Gauss-Bonnet formulae and rotational integrals in constant curvature spaces

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Abstract

We obtain generalizations of the main result in [18], and then provide geometric interpretations of linear combinations of the mean curvature integrals that appear in the Gauss-Bonnet formula for hypersurfaces in space forms M^n_λ. Then, we combine these results with classical Morse theory to obtain new rotational integral formulae for the $k-$th mean curvature integrals of a hypersurface in M^n_λ.

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1. Introduction

Let M^n_λ denote a simply connected Riemannian manifold of constant sectional curvature λ. Further, let L^n_r denote a $r-$plane, $(r \leq n)$, namely a totally geodesic submanifold of dimension r in M^n_λ, and let $\mathrm{d}L^n_r$ denote the corresponding density, invariant under the group of Euclidean and non Euclidean motions. A $r-$plane through a fixed point O in M^n_λ, and its invariant density, are denoted by $L^n_{r[0]}$ and $\mathrm{d}L^n_{r[0]}$, respectively [16].

In [8] a new expression for the density of $r-$planes in M^n_λ has been obtained in terms of the density $\mathrm{d}L^n_{r+1[0]}$, of the density $\mathrm{d}L^{r+1}_r$ of $r-$planes in $L^n_{r+1[0]}$ and the distance ρ from O to L^{r+1}_r. Thus, an invariant $r-$plane in M^n_λ may be generated by taking first an isotropic $(r+1)-$plane through a fixed point O and then an invariant $r-$plane within this $(r+1)-$plane, weighted by a function of ρ.

This construction, called the invariator principle in M_λ^n ([19]), has opened the way to solve rotational integral equations for different quantities as the volume of a $k-$dimensional submanifold in M_λ^n [8], the $k-$th mean curvature integrals or $k-$th intrinsic volumes ([10] and [1], and different curvature measures ([19] for $\lambda = 0$)). The solutions of these equations allow to express these quantities as the integral of some functionals defined in sections produced by isotropic planes through a fixed point. Moreover, in [19], the authors, using classical Morse theory, rewrite the volume of compact submanifolds in $\mathbb{R}^n$ of dimension $n-r$, in terms of critical values of the sectioned object with $(r+1)-$planes; and in [9] related generalizations valid for submanifolds in space forms of constant curvature are obtained.

On the other hand, in [18] it is proved that the Gauss-Bonnet defect of a hypersurface in M_λ^n is the measure of planes L_{n-2}^n meeting it, counted with multiplicity. From this result an integral-geometric proof of the Gauss-Bonnet theorem for hypersurfaces in M_λ^n is given.

The purpose of this paper is twofold: to obtain generalizations of the main result in [18], following a completely different route; and to combine these results with classical Morse theory to obtain new rotational integral formulae for the $k-$th mean curvature integrals of a hypersurface in M_λ^n.

2. The Gauss-Bonnet theorem in M_λ^n

Let $Q \subset M_\lambda^n$ be a compact domain with smooth boundary $S = \partial Q$. Let V denote the volume of of Q, F the $(n-1)-$surface area of S, $\chi(Q)$ the Euler-Poincaré characteristic of Q, and M_i the $i-$th integral of mean curvature of S. The Gauss-Bonnet formula for S states that [16]

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-2}{2}} c_1 M_1 + \lambda^{\frac{n}{2}} V = \frac{1}{2} O_n \chi(Q), \qquad (1)$$

for n even, where $O_k = \mathrm{vol}(\mathbb{S}^k)$ (surface area of the $k-$dimensional unit sphere), and

$$c_{n-1}M_{n-1} + \lambda c_{n-3}M_{n-3} + \cdots + \lambda^{\frac{n-3}{2}} c_2 M_2 + \lambda^{\frac{n-1}{2}} c_0 F = \frac{1}{2} O_n \chi(Q), \qquad (2)$$

for n odd, where

$$c_h = \left(\begin{array}{c} n-1 \\ h \end{array} \right) \frac{O_n}{O_h O_{n-1-h}}. \tag{3}$$

If n is odd, we can use the equality $2\chi(Q) = \chi(S)$, and for $\lambda = 0$, in any case, we obtain $M_{n-1} = O_{n-1}\chi(Q)$.

Let $\mathcal{L}_r$ be the space of $r-$dimensional totally geodesic submanifolds of M_λ^n. Our first result is the following theorem, which is a generalization of the main result in [18].

