Frequently hypercyclic translation semigroups

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Dedicated to José Bonet on the occasion of his 60th birthday

Abstract

Frequent hypercyclicity for translation C_0-semigroups on weighted spaces of continuous functions is studied. The results are achieved by establishing an analogy between frequent hypercyclicity for translation semigroups and for weighted pseudo-shifts and by characterizing frequently hypercyclic weighted pseudo-shifts on spaces of vanishing sequences. Frequently hypercyclic translation semigroups on weighted L^p-spaces are also characterized.

1 Introduction and preliminaries

A continuous linear operator T on a separable Banach space X is called *hypercyclic* if there is an element $x \in X$, called a *hypercyclic vector*, such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in X. The first historically known examples of hypercyclic operators are due to Birkhoff, MacLane and Rolewicz. In

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particular, the last author studied hypercyclicity of weighted shift operators on l^p and c_0. The interest in the study of linear dynamics of shift operators is nowadays still alive, since many classical operators (e.g. derivative operators in spaces of entire functions) can be viewed as such operators. We refer to the recent monographs [11] and [24] for a complete overview on the subject.

In 2005, motivated by Birkhoff's ergodic theorem, Bayart and Grivaux [9] introduced the notion of frequently hypercyclic operators, trying to quantify how "often" an orbit meets non-empty open sets. More precisely, if the *lower density* of a set $A \subseteq \mathbb{N}$ is defined as

$$\underline{\text{dens}}(A) := \liminf_{N \to \infty} \#\{n \leq N : n \in A\}/N,$$

an operator $T \in L(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ (called a *frequently hypercyclic vector*) such that, for every non-empty open subset $U \subseteq X$,

$$\underline{\text{dens}}(\{n \in \mathbb{N} : T^n x \in U\}) > 0.$$

This notion has been deeply investigated by various authors: see e.g. [22, 14, 18]. In particular frequently hypercyclic weighted shifts have been investigated in [10, 14]; their behaviour in l^p and c_0 has been completely characterized by Bayart and Ruzsa [12].

In parallel with the theory for linear operators, since the seminal paper by Desch, Schappacher and Webb [20], many researchers turned their attention to the hypercyclic behaviour of strongly continuous semigroups. Actually hypercyclicity appears in solution semigroups of evolution problems associated with "birth and death" equations for cell populations, transport equations, first order partial differential equations, Black–Scholes equation, and diffusion operators like Ornstein–Uhlenbeck operators [2, 4, 5, 6, 8, 13, 15, 17, 21, 25, 27].

We recall that, if X is a separable infinite-dimensional Banach space, a C_0-semigroup $(T_t)_{t\geq 0}$ of continuous linear operators on X is said to be *hypercyclic* if there exists $x \in X$ (called a *hypercyclic vector* for the semigroup) such that the set $\{T_t x : t \geq 0\}$ is dense in X. An element $x \in X$ is said to be a *periodic point* for the semigroup if there exists $t > 0$ such that $T_t x = x$. A semigroup $(T_t)_{t\geq 0}$ is called *chaotic* if it is hypercyclic and the set of periodic points is dense in X.

The role of a "test" class, which is played by weighted shifts in the setting of discrete linear dynamical systems, is taken over by translation semigroups in the setting of continuous linear dynamical systems.

Let $I = \mathbb{R}$ or $I = [0, \infty[$. An *admissible weight function* on I is a measurable function $\rho : I \to]0, \infty[$ for which there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $\rho(\tau) \leq M e^{\omega t} \rho(\tau + t)$ for all $\tau \in I$ and $t > 0$.

