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# PROPERTIES OF A MATRIX GROUP ASSOCIATED TO A $\{K, S+1\}$-POTENT MATRIX*

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**Abstract.** In a previous paper, the authors introduced and characterized a new kind of matrices called $\{K, s+1\}$-potent. In this paper, an associated group to a $\{K, s+1\}$-potent matrix is explicitly constructed and its properties are studied. Moreover, it is shown that the group is a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_{(s+1)^2-1}$. For some values of $s$, more specifications on the group are derived. In addition, some illustrative examples are given.

**Key words.** Involutory matrix, $\{K, s+1\}$-potent matrix, Group.

**AMS subject classifications.** 15A30, 15A24.

**1. Introduction.** Let $K \in \mathbb{C}^{n\times n}$ be an involutory matrix, that is $K^2 = I_n$, where $I_n$ denotes the $n \times n$ identity. In [5], the authors introduced and characterized a new kind of matrices called $\{K, s+1\}$-potent matrices where $K$ is involutory. We recall that for an involutory matrix $K \in \mathbb{C}^{n\times n}$ and $s \in \{0, 1, 2, 3, \ldots\}$, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{K, s+1\}$-potent if

$$KA^{s+1}K = A. \tag{1.1}$$

These matrices generalize all the following classes of matrices: $k$-potent matrices, idempotent matrices, periodic matrices, involutory matrices, centrosymmetric matrices, mirror symmetric matrices, circulant matrices, etc. Several applications of these matrices can be found in the literature [1, 9, 13]. The class of $\{K, s+1\}$-potent matrices was linked to other kind of matrices (as $\{s+1\}$-generalized projectors, $\{K\}$-Hermitian matrices, normal matrices, etc.) in [6]. Throughout this paper, we consider $K \in \mathbb{C}^{n\times n}$ to be an involutory matrix.

Some results on a similar class of $2\times 2$ matrices and $n\times n$ invertible matrices have been presented in [2]. On the other hand, matrices commuting with a permutation and $\{K\}$-centrosymmetric matrices (that correspond to $s = 0$) have received increasing

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interest in the last twenty years. Some of their properties can be found in [1, 8, 9, 13]. Furthermore, matrices with $k$-involutory symmetries have been studied in [11, 12]. Moreover, some spectral properties related to similar classes of matrices are given in [3, 5, 10].

Related to the group theory, we recall that if $G$ is a finite group with identity element $e$ and $a \in G$ then $a^m = e$ implies that the order of $a$ divides to $m$ for any natural power $m$.

Motivated by the fact that the definition of $\{K, s+1\}$-potent matrices involves products of the two matrices $A$ and $K$, we wonder if there are any other relationships between products where these matrices appear. As a particular case, when $s$ is the smallest positive integer such that $A^{s+1} = A$, it is clear that $\{A, A^2, A^3, \dots, A^s\}$ is a cyclic group (and, therefore, commutative and normal) of order $s$. This leads to our main aim, which is to extend these results to $\{K, s+1\}$-potent matrices.

This paper is organized as follows. First, properties of a $\{K, s+1\}$-potent matrix $A$ are studied in Section 2 involving products and powers of $A$ and $K$. These properties are necessary to construct, in Section 3, a finite group $G$ from a given $\{K, s+1\}$-potent matrix. As a consequence, this group is a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_{(s+1)^2-1}$ where $\mathbb{Z}_r$ is the group of integers modulo $r$. Moreover, the group $G$ is calculated in some simple cases. The case $A^k = A$ for some $k < (s+1)^2$ is also analyzed in Section 4. Finally, in Section 5, some illustrative examples are given.

## 2. Basic properties of $\{K, s+1\}$-potent matrices.

It is clear that for each $n \in \{1, 2, 3, \dots\}$, there exists at least one matrix $A \in \mathbb{C}^{n\times n}$ such that $A$ is $\{K, s+1\}$-potent for each involutory matrix $K$ and for each $s \in \{1, 2, 3, \dots\}$. It is also easy to see that such a matrix is not unique [5].

