Strong mixing measures for linear operators and frequent hypercyclicity

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Abstract

We construct strongly mixing invariant measures with full support for operators on F-spaces which satisfy the Frequent Hypercyclicity Criterion. For unilateral backward shifts on sequence spaces, a slight modification shows that one can even obtain exact invariant measures.

1 Introduction

We recall that an operator T on a topological vector space X is called *hypercyclic* if there is a vector x in X such that its *orbit* $\mathrm{Orb}(x,T) = \{x, Tx, T^2x, \dots\}$ is dense in X. The recent books [5] and [15] contain the theory and most of the recent advances on hypercyclicity and linear dynamics, especially in topological dynamics.

Here we are concerned with measure theoretic properties. Let $(X, \mathfrak{B}, \mu)$ be a probability space, where X is a topological space and $\mathfrak{B}$ denotes the σ-algebra of Borel subsets of X. We will say that a Borel probability measure μ has *full support* if for all non-empty open set $U \subset X$ we have $\mu(U) > 0$. A measurable map $T : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is called a *measure-preserving* transformation if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathfrak{B}$). T is said to be *strongly mixing* with respect to μ if

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B) \qquad (A, B \in \mathfrak{B}),$$

and it is *exact* if given $A \in \bigcap_{n=0}^{\infty} T^{-n}\mathfrak{B}$ then either $\mu(A) = 0$ or $\mu(A) = 1$. The interested reader is referred to [20, 10] for a detailed account on the above properties.

Ergodic theory was first used for the dynamics of linear operators by Rudnicki [18] and Flytzanis [11]. During the last years it has deserved special attention thanks to the work of Bayart and Grivaux [2, 3]. For instance, the papers [1, 4, 6, 9, 13, 19] contain recent advances on the subject.

The concept of frequently hypercyclicity was introduced by Bayart and Grivaux [3] inspired by Birkhoff's ergodic theorem. They also gave the first version of a Frequent Hypercyclicity Criterion, although we will consider the formulation of Bonilla and Grosse-Erdmann [8] for operators on separable F-spaces. Another (probabilistic) version of it was given by Grivaux [12].

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Under the hypothesis of Bonilla and Grosse-Erdmann we derive a stronger result by showing that a T-invariant mixing measure can be obtained. Recently, Bayart and Matheron gave very general conditions expressed on eigenvector fields associated to unimodular eigenvalues under which an operator T admits a T-invariant mixing measure [6]. Actually, on the one hand our results can be deduced from [6] in the context of complex Fréchet spaces, and on the other hand we only need rather elementary tools.

From now on, T will be an operator defined on a separable F-space X.

2 Invariant measures and the frequent hypercyclicity criterion

We recall that a series $\sum_n x_n$ in X *converges unconditionally* if it converges and, for any 0-neighbourhood U in X, there exists some $N \in \mathbb{N}$ such that $\sum_{n\in F} x_n \in U$ for every finite set $F \subset \{N, N+1, N+2, \dots\}$.

We are now ready to present our main result. The idea behind the proof is to construct a "model" probability space $(Z, \overline{\mu})$ and a (Borel) measurable map $\Phi : Z \to X$, where $Z \subset \mathbb{N}^{\mathbb{Z}}$ is such that $\sigma(Z) = Z$ for the Bernoulli shift $\sigma(\dots, n_{-1}, n_0, n_1, \dots) = (\dots, n_0, n_1, n_2, \dots)$, $\overline{\mu}$ is a σ^{-1}-invariant strongly mixing measure, $Y := \Phi(Z)$ is a T-invariant dense subset of X, $\Phi\sigma^{-1} = T\Phi$, and then the Borel probability measure μ on X defined by $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$, $A \in \mathfrak{B}(X)$, is T-invariant and strongly mixing. We will use the slight generalization of the Frequent Hypercyclicity Criterion for operators given in [15, Remark 9.10].

Theorem 1. *Let T be an operator on a separable F-space X. If there is a dense subset X_0 of X and a sequence of maps $S_n : X_0 \to X$ such that, for each $x \in X_0$,*

(i) $\sum_{n=0}^{\infty} T^n x$ *converges unconditionally,*

(ii) $\sum_{n=0}^{\infty} S_n x$ *converges unconditionally, and*

(iii) $T^n S_n x = x$ *and* $T^m S_n x = S_{n-m} x$ *if* $n > m$,

then there is a T-invariant strongly mixing Borel probability measure μ on X with full support.