If ρ is an admissible weight function, then for every $l > 0$ there exist $A, B > 0$ such that for all $\sigma \in I$ and $t \in [\sigma, \sigma + l]$,

$$A\rho(\sigma) \leq \rho(t) \leq B\rho(\sigma + l). \tag{1.1}$$

For any $1 \leq p < \infty$, consider the following function spaces:

$$L_p^\rho(I) = \{u : I \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_p^\rho < \infty\},$$

where $\|u\|_p^\rho = (\int_I |u(t)|^p \rho(t)\, dt)^{1/p}$, and

$$C_0^\rho(I) = \left\{u : I \to \mathbb{R} \;\middle|\; u \text{ is continuous and } \lim_{x \to \pm\infty} u(x)\rho(x) = 0\right\},$$

with $\|u\|_\infty^\rho = \sup_{t \in I} |u(t)|\rho(t)$.

If X is any of the spaces above and ρ is an admissible weight function, the *translation semigroup* $\mathcal{T} = (T_t)_{t \geq 0}$ is defined as usual by

$$T_t f(x) = f(x + t), \qquad t \geq 0,\ f \in X,\ x \in \mathbb{R},$$

and it is a C_0-semigroup (see e.g. [20]).

Hypercyclicity and chaos for translation semigroups have been characterized in [20, 27]. In particular, if X is one of the spaces $L_p^\rho(\mathbb{R})$ or $C_0^\rho(\mathbb{R})$ with an admissible weight function ρ, then the translation semigroup $\mathcal{T}$ on X is hypercyclic if and only if for each $\theta \in \mathbb{R}$ there exists a sequence $(t_j)_j$ of positive real numbers tending to ∞ such that

$$\lim_{j \to \infty} \rho(t_j + \theta) = \lim_{j \to \infty} \rho(-t_j + \theta) = 0.$$

If $X = C_0^\rho(\mathbb{R})$, then the translation semigroup $\mathcal{T}$ on X is chaotic if and only if $\lim_{x \to \pm\infty} \rho(x) = 0$.

If $X = L_p^\rho(\mathbb{R})$, then $\mathcal{T}$ is chaotic if and only if for all $\varepsilon, l > 0$ there exists $P > 0$ such that

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \rho(l + kP) < \varepsilon.$$

The concept of frequent hypercyclicity was extended to C_0-semigroups in [3].

The *lower density* of a measurable set $M \subseteq \mathbb{R}_+$ is defined by

$$\underline{\text{Dens}}(M) := \liminf_{N \to \infty} \mu(M \cap [0, N])/N,$$

where μ is the Lebesgue measure on $\mathbb{R}_+$.

A C_0-semigroup $(T_t)_{t\geq 0}$ on a separable Banach space X is said to be *frequently hypercyclic* if there exists $x \in X$ (called a *frequently hypercyclic vector* for the semigroup) such that $\underline{\text{Dens}}(\{t \in \mathbb{R}_+ : T_t x \in U\}) > 0$ for any non-empty open set $U \subseteq X$. In [16, 26], it was proved that $x \in X$ is a (frequently) hypercyclic vector for $(T_t)_{t\geq 0}$ if and only if x is a (frequently) hypercyclic vector for each single operator T_t, $t > 0$. However, this is not the case in general if we consider the chaos property [8].

In [26], a continuous version of the Frequent Hypercyclicity Criterion was proved, based on the Pettis integral and the fact that chaotic translation semigroups on weighted spaces of integrable functions are frequently hypercyclic.

Moreover, in [28], it is proved that the Frequent Hypercyclicity Criterion for semigroups implies the existence of strongly-mixing Borel probability measures with full support.

In this paper we characterize, in the spirit of [12], frequently hypercyclic translation semigroups on $L^p_\rho(I)$ and, for $\sup_k \rho(k+1)/\rho(k) < \infty$, on $C^\rho_0(I)$. The main results are Theorems 3.2 and 3.9, proved in the last section. In particular, Theorem 3.2 will be a consequence of Theorem 2.1 which characterizes frequent hypercyclicity of the so-called pseudo-shifts on $c_0(I)$ spaces, where I is a countably infinite set.

2 Frequently hypercyclic weighted pseudo-shift

We recall the concept of weighted pseudo-shift introduced by Grosse-Erdmann [23].