Throughout this section, we consider $s \in \{1, 2, 3, \dots\}$. It is well-known [5] that a matrix $A \in \mathbb{C}^{n\times n}$ is $\{K, s+1\}$-potent if and only if any of the following conditions are (trivially) equivalent: $KAK = A^{s+1}$, $KA = A^{s+1}K$, and $AK = KA^{s+1}$.

We now establish properties regarding $\{K, s+1\}$-potent matrices.

Lemma 2.1. *If $A \in \mathbb{C}^{n\times n}$ is a $\{K, s+1\}$-potent matrix then the following properties hold*

*(a)* $KA^{s+2} = A^{s+2}K$ *and* $KA^{s+2}K = A^{s+2}$.
*(b)* $A^{s+2} = (KA)^2 = (AK)^2$.
*(c)* $A^{(s+1)^2} = A$.
*(d)* $(A^{(s+1)^2-1})^k = A^{(s+1)^2-1}$ *for every* $k \in \{1, 2, 3, \dots\}$.
*(e)* $(A^{s+2})^{s+1} = A^{s+2}$.
*(f)* $(KA)^{2s+1} = KA$ *and* $(AK)^{2s+1} = AK$.

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*(g)* $KA^jK = A^{j(s+1)}$ *and* $A^jK = KA^{j(s+1)}$ *for every* $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.
*(h)* $(KA^j)^2 = A^{j(s+2)}$ *for all* $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.
*(i)* $A^jA^{(s+1)^2-1} = A^{(s+1)^2-1}A^j = A^j$ *and* $(KA^j)A^{(s+1)^2-1} = A^{(s+1)^2-1}(KA^j) = KA^j$, *for all* $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.
*(j)* *For each* $j \in \{1, 2, \dots, (s+1)^2 - 1\}$, *one has* $(KA^j)(KA^k) = A^{(s+1)^2-1}$, *where* $k$ *is the unique element of* $\{1, 2, \dots, (s+1)^2 - 1\}$ *such that* $k \equiv -j(s+1)$ $[\mathrm{mod}\ ((s+1)^2 - 1)]$.
*(k)* $K(KA^j)^{s+1}K = \begin{cases} KA^{j\frac{s}{2}(s+4)+j} & \text{if } s \text{ is even} \\ \left(A^{j(s+2)}\right)^{\frac{s+1}{2}} & \text{if } s \text{ is odd} \end{cases}$ *for all* $j \in \{1, 2, \dots, (s+1)^2 - 1\}$.

*Proof.* $(a)$ One has $KA^{s+2} = KA^{s+1}A = AKA = AA^{s+1}K = A^{s+2}K$. The second equality can be deduced post-multiplying both sides by $K$.

$(b)$ From $(a)$ and the definition, we have $A^{s+2} = KA^{s+2}K = KAA^{s+1}K = KAKA = (KA)^2$. The other equality in $(b)$ can be similarly deduced.

$(c)$ By definition we have $A^{(s+1)^2} = (A^{s+1})^{s+1} = (KAK)^{s+1} = KA^{s+1}K = A$.

$(d)$ Using Property $(c)$ we get

$$(A^{(s+1)^2-1})^2 = A^{(s+1)^2}A^{(s+1)^2-2} = AA^{(s+1)^2-2} = A^{(s+1)^2-1},$$

and now Property $(d)$ can be easily shown by induction.

$(e)$ From $(c)$ we get $(A^{s+2})^{s+1} = (A^{s+1}A)^{s+1} = (A^{s+1})^{s+1}A^{s+1} = A^{(s+1)^2}A^{s+1} = AA^{s+1} = A^{s+2}$.

$(f)$ From $(b)$ and $(c)$ the equalities $(KA)^{2s+1} = KA(KA)^{2s} = KA(A^{s+2})^s = KA^{s^2+2s+1} = KA^{(s+1)^2} = KA$ hold, and in a similar way it can be shown the equality $(AK)^{2s+1} = AK$.

