tilde{\mathrm{I}}^0_\rho)$ *and* $(\tilde{\mathrm{I}}^1_\rho)$ *respectively.*

We now illustrate how the behavior of the deviated argument affects the allowed growth of the nonlinearity f.

Example 9.3.3. Take $\eta = 1/5$, $\beta = 3/5$. It was proven in [94] that the optimal interval for such a choice of parameters is $[\hat{a}, \hat{b}] = [0, 3/5]$, for which $M_{opt} = 5$, $m = 50/31$, $c_1 = 1/4$. Consider $\sigma(t) = 11t - 101t^2 + 318t^3 - 394t^4 + 167t^5$. σ satisfies $\sigma([0,1]) = [0,1]$ and $\sigma([0,2/5]) \subseteq [0,2/5]$ as it is shown in Figure 9.3.2.

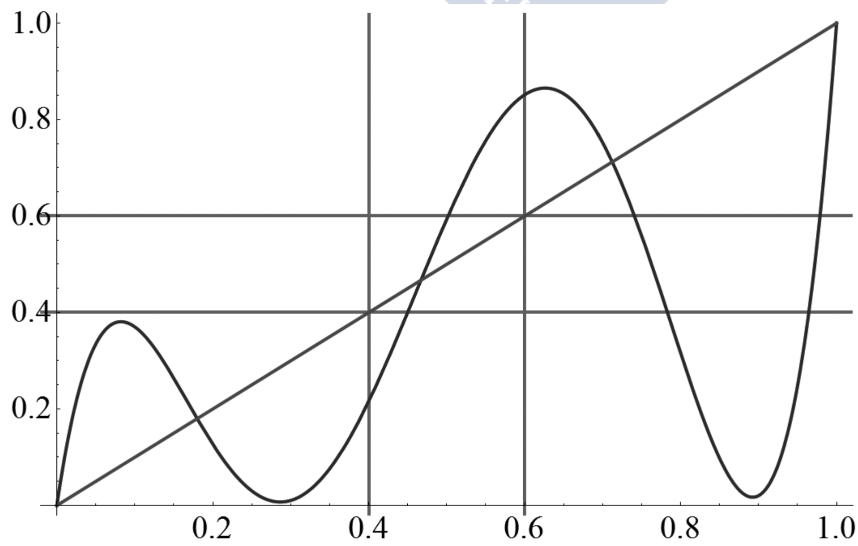


Figure 9.3.2: Plot of the function σ and the identity.

Remember that the condition $(\tilde{\mathrm{I}}^0_\rho)$ is of the form

$$f_{\rho,\rho/c}(\hat{a},\hat{b})\ \left(p(\alpha)q(\hat{a},\hat{b})+r(\hat{a},\hat{b})\right)>1$$

where

$$p(\alpha)=\frac{\|\gamma\|}{1-\alpha[\gamma]},\quad q(\hat{a},\hat{b})=c_2(\hat{a},\hat{b})\int_{\hat{a}}^{\hat{b}}\mathcal{K}_A(s)g(s)\,\mathrm{d}s\quad\text{and}\quad r(\hat{a},\hat{b})=\frac{1}{M(\hat{a},\hat{b})}.$$

Now, picking up Remark 9.1.6, the questions is: Is it worth it to take $[\hat{a},\hat{b}]=[0,3/5]$ or it is preferable to take $[\hat{a},\hat{b}]=[0,2/5]$? Observe that, as mentioned, $\sigma([0,2/5])\subseteq[0,2/5]$ but $\sigma([0,3/5])\not\subseteq[0,3/5]$, which means that the value of $f_{\rho,\rho/c}(\hat{a},\hat{b})$ can vary considerably from one case to the other. It will be preferable to take $[\hat{a},\hat{b}]=[0,2/5]$ if and only if

$$\frac{f_{\rho,\rho/c}(0,2/5)}{f_{\rho,\rho/c}(0,3/5)}>\frac{p(\gamma,\alpha)q(0,3/5)+r(0,3/5)}{p(\gamma,\alpha)q(0,2/5)+r(0,2/5)}.$$

We can compute, a priori, $q(0,3/5)$, $q(0,2/5)$, $r(0,2/5)$ and $r(0,3/5)$, but $f_{\rho,\rho/c}(0,2/5)$ and $f_{\rho,\rho/c}(0,3/5)$ will depend on f and $p(\gamma,\alpha)$ on α. As a simple example, if f is zero at a subset of $(2/3,5/3]$ of positive measure, we have that the choice to make is $[\hat{a},\hat{b}]=[0,2/5]$.

Example 9.3.4. Continuing with last example, assume now $\alpha[u]=\lambda\,u(2/5)$ for some $\lambda\in(0,5/2)$. (C_1) and (C_2) are satisfied by the properties of the kernel and by the choice of c_1. We assume (C_6) is satisfied for the nonlinearity chosen. (C_4) and (C_7) are obviously satisfied. $\mathcal{K}_\alpha(s)=k((2\lambda)/5,s)>0$ for every $s\in[0,1]$ by the properties of the kernel, so (C_3) is also satisfied. Last, $0\le\alpha[4/5-t]=(2\lambda)/5<1$ and, by the choice of c_2, (C_7) is satisfied as well. In this case we have $m_\alpha=25/26$, and it is independent of the choice of $[\hat{a},\hat{b}]$. Let us compare the intervals $[0,2/5]$ and $[0,3/5]$.

$$\frac{1}{M_\alpha(0,\hat{b})}=\frac{4-5\hat{b}}{1-2\lambda}\int_0^{\hat{b}}k((2\lambda)/5,s)\,\mathrm{d}s+\inf_{t\in(0,\hat{b}]}\int_0^{\hat{b}}k(t,s)\,\mathrm{d}s.$$

It was proven in [94] that, for $0\le\hat{a}<\hat{b}<\beta+\eta$,

$$\inf_{t\in(0,\hat{b}]}\int_0^{\hat{b}}k(t,s)\,\mathrm{d}s=\int_0^{\hat{b}}k(\hat{b},s)\,\mathrm{d}s.$$

Hence,

$$M_\alpha(0,2/5)=\begin{cases}\frac{50(1-2\lambda)}{43+2\lambda} & \text{if}\quad\lambda\in[1,5/2),\\ \frac{50(1-2\lambda)}{(7-2\lambda)(5+4\lambda)} & \text{if}\quad\lambda\in(0,1),\end{cases}$$

$$M_\alpha(0,3/5)=\begin{cases}\frac{25+50\lambda}{19+4\lambda} & \text{if}\quad\lambda\in[1,5/2),\\ \frac{50(1+2\lambda)}{29+20\lambda-4\lambda^2} & \text{if}\quad\lambda\in(0,1).\end{cases}$$

Figure 9.3.3 shows how these two values vary depending on λ.

If we take an specific value for λ, say $\lambda=1$, we get $M_\alpha(0,2/5)=M_\alpha(0,3/5)=10/3$, and so it is more convenient to take $[\hat{a},\hat{b}]=[0,2/5]$. The reason for this is that $f_{\rho,\rho/c}(0,2/5)\ge f_{\rho,\rho/c}(0,3/5)$ independently of f, and so I^0_ρ is more easily satisfied.

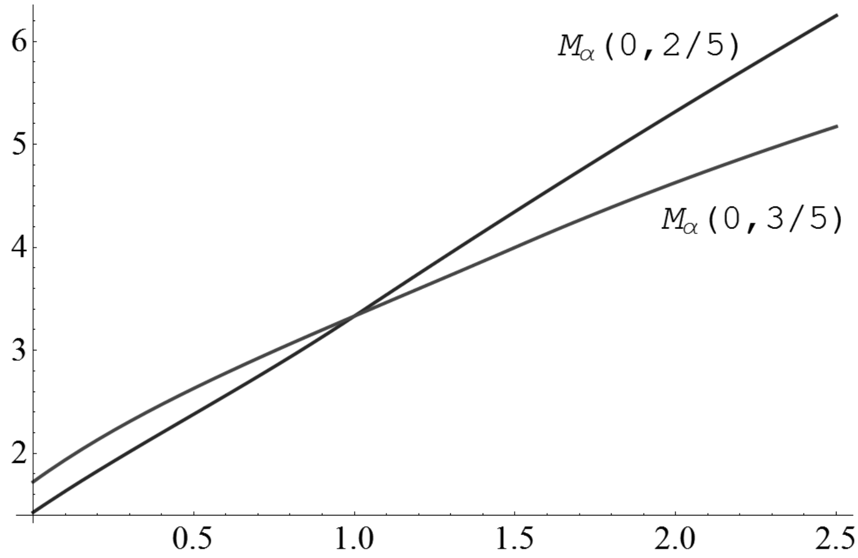


Figure 9.3.3: Plot of $M_\alpha(0,2/5)$ and $M_\alpha(0,3/5)$ depending on λ.

Observe in Figure 9.3.3 that the graphs of $M_\alpha(0,2/5)(\lambda)$ and $M_\alpha(0,3/5)(\lambda)$ cross at $\lambda = 1$. If f is continuous and $f_{\rho,\rho/c}(0,2/5) > f_{\rho,\rho/c}(0,3/5)$, since $M_\alpha(0,2/5)(1)$ is a better choice than $M_\alpha(0,3/5)(1)$, by the continuity of f, so it will be in a neighborhood of 1. That shows that the condition $M_\alpha(0,2/5)(\lambda) < M_\alpha(0,3/5)(\lambda)$ may help but is not deciding when choosing the interval.

10. Nonlocal boundary conditions

In this chapter we discuss the existence, localization, multiplicity and nonexistence of nontrivial solutions of the second order differential equation

$$u''(t) + h(t, u(t)) = 0,\ t \in (0,1), \tag{10.0.1}$$

subject to (local) Neumann boundary conditions

$$u'(0) = u'(1) = 0, \tag{10.0.2}$$

or to nonlocal boundary conditions of Neumann type

$$u'(0) = \alpha[u], \quad u'(1) = \beta[u], \tag{10.0.3}$$

where $\alpha[\cdot]$, $\beta[\cdot]$ are linear functionals given by Stieltjes integrals, namely

$$\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s), \quad \beta[u] = \int_0^1 u(s)\,\mathrm{d}B(s).$$

The local boundary value problem (10.0.1)–(10.0.2) has been studied by Miciano and Shivaji in [133], where the authors proved the existence of multiple positive solutions, by means of the quadrature technique; using Morse theory, Li [124] proved the existence of positive solutions and Li and co-authors [125] continued the study of [124] and proved the existence of multiple solutions. Multiple positive solutions were also investigated by Boscaggin [19] via shooting-type arguments.

Note that, since $\lambda = 0$ is an eigenvalue of the associated linear problem

$$u''(t) + \lambda u(t) = 0, \quad u'(0) = u'(1) = 0,$$

the corresponding Green's function does not exist. Therefore we use a shift argument similar to the ones in [85, 167, 184] and previous chapters and we study two related boundary value problems for which the Green's function can be constructed, namely

$$-u''(t) - \omega^2 u(t) = f(t, u(t)) := h(t, u(t)) - \omega^2 u(t), \quad u'(0) = u'(1) = 0, \tag{10.0.4}$$

and (with an abuse of notation)

$$-u''(t) + \omega^2 u(t) = f(t, u(t)) := h(t, u(t)) + \omega^2 u(t), \quad u'(0) = u'(1) = 0. \tag{10.0.5}$$

The boundary value problems (10.0.4) and (10.0.5) have been recently object of interest by a number of authors, see for example [18, 59, 67, 159, 160, 170–172, 194, 195, 197–199]; in Section 10.4 we study in details the properties of the associated Green's functions and we improve and complement some estimates that occur in earlier papers, see Remark 10.4.2.

As we have mentioned in Chapter 9 The formulation of the nonlocal boundary conditions in terms of linear functionals is fairly general and includes, as special cases, multi-point and integral conditions, namely

$$\alpha[u] = \sum_{j=1}^{m} \alpha_j u(\eta_j) \quad \text{or} \quad \alpha[u] = \int_0^1 \phi(s) u(s)\,\mathrm{d}s.$$

One motivation for studying nonlocal problems in the context of Neumann problems is that they occur naturally when modeling heat-flow problems.

For example, the four point boundary value problem

$$u''(t)+h(t,u(t))=0, \quad u'(0)=\alpha u(\xi),\ u'(1)=\beta u(\eta),\ \xi,\eta\in[0,1],$$

(an specific case of the one studied in Chapter 9) models a thermostat where two controllers at $t=0$ and $t=1$ add or remove heat according to the temperatures detected by two sensors at $t=\xi$ and $t=\eta$. In particular Webb [179] studied the existence of *positive* solutions of the boundary value problem

$$u''(t)+h(t,u(t))=0, \quad u'(0)=\alpha[u],\ u'(1)=-\beta[u].$$

The methodology in [179] is somewhat different from ours and relies on a careful rewriting of the associated Green's function, due to the presence of the term $-\beta[u]$ in the boundary conditions. The existence of solutions that *change sign* have been investigated by Fan and Ma [66], in the case of the boundary value problem

$$u''(t)+h(t,u(t))=0, \quad u'(0)=\alpha u(\xi),\ u'(1)=-\beta u(\eta),\ \xi,\eta\in[0,1],$$

and in [30, 94, 100] for the boundary value problem

$$u''(t)+h(t,u(t))=0, \quad u'(0)=-\alpha[u],\ u'(1)=-\beta u(\eta),\ \eta\in[0,1].$$

A common feature of the papers [30, 66, 94, 100] is that a direct construction of a Green's function is possible due to the term $-\beta u(\eta)$.

In Section 10.1 we develop a fairly general theory for the existence and multiplicity of nontrivial solutions of the perturbed Hammerstein integral equation of the form

$$u(t)=\gamma(t)\alpha[u]+\delta(t)\beta[u]+\int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \qquad (10.0.6)$$

that covers, as *special cases*, the boundary value problem (10.0.1)–(10.0.3) and the boundary value problem (10.0.1)–(10.0.2) –in this last case, when α and β are the trivial functionals. We recall that the existence of *positive solutions* of this type of integral equations has been investigated by Webb and Infante in [180], under a nonnegativity assumption on the terms γ,δ,k, by working on a suitable cone of positive functions that takes into account the functionals α,β.

In Section 10.2 we provide some sufficient conditions on the nonlinearity f for the nonexistence of solutions of the equation (10.0.6), this is achieved via an associated Hammerstein integral equation

$$u(t)=\int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s,$$

whose kernel k_S is allowed to change sign and is constructed in the line of [180], where the authors dealt with positive kernels.

In Section 10.3 we provide a number of results that link the existence of nontrivial solutions of the equation (10.0.6) with the spectral radius of some associated linear integral operators. The main tool here is the celebrated Krein-Rutman Theorem, combined with some ideas from the paper of Webb and Lan [183]; here due to the nonconstant sign of the Green's function the situation is more delicate than the one in [183] and we introduce a number of different linear operators that yield different growth restrictions on the nonlinearity f.

In Section 10.5 we illustrate the applicability of our theory in three examples, two of which deal with solutions that change sign. The third example is taken from an interesting paper by Bonanno and Pizzimenti [18], where the authors proved the existence, with respect to the parameter λ, of positive solutions of the following boundary value problem

$$-u''(t) + u(t) = \lambda t e^{u(t)}, \quad u'(0) = u'(1) = 0.$$

The methodology used in [18] relies on a critical point Theorem of Bonanno [17]. Here we enlarge the range of the parameters and provide a sharper localization result. We also prove a nonexistence result for this boundary value problem.

Our results complement the ones of [180], focusing the attention on the existence of solutions that are allowed to *change sign*, in the spirit of the earlier works [97, 98, 100]. The approach that we use is topological, relies on classical fixed point index theory and we make use of ideas from the papers [30, 98, 178, 180, 183]. The results in this Chapter were published in [96].

10.1 Nonzero solutions of perturbed Hammerstein integral equations

In this Section we study the existence of solutions of the perturbed Hammerstein equations of the type

$$u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s := Tu(t), \tag{10.1.1}$$

where

$$\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s), \;\; \beta[u] = \int_0^1 u(s)\,\mathrm{d}B(s),$$

and A and B are functions of bounded variation. If we set

$$Fu(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s$$

we can write

$$Tu(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + Fu(t),$$

that is, we consider T as a perturbation of the simpler operator F.

We work in the space $C([0,1])$ of the continuous functions on $[0,1]$ endowed with the usual supremum norm.

We make the following assumptions on the terms that occur in (10.1.1).

(C_1) $k : [0,1] \times [0,1] \to \mathbb{R}$ is measurable, and for every $\tau \in [0,1]$ we have

$$\lim_{t\to\tau} |k(t,s) - k(\tau,s)| = 0 \text{ for almost every } s \in [0,1].$$

(C_2) There exist a subinterval $[a,b] \subseteq [0,1]$, a function $\Phi \in \mathrm{L}^\infty([0,1])$, and a constant $c_1 \in (0,1]$ such that

$$|k(t,s)| \le \Phi(s) \text{ for } t \in [0,1] \text{ and almost every } s \in [0,1],$$
$$k(t,s) \ge c_1 \Phi(s) \text{ for } t \in [a,b] \text{ and almost every } s \in [0,1].$$

(C_3) g is measurable, $g\Phi \in \mathrm{L}^1([0,1])$, $g(s) \ge 0$ for almost every $s \in [0,1]$, and $\int_a^b \Phi(s) g(s)\,\mathrm{d}s > 0$.

(C_4) The nonlinearity $f : [0,1] \times (-\infty,\infty) \to [0,\infty)$ satisfies L^∞-Carathéodory conditions, that is, $f(\cdot,u)$ is measurable for each fixed $u \in (-\infty,\infty)$, $f(t,\cdot)$ is continuous for almost every $t \in [0,1]$, and for each $r > 0$, there exists $\phi_r \in \mathrm{L}^\infty([0,1])$ such that

$$f(t,u) \le \phi_r(t) \quad \text{for all } \ u \in [-r,r], \ \text{ and almost every } \ t \in [0,1].$$

(C_5) A, B are functions of bounded variation and $\mathcal{K}_A(s),\ \mathcal{K}_B(s) \ge 0$ for almost every $s \in [0,1]$, where

$$\mathcal{K}_A(s) := \int_0^1 k(t,s)\,\mathrm{d}A(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 k(t,s)\,\mathrm{d}B(t).$$

(C_6) $\gamma \in C[0,1],\ 0 \le \alpha[\gamma] < 1,\ \ \beta[\gamma] \ge 0.$
There exists $c_2 \in (0,1]$ such that $\gamma(t) \ge c_2\|\gamma\|$ for $t \in [a,b]$.

(C_7) $\delta \in C[0,1],\ 0 \le \beta[\delta] < 1,\ \ \alpha[\delta] \ge 0.$
There exists $c_3 \in (0,1]$ such that $\delta(t) \ge c_3\|\delta\|$ for $t \in [a,b]$.

(C_8) $D := (1-\alpha[\gamma])(1-\beta[\delta]) - \alpha[\delta]\beta[\gamma] > 0.$

From (C_6)-(C_8) it follows that, for $\lambda \ge 1$,

$$D_\lambda := (\lambda-\alpha[\gamma])(\lambda-\beta[\delta]) - \alpha[\delta]\beta[\gamma] \ge D > 0.$$

The assumptions above allow us to work in the cone

$$K := \{u \in C[0,1] : \min_{t\in[a,b]} u(t) \ge c\|u\|,\ \alpha[u], \beta[u] \ge 0\}$$

where $c = \min\{c_1, c_2, c_3\}$.

The cone K allows the use of signed measures, taking into account two functionals.

We denote by P the cone of positive functions

$$P := \{u \in C[0,1] : u(t) \ge 0, t \in [0,1]\}.$$

First of all we prove that T leaves K invariant and is compact and continuous.

Lemma 10.1.1. *The operator* (10.1.1) *maps K into K and is compact and continuous.*

Proof. Let $u \in K$. First of all, we observe that $Tu(t) \geq 0$ for $t \in [a,b]$. We have, for $t \in [0,1]$,

$$|Tu(t)| \leq |\gamma(t)|\alpha[u] + |\delta(t)|\beta[u] + \int_0^1 |k(t,s)|g(s)f(s,u(s))\,\mathrm{d}s,$$

therefore, taking the supremum on $t \in [0,1]$, we get

$$\|Tu\| \leq \|\gamma\|\alpha[u] + \|\delta\|\beta[u] + \int_0^1 \Phi(s)g(s)f(s,u(s))\,\mathrm{d}s,$$

and, combining this fact with (C_2), (C_6) and (C_7),

$$\begin{aligned}\min_{t\in[a,b]} Tu(t) &\geq c_2\|\gamma\|\alpha[u] + c_3\|\delta\|\beta[u] + c_1\int_0^1 \Phi(s)g(s)f(s,u(s))\,\mathrm{d}s\\ &\geq c\|Tu\|.\end{aligned}$$

Furthermore, by (C_3) and (C_5)-(C_7),

$$\alpha[Tu] = \alpha[\gamma]\alpha[u] + \alpha[\delta]\beta[u] + \int_0^1 \mathcal{K}_A(s)g(s)f(s,u(s))\,\mathrm{d}s \geq 0$$

and

$$\beta[Tu] = \beta[\gamma]\alpha[u] + \beta[\delta]\beta[u] + \int_0^1 \mathcal{K}_B(s)g(s)f(s,u(s))\,\mathrm{d}s \geq 0.$$

Hence we have $Tu \in K$.

The compactness and continuity are derived from Lemma 8.1.4. ■

For $\rho > 0$ we recall the following open subsets of K:

$$K_\rho := \{u \in K : \|u\| < \rho\}, \quad V_\rho := \{u \in K : \min_{t\in[a,b]} u(t) < \rho\}.$$

We have $K_\rho \subset V_\rho \subset K_{\rho/c}$.

We state now some useful facts concerning real 2×2 matrices.

Definition 10.1.2. [180] A 2×2 matrix Q is said to be order preserving (or nonnegative) if $p_1 \geq p_0, q_1 \geq q_0$ imply

$$Q\begin{pmatrix}p_1\\ q_1\end{pmatrix} \geq Q\begin{pmatrix}p_0\\ q_0\end{pmatrix},$$

in the sense of components.

We have the following property, as stated in [180], whose proof is straightforward.

Lemma 10.1.3. *Let*

$$Q = \begin{pmatrix} a & -b\\ -c & d\end{pmatrix}$$

with $a,b,c,d \geq 0$ and $\det Q > 0$. Then Q^{-1} is order preserving.

Remark 10.1.4. It is a consequence of Lemma 10.1.3 that if

$$\mathcal{N} = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix},$$

satisfies the hypotheses of Lemma 10.1.3, $p \geq 0, q \geq 0$ and $\mu > 1$ then

$$\mathcal{N}_\mu^{-1} \begin{pmatrix} p \\ q \end{pmatrix} \leq \mathcal{N}^{-1} \begin{pmatrix} p \\ q \end{pmatrix},$$

where

$$\mathcal{N}_\mu = \begin{pmatrix} \mu-a & -b \\ -c & \mu-d \end{pmatrix}.$$

We now give a sufficient condition on the growth of the nonlinearity that provides that the index is 1 on K_ρ.

Lemma 10.1.5. *Assume that*

(I_ρ^1) *there exists* $\rho > 0$ *such that*

$$\begin{aligned} f^{-\rho,\rho} \Bigg(\sup_{t\in[0,1]} \Bigg\{ \Bigg(\frac{|\gamma(t)|}{D}(1-\beta[\delta]) + \frac{|\delta(t)|}{D}\beta[\gamma] \Bigg) \int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s \\ + \Bigg(\frac{|\gamma(t)|}{D}\alpha[\delta] + \frac{|\delta(t)|}{D}(1-\alpha[\gamma]) \Bigg) \int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s \\ + \max\Big\{ \int_0^1 k^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k^-(t,s)g(s)\,\mathrm{d}s \Big\}\Bigg\}\Bigg) < 1. \end{aligned} \tag{10.1.2}$$

where

$$f^{-\rho,\rho} := \operatorname{ess\,sup}\Big\{ \frac{f(t,u)}{\rho} : \ (t,u) \in [0,1]\times[-\rho,\rho] \Big\}.$$

Then we have $i_K(T,K_\rho) = 1$.

Proof. We show that $Tu \neq \lambda u$ for all $\lambda \geq 1$ when $u \in \partial K_\rho$, which implies that $i_K(T,K_\rho) = 1$. In fact, if this does not happen, then there exist u with $\|u\| = \rho$ and $\lambda \geq 1$ such that $\lambda u(t) = Tu(t)$, that is

$$\lambda u(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + Fu(t). \tag{10.1.3}$$

Therefore we obtain

$$\lambda\alpha[u] = \alpha[\gamma]\alpha[u] + \alpha[\delta]\beta[u] + \alpha[Fu]$$

and

$$\lambda\beta[u] = \beta[\gamma]\alpha[u] + \beta[\delta]\beta[u] + \beta[Fu].$$

Thus we have

$$\begin{pmatrix} \lambda-\alpha[\gamma] & -\alpha[\delta] \\ -\beta[\gamma] & \lambda-\beta[\delta] \end{pmatrix} \begin{pmatrix} \alpha[u] \\ \beta[u] \end{pmatrix} = \begin{pmatrix} \alpha[Fu] \\ \beta[Fu] \end{pmatrix}. \tag{10.1.4}$$

Note that the matrix that occurs in (10.1.4), due to (C_6)–(C_8), satisfies the hypothesis of Lemma 10.1.3, so its inverse is order preserving. Then, applying its inverse matrix to both sides of (10.1.4), we have

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} = \frac{1}{D_\lambda}\begin{pmatrix}\lambda-\beta[\delta] & \alpha[\delta]\\ \beta[\gamma] & \lambda-\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}.$$

By Remark 10.1.4, we obtain that

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} \le \frac{1}{D}\begin{pmatrix}1-\beta[\delta] & \alpha[\delta]\\ \beta[\gamma] & 1-\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}. \tag{10.1.5}$$

Hence, from (10.1.3) and (10.1.5) we get

$$\begin{aligned}\lambda|u(t)| \le{}& \frac{|\gamma(t)|}{D}((1-\beta[\delta])\alpha[Fu]+\alpha[\delta]\beta[Fu])\\ &+\frac{|\delta(t)|}{D}((1-\alpha[\gamma])\beta[Fu])+\beta[\gamma]\alpha[Fu])+|Fu(t)|.\end{aligned}$$

Taking the supremum over $[0,1]$ gives

$$\begin{aligned}\lambda\rho \le \rho f^{-\rho,\rho}\Bigg(\sup_{t\in[0,1]}\Bigg\{&\left(\frac{|\gamma(t)|}{D}(1-\beta[\delta])+\frac{|\delta(t)|}{D}\beta[\gamma]\right)\int_0^1\mathcal{K}_A(s)g(s)\,\mathrm{d}s\\ &+\left(\frac{|\gamma(t)|}{D}\alpha[\delta]+\frac{|\delta(t)|}{D}(1-\alpha[\gamma])\right)\int_0^1\mathcal{K}_B(s)g(s)\,\mathrm{d}s\\ &+\max\left\{\int_0^1 k^+(t,s)g(s)\,\mathrm{d}s,\int_0^1 k^-(t,s)g(s)\,\mathrm{d}s\right\}\Bigg\}\Bigg).\end{aligned}$$

From (10.1.2) we obtain that $\lambda\rho<\rho$, contradicting the fact that $\lambda\ge 1$. ■

Remark 10.1.6. In similar way as in [180] (where the positive case was studied) we point out that a stronger (but easier to check) condition than (I_ρ^1) is given by the following.

$$\begin{aligned}f^{-\rho,\rho}\Bigg[&\left(\frac{\|\gamma\|}{D}(1-\beta[\delta])+\frac{\|\delta\|}{D}\beta[\gamma]\right)\int_0^1\mathcal{K}_A(s)g(s)\,\mathrm{d}s\\ &+\left(\frac{\|\gamma\|}{D}\alpha[\delta]+\frac{\|\delta\|}{D}(1-\alpha[\gamma])\right)\int_0^1\mathcal{K}_B(s)g(s)\,\mathrm{d}s+\frac{1}{m}\Bigg]<1.\end{aligned}$$

where

$$\frac{1}{m} := \sup_{t\in[0,1]}\left\{\max\left\{\int_0^1 k^+(t,s)g(s)\,\mathrm{d}s,\int_0^1 k^-(t,s)g(s)\,\mathrm{d}s\right\}\right\}.$$

Note that, since $\max\{k^+,k^-\}\le|k|$, the constant m provides a better estimate on the growth of the nonlinearity f than the constant

$$\sup_{t\in[0,1]}\int_0^1|k(t,s)|g(s)\,\mathrm{d}s,$$

used in [30, 34, 66, 69, 70, 87, 93, 94, 97, 99, 100, 134].