Given topological sequence spaces X, Y over countably infinite sets I and J respectively, a continuous linear operator $T : X \to Y$ is called a *weighted pseudo-shift* if there is a sequence $(b_j)_{j\in J}$ of non-zero scalars and an injective mapping $\phi : J \to I$ such that

$$T[(x_i)_{i\in I}] = (b_j x_{\phi(j)})_{j\in J} \quad \text{for } (x_i)_{i\in I} \in X.$$

We will be interested in weighted pseudo-shifts acting on spaces of vanishing sequences. More precisely, given a countable set I, we consider the space

$$c_0(I) = \{(x_i)_{i\in I} \in \mathbb{R}^I \mid \forall \varepsilon > 0 \ \exists J \subseteq I,\ J \text{ finite } \forall i \in I \setminus J : |x_i| < \varepsilon\},$$

endowed with the norm $\|(x_i)_{i\in I}\| = \sup_{i\in I} |x_i|$.

Obviously, if $(W_p)_{p\in\mathbb{N}}$ is an increasing sequence of finite subsets of I such that $I = \bigcup_{p=1}^{\infty} W_p$, then

$$c_0(I) = \{(x_i)_{i\in I} \in \mathbb{R}^I \mid \forall\varepsilon > 0\ \exists n \in \mathbb{N}\forall i \in I \setminus W_n : |x_i| < \varepsilon\}. \tag{2.1}$$

The first result that we prove is a characterization of frequently universal sequences of weighted pseudo-shifts on $c_0(I)$.

We recall that a sequence $(T_n)_{n\in\mathbb{N}}$ of continuous mappings between topological spaces X and Y is said to be *frequently universal* if there exists $x \in X$, called a *frequently universal vector* for the sequence, such that for every non-empty open set $U \subseteq Y$,

$$\underline{\text{dens}}(\{n \in \mathbb{N} : T_n x \in U\}) > 0.$$

Following the idea of Bayart and Ruzsa [12] for weighted backward shifts on $c_0(\mathbb{Z})$, we first obtain a characterization for weighted pseudo-shifts.

Theorem 2.1. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of weighted pseudo-shifts on $c_0(I)$ defined by $T_n[(x_i)_{i\in I}] = (b_i^n x_{\phi_n(i)})_{i\in I}$, where the b_i^n are positive real numbers. Assume that:*

(i) *$(\phi_n)_n$ is a run-away sequence, i.e. for any finite subsets $I_0, J_0 \subseteq I$ there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $\phi_n(J_0) \cap I_0 = \emptyset$,*

(ii) *there exists $\rho > 1$ such that $1/\rho^{|n-m|} \leq b_s^n / b_t^m$ for all $n, m \in \mathbb{N}$ and $s, t \in I$ such that $\phi_n(s) = \phi_m(t)$,*

(iii) *there exists $g : I \to \mathbb{R}$ such that $|n - m| \leq |g(s) - g(t)|$ for all $n, m \in \mathbb{N}$ and $s, t \in I$ such that $\phi_n(s) = \phi_m(t)$,*

(iv) *$(W_p)_{p\in\mathbb{N}}$ is an increasing sequence of finite subsets of I such that $I = \bigcup_{p=1}^{\infty} W_p$.*

Then $(T_n)_{n\in\mathbb{N}}$ is frequently universal on $c_0(I)$ if and only if there exist a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that:

(a) *for any $p \geq 1$, $\underline{\text{dens}}(E_p) > 0$,*
(b) *for any distinct $p, q \geq 1$, $n \in E_p$ and $m \in E_q$, $\phi_n(W_p) \cap \phi_m(W_q) = \emptyset$,*
(c) *for every $p \geq 1$ and every $s \in W_p$, $\lim_{n\to\infty, n\in E_p} b_s^n = \infty$,*

(d) *for any* $p, q \geq 1$, $n \in E_p$, $m \in E_q$ *with* $n \neq m$, $t \in W_q$ *and* $s \in I$ *such that* $\phi_n(s) = \phi_m(t)$,

$$\frac{b_s^n}{b_t^m} \leq \frac{1}{M(p)M(q)}.$$

Moreover, one can replace "there exists a sequence $(M(p))_{p\in\mathbb{N}}$ *and a sequence* $(E_p)_{p\in\mathbb{N}}$*" by "for any sequence* $(M(p))_{p\in\mathbb{N}}$ *there exists a sequence* $(E_p)_{p\in\mathbb{N}}$*".*