$(g)$ We proceed by recurrence. In fact, by definition we have

$$KAK = A^{s+1}. \tag{2.1}$$

Then Equality (2.1) yields $KA^2K = KAAK = A^{s+1}KAK = A^{s+1}A^{s+1} = A^{2(s+1)}$. Following a similar reasoning it can be proven that $KA^jK = A^{j(s+1)}$ for all $j \in \{1, 2, \dots, s\}$. Now, by using the definition and $A^{(s+1)^2} = A$ we get the property for $j = s+1$ as follows: $KA^{s+1}K = A = A^{(s+1)^2} = A^{(s+1)(s+1)}$. From now on, following a similar reasoning as before it can be proven that $KA^jK = A^{j(s+1)}$ for all $j \in \{1, 2, \dots, (s+1)^2 - 1\}$, and the other equality in (g) is easily obtained from $K^2 = I_n$.

$(h)$ Using (g) one has

$$(KA^j)^2 = (A^{j(s+1)}K)^2 = A^{j(s+1)}(KA^{j(s+1)}K) = A^{j(s+1)}A^j = A^{j(s+2)}.$$

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$(i)$ Follows from (c) and (g).

$(j)$ Let $k \geq 1$. One has $KA^jKA^k = A^{j(s+1)+k}$. The right hand side is $A^{(s+1)^2-1}$ if $k = -j(s+1)[\text{mod } ((s+1)^2-1)]$.

$(k)$ Case 1. $s$ is even. Using Properties $(g)$ and $(c)$, we get

$$\begin{aligned} K(KA^j)^{s+1}K &= A^jA^{j(s+1)\frac{s}{2}}A^{j\frac{s}{2}}K = A^{j(s+2)\frac{s}{2}}A^jK = A^{j(s+2)\frac{s}{2}}KA^{j(s+1)} \\ &= KA^{(s+1)\left(j(s+2)\frac{s}{2}\right)}A^{j(s+1)} = KA^{j(s+1)\left((s+1)\frac{s}{2}+\frac{s}{2}\right)}A^{j(s+1)} \\ &= K\left(A^{(s+1)^2}\right)^{j\frac{s}{2}}A^{\left(\frac{s}{2}+1\right)j(s+1)} = KA^{j\frac{s}{2}(s+4)+j}. \end{aligned}$$

Case 2. $s$ is odd. Using Property $(g)$, we get

$$K(KA^j)^{s+1}K = A^jA^{j(s+1)\frac{s+1}{2}}A^{j\frac{s-1}{2}} = A^{j(s+1)\frac{s+1}{2}+j\frac{s+1}{2}} = \left(A^{j(s+2)}\right)^{\frac{s+1}{2}}. \qquad \square$$

**3. Construction of a matrix group.** Firstly, we note that, from a $\{K, s+1\}$-potent matrix, Lemma 2.1 allows us to construct a group containing a cyclic subgroup of $\{K, s+1\}$-potent matrices. Throughout this section we assume that $s \geq 1$.

THEOREM 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a $\{K, s+1\}$-potent matrix. If $A^i \neq A^j$ for all distinct $i, j \in \{1, 2, \ldots, (s+1)^2-1\}$ then the set*

$$G = \{A, A^2, A^3, \ldots, A^{(s+1)^2-1}, KA, KA^2, KA^3, \ldots, KA^{(s+1)^2-1}\}$$

*is a group with respect to the matrix product satisfying the following properties:*

*(a) $A$ is an element of order $(s+1)^2-1$, and then the set*

$$S_A = \{A, A^2, A^3, \ldots, A^{(s+1)^2-1}\} \tag{3.1}$$

*is a cyclic subgroup of $G$.*

*(b) $KA^s$ and $KA^{(s+1)^2-1}$ are elements of order 2 of $G$.*

*(c) $(KA^s)A(KA^s) = A^{s+1}$.*

*(d) The set $S_A$ is a normal subgroup of $G$ and all its elements are $\{K, s+1\}$-potent matrices.*

*(e) The order of $G$ is:*

- *$(s+1)^2-1$ if $KA = A^j$ for some $j \in \{1, 2, \ldots, (s+1)^2-1\}$ and, in this case, the group $G$ is commutative.*
- *$2((s+1)^2-1)$ if $KA \neq A^j$ for all $j \in \{1, 2, \ldots, (s+1)^2-1\}$ and, in this case, the group $G$ is noncommutative.*

*(f) For every $j \in \{1, 2, \ldots, (s+1)^2-1\}$, the element $KA^j$ of the set $G \setminus S_A$ (when it is nonempty) is $\{K, s+1\}$-potent if and only if $s$ is even and one of the following conditions $\{4|s, \frac{s}{2}+1|j\}$ or $\{4\not| s, s+2|j\}$ holds.*

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*Proof.* From the properties given in Lemma 2.1, it can be checked that $A^{(s+1)^2-1}$ is the identity element of the group $G$.