Theorem 2.1. *For n and r even, or n and r odd, we have*

$$\begin{aligned} \frac{1}{2} O_n \chi(Q) - c_{n-1} M_{n-1} - \lambda c_{n-3} M_{n-3} - \cdots - \lambda^{\frac{n-r-2}{2}} c_{r+1} M_{r+1} \\ = \lambda^{\frac{n-r}{2}} \frac{O_r \ldots O_1}{O_{n-1} \ldots O_{n-r}} \int_{\mathcal{L}_r} \chi(Q \cap L_r^n) \mathrm{d}L_r^n. \end{aligned} \tag{4}$$

Proof. We begin assuming that n and r are both even numbers. Given a $r-$plane L_r^n of M_λ^n, $Q_r = L_r^n \cap Q$ is, in general, a domain of dimension r in L_r^n. Applying Eq.(1) to Q_r we obtain

$$c'_{r-1} M'_{r-1} + \lambda c'_{r-3} M'_{r-3} + \cdots + \lambda^{\frac{r-2}{2}} c'_1 M'_1 + \lambda^{\frac{r}{2}} V(Q_r) = \frac{1}{2} O_r \chi(Q_r), \tag{5}$$

where M'_i is the $i-$th integral of mean curvature of ∂Q_r and

$$c'_h = \left(\begin{array}{c} r-1 \\ h \end{array} \right) \frac{O_r}{O_h O_{r-1-h}}. \tag{6}$$

Eq.(14.69) for $q = n$ and Eq.(14.78) of [16], which are valid for M_λ^n, are

$$\int_{\mathcal{L}_r} V(Q_r) \mathrm{d}L_r^n = \frac{O_{n-1} \ldots O_{n-r}}{O_{r-1} \ldots O_0} V(Q) \tag{7}$$

and

$$\int_{\mathcal{L}_r} M'_i \, \mathrm{d}L_r^n = \frac{O_{n-2} \ldots O_{n-r} O_{n-i}}{O_{r-2} \ldots O_0 O_{r-i}} M_i. \tag{8}$$

Now, having the preceding equalities in mind, we integrate Eq.(5) and we obtain

$$\begin{aligned} d_{r-1} M_{r-1} + \lambda d_{r-3} M_{r-3} + \cdots + \lambda^{\frac{r-2}{2}} d_1 M_1 + \lambda^{\frac{r}{2}} d_0 V \\ = \frac{1}{2} O_r \int_{\mathcal{L}_r} \chi(Q_r) \mathrm{d}L_r^n, \end{aligned} \tag{9}$$

where

$$d_i = \left(\begin{array}{c} r-1 \\ i \end{array} \right) \frac{O_r}{O_i O_{r-1-i}} \frac{O_{n-2} \dots O_{n-r} O_{n-i}}{O_{r-2} \dots O_0 O_{r-i}}; \quad i = 1, 3, \dots, r-1; \tag{10}$$

$$d_0 = \frac{O_{n-1} \dots O_{n-r}}{O_{r-1} \dots O_0}. \tag{11}$$

We multiply Eq.(9) by $\frac{\lambda^{(n-r)/2}}{d_0}$ to obtain

$$\begin{aligned} \lambda^{\frac{n-r}{2}} k_{r-1} M_{r-1} + \lambda^{\frac{n-r+2}{2}} k_{r-3} M_{r-3} + \dots + \lambda^{\frac{n-2}{2}} k_1 M_1 + \lambda^{\frac{n}{2}} V \\ = \frac{1}{2} \lambda^{\frac{n-r}{2}} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) \mathrm{d}L_r^n, \end{aligned} \tag{12}$$

where

$$k_i = \left(\begin{array}{c} r-1 \\ i \end{array} \right) \frac{O_r O_{r-1} O_{n-i}}{O_i O_{n-1} O_{r-i} O_{r-i-1}}. \tag{13}$$

If we compare the constants k_i and c_i in Eq.(1), using the equality $(k-1)O_k = O_1 O_{k-2}$, we have that

$$k_i = c_i; \tag{14}$$

then, Eq.(12) can be written as

$$\begin{aligned} \lambda^{\frac{n-r}{2}} c_{r-1} M_{r-1} + \lambda^{\frac{n-r+2}{2}} c_{r-3} M_{r-3} + \dots + \lambda^{\frac{n-2}{2}} c_1 M_1 + \lambda^{\frac{n}{2}} V \\ = \frac{1}{2} \lambda^{\frac{n-r}{2}} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) \mathrm{d}L_r^n, \end{aligned} \tag{15}$$

and, from Eq.(1) we obtain the result for the case n and r even.