Proof. We first observe that if properties (a) to (d) hold for some sequence $(M(p))_{p\in\mathbb{N}}$, then they are also satisfied for any sequence $(M(p))_{p\in\mathbb{N}}$, on considering, if necessary, a subsequence of $(E_p)_{p\in\mathbb{N}}$.

"$\Rightarrow$": Let $x \in c_0(I)$ be a frequently universal vector for $(T_n)_{n\in\mathbb{N}}$. Let $(\alpha_p)_{p\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers such that $\alpha_1 = 2$ and for all $p \geq 2$, $\alpha_p > 4\alpha_{p-1}\rho^{2\Psi(p)}$, where $\Psi(p) = \max\{|g(t)| : t \in W_p\}$. Define

$$F_p = \Big\{ n \in \mathbb{N} : \|T_n x - \alpha_p \sum_{i\in W_p} e_i\| < 1/p \Big\}.$$

If $F_p = \{n_k^p : k \in \mathbb{N}\}$, where $(n_k^p)_{k\in\mathbb{N}}$ is an increasing sequence of natural numbers, we define $E_p = \{n^p_{(2[\Psi(p)]+3)k} : k \in \mathbb{N}\}$ where $[\Psi(p)]$ is the integer part of $\Psi(p)$.

Clearly $\underline{\mathrm{dens}}(E_p) > 0$ and the distance between two different elements of E_p is greater than $2\Psi(p)$. Moreover

$$\forall p \in \mathbb{N}\ \forall s \in W_p\ \forall n \in E_p : \quad \alpha_p/2 \leq |b_s^n x_{\phi_n(s)}| < 2\alpha_p. \tag{2.2}$$

Indeed, $b_s^n x_{\phi_n(s)}$ is the sth coefficient of $T_n x$, so

$$|b_s^n x_{\phi_n(s)}| \leq \Big\| T_n x - \alpha_p \sum_{i\in W_p} e_i \Big\| + \alpha_p \Big\| \sum_{i\in W_p} e_i \Big\| < \frac{1}{p} + \alpha_p < 2\alpha_p,$$

while

$$\begin{aligned} |b_s^s x_{\phi_n(s)}| &\geq \alpha_p - |b_s^n x_{\phi_n(s)} - \alpha_p| \geq \alpha_p - \Big\| T_n x - \alpha_p \sum_{i\in W_p} e_i \Big\| \\ &\geq \alpha_p - 1/p \geq \alpha_p/2. \end{aligned} \tag{2.3}$$

In particular,

$$\forall p \in \mathbb{N}\ \forall s \in W_p\ \forall n \in E_p : \quad x_{\phi_n(s)} \neq 0. \tag{2.4}$$

In order to prove (b), fix $p \neq q$, with $p < q$, $n \in E_p, m \in E_q$ and assume by contradiction, that there exist $s \in W_p$ and $t \in W_q$ such that $\phi_n(s) = \phi_m(t)$. Then, by (2.2),

$$\frac{1}{\rho^{2\Psi(q)}} \leq \frac{1}{\rho^{|n-m|}} \leq \frac{|b_s^n x_{\phi_n(s)}|}{|b_t^m x_{\phi_m(t)}|} \leq 2\alpha_p \frac{2}{\alpha_q} \leq 4\frac{\alpha_{q-1}}{\alpha_q},$$

contradicting the choice of $(\alpha_p)_p$.