$(a)$ $G$ contains clearly a cyclic subgroup generated by the element $A$ of order $(s+1)^2-1$.

$(b)$ Using Property (h) of Lemma 2.1, one has $(KA^s)^2 = A^{s(s+2)} = A^{(s+1)^2-1}$. Similarly, $(KA^{(s+1)^2-1})^2 = A^{(s+1)^2-1}$.

$(c)$ Using the definition we get $(KA^s)A(KA^s) = KA^{s+1}KA^s = AA^s = A^{s+1}$.

$(d)$ The set $S_A$ is a subgroup of $G$ of index 2. Then it is normal. As a direct consequence of Property $(f)$ of Lemma 2.1 we obtain the second part of Property $(d)$ since $A^j$ are $\{K, s+1\}$-potent matrices for all $j \in \{1, 2, \ldots, (s+1)^2-1\}$.

$(e)$ Case 1. There exists $j \in \{1, 2, \ldots, (s+1)^2-1\}$ such that $KA = A^j$. Then $KA^i = A^{j+i-1}$, that is, it is also one of the $(s+1)^2-1$ powers of $A$ indicated in (3.1) for all $i \in \{2, 3, \ldots, (s+1)^2-1\}$. Thus, the order of $G$ is $(s+1)^2-1$ and then $G$ is commutative.

Case 2. For every $j \in \{1, 2, \ldots, (s+1)^2-1\}$ one has $KA \neq A^j$. It is clear that the order of $G$ is $2((s+1)^2-1)$ because if $i, j \in \{1, 2, \ldots, (s+1)^2-1\}$ such that $KA^i = KA^j$ with $i \neq j$ exist, then $A^i = K^2A^i = K^2A^j = A^j$, which is impossible. Assume that $G$ is commutative. Then $(KA)(KA^{s+1}) = (KA^{s+1})(KA)$, that implies $A^{2s+1} = A$. We obtain a contradiction.

$(f)$ Assume that $G \setminus S_A \neq \emptyset$ and $KA^j \in G \setminus S_A$. Then the fact that $KA^j$ is $\{K, s+1\}$-potent implies that $s$ is even, by Property $(k)$ of Lemma 2.1. This same property assures that, in this case, $K(KA^j)^{s+1}K = KA^{j\frac{s}{2}(s+4)+j}$. Then,

$$KA^j \text{ is } \{K, s+1\}-\text{potent} \iff KA^{j\frac{s}{2}(s+4)+j} = KA^j \iff A^{j\frac{s}{2}(s+4)+j} = A^j$$

$$\iff A^{j\frac{s}{2}(s+4)+j}A^{(s+1)^2-1-j} = A^{(s+1)^2-1} \iff A^{j\frac{s}{2}(s+4)} = A^{(s+1)^2-1}$$

and so, this is equivalent to the statement $(s+1)^2-1 = s(s+2)$ divides $j\frac{s}{2}(s+4)$. Now, it can be checked that $s(s+2)|j\frac{s}{2}(s+4)$ is equivalent to $\{4|s, \frac{s}{2}+1|j\}$ or $\{4 \nmid s, s+2|j\}$. In the first case, there are $2s$ such $j$ and in the second one, there are $s$ such $j$. ∎

Corollary 3.2. *The group $G$ is a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_{(s+1)^2-1}$ when $G$ has order $2((s+1)^2-1)$.*

*Proof.* We consider a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_r$. Then its presentation is in the form $\langle a, b|\, a^2 = e, b^r = e, aba = b^m\rangle$ where $m, r$ are coprime. Here $r = (s+1)^2-1$, $a = KA^s$, $b = A$, $m = s+1$. ∎

Now, we need the following definition.

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Definition 3.3 ([7]).