If we consider that n and r are both odd numbers the proof is similar to the preceding one but considering, instead of Eq.(7), the following equality (Eq.(14.69) of [16] with $q = n-1$):

$$\int_{\mathcal{L}_r} F(\partial Q_r) \mathrm{d}L_r^n = \frac{O_n \dots O_{n-r} O_{r-1}}{O_r \dots O_0 O_{n-1}} F, \tag{16}$$

where $F(\partial Q_r)$ is the $(r-1)-$surface area of $\partial Q \cap L_r^n = \partial(Q \cap L_r^n)$. $\Box$

Remark. For $r = n-2$, Theorem 2.1 gives Theorem 1 of [18] and, as a result of Theorem 2.1, we obtain the following corollary which is equivalent to Proposition 7 of [18].

Corollary 2.2. *Let Q be a compact domain in M^n_λ and $L_r \in \mathcal{L}_r$, we have*

$$\begin{aligned} M_r = {} & \frac{(n-r-1)O_r \dots O_0}{O_{n-2} \dots O_{n-r-2}} \int_{\mathcal{L}_{r+1}} \chi(Q \cap L^n_{r+1}) \mathrm{d}L^n_{r+1} \\ & - \lambda \frac{rO_{r-2} \dots O_0}{O_{n-2} \dots O_{n-r}} \int_{\mathcal{L}_{r-1}} \chi(Q \cap L^n_{r-1}) \mathrm{d}L^n_{r-1}. \end{aligned} \tag{17}$$

Proof. When r is an even number, Eq.(15) divided by $\lambda^{\frac{n-r}{2}}$ is

$$c_{r-1}M_{r-1} + \lambda c_{r-3}M_{r-3} + \dots + \lambda^{\frac{r-2}{2}} c_1 M_1 + \lambda^{\frac{r}{2}} V = \frac{1}{2} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) \mathrm{d}L^n_r; \tag{18}$$

and the corresponding equation to Eq.(15) divided by $\lambda^{\frac{n-r}{2}}$ when r is an odd number is

$$c_{r-1}M_{r-1} + \lambda c_{r-3}M_{r-3} + \dots + \lambda^{\frac{r-3}{2}} c_2 M_2 + \lambda^{\frac{r-1}{2}} c_0 F = \frac{1}{2} \frac{O_r}{d_0} \int_{\mathcal{L}_r} \chi(Q_r) \mathrm{d}L^n_r. \tag{19}$$

If r is odd, subtracting each part of Eq.(18), with $r \longrightarrow r+1$, minus the corresponding part of λ multiplied by Eq.(18) with $r \longrightarrow r-1$ we obtain the result. If r is even, we proceed in the same way but using Eq.(19) instead of the Eq.(18). □

Remark. For $\lambda = 0$, Eq.(17) coincides with Eq.(14.79) of [16].

3. Rotational integrals and Morse representations for M_r

From rotational integral formulae we obtain quantitative properties (as M_r) of differential manifolds in M^n_λ, from the intersection of the manifold with planes (totally geodesic submanifolds) through a fixed point O. In this context, from Eq.(17), we will find measurement functions α_r defined on $L^n_{r+2[0]} \cap Q$ with rotational average equal to M_r, that is,

$$M_r = \int_{L^n_{r+2[0]} \cap Q \neq \emptyset} \alpha_r (L^n_{r+2[0]} \cap Q) \mathrm{d}L^n_{r+2[0]}. \tag{20}$$

Theorem 3.1. *Let $Q \subset M^n_\lambda$ be a compact domain with smooth boundary $S = \partial Q$. The measurement functions α_r corresponding to the $r-$th integral* *of mean curvature of S, M_r, can be expressed as*

$$\alpha_r(L^n_{r+2[0]} \cap Q) = \frac{O_{r-2}\dots O_0}{O_{n-2}\dots O_{n-r-2}}$$
$$\Big[(n-r-1)O_r O_{r-1}\int \chi((Q\cap L^n_{r+2[0]})\cap L^{r+2}_{r+1})s_\lambda^{n-r-2}(\rho)\mathrm{d}L^{r+2}_{r+1} \tag{21}$$
$$-\lambda r O_1 O_0 \int \chi(((Q\cap L^n_{r+2[0]})\cap L^{r+2}_{r[0]})\cap L^r_{r-1}))s_\lambda^{n-r}(\rho)\mathrm{d}L^r_{r-1}\mathrm{d}L^{r+2}_{r[0]}\Big],$$

where, in both integrals, ρ is the distance from O to the planes L^{r+2}_{r+1} and L^r_{r-1}, respectively; and

$$s_\lambda(\rho) = \begin{cases} \lambda^{-1/2}\sin(\rho\sqrt{\lambda}), & \lambda > 0 \\ \rho, & \lambda = 0\ . \\ |\lambda|^{-1/2}\sinh(\rho\sqrt{|\lambda|}), & \lambda < 0 \end{cases} \tag{22}$$