Now let $p \geq 1$ and $s \in W_p$. Let $M > 0$. Given $\varepsilon = \alpha_p/(2M)$, since $x \in c_0(I)$, there exists $J \subseteq I$ finite such that $|x_i| < \varepsilon$ for all $i \in I \setminus J$. Since (ϕ_n) is a run-away sequence, there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n > n_0$ and all $s \in W_p$ we have $\phi_n(s) \notin J$, and so $|x_{\phi_n(s)}| < \varepsilon$. Hence, for all $n \in E_p$ with $n \geq n_0$, by (2.2) and (2.4),

$$b_s^n \geq \frac{\alpha_p}{2|x_{\phi_n(s)}|} \geq \frac{\alpha_p}{2\varepsilon} = M.$$

So, we have proved (c).

Finally, let $p, q \geq 1$, $n \in E_p$, $m \in E_q$ with $n \neq m$, $t \in W_q$ and $s \in I$ be such that $\phi_n(s) = \phi_m(t)$. Then $p \neq q$, as otherwise, since $n \neq m$, by the definition of E_p we have $|n - m| > 2\Psi(p)$; on the other hand, (iii) yields $|n - m| \leq 2\Psi(p)$, a contradiction. Therefore, since $p \neq q$, we can apply (b) to get $s \notin W_p$ and so the s-coefficient of $T_n x - \alpha_p \sum_{i \in W_p} e_i$ is $b_s^n x_{\phi_n(s)}$. Hence, by (2.2),

$$\frac{b_s^n}{b_t^m} = \frac{|b_s^n x_{\phi_n(s)}|}{|b_t^m x_{\phi_m(t)}|} \leq \frac{1}{p}\frac{2}{\alpha_q} \leq \frac{1}{p}\frac{1}{q}.$$

Hence (d) holds with $M(p) = p$.

"$\Leftarrow$": We first observe that if (b) holds, then

$$\forall p, q \in \mathbb{N},\ p \neq q: \quad E_p \cap E_q = \emptyset. \tag{2.5}$$

Indeed, assume $p < q$; if there exists $n \in E_p \cap E_q$, then for any $s \in W_p \subseteq W_q$, one gets $\phi_n(s) \in \phi_n(W_p) \cap \phi_n(W_q)$, contradicting (b).

As properties (a) to (d) hold true for any sequence $(M(p))_{p\in\mathbb{N}}$, we may assume that $M(p) \geq \rho^{4p}$ for any $p \geq 1$.

We set

$$E_p' = E_p \setminus \bigcup_{s \in W_p} \{n \in \mathbb{N} : b_s^n \leq \rho^{4p}\}.$$

By (c), E_p' is a cofinite subset of E_p, hence $\underline{\text{dens}}(E_p') > 0$. If $E_p' = \{n_k^p : k \in \mathbb{N}\}$, where $(n_k^p)_k$ is an increasing sequence of natural numbers, we consider the set $G_p = \{n^p_{(2[\Psi(p)]+3)k} : k \in \mathbb{N}\}$. It has positive lower density and moreover the distance between two different elements of G_p is greater than $2\Psi(p)$.

Let $(y^p)_{p\geq 0}$ be a dense sequence in $c_0(I)$ such that $\text{supp}(y^p) \subseteq W_p$ and $\|y^p\| < \rho^p$. We define $x \in \mathbb{R}^I$ by setting

$$x_i = \begin{cases} \frac{1}{b_s^n} y^p(s) & \text{if } i = \phi_n(s),\ n \in G_p, s \in W_p, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

This definition is correct, because if $i = \phi_n(s) = \phi_m(t)$ with $n \in G_p$, $s \in W_p$, $m \in G_q$ and $t \in W_q$, then, by (b), $p = q$, and assumption (iii) yields $|n - m| \le |g(s) - g(t)| \le 2\Psi(p)$; hence, by the definition of G_p, $n = m$ and so $s = t$, by the injectivity of ϕ_n.

We have $x \in c_0(I)$. Indeed, given $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that for $p \ge p_0$ and $n \in G_p$, $s \in W_p$, $i = \phi_n(s)$,

$$|x_i| \le \frac{\rho^p}{\rho^{4p}} \le \varepsilon.$$

If $p \le p_0$, then

$$|x_i| \le \frac{\rho^{p