Proof. We suppose $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$ with $x_1 = 0$ and $S_n 0 = 0$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. By (i) and (ii), there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} T^k x_{m_k} \in U_{n+1} \text{ and } \sum_{k>N_n} S_k x_{m_k} \in U_{n+1}, \text{ if } m_k \le 2l, \text{ for } N_l < k \le N_{l+1}, \ l \ge n. \quad (1)$$

Actually, this is a consequence of the completeness of X and the fact that, for each 0-neighbourhood U and for all $l \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that $\sum_{k\in F} T^k x \in U$ and $\sum_{k\in F} S_k x \in U$ for any finite subset $F \subset]N, +\infty[$ and for each $x \in \{x_1, \dots, x_{2l}\}$.

1.-The model probability space $(Z, \overline{\mu})$.

We define $K = \prod_{k\in\mathbb{Z}} F_k$ where

$$F_k = \{1, \dots m\} \text{ if } N_m < |k| \le N_{m+1}, \ m \in \mathbb{N}, \text{ and } F_k = \{1\}, \text{ if } |k| \le N_1.$$

Let $K(s) := \sigma^s(K)$, $s \in \mathbb{Z}$, where $\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$ is the backward shift. $K(s)$ is a compact space when endowed with the product topology inherited from $\mathbb{N}^{\mathbb{Z}}$, $s \in \mathbb{Z}$.

We consider in $\mathbb{N}^{\mathbb{Z}}$ the product measure $\overline{\mu} = \bigotimes_{k\in\mathbb{Z}} \overline{\mu_k}$, where $\overline{\mu_k}(\{n\}) = p_n$ for all $n \in \mathbb{N}$ and $\overline{\mu_k}(\mathbb{N}) = \sum_{n=1}^{\infty} p_n = 1$, $k \in \mathbb{Z}$. The values of $p_n \in]0,1[$ are selected such that, if

$$\beta_j := \left(\sum_{i=1}^{j} p_i\right)^{N_{j+1}-N_j}, \ j \in \mathbb{N}, \text{ then } \prod_{j=1}^{\infty} \beta_j > 0.$$

Let $Z = \bigcup_{s\in\mathbb{Z}} K(s)$. We have

$$\overline{\mu}(Z) \geq \overline{\mu}(K) = \prod_{|k|\leq N_1} \overline{\mu}_k(\{1\}) \prod_{l=1}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\ldots,l\}) \right) = p_1^{2N_1+1} (\prod_{l=1}^{\infty} \beta_l)^2 > 0.$$

It is well-known [20] that $\overline{\mu}$ is a σ^{-1}-invariant strongly mixing Borel probability measure. Since $\sigma(Z) = Z$ and it has positive measure, then $\overline{\mu}(Z) = 1$.

2.-The map Φ.

Given $s \in \mathbb{Z}$ we define the map $\Phi : K(s) \to X$ by

$$\Phi((n_k)_{k\in\mathbb{Z}}) = \sum_{k<0} S_{-k} x_{n_k} + x_{n_0} + \sum_{k>0} T^k x_{n_k}. \tag{2}$$

Φ is well-defined since, given $(n_k)_{k\in\mathbb{Z}} \in K(s)$ and for $l \geq |s|$, we have $n_k \leq 2l$ if $N_l < |k| \leq N_{l+1}$, which shows the convergence of the series in (2) by (1).

Φ is also continuous. Indeed, let $(\alpha(j))_j$ be a sequence of elements of $K(s)$ that converges to $\alpha \in K(s)$ and fix any $n \in \mathbb{N}$ with $n > |s|$. We will find $n_0 \in \mathbb{N}$ such that $\Phi(\alpha(j)) - \Phi(\alpha) \in U_n$ for $n \geq n_0$. To do this, by definition of the topology in $K(s)$ there exists $n_0 \in \mathbb{N}$ such that

$$\alpha(j)_k = \alpha_k \quad \text{if} \quad |k| \leq N_{n+1} \text{ and } j \geq n_0.$$

By (1) we have

$$\Phi(\alpha(j)) - \Phi(\alpha) = \sum_{k<-N_{n+1}} S_{-k}(x_{\alpha(j)_k} - x_{\alpha_k}) + \sum_{k>N_{n+1}} T^k(x_{\alpha(j)_k} - x_{\alpha_k}) \in U_n$$

for all $j \geq n_0$. This shows the continuity of $\Phi : K(s) \to X$ for every $s \in \mathbb{Z}$.

The map Φ is then well-defined on Z, and $\Phi : Z \to X$ is measurable (i.e., $\Phi^{-1}(A) \in \mathfrak{B}(Z)$ for every $A \in \mathfrak{B}(X)$).