Proof. The idea of the proof consists in generating the planes L^n_{r+1} and L^n_{r-1}, which appear in Eq.(17), by taking first an isotropic plane through O and then an invariant plane within this isotropic plane, weighted by a function of ρ; that is, from Corollary 3.1 of [8] we have the identity

$$\mathrm{d}L^n_{r+1} = s_\lambda^{n-r-2}(\rho)\mathrm{d}L^{r+2}_{r+1}\mathrm{d}L^n_{r+2[0]}, \tag{23}$$

and also

$$\mathrm{d}L^n_{r-1}\mathrm{d}L^n_{r+2[r]} = s_\lambda^{n-r}(\rho)\mathrm{d}L^r_{r-1}\mathrm{d}L^n_{r+2[r]}\mathrm{d}L^n_{r[0]}, \tag{24}$$

where $\mathrm{d}L^n_{r+2[r]}$ denotes the density for $(r+2)-$planes about a about a $r-$plane L^n_r (see page 202 of [16]).

As justified in [16], p. 309, before Eq. (17.55), from the expressions of the densities of planes in M^n_λ it follows that some density decompositions (such as [16], Eq. (12.53)) have the same form whatever the sign of λ. Then, from Eq.(12.53) of [16], Eq.(24), can be expressed as

$$\mathrm{d}L^n_{r-1}\mathrm{d}L^n_{r+2[r]} = s_\lambda^{n-r}(\rho)\mathrm{d}L^r_{r-1}\mathrm{d}L^{r+2}_{r[0]}\mathrm{d}L^n_{r+2[0]}. \tag{25}$$

Finally, substituting Eq.(23) and Eq.(25) in Eq.(17), having in mind that

$$\int \mathrm{d}L^n_{r+2[r]} = \frac{O_{n-r-1}O_{n-r-2}}{O_1 O_0}, \tag{26}$$

we obtain the result. □

Remark. For $\lambda = 0$, Eq.(21) coincides, up to a constant factor, with Eq.(18) of [10].

3.1. Morse representations for M_r

In this section a geometric interpretation is given of Eq.(21) in terms of the critical points of height functions. In particular, and in order to simplify, we will give a geometric interpretation of the function

$$\beta_r = \int \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r) s^{n-r-1}_\lambda(\rho) \mathrm{d}L^{r+1}_r. \quad (27)$$

The density $\mathrm{d}L^{r+1}_r$ may be decomposed as follows,

$$\mathrm{d}L^{r+1}_r = c^r_\lambda(\rho)\mathrm{d}\rho\, \mathrm{d}u_r, \quad (28)$$

where $\mathrm{d}u_r$ denotes the surface area element of the $r-$dimensional unit sphere and $c_\lambda(\rho) = \frac{\mathrm{d}}{\mathrm{d}\rho} s_\lambda(\rho)$. Note that $\rho \geq 0$ for the cases $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic); however, for the case $\lambda > 0$ (spherical) ρ varies from 0 (which corresponds to the point O) to $\frac{\pi}{\sqrt{\lambda}}$ (which corresponds to the cut locus of O (i.e., the antipodal point of O).

Therefore, for the cases $\lambda = 0$ (Euclidean) and $\lambda < 0$ (hyperbolic), we may write,

$$\beta_r = \int_{\mathbb{S}^r} \mathrm{d}u_r \int_0^\infty s^{n-r-1}_\lambda(\rho)\, c^r_\lambda(\rho)\, \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r)\mathrm{d}\rho, \quad (29)$$

whereas, for the case $\lambda > 0$ (spherical),

$$\beta_r = \int_{\mathbb{S}^r} \mathrm{d}u_r \int_0^{\frac{\pi}{\sqrt{\lambda}}} s^{n-r-1}_\lambda(\rho)\, c^r_\lambda(\rho)\, \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r)\mathrm{d}\rho, \quad (30)$$

where L^{r+1}_r is the $r-$plane expressed in terms of its distance ρ from the fixed point O, perpendicular to the geodesic defined from the direction u_r from O, and $\chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r) = 0$ whenever $(Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r = \emptyset$.