*(i) The dihedral group, denoted by $D_m$, is generated by an element $a$ of order $m$ and another element $b$ of order 2 such that $b^{-1}ab = a^{-1}$. It has order $2m$.*

*(ii) The quasi-dihedral group, denoted by $Q_{2^m}$ for $m \geq 4$, is generated by an element $a$ of order $2^{m-1}$ and another element $b$ of order 2 such that $b^{-1}ab = a^{2^{m-2}-1}$. It has order $2^m$.*

We now classify the group $G$ defined in Theorem 3.1. It is clear that $G$ depends on $A$, $K$, and $s$. That is why, for the following result, we will denote $G$ by $G_s$ and so we stress on the parameter $s$. We shall see a relation between the group $G_s$ and dihedral and quasi-dihedral groups in the following result.

Proposition 3.4. *Assume that $G_s$ has order $2((s+1)^2-1)$. One has*

*(a) $G_s$ is a dihedral group if and only if $s = 1$. In this case, $G_1 \simeq D_3$.*

*(b) Let $s > 1$. Then $G_s$ is a quasi-dihedral group if and only if $s = 2$. In this case, $G_2 \simeq Q_{16}$.*

*Proof.* $(a)$ If $s = 1$, the proof for case $(a)$ can be deduced from the fact that $G_s$ is generated by the element $A$ of order 3, the element $KA$ of order 2, and furthermore, the relation $(KA)A(KA) = A^2$ holds, where $A^2$ is the inverse of $A$.

Let $s > 1$. In this case, it is known that the group $G_s$ has an element $A$ of order $(s+1)^2-1$. On the other hand, we check that for any $\alpha \in \{1, 2, \ldots, (s+1)^2-1\}$ the element $KA^\alpha$ has order 2 if and only if $\alpha$ is a multiple of $s$. Indeed,

$$A^{(s+1)^2-1} = (KA^\alpha)^2 \Longleftrightarrow A^{s(s+2)} = KA^\alpha KA^\alpha = A^{\alpha(s+1)}A^\alpha = A^{\alpha(s+2)}$$

$$\Longleftrightarrow s(s+2) \text{ divides } \alpha(s+2) \Longleftrightarrow s \text{ divides } \alpha,$$

and hence all the elements of order 2 of $G_s$ have the form $KA^{st}$. However, for any element of the form $KA^{st}$, where

$$t \in \left\{\frac{1}{s}, \frac{2}{s}, \ldots, \frac{(s+1)^2-1}{s}\right\} \cap \mathbb{N} = \{1, 2, 3, \ldots, s+2\},$$

the equality $(KA^{st})A(KA^{st}) = A^{(s+1)^2-2}$ is not satisfied since

$$(KA^{st})A(KA^{st}) = (KA^{st+1}K)A^{st} = A^{((s+1)^2-1)t+(s+1)} = (A^{(s+1)^2-1})^t A^{s+1} = A^{s+1}.$$

Finally $s > 1$ implies that $s+1 < (s+1)^2-2$, a contradiction.

$(b)$ It is necessary to take into account the fact that the properties of $G_s$ for $s = 2$ coincide with those of $Q_{2^m}$ for $m = 4$. In fact, if $s = 2$ then $G_s$ has an element $A$ of order $8 = 2^{4-1}$, another element $KA^2$ of order 2 and, moreover, by Property $(c)$

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of Theorem 3.1 we have $(KA^2)A(KA^2) = A^3 = A^{2^{4-2}-1}$ and therefore $G_2 \simeq Q_{2^4}$. Conversely, if $G_s \simeq Q_{2^m}$ then we have $2^{m-1} = (s+1)^2 - 1 = s(s+2)$. Thus $s$ is a power of 2 and $s = 2$, $m = 4$. ∎

**4. What about the group when $A^k = A$ for $k < (s+1)^2$?.** We can ask: what would happen if a power $k$ of $A$ less than $(s+1)^2$ and such that $A^k = A$ existed?