3.-The measure μ on X.

$L(s) := \Phi(K(s))$ is compact in X, $s \in \mathbb{Z}$, and $Y := \bigcup_{s\in\mathbb{Z}} L(s)$ is a T-invariant Borel subset of X because $\Phi\sigma^{-1} = T\Phi$.

We then define in X the measure $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$ for all $A \in \mathfrak{B}(X)$. Obviously, μ is well-defined and it is a T-invariant strongly mixing Borel probability measure. The proof is completed by showing that μ has full support. Given a non-empty open set U in X, we pick $n \in \mathbb{N}$ satisfying $x_n + U_n \subset U$. Thus

$$\mu(U) \geq \mu(\{x = x_n + \sum_{k>N_n} T^k x_{m_k} + \sum_{k>N_n} S_k x_{m_k} \ ; \ m_k \leq 2l \text{ for } N_l < k \leq N_{l+1}, \ l \geq n\})$$

$$\geq \overline{\mu}_0(\{n\}) \prod_{0<|k|\leq N_n} \overline{\mu}_k(\{1\}) \prod_{l=n}^{\infty} \left(\prod_{N_l<|k|\leq N_{l+1}} \overline{\mu}_k(\{1,\dots,2l\}) \right) > p_n p_1^{2N_n} (\prod_{l=n}^{\infty} \beta_l)^2 > 0.$$

□

As we mentioned in the Introduction, Theorem 1 can be deduced from [6, Corollary 1.3] when dealing with operators on separable complex Fréchet spaces. Indeed, the argument of É. Matheron is the following (we thank S. Grivaux for letting us know about it):

Let $T : X \to X$ be an operator on a separable complex Fréchet space X satisfying the hypothesis of the Frequent Hypercyclicity Criterion given in Theorem 1, and suppose $X_0 = \{x_n \ ; \ n \in \mathbb{N}\}$. We define the following family of continuous $\mathbb{T}$-eigenvector fields for T

$$E_m(\lambda) = \sum_{n\geq 0} \lambda^{-n} T^n x_m + \sum_{n\in\mathbb{N}} \lambda^n S_n x_m, \quad \lambda \in \mathbb{T}, \quad m \in \mathbb{N}.$$

They span X since, for any functional x^* that vanishes on $E_m(\lambda)$ for each $\lambda \in \mathbb{T}$ and $m \in \mathbb{N}$, the equality $\langle x^*, E_m(\lambda)\rangle = 0$ for fixed m and for all $\lambda \in \mathbb{T}$ implies that $\langle x^*, T^n x_m\rangle = 0$ for every $n \geq 0$. Thus, $\langle x^*, x_m\rangle = 0$ for each $m \in \mathbb{N}$, and by density $x^* = 0$.

The previous Theorem can be applied to different classes of operators. A distinguished one is the class of weighted shifts on sequence F-spaces.

By a *sequence space* we mean a topological vector space X which is continuously included in ω, the countable product of the scalar field $\mathbb{K}$. A *sequence F-space* is a sequence space that is also an F-space. Given a sequence $w = (w_n)_n$ of non-zero weights, the associated *unilateral* (respectively, *bilateral*) *weighted backward shift* $B_w : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ is defined by $B_w(x_1, x_2, \dots) = (w_2x_2, w_3x_3, \dots)$ (respectively, $B_w : \mathbb{K}^{\mathbb{Z}} \to \mathbb{K}^{\mathbb{Z}}$ is defined by $B_w(\dots, x_{-1}, x_0, x_1, \dots) = (\dots, w_0x_0, w_1x_1, w_2x_2, \dots)$). When a sequence F-space X is invariant under certain weighted backward shift T, then T is also continuous on X by the closed graph theorem.

We refer the reader to, e.g., Chapter 4 of [15] for more details about hypercyclic and chaotic weighted shifts on Fréchet sequence spaces. In particular, Theorems 4.6 and 4.12 in [15] (we refer to [14] for the original results) remain valid for F-spaces, and a bilateral (respectively, unilateral) weighted backward shift $T : X \to X$ on a sequence F-space X in which the canonical unit vectors $(e_n)_{n\in\mathbb{Z}}$ (respectively, $(e_n)_{n\in\mathbb{N}}$) form an unconditional basis is chaotic if, and only if, $\sum_{n\in\mathbb{Z}} e_n$ (respectively, $\sum_{n\in\mathbb{N}} e_n$) converges unconditionally.