Since we want to give a geometrical interpretation of β_r, based on critical points of height functions, from now on we will consider that ρ means signed distance and we will rewrite β_r as:

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} \mathrm{d}u_r \int_{-\infty}^\infty s^{n-r-1}_\lambda(|\rho|)\, c^r_\lambda(\rho)\, \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r)\mathrm{d}\rho, \quad \lambda \leq 0; \quad (31)$$

$$\beta_r = \frac{1}{2} \int_{\mathbb{S}^r} \mathrm{d}u_r \int_{\frac{-\pi}{\sqrt{\lambda}}}^{\frac{\pi}{\sqrt{\lambda}}} s^{n-r-1}_\lambda(|\rho|)\, c^r_\lambda(\rho)\, \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r)\mathrm{d}\rho \quad \lambda > 0. \quad (32)$$

Let u_r denote a unit vector in $\mathbb{S}^r \subset T_O L^n_{r+1[0]}$. The geodesic $\gamma_{u_r} : \mathbb{R} \longrightarrow L^n_{r+1[0]}$, with $\gamma_{u_r}(0) = O$ and $\gamma'(0) = u_r$ is given by $\gamma_{u_r}(t) = c_\lambda(t)O + s_\lambda(t)u_r$, where $c_\lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t} s_\lambda(t)$. Given u_r, let $h_{u_r} : L^n_{r+1[0]} \longrightarrow \mathbb{R}$ be the height function whose level hypersurfaces are just the $r-$planes L^{r+1}_r perpendicular to the geodesic $\gamma_{u_r}(t)$. Note that in the Euclidean case ($\lambda = 0$) this height function coincides with the standard height function considered in [19]. We suppose that the level hypersurface L^{r+1}_r is oriented in such a way that the unit vector $\nu(p)$, perpendicular to the level set $L^{r+1}_r \subset L^n_{r+1[0]}$ at p is given by $\nu(p) = \mathrm{grad}(h_{u_r})(p)/||\mathrm{grad}(h_{u_r})(p)||$.

Let us denote $Q_{r+1} = Q \cap L^n_{r+1[0]}$ which is, in general, a domain with boundary in $L^n_{r+1[0]}$ (see Appendix A of [10]). In Section 5 (Appendix) we show that in Euclidean and hyperbolic cases; and in the spherical case, if the domain Q is contained in the hemisphere of M^n_λ with pole O, $h_{u_r}|_{Q_{r+1}}$ is a strong Morse function for almost all $u_r \in \mathbb{S}^r$, it means that all of the critical points in the direction u_r from O are non-degenerate, and no two of them lie on the same level hypersurface (i.e. they have different critical values). In particular, $h_{u_r}|_{Q_{r+1}}$ has not critical points in Q_{r+1}. Let $p_i \in \mathrm{Crit}(h_{u_r}|_{\partial Q_{r+1}})$, $i = 1, \dots, m$, be the set of critical points, and

$$\rho_1 < \rho_2 < \cdots < \rho_m, \quad (\text{with} \quad \frac{-\pi}{2\sqrt{\lambda}} \leq \rho_1, \quad \rho_m \leq \frac{\pi}{2\sqrt{\lambda}} \quad \text{for} \quad \lambda > 0)$$

the corresponding critical values ($h_{u_r}(p_i) = \rho_i$). To each critical point p_i we assign an index

$$\epsilon_i = \chi(Q_{r+1} \cap L^{r+1}_r(\rho_i - \varepsilon)) - \chi(Q_{r+1} \cap L^{r+1}_r(\rho_i + \varepsilon)), \tag{33}$$

where $L^{r+1}_r(\rho_i + \varepsilon)$ denotes the $r-$plane defined from the direction u_r at a signed distance $\rho_i + \varepsilon$ from O; and ε is small enough to ensure that there are no critical points of $\mathrm{Crit}(h_{u_r}|_{\partial Q_{r+1}})$ whose height function belongs to $(\rho_i - \varepsilon, \rho_i + \varepsilon)$.

For $r < n \in \{1, 2, \dots\}$, define:

$$\begin{aligned} I_{n-r-1,r}(\rho) &= \int s_\lambda^{n-r-1}(|\rho|)\, c_\lambda^r(\rho)\, \mathrm{d}\rho \\ &= \begin{cases} \int s_\lambda^{n-r-1}(\rho)\, c_\lambda^r(\rho)\, \mathrm{d}\rho, & \rho \geq 0, \\ (-1)^{n-r-1} \int s_\lambda^{n-r-1}(\rho)\, c_\lambda^r(\rho)\, \mathrm{d}\rho, & \rho < 0. \end{cases} \end{aligned} \tag{34}$$

Then, for $\lambda = 0$,

$$I_{n-r-1,r}(\rho) = \int |\rho|^{n-r-1}\,\mathrm{d}\rho = \begin{cases} \frac{\rho^{n-r}}{n-r}, & \rho \geq 0, \\ (-1)^{n-r-1}\frac{\rho^{n-r}}{n-r}, & \rho < 0. \end{cases} \tag{35}$$

For $\lambda \neq 0$, and for any given pair (n, r), the integral $I_{n-r-1,r}(\rho)$ may be evaluated explicitly from [13], pages 114 and 159, or with the aid of a mathematical software package such as Mathematica®.