If $A \in \mathbb{C}^{n\times n}$ is a $\{K, s+1\}$-potent matrix then Property $(b)$ of Lemma 2.1 allows us to construct, by Theorem 3.1, the group $G$ considering the subgroup $S_A$ of order $(s+1)^2-1$ when all the powers of $A$ are different. But, it may occur that there exists an integer $k$ such that $A^k = A$ with $2 \le k < (s+1)^2$. In this case, it is also possible to consider the group $G_{s,k} = \{A, A^2, \dots, A^{k-1}, KA, KA^2, \dots, KA^{k-1}\}$ associated to the matrix $A$. Therefore, the subset $S_A^k = \{A, A^2, \dots, A^{k-1}\}$ is a (cyclic) subgroup of the group $S_A$ and then $k-1$ has to divide $(s+1)^2 - 1 = s(s+2)$.

How many groups $G_{s,k}$ can we construct in this way? One only: the group corresponding to the smallest power $k$ such that $A^k = A$ (otherwise, we obtain exactly the same group $G_{s,k}$). Consequently, the only possibilities for the order of the group are: $(s+1)^2 - 1$, $2((s+1)^2 - 1)$, $k-1$ or $2(k-1)$ (if such $k$ exists).

For some values of $s$ and $k$, more specifications on the group are given in the following result. In order to analyze these special cases we recall the following definition.

DEFINITION 4.1. *[4] The quaternion group, denoted by $Q$, is generated by three elements $a, b, c$ of order 4 such that $a^2 = b^2 = c^2$ and $bab^{-1} = a^{-1}$. It has order 8.*

PROPOSITION 4.2. *Assuming that $G_{s,k}$ has order $2(k-1)$, the following statements hold.*

*(a) Let $s = 1$. Then $k = 2$. In this case, $G_{1,2} \simeq \mathbb{Z}_2$.*
*(b) Let $s = 2$. Then one of the following statements hold:*
- *(i) $k = 2$. In this case, $G_{2,2} \simeq \mathbb{Z}_2$.*
- *(ii) $k = 3$. In this case, $G_{2,3} \simeq D_2$.*
- *(iii) $k = 5$. In this case, $G_{2,5} \simeq D_4$ or $G_{2,5} \simeq Q$.*

*(c) Let $s > 2$. Then*
- *(i) $G_{s,s+1} \simeq \mathbb{Z}_{2s}$ when $s$ is prime.*
- *(ii) $G_{s,s+3} \simeq D_{s+2}$.*

*Proof.* (a) Let $s = 1$. If $k$ is an integer such that $2 \le k \le 3$ and $A^k = A$, it must be $k = 2$ because $k-1$ divides 3. Thus, $KAK = A^2 = A$ and then the group $G_{1,2} = \left\{KA, (KA)^2 = A\right\}$ is generated by the element $KA$ of order 2. Hence, $G_{1,2} \simeq \mathbb{Z}_2$.

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(b) Let $s = 2$. If $k$ is an integer such that $2 \leq k \leq 8$ and $A^k = A$, it must be $k = 2$, $k = 3$ or $k = 5$ because $k - 1$ divides 8. We now analyze these three cases:

- $k = 2$: the same reasoning as in (a).
- $k = 3$: in this case, $G_{2,3} = \{A, A^2, KA, KA^2\}$ where $A$ and $KA^2$ have order 2 and $KA^2AKA^2 = A$. Hence, $G_{2,3} \simeq D_2$.
- $k = 5$: in this case, $G_{2,5} = \{A, A^2, A^3, A^4, KA, KA^2, KA^3, KA^4\}$ is a non-commutative group (for instance, $A(KA^2) \neq (KA^2)A$). Then, Proposition 6.3 in [4] assures that $G_{2,5} \simeq D_4$ or $G_{2,5} \simeq Q$.

(c) Let $s > 2$. If $k$ is an integer such that $2 \leq k \leq s(s + 2)$ and $A^k = A$, we get $k = s + 1$ or $k = s + 3$ as particular values of $k$ because $k - 1$ divides $s(s + 2)$. Now, we analyze these two cases:

- $k = s + 1$: in this case, $G_{s,s+1} = \{A, A^2, \ldots, A^s, KA, KA^2, \ldots, KA^s\}$ is a commutative group of order $2s$. Then, Corollary 6.2 in [4] assures that $G_{s,s+1} \simeq \mathbb{Z}_{2s}$.
- $k = s + 3$: in this case, $G_{s,s+3} = \{A, A^2, , \ldots, A^{s+2}, KA, KA^2, \ldots, KA^{s+2}\}$ is a non-commutative group (for example, $A(KA^{s+1}) \neq (KA^{s+1})A$). $A$ is of order $s + 2$, $KA^s$ of order 2 and $KA^sAKA^s = A^{s+1}$. Hence, $G_{s,s+3} \simeq D_{s+2}$.