Corollary 2. *Let $T : X \to X$ be a chaotic bilateral weighted backward shift on a sequence F-space X in which $(e_n)_{n\in\mathbb{Z}}$ is an unconditional basis. Then there exists a T-invariant strongly mixing Borel probability measure on X with full support.*

Remark 3. The preceding result can be improved if T is a unilateral backward shift operator on a sequence F-space. In that case, there exists a T-invariant exact Borel probability measure on X with full support.

Proof. Let $M = \{z_n \ ; \ n \in \mathbb{N}\}$ be a countable dense set in $\mathbb{K}$ with $z_1 = 0$. Let $(U_n)_n$ be a basis of balanced open 0-neighbourhoods in X such that $U_{n+1} + U_{n+1} \subset U_n$, $n \in \mathbb{N}$. Since T is chaotic, $\sum_{n=1}^{\infty} e_n$ converges unconditionally, so there exists an increasing sequence of positive integers $(N_n)_n$ with $N_{n+2} - N_{n+1} > N_{n+1} - N_n$ for all $n \in \mathbb{N}$ such that

$$\sum_{k>N_n} \alpha_k e_k \in U_{n+1}, \text{ if } \alpha_k \in \{z_1, \dots, z_{2m}\}, \text{ for } N_m < k \leq N_{m+1}, \ m \geq n. \quad (3)$$

We define $K = \prod_{k\in\mathbb{N}} F_k$ where

$$F_k = \{z_1, \dots z_m\} \text{ if } N_m < k \leq N_{m+1},\ m \in \mathbb{N}, \text{ and } F_k = \{z_1\}, \text{ if } k \leq N_1.$$

Let $K(s) := \sigma^s(K)$, $s \geq 0$. $K(s)$ is a compact space when endowed with the product topology inherited from $M^{\mathbb{N}}$, $s \geq 0$. We consider in $M^{\mathbb{N}}$ the product measure $\overline{\mu} = \bigotimes_{k\in\mathbb{N}} \overline{\mu_k}$, where $\overline{\mu_k}(\{z_n\}) = p_n$ for all $n \in \mathbb{N}$ and $\overline{\mu_k}(M) = \sum_{n=1}^{\infty} p_n = 1$, $k \in \mathbb{N}$. As before, we select the sequence $(p_n)_n$ of positive numbers such that, if

$$\beta_j = \left(\sum_{i=1}^{j} p_i\right)^{N_{j+1}-N_j}, \text{ then } \prod_{j=1}^{\infty} \beta_j > 0.$$

It is known [20, §4.12] that $\overline{\mu}$ is a σ-invariant exact Borel probability measure. By setting $Z = \bigcup_{s\geq 0} K(s)$, we have $\overline{\mu}(Z) = 1$.

Now we define the map $\Phi : K(s) \to X$ given by

$$\Phi((\alpha_k)_{k\in\mathbb{N}}) = \sum_{k=1}^{\infty} \alpha_k e_k.$$

Φ is (well-defined and) continuous, $s \geq 0$. We have that $\Phi : Z \to X$ is measurable. $L(s) := \Phi(K(s))$ is compact in X, $s \geq 0$, and $Y := \bigcup_{s\geq 0} L(s) = \Phi(Z)$ is a T-invariant Borel subset of X.

We then define on X the measure $\mu(A) = \overline{\mu}(\Phi^{-1}(A))$ for all $A \in \mathfrak{B}(X)$. As in Theorem 1, we conclude that μ is well-defined, and now it is a T-invariant exact Borel probability measure with full support. □

Devaney chaos is therefore a sufficient condition for the existence of strongly mixing measures within the framework of weighted shift operators on sequence F-spaces. In some natural spaces it is even a characterization of this fact. For instance, F. Bayart and I. Z. Ruzsa [7] recently proved that weighted shift operators on ℓ^p, $1 \leq p < \infty$, are frequently hypercyclic if, and only if, they are Devaney chaotic. It turns out that this is equivalent to the existence of an invariant strongly mixing Borel probability measure with full support on ℓ^p. Also, for the space ω, every weighted shift operator is chaotic [14]. In particular, for the unilateral case we obtain exact measures.

Example 4. Every unilateral weighted backward shift operator on $\omega = \mathbb{K}^{\mathbb{N}}$ admits an invariant exact Borel probability measure with full support on ω.

We finish the paper by mentioning that a continuous-time version of Theorem 1 can be given by using the Frequent Hypercyclicity Criterion for C_0-semigroups introduced in [16]. This is part of a forthcoming paper [17].

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