Theorem 3.2. *Let O be a point in M^n_λ and $Q \subset M^n_\lambda$ a compact domain which is contained in the hemisphere of M^n_λ with pole O when $\lambda > 0$. Let $Q_{r+1} = Q \cap L^n_{r+1[0]}$ be the domain with boundary in $L^n_{r+1[0]}$. Then, for $r \in \{0, 1, \ldots, n-2\}$,*

$$\beta_r = \frac{1}{2}\int_{\mathbb{S}^r} \left(\sum_{k=1}^{m} \epsilon_k\, I_{n-r-1,r}(\rho_k) \right) \mathrm{d}u_r, \tag{36}$$

where m represents the number of points $\mathrm{Crit}(h_{u_r}|_{\partial Q_{r+1}})$ corresponding to the direction u_r.

Proof. The fact that Q_{r+1} will be a domain with boundary in $L^n_{r+1[0]}$, for a generic $(r+1)-$space $L^n_{r+1[0]}$, follows from Theorem A.1 of [10], and the fact that $h_{u_r}|_{Q_{r+1}}$ will in general be a strong Morse function for almost all $u_r \in \mathbb{S}^r$ follows from the appendix, having in mind that Q_{r+1} is contained in the hemisphere of $L^n_{r+1[0]}$ with pole O.

Then Eq.(31) and Eq.(32) may be written as follows,

$$\beta_r = \frac{1}{2}\int_{\mathbb{S}^r} \mathrm{d}u_r \sum_{k=1}^{m-1} \int_{\rho_k}^{\rho_{k+1}} s_\lambda^{n-r-1}(|\rho|)\, c_\lambda^r(\rho)\, \chi((Q \cap L^n_{r+1[0]}) \cap L^{r+1}_r)\mathrm{d}\rho, \tag{37}$$

Thus,

$$\begin{aligned} \beta_r &= \frac{1}{2}\int_{\mathbb{S}^r} \mathrm{d}u_r \sum_{k=1}^{m-1} (I_{n-r-1,r}(\rho_{k+1}) - I_{n-r-1,r}(\rho_k)) \sum_{j=k+1}^{m} \epsilon_j \\ &= \frac{1}{2}\int_{\mathbb{S}^r} \left(\sum_{k=2}^{m} \epsilon_k\, I_{n-r-1,r}(\rho_k) - I_{n-r-1,r}(\rho_1) \sum_{k=2}^{m} \epsilon_k \right) \mathrm{d}u_r. \end{aligned} \tag{38}$$

Finally, since $\sum_{k=1}^{m} \epsilon_k = 0$, it means $\sum_{k=2}^{m} \epsilon_k = -\epsilon_1$, and the proposed result is obtained. □

4. Applications

Let $Q \subset M_\lambda^3$ ($\lambda \neq 0$) be a compact domain with smooth boundary $S = \partial Q$; then, from Theorem 2.1 with $n = 3$ and $r = 1$, we have

$$2\pi\chi(S) - \int_S K(x)\mathrm{d}x = \frac{2\lambda}{\pi} \int_{\mathcal{L}} \chi(Q \cap L_1^3)\mathrm{d}L_1^3, \tag{39}$$

where $K(x)$ is the Gauss curvature of S at x, and χ denotes Euler characteristic.

Now, from Eq.(23) and the definition of β_1 (Eq.(27)), a rotational formula of the *defect* of the surface in $M^3(\lambda)$ is gven by

$$2\pi\chi(S) - \int_S K(x)\mathrm{d}x = \frac{2\lambda}{\pi} \int_{Q \cap L_{2[0]}^3 \neq \emptyset} \beta_1(Q \cap L_{2[0]}^3)\mathrm{d}L_{2[0]}^3, \tag{40}$$

where, using Theorem 3.2,

$$\beta_1(Q \cap L_{2[0]}^3) = \frac{1}{2} \int_{\mathbb{S}^2 \cap L_{2[0]}^3} \sum_{k=1}^{m} \epsilon_k \, I_{1,1}(\rho_k)\mathrm{d}u. \tag{41}$$

Example. Let S be a geodesic sphere of radius ρ centered at O in $M^3(\lambda)$; then, $\chi(S) = 2$, and $\int_{M^2} K(x)\mathrm{d}x = 4\pi c_\lambda^2(\rho)$.