□

Remark 4.3. *If $G_{s,k}$ has order $2(k - 1)$ then $G_{s,k}$ is a semidirect product of $\mathbb{Z}_2$ acting on $\mathbb{Z}_{k-1}$, when $k - 1, s + 1$ are coprime. Its proof is similar to the proof of Corollary 3.2 where $a = KA^{k-1}$, $b = A$, $r = k - 1$, $m = s + 1$. The Property (h) of Lemma 2.1 allows us to show that $KA^{k-1}$ has order 2. In fact, $a^2 = (KA^{k-1})^2 = A^{(k-1)(s+2)} = A^{k-1} = e$. Moreover, $aba = (KA^{k-1})A(KA^{k-1}) = KA^kKA^{k-1} = A^{s+1}A^{k-1} = A^{s+1} = b^m$.*

Remark 4.4. *We observe that $G_{2,5}$ is isomorphic to $D_4$ or $Q$ because there are (up to isomorphism) exactly two distinct non-commutative groups of order 8. We will see, in Example 2, that these two possibilities can be realized.*

Remark 4.5. *We observe that if $s > 2$ and $k = s+1$, then $A$ satisfies $KA = AK$ since $A^{s+1} = A$. Such a matrix is said to be $\{K\}$-centrosymmetric [9]. In this case, the group associated to such a matrix $A$ has order $2s$, commutative and then isomorphic to $\mathbb{Z}_{2s}$ if $s$ is prime.*

We close this section with the following remark.

Remark 4.6. *(The case $s = 0$). It corresponds to $\{K\}$-centrosymmetric matrices. It is sometimes also possible to construct a similar group as before. Observe that the condition $A^{(s+1)^2} = A$ does not give any information for $s = 0$. So, if we assume that $A^t = A$ for some positive integer $t$ (where $A^l \neq A^m$ for all $l, m < t$, $l \neq m$),*

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*then* $G = \{A, A^2, \ldots, A^{t-1}, KA, KA^2, \ldots, KA^{t-1}\}$ *is a group with the same features mentioned before.*

*However, when the assumption is not fulfilled, since the powers of* $A$ *cannot be the identity element, the group does not exist. An example will clarify this situation. For the matrices*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

*it is impossible to construct such a group because*

$$A^{m+1} = \begin{bmatrix} 2^m & 0 & 2^m \\ 0 & 1 & 0 \\ 2^m & 0 & 2^m \end{bmatrix} \quad \text{for every } m \geq 1.$$

**5. Examples.** Now we present some more examples illustrating the results we have obtained.

Example 5.1. *Let*

$$A = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$A$ *is* $\{K, 2\}$*-potent as the authors showed in [5].* $G = \{A, A^2, A^3, KA, KA^2, KA^3\}$ *is a group of order* $6$ *because* $A^3 = I_3$ *and, in this case,* $G \simeq D_3$ *by Proposition 3.4.*

Example 5.2.

*(1) Let*

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$A$ *is* $\{K, 3\}$*-potent and* $A^2 = -I_2$*. This example shows a matrix of the class considered in Section 4. For this matrix,* $A^5 = A$ *holds and so it is possible to construct a group with similar features as in Proposition 4.2. Since* $s = 2$, $k = 5$, $G_{2,5}$ *has order* $4$ *or* $8$*. In this example, the group is* $G_{2,5} = \{\pm I_2, \pm A, \pm K, \pm KA\}$ *and, in this case,* $A$ *is an element of order* $4$, $KA^2 = -K$ *is an element of order* $2$*, and* $(KA^2)A(KA^2) = A^3$*. So* $G_{2,5}$ *is isomorphic to* $D_4$*.*