Remark 10.1.7. If the functions γ, δ, k are nonnegative on $[0,1]$, we can work within the cone $K \cap P$, regaining the condition given in [180], namely

$$f^{0,\rho}\Big(\sup_{t\in[0,1]}\Big\{\Big(\frac{\gamma(t)}{D}(1-\beta[\delta])+\frac{\delta(t)}{D}\beta[\gamma]\Big)\int_0^1 \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \Big(\frac{\gamma(t)}{D}\alpha[\delta]+\frac{\delta(t)}{D}(1-\alpha[\gamma])\Big)\int_0^1 \mathcal{K}_B(s)g(s)\,\mathrm{d}s+\int_0^1 k(t,s)g(s)\,\mathrm{d}s\Big\}\Big)<1,$$

where

$$f^{0,\rho} := \operatorname{ess\,sup}\Big\{\frac{f(t,u)}{\rho}:\ (t,u)\in[0,1]\times[0,\rho]\Big\}.$$

Lemma 10.1.8. *Assume that*

(I^0_ρ) *There exists $\rho>0$ such that*

$$f_{\rho,\rho/c}\left(\inf_{t\in[a,b]}\left\{\left(\frac{\gamma(t)}{D}(1-\beta[\delta])+\frac{\delta(t)}{D}\beta[\gamma]\right)\int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s + \left(\frac{\gamma(t)}{D}\alpha[\delta]+\frac{\delta(t)}{D}(1-\alpha[\gamma])\right)\int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s+\int_a^b k(t,s)g(s)\,\mathrm{d}s\right\}\right)>1, \tag{10.1.6}$$

where

$$f_{\rho,\rho/c} := \operatorname{ess\,inf}\left\{\frac{f(t,u)}{\rho}:\ (t,u)\in[a,b]\times[\rho,\rho/c]\right\}.$$

Then we have $i_K(T,V_\rho)=0$.

Proof. Let $e(t)=\int_0^1 k(t,s)\,\mathrm{d}s$ for $t\in[0,1]$. Then, according to (C_2), (C_3) and (C_5), we have $e\in K\setminus\{0\}$. We show that $u\neq Tu+\lambda e$ for all $\lambda\geq 0$ and $u\in\partial V_\rho$ which implies that $i_K(T,V_\rho)=0$. In fact, if this does not happen, there are $u\in\partial V_\rho$ (and so for $t\in[a,b]$ we have $\min u(t)=\rho$ and $\rho\leq u(t)\leq\rho/c$), and $\lambda\geq 0$ with

$$u(t)=Tu(t)+\lambda e(t)=\gamma(t)\alpha[u]+\delta(t)\beta[u]+Fu(t)+\lambda e(t).$$

Applying α and β to both sides of the previous equation we get

$$\begin{pmatrix}1-\alpha[\gamma] & -\alpha[\delta]\\ -\beta[\gamma] & 1-\beta[\delta]\end{pmatrix}\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}=\begin{pmatrix}\alpha[Fu]+\lambda\alpha[e]\\ \beta[Fu]+\lambda\beta[e]\end{pmatrix}\geq\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}. \tag{10.1.7}$$

Note that the matrix that occurs in (10.1.7), due to (C_6)-(C_8), satisfies the hypothesis of Lemmas 10.1.3, so its inverse is order preserving. Then applying the inverse matrix to both sides of (10.1.7) we have

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}\geq\frac{1}{D}\begin{pmatrix}1-\beta[\delta] & \alpha[\delta]\\ \beta[\gamma] & 1-\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}.$$

Therefore, for $t \in [a,b]$, we obtain

$$\begin{aligned} u(t) \geq & \left(\frac{\gamma(t)}{D}(1-\beta[\delta]) + \frac{\delta(t)}{D}\beta[\gamma] \right) \alpha[Fu] \\ & + \left(\frac{\gamma(t)}{D}\alpha[\delta] + \frac{\delta(t)}{D}(1-\alpha[\gamma]) \right) \beta[Fu] + Fu(t) + \lambda e(t) \\ \geq & \left(\frac{\gamma(t)}{D}(1-\beta[\delta]) + \frac{\delta(t)}{D}\beta[\gamma] \right) \int_a^b \mathcal{K}_A(s)g(s)f(s,u(s))\,\mathrm{d}s \\ & + \left(\frac{\gamma(t)}{D}\alpha[\delta] + \frac{\delta(t)}{D}(1-\alpha[\gamma]) \right) \int_a^b \mathcal{K}_B(s)g(s)f(s,u(s))\,\mathrm{d}s \\ & + \int_a^b k(t,s)g(s)f(s,u(s))\,\mathrm{d}s. \end{aligned}$$

Taking the infimum for $t \in [a,b]$ then gives

$$\begin{aligned} & \rho = \min u(t) \\ & \geq \rho f_{\rho,\rho/c} \left(\inf_{t\in[a,b]} \left\{ \left(\frac{\gamma(t)}{D}(1-\beta[\delta]) + \frac{\delta(t)}{D}\beta[\gamma] \right) \int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s \right.\right. \\ & \left.\left. + \left(\frac{\gamma(t)}{D}\alpha[\delta] + \frac{\delta(t)}{D}(1-\alpha[\gamma]) \right) \int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \int_a^b k(t,s)g(s)\,\mathrm{d}s \right\} \right), \end{aligned}$$

contradicting (10.1.6). ■

Remark 10.1.9. We point out, in similar way as in [180], that a stronger (but easier to check) condition than (I^0_ρ) is given by the following.

$$\begin{aligned} f_{\rho,\rho/c} \Big(\Big(& \frac{c_2\|\gamma\|}{D}(1-\beta[\delta]) + \frac{c_3\|\delta\|}{D}\beta[\gamma] \Big) \int_a^b \mathcal{K}_A(s)g(s)\,\mathrm{d}s \\ & + \Big(\Big(\frac{c_2\|\gamma\|}{D}\alpha[\delta] + \frac{c_3\|\delta\|}{D}(1-\alpha[\gamma]) \Big) \int_a^b \mathcal{K}_B(s)g(s)\,\mathrm{d}s + \frac{1}{M(a,b)} \Big) > 1, \end{aligned}$$

where

$$\frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,\mathrm{d}s.$$

We now combine the results above in order to prove a Theorem regarding the existence of one, two or three nontrivial solutions. The proof is a direct consequence of the properties of the fixed point index and is omitted. It is possible to state a result for the existence of four or more solutions, we refer to Lan [123] for similar statements.

Theorem 10.1.10. *Assume conditions (C_1)–(C_8) are satisfied. The integral equation* (10.1.1) *has at least one nonzero solution in K if any of the following conditions hold.*

(S_1) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1/c < \rho_2$ such that $(\mathrm{I}^0_{\rho_1})$ and $(\mathrm{I}^1_{\rho_2})$ hold.*

(S_2) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $(\mathrm{I}^1_{\rho_1})$ and $(\mathrm{I}^0_{\rho_2})$ hold.*

The integral equation (10.1.1) *has at least two nonzero solutions in* K *if one of the following conditions hold.*

(S_3) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1/c < \rho_2 < \rho_3$ *such that* $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$ *and* $(\mathrm{I}^0_{\rho_3})$ *hold.*

(S_4) *There exist* $\rho_1, \rho_2, \rho_3 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *and* $\rho_2/c < \rho_3$ *such that* $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$ *and* $(\mathrm{I}^1_{\rho_3})$ *hold.*

The integral equation (10.1.1) *has at least three nonzero solutions in* K *if one of the following conditions hold.*

(S_5) *There exist* $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ *with* $\rho_1/c < \rho_2 < \rho_3$ *and* $\rho_3/c < \rho_4$ *such that* $(\mathrm{I}^0_{\rho_1})$, $(\mathrm{I}^1_{\rho_2})$, $(\mathrm{I}^0_{\rho_3})$ *and* $(\mathrm{I}^1_{\rho_4})$ *hold.*

(S_6) *There exist* $\rho_1, \rho_2, \rho_3, \rho_4 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *and* $\rho_2/c < \rho_3 < \rho_4$ *such that* $(\mathrm{I}^1_{\rho_1})$, $(\mathrm{I}^0_{\rho_2})$, $(\mathrm{I}^1_{\rho_3})$ *and* $(\mathrm{I}^0_{\rho_4})$ *hold.*

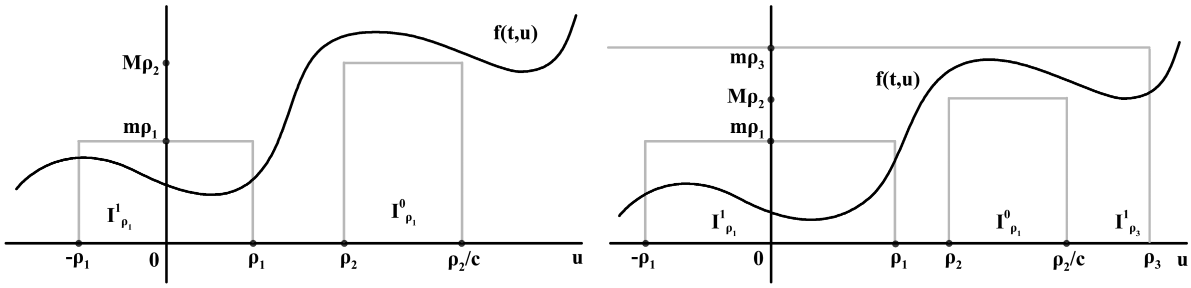


Figure 10.1.1: Illustration of conditions (S_2) (left) and (S_3) (right).

10.2 Some nonexistence results

We now consider the auxiliary Hammerstein integral equation

$$u(t) = \int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s := Su(t), \tag{10.2.1}$$

where the kernel k_S is given by the formula

$$\begin{aligned} k_S(t,s) =& \frac{\gamma(t)}{D}\left[(1-\beta[\delta])\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)\right] \\ &+ \frac{\delta(t)}{D}\left[\beta[\gamma]\mathcal{K}_A(s) + (1-\alpha[\gamma])\mathcal{K}_B(s)\right] + k(t,s). \end{aligned}$$

The operator S shares a number of useful properties with T, firstly the cone invariance, continuity and compactness, the proof follows directly from (C_1)-(C_8) and is omitted.

Lemma 10.2.1. *The operator S defined in* (10.2.1) *maps K into K and is compact and continuous.*

A key property that is also useful is the one given by the following Theorem; the proof is similar to the one in [180, Lemma 2.8 and Therem 2.9] and is omitted.

Lemma 10.2.2. *The operators S and T have the same fixed points in K. Furthermore if $u \neq Tu$ for $u \in \partial D_K$, then $i_K(T, D_K) = i_K(S, D_K)$.*

We define the constants

$$\frac{1}{m_S} := \sup_{t\in[0,1]} \left\{ \max \left\{ \int_0^1 k_S^+(t,s)g(s)\,\mathrm{d}s, \int_0^1 k_S^-(t,s)g(s)\,\mathrm{d}s \right\} \right\},$$

$$\frac{1}{M_S(a,b)} = \frac{1}{M_S} := \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)\,\mathrm{d}s,$$

and we prove the following nonexistence results.

Theorem 10.2.3. *Assume that one of the following conditions holds:*

(1) *$f(t,u) < m_S|u|$ for every $t \in [0,1]$ and $u \in \mathbb{R}\backslash\{0\}$,*

(2) *$f(t,u) > M_S u$ for every $t \in [a,b]$ and $u \in \mathbb{R}^+$.*

Then the equations (10.1.1) *and* (10.2.1) *have no nontrivial solution in K.*

Proof. In view of Lemma 10.2.2 we prove the Theorem using the operator S.

(1) Assume, on the contrary, that there exists $u \in K$, $u \not\equiv 0$ such that $u = Su$ and let $t_0 \in [0,1]$ such that $\|u\| = |u(t_0)|$. Then we have

$$\begin{aligned}
\|u\| =& |u(t_0)| = \left| \int_0^1 k_S(t_0,s)g(s)f(s,u(s))\,\mathrm{d}s \right| \\
&\le \max \left\{ \int_0^1 k_S^+(t_0,s)g(s)f(s,u(s))\,\mathrm{d}s, \int_0^1 k_S^-(t_0,s)g(s)f(s,u(s))\,\mathrm{d}s \right\} \\
&< \max \left\{ \int_0^1 k_S^+(t_0,s)g(s)m_S|u(s)|\,\mathrm{d}s, \int_0^1 k_S^-(t_0,s)g(s)m_S|u(s)|\,\mathrm{d}s \right\} \\
&\le \max \left\{ \int_0^1 k_S^+(t_0,s)g(s)\,\mathrm{d}s, \int_0^1 k_S^-(t_0,s)g(s)\,\mathrm{d}s \right\} m_S\|u\| \le \|u\|,
\end{aligned}$$

a contradiction.

(2) Assume, on the contrary, that there exists $u \in K$, $u \not\equiv 0$ such that $u = Su$ and let $\eta \in [a,b]$ be such that $u(\eta) = \min_{t\in[a,b]} u(t)$. For $t \in [a,b]$ we have

$$\begin{aligned}
u(t) =& \int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s \ge \int_a^b k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s \\
>& M_S \int_a^b k_S(t,s)g(s)u(s)\,\mathrm{d}s.
\end{aligned}$$

Taking the infimum for $t \in [a,b]$, we have

$$\min_{t\in[a,b]} u(t) > M_S \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)u(s)\,\mathrm{d}s.$$

Thus we obtain

$$u(\eta) > M_S u(\eta) \inf_{t\in[a,b]} \int_a^b k_S(t,s)g(s)\,\mathrm{d}s = u(\eta),$$

a contradiction. ■

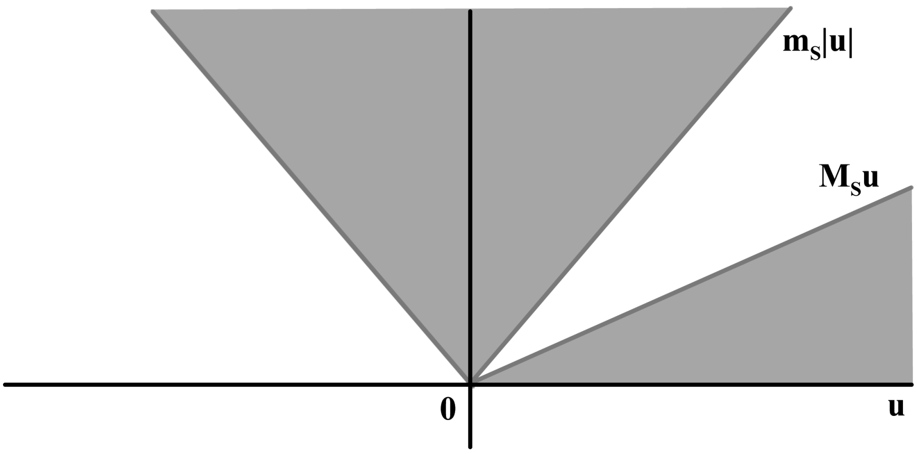


Figure 10.2.1: Illustration of the conditions in Theorem 10.2.3. f cannot intersect one of the shaded areas in each case.

10.3 Eigenvalue criteria for the existence of nontrivial solutions

In this Section we assume the additional hypothesis that the functionals α and β are given by *positive measures*.

In order to state our eigenvalue comparison results, we consider the following operators on $C[0,1]$.

$$Lu(t) := \int_0^1 |k_S(t,s)|g(s)u(s)\,\mathrm{d}s, \quad \tilde{L}u(t) := \int_a^b k_S^+(t,s)g(s)u(s)\,\mathrm{d}s.$$

By similar proofs of [180, Lemma 2.6 and Theorem 2.7], we study the properties of those operators.

Theorem 10.3.1. *Assume properties (C_1)–(C_8) hold. The operators L and $\tilde{L}$ are compact and continuous and map P into $P \cap K$.*

Proof. Note that the operators L and $\tilde{L}$ map P into P (because they have a positive integral kernel) and are compact. We now show that they map P into $P \cap K$. We do this for the operator L, a similar proof works for $\tilde{L}$.

Firstly we observe that

$$\begin{aligned}
|k_S(t,s)| \leq & \frac{|\gamma(t)|}{D}((1-\beta[\delta])\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)) + \frac{|\delta(t)|}{D}(\beta[\gamma]\mathcal{K}_A(s) \\
& + (1-\alpha[\gamma])\mathcal{K}_B(s)) + |k(t,s)| \\
\leq & \frac{\|\gamma\|}{D}((1-\beta[\delta])\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)) + \frac{\|\delta\|}{D}(\beta[\gamma]\mathcal{K}_A(s) \\
& + (1-\alpha[\gamma])\mathcal{K}_B(s)) + |k(t,s)| \\
\leq & \Upsilon(s) + \Phi(s) =: \Psi(s),
\end{aligned}$$

where

$$\Upsilon(s) = \frac{\|\gamma\|}{D}\left((1-\beta[\delta])\mathcal{K}_A(s) + \alpha[\delta]\mathcal{K}_B(s)\right)$$

$$+\frac{\|\delta\|}{D}\left(\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)\right).$$

Moreover, we have, for $t\in[a,b]$,

$$\begin{aligned}|k_S(t,s)|=k_S(t,s)\geq&\frac{c_2\|\gamma\|}{D}\left[(1-\beta[\delta])\mathcal{K}_A(s)+\alpha[\delta]\mathcal{K}_B(s)\right]\\&+\frac{c_3\|\delta\|}{D}\left[\beta[\gamma]\mathcal{K}_A(s)+(1-\alpha[\gamma])\mathcal{K}_B(s)\right]+c_1\Phi(t)\geq c\Psi(s),\end{aligned}$$

and thus

$$\min_{t\in[a,b]}k_S(t,s)\geq c\Psi(s). \tag{10.3.1}$$

Also we have $g\,\Psi\in \mathrm{L}^1([0,1])$ and we obtain that, for $u\in P$ and $t\in[0,1]$,

$$Lu(t)\leq\int_0^1\Psi(s)g(s)u(s)\,\mathrm{d}s,$$

in such a way that, taking the supremum on $t\in[0,1]$, we get

$$\|Lu\|\leq\int_0^1\Psi(s)g(s)u(s)\,\mathrm{d}s.$$

On the other hand,

$$\min_{t\in[a,b]}Lu(t)\geq c\int_0^1\Psi(s)g(s)u(s)\,\mathrm{d}s\geq c\|Lu\|.$$

Furthermore, since α and β are given by positive measures,

$$\alpha[Lu]=\int_0^1\int_0^1|k_S(t,s)|g(s)u(s)\,\mathrm{d}s\,\mathrm{d}A(t)\geq 0$$

and

$$\beta[Lu]=\int_0^1\int_0^1|k_S(t,s)|g(s)u(s)\,\mathrm{d}s\,\mathrm{d}B(t)\geq 0.$$

Hence we have $Lu\in K$. ■

We recall that λ is an *eigenvalue* of a linear operator Γ with corresponding eigenfunction φ if $\varphi\neq 0$ and $\lambda\varphi=\Gamma\varphi$. The reciprocals of nonzero eigenvalues are called *characteristic values* of Γ. We will denote the *spectral radius* of Γ by $r(\Gamma):=\lim_{n\to\infty}\|\Gamma^n\|^{\frac{1}{n}}$ and its *principal characteristic value* (the reciprocal of the spectral radius) by $\mu(\Gamma)=1/r(\Gamma)$.

The following Theorem is analogous to the ones in [180,183] and is proven by using the facts that the considered operators leave P invariant, that P is reproducing, that is $C(I)=P-P$, combined with the well-known Krein-Rutman Theorem. The condition (C_3) is used to show that $r(L)>0$.

Theorem 10.3.2. *The spectral radius of L is nonzero and is an eigenvalue of L with an eigenfunction in P. A similar result holds for $\tilde{L}$.*

Remark 10.3.3. As a consequence of the two previous theorems, we have the above mentioned eigenfunction is in $P\cap K$.

We use the following operator on $\mathcal{C}([a,b])$ defined by, for $t \in [a,b]$,

$$\bar{L}u(t) := \int_a^b k_S^+(t,s)g(s)u(s)\,\mathrm{d}s$$

and the cone $P_{[a,b]}$ of positive functions in $\mathcal{C}([a,b])$.

In the recent papers [177, 178], Webb developed a theory valid for u_0-positive linear operators relative to two cones. It turns out that our operator $\bar{L}$ fits within this setting and, in particular, satisfies the assumptions of Theorem 3.4 of [178]. We state here a special case of Theorem 3.4 of [178] that can be used for $\bar{L}$.

Theorem 10.3.4. *Suppose that there exist $u \in P_{[a,b]}\backslash\{0\}$ and $\lambda > 0$ such that*

$$\lambda u(t) \geq \bar{L}u(t), \text{ for } t \in [a,b].$$

Then we have $r(\bar{L}) \leq \lambda$.

We define the following extended real numbers.

$$f^0 = \overline{\lim_{u\to 0}} \frac{\operatorname{ess}\sup_{t\in[0,1]} f(t,u)}{|u|}, \quad f_0 = \underline{\lim_{u\to 0^+}} \frac{\operatorname{ess}\inf_{t\in[a,b]} f(t,u)}{u},$$

$$f^\infty = \overline{\lim_{|u|\to +\infty}} \frac{\operatorname{ess}\sup_{t\in[0,1]} f(t,u)}{|u|}, \quad f_\infty = \underline{\lim_{u\to +\infty}} \frac{\operatorname{ess}\inf_{t\in[a,b]} f(t,u)}{u}.$$

In order to prove the following Theorem, we adapt some of the proofs of [183, Theorems 3.2-3.5] to this new context.

Theorem 10.3.5. *We have the following.*

(1) *If $0 \leq f^0 < \mu(L)$, then there exists $\rho_0 > 0$ such that $i_K(T,K_\rho) = 1$ for each $\rho \in (0,\rho_0]$.*

(2) *If $0 \leq f^\infty < \mu(L)$, then there exists $R_0 > 0$ such that $i_K(T,K_R) = 1$ for each $R > R_0$.*

(3) *If $\mu(\tilde{L}) < f_0 \leq \infty$, then there exists $\rho_0 > 0$ such that $i_K(T,K_\rho) = 0$ for each $\rho \in (0,\rho_0]$.*

(4) *If $\mu(\tilde{L}) < f_\infty \leq \infty$, then there exists $R_1 > 0$ such that $i_K(T,K_R) = 0$ for each $R \geq R_1$.*

Proof. We show the statements for the operator S instead of T, in view of Lemma 10.2.2.

(1) Let $\tau \in \mathbb{R}^+$ be such that $f^0 \leq \mu(L) - \tau$. Then there exists $\rho_0 \in (0,1)$ such that for all $u \in [-\rho_0,\rho_0]$ and almost every $t \in [0,1]$ we have

$$f(t,u) \leq (\mu(L) - \tau)|u|.$$

Let $\rho \in (0,\rho_0]$. We prove that $Su \neq \lambda u$ for $u \in \partial K_\rho$ and $\lambda \geq 1$, which implies $i_K(S,K_\rho) = 1$. In fact, if we assume otherwise, then there exist $u \in \partial K_\rho$ and $\lambda \geq 1$ such that $\lambda u = Su$. Therefore,

$$|u(t)| \leq \lambda|u(t)| = |Su(t)| = \left|\int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s\right|$$

$$\leq \int_0^1 |k_S(t,s)|g(s)f(s,u(s))\,\mathrm{d}s$$
$$\leq (\mu(L)-\tau)\int_0^1 |k_S(t,s)|g(s)|u(s)|\,\mathrm{d}s = (\mu(L)-\tau)L|u|(t).$$

Thus, we have, for $t \in [0,1]$,

$$|u|(t) \leq (\mu(L)-\tau)L[(\mu(L)-\tau)L|u|(t)]$$
$$=(\mu(L)-\tau)^2L^2|u|(t) \leq \cdots \leq (\mu(L)-\tau)^nL^n|u|(t),$$

thus, taking the norms, $1 \leq (\mu(L)-\tau)^n\|L^n\|$, and then

$$1 \leq (\mu(L)-\tau)\lim_{n\to\infty}\|L^n\|^{\frac{1}{n}} = \frac{\mu(L)-\tau}{\mu(L)} < 1,$$

a contradiction.

(2) Let $\tau \in \mathbb{R}^+$ such that $f^\infty < \mu(L)-\tau$. Then there exists $R_1 > 0$ such that for every $|u| \geq R_1$ and almost every $t \in [0,1]$

$$f(t,u) \leq (\mu(L)-\tau)|u|.$$

Also, by (C_4) there exists $\phi_{R_1} \in \mathrm{L}^\infty([0,1])$ such that $f(t,u) \leq \phi_{R_1}(t)$ for all $u \in [-R_1,R_1]$ and almost every $t \in [0,1]$. Hence,

$$f(t,u) \leq (\mu(L)-\tau)|u| + \phi_{R_1}(t) \text{ for all } u \in \mathbb{R} \text{ and almost every } t \in [0,1]. \quad (10.3.2)$$

Denote by Id the identity operator and observe that $\mathrm{Id}-(\mu(L)-\tau)L$ is invertible since $(\mu(L)-\tau)L$ has spectral radius less than one. Furthermore, by the Neumann series expression,

$$[\mathrm{Id}-(\mu(L)-\tau)L]^{-1} = \sum_{k=0}^\infty [(\mu(L)-\tau)L]^k$$

therefore, $[\mathrm{Id}-(\mu(L)-\tau)L]^{-1}$ maps P into P, since L does.

Let

$$C := \int_a^b \Phi(s)g(s)\phi_{R_1}(s)\,\mathrm{d}s \text{ and } R_0 := \|[\mathrm{Id}-(\mu(L)-\tau)L]^{-1}C\|.$$

Now we prove that for each $R > R_0$, $Su \neq \lambda u$ for all $u \in \partial K_R$ and $\lambda \geq 1$, which implies $i_K(S,K_R) = 1$. Assume otherwise: there exists $u \in \partial K_R$ and $\lambda \geq 1$ such that $\lambda u = Su$. Taking into account the inequality (10.3.2), we have for $t \in [0,1]$

$$|u(t)| \leq \lambda|u(t)| = |Su| = \left|\int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s\right| \leq \int_0^1 |k_S(t,s)|g(s)f(s,u(s))\,\mathrm{d}s$$
$$\leq (\mu(L)-\tau)\int_0^1 |k_S(t,s)|g(s)|u(s)|\,\mathrm{d}s + C = (\mu(L)-\tau)L|u|(t) + C,$$

which implies

$$[\mathrm{Id}-(\mu(L)-\tau)L]|u|(t) \leq C.$$

Since $(\mathrm{Id}-(\mu(L)-\tau)L)^{-1}$ is nonnegative, we have

$$|u|(t) \leq [\mathrm{Id}-(\mu(L)-\tau)L]^{-1}C \leq R_0.$$

Therefore, we have $\|u\| \le R_0 < R$, a contradiction.