On the other hand, $S \cap L_{2[0]}^3$ is a geodesic circle (boundary of a geodesic ball) in $L_{2[0]}^3$; that is, all the points in $S \cap L_{2[0]}^3$ are a distance ρ apart from O. Then, for all directions $u \in \mathbb{S}^1$, $m = 2$, $\epsilon_1 = 1$, $\epsilon_2 = -1$, $I_{1,1}(\rho_1) = I_{1,1}(\rho) = \frac{1}{2}s_\lambda^2(\rho)$ and $I_{1,1}(\rho_2) = I_{1,1}(-\rho) = -\frac{1}{2}s_\lambda^2(\rho)$, $\beta_1(S \cap L_{2[0]}^3) = \pi s_\lambda^2(\rho)$; and Eq.(40) is satisfied.

If we consider a domain Q in $\mathbb{R}^3$ ($\lambda = 0$), Corollary 2.2, with $r = 1$ and $n = 3$, coincides with Eq.(12) of [6], Theorem 2.1 coincides with Eq.(12) of [6], and, since

$$2\chi(Q_2 \cap L_1^2) = N(\partial Q_2 \cap L_1^2), \tag{42}$$

where N denotes number, Theorem 3.2 coincides with the integrand of Eq.(50) in [6]; but now, for each axial direction $u \in [0, 2\pi)$ in the pivotal plane $L^3_{2[0]}$, the pivotal section is scanned entirely from top to bottom by a sweeping straight line parallel to the axis Ou, in search of critical points.

5. Appendix

Let X be a smooth manifold with boundary. We say that a smooth function $f : X \to \mathbb{R}$ is a *strong Morse function* if

1. all critical points of $f : X \to \mathbb{R}$ are non-degenerate and are contained in the interior of X,
2. all critical points of the restriction $f : \partial X \to \mathbb{R}$ are also non-degenerate,
3. if $x, y \in X$ are distinct critical points of either $f : X \to \mathbb{R}$ or $f : \partial X \to \mathbb{R}$, then $f(x) \neq f(y)$.

5.1. Preliminary results for the Euclidean case ($\lambda = 0$)

Assume now that $X \subset \mathbb{R}^n$ is a submanifold with boundary and for each unit vector $v \in \mathbb{S}^{n-1}$, let us denote by $h_v : X \to \mathbb{R}$ the height function defined as $h_v(x) = \langle x, v \rangle$.

Theorem 5.1. *Let $X \subset \mathbb{R}^n$ be a compact submanifold with boundary. For almost any $v \in S^{n-1}$, $h_v : X \to \mathbb{R}$ is a strong Morse function.*

Proof. We consider $S = X$ or $S = \partial X$ which are compact spaces in $\mathbb{R}^n$. From Theorem 3 of [14], since $(1, p)$ is in the nice range for all $p = dim(S)$, the linear map $h_a : S \to \mathbb{R}$ given by $h_a(x) = \sum_i a_i x_i$ is stable for almost any $a \in \mathbb{R}^n \setminus \{0\}$.

Let $W \subset \mathbb{R}^n \setminus \{0\}$ be the set of points a such that $h_a : S \to \mathbb{R}$ is not stable. Since W is a null set in $\mathbb{R}^n \setminus \{0\}$, $p(W)$ is a null set in $\mathbb{S}^{n-1}$, where $p : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1}$ is the normalization map. Then, for any $v \in \mathbb{S}^{n-1} \setminus p(W)$, $h_v : S \to \mathbb{R}$ is stable.

In the case of functions, it is well known that stability is equivalent to that all critical points are non-degenerate with distinct critical values (see [4]). Therefore $h_v : X \to \mathbb{R}$ and $h_v : \partial X \to \mathbb{R}$ are Morse functions with distinct critical values for almost any $v \in \mathbb{S}^{n-1}$. Since $h_v : X \to \mathbb{R}$ has not critical points, critical values of $h_v : \partial X \to \mathbb{R}$ cannot coincide with critical values of $h_v : X \to \mathbb{R}$. Then, $h_v : X \to \mathbb{R}$ is a strong Morse function for almost any $v \in \mathbb{S}^{n-1}$. $\square$

Corollary 5.2. *Let $Q \subset \mathbb{R}^n$ be a compact domain with boundary. For almost any $v \in \mathbb{S}^{n-1}$, $h_v : Q \to \mathbb{R}$ is a strong Morse function.*

5.2. General case M^n_λ ($\lambda \neq 0$)

Lemma 5.3. *Let $X \subset M^n_\lambda$ be a submanifold and let $\psi : I \to \mathbb{R}$ be a diffeomorphism, where I is an open interval in $\mathbb{R}$. If $f : X \to I$ is a strong Morse function, then $g := \psi \circ f$ is a strong Morse function.*

Proof. Since ψ is a diffeomorphism and f is a strong Morse function, it is deduced that g is also a strong Morse function. Note that the critical points of f coincide with the critical points of g. □

Let $Q \subset M^n_\lambda$ be a compact domain with boundary, $O \in M^n_\lambda$ and v denote a unit vector in $\mathbb{S}^{n-1} \subset T_O Q$. The geodesic $\gamma_v : I \subset \mathbb{R} \to Q$ is given by $\gamma_v = c_\lambda(t)O + s_\lambda(t)v$, where $I =]-\frac{\pi}{\sqrt{\lambda}}, \frac{\pi}{\sqrt{\lambda}}[$ for $\lambda > 0$ and $I = \mathbb{R}$ for $\lambda < 0$.