*(2) Let* $A$ *be a* $\{K, 3\}$*-potent matrix given by*

$$A = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1+i & 1-i \\ 0 & 0 & 1-i & 1+i \\ 1+i & 1-i & 0 & 0 \\ 1-i & 1+i & 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \frac{1}{8}\begin{bmatrix} 9 & -1 & 3 & -3 \\ -1 & 9 & -3 & 3 \\ -3 & 3 & -1 & 9 \\ 3 & -3 & 9 & -1 \end{bmatrix}.$$

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*$A^4 = I_4$ and the associated matrix group (of order 8) is*

$$G_{2,5} = \{A, A^2, A^3, I_4, KA, KA^2, KA^3, K\}.$$

*$G_{2,5}$ is generated by the three elements $A^2, KA, KA^3$ of order 4 such that $(A^2)^2 = (KA)^2 = (KA^3)^2$ and $(KA)(A^2)(KA)^3 = A^2$. So $G_{2,5}$ is isomorphic to $Q$.*

Example 5.3. *Let*

$$A = \begin{bmatrix} \frac{\sqrt{5}-1}{2} & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ 0 & -1 \end{bmatrix}.$$

*$A$ is $\{K, 4\}$-potent and $A^5 = I_2$. The associated group (of order 10) is*

$$G = \{A, A^2, A^3, A^4, I_2, KA, KA^2, KA^3, KA^4, K\}.$$

*$G$ is generated by $A$ of order 5, $KA^2$ of order 2, and $(KA^2)A(KA^2) = A^4$. Then $G$ is isomorphic to $D_5$.*

Example 5.4. *Let $A$ be a $\{K, 5\}$-potent matrix given by*

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}.$$

*The associated group (of order 6) is $G = \{A, A^2, I_2, KA, KA^2, K\}$ since $A^3 = I_2$. As $G$ is generated by $A$ of order 3, $KA^2$ of order 2 and $(KA^2)A(KA^2) = A^2$, then $G$ is isomorphic to $D_3$.*

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## REFERENCES

[1] A. Andrew. Centrosymmetric matrices. *SIAM Review*, 40:697–698, 1998.
[2] G. Bourgeois. Algebraic matrix equations in two unknowns. *arXiv:1103.4203v3 [math.RA]*, 2011.
[3] H. K. Du and Y. Li. The spectral characterization of generalized projections. *Linear Algebra and its Applications*, 400:313–318, 2005.
[4] T.W. Hungerford. *Algebra*. Springer-Verlag, Berlin, 1989.
[5] L. Lebtahi, O. Romero, and N. Thome. Characterizations of $\{K, s+1\}$-potent matrices and applications. *Linear Algebra and its Applications*, 436:293–306, 2012.
[6] L. Lebtahi, O. Romero, and N. Thome. Relations between $\{K, s+1\}$-potent matrices and different classes of complex matrices. *Linear Algebra and its Applications*, doi:10.1016/j.laa.2011.10.042.
[7] J.S. Rose. *A Course on Group Theory.* Cambridge, 1978.

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[8] J.L. Stuart and J.R. Weaver. Matrices that commute with a permutation matrix. *Linear Algebra and its Applications*, 150:255–265, 1991.

[9] D. Tao and M. Yasuda. A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices. *SIAM J. Matrix Analysis Applications*, 23:885–895, 2002.

[10] W.F. Trench. Characterization and properties of matrices with generalized symmetry or skew symmetry. *Linear Algebra and its Applications*, 377:207–218, 2004.

[11] W.F. Trench. Characterization and properties of matrices with k-involutory symmetries. *Linear Algebra and its Applications*, 429:2278–2290, 2008.

[12] W.F. Trench. Characterization and properties of matrices with k-involutory symmetries II. *Linear Algebra and its Applications*, 432:2782–2797, 2010.

[13] J. Weaver. Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors. *American Mathematical Monthly*, 92:711–717, 1985.

[14] Hans Schneider. Theorems on M-splittings of a singular M-matrix which depend on graph structure. *Linear Algebra and its Applications*, 58:407–424, 1984.

[15] Richard S. Varga. *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[16] S. Friedland and H. Schneider. Spectra of expansion graphs. *Electronic Journal of Linear Algebra*, 6:2–10, 1999.