(3) There exists $\rho_0 > 0$ such that for all $u \in [0, \rho_0]$ and all $t \in [a, b]$ we have

$$f(t, u) \ge \mu(\tilde{L})u.$$

Let $\rho \in (0, \rho_0]$. Let us prove that $u \ne Su + \lambda\varphi_1$ for all u in ∂K_ρ and $\lambda \ge 0$, where $\varphi_1 \in K \cap P$ is the eigenfunction of $\tilde{L}$ with $\|\varphi_1\| = 1$ corresponding to the eigenvalue $1/\mu(\tilde{L})$. This implies that $i_K(S, K_\rho) = 0$.

Assume, on the contrary, that there exist $u \in \partial K_\rho$ and $\lambda \ge 0$ such that $u = Su + \lambda\varphi_1$. We distinguish two cases.

Firstly we discuss the case $\lambda > 0$. We have, for $t \in [a, b]$,

$$\begin{aligned} u(t) &= \int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s + \lambda\varphi_1(t) \\ &\ge \int_a^b k_S^+(t,s)g(s)f(s,u(s))\,\mathrm{d}s + \lambda\varphi_1(t) \\ &\ge \mu(\tilde{L}) \int_a^b k_S^+(t,s)g(s)u(s)\,\mathrm{d}s + \lambda\varphi_1(t) = \mu(\tilde{L})\tilde{L}u(t) + \lambda\varphi_1(t). \end{aligned}$$

Moreover, we have $u(t) \ge \lambda\varphi_1(t)$ in $[a, b]$ and then $\tilde{L}u(t) \ge \lambda\tilde{L}\varphi_1(t) = \dfrac{\lambda}{\mu(\tilde{L})}\varphi_1(t)$ in $[a, b]$ in such a way that we obtain

$$u(t) \ge \mu(\tilde{L})\tilde{L}u(t) + \lambda\varphi_1(t) \ge 2\lambda\varphi_1(t), \text{ for } t \in [a, b].$$

By iteration, we deduce that, for $t \in [a, b]$, we get

$$u(t) \ge n\lambda\varphi_1(t) \text{ for every } n \in \mathbb{N},$$

a contradiction because $\|u\| = \rho$.

Now we consider the case $\lambda = 0$. Let $\varepsilon > 0$ be such that for all $u \in [0, \rho_0]$ and almost every $t \in [a, b]$ we have

$$f(t, u) \ge (\mu(\tilde{L}) + \varepsilon)u.$$

We have, for $t \in [a, b]$,

$$\begin{aligned} u(t) &= \int_0^1 k_S(t,s)g(s)f(s,u(s))\,\mathrm{d}s \\ &\ge \int_a^b k_S^+(t,s)g(s)f(s,u(s))\,\mathrm{d}s \ge (\mu(\tilde{L}) + \varepsilon)\tilde{L}u(t). \end{aligned}$$

Since $\tilde{L}\varphi_1(t) = r(\tilde{L})\varphi_1(t)$ for $t \in [0, 1]$, we have, for $t \in [a, b]$,

$$\bar{L}\varphi_1(t) = \tilde{L}\varphi_1(t) = r(\tilde{L})\varphi_1(t),$$

and we obtain $r(\bar{L}) \ge r(\tilde{L})$. On the other hand, we have, for $t \in [a, b]$,

$$u(t) \ge (\mu(\tilde{L}) + \varepsilon)\tilde{L}u(t) = (\mu(\tilde{L}) + \varepsilon)\bar{L}u(t),$$

where $u(t) > 0$ in $[a, b]$. Thus, using Theorem 10.3.4, we have

$$r(\bar{L}) \le \frac{1}{\mu(\tilde{L}) + \varepsilon} \text{ and, therefore, } r(\tilde{L}) \le \frac{1}{\mu(\tilde{L}) + \varepsilon}.$$

This gives $\mu(\tilde{L}) + \varepsilon \le \mu(\tilde{L})$, a contradiction.

(4) Let $R_1 > 0$ such that
$$f(t,u) > \mu(\tilde{L})u$$
for all $u \ge cR_1$, c as in (10.3.1), and all $t \in [a,b]$.
Let $R \ge R_1$. We prove that $u \neq Su + \lambda\varphi_1$ for all u in ∂K_R and $\lambda \ge 0$, which implies $i_K(S,K_R) = 0$.

Assume now, on the contrary, that there exist $u \in \partial K_R$ and $\lambda \ge 0$ such that $u = Su + \lambda\varphi_1$. Observe that for $u \in \partial K_R$, we have $u(t) \ge c\|u\| = cR \ge cR_1$ for $t \in [a,b]$. Hence, we have $f(t,u(t)) > \mu(\tilde{L})u(t)$ for $t \in [a,b]$. This implies, proceeding as in the proof of the statement (3) for the case $\lambda > 0$, that
$$u(t) \ge \mu(\tilde{L})\tilde{L}u(t) + \lambda\varphi_1(t) \ge 2\lambda\varphi_1(t), \text{ for } t \in [a,b].$$
Then, for $t \in [a,b]$, we have $u(t) \ge n\lambda\varphi_1(t)$ for every $n \in \mathbb{N}$, a contradiction because $\|u\| = R$. The proof in the case $\lambda = 0$ is treated as in the proof of the statement (3). ■

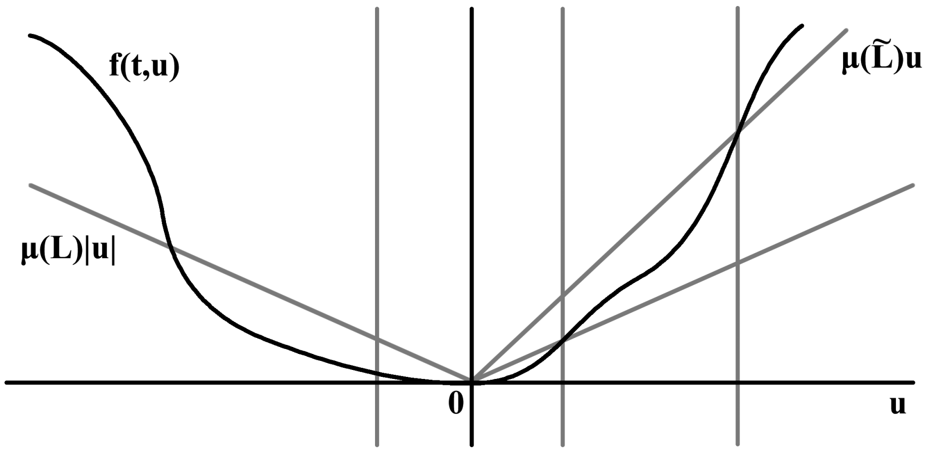


Figure 10.3.1: Illustration of conditions (1) and (4) of Theorem 10.3.5 being satisfied simultaneously.

The following Theorem, in the line of [180, 184], applies the index results in Lemmas 10.1.5 and 10.1.8 and Theorem 10.3.5 in order to get some results on existence of multiple nontrivial solutions for the equation (10.1.1).

Theorem 10.3.6. *Assume that conditions $(C1)$-$(C8)$ hold with α, β given by positive measures.*
The integral equation (10.1.1) *has at least one nontrivial solution in K if one of the following conditions holds.*

(H_1) $0 \le f^0 < \mu(L)$ *and* $\mu(\tilde{L}) < f_\infty \le \infty$.

(H_2) $0 \le f^\infty < \mu(L)$ *and* $\mu(\tilde{L}) < f_0 \le \infty$.

The integral equation (10.1.1) *has at least two nontrivial solutions in K if one of the following conditions holds.*

(Z_1) $0 \le f^0 < \mu(L)$, $f_{\rho,\rho/c} > M_S(a,b)$ *for some* $\rho > 0$, *and* $0 \le f^\infty < \mu(L)$.

(Z_2) $\mu(\tilde{L}) < f_0 \le \infty$, $f^{-\rho,\rho} < m_S$ *for some* $\rho > 0$, *and* $\mu(\tilde{L}) < f_\infty \le \infty$.

The integral equation (10.1.1) *has at least three nontrivial solutions in* K *if one of the following conditions holds.*

(T_1) *There exist* $0 < \rho_1 < \rho_2 < \infty$, *such that*

$$\mu(\tilde{L}) < f_0 \le \infty, \ \ f^{-\rho_1,\rho_1} < m_S, \ \ f_{\rho_2,\rho_2/c} > M_S(a,b), \ \ 0 \le f^\infty < \mu(L).$$

(T_2) *There exist* $0 < \rho_1 < c\rho_2 < \infty$, *such that*

$$0 \le f^0 < \mu(L), \ \ f_{\rho_1,\rho_1/c} > M_S(a,b), \ \ f^{-\rho_2,\rho_2} < m_S, \ \ \mu(\tilde{L}) < f_\infty \le \infty.$$

It is possible to give criteria for the existence of an arbitrary number of nontrivial solutions by extending the list of conditions. We omit the routine statement of such results.

The following Lemma sheds some light on the relation between some of these constants.

Lemma 10.3.7. *The following relations hold*

$$M_S(a,b) \ge \mu(\tilde{L}) \ge \mu(L) \ge m_S.$$

Proof. The fact that $\mu(L) \ge m_S$ essentially follows from Theorem 2.8 of [183]. The comment that follows after Theorem 3.4 of [183] also applies in our case, giving $\mu(\tilde{L}) \ge \mu(L)$.

We now prove $M_S(a,b) \ge \mu(\tilde{L})$. Let $\varphi \in P \cap K$ be a corresponding eigenfunction of norm 1 of $1/\mu(\tilde{L})$ for the operator $\tilde{L}$, that is $\varphi = \mu(\tilde{L})\tilde{L}(\varphi)$ and $\|\varphi\| = 1$. Then, for $t \in [a,b]$ we have

$$\varphi(t) = \mu(\tilde{L}) \int_a^b k_S(t,s) g(s) \varphi(s) \, \mathrm{d}s \ge \mu(\tilde{L}) \min_{t\in[a,b]} \varphi(t) \int_a^b k_S(t,s) g(s) \, \mathrm{d}s.$$

Taking the infimum over $[a,b]$, we obtain

$$\min_{t\in[a,b]} \varphi(t) \ge \mu(\tilde{L}) \min_{t\in[a,b]} \varphi(t)/M_S(a,b),$$

that is $M_S(a,b) \ge \mu(\tilde{L})$. ∎

In order to present an index zero result of a different nature, we introduce the following operator

$$L_+ u(t) := \int_0^1 k_S^+(t,s) g(s) u(s) \, \mathrm{d}s,$$

for which a result similar to Theorems 10.3.1 and 10.3.2 hold.

In the next Theorem we use the following notation, with c as in (10.3.1),

$$\tilde{f}_0 = \varliminf_{u\to 0} \frac{\operatorname{ess}\inf\limits_{t\in[0,1]} f(t,u)}{|u|}, \quad \tilde{c} := \frac{1}{c} \sup_{t\in[0,1]} \frac{\int_0^1 k_S^-(t,s) g(s) \, \mathrm{d}s}{\int_a^b k_S^+(t,s) g(s) \, \mathrm{d}s}.$$

Theorem 10.3.8. *If $\mu(L_+) < \tilde{f}_0 - \tilde{c} f^0$, then there exists $\rho_0 > 0$ such that for each $\rho \in (0, \rho_0]$, if $u \neq Tu$ for $u \in \partial K_\rho$, it is satisfied that $i_K(T, K_\rho) = 0$.*

Proof. Firstly, since $u \in K$ we have, for $t \in [0,1]$,

$$\begin{aligned}\int_0^1 k_S^-(t,s) g(s) |u(s)| \,\mathrm{d}s \le \int_0^1 k_S^-(t,s) g(s) \|u\| \,\mathrm{d}s \le \tilde{c} \int_a^b k_S^+(t,s) g(s) c \|u\| \,\mathrm{d}s \\ \le \tilde{c} \int_a^b k_S^+(t,s) g(s) |u(s)| \,\mathrm{d}s \le \tilde{c} L_+ |u|(t).\end{aligned}$$

Observe that the hypothesis $\mu(L_+) < \tilde{f}_0 - \tilde{c} f^0$ implies $\tilde{f}_0, f^0 < \infty$. Let $\rho_0 > 0$ such that

$$f(t,u) \ge (\mu(L_+) + \tilde{c} f^0)|u| \text{ and } f(t,u) \le (f^0 + \mu(L_+)/2)|u|$$

for all $u \in [-\rho_0, \rho_0]$ and almost all $t \in [0,1]$.
Let $\rho \le \rho_0$. We will prove that $u \neq Su + \lambda \varphi_+$ for all u in ∂K_ρ and $\lambda > 0$ where $\varphi_+ \in K$ is an eigenfunction of L_+ related to the eigenvalue $1/\mu(L_+)$ such that $\|\varphi_+\| = 1$.

Assume now, on the contrary, that there exist $u \in \partial K_\rho$ and $\lambda > 0$ such that $u(t) = Su(t) + \lambda \varphi_+(t)$ for all $t \in [0,1]$. Hence, we have

$$u(t) = -\int_0^1 k_S^-(t,s) g(s) f(s,u(s)) \,\mathrm{d}s + \int_0^1 k_S^+(t,s) g(s) f(s,u(s)) \,\mathrm{d}s + \lambda \varphi_+(t).$$

On one hand, we have

$$\begin{aligned}u(t) + \int_0^1 k_S^-(t,s) g(s) f(s,u(s)) \,\mathrm{d}s \le |u(t)| + [f^0 + \tfrac{1}{2}\mu(L_+)] \int_0^1 k_S^-(t,s) g(s) |u(s)| \,\mathrm{d}s \\ \le |u(t)| + \tilde{c}[f^0 + \tfrac{1}{2}\mu(L_+)] L_+ |u|(t).\end{aligned}$$

On the other hand, we have

$$\int_0^1 k_S^+(t,s) g(s) f(s,u(s)) \,\mathrm{d}s + \lambda \varphi_+(t) \ge (\mu(L_+) + \tilde{c} f^0) L_+ |u|(t) + \lambda \varphi_+(t).$$

Therefore, we obtain

$$(\mu(L_+) + \tilde{c} f^0) L_+ |u|(t) + \lambda \varphi_+ \le |u(t)| + \tilde{c}\, [f^0 + \tfrac{1}{2}\mu(L_+)] L_+ |u|(t),$$

or, equivalently,

$$\tfrac{1}{2}\mu(L_+) L_+ |u|(t) + \lambda \varphi_+(t) \le |u(t)|.$$

Hence we get

$$\lambda \varphi_+(t) \le |u(t)|.$$

Reasoning as in the proof of (3) of Theorem 10.3.5, we obtain

$$|u(t)| \ge \lambda \tfrac{1}{2}\mu(L_+) L_+ \varphi_+(t) + \lambda \varphi_+(t) = \tfrac{3}{2} \lambda \varphi_+(t).$$

By induction we deduce that $|u(t)| \ge (\frac{n}{2} + 1)\lambda \varphi_+(t)$ for every $n \in \mathbb{N}$, a contradiction since $\|u\| = \rho$. ■

As in Theorem 10.3.6, results on existence of multiple nontrivial solutions can be established. We omit the statement of such results.

Remark 10.3.9. The hypothesis in the Theorem 10.3.8 imply that $\tilde{c} \in (0,1)$. Also, if $\tilde{f}_0 = f^0 = f_0$ then the hypothesis in the Theorem 10.3.8 is equivalent to $\mu(L^+)/(1 - \tilde{c}) < \tilde{f}_0 < \infty$. Furthermore, if $[a,b] = [0,1]$, then $L = L^+ = \tilde{L}$ and the growth condition becomes $\mu(L) < \tilde{f}_0 < \infty$, which is condition (3) in the Theorem 10.3.5 for $f_0 < \infty$.

10.4 Related boundary value problems

In this Section we study the properties of the Green's function of the boundary value problem

$$\epsilon u''(t) + \omega^2 u(t) = y(t), \quad u'(0) = u'(1) = 0,$$

where $y \in \mathrm{L}^1([0,1])$, $\epsilon = \pm 1$ and $\omega \in \mathbb{R}^+$. We discuss separately two cases.

10.4.1 CASE $\epsilon = -1$

The Green's function k of boundary value problem

$$-u''(t) + \omega^2 u(t) = y(t), \quad u'(0) = u'(1) = 0,$$

is given by (see for instance [170] or [195]),

$$\omega \sinh \omega\, k(t,s) := \begin{cases} \cosh \omega(1-t) \cosh \omega s, & 0 \le s \le t \le 1, \\ \cosh \omega(1-s) \cosh \omega t, & 0 \le t \le s \le 1. \end{cases}$$

Note that k is continuous, positive and satisfies some symmetry properties such as

$$k(t,s) = k(s,t) = k(1-t, 1-s).$$

Observe that $\dfrac{\partial k}{\partial t}(t,s) < 0$ for $s < t$ and $\dfrac{\partial k}{\partial t}(t,s) > 0$ for $s > t$. Therefore we choose

$$\Phi(s) := \sup_{t\in[0,1]} k(t,s) = k(s,s).$$

For a fixed $[a,b] \subset [0,1]$ we have

$$c(a,b) := \min_{t\in[a,b]} \min_{s\in[0,1]} \frac{k(t,s)}{\Phi(s)} = \frac{\min\{\cosh \omega a, \cosh \omega(1-b)\}}{\cosh \omega}.$$

The choice of $g \equiv 1$ gives

$$\frac{1}{m} = \sup_{t\in[0,1]} \int_0^1 k(t,s)\,\mathrm{d}s,$$

and, by direct calculation, we obtain that $m = \omega^2$.

The constant M can be computed as follows

$$\begin{aligned}\frac{1}{M(a,b)} &:= \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s \\ &= \frac{1}{\omega^2} - \sup_{t\in[a,b]} \frac{\sinh \omega a \cosh \omega(1-t) + \sinh \omega(1-b) \cosh \omega t}{\omega^2 \sinh \omega}.\end{aligned}$$

Let $\xi_1(t) := \sinh \omega a \cosh \omega(1-t) + \sinh \omega(1-b) \cosh \omega t$. Then we have $\xi_1''(t) = \omega^2 \xi(t) \ge 0$. Therefore the supremum of ξ_1 must be attained in one of the endpoints of the interval $[a,b]$. Thus we have

$$\frac{1}{M(a,b)} = \frac{1}{\omega^2} - \frac{\max\{\xi_1(a), \xi_1(b)\}}{\omega^2 \sinh \omega}.$$

Note that

$$\xi_1(b) - \xi_1(a) = -2\sinh^2\left(\frac{b-a}{2}\omega\right)\sinh\omega(a+b-1),$$

and therefore, $\xi_1(b) \geq \xi_1(a)$ if and only if $a+b \leq 1$. Hence, we have that $1/M(a,b) =$

$$\frac{1}{\omega^2} - \frac{1}{\omega^2\sinh\omega}\begin{cases}\sinh\omega a\cosh\omega(1-b) + \sinh\omega(1-b)\cosh\omega b, & a+b\leq 1,\\ \sinh\omega a\cosh\omega(1-a) + \sinh\omega(1-b)\cosh\omega a, & a+b>1.\end{cases}$$

10.4.2 CASE $\epsilon = 1$

The Green's function k of the boundary value problem

$$u''(t) + \omega^2 u(t) = y(t), \quad u'(0) = u'(1) = 0,$$

is given by

$$\omega\sin\omega\, k(t,s) := \begin{cases}\cos\omega(1-t)\cos\omega s, & 0\leq s\leq t\leq 1,\\ \cos\omega(1-s)\cos\omega t, & 0\leq t\leq s\leq 1.\end{cases}$$

In the following Lemma we describe the sign properties of this Green's function with respect to the parameter ω. The proof is straightforward and is omitted.

Lemma 10.4.1. *We have the following.*

(1) k is positive for $\omega \in (0,\pi/2)$.

(2) k is positive for $\omega = \pi/2$ except at the points $(0,0)$ and $(1,1)$ where it is zero.

(3) k is positive on the strip $(1-\pi/(2\omega), \pi/(2\omega)) \times [0,1]$ if $\omega \in (\pi/2,\pi)$.

(4) if $\omega > \pi$, there is no strip of the form $(a,b)\times[0,1]$ where k is positive.

Consider $\omega \in (0,\pi)$. Fix $s \in [0,1]$ and note that $\dfrac{\partial k}{\partial t}(t,s)$ never changes sign for $t \in [0,s)$ nor for $t \in (s,1]$. Thus we can take

$$\begin{aligned}\Phi(s) := \sup_{t\in[0,1]} |k(t,s)| &= \max\{|k(0,s)|, |k(1,s)|, |k(s,s)|\}\\ &= \frac{\max\{|\cos\omega(1-s)|, |\cos\omega s|, |\cos\omega s\cos\omega(1-s)|\}}{\omega\sin\omega}\\ &= \frac{\max\{\cos\omega(1-s), \cos\omega s\}}{\omega\sin\omega}.\end{aligned}$$

The last equality holds because $\cos(\omega s) \geq -\cos\omega(1-s) \geq 0$ for $s \leq 1-\pi/(2\omega)$ and $\cos(1-\omega s) \geq -\cos\omega s \geq 0$ for $s \geq \pi/(2\omega)$.
On the other hand, for $[a,b] \subset (\max\{0, 1-\pi/(2\omega)\}, \min\{1, \pi/(2\omega)\})$, we have

$$\inf_{t\in[a,b]} k(t,s) = \begin{cases}\min\{k(a,s), k(b,s)\}, & s\in[0,1]\backslash[a,b],\\ \min\{k(a,s), k(s,s), k(b,s)\}, & s\in[a,b].\end{cases}$$

Now, we study the three intervals $[0,a)$, $[a,b]$ and $(b,1]$ separately.

If $s\in[0,a)$, we have

$$\begin{aligned}
&\inf_{s\in[0,a)}\frac{\min\{k(a,s),k(b,s)\}}{\Phi(s)}\\
&=\inf_{s\in[0,a)}\frac{\min\{\cos\omega(1-a)\cos\omega s,\cos\omega(1-b)\cos\omega s\}}{\max\{\cos\omega(1-s),\cos\omega s\}}\\
&=\inf_{s\in[0,a)}\min\left\{\cos\omega(1-a),\cos\omega(1-b),\cos\omega(1-a)\frac{\cos\omega s}{\cos\omega(1-s)},\right.\\
&\quad\left.\cos\omega(1-b)\frac{\cos\omega s}{\cos\omega(1-s)}\right\}\\
&=\min\left\{\cos\omega(1-a),\cos\omega(1-b),\cos\omega a,\cos\omega(1-b)\frac{\cos\omega a}{\cos\omega(1-a)}\right\}\\
&=\min\left\{\cos\omega(1-a),\cos\omega(1-b),\cos\omega a\right\},
\end{aligned}$$

where these equalities hold because $\dfrac{\cos\omega s}{\cos\omega(1-s)}$ is a decreasing function for $s\in[\max\{0,1-\pi/(2\omega)\},1]$ and the function cosine is decreasing in $[0,\pi]$.

If $s\in[a,b]$, we have

$$\begin{aligned}
&\inf_{s\in[a,b]}\frac{\min\{k(a,s),k(s,s),k(b,s)\}}{\Phi(s)}\\
&=\inf_{s\in[a,b]}\frac{\min\{\cos\omega a\cos\omega(1-s),\cos\omega s\cos\omega(1-s),\cos\omega(1-b)\cos\omega s\}}{\max\{\cos\omega(1-s),\cos\omega s\}}\\
&=\inf_{s\in[a,b]}\min\Big\{\cos\omega a,\cos\omega(1-b),\cos\omega s,\cos\omega(1-s),\cos\omega a\frac{\cos\omega(1-s)}{\cos\omega s},\\
&\quad\cos\omega(1-b)\frac{\cos\omega s}{\cos\omega(1-s)}\Big\}\\
&=\min\left\{\cos\omega a,\cos\omega(1-b),\cos\omega b,\cos\omega(1-a)\right\}.
\end{aligned}$$

If $s\in(b,1])$, we have

$$\begin{aligned}
&\inf_{s\in(b,1]}\frac{\min\{k(a,s),k(b,s)\}}{\Phi(s)}\\
&=\inf_{s\in(b,1]}\frac{\min\{\cos\omega a\cos\omega(1-s),\cos\omega b\cos\omega(1-s)\}}{\max\{\cos\omega(1-s),\cos\omega s\}}\\
&=\inf_{s\in(b,1]}\min\left\{\cos\omega a,\cos\omega b,\cos\omega a\frac{\cos\omega(1-s)}{\cos\omega s},\cos\omega b\frac{\cos\omega(1-s)}{\cos\omega s}\right\}\\
&=\min\left\{\cos\omega a,\cos\omega b,\cos\omega a\frac{\cos\omega(1-b)}{\cos\omega b},\cos\omega(1-b)\right\}\\
&=\min\left\{\cos\omega a,\cos\omega b,\cos\omega(1-b)\right\}.
\end{aligned}$$

Therefore, taking into account these three infima, we obtain that

$$c(a,b):=\inf_{s\in[0,1]}\frac{\inf_{t\in[a,b]}k(t,s)}{\Phi(s)}=\min\left\{\cos\omega a,\cos\omega(1-a),\cos\omega b,\cos\omega(1-b)\right\}.$$

In order to compute the constant m we use Lemma 10.4.1 and the fact that $k(t,s) = k(s,t)$ for all $t,s \in [0,1]$.
If $\omega \in (0,\pi/2)$, the function k is positive and therefore

$$m = \omega^2.$$

If $\omega \in [\pi/2,\pi)$, we have

$$\zeta(t) := \int_0^1 k^+(t,s)\,\mathrm{d}s = \begin{cases} \int_{1-\frac{\pi}{2\omega}}^1 k(t,s)\,\mathrm{d}s = \dfrac{1}{\omega^2}\dfrac{\cos\omega t}{\sin\omega}, & t \in [0,1-\frac{\pi}{2\omega}),\\ \dfrac{1}{\omega^2}, & t \in [1-\frac{\pi}{2\omega},\frac{\pi}{2\omega}],\\ \int_0^{\frac{\pi}{2\omega}} k(t,s)\,\mathrm{d}s = \dfrac{1}{\omega^2}\dfrac{\cos\omega(1-t)}{\sin\omega}, & t \in (\frac{\pi}{2\omega},1]. \end{cases}$$

Since

$$0 < \frac{1}{\omega^2} = \int_0^1 k(t,s)\,\mathrm{d}s = \int_0^1 k^+(t,s)\,\mathrm{d}s - \int_0^1 k^-(t,s)\,\mathrm{d}s,$$

we obtain that $\int_0^1 k^+(t,s)\,\mathrm{d}s > \int_0^1 k^-(t,s)\,\mathrm{d}s$, in such a way that

$$m = 1/\max_{t\in[0,1]} \zeta(t) = \omega^2 \sin\omega.$$

Also we have

$$\frac{1}{M(a,b)} = \frac{1}{\omega^2} - \sup_{t\in[a,b]} \frac{\cos\omega(1-t)\sin\omega a + \cos\omega t\sin\omega(1-b)}{\omega^2\sin\omega}.$$

Denote by

$$\xi_3(t) := \cos\omega(1-t)\sin\omega a + \cos\omega t\sin\omega(1-b),$$

and observe that

$$\xi_3(t) = \omega^2\sin\omega\left(\int_0^1 k(t,s)\,\mathrm{d}s - \int_a^b k(t,s)\,\mathrm{d}s\right),$$

and therefore we have $\xi_3(t) \ge 0$ for $t \in [a,b]$. Then, we have $\xi_3'(a)\xi_3'(b) =$

$$-4\omega^2\cos\left[\frac{\omega}{2}(2-a+b)\right]\cos\left[\frac{\omega}{2}(a+b)\right]\sin^2\left[\frac{\omega}{2}(a-b)\right]\sin\omega(1-b)\sin\omega a.$$

Now, $\xi_3'(a)\xi_3'(b) < 0$ if and only if $2-\pi/\omega < a+b < \pi/\omega$, which is always satisfied for $[a,b] \subset (1-\pi/(2\omega),\pi/(2\omega))$. In such a case, ξ_3 has a maximum in $[a,b]$, precisely at the unique point t_0 satisfying

$$\sin\omega t_0 = \frac{\sin\omega\sin\omega a}{\cos\omega\sin\omega a + \sin\omega(1-b)}\cos\omega t_0.$$

Thus we obtain $\xi_3(t_0) =$

$$\cos\omega\cos\omega b\cos\omega t_0 + \cos\omega\sin\omega a\cos\omega t_0$$

$$
\begin{aligned}
&-\cos\omega\sin\omega b\cos\omega t_0+\sin\omega\sin\omega a\sin\omega t_0\\
=&\left(\cos\omega\cos\omega b+\cos\omega\sin\omega a-\cos\omega\sin\omega b+\frac{(\sin\omega\sin\omega a)^2}{\cos\omega\sin\omega a+\sin\omega(1-b)}\right)\\
&\cdot\cos\omega t_0\\
=&\frac{\left|\cos\omega\cos\omega b+\cos\omega\sin\omega a-\cos\omega\sin\omega b+\dfrac{(\sin\omega\sin\omega a)^2}{\cos\omega\sin\omega a+\sin\omega(1-b)}\right|}{\sqrt{\left(\dfrac{\sin\omega\sin\omega a}{\cos\omega\sin\omega a+\sin\omega(1-b)}\right)^2+1}}
\end{aligned}
$$

Remark 10.4.2. In the particular case $a+b=1$, we have $\xi_3(t)=\sin\omega a[\cos\omega(1-t)+\cos\omega t]$. In this case, observe that $\xi_3(t)=\xi_3(1-t)$ and recall that $\xi_3''(t)=-\omega^2\xi_3(t)\geq 0$ (ξ_3 is not constantly zero in any open subinterval). Therefore the maximum is reached at the only point where $t=1-t$, that is, $t=1/2$. Hence we obtain

$$\frac{1}{M(a,b)}=\frac{1-2\cos\dfrac{\omega}{2}\sin\omega a}{\omega^2\sin\omega}.$$

Remark 10.4.3. The constants $m, M(a,b), c(a,b)$ and the function Φ improve and complement some of the ones used in [159–161, 170, 171, 194, 195].

10.5 Examples

In this Section we present some examples in order to illustrate some of the constants that occur in our theory and the applicability of our theoretical results. Note that the constants that occur are rounded to the third decimal place unless exact.

In the first example we study the existence of multiple nontrivial solutions of a (local) Neumann boundary value problem.

Example 10.5.1. Consider the boundary value problem

$$u''(t)+\left(\frac{7\pi}{12}\right)^2u(t)=\frac{\tau_1u^2(t)}{1+t^2}e^{-\tau_2|u(t)|},\ t\in[0,1],\ u'(0)=u'(1)=0, \tag{10.5.1}$$

where $\tau_1,\tau_2>0$.

In this case $\omega=\frac{7\pi}{12}$ and, by Lemma 10.4.1, the Green's function is positive on the strip $(1/7,6/7)\times[0,1]$. We illustrate the Remark 10.4.2 by choosing $[1/4,3/4]\subset(1/7,6/7)$ and we prove, by means of Theorem 10.3.6, the existence of two nontrivial solutions of the boundary value problem (10.5.1) which are (strictly) positive on the interval $[1/4,3/4]$.

In order to do this, note that in our case we have $f(t,u)=\dfrac{\tau_1u^2}{1+t^2}e^{-\tau_2|u|}$ and $f^0=f^\infty=0$. Furthermore, using the results in the previous Section, we have

$$c(1/4,3/4)=\cos\left(\frac{7\pi}{16}\right)=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}=0.195, \tag{10.5.2}$$

and

$$M = M(1/4, 3/4) = 7.029.$$

Henceforth we work in the cone

$$K = \{u \in C[0,1] : \min_{t\in[1/4,3/4]} u(t) \geq c\|u\|\},$$

with c given by (10.5.2).

We set

$$\hat{f}_0 := 2\frac{c-1}{\ln c} c^{\frac{c}{c-1}} M = 10.289.$$

We now prove that if $\tau_1/\tau_2 > \hat{f}_0$, then the condition (Z_1) is satisfied. Let

$$\hat{f}(u) := \inf_{t\in[0,1]} \frac{\tau_1 u^2}{1+t^2} e^{-\tau_2 u} = \frac{1}{2}\tau_1 u^2 e^{-\tau_2 u},\ u \in [0, +\infty).$$

Note that $\hat{f}'$ only vanishes at 0 and $2/\tau_2$, $\hat{f}$ is strictly increasing in the interval $(0, \frac{2}{\tau_2})$ and is strictly decreasing in the interval $(\frac{2}{\tau_2}, +\infty)$. Thus $\hat{f}$ assumes the maximum in the unique point $2/\tau_2$ and, since $\hat{f}(0) = 0$ and $\lim_{x\to+\infty} \hat{f}(x) = 0$, the inverse image by $\hat{f}$ of any strictly positive real number different to $\hat{f}(\frac{2}{\tau_2})$ has either 2 or no points. Let for $x \in [0, +\infty)$

$$l(x) := \hat{f}(x) - \hat{f}(x/c).$$

Take $\varepsilon \in (0, \frac{2c}{\tau_2})$ and note that $l(\varepsilon) < 0$ in view of the strict monotonicity of $\hat{f}$. Moreover, if $\eta > \frac{2}{\tau_2}$, then $l(\eta) > 0$. Since the function l is continuous, there exists a point $\bar{x} \in (\varepsilon, \eta)$ such that $l(\bar{x}) = 0$, that is, $\hat{f}(\bar{x})$=$\hat{f}(\bar{x}/c) = p$. From the type of monotonicity of f, for $x \in [\bar{x}, \bar{x}/c]$ we have $p \leq \hat{f}(x)$. Hence we have

$$\hat{f}(\bar{x}) = \hat{f}(\bar{x}/c) \Rightarrow \bar{x} = e^{\tau_2(\bar{x}/c-\bar{x})}\bar{x}/c \Rightarrow \bar{x} = \frac{2c\ln c}{\tau_2(c-1)}, \bar{x}/c = \frac{2\ln c}{\tau_2(c-1)}.$$

Thus, if we impose $p > M\bar{x}$, we obtain

$$M\frac{2c\ln c}{\tau_2(c-1)} = M\bar{x} < \hat{f}(\bar{x}) = \hat{f}(\bar{x}/c) = \tau_1\left(\frac{2\ln c}{\tau_2(c-1)}\right)^2 c^{-\frac{2}{c-1}},$$

that is, $\tau_1/\tau_2 > \hat{f}_0$.

We now present an example for a boundary value problem subject to two nonlocal boundary conditions.

Example 10.5.2. Consider the boundary value problem

$$\begin{aligned} u''(t) + \omega^2 u(t) &= e^{-|u(t)|},\ t \in [0,1], \\ u'(0) &= u(0) + u(1), \\ u'(1) &= \int_0^1 u(t) \sin \pi t \,\mathrm{d}t, \end{aligned} \tag{10.5.3}$$

where $\omega \in (\pi/2, \pi)$. We rewrite the boundary value problem (10.5.3) in the integral form

$$Tu(t) = \gamma(t)\alpha[u] + \delta(t)\beta[u] + \int_0^1 k(t,s)f(s,u(s))\,\mathrm{d}s,$$

where

$$\begin{aligned}\gamma(t) &= \cos\omega(1-t)/(\omega\sin\omega), \quad \delta(t) = \cos(\omega t)/(\omega\sin\omega),\\ \alpha[u] &= u(0)+u(1), \quad \beta[u] = \int_0^1 u(t)\sin\pi t\,\mathrm{d}t.\end{aligned}$$

In order to verify condition (S_1) of Theorem 10.1.10, we take $[a,b] \subset (1-\pi/(2\omega), \pi/(2\omega))$ and let $f(u) = e^{-|u|}$.

Note that the condition $f^\infty = 0$ implies that the condition (I^1_ρ) is satisfied for ρ sufficiently large (hence $i_K(T, K_R) = 1$ for R big enough).

Now it is left to prove that $i_K(T, V_\rho) = 0$ for ρ small enough (condition (I^0_ρ)).

We have

$$\begin{aligned}\alpha[\gamma] &= \alpha[\delta] = \sqrt{2}\frac{\sin\left(\frac{\pi}{4}+\omega\right)}{\omega\sin\omega}, \ \beta[\gamma] = \beta[\delta] = \frac{\pi\cot\left(\frac{\omega}{2}\right)}{\pi^2\omega-\omega^3},\\ D := D(\omega) &= \frac{\left(\pi^2\omega-\omega^3\right)\sin(\omega/2) - \left(\pi+\pi^2-\omega^2\right)\cos\left(\frac{\omega}{2}\right)}{\left(\pi^2\omega-\omega^3\right)\sin(\omega/2)},\\ \mathcal{K}_A(s) &= \frac{\cos\omega s + \cos(\omega[1-s])}{\omega\sin\omega},\\ \mathcal{K}_B(s) &= \frac{\pi\cos\omega s\cot(\omega/2) - \omega\sin\pi s + \pi\sin\omega s}{\pi^2\omega-\omega^3}.\end{aligned}$$

Observe that $\alpha[\gamma], \alpha[\delta], \beta[\gamma], \beta[\delta], \mathcal{K}_A(s), \mathcal{K}_B(s) \geq 0$ and $\alpha[\gamma], \beta[\delta] < 1$ for $\omega \in (\pi/2, \pi)$.
Also, we have $D(\omega) > 0$ for $\omega \in (\pi/2, \pi)$. In fact, $D(\omega)$ is a strictly increasing function (since $D'(\omega) > 0$ for $\omega \in (0,\pi)$), $\lim\limits_{\omega\to 0^+} D(\omega) = -\infty$ and $D(\pi) = 1 - \frac{1}{4\pi} > 0$, so there is a unique zero ω_0 of D in $(0,\pi)$ but $\omega_0 = 1.507\ldots < \pi/2$.

Now, γ is increasing and δ is decreasing, therefore $c_2 = \gamma(a)/\gamma(1) = \cos(\omega[1-a])$, $c_3 = \delta(b)/\delta(0) = \cos\omega b$. On the other hand, we have

$$\begin{aligned}f_{\rho,\rho/c} &= f(\rho/c)/(\rho/c) = e^{-\rho/c}c/\rho,\\ c(a,b) &= \min\{\cos\omega a, \cos\omega(1-a), \cos\omega b, \cos\omega(1-b)\},\\ \int_a^b \mathcal{K}_A(s)\,\mathrm{d}s &= \frac{\sin\omega b - \sin\omega a + \sin\omega(1-a) - \sin\omega(1-b)}{\omega^2\sin\omega},\\ \omega^2\left(\pi^3-\pi\omega^2\right)\int_a^b \mathcal{K}_B(s)\,\mathrm{d}s &= \pi^2\cot\left(\frac{\omega}{2}\right)(\sin(b\omega)-\sin(a\omega)) + \pi^2\cos(a\omega)\\ &\quad - \pi^2\cos(b\omega) - \omega^2\cos(\pi a) + \omega^2\cos(\pi b).\end{aligned}$$

Taking $a + b = 1$, we obtain

$$\int_a^b \mathcal{K}_A(s)\,\mathrm{d}s = \frac{2\csc\left(\frac{\omega}{2}\right)\sin\left(\frac{1}{2}(\omega-2a\omega)\right)}{\omega^2},$$

$$\int_a^b \mathcal{K}_B(s)\,\mathrm{d}s = -\frac{2\left(\omega^2\cos(\pi a) - \pi^2\cos(a\omega) + \pi^2\cot\left(\frac{\omega}{2}\right)\sin(a\omega)\right)}{\omega^2\left(\pi^3 - \pi\omega^2\right)},$$
$$c = \cos\omega a.$$

Condition (I_ρ^0) is equivalent to

$$f_{\rho,\rho/c}\cdot\inf_{t\in[a,b]}\left\{q(t,\omega,a) + \int_a^b k(t,s)\,\mathrm{d}s\right\} > 1,$$

where $q(t,\omega,a) :=$

$$\frac{2\csc(\omega)\left(\pi\csc\left(\frac{\omega}{2}\right)\sin\left(\frac{1}{2}(\omega-2a\omega)\right)(\pi\cos(t\omega) + (\pi-\omega)(\omega+\pi)\cos(\omega-t\omega))\right)}{\pi\omega^2\left((\pi-\omega)\omega(\omega+\pi) - (-\omega^2+\pi^2+\pi)\cot\left(\frac{\omega}{2}\right)\right)}$$
$$-\frac{2\omega\csc(\omega)\cos(\pi a)(\sin(t\omega) - \sin(\omega-t\omega) + \omega\cos(t\omega))}{\pi\omega^2\left((\pi-\omega)\omega(\omega+\pi) - (-\omega^2+\pi^2+\pi)\cot\left(\frac{\omega}{2}\right)\right)}.$$

Using Remark 10.1.9, it is enough to check

$$f_{\rho,\rho/c}\cdot\left(\inf_{t\in[a,b]} q(t,\omega,a) + \frac{1}{M(a,b)}\right) > 1.$$

It can be checked that $\inf_{t\in[a,b]} q(t,\omega,a) = q(a,\omega,a)$. Hence, we need

$$\frac{e^{-\rho/\cos\omega a}\cos\omega a}{\rho}\left(q(a,\omega,a) + \frac{1-2\cos\frac{\omega}{2}\sin\omega a}{\omega^2\,\sin\omega}\right) > 1.$$

Since $\lim_{\rho\to 0} e^{-\rho/\cos\omega a}/\rho = +\infty$, the inequality is satisfied for ρ small enough and, hence, we have proved that the boundary value problem (10.5.3) has at least a nontrivial solution in the cone K.

We now study an example that occurs in an earlier article by Bonanno and Pizzimenti [18].

Example 10.5.3. Consider the boundary value problem

$$-u''(t) + u(t) = \lambda t e^{u(t)}, t\in[0,1], \quad u'(0) = u'(1) = 0. \tag{10.5.4}$$

In [18] the authors establish the existence of at least one positive solution such that $\|u\| < 2$ for $\lambda \in (0, 2e^{-2})$.

The boundary value problem (10.5.4) is equivalent to the following integral problem

$$u(t) = \int_0^1 k(t,s)g(s)f(u(s))\,\mathrm{d}s,$$

where

$$g(s) = s,\ f(u) = \lambda e^u$$

and

$$k(t,s) := \frac{1}{\sinh(1)} \begin{cases} \cosh(1-t)\cosh s, & 0 \le s \le t \le 1, \\ \cosh(1-s)\cosh t, & 0 \le t \le s \le 1. \end{cases}$$

The kernel k is positive and, by the results provided in this section, conditions (C_1)-(C_8) are satisfied with $[a,b] = [0,1]$. Thus we work in the cone

$$K = \{u \in C[0,1] : \min_{t\in[0,1]} u(t) \ge c\|u\|\},$$

where

$$c = c(0,1) = 1/\cosh 1 = 0.648.$$

We can compute the following constants

$$\begin{aligned} m =& \frac{e+1}{2} = 1.859, \\ M(0,1) =& \frac{e+1}{e-1} = 2.163, \\ f^{0,\rho} =& f_{\rho,\rho/c} = \lambda e^{\rho}/\rho. \end{aligned}$$

Taking $\rho_2 = 2$ we have $(I^1_{\rho_2})$ is satisfied for $\lambda < (e+1)e^{-2}$, and taking $0 < \rho_1 < c/2$ we have $(I^0_{\rho_1})$ for $\lambda > [(e+1)/(e-1)]\rho_1 e^{-\rho_1}$.

Hence, the condition (S_1) of Theorem 10.1.10 is satisfied whenever

$$\lambda \in \left(0, \frac{e+1}{e^2}\right) \supset (0, 2e^{-2}).$$

Furthermore, reasoning as in [95], when $\lambda = \dfrac{1}{4}$ the choice of $\rho_2 = 0.16$ and $\rho_1 = 0.1$ gives the following localization for the solution

$$0.064 \le u(t) \le 0.16,\ t \in [0,1].$$

An application of Theorem 10.2.3 gives that for

$$\lambda > \frac{e+1}{e(e-1)} = 0.797,$$

there are no solutions in K (the trivial solution does not satisfy the differential equation). Furthermore note that $T : P \to K$; this shows that there are no positive solutions for the boundary value problem (10.5.4) when $\lambda > \dfrac{e+1}{e(e-1)}$.

11. General nonlocal operators

In Chapters 8-10 we have dealt with linear conditions where in terms of Stieltjes integrals, which are fairly general and include, as special cases, multi-point and integral conditions.

Webb and Infante [182] gave a unified method for establishing the existence of *positive* solutions of a large class of ordinary differential equations of arbitrary order, subject to nonlocal boundary conditions. The methodology in [182] involves the fixed point index and, in particular deals with the integral equation

$$u(t) = \sum_{i=1}^{N} \gamma_i(t)\alpha_i[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s. \tag{11.0.1}$$

Here the functions γ_i are nonnegative and the *linear* functionals $\alpha_i[\cdot]$ are of the type $\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s)$. The results of [182] are well suited for dealing with differential equations of arbitrary order with many nonlocal terms. These results were applied to the study of fourth order problems that model the deflection of an elastic beam.

An important feature of the integral equation (11.0.1) is the fact that it is designed to deal with boundary value problems where the boundary conditions involve at most *affine* functionals. In physical models this corresponds to feedback controllers having a *linear* response. Nevertheless, in a number of applications, the response of the feedback controller can be *nonlinear*; for example the nonlocal boundary value problem

$$u^{(4)}(t) = g(t)f(t,u(t)),\ u(0) = u'(0) = u''(1) = 0,\ u'''(1) + \hat{B}(u(\eta)) = 0, \tag{11.0.2}$$

describes a cantilever equation with a feedback mechanism, where a spring reacts (in a nonlinear manner) to the displacement registered in a point η of the beam. Positive solutions of the boundary value problem (11.0.2) were investigated by Infante and Pietramala in [92] by means of the perturbed integral equation

$$u(t) = \gamma(t)\hat{B}(\hat{\alpha}[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s,$$

where $\hat{B} : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, possibly nonlinear function.

Note that the idea of using perturbed Hammerstein integral equations in order to deal with the existence of solutions of boundary value problems with nonlinear boundary conditions has been used with success in a number of papers, see, for example, the manuscripts of Alves and co-authors [3], Cabada [27], Franco et al. [71], Goodrich [75–79], Infante [88], Karakostas [111], Pietramala [144], Yang [192, 193] and references therein.

The existence of *nontrivial* solutions of the boundary value problem

$$-u''(t) = g(t)f(u(t)), u'(0) + \hat{B}(\hat{\alpha}[u]) = 0,\ \beta u'(1) + u(\eta) = 0,$$

that models a heat-flow problem with a nonlinear controller, were discussed by Infante [89], by means of the perturbed integral equation

$$u(t) = \gamma(t)\hat{B}(\hat{\alpha}[u]) + \int_0^1 k(t,s)g(s)f(u(s))\,\mathrm{d}s.$$

In this chapter we generalize the previous ones insofar as we consider nonlinear boundary conditions and functional terms. To be precise, we discuss the existence of multiple nontrivial solutions of perturbed Hammerstein integral equations of the kind

$$u(t) = Bu(t) + \int_0^1 k(t,s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s,$$

where $B : C(I) \to C(I)$ is a compact and continuous map, $D : C(I) \to \mathrm{L}^\infty(I)$, a continuous map and f is a nonnegative L^∞-Carathéodory function. In our setting B and D are possibly nonlinear. This type of integral equation arises naturally when dealing with a boundary value problem where nonlocal terms occur in the differential equation and in the boundary conditions. Here we prove the existence of multiple solutions that are allowed to *change sign*, in the spirit of the earlier chapters.

At the end of the chapter we study, for illustrative purposes and in two examples, the nonlocal differential equation

$$-u''(t) = f(t,u(t)) + \gamma(t)u(\eta(t)),$$

subject to the boundary conditions

$$u(0) = 0,\ u(1) = \theta\|u\| \text{ or } u(0) = u'(1),\ u'(0) = u(1),$$

showing that the constants occurring in our theoretical results can be computed.

11.1 The integral operator

Let $I := [0,1]$. In this section we obtain results for the fixed points of the integral operator

$$Tu(t) = Bu(t) + \int_0^1 k(t,s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s, \tag{11.1.1}$$

where $B : C(I) \to C(I)$ is a continuous and compact map, $D : C(I) \to \mathrm{L}^\infty(I)$, a continuous map and f is a nonnegative L^∞-Carathéodory function. B and D are not necessarily linear.

Recall that P be the cone of nonnegative functions in $C(I)$. We make the following assumptions.

(C_1) $k : I \times I \to \mathbb{R}$ is measurable, and for every $\tau \in I$ we have

$$\lim_{t\to\tau} |k(t,s) - k(\tau,s)| = 0 \text{ for a.e. } s \in I.$$

(C_2) There exist a subinterval $[a,b] \subseteq I$, a function $\Phi \in \mathrm{L}^1(I)$, and a constant $c_1 \in (0,1]$ such that

$$|k(t,s)| \le \Phi(s) \text{ for } t \in I \text{ and almost every } s \in I,$$
$$k(t,s) \ge c_1\Phi(s) \text{ for } t \in [a,b] \text{ and almost every } s \in I.$$

(C_3) $g, g\,\Phi \in \mathrm{L}^1(I), g(t) \ge 0$ for a.e. $t \in I$, and $\int_a^b \Phi(s)g(s)\,\mathrm{d}s > 0$.

(C_4) Consider functions $f_i : I \times \mathbb{R} \to [0,\infty)$, $\gamma_{ij} : I \to \mathbb{R}$, $j = 1,\dots,m_i$, $\delta_{ij} : I \to \mathbb{R}$, $j = 1,\dots,n_i$ and continuous functionals $\alpha_{ij} : C(I) \to \mathbb{R}$, $j = 1,\dots,m_i$ and $\beta_{ij} : C(I) \to \mathbb{R}$, $j = 1,\dots,n_i$, $i = 1,2$, a constant $c \in (0,c_1]$ and a cone

$$K := \{u \in C(I) \ : \ \min_{t\in[a,b]} u(t) \geq c\|u\|, \ \alpha_{ij}[u], \beta_{ij}[u] \geq 0\}$$

such that, for all $u \in K$, the following inequalities hold:

$$\sum_{j=1}^{m_1} \gamma_{1j}(t)\alpha_{1j}[u] + f_1(t,u(t)) \leq f(t,u(t),Du(t)), \text{ for every } t \in [a,b],$$

$$\sum_{j=1}^{n_1} \delta_{1j}(t)\beta_{1j}[u] \leq Bu(t), \text{ for every } t \in [a,b],$$

$$f(t,u(t),Du(t)) \leq \sum_{j=1}^{m_2} \gamma_{2j}(t)\alpha_{2j}[u] + f_2(t,u(t)), \text{ for every } t \in I$$

and

$$Bu(t) \leq \sum_{j=1}^{n_2} \delta_{2j}(t)\beta_{2j}[u], \text{ for every } t \in I.$$

(C_5) The nonlinearities $f : I \times \mathbb{R}^2 \to [0,+\infty)$, $f_1 : I \times \mathbb{R} \to [0,+\infty)$ and $f_2 : I \times \mathbb{R} \to [0,+\infty)$ satisfy L^∞-Carathéodory conditions, that is $f(\cdot,u,v)$, $f_i(\cdot,u)$ are measurable for each fixed $u,v \in \mathbb{R}$; $f(t,\cdot,\cdot)$, $f_i(t,\cdot)$ are continuous for a. e. $t \in I$, and for each $r > 0$, there exists $\phi_r \in \mathrm{L}^\infty(I)$ such that

$$f(t,u,v), f_i(t,u) \leq \phi_r(t) \ \text{ for all } \ u,v \in [-r,r], \ \text{ and a. e. } \ t \in I.$$

(C_6) $\gamma_{ij} \in C(I)$. Let $\tilde{\gamma}_{ij}(t) := \int_0^1 k^+(t,s)g(s)\gamma_{ij}(s)\,\mathrm{d}s$. Assume the families of functions $\{\tilde{\gamma}_{ij}, \delta_{ij}\}_{i,j}$ belong to $K\setminus\{0\}$.

(C_7) Define $\varphi_i = (\alpha_{i1},\dots,\alpha_{im_i},\beta_{i1},\dots,\beta_{in_i})$, $\psi_i = (\tilde{\gamma}_{i1},\dots,\tilde{\gamma}_{im_i},\delta_{i1},\dots,\delta_{in_i})$ and denote by φ_{ij} and ψ_{ij} the j-th element of φ_i and ψ_i respectively. We have the following inequalities.

$$\begin{gathered}\varphi_{ij}[\tau_1 u + \tau_2 v] \geq \tau_1\varphi_{ij}[u] + \tau_2\varphi_{ij}[v], \\ \tau_1,\tau_2 \in \mathbb{R}^+, \ u,v \in K, \ j = 1,\dots,m_i+n_i, \ i = 1,2,\end{gathered} \tag{11.1.2}$$

$$\begin{gathered}\varphi_{2j}[\tau_1 u + \tau_2 v] \leq |\tau_1||\varphi_{2j}[u]| + |\tau_2||\varphi_{2j}[v]|, \\ \tau_1,\tau_2 \in \mathbb{R}, \ u,v \in K, \ j = 1,\dots,m_2+n_2.\end{gathered} \tag{11.1.3}$$

Let $\mathcal{K}_{\varphi_{1j}}(s) := \varphi_{1j}[k(\cdot,s)] \geq 0$, $\mathcal{K}_{\varphi_{2j}}(s) := \varphi_{2j}[k^+(\cdot,s)] \geq 0$ for a. e. $s \in I$, and assume $\mathcal{K}_{\varphi_{ij}} \in \mathrm{L}^\infty(I)$ for every $j = 1,\dots,m_i+n_i$, $i = 1,2$.

$$\begin{aligned}\varphi_{1j}\left[\int_a^b k(\cdot,s)g(s)f_1(s,u(s))\,\mathrm{d}s\right] \geq \int_a^b \varphi_{1j}[k(\cdot,s)]g(s)f_1(s,u(s))\,\mathrm{d}s, \\ u \in K, \ j = 1,\dots,m_1+n_1,\end{aligned} \tag{11.1.4}$$

$$\varphi_{2j}\left[\int_0^1 k^+(\cdot,s)g(s)f_2(s,u(s))\,\mathrm{d}s\right] \le \int_0^1 |\varphi_{2j}[k^+(\cdot,s)]|g(s)f_2(s,u(s))\,\mathrm{d}s,$$
$$u \in K,\ j=1,\dots,m_2+n_2, \tag{11.1.5}$$

(C_8) Define $M_k = (\varphi_{ki}[\psi_{kj}])_{i,j=1}^{m_k+n_k} \in \mathcal{M}_{m_k+n_k}(\mathbb{R})$, $k=1,2$. Assume that their respective spectral radii r satisfy that $r(M_1) < 1/c_1$ and $r(M_2) < 1$.

(C_9) Let c and K be given in (C_4). Then

$$\sum_{j=1}^{n_1} \delta_{1j}(t)\beta_{1j}[u] \ge c\sum_{j=1}^{n_2} \|\delta_{2j}\|\beta_{2j}[u] \text{ for every } t\in[a,b] \text{ and } u \in K.$$

(C_{10}) $\varphi_{1j}[u] \ge \varphi_{1j}[v]$ for every $u,v\in K$ such that $u(t)\ge v(t)$ for all $t\in[a,b]$, $\varphi_{2j}[u] \ge \varphi_{2j}[v]$ for every $u,v\in K$ such that $u(t)\ge v(t)$ for all $t\in I$ and $\varphi_{ij}[u] \ge 0$ for every $u\in P$.

Also, $\varphi_{ij}[Tu], \varphi_{ij}[F_1u], \varphi_{ij}[F_2u], \varphi_{ij}[L_1u] \ge 0$ for every $u \in K$ where, for $t \in [0,1]$,

$$F_1u(t) := \int_a^b k(t,s)g(s)f_1(s,u(s))\,\mathrm{d}s,$$
$$F_2u(t) := \int_0^1 k^+(t,s)g(s)f_2(s,u(s))\,\mathrm{d}s,$$
$$L_1u(t) := \int_a^b k^+(t,s)g(s)u(s)\,\mathrm{d}s.$$

Remark 11.1.1. Observe that from conditions (C_6) and (C_8) we know that $\psi_{ij} \in K$ and M_k has positive entries for $k=1,2$. Also, if the φ_{ij} are linear functionals defined as integrals with respect to a measure of bounded variation, properties (11.1.2)–(11.1.5) hold.

Remark 11.1.2. Condition (11.1.3) is some sort of triangle inequality. In particular, it implies a kind of *second triangle inequality*. Indeed, let $u,v\in K$, Then we have

$$\varphi_{2j}[u] = \varphi_{2j}[(u+v)-v] \le |\varphi_{2j}[u+v]| + |\varphi_{2j}[v]|.$$

Hence we obtain

$$\varphi_{2j}[u] - |\varphi_{2j}[v]| \le |\varphi_{2j}[u+v]|.$$

Interchanging u and v we get

$$\varphi_{2j}[v] - |\varphi_{2j}[u]| \le |\varphi_{2j}[u+v]|,$$

which implies, in particular,

$$|\varphi_{2j}[u] - \varphi_{2j}[v]| \le |\varphi_{2j}[u+v]|.$$

Therefore we obtain

$$|\varphi_{2j}[u] - \varphi_{2j}[v]| \le |\varphi_{2j}[u-v]|.$$

Remark 11.1.3. By (C_{10}), if $u\in K$, then $u^+, |u| \in K$.

Remark 11.1.4. Let $\tilde{K} = \{u|_{[a,b]} \,:\, u \in K\}$ and $\iota : C([a,b]) \to C(I)$ such that $\iota[u](t) = u(t)$ for $t \in [a,b]$, $\iota[u](t) = u(a)$ for $t \in [0,a]$ and $\iota[u](t) = u(b)$ for $t \in [b,1]$. The first part of condition (C_{10}) implies that, if $u, v \in K$ satisfy $u|_{[a,b]} = v|_{[a,b]}$, then $\varphi_{1j}[u] = \varphi_{1j}[v]$. Hence, there exists $\tilde{\varphi}_{1j} : \tilde{K} \to \mathbb{R}$ such that $\varphi_{1j}|_K = \tilde{\varphi}_{1j} \circ \iota$.

Lemma 11.1.5. *The operator T defined in* (11.1.1) *maps K into K and is continuous and compact.*

Proof. Take $u \in K$. Then, by (C_2), (C_4) and (C_5), we have

$$\begin{aligned} Tu(t) =& Bu(t) + \int_0^1 k(t,s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s \\ \le& \sum_{j=1}^{n_2} \delta_{2j}(t)\beta_{2j}[u] + \int_0^1 \Phi(s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s. \end{aligned}$$

Hence, we obtain

$$\|Tu\| \le \sum_{j=1}^{n_2} \|\delta_{2j}\|\beta_{2j}[u] + \int_0^1 \Phi(s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s.$$

Combining this fact with (C_2), (C_5), (C_6) and (C_9), for $t \in [a,b]$, we get

$$\begin{aligned} Tu(t) \ge& \sum_{j=1}^{n_1} \delta_{1j}(t)\beta_{1j}[u] + c_1 \int_0^1 \Phi(s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s \\ \ge& c \sum_{j=1}^{n_2} \|\delta_{2j}\|\beta_{2j}[u] + c \int_0^1 \Phi(s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s \ge c\|Tu\|. \end{aligned}$$

Furthermore, by (C_{10}), $\varphi_{ij}[Tu] \ge 0$. Hence we have $Tu \in K$.

Now, we have that the operator $N_f : C(I) \times \mathrm{L}^\infty(I) \to C(I)$ such that $N_f(u,v)(t) = \int_0^1 k(t,s)g(s)f(s,u(s),v(s))\,\mathrm{d}s$ is compact.

Since D is continuous, $\mathrm{Id} \times D$ is also continuous so $N_f \circ (\mathrm{Id} \times D)$ is compact. Since T is the sum of two compact operators, it is compact. The continuity is proved in a similar way. ■

Remark 11.1.6. Similarly, from condition (C_2), we observe here that F_1, F_2 and L_1 map K to K. To see this, observe that for all $t \in [a,b]$ and $u \in K$ the following properties hold:

$$\begin{aligned} F_1u(t) &= \int_a^b k(t,s)g(s)f_1(s,u(s))\,\mathrm{d}s \ge c\int_a^b \Phi(s)g(s)f_1(s,u(s))\,\mathrm{d}s \ge c\|F_1u\|, \\ F_2u(t) &= \int_0^1 k^+(t,s)g(s)f_2(s,u(s))\,\mathrm{d}s \ge c\int_0^1 \Phi(s)g(s)f_2(s,u(s))\,\mathrm{d}s \ge c\|F_2u\|, \\ L_1u(t) &= \int_a^b k^+(t,s)g(s)u(s)\,\mathrm{d}s \ge c\int_a^b \Phi(s)g(s)u(s)\,\mathrm{d}s \ge c\|L_1u\|. \end{aligned}$$

Also, $\varphi_{ij}[F_1u], \varphi_{ij}[F_2u], \varphi_{ij}[L_1u] \ge 0$ by (C_{10}).

On the other hand, L_1 maps P to P, but also maps P to K. The proof goes as above.

11.2 Fixed point index calculations

Let us define, in a similar way to the previous times,

$$K_\rho := \{u \in K : \|u\| < \rho\}, \quad V_\rho := \{u \in K : \min_{t\in[a,b]} u(t) < \rho\}.$$

If u, v are vectors, we denote by $[u]_j$ the j-th component of u and if we write $u \le v$ the inequality is to be interpreted component-wise. Also, we denote by $\mathcal{K}_{\varphi_i} := \left(\mathcal{K}_{\varphi_{ij}}\right)_{j=1}^{m_i+n_i}$, $i = 1, 2$ ($\mathcal{K}_{\varphi_{ij}}$ as defined in (C_7)).

Lemma 11.2.1. *Assume that*

(I_ρ^1) *there exists* $\rho > 0$ *such that*

$$f_2^{-\rho,\rho} \cdot \sup_{t\in I}\left(\sum_{j=1}^{m_2+n_2} |\psi_{2j}(t)| \left[(\mathrm{Id} - M_2)^{-1} \int_0^1 |\mathcal{K}_{\varphi_2}(s)| g(s) ds\right]_j + \sigma(t)\right) < 1, \tag{11.2.1}$$

where

$$f_2^{-\rho,\rho} := \operatorname{ess\,sup}\Big\{\frac{f_2(t,u)}{\rho} : (t,u) \in I \times [-\rho,\rho]\Big\}$$

and

$$\sigma(t) := \max\left\{\int_0^1 k^+(t,s) g(s)\, \mathrm{d}s, \int_0^1 k^-(t,s) g(s)\, \mathrm{d}s\right\}.$$

Then we have $i_K(T, K_\rho) = 1$.

Proof. We show that $Tu \ne \lambda u$ for all $\lambda \ge 1$ when $u \in \partial K_\rho$, which implies that $i_K(T, K_\rho) = 1$. In fact, if this does not happen, then there exist u with $\|u\| = \rho$ and $\lambda \ge 1$ such that $\lambda u(t) = Tu(t)$. Therefore, by (C_4) and (C_6),

$$\lambda u(t) \le \sum_{j=1}^{m_2+n_2} \psi_{2j}(t)\varphi_{2j}[u] + F_2 u(t), \; t \in I, \tag{11.2.2}$$

so, from (C_6), and Remark 11.1.6, we have that both sides of the inequality are in K. As a consequence, from (11.1.3), we deduce

$$\lambda \varphi_{2i}[u] \le \sum_{j=1}^{m_2+n_2} \varphi_{2i}[\psi_{2j}]\varphi_{2j}[u] + |\varphi_{2i}[F_2 u]|,$$

which, expressed in matrix notation, is

$$\lambda \varphi_2[u] \le M_2 \varphi_2[u] + |\varphi_2[F_2 u]|.$$

Hence, we have

$$(\mathrm{Id} - M_2)\,\varphi_2[u] \le (\lambda\, \mathrm{Id} - M_2)\,\varphi_2[u] \le |\varphi_2[F_2 u]|.$$

Since $r(M_2) < 1$, $\mathrm{Id} - M_2$ is invertible and $(\mathrm{Id} - M_2)^{-1} = \sum_{k=0}^{\infty} M_2^k$, so $(\mathrm{Id} - M_2)^{-1}$ is positive and thus

$$\varphi_2[u] \le (\mathrm{Id} - M_2)^{-1} |\varphi_2[F_2 u]|. \tag{11.2.3}$$

Therefore, from (11.1.5), (11.2.2) and (11.2.3) we obtain, for all $t \in I$,

$$\begin{aligned}
\lambda |u(t)| &\le \sum_{j=1}^{m_2+n_2} |\psi_{2j}(t)| \varphi_{2j}[u] + |F_2 u(t)| \\
&\le \sum_{j=1}^{m_2+n_2} |\psi_{2j}(t)| \left[(\mathrm{Id} - M_2)^{-1} |\varphi_2[F_2 u]| \right]_j + |F_2 u(t)| \\
&\le \rho f_2^{-\rho,\rho} \sup_{t\in I} \left(\sum_{j=1}^{m_2+n_2} |\psi_{2j}(t)| \left[(\mathrm{Id} - M_2)^{-1} \int_0^1 |\mathcal{K}_{\varphi_2}(s)| g(s)\, \mathrm{d}s \right]_j + \sigma(t) \right).
\end{aligned}$$

Taking the supremum on $t \in I$,

$$\lambda \rho \le \rho f_2^{-\rho,\rho} \sup_{t\in I} \left(\sum_{j=1}^{m_2+n_2} |\psi_{2j}(t)| \left[(\mathrm{Id} - M_2)^{-1} \int_0^1 |\mathcal{K}_{\varphi_2}(s)| g(s)\, \mathrm{d}s \right]_j + \sigma(t) \right).$$

From (11.2.1) we obtain $\lambda\rho < \rho$, contradicting the fact that $\lambda \ge 1$. ■

Remark 11.2.2. We point out, in similar way as in Chapter 10, that a stronger (but easier to check) condition than (I_ρ^1) is given by the following.

$$f_2^{-\rho,\rho} \left(\sum_{j=1}^{m_2+n_2} \|\psi_{2j}\| \left[(\mathrm{Id} - M_2)^{-1} \int_0^1 \left|\mathcal{K}_{\varphi_2}(s)\right| g(s)\, \mathrm{d}s \right]_j + \frac{1}{m} \right) < 1.$$

where

$$\frac{1}{m} := \sup_{t\in I} \sigma(t). \tag{11.2.4}$$

Lemma 11.2.3. *Assume that*

(I_ρ^0) *There exists $\rho > 0$ such that*

$$\begin{aligned}
f_{1,\rho,\rho/c} \cdot \inf_{t\in[a,b]} \Bigg(\sum_{j=1}^{m_1+n_1} \psi_{1j}(t) \left[(\mathrm{Id} - c_1 M_1)^{-1} \int_a^b \mathcal{K}_{\varphi_1}(s) g(s)\, \mathrm{d}s \right]_j \\
+ \int_a^b k(t,s) g(s)\, \mathrm{d}s \Bigg) > 1,
\end{aligned}$$

where

$$f_{1,\rho,\rho/c} := \operatorname{ess\,inf} \left\{ \frac{f_1(t,u)}{\rho} : \ (t,u) \in [a,b] \times [\rho, \rho/c] \right\}.$$

Then we have $i_K(T,V_\rho) = 0$.

Proof. Take $e \in K\backslash\{0\}$. We will show that $u \neq Tu + \lambda e$ for all $\lambda \geq 0$ and $u \in \partial V_\rho$ which implies that $i_K(T,V_\rho) = 0$. In fact, if this does not happen, there are $u \in \partial V_\rho$ (and so we have $\min_{t\in[a,b]} u(t) = \rho$ and $\rho \leq u(t) \leq \rho/c$ for all $t \in [a,b]$), and $\lambda \geq 0$ with

$$u(t) = Tu(t) + \lambda e.$$

Therefore, for $t \in [a,b]$, by (C_2), (C_4), (C_6) and Remark 11.1.6, we have

$$u(t) \geq \sum_{j=1}^{m_1+n_1} \psi_{1j}(t)\varphi_{1j}[u] + F_1u(t) + \lambda e(t) \in K. \tag{11.2.5}$$

Thus we obtain, using (11.1.2),

$$\begin{aligned}\varphi_{1i}[u] &\geq \sum_{j=1}^{m_1+n_1} \varphi_{1i}[\psi_{1j}]\varphi_{1j}[u] + \varphi_{1i}[F_1u] + \lambda\varphi_{1i}[e] \\ &\geq c_1\left(\sum_{j=1}^{m_1+n_1} \varphi_{1i}[\psi_{1j}]\varphi_{1j}[u] + \varphi_{1i}[F_1u]\right),\end{aligned}$$

which, expressed in matrix notation, is

$$\varphi_1[u] \geq c_1\left(M_1\varphi_1[u] + \varphi_1[F_1u]\right).$$

Hence we get

$$(\mathrm{Id} - c_1M_1)\varphi_1[u] \geq \varphi_1[F_1u].$$

Since $r(M_1) < 1/c_1$, $\mathrm{Id} - c_1M_1$ is invertible and $(\mathrm{Id} - c_1M_1)^{-1} = \sum_{k=0}^{\infty}(c_1M_1)^k$, so $(\mathrm{Id} - c_1M_1)^{-1}$ is positive and hence

$$\varphi_1[u] \geq (\mathrm{Id} - c_1M_1)^{-1}\varphi_1[F_1u]. \tag{11.2.6}$$

Therefore, from (11.1.2), (11.2.5) and (11.2.6) we obtain, for $t \in [a,b]$,

$$\begin{aligned}u(t) &\geq \sum_{j=1}^{m_1+n_1} \psi_{1j}(t)\varphi_{1j}[u] + F_1u(t) \\ &\geq \sum_{j=1}^{m_1+n_1} \psi_{1j}(t)\left[(\mathrm{Id} - c_1M_1)^{-1}\varphi_1[F_1u]\right]_j + F_1u(t) \\ &\geq \inf_{t\in[a,b]}\left(\sum_{j=1}^{m_1+n_1} \psi_{1j}(t)\left[(\mathrm{Id} - c_1M_1)^{-1}\int_a^b \mathcal{K}_{\varphi_1}(s)g(s)\,\mathrm{d}s\right]_j + \int_a^b k(t,s)g(s)\,\mathrm{d}s\right) \\ &\quad \cdot \rho f_{1,\rho,\rho/c}.\end{aligned}$$

Taking the infimum on $t \in [a,b]$, gives

$$\rho \geq \inf_{t\in[a,b]}\left(\sum_{j=1}^{m_1+n_1} \psi_{1j}(t)\left[(\mathrm{Id} - c_1M_1)^{-1}\int_a^b \mathcal{K}_{\varphi_1}(s)g(s)\,\mathrm{d}s\right]_j + \int_a^b k(t,s)g(s)\,\mathrm{d}s\right) \rho f_{1,\rho,\rho/c}.$$

which contradicts the hypothesis. ■

Remark 11.2.4. Again, a stronger condition than (I_{ρ}^{0}) is given by the following.

$$f_{1,\rho,\rho/c}\left(\inf_{t\in[a,b]}\sum_{j=1}^{m_1+n_1}\psi_{1j}(t)\left[(\mathrm{Id}-c_1M_1)^{-1}\int_a^b\mathcal{K}_{\varphi_1}(s)g(s)\,\mathrm{d}s\right]_j+\frac{1}{M(a,b)}\right)>1,$$

where

$$\frac{1}{M(a,b)}:=\inf_{t\in[a,b]}\int_a^b k(t,s)g(s)\,\mathrm{d}s. \tag{11.2.7}$$

The results above can be used in order to prove the existence of at least one, two or three nontrivial solutions.

Theorem 11.2.5. *Assume conditions $(C_1)-(C_{10})$ are satisfied. The integral equation* (11.1.1) *has at least one nonzero solution in K if any of the following conditions hold.*

(S_1) *There exist $\rho_1,\rho_2\in(0,\infty)$ with $\rho_1/c<\rho_2$ such that $(\mathrm{I}_{\rho_1}^{0})$ and $(\mathrm{I}_{\rho_2}^{1})$ hold.*

(S_2) *There exist $\rho_1,\rho_2\in(0,\infty)$ with $\rho_1<\rho_2$ such that $(\mathrm{I}_{\rho_1}^{1})$ and $(\mathrm{I}_{\rho_2}^{0})$ hold.*

The integral equation (11.1.1) *has at least two nonzero solutions in K if one of the following conditions hold.*

(S_3) *There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$ with $\rho_1/c<\rho_2<\rho_3$ such that $(\mathrm{I}_{\rho_1}^{0})$, $(\mathrm{I}_{\rho_2}^{1})$ and $(\mathrm{I}_{\rho_3}^{0})$ hold.*

(S_4) *There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$ with $\rho_1<\rho_2$ and $\rho_2/c<\rho_3$ such that $(\mathrm{I}_{\rho_1}^{1})$, $(\mathrm{I}_{\rho_2}^{0})$ and $(\mathrm{I}_{\rho_3}^{1})$ hold.*

The integral equation (11.1.1) *has at least three nonzero solutions in K if one of the following conditions hold.*

(S_5) *There exist $\rho_1,\rho_2,\rho_3,\rho_4\in(0,\infty)$ with $\rho_1/c<\rho_2<\rho_3$ and $\rho_3/c<\rho_4$ such that $(\mathrm{I}_{\rho_1}^{0})$, $(\mathrm{I}_{\rho_2}^{1})$, $(\mathrm{I}_{\rho_3}^{0})$ and $(\mathrm{I}_{\rho_4}^{1})$ hold.*

(S_6) *There exist $\rho_1,\rho_2,\rho_3,\rho_4\in(0,\infty)$ with $\rho_1<\rho_2$ and $\rho_2/c<\rho_3<\rho_4$ such that $(\mathrm{I}_{\rho_1}^{1})$, $(\mathrm{I}_{\rho_2}^{0})$, $(\mathrm{I}_{\rho_3}^{1})$ and $(\mathrm{I}_{\rho_4}^{0})$ hold.*

11.2.1 Nonexistence results

For this epigraph we will assume that the operators φ_{ij} are *linearly bounded*.

Definition 11.2.6. An operator $A:X\to Y$ between two normed spaces $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$† is linearly bounded if there exists $M\in\mathbb{R}^+$ such that $\|Ax\|\le M\|x\|$ for every $x\in X$. We then define the *norm* of A as

$$\|A\|:=\inf\{M\in\mathbb{R}^+\ :\ \|Ax\|\le M\|u\|,\ x\in X\}.$$

†Although they may be different, we use the same notation for the norms in X and Y to simplify the notation.

Observe that for linear operators this is the usual norm. We denote by $\mathrm{LB}(X,Y)$ the space of linearly bounded operators from X to Y (and by $\mathrm{LB}(X)$ if $X = Y$). For operators $A \in \mathrm{LB}(X)$ we can define the *spectral radius* of A as $r(A) = \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$. We define the *principal characteristic value* as $\mu(A) := 1/r(A)$.

For more properties on this generalized spectral value we refer the reader to [22, 200].

We now offer some nonexistence results for the integral equation (11.1.1).

Theorem 11.2.7. *Let m and $M(a,b)$ be defined in* (11.2.4) *and* (11.2.7) *respectively and the φ_{ij} be linearly bounded. If one of the following conditions holds,*

(1) $f_2(t,u) < m\left(1 - \sum_{j=1}^{m_2+n_2} \|\psi_{2j}\|\|\varphi_{2j}\|\right)|u|$, for every $t \in I$ and $u \in \mathbb{R}\backslash\{0\}$,

(2) $f_1(t,u) > M(a,b)\,u$ for every $t \in [a,b]$ and $u \in \mathbb{R}^+$,

then there is no nontrivial solution of the problem (11.1.1) *in K.*

Proof. (1) Assume, on the contrary, that there exists $u \in K$, $u \not\equiv 0$ such that $u = Tu$ and let $t_0 \in I$ such that $\|u\| = |u(t_0)|$. Then, arguing as in the proof of Lemma 11.2.1,

$$
\begin{aligned}
&\|u\| = |u(t_0)| \\
&\leq \sum_{j=1}^{m_2+n_2} \|\psi_{2j}\|\varphi_{2j}[u] \\
&\quad + \max\left\{\int_0^1 k^+(t_0,s)g(s)f_2(s,u(s))\,\mathrm{d}s, \int_0^1 k^-(t_0,s)g(s)f_2(s,u(s))\,\mathrm{d}s\right\} \\
&< \sum_{j=1}^{m_2+n_2} \|\psi_{2j}\|\|\varphi_{2j}\|\|u\| \\
&\quad + \max\left\{\int_0^1 k^+(t_0,s)g(s)\,\mathrm{d}s, \int_0^1 k^-(t_0,s)g(s)\,\mathrm{d}s\right\} m\left(1 - \sum_{j=1}^{m_2+n_2} \|\psi_{2j}\|\|\varphi_{2j}\|\right)\|u\| \\
&= \sum_{j=1}^{m_2+n_2} \|\psi_{2j}\|\|\varphi_{2j}\|\|u\| + \left(1 - \sum_{j=1}^{m_2+n_2} \|\psi_{2j}\|\|\varphi_{2j}\|\right)\|u\| = \|u\|,
\end{aligned}
$$

a contradiction, thus there is no nontrivial solution of problem (11.1.1) in K.

(2) Assume, on the contrary, that there exists $u \in K$, $u \not\equiv 0$ such that $u = Tu$ and let $t_0 \in [a,b]$ such that $u(t_0) = \min_{t\in[a,b]} u(t)$. Then, as in the proof of Lemma 11.2.3,

$$
\begin{aligned}
u(t_0) = Tu(t_0) &\geq \sum_{j=1}^{m_2+n_2} \psi_{1j}(t_0)\varphi_{1j}[u] + \int_0^1 k(t_0,s)g(s)f_1(s,u(s))\,\mathrm{d}s \\
&> \int_a^b k(t_0,s)g(s)M(a,b)u(s)\,\mathrm{d}s \\
&\geq M(a,b)u(t_0)\int_a^b k(t_0,s)g(s)\,\mathrm{d}s \geq u(t_0),
\end{aligned}
$$

a contradiction, thus there is no nontrivial solution of problem (11.1.1) in K. ■

11.3 The spectral radius and the existence of multiple solutions

In order to prove the results to come we will need different requirements on the functionals $\varphi_{i,j}$ than being linearly bounded. We introduce now some definitions [57, 58]. Let X, Y be two real normed spaces. Let $\operatorname{Lip}(X,Y)$ be the set of operators between X and Y such that satisfy the *Lipschitz property*, that is,

$$\operatorname{Lip}(X,Y) := \{N : X \to Y \ : \ \exists M \in \mathbb{R}^+, \|Nx - Ny\| \leq M\|x-y\|, \ \ \forall x,y \in X\}.$$

Define the function

$$\|N\|^* := \inf\{M \in \mathbb{R}^+ \ : \ \|Nx - Ny\| \leq M\|x-y\|, \ \ \forall x,y \in X\}, \ N \in \operatorname{Lip}(X,Y).$$

We denote by $\operatorname{Lip}(X) \equiv \operatorname{Lip}(X,X)$. $\operatorname{Lip}(X,Y)$ is a real vector space and $\|\cdot\|^*$ is a seminorm on $\operatorname{Lip}(X,Y)$ (in fact, $(\|\cdot\|^*)^{-1}(\{0\}) = \mathbb{R}$). Also, observe that

$$\|N - N(0)\| = \sup_{\substack{x\in X,\\ x\neq 0}} \frac{\|N(x) - N(0)\|}{\|x\|} \leq \sup_{\substack{x,y\in X,\\ x\neq y}} \frac{\|N(x) - N(y)\|}{\|x-y\|} = \|N\|^*,$$

so in particular $N - N(0)$ is linearly bounded for every $N \in \operatorname{Lip}(X,Y)$. On the other hand if $N(0) \neq 0, N$ is not linearly bounded, for the definition of linearly bounded operators implies that they vanish at zero. With these considerations we can define then

$$\operatorname{Lip}_0(X,Y) := \operatorname{Lip}(X,Y) \cap \operatorname{LB}(X,Y) = \{N \in \operatorname{Lip}(X,Y) \ : \ N(0) = 0\}.$$

$\|\cdot\|^*$ is a norm on $\operatorname{Lip}_0(X,Y)$.

The following Theorem from [58] characterizes invertibility of the operators between X and Y.

Theorem 11.3.1. *[58, Theorem 2] Let X a real normed space and Y a real Banach space. Let $N : X \to Y$ be an operator. Then N is invertible if and only if there exists an invertible operator $J : Y \to X$ such that $(N-J)J^{-1} \in \operatorname{Lip}(Y)$ and $\|(N-J)J^{-1}\|^* < 1$.*

In such a case, $\|N^{-1}\|^* \leq \|J^{-1}\|^*/(1 - \|(N-J)J^{-1}\|^*)$.

The following consequence (in the line of [57, Corollary 2]) can be obtained by taking $X = Y, N = \operatorname{Id} - Q, J = \operatorname{Id}$.

Corollary 11.3.2. *Let X be a real Banach space and $Q \in \operatorname{Lip}(X)$ such that $\|Q\|^* < 1$. Then $\operatorname{Id} - Q$ is an invertible operator and $\|(\operatorname{Id} - Q)^{-1}\|^* \leq 1/(1 - \|Q\|^*)$.*

Remark 11.3.3. Assume $Q \in \operatorname{Lip}(X)$, $Q(X)$ closed for the sum, $\|Q\|^* < 1$. Then

$$(\operatorname{Id} - Q)^{-1}|_{Q(X)} : Q(X) \to Q(X).$$

To see this take $x \in X$ and define $y = (\operatorname{Id} - Q)^{-1}Qx$. Then $y = Qx + Qy \in Q(X)$.

We now present a result which is an straightforward generalization to the case of linearly bounded operators of a classical result on linear operators. Let us define the following operators and constants.

$$
\begin{aligned}
H_1 u(t) &:= \sum_{j=1}^{m_1+n_1} \psi_{1j}(t)\varphi_{1j}[u],\\
H_2 u(t) &:= \sum_{j=1}^{m_2+n_2} |\psi_{2j}(t)||\varphi_{2j}[u]|,\\
L_2 u(t) &:= \int_0^1 |k(t,s)|g(s)u(s)\,\mathrm{d}s,\\
f_2^0 &:= \overline{\lim_{u\to 0}}\ \operatorname{ess\,sup}_{t\in I}\frac{f_2(t,u)}{|u|},\ f_{1,0} := \underline{\lim_{u\to 0^+}}\ \operatorname{ess\,inf}_{t\in[a,b]}\frac{f_1(t,u)}{u},\\
f_2^\infty &:= \overline{\lim_{|u|\to \infty}}\ \operatorname{ess\,sup}_{t\in I}\frac{f_2(t,u)}{|u|},\ f_{1,\infty} := \underline{\lim_{u\to \infty}}\ \operatorname{ess\,inf}_{t\in[a,b]}\frac{f_1(t,u)}{u}.
\end{aligned}
$$

Lemma 11.3.4. *Assume that condition* (11.1.3) *holds for every* $u,v\in C(I)$ *and* $H_2\in \mathrm{LB}(C(I))$, *then* $H_2\in \mathrm{Lip}_0(C(I))$.

Proof. Let $u,v\in C(I)$. Using inequality (11.1.3) and Remark 11.1.2.

$$
\begin{aligned}
|H_2u-H_2v| &= \left|\sum_{j=1}^{m_2+n_2}|\psi_{2j}(t)|\varphi_{2j}[u]-\sum_{j=1}^{m_2+n_2}|\psi_{2j}(t)|\varphi_{2j}[v]\right|\\
&= \left|\sum_{j=1}^{m_2+n_2}|\psi_{2j}(t)|\left(\varphi_{2j}[u]-\varphi_{2j}[v]\right)\right|\\
&\le \sum_{j=1}^{m_2+n_2}\|\psi_{2j}\|\left|\varphi_{2j}[u]-\varphi_{2j}[v]\right| \le \sum_{j=1}^{m_2+n_2}\|\psi_{2j}\|\left|\varphi_{2j}[u-v]\right|\\
&\le \sum_{j=1}^{m_2+n_2}\|\psi_{2j}\|\|\varphi_{2j}\|\|u-v\|.
\end{aligned}
$$

Hence, $H_2\in\mathrm{Lip}(C(I))$ and $\|H_2\|^*\le\sum_{j=1}^{m_2+n_2}\|\psi_{2j}\|\|\varphi_{2j}\|$. Also, since $H_2\in\mathrm{LB}(C(I))$, $H_2(0)=0$, so $H_2\in\mathrm{Lip}_0(C(I))$.

■

We recall now the following Theorem and Remark from [178], applied to our particular setting.

Theorem 11.3.5. *[178, Theorem 2.4] Let K_1 be a cone in a Banach space X, and let $\preceq$ denote the partial order in K_1. Suppose that a bounded linear operator $N : X\to X$ maps K_1 to K_1. Let there exist $\lambda_0>0$ and $u\in X$ such that $Nu\succeq\lambda_0 u$ where $-u\notin K_1$ and $u=u_1-u_2$ with $u_1,u_2\in K_1$. Then, if $r(N)<\lambda_0$, there exist $\lambda\ge\lambda_0$ and $v\in K_1\backslash\{0\}$ such that $Nv=\lambda v$.*

Remark 11.3.6. [178, Remark 2.5] If K_1 is a total cone, that is, $\overline{K_1-K_1}=X$, N is compact and continuous and $r(N)>0$, then $r(N)$ is an eigenvalue of N with an eigenvector in K_1.

Corollary 11.3.7. *The spectral radius of L_1 is an eigenvalue of L_1 with an eigenfunction in $P\cap K$.*

Proof. Recall that L_1 maps P to $P \cap K$ (see Remark 11.1.6). Choose, for the previous Theorem and Remark $K_1 = P$ and $N = L_1$. It is not difficult to verify that L_1 is compact and continuous and $r(L_1) > 0$. Also, P is a total cone.

Let $u \equiv 1$. Then $u \in P$ and we have by (C_3) that

$$L_1 u(t) = \int_a^b k^+(t,s) g(s) u(s)\,\mathrm{d}s \geq c \int_a^b \Phi(s) g(s)\,\mathrm{d}s = c \int_a^b \Phi(s) g(s)\,\mathrm{d}s\, u(t), t \in I,$$

that is $L_1 u \geq \lambda_0 u$ for $\lambda_0 = c \int_a^b \Phi(s) g(s)\,\mathrm{d}s$. Therefore, the hypothesis of Theorem 11.3.5 are satisfied and therefore there is $v \in P$ such that $L_1 v = r(L_1) v$. Since $L_1 : P \to P \cap K$, $v \in P \cap K$. ■

In order to prove the next result, we use the following operator on $C([a,b])$ defined by

$$\bar{L}u(t) := \int_a^b k^+(t,s) g(s) u(s)\,\mathrm{d}s,\ t \in [a,b]$$

and the cone $P_{[a,b]}$ of positive functions in $C([a,b])$.

Theorem 11.3.8. *We have the following.*

(1) *If $H_2 \in \mathrm{Lip}_0(C(I))$, $\|H_2\|^* < 1$, $(\mathrm{Id} - H_2)^{-1} L_2 \in \mathrm{LB}(C(I))$, $(\mathrm{Id} - H_2)^{-1} : P \cap K \to P \cap K$ order preserving, $(\mathrm{Id} - H_2)^{-1}(\lambda u) \leq \lambda (\mathrm{Id} - H_2)^{-1} u$ for every $\lambda \in \mathbb{R}^+$ and $u \in K \cap P$, and $0 \leq f_2^0 < \mu((\mathrm{Id} - H_2)^{-1} L_2)$, then there exists $\rho_0 \in \mathbb{R}^+$ such that*
$$i_K(T, K_\rho) = 1 \text{ for each } \rho \in (0, \rho_0].$$

(2) *If $\mu(L_1) < f_{1,0} \leq \infty$, then there exists $\rho_0 \in \mathbb{R}^+$ such that for each $\rho \in (0, \rho_0]$*
$$i_K(T, K_\rho) = 0.$$

(3) *If $\mu(L_1) < f_{1,\infty} \leq \infty$, then there exists R_1 such that for each $R \geq R_1$*
$$i_K(T, K_R) = 0.$$

Proof. (1)

Let $\xi = \mu((\mathrm{Id} - H_2)^{-1} L_2)$. By the hypotheses, there exist $\rho_0, \tau \in (0,1)$ such that

$$f_2(t,u) \leq (\xi - \tau)|u|$$

for all $u \in [-\rho_0, \rho_0]$ and almost every $t \in I$.

Let $\rho \in (0, \rho_0]$, we prove that $Tu \neq \lambda u$ for $u \in \partial K_\rho$ and $\lambda \geq 1$, which implies the result by Lemma 8.1.2. In fact, if we assume otherwise, then there exists $u \in \partial K_\rho$ and $\lambda \geq 1$ such that $\lambda u = Tu$. Observe that if $u \in K$, $|u| \in K \cap P$ and for $t \in I$,

$$\begin{aligned} |u(t)| \leq& \lambda |u(t)| = |Tu(t)| \leq H_2 u(t) + \int_0^1 |k(t,s)| g(s) f_2(s, u(s))\,\mathrm{d}s \\ \leq& H_2 |u|(t) + (\xi - \tau) L_2 |u|(t). \end{aligned}$$

Now,

$$|u|(t) \leq (\mathrm{Id}-H_2)^{-1}(\xi-\tau)L_2|u|(t) \leq (\xi-\tau)(\mathrm{Id}-H_2)^{-1}L_2|u|(t).$$

Iterating, that is, substituting the left hand side into the right hand side, for $n \in \mathbb{N}$,

$$|u|(t) \leq \ldots \leq \left[(\xi-\tau)(\mathrm{Id}-H_2)^{-1}L_2\right]^n |u|(t).$$

So, taking norms,

$$\|u\| \leq \|\left[(\xi-\tau)(\mathrm{Id}-H_2)^{-1}L_2\right]^n |u|\|,$$

which implies

$$1 \leq \|\left[(\xi-\tau)(\mathrm{Id}-H_2)^{-1}L_2\right]^n\|,$$

or

$$1 \leq (\xi-\tau)\|\left[(\mathrm{Id}-H_2)^{-1}L_2\right]^n\|^{\frac{1}{n}}.$$

Taking the limit on both sides we arrive to a contradiction,

$$1 \leq \frac{\xi-\tau}{\xi} < 1.$$

(2) There exists $\rho_0 > 0$ such that $f_1(t,u) \geq \mu(L_1)u$ for all $u \in [0,\rho_0]$ and almost all $t \in I$.

Let $\rho \in [0,\rho_0]$ and let us prove that $u \neq Tu + \lambda v_1$ for all u in ∂K_ρ and $\lambda \geq 0$, where $v_1 \in K$ is the eigenfunction (cf. Corollary 11.3.7) of L_1 with $\|v_1\| = 1$ corresponding to the eigenvalue $1/\mu(L_1)$, which would imply the result.

We distinguish now two cases, $\lambda \in \mathbb{R}^+$ and $\lambda = 0$. Assume, on the contrary, that there exist $u \in \partial K_\rho$ and $\lambda \in \mathbb{R}^+$ such that $u = Tu + \lambda v_1$. Since $Tu \geq 0$ in $[a,b]$, this implies $u \geq \lambda v_1$ in $[a,b]$ and $L_1 u \geq \lambda L_1 v_1 \geq [\lambda/\mu(L_1)]v_1$ in $[a,b]$. Using this and the previous estimate for f we have, by (C_4) and (C_6),

$$u \geq \mu(L_1)L_1 u + \lambda v_1 \geq \lambda\mu(L_1)L_1 v_1 + \lambda v_1 = 2\lambda v_1, \text{ in } [a,b].$$

Through induction we deduce that $\rho \geq u \geq n\lambda v_1$ in $[a,b]$ for every $n \in \mathbb{N}$, a contradiction because $v_1 \in K$.

Now we consider the case $\lambda = 0$. Let $\varepsilon > 0$ be such that for all $u \in [0,\rho_0]$ and almost every $t \in [a,b]$ we have

$$f_1(t,u) \geq (\mu(L_1)+\varepsilon)u.$$

Arguing as in the previous cases, we have, for $t \in [a,b]$,

$$u(t) \geq (\mu(L_1)+\varepsilon)L_1 u(t).$$

Since $L_1 v_1(t) = r(L_1)v_1(t)$ for $t \in [0,1]$, we have, for $t \in [a,b]$,

$$\bar{L}v_1(t) = L_1 v_1(t) = r(L_1)v_1(t),$$

and we obtain $r(\bar{L}) \geq r(L_1)$. On the other hand, we have, for $t \in [a,b]$,

$$\begin{aligned} u(t) =& Tu = Bu(t) + \int_0^1 k(t,s)g(s)f(s,u(s),Du(s))\,\mathrm{d}s \\ \geq& (\mu(L_1)+\varepsilon)\int_a^b k(t,s)g(s)u(s)\,\mathrm{d}s = (\mu(L_1)+\varepsilon)L_1u(t) = (\mu(L_1)+\varepsilon)\bar{L}u(t). \end{aligned}$$

where $u(t) > 0$ in $[a,b]$. Thus, using Theorem 10.3.4, we have $r(\bar{L}) \leq 1/(\mu(L_1)+\varepsilon)$ and therefore $r(L_1) \leq 1/(\mu(L_1)+\varepsilon)$. This gives $\mu(L_1)+\varepsilon \leq \mu(L_1)$, a contradiction.

(3) Take v_1 as in part (2). Let $R_1 \in \mathbb{R}^+$ such that $f_1(t,u) > \mu(L_1)u$ for all $u \geq cR_1$, c as in (C_4), and almost all $t \in I$. We will prove that $u \neq Tu + \lambda v_1$ for all u in ∂K_R and $\lambda \in \mathbb{R}^+$ when $R > R_1$. Observe that for $u \in \partial K_R$, we have $u(t) \geq c\|u\| \geq cR_1$ for all $t \in [a,b]$, so $f_1(t,u) > \mu(L_1)u$ on $[a,b]$.

Assume now, on the contrary, that there exist $u \in \partial K_R$ and $\lambda \in \mathbb{R}^+$ (the proof in the case $\lambda = 0$ is treated as in the proof of the statement (2)) such that $u = Tu + \lambda v_1$. This implies $u \geq \lambda v_1$ in $[a,b]$ and $L_1 u \geq \lambda L_1 v_1 \geq [\lambda/\mu(L_1)]v_1$ in $[a,b]$. Using this and the previous estimate for f we have

$$u \geq \mu(L_1)L_1u + \lambda v_1 \geq \lambda\mu(L_1)L_1v_1 + \lambda v_1 = 2\lambda v_1, \text{ in } [a,b].$$

Through induction we deduce that $R \geq u \geq n\lambda v_1$ for every $n \in \mathbb{N}$, a contradiction because $v_1 \in K$. ■

Remark 11.3.9. In the previous Theorem, in point (1), it is enough to ask for $L_2 \in \mathrm{LB}(C(I))$ in order to have $(\mathrm{Id}-H_2)^{-1}L_2 \in \mathrm{LB}(C(I))$ since $(\mathrm{Id}-H_2)^{-1} \in \mathrm{Lip}(C(I))$.

Remark 11.3.10. It can be checked that the spectral radius of a linearly bounded operator is bounded from above by the norm $\|\cdot\|$. Hence, in the previous Theorem, in point (1) the condition $0 \leq f_2^0 < \mu((\mathrm{Id}-H_2)^{-1}L_2)$ can be strenghtened to $0 \leq f_2^0 < 1/\|(\mathrm{Id}-H_2)^{-1}L_2\|^*$, and even further, through Corollary 11.3.2, to $0 \leq f_2^0 < (1-\|H_2\|^*)/\|L_2\|$.

Remark 11.3.11. In the previous Theorem, the conditions $\mu(L_1) < f_{1,0} \leq \infty$ and $\mu(L_1) < f_{1,\infty} \leq \infty$ in (2) and (3) respectively can be strengthen in order to avoid the computation of the spectral value of L_1. As it is shown in [183], the new conditions would be

$$1/\inf_{t\in[0,1]}\int_a^b k(t,s)g(s)\,\mathrm{d}s < f_{1,0} \leq \infty$$

and

$$1/\inf_{t\in[0,1]}\int_a^b k(t,s)g(s)\,\mathrm{d}s < f_{1,\infty} \leq \infty.$$

11.4 An applicaton

In order to prove the usefulness of our theory, we present a simple but yet fairly general application in this Section. Consider the boundary value problem

$$-u''(t) = f(t,u(t)) + \gamma(t)u(\eta(t)),\ t \in [0,1] \quad u(0) = 0, u(1) = \theta\|u\|. \tag{11.4.1}$$

where f satisfies the L^∞-Carathéodory conditions (see (C_5)), $\gamma \in C(I)$, $\gamma \geq 0$, $\theta \in (0,1)$ and $\eta : I \to I$ is a measurable function such that for a fixed $[a,b] \subset (0,1)$ such that $\eta([a,b]) \subset [a,b]$. Note that $u \circ \eta$ is in $\mathrm{L}^\infty(I)$.

We could consider more complex boundary conditions or nonlinearities, but for the sake of simplicity and insight we will keep it this way. Observe that this problem is equivalent to

$$u(t) = \int_0^1 k(t,s)\, [f(s,u(s)) + \gamma(s)u(\eta(s))]\, \mathrm{d}s + \theta t\|u\|,$$

where

$$k(t,s) := \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Observe that k is nonnegative. Take $\Phi(s) = \sup_{t\in I} k(t,s) = s(1-s)$. By direct calculation we obtain

$$\tilde{\Phi}(s) := \inf_{t\in[a,b]} k(t,s) = \begin{cases} s(1-b), & 0 \leq s \leq \frac{a}{1-(b-a)}, \\ a(1-s), & \frac{a}{1-(b-a)} \leq s \leq 1. \end{cases}$$

Thus, $\inf_{s\in I} \tilde{\Phi}(s)/\Phi(s) = \min\{a, 1-b\}$, so we have to take $c \leq \min\{a,1-b\}$. Fix $c = \min\{a,1-b\}$. Observe that, for $u \in K$,

$$\begin{aligned} f(t,u(t)) + \gamma(t)u(\eta(t)) \leq& f(t,u(t)) + \gamma(t)\|u\|,\ t \in I, \\ f(t,u(t)) + \gamma(t) \min_{t\in[a,b]} u(t) \leq& f(t,u(t)) + \gamma(t)u(\eta(t)),\ t \in [a,b]. \end{aligned}$$

Hence, take $g \equiv 1$, $f_i = f$, $m_i = n_i = 1$, $i = 1,2$, $\varphi_1[u] = \min_{t\in[a,b]} u(t)$, $\varphi_2[u] = \|u\|$, $\tilde{\gamma}_1(t) = \tilde{\gamma}_2(t) = \tilde{\gamma}(t) = \int_0^1 k(t,s)\gamma(s)\,\mathrm{d}s + \theta t$.

Observe that, with these definitions, conditions (C_1)–(C_5), (C_7), (C_9) and (C_{10}) are satisfied. Assume now that $\tilde{\gamma}(t) \geq c\|\tilde{\gamma}\|$ for $t \in [a,b]$ and $\|\tilde{\gamma}\| < 1$. Then we have that (C_6) and (C_8) are also satisfied.

If we write condition (I_ρ^1) in terms of the choices we have made, we get

$$f^{-\rho,\rho} \sup_{t\in I} \left(\frac{\tilde{\gamma}(t)}{6(1-\|\tilde{\gamma}\|)} + \frac{1}{2}t(1-t) \right) < 1.$$

Of course, a sufficient condition in order for (I_ρ^1) to be satisfied, which is easier to check, is

$$f^{-\rho,\rho} \left(\frac{\|\tilde{\gamma}\|}{6(1-\|\tilde{\gamma}\|)} + \frac{1}{8} \right) < 1.$$

If we write condition (I_ρ^0) in terms of the choices we have made, we get

$$f_{\rho,\rho/c} \left(\frac{\|\tilde{\gamma}\|}{1-\|\tilde{\gamma}\|} \cdot \frac{a(1-b)[2-(b-a)]}{2c[1-(b-a)]} + \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s \right) > 1.$$

Example 11.4.1. Let us now consider a particular case. Take $f(t,u) = tu^2$, $\gamma(t) = t(1-t) + \frac{1}{4}$, $\theta = 1/2$ in the boundary value problem (11.4.1). Fix $\rho_1 = 5/2$, $\rho_2 = 4$, $a = 1/4$, $b = 3/4$. With this data, we have $c = 1/4$, $f^{-\rho_1,\rho_1} = \rho_1^2 = 25/4$, $f_{\rho_2,\rho_2/c} = 4$.

Also, $\tilde{\gamma}(t) = \frac{1}{24}t\left(17-3t-4t^2+2t^3\right)$, $\|\tilde{\gamma}\| = 1/2$ and

$$\begin{aligned}&\sup_{t\in I}\left(\frac{\tilde{\gamma}(t)}{6(1-\|\tilde{\gamma}\|)}+\frac{1}{2}t(1-t)\right)\\ =&\frac{1}{2}\left(1+5\sqrt{2}\cos\left(\frac{1}{3}\cot^{-1}\left(\frac{3}{\sqrt{31241}}\right)\right)-5\sqrt{6}\sin\left(\frac{1}{3}\cot^{-1}\left(\frac{3}{\sqrt{31241}}\right)\right)\right)\\ =&0.540002\dots\end{aligned}$$

Hence, condition $(I^1_{\rho_1})$ is satisfied.

Also,

$$\left(\frac{\|\tilde{\gamma}\|}{1-\|\tilde{\gamma}\|}\cdot\frac{a(1-b)[2-(b-a)]}{2c[1-(b-a)]}+\inf_{t\in[a,b]}\int_a^b k(t,s)\,\mathrm{d}s\right)=\frac{7}{16},$$

so condition $(I^0_{\rho_2})$ is satisfied. Therefore (S_2) in Theorem 11.2.5 is satisfied and problem (11.4.1) has at least a solution which is positive in $[1/4, 3/4]$.

We now apply Theorem 11.3.8 to the boundary value problem

$$-u''(t) = f(t,u(t))+\gamma(t)u(\eta(t)),\ u(0)=u'(1), u'(0)=u(1), \tag{11.4.2}$$

rewriting sufficient conditions according to Remarks 11.3.9–11.3.11, for the points $(1)-(3)$ to be satisfied. First, let us bound $\|L_2\|$ from above.

$$k(t,s)=\begin{cases}1+(1-s)t, & 0\le s\le t\le 1,\\ 1-s+(2-s)t, & 0<t<s\le 1,\end{cases}$$

$$L_2u(t)=\int_0^1|k(t,s)|u(s)\,\mathrm{d}s\le\int_0^1|k(t,s)|\,\mathrm{d}s\|u\|.$$

Hence we obtain

$$\|L_2\|\le\sup_{t\in[0,1]}\int_0^1|k(t,s)|\,\mathrm{d}s=\frac{3}{2}.$$

Also, assuming $\|\tilde{\gamma}\|<1$, take $H_2u(t)=\|\tilde{\gamma}\|\|u\|\quad\forall t\in I$.

In this case $H_2(K\cap P)=\{r\|\tilde{\gamma}\| : r\in[0,+\infty)\}$ is a cone and therefore closed for the sum, which means, by Remark 11.3.3, that $(\mathrm{Id}-H_2)^{-1}$ maps $K\cap P$ to itself. Furthermore, we have that

$$(\mathrm{Id}-H_2)^{-1}u(t)=u(t)+\frac{\|\tilde{\gamma}\|}{1-\|\tilde{\gamma}\|}\|u\|,$$

for $\|\tilde{\gamma}\|\le 1/2$, which satisfies $(\mathrm{Id}-H_2)^{-1}u\le(\mathrm{Id}-H_2)^{-1}v$, $(\mathrm{Id}-H_2)^{-1}(\lambda u)\le\lambda(\mathrm{Id}-H_2)^{-1}u$ for every $u\le v$, $u,v\in P\cap K$, $\lambda\in\mathbb{R}^+$.

On the other hand, we have

$$\inf_{t\in[0,1]}\int_a^b k(t,s)\,\mathrm{d}s=\frac{1}{2}(b-a)(2-a-b).$$

With these values, we have

(1) $\|\tilde{\gamma}\| \le \frac{1}{2}$, $0 \le f_2^0 < \frac{2}{3}(1 - \|\tilde{\gamma}\|)$,

(2) $\|\tilde{\gamma}\| < 1$, $0 \le 2/[(b-a)(2-a-b)] < f_{1,0} \le \infty$,

(3) $\|\tilde{\gamma}\| < 1$, $0 \le 2/[(b-a)(2-a-b)] < f_{1,\infty} \le \infty$.

Example 11.4.2. Consider again $f(t,u) = tu^2$, $\gamma(t) = t(1-t) + \frac{1}{4}$, $\theta = 1/2$, $a = 1/4$, $b = 3/4$; this time in the boundary value problem (11.4.2). We have that $f_2^0 = f_0 = 0$ and $f^\infty = f_\infty = +\infty$. Hence, the conditions (1) and (3) in Theorem 11.3.8 are satisfied and therefore, by Lemma 8.1.2, the boundary value problem (11.4.2) has at least a solution.

A. A Hyperbolic Analog of the Phasor Addition Formula

A.1 Introduction

The idea for this chapter was born when the author was confronted with the need of simplifying linear combinations of hyperbolic sines and cosines with the same argument into a single trigonometric expression in order to solve for that argument (see Section 4.3). In the usual euclidean case, there are very well know formulae for the sum of linear combinations of sines and cosines. In particular, we have the phasor addition formula (equations (A.3.1)–(A.3.2) are some of its incarnations) which, somehow, is a generalization of the standard formula $\cos x + \sin x = \sqrt{2}\sin(x + \pi/4)$. Nevertheless, similar formulae for the hyperbolic case seem to be absent from the literature, thus the results of this Chapter were published in [166].

It is interesting to note that something that seems so trivial as a mere algebraic manipulation has profound (and very well studied) roots in physics, where these linear combinations (in the euclidean case) occur naturally when studying *phasors*. This chapter is written with the intention of introducing the reader to the usual phasor formalism used in physics and the motivation behind it, containing all the rigor expected by a mathematician. It will also generalize the formulae previously derived for the hyperbolic case with the hope they may eventually become handy for the reader.

A.2 Phasors in physics

To be more precise, phasors appear in Physics from the need of establishing some kind of arithmetic for the set of functions

$$\mathcal{F} := \{f : \mathbb{R} \to \mathbb{R} \ : f(t) = a\cos(\omega t + \varphi),\ a \in \mathbb{R},\ \varphi \in \mathbb{R}|_{\sim}\},$$

for some fixed $\omega \in \mathbb{R}\backslash\{0\}$ and where $\varphi_1 \sim \varphi_2$ if and only if $\varphi_1 - \varphi_2 \in 2\pi\mathbb{Z}$ for any $\varphi_1, \varphi_2 \in \mathbb{R}$. The parameters present in the functions of $\mathcal{F}$ are called, respectively, *amplitude* (a), *frequency* (ω) and *phase* (φ). The functions in $\mathcal{F}$ occur mostly in problems related to Mechanics and Electronics (see, for instance, [56, 107, 140]), but their origin is rooted in arguably the most important problem in Physics: the *harmonic oscillator*.

If we consider one space variable x and a time variable t, the *Euler-Lagrange equation* of motion (a fundamental principle of Dynamics) implies that the equation of motion of a free particle is given by

$$m\,x''(t) + V'(x(t)) = 0, \tag{A.2.1}$$

where m is the mass of the particle and $V : \mathbb{R} \to \mathbb{R}$ is a given potential. Equation (A.2.1) is, basically, *Newton's second Law of motion* for the potential V.

In many problems of Physics it is common to chose as potential a quadratic function of the kind $V(x) = \frac{1}{2}kx^2$ with $k > 0$. This is the case, for instance, of *Hook's Law* on the force of a spring, but this kind of potential also occurs in problems concerning pendula (when the angle of displacement is considered to be small), RLC circuits, or acoustical systems. If fact, this potential appears naturally when taking a 'first order' approximation for small perturbations on a mass in a stable equilibrium with respect to the forces it is subject to.

Hence, considering $V(x) = \frac{1}{2}kx^2$, and defining $\omega = \sqrt{k/m}$, we have that equation (A.2.1) can be expressed as

$$x''(t) + \omega^2 x(t) = 0,$$

which is known as the equation of the *harmonic oscillator*.

The set of solutions of this equation is precisely

$$\{a \cos \omega t + b \cos(\omega t + \pi/2) \, : \, a, b \in \mathbb{R}\}$$

(observe that $-\cos(\omega t + \pi/2) = \sin \omega t$). Therefore, the need for adding functions in $\mathcal{F}$ appears in a natural way, because they are the solutions of one or more harmonic oscillators with the same constant k.

Now, the question that almost any mathematician would ask is, 'what happens when $k < 0$?' This situation has to do with the theory of critical phenomena [157]. Briefly speaking, the potential has a critical point at $k = 0$ and for $k < 0$ the physical laws change qualitatively. This is the case of phase transitions in matter, for instance, the change from liquid to vapor or from being a normal conductor to being a superconductor.

In this new scenario, we can define $\omega = \sqrt{-k/m}$ and the equation derived from equation (A.2.1) is

$$x''(t) - \omega^2 x(t) = 0, \tag{A.2.2}$$

which has

$$\{a \cosh \omega t + b \sinh \omega t \, : \, a, b \in \mathbb{R}\}$$

as set of solutions. Now, can we develop a hyperbolic version of the phasor understanding of equation (A.2.2)? Section A.4 will answer this question and in Section A.3 we establish the basics of the phasor formalism. Finally, Section A.5 is a brief note on the possible extensions of the phasor addition formula and a new way of obtaining it.

A.3 The phasor addition formula

Fix $\omega \in \mathbb{R}$. First of all, we will show that $\mathcal{F}$ is a group using some basic group algebra. Let

$$\mathcal{F}_{\mathbb{C}} := \{f : \mathbb{R} \to \mathbb{C} \, : \, f(t) = ze^{i\omega t}, \, z \in \mathbb{C}\}.$$

The functions in $\mathcal{F}$ are called *phasors*. Observe that the map $P : \mathcal{C}(\mathbb{R}, \mathbb{C}) \to \mathcal{C}(\mathbb{R}, \mathbb{C})$ such that $Pf(t) = f(t)/e^{i\omega t}$ is a group isomorphism with respect to the sum. We have that $\mathcal{F}$ is a subset of $\mathcal{C}(\mathbb{R}, \mathbb{C})$ and $(\mathbb{C}, +)$, identified with the set of constant functions of $\mathcal{C}(\mathbb{R}, \mathbb{C})$, is a subgroup of $\mathcal{C}(\mathbb{R}, \mathbb{C})$. Furthermore, $P|_{\mathcal{F}_{\mathbb{C}}} : \mathcal{F}_{\mathbb{C}} \to \mathbb{C}$ is bijective. Hence, $(\mathcal{F}_{\mathbb{C}}, +)$ is a group. To see this it suffices to see that $x + y \in \mathcal{F}_{\mathbb{C}}$ for any given $x, y \in \mathcal{F}_{\mathbb{C}}$. $P(x), P(y) \in \mathbb{C}$ and, since $(\mathbb{C}, +)$ is a group, $P(x) + P(y) \in \mathbb{C}$. Thus, $P^{-1}(P(x) + P(y)) = x + y \in \mathcal{F}_{\mathbb{C}}$.

On the other hand, consider the real part operator $\mathfrak{R} : \mathcal{C}(\mathbb{R},\mathbb{C}) \to \mathcal{C}(\mathbb{R},\mathbb{R})$. $\mathfrak{R}$ is a surjective homomorphism and $\mathfrak{R}|_{\mathcal{F}_\mathbb{C}} : \mathcal{F}_\mathbb{C} \to \mathcal{F}$ is a surjective function. Thus, $\mathcal{F}$ is also a group. To see this, let $x, y \in \mathcal{F}$ and $x', y' \in \mathcal{F}_\mathbb{C}$ such that $\mathfrak{R}(x') = x$, $\mathfrak{R}(y') = y$. Hence, $x'+y' \in \mathcal{F}_\mathbb{C}$ and $\mathfrak{R}(x'+y') = x+y \in \mathcal{F}$. Due to these homomorphisms between the considered groups, to study the sum in $\mathcal{F}$, it is enough to study the sum in $\mathbb{C}$.

Let $a\,e^{i\varphi}, b\,e^{i\psi} \in \mathbb{C}\backslash\{0\}$. Then $a\,e^{i\varphi} + b\,e^{i\psi} = c\,e^{i\theta}$ for some $c \in \mathbb{R}^+$ and $\theta \in \mathbb{R}|_\sim$. Observe that

$$a\,e^{i\varphi} = a\cos\varphi + ia\sin\varphi, \quad b\,e^{i\psi} = b\cos\psi + ib\sin\psi,$$

so

$$a\,e^{i\varphi} + b\,e^{i\psi} = a\cos\varphi + b\cos\psi + i(a\sin\varphi + b\sin\psi).$$

Therefore, using the law of cosines,

$$\begin{aligned} c = |a\,e^{i\varphi} + b\,e^{i\psi}| &= \sqrt{(a\cos\varphi + b\cos\psi)^2 + (a\sin\varphi + b\sin\psi)^2} \\ &= \sqrt{a^2 + b^2 + 2\,a\,b\cos(\varphi - \psi)}. \end{aligned}$$

In order to get θ, we consider the principal argument function arg[†] such that, for every $z = x + iy \in \mathbb{C}$, $\arg(z) = \alpha$ where α is the only angle in $[-\pi, \pi)$ satisfying $\sin\alpha = y/\sqrt{x^2+y^2}$ and $\cos\alpha = x/\sqrt{x^2+y^2}$.

Therefore, $\theta = \arg(a\cos\varphi + b\cos\psi + i(a\sin\varphi + b\sin\psi))$. So we can conclude that

$$a\,e^{i\varphi} + b\,e^{i\psi} = \sqrt{a^2+b^2+2\,a\,b\cos(\varphi-\psi)}\,e^{i\arg(a\cos\varphi + b\cos\psi + i(a\sin\varphi + b\sin\psi))}. \tag{A.3.1}$$

Equation (A.3.1) is called the *phasor addition formula*.

If we want to write equation (A.3.1) in terms of the elements of $\mathcal{F}$, we just have to take the real part on both sides of the equation:

$$\begin{aligned} a\cos(\omega t + \varphi) + b\cos(\omega t + \psi) = &\sqrt{a^2+b^2+2\,a\,b\cos(\varphi-\psi)} \\ &\cdot \cos[\omega t + \arg(a\cos\varphi + b\cos\psi + i(a\sin\varphi + b\sin\psi))]. \end{aligned}$$

In particular,

$$\begin{aligned} &a\cos(\omega t) + b\sin(\omega t) = a\cos(\omega t) - b\cos(\omega t + \pi/2) \\ =&\sqrt{a^2+b^2}\cos[\omega t + \arg(a - ib)] = \sqrt{a^2+b^2}\sin[\omega t + \arg(b + ia)]. \end{aligned} \tag{A.3.2}$$

From this last formula, we can recover the phasor addition formula just by observing the classical trigonometric identities $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ and $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.

There is an straightforward geometrical representation of the phasor addition formula in the euclidean case as Figure 1 shows. The key to this graphical representation is that, on $\mathcal{F}_\mathbb{C}$,

[†] The principal argument function is basically the `atan2` function common to the math libraries of many computer languages such as FORTRAN [138, p. 42], C, Java, Python, Ruby or Pearl. The principal advantage of having two arguments instead of one, unlike in the traditional definition of the arctan function, is that it returns the appropriate quadrant of the angle, something that cannot be achieved with the arctan. Some more basic information on the `atan2` function and its usage can be found at `http://en.wikipedia.org/wiki/Atan2`.

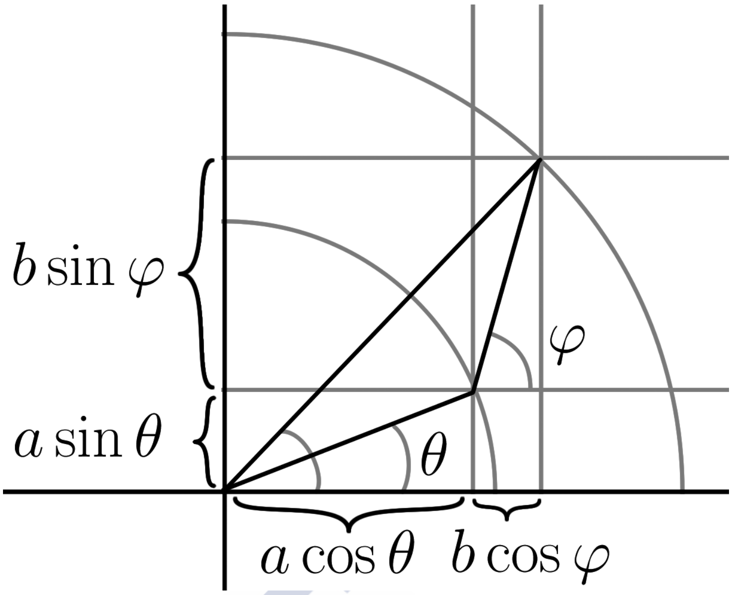


Figure A.3.1: Graphical representation of $a\cos\theta + b\cos\varphi$ and $a\sin\theta + b\sin\varphi$

the sum is the sum of vectors on the plane. Then we just have to take the real part of this sum, that is, the projection onto the OX axis, to obtain the desired result.

A.4 The hyperbolic version of the phasor addition formula

We now obtain a hyperbolic counterpart of the phasor addition formula as expressed in equation (A.3.2).

Let

$$\mathcal{G} := \{f : \mathbb{R} \to \mathbb{R} \ : \ f(t) = a\cosh\omega t + b\sinh\omega t;\ a, b \in \mathbb{R}\}.$$

It is straightforward to check that $(\mathcal{G}, +)$ is a group (and a 2-dimensional real vector space). Taking into account the identities

$$\begin{aligned}\cosh(x+y) &= \sinh x\sinh y + \cosh x\cosh y,\\ \sinh(x+y) &= \cosh x\sinh y + \sinh x\cosh y,\end{aligned}$$

it is clear that

$$\begin{aligned}&a\cosh(\omega t+\varphi) + b\sinh(\omega t+\psi)\\ =&(a\cosh\varphi + b\sinh\psi)\cosh\omega t + (a\sinh\varphi + b\cosh\psi)\sinh\omega t \in \mathcal{G}.\end{aligned}$$

It is also clear that

$$\begin{aligned}&a\cosh(\omega t+\varphi) + b\cosh(\omega t+\psi)\\ =&(a\cosh\varphi + b\cosh\psi)\cosh\omega t + (a\cosh\varphi + b\cosh\psi)\sinh\omega t \in \mathcal{G},\end{aligned}$$

and

$$a\sinh(\omega t+\varphi) + b\sinh(\omega t+\psi)$$

$$=(a\sinh\varphi + b\sinh\psi)\cosh\omega t + (a\sinh\varphi + b\sinh\psi)\sinh\omega t \in \mathcal{G}.$$

So we can reduce the general sums $a\cosh(\omega t+\varphi)+b\sinh(\omega t+\psi)$, $a\cosh(\omega t+\varphi)+b\cosh(\omega t+\psi)$ and $a\sinh(\omega t+\varphi)+b\sinh(\omega t+\psi)$ to the more simple case of $\alpha\cosh\omega t+\beta\sinh\omega t$.

Now we prove the following hyperbolic version of the phasor addition formula.

Lemma A.4.1. *Let a, b, $t\in\mathbb{R}$. Then*

$$a\cosh\omega t + b\sinh\omega t = \begin{cases} \sqrt{|a^2-b^2|}\cosh\left(\frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|+\omega t\right) & \text{if} \quad a>|b|,\\ -\sqrt{|a^2-b^2|}\cosh\left(\frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|+\omega t\right) & \text{if} \quad -a>|b|,\\ \sqrt{|a^2-b^2|}\sinh\left(\frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|+\omega t\right) & \text{if} \quad b>|a|,\\ -\sqrt{|a^2-b^2|}\sinh\left(\frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|+\omega t\right) & \text{if} \quad -b>|a|,\\ a\,e^{\omega t} & \text{if} \quad a=b,\\ a\,e^{-\omega t} & \text{if} \quad a=-b. \end{cases} \tag{A.4.1}$$

Proof. For convenience, let $c=e^{\omega t}$. We prove the case $a>|b|$. The rest of the cases are proved in an analogous fashion.

Observe that, if $a>|b|$, then $a+b, a-b>0$. Thus,

$$\begin{aligned} &a\cosh\omega t + b\sinh\omega t\\ =&\frac{a}{2}(c+c^{-1})+\frac{b}{2}(c-c^{-1})=\frac{a+b}{2}c+\frac{a-b}{2}c^{-1}\\ =&\frac{\sqrt{a^2-b^2}}{2}\left(\sqrt{\frac{a+b}{a-b}}c+\sqrt{\frac{a-b}{a+b}}c^{-1}\right)=\frac{\sqrt{a^2-b^2}}{2}\left(e^{\ln\sqrt{\frac{a+b}{a-b}}}c+e^{-\ln\sqrt{\frac{a+b}{a-b}}}c^{-1}\right)\\ =&\frac{\sqrt{a^2-b^2}}{2}\left(e^{\frac{1}{2}\ln\frac{a+b}{a-b}}c+e^{-\frac{1}{2}\ln\frac{a+b}{a-b}}c^{-1}\right)\\ =&\frac{\sqrt{a^2-b^2}}{2}\left(e^{\frac{1}{2}\ln\frac{a+b}{a-b}+\omega t}+e^{-\left(\frac{1}{2}\ln\frac{a+b}{a-b}+\omega t\right)}\right)=\sqrt{a^2-b^2}\cosh\left(\frac{1}{2}\ln\frac{a+b}{a-b}+\omega t\right). \end{aligned}$$

■

Remark A.4.2. One of the crucial differences between the hyperbolic and euclidean cases is that in the hyperbolic case there is not periodicity[†], what is more, we cannot relate the hyperbolic sine and cosine by a phase displacement, which implies that we may or may not be able to express an element of $\mathcal{G}$ in the form of a hyperbolic cosine depending on the values of a and b, as Lemma A.4.1 shows.

Also, comparing it with formula (A.3.2), we observe two common elements. First, the argument of the function (euclidean or hyperbolic) involved is ωt plus a displacement depending on the parameters a and b. The second similitude is that, multiplying such function, there is a metric applied to the vector (a,b). In the euclidean case case, it is just the euclidean norm $\|(a,b)\|=\sqrt{a^2+b^2}$, that is, the square root of the metric $\mu(a,b)=a^2+b^2$ on $\mathbb{R}^2$. In the

[†]Not, at least, when we consider those functions as defined on the real numbers. Hyperbolic functions are periodic when defined on the complex plane.

hyperbolic case, however, we have what is called the *Minkowski norm* $\|(a,b)\|_M = \sqrt{|\nu(a,b)|}$ where $\nu(a,b) = a^2 - b^2$ is the *Minkowski metric* on $\mathbb{R}^2$ of signature $(1,-1)$. The Minkowski norm is not a norm in the usual sense (it is not subadditive), but it provides a useful generalization of the concept of 'length' in the Minkowski plane[†].

The vectors $w = (a,b)$ are called *timelike* when $\nu(w,w) < 0$, *spacelike* when $\nu(w,w) > 0$ and **null**, or *lightlike* when $\nu(w,w) = 0$. Observe that the two first cases of equation (A.4.1) are for spacelike vectors, the two following ones for timelike vectors, and the two last ones for lightlike vectors.

It is also possible to give a geometrical representation of linear combinations of hyperbolic sines and cosines but, due to the euclidean nature of the plane, it is not as straightforward as in the euclidean case. In Figure 2 we illustrate how $a\cosh u + b\sinh u$ can be computed graphically.

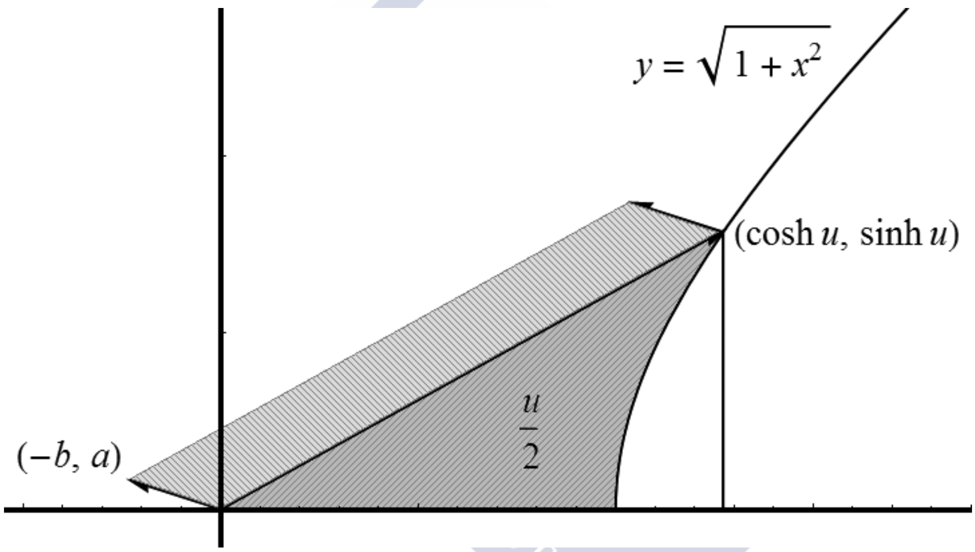


Figure A.4.1: Graphical representation of $a\cosh u + b\sinh u$

Consider $a, b, u > 0$. The graph of the hyperbola $y^2 - x^2 = 1$ satisfies that its points are of the form $(\cosh u, \sinh u)$. Furthermore, the area between the vector $(\cosh u, \sinh u)$, the hyperbola and the OX axis is half the **hyperbolic angle** u. Now, if we draw the vector $(-b, a)$ and consider the parallelogram formed by the vectors $(\cosh u, \sinh u)$ and $(-b, a)$, the area of this parallelogram is precisely $a\cosh u + b\sinh u$. The reason for this is given by the cross product formula for the area of the parallelogram and the fact that $u > 0$:

$$\begin{aligned}&|(\cosh u, \sinh u, 0) \times (-b, a, 0)| = |(0, 0, a\cosh u + b\sinh u)|\\ =&|a\cosh u + b\sinh u| = a\cosh u + b\sinh u.\end{aligned}$$

[†]For more information on this topic, the book [47] has a whole chapter on the trigonometry of the Minkowski plane.

A.5 A final note: extending the formula

If there is anything powerful behind the concept of exponential, hyperbolic sine, hyperbolic cosine, and other trigonometric functions, it is their wide range of definition. By this, we mean that they are defined in any Banach algebra with unity[†]. Let $\mathcal{A}$ be a Banach algebra and $x \in \mathcal{A}$. We define, as usual,

$$
\begin{aligned}
e^x &:= \sum_{k=0}^{\infty} \frac{x^k}{k!}, \\
\cosh x &:= \frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \\
\sinh x &:= \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.
\end{aligned}
$$

Clearly, $\cosh$ is just the even part of the exponential and $\sinh$ its odd part, so $e^x = \cosh x + \sinh x$. If we go back to the proof of Lemma A.4.1, we observe that it relies only on these kind of definitions, so it is valid for every $a, b \in \mathbb{R}$ and any $\gamma = \omega t$ in a real Banach algebra with unity $\mathcal{A}$, in particular for $\gamma \in \mathbb{C}$. This is consistent with the euclidean phasor addition formula as we show next. Let $a, b, x \in \mathbb{R}$, assume, for instance, $a > |b|$ and consider $a \cosh ix + b \sinh ix$. Then, using Lemma A.4.1,

$$
\begin{aligned}
& a \cosh ix + b \sinh ix = \sqrt{a^2 - b^2} \cosh\left(\frac{1}{2} \ln \frac{a+b}{a-b} + ix\right) \\
=& \frac{\sqrt{a^2 - b^2}}{2} \left(\sqrt{\frac{a+b}{a-b}} e^{ix} + \sqrt{\frac{a-b}{a+b}} e^{-ix}\right) \\
=& \frac{1}{2}\left[(a+b)(\cos x + i \sin x) + (a-b)(\cos x - i \sin x)\right] = a \cos x + ib \sin x
\end{aligned}
$$

which is expected from the known fact that $\cosh ix = \cos x$, $\sinh ix = i \sin x$.

This observation relating the generality of the definitions of the trigonometric functions suggests yet another question. Is there a way to derive the hyperbolic phasor addition formula in the same way we derived it for the euclidean case? Or, to be more precise, is there a Banach algebra which would fulfill the role $\mathbb{C}$ played in the euclidean case? The answer is yes. Remember the traditional definition of the complex numbers:

$$\mathbb{C} = \{x + iy \ : \ x, y \in \mathbb{R}, \ i \notin \mathbb{R}, \ i^2 = -1\}.$$

In the same way, we can define the hyperbolic numbers[‡]:

$$\mathbb{D} = \{x + jy \ : \ x, y \in \mathbb{R}, \ j \notin \mathbb{R}, \ j^2 = 1\}.$$

[†] A Banach algebra $\mathcal{A}$ is just an algebra endowed with a norm $\|\cdot\|$ that makes it a Banach space such that $\|xy\| \leq \|x\|\|y\|$ for every $x, y \in \mathcal{A}$.

[‡] See [6, 47] for an extended description on hyperbolic number arithmetic, calculus and geometry. It is also interesting to point out that hyperbolic numbers are a natural setting for the Theory of Relativity.

We introduced the the hyperbolic numbers in Section 5.4.1. Here we recall that, as in the case of the complex numbers, the arithmetic in $\mathbb{D}$ is the natural extension assuming the distributive, associative, and commutative properties for the sum and product. Several definitions appear in a natural way, parallel to the case of $\mathbb{C}$.

Let $w \in \mathbb{D}$, with $w = x + jy$. Hence

$$\overline{w} := x - jy, \quad \mathfrak{R}(w) := x, \quad \mathfrak{I}(w) := y,$$

and since $w\overline{w} = x^2 - y^2 \in \mathbb{R}$, we can define

$$|w| := \sqrt{|w\overline{w}|},$$

which is *precisely* the Minkowski norm. It follows that $|w_1 w_2| = |w_1||w_2|$ for every $w_1, w_2 \in \mathbb{D}$ and, if $|w| \neq 0$, then $w^{-1} = \overline{w}/|w|^2$. If we define

$$\|w\| = \sqrt{2(x^2 + y^2)},$$

we have that $\|\cdot\|$ is a norm and $(\mathbb{D}, \|\cdot\|)$ is a Banach algebra, so the exponential and the hyperbolic trigonometric functions are well defined. Also, it is clear from the definitions that

$$e^{jw} = \cosh w + j \sinh w,$$

and $|e^{jx}| = 1$ for $x \in \mathbb{R}$.

The only important difference with respect to $\mathbb{C}$ is that $\mathbb{D}$ is not a division algebra (not every nonzero element has an inverse).

Now, let $a, b \in \mathbb{R}$ and $\gamma = \gamma_1 + j\gamma_2 \in \mathbb{D}$ with $\gamma_1, \gamma_2 \in \mathbb{R}$. Observe that

$$\mathfrak{R}([a + jb]e^{j\gamma}) = a \cosh \gamma + b \sinh \gamma.$$

We try, as we do with complex numbers, to rewrite $(a + jb)e^{j\gamma}$ as $r\,e^{j\theta}$, where $r \in [0, +\infty)$ and $\theta \in \mathbb{R}$. Assume $|a + jb| \neq 0$. Then

$$r = |(a + jb)e^{j\gamma}| = |a + jb|e^{\gamma_2},$$

and

$$\begin{aligned}(a + jb)e^{j\gamma} =& e^{\gamma_2}[a \cosh \gamma_1 + b \sinh \gamma_1 + j(a \sinh \gamma_1 + b \cosh \gamma_1)] \\ =& |a + jb|e^{\gamma_2} \cosh \theta + j|a + jb|e^{\gamma_2} \sinh \theta = r\,e^{j\theta}.\end{aligned}$$

Therefore,

$$a \cosh \gamma_1 + b \sinh \gamma_1 = |a + jb| \cosh \theta \quad \text{and} \quad b \cosh \gamma_1 + a \sinh \gamma_1 = |a + jb| \sinh \theta.$$

That is, assuming $a > |b|$ and defining $\sigma = \operatorname{arctanh}(b/a)$,

$$\tanh \theta = \frac{b \cosh \gamma_1 + a \sinh \gamma_1}{a \cosh \gamma_1 + b \sinh \gamma_1} = \frac{\frac{b}{a} + \tanh \gamma_1}{1 + \frac{b}{a} \tanh \gamma_1} = \tanh(\sigma + \gamma_1),$$

so

$$\theta = \operatorname{arctanh}\frac{b}{a} + \gamma_1 = \frac{1}{2}\ln\frac{1+\frac{b}{a}}{1-\frac{b}{a}} + \gamma_1 = \frac{1}{2}\ln\frac{a+b}{a-b} + \gamma_1.$$

Hence,

$$\begin{aligned} a\cosh\gamma + b\sinh\gamma =& |a+jb|e^{\gamma_2}\Re\left(e^{j\left(\frac{1}{2}\ln\frac{a+b}{a-b}+\gamma_1\right)}\right) \\ =& |a+jb|e^{\gamma_2}\cosh\left(\frac{1}{2}\ln\frac{a+b}{a-b}+\gamma_1\right). \end{aligned}$$

For $\gamma \in \mathbb{R}$, we recover the first case of Lemma A.4.1.

B. A *Mathematica* Implementation

Now we present the complete code of the program introduced in Chapter 7. The reader may download the *Mathematica* notebook and a brief user's guide from the Wolfram Library Archive at `http://library.wolfram.com/infocenter/MathSource/9087/`.

```
Clear["Global`*"];
mess="done";
result="";

CLength[x_] := Module[{y}, y = x;
  While[y[[Length[y]]] == 0, y = y[[1 ;; Length[y] - 1]]];
  Length[y]]

NFill[x_, n_] := If[TrueQ[n > Length[x]], Join[x, Table[0, {n - Length[x]}]], x
    ];

Start[c1_,c2_,T_,cc1_]:= Module[
  {asdf,aa,n,bb,bcn,rango2,m, Graphic,opred,Opp,cadenatexto,equation, ecuacion,
     condcont, condcont2, ecinicial, ecu, eqaux,ViaLibre,c,lc,c1b,c2b},
  Off[];
  ecu=0;
  aa=-T;
  bb=T;
  mess="Processing␣data...";
  n=Max[CLength[c1],CLength[c2]]-1;
  If[TrueQ[c1 \[Element] Reals && c2 \[Element] Reals && T \[Element] Reals],
    Graphic = True, Graphic = False];
  ViaLibre = True;
  If[Not[n \[Element] Integers && n > 0], MessageDialog["Order␣must␣be␣a␣
    positive␣integer"];
     ViaLibre = False;
  ];
  If[Not[T > 0] && ViaLibre,
     MessageDialog["T␣must␣be␣a␣positive␣real␣number"];
     ViaLibre = False;
  ];
  If[Not[n + 1 == Length[c1] && n + 1 == Length[c2]] && ViaLibre,
     MessageDialog[
     "Vector␣of␣coefficients:␣LENGTH␣INCORRECT"];
     ViaLibre = False;
  ];
  L[f_][x_] := Sum[c1b[[k + 1]] Derivative[k][f][-x] + c2b[[k + 1]] Derivative[
    k][f][x], {k, 0, n}];
  R[f_][x_] := Sum[c1b[[k + 1]] Derivative[k][f][-x] -(-1)^k c2b[[k + 1]]
    Derivative[k][f][x], {k, 0, n}];
```

```
Clear[aux];
Opp=1;
If[Opp==1,  aux[var_]:= cc1 /.u->var,aux[var_]:= cc1 /.{u->var,T->-T}];
EG=False;
If[Not[TrueQ[Norm[c1]*Norm[c2]==0]],
  m=2 n;
  c1b=NFill[c1,m+1];
  c2b=NFill[c2,m+1];
  bcn=2*CLength[cc1];
  c=Table[0,{m+1}];
  Do[c[[j + 1]] = Sum[(-1)^i*(c1b[[i + 1]]*c1b[[j - i + 1]] - c2b[[i + 1]]*c2b[[j - i + 1]]), {i, 0, j}], {j, 0, m}];
  aux2[u_]:= Join[aux[u], Expand[aux[R[u]]]];
,
  m=n;
  c1b=NFill[c1,m+1];
  c2b=NFill[c2,m+1];
  EG=True;
  bcn=CLength[cc1];
  aux2[var_]:= aux[var];
  If[TrueQ[Norm[c1]==0],
    c=c2;
  ];
  If[TrueQ[Norm[c2]==0],
    c=c1;
    Opp=-1;
  ];
];
lc = CLength[c]-1;
If[TrueQ[c[[m+1]]==0],
  MessageDialog["The reduced problem is of order less than 2 n"];
  G[t_, s_] = "Undetermined";
,
  If[ViaLibre == True,
    Do[alfa[i, j] = Coefficient[aux2[u][[i]], Derivative[j][u][-T]], {j, 0, m - 1}, {i, 1, bcn}];
    Do[beta[i, j] = Coefficient[aux2[u][[i]], Derivative[j][u][T]], {j, 0, m - 1}, {i, 1, bcn}];
    Do[U[i][u_]= Sum[alfa[i, j]*Derivative[j][u][-T] + beta[i, j]*Derivative[j][u][T], {j, 0, m - 1}], {i, 1, bcn}];
    condcont2 = Sort[Table[Expand[U[i][u]] == 0, {i, 1, bcn}]];
    condcont = Sort[Table[Expand[aux2[u][[i]]] == 0, {i, 1, bcn}]];
    If[TrueQ[Chop[condcont] == Chop[condcont2]],
      opred[u_][t_]:=Sum[c[[k+1]]Derivative[k][u][t], {k,0, lc}];
      D0[k_][u_]:=Derivative[k][u][0];
      equation=Join[{opred[y][t] == 0}, Table[D0[i][y] == 0, {i, 0, lc - 2}], {D0[lc-1][y] == 1}];
      mess="Solving homogeneous equation...";
      ecinicial = DSolve[equation, y, t];
```

```
      cadenatexto = ToString[ecinicial];
      If[Not[StringMatchQ[cadenatexto, "*Root*"]],
        Result=Style[Column[{
          Style["␣", Bold],
          Style["PROBLEM:␣", Bold], Style["␣", Bold],
          Row[{TraditionalForm[L[u][t]] == \[Sigma][t], ",␣␣␣␣t␣\[Element]␣[", -T, ",", T, "]"}],
          Style["␣", Bold],
          Style["with␣boundary␣conditions", Bold],
          Style["␣", Bold], Table[aux[u][[i]] == 0, {i, 1,Length[aux[u]]}],
          Style["␣", Bold],
          Style["The␣Green\.b4s␣function␣is␣giving␣by:␣", Bold],
          r = ComplexExpand[Re[y /. ecinicial[[1]]]];,
          mess="Computing␣fundamental␣matrix...";
          If[TrueQ[c \[Element] Reals && T \[Element] Reals],
            Do[soluci[k] = DSolve[Join[{opred[y][t] == 0}, Table[D0[i][y] == 0, {i, 0, k-2}], {D0[k-1][y] == 1}, Table[D0[i][y]== 0, {i, k, lc-1}]], y[t], t];
              yk[k][t_]=FullSimplify[ComplexExpand[y[t]/.soluci[k][[1]]]];
            , {k, 1, lc}];
          ,
            Do[yk[k][t_] =  Sum[c[[lc + 1 - j]] Derivative[j - k][r][t], {j, k, lc}];, {k, 1, lc}];
          ];
          rango2=MatrixRank[Table[U[i][yk[j]], {i, 1, bcn}, {j, 1, lc}]];
          If[TrueQ[Not[rango2 == lc]],
            MessageDialog["There␣is␣no␣Green's␣function␣for␣the␣reduced␣problem"];
            Graphic = False;
            G[t_, s_] = "There␣is␣no␣unique␣solution";
          ,
            eqaux=Table[Sum[beta[i, j]*Derivative[j][r][T - s]  , {j, 0, lc - 1}] + Sum[d[j][s] U[i][yk[j]],{j, 1, lc}]==0, {i, 1, bcn}];
            ecuacion =Solve[eqaux , Table[d[j][s], {j, 1, lc}]];
            If[ecuacion == {},
              MessageDialog["There␣is␣no␣Green's␣function"];
              Graphic = False;
              G[t_, s_] = "There␣is␣no␣unique␣solution";
            ,
              ecu = 1;
              mess="Constructing␣Green's␣function...␣(100␣s␣max)";
              asdf=ecuacion/.Rule[a_,b_]:>b;
              Do[e[j][s_]= d[j][s] /. {d[j][s]->asdf[[1]][[j]]}, {j, 1, lc}];
              h[t_, s_] = Simplify[Sum[e[i][s]*yk[i][t], {i, 1, lc}], TimeConstraint->15];
              G1[t_, s_] = Simplify[TrigFactor[Chop[r[t - s]] + h[t, s]], TimeConstraint->15];
              G2[t_, s_] = Simplify[TrigFactor[h[t, s]],TimeConstraint->15];
```

```
              Gb[s_][t_] = Piecewise[{{G1[t, s], -T <= s<=T&&-T <= t<=T&&s <= t},{G2[t, s], -T <= s<=T&&-T <= t<=T&&t<s},{0, -T>s||-T>t||T<s||T<t}}];
              If[Not[EG],
                Gb1[t_,s_]=PiecewiseExpand[R[Gb[s]][t], TimeConstraint -> 15];
                G[t_,s_]=Piecewise[{{Simplify[Gb1[t,s],-T<=s<=T\[And]-T<=t<=T\[And]s-t<=0\[And]s+t<=0],-T<=s<=T\[And]-T<=t<=T\[And]s-t<=0\[And]s+t<=0},{Simplify[Gb1[t,s],-T<=s<=T\[And]-T<=t<=T\[And]s-t>0\[And]s+t<=0],-T<=s<=T\[And]-T<=t<=T\[And]s-t>0\[And]s+t<=0},{Simplify[Gb1[t,s],-T<=s<=T\[And]-T<=t<=T\[And]s-t<=0\[And]s+t>0],-T<=s<=T\[And]-T<=t<=T\[And]s-t<=0\[And]s+t>0},{Simplify[Gb1[t,s],-T<=s<=T\[And]-T<=t<=T\[And]s-t>0\[And]s+t>0],-T<=s<=T\[And]-T<=t<=T\[And]s-t>0\[And]s+t>0}}];
                G[t_,s_]=Chop[PiecewiseExpand[G[t,s]/c[[m+1]], TimeConstraint -> 15]];
              ,
                If[Opp==1,
                  G[t_,s_]=Chop[PiecewiseExpand[Gb[s][t]/c[[m+1]], TimeConstraint -> 15]];
                ,
                  G[t_,s_]=Chop[PiecewiseExpand[Gb[-s][t]/c[[m+1]], TimeConstraint -> 15]];
                ];
              ];
            ];
          ];
        Row[{Style["G[t,s]=␣", Bold], TraditionalForm[G[t, s]]}],
        Style["␣", Bold],
        Style["␣", Bold],
        Style["␣", Bold],
        If[TrueQ[Graphic],
        If[ecu == 1,
          Plot3D[G[t, s], {s, aa, bb}, {t, aa, bb}]
          ,
          Print["Cannot␣show␣the␣graphic"]]]
        },{Frame->True,Alignment->Center}]]
      ,
      MessageDialog["Green's␣Function␣with␣a␣complex␣expression"];
      ];
    ,
      MessageDialog["The␣boundary␣conditions␣are␣not␣valid"];
    ];
  ];
 ];
mess="done";
];

F[]:=(c1={1, 0, 1};c2={0, 0, 0};lim=1;cc1={u[1],u[-1]}; Nap=False;Framed[Column[{
Style["Program␣to␣compute␣the␣Green's␣␣function␣of␣the␣equation:␣␣␣␣", Bold],
```

```
Style[Row[{TraditionalForm[Sum[Subscript[a, j] Derivative[j][u][-t], {j, 0, n}]
     + Sum[Subscript[b, j] Derivative[j][u][t] , {j, 0, n}] == \[Sigma] [t]],",
    ␣␣␣␣t\[Element][-T,T]"}], Bold],
Style[Column[{"␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣␣"}, Center], Bold],
Style["with␣boundary␣conditions:␣␣␣␣", Bold],
Style[Row[{TraditionalForm[Subscript[U,i][u]=Sum[Subsuperscript[\[Alpha],i,j]
    Superscript[u,(j)][-T],{j,0,n-1}]+Sum[Subsuperscript[\[Beta],i,j]
    Superscript[u,(j)][T],{j,0,n-1}]==0],",␣␣␣␣i=1,...,n"}],  Bold],
Style["␣", Bold],
Column[{Panel[Grid[{
     {"Coefficients␣\!\(\*SubscriptBox[\(a\),␣\(i\)]\)", InputField[Dynamic[c1
    ]]},
     {"Coefficients␣\!\(\*SubscriptBox[\(b\),␣\(i\)]\)", InputField[Dynamic[c2
    ]]},
     {"T", InputField[Dynamic[lim]]},
     {"Boundary␣conditions", InputField[Dynamic[cc1]]},
     {"Numerical␣Approximation", Checkbox[Dynamic[Nap],Appearance->Large]}
 },Alignment -> {{Right, Left}}]],
 Button["Enter", If[Nap,
      If[Element[c1,Reals],c1= N[c1]];
      If[Element[c2,Reals],c2=N[c2]];
     ];Start[c1, c2, lim, cc1],ImageSize->150,Method -> "Queued"]},Alignment->
    Center],
Column[{Framed[Style[Row[{"Progress:␣",Dynamic[mess]}],Bold]],Dynamic[Result]},
    Alignment->Left]}]])

F[]
```

C. Resumen en castellano

La presente Tesis contiene la mayoría del trabajo llevado a cabo por el autor en los últimos años. Es, de hecho, una aventura investigadora en el ámbito de las soluciones de ecuaciones diferenciales, de ahí el título *«Existencia y Multiplicidad de Soluciones de Ecuaciones diferenciales Funcionales»*. Sin embargo, ¿cómo aproximarse al estudio de un área tan amplia? En tanto a lo que las soluciones son a las ecuaciones diferenciales, podemos optar por una aproximación bastante sencilla: existen dos posibilidades, o bien hay soluciones o no las hay y, si las hay, puede haber una o muchas. De este simple hecho surge este trabajo y las publicaciones que se han realizado durante la elaboración del mismo [34, 35, 39–44, 96, 165, 166].

C.1 Primera Parte

Que queramos demostrar que hay una –unicidad de solución– o muchas –multiplicidad de solución– es lo que determina que usemos un método u otro a la hora de tratar cada problema considerado. La existencia se ha obtenido tradicionalmente en una de dos maneras: o bien a través de la construcción directa de la solución, o bien usando métodos topológicos, estos últimos, en la mayoría de los casos, concerniendo contracciones globales como el teorema de contracción de Banach.

En la primera parte de esta memoria nos ocuparemos de la unicidad por medio de la construcción directa usando lo que se conoce como función de Green, esto es, la obtención de la solución de un problema del tipo $Lu = h$, $u \in H$, donde H es un espacio de funciones, L un operador lineal definido en H y $h \in L(H)$, expresándola, de ser posible, de la forma

$$u(t) = \int G(t,s)h(s)\,\mathrm{d}s,$$

con los extremos de integración adecuados para el problema. Se entiende entonces que esta expresión proporciona los que se conocen como principios del máximo y del anti-máximo, los cuales, en pocas palabras, recogen la idea de que, si G es positiva y h es positiva entonces u es positiva –principio del anti-máximo– y que si G es negativa y h es positiva entonces u es negativa –principio del máximo–.

Estas son sólo algunas de las notables propiedades de las funciones de Green pero, como suele suceder con las estructuras matemáticas más útiles, estas son a menudo también las más difíciles de obtener. El caso de las ecuaciones funcionales no es una excepción a esta regla y a través de los siete primeros capítulos de esta memoria exploraremos la construcción de estas funciones y sus diferentes aplicaciones. Centraremos nuestra atención en el caso de las ecuaciones con involuciones, un campo particular de las ecuaciones diferenciales funcionales donde podemos *reducir* –de una manera específica que detallaremos en su momento– el problema estudiado a un problema con ecuaciones diferenciales ordinarias. Además escribiremos un programa de ordenador en *Mathematica* que nos permitirá calcular automáticamente las funciones de Green para el caso de coeficientes constantes y condiciones de contorno de dos puntos.

Pero, ¿qué son las involuciones? Este tipo particular de funciones ha constituido un área de investigación de interés desde que Rothe calculó por primera vez, en 1800, el número de involuciones diferentes que es posible encontrar sobre conjuntos finitos [152]. Después de eso, Babbage publicó en 1815 [7] el trabajo fundacional en el cual las ecuaciones funcionales se consideraban por primera vez, en particular aquellas de la forma $f \circ f = \mathrm{Id}$, cuyas soluciones distintas de la identidad son, precisamente, lo que llamamos involuciones[†].

A pesar de los progresos en el estudio de las ecuaciones funcionales, tenemos que esperar hasta 1940, cuando Silberstein [156] resolvió por primera vez una ecuación diferencial con involución. El interés por las ecuaciones diferenciales con involuciones es retomado por Wiener en 1969 [186]. Wiener, junto con Watkins, liderarán los descubrimientos en esta dirección en las décadas venideras [1, 155, 173, 174, 186–189]. Muchos autores han llevado a cabo una gran cantidad de trabajo desde entonces en este campo. Hacemos una breve reseña al respecto en el Capítulo 2. En el año 2013 aparecieron de la mano del autor y su director de Tesis los primeros resultados sobre funciones de Green para ecuaciones diferenciales [39] y estos estudios se continúan en [40, 41, 43, 44]. La primera parte de la Tesis recoge estos descubrimientos relacionados con funciones de Green. En el primer capítulo repasamos algunos resultados generales sobre involuciones que nos ayudarán a entender sus sorprendentes propiedades analíticas y algebraicas.

El Capítulo 2, como ya hemos dicho, está dedicado a aquellos resultados con involución no directamente asociados a funciones de Green. Las demostraciones de esos resultados se pueden encontrar en la bibliografía citada en cada caso. No se profundiza en los mismos, pero se resumen a conveniencia del lector, quien puede consultar asimismo el libro de Wiener [187] que, a pesar de haber sido escrito hace más de veinte años, sigue siendo un buen punto de partida en lo que a este tipo de resultados generales se refiere. En este capítulo, es interesante observar la progresión y los distintos tipos de resultados recogidos con aquellos relacionados con funciones de Green que aparecen en los capítulos posteriores.

En el siguiente capítulo, el 3, empezamos a trabajar con la teoría de funciones de Green para ecuaciones diferenciales funcionales con involuciones en aquellos casos más sencillos: problemas de orden uno con coeficientes constantes y reflexión. En él resolvemos el problema asociado al operador $x'(t) + m\,x(-t)$ y describimos sus autovalores, obteniendo la función de Green en el caso no resonante y el rango de valores del parámetro real m para el cual el núcleo integral –la función de Green–, que proporciona la única solución, tiene signo constante. Esto nos permite derivar de manera automática principios del máximo y del anti-máximo. Este estudio se lleva a cabo con diferentes condiciones de contorno, analizando las características específicas que aparecen cuando consideramos condiciones periódicas, anti-periódicas, iniciales o lineales arbitrarias. Además aplicamos algunas técnicas muy conocidas –sub y sobresoluciones, el teorema de contracción-expansión de Krasnosel'skiĭ...– para obtener nuevos resultados que son ilustrados con diversos ejemplos.

Calcular las funciones de Green de manera explícita en el caso de un problema con coeficientes no constantes no es sencillo, ni siquiera cuando estamos tratando con ecuaciones

[†]Babbage, en el prefacio de su trabajo [7], describió muy bien la importancia de las involuciones: *«Muchos de los cálculos con los que estamos familiarizados consisten de dos partes, una directa y su inversa; así, cuando consideramos el exponente de una cantidad, esto es, elevarla a una potencia, esa es la operación directa; cuando tomamos la raíz de una cantidad, ese es el método inverso [...] En todos los casos el método inverso es con diferencia el más difícil y también podríamos añadir que el más útil»*.

diferenciales ordinarias. Siguiendo los resultados publicados en [41], nos enfrentamos a estos obstáculos en el Capítulo 4, donde reducimos un nuevo problema general con coeficientes no constantes e involuciones diferenciables arbitrarias al caso estudiado en el Capítulo 3. Para poner esto en práctica llevamos a cabo un triple artificio tomando como punto de partida los conocimientos del capítulo anterior. Primero añadimos un término que depende de $x(t)$ que hace que la situación no cambie demasiado con respecto a la estudiada en el Capítulo 3 para luego reducir el caso de una involución general al caso de la reflexión usando algo del conocimiento adquirido en el Capítulo 1 . El último paso, ir del caso constante al no constante, es un tema aparte. Tenemos que usar un cambio especial de variable –sólo válido en determinados casos– que nos permitirá obtener la función de Green para aquellos problemas con coeficientes no constantes a partir de la función de Green de problemas análogos con coeficientes constantes. En este mismo capítulo estudiamos además aquellos casos en los que dicho cambio de variable no es posible, demostrando que, cuando se presentan, puede ocurrir que exista solución única, múltiple o que no exista solución.

Para terminar esta parte del trabajo más teórica, tenemos el Capítulo 5, en el que profundizamos en la naturaleza algebraica de las reflexiones y extrapolamos estas propiedades a otras álgebras. De esta manera, no sólo generalizamos los resultados del capítulo 3 al caso de problemas de orden n y condiciones de contorno de dos puntos generales, sino que además resolvemos problemas diferenciales funcionales en los que participa la transformada de Hilbert y / u otros operadores adecuados, escogidos por sus propiedades algebraicas. En este capítulo reducimos los problemas en cuestión a ecuaciones diferenciales ordinarias para poder resolverlos y describimos un método general para obtener funciones de Green de problemas funcionales (diferenciales o no) generales. La utilidad de este método se ilustra con el caso de problemas con condiciones de contorno homogéneas con reflexión y varios ejemplos específicos.

Es necesario apuntar que las transformaciones necesarias en este proceso en el que reducimos un problema funcional a uno ordinario son de naturaleza puramente algebraica. Esta teoría, publicada en [44], es por tanto, y en ese sentido, similar a lo que se conoce como *análisis algebraico*, una teoría con la cual, a través del estudio de álgebras y módulos de Ore, se obtiene información importante acerca de algunos problemas funcionales, incluyendo soluciones explícitas [21, 50]. Sin embargo, las estructuras algebraicas con las que lidiamos aquí son en cierto modo diferentes, es decir, en general no son álgebras de Ore[†].

Cabe destacar que de entre las ecuaciones diferenciales funcionales reducibles, aquellas con reflexión han generado un interés más allá del mero formalismo matemático. Algunas por sus aplicaciones a la mecánica cuántica supersimétrica [73, 147, 153] y otras por su uso en otras áreas de las matemáticas, como son los métodos topológicos de los que tratamos en la segunda parte de la Tesis.

El final de la primera parte de la memoria coincide con dos capítulos dedicados a aplicar los resultados obtenidos anteriormente a algunos problemas relacionados. Para empezar, en el Capítulo 6 obtenemos algunos resultados relativos a la periodicidad de las soluciones de aquel primer problema con reflexión. Esto se hace recogiendo de nuevo una interesante relación entre una ecuación con reflexión y una ecuación con un φ-laplaciano expuesta en el Capítulo 3 que nos permite deducir la existencia de solución en un caso partiendo del otro y viceversa. El estudio de esta periodicidad de problemas de valor inicial se lleva a cabo poniendo el foco

[†]Remitimos al lector a [118, 149–151] para una aproximación algebraica a la teoría abstracta de problemas de contorno y sus aplicaciones a la computación simbólica.

sobre el cálculo explícito del período, lo que resulta interesante ya que nos permitirá estudiar su variación en función de varios parámetros.

El último capítulo de la primera parte, el Capítulo 7, nos devuelve a una situación más práctica para poder aplicar, en situaciones concretas, el método desarrollado en el Capítulo 5 para obtener funciones de Green asociadas a ecuaciones diferenciales con reflexión, coeficientes constantes y condiciones de contorno de dos puntos. Es del máximo interés poder disponer de programas de ordenador adecuados que nos permitan obtener las funciones de Green mencionadas dado que, en general, los cálculos necesarios para derivarlas son muy complicados. Siendo así, presentamos en este capítulo un algoritmo para el caso implementado en *Mathematica*. Además añadimos algunas consideraciones que nos podrían ayudar a simplificar los cálculos a realizar, y por lo tanto el tiempo necesario para ejecutar el programa, en un futuro. El lector puede encontrar en el Apéndice B el código exacto del programa en cuestión.

C.2 Segunda Parte

La fortaleza del método de las funciones de Green reside en que estas son los núcleos integrales del operador inverso que nos proporciona la única solución del problema en cuestión pero, por supuesto, este no es el camino a tomar cuando lo esperable es que existan varias soluciones. En la segunda parte de la memoria exploramos un tipo particular de métodos topológicos que nos permiten demostrar la existencia de múltiples soluciones e incluso localizarlas dentro de un cono meticulosamente definido. Los problemas a los que vamos a aplicar esta técnica contendrán una no-linealidad, esto es, una relación funcional no lineal entre las derivadas de la solución y la propia solución. El punto clave de este método se encuentra en un perfeccionamiento del teorema clásico de Guo-Krasnosel'skiĭ para la contracción / expansión en conos. La no linealidad, que toma valores reales, oscilará de una determinada manera, sobrepasando y quedando por debajo, alternativamente, de ciertos valores dependientes de las variables y estas ondas causarán, precisamente, la existencia de muchas soluciones. Esta situación es similar a la que ocurre cuando agitamos un cubo con agua. Si hacemos una pequeña marca un poco por encima del nivel del agua y agitamos el cubo, empiezan a aparecer ondas sobre la superficie y, cuando llegan a una altura suficiente, alcanzan la línea que habíamos marcado. Cuantas más ondas hay, tantas más veces el agua alcanza el nivel marcado.

Sencillo como pueda parecer, las condiciones que se tienen que satisfacer para poder aplicar esta técnica pueden llegar a ser, como se puede apreciar en esta parte, muy complicadas. Además, esta complejidad crece a medida que los problemas a estudiar aumentan en generalidad.

Como decíamos, antes de llegar a esta parte se habían estudiado, eminentemente, las situaciones de unicidad de solución en casos lineales pero, cuando hay no-linealidades involucradas, los problemas se escapan a la construcción directa de soluciones y otros métodos diferentes se hacen necesarios.

Los métodos topológicos se vuelven útiles en estas situaciones, en particular aquellos relacionados con el índice de punto fijo. En los cuatro capítulos de esta parte usamos esta técnica para resolver cuatro problemas crecientes en dificultad. La estructura del método es bastante consistente y se desarrolla como sigue.

(1) Se establece la *naturaleza del problema* a ser estudiado y sus características específicas.

(2) Se elabora una lista de *propiedades*, a tener por parte de los elementos considerados en el problema, que son necesarias para poder garantizar que los resultados de existencia / multiplicidad / no existencia de soluciones se pueden aplicar. Por ejemplo el operador F del cual los puntos fijos serán las soluciones a nuestro problema tiene que ser continuo y compacto.

(3) Se define un cono apropiado K en el cual localizaremos las soluciones del problema. Aquí tenemos que tomar una importante decisión: los conos grandes permiten encontrar más soluciones pero, al mismo tiempo, no proporcionan buenos resultados de localización.

(4) Se demuestra que el operador F es *compacto*, *continuo*, y *lleva* K en K.

(5) Se encuentran *condiciones suficientes* para las cuales el índice de punto fijo del operador F es 0 y ± 1 respectivamente en –al menos– dos subconjuntos del cono anidados. Si encontramos n subconjuntos del cono anidados para los cuales el índice alterna el valor 0 con los valores ± 1, entonces podemos garantizar la existencia de al menos $n-1$ soluciones no triviales diferentes (cf. [123]).

   Haciendo el cono más pequeño trocamos un mayor número de soluciones por condiciones más simples. Por otra parte, también podemos usar condiciones para el índice relacionadas con los autovalores de algunos de los operadores involucrados –véanse los Capítulos 10 y 11–.

(6) Finalmente, podemos aplicar los resultados obtenidos a una enorme variedad de problemas e ilustrar así su aplicación con algunos ejemplos.

Como se puede observar, las particularidades de cada problema hacen que sea imposible tomar una aproximación común a todos. Sin embargo, se presentan importantes similitudes que nos llevarán a la obtención de resultados comparables. Los resultados presentados en los Capítulos 8, 9 y 10 han sido publicados, respectivamente, en [34], [34] y [96]. Los del Capítulo 11 ya están listos para ser enviados pronto para publicación.

En el Capítulo 8 se prueban nuevos resultados relativos a la existencia de soluciones no triviales de una ecuación integral de Hammerstein –que nos sirve como modelo para los siguientes capítulos– que incluye una reflexión, con la particularidad de que al núcleo integral en cuestión le es permitido cambiar de signo fuera de un intervalo del dominio. Resolver este problema nos permitirá aplicar los resultados obtenidos a una ecuación diferencial con reflexión estudiada en el Capítulo 3. Además, realizamos el estudio en diferentes conos, observando como los resultados van variando según el contexto.

El Capítulo 8 abre la puerta a modelos más generales. En el Capítulo 9 cambiamos la reflexión por una función continua cualquiera, lo que nos permite estudiar el modelo de un termostato con argumento desviado. Este modelo tiene en cuenta todos los efectos físicos relevantes que pueden darse en el mundo real, lo cual lo hace demasiado complicado para estudiarlo mediante un método convencional. Además, añadimos al problema la presencia, en las condiciones de contorno, de un funcional lineal arbitrario, lo cual permite adaptar el modelo a sistemas de control muy variados.

El hecho de haber contribuido con la presencia de un funcional en las condiciones de contorno hace que en el Capítulo 10 se estudie otra vez el problema integral de Hammerstein, pero en este caso con la peculiaridad de estar sometido a dos funcionales lineales distintos en las condiciones de contorno que, por otra parte, son de tipo Neumann. A mayores se ofrecen por primera vez resultados para el cálculo del índice de punto fijo relacionados con el radio espectral de los operadores asociados lo cual, en muchos casos, resulta ventajoso a la hora de obtener resultados sin realizar demasiados cálculos.

Finalmente, corona la segunda parte de esta memoria el Capítulo 11. Este destaca sobre los anteriores en tanto a que la complejidad del problema estudiado es muy superior. Esto se debe a la presencia de funcionales y operadores no lineales, tanto en la ecuación como en las condiciones de contorno. Tal generalidad obliga a la aparición de una gran profusión de condiciones a ser satisfechas y resultados muy interesantes. En particular, se aplica la generalización de la definición del radio espectral a operadores acotados para poder obtener resultados de índice de punto fijo sencillos.

Más allá de las dos partes que constituyen el núcleo del trabajo realizado, encontramos dos apéndices. El primero profundiza en un tema que se mencionó en el Capítulo 5, la obtención de una versión hiperbólica de la fórmula para la suma de fasores. La obtención de dicha fórmula da lugar a un capítulo muy didáctico –publicado en [166]– en el cual se desgrana, desde el punto de vista matemático, el formalismo de fasores tan comúnmente utilizado en el ámbito de la física y la ingeniería eléctrica. El segundo apéndice contiene el código del programa de *Mathematica* desarrollado en el Capítulo 7 y una referencia a la biblioteca electrónica Wolfram Library Archive desde el cual se puede descargar.

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