Then, given v, let $h_v : Q \subset M^n_\lambda \to \mathbb{R}$ be the height function in M^n_λ, whose level hypersurfaces are perpendicular to the geodesic γ_v.

Theorem 5.4. *Let $Q \subset M^n_\lambda$ be a compact domain with boundary which, for $\lambda > 0$, it is contained in the hemisphere of M^n_λ with pole O. Then, for almost any $v \in \mathbb{S}^{n-1}$, $h_v : Q \to \mathbb{R}$ is a strong Morse function.*

Proof. It is useful to consider the embedding of the space form M^n_λ into $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_\lambda)$ as follows:

$$\begin{cases} x_0 = 1, & \lambda = 0, \\ x_0^2 + x_1^2 + \ldots + x_n^2 = \frac{1}{\lambda}, & \lambda > 0, \\ -x_0^2 + x_1^2 + \ldots + x_n^2 = \frac{1}{\lambda}, \quad x_0 > 0, & \lambda < 0, \end{cases} \tag{43}$$

where $(x_0, x_1, \ldots, x_n)$ denote the coordinates of a point in $\mathbb{R}^{n+1}$, and $\langle \cdot, \cdot \rangle_\lambda$ is the appropriate metric to the embedding, which depends on the sign of λ.

Using this embedding, $Q \subset M^n_\lambda \subset \mathbb{R}^{n+1}$ can be considered as a compact submanifold with boundary in $\mathbb{R}^{n+1}$. Then, the height function of $\mathbb{R}^{n+1}$ with respect to the direction v, restricted to Q is:

$$h^{\mathbb{R}^{n+1}}_{v,\lambda}: \qquad \begin{aligned} Q &\longrightarrow \mathbb{R} \\ x &\longrightarrow \langle x, v\rangle_\lambda \end{aligned} \tag{44}$$

From Theorem 5.1, $h^{\mathbb{R}^{n+1}}_{v,\lambda}$ is a strong Morse function for almost any $v \in S^{n-1}_+$. Moreover, we note $h^{\mathbb{R}^{n+1}}_{v,\lambda}(Q) \subset \overline{I}$.

Since $\langle v, O\rangle_\lambda = 0$, we have that,

$$h^{\mathbb{R}^{n+1}}_{v,\lambda}(\gamma_v(\rho)) = \langle \gamma_v(\rho), v\rangle_\lambda = s_\lambda(\rho) = \begin{cases} \lambda^{-1/2}\sin(\rho\sqrt{\lambda}), & \lambda > 0, \\ |\lambda|^{-1/2}\sinh(\rho\sqrt{|\lambda|}), & \lambda < 0. \end{cases} \tag{45}$$

Eq.(45) gives a relation between the height function $h_v(\gamma_v(\rho)) = \rho$ of Q in M^n_λ and the height function $h^{\mathbb{R}^{n+1}}_{v,\lambda}$ of Q in $\mathbb{R}^{n+1}$. That is,

$$h_v(x) = \psi(h^{\mathbb{R}^{n+1}}_{v,\lambda}(x)) = \begin{cases} \frac{1}{\sqrt{\lambda}}\arcsin(\sqrt{\lambda}\, h^{\mathbb{R}^{n+1}}_{v,\lambda}(x)), & \lambda > 0, \\ \frac{1}{\sqrt{-\lambda}}\operatorname{arcsinh}(\sqrt{-\lambda}\, h^{\mathbb{R}^{n+1}}_{v,\lambda}(x)), & \lambda < 0. \end{cases} \tag{46}$$

Finally, since Q is contained in the hemisphere of M^n_λ with pole O for $\lambda > 0$, we have that ψ is a diffeomorphism from I to $\mathbb{R}$ when $I =]-\frac{\pi}{\sqrt{\lambda}}, \frac{\pi}{\sqrt{\lambda}}[$ for $\lambda > 0$ and when $I = \mathbb{R}$ for $\lambda < 0$; therefore from Lemma 5.3 we obtain the result. $\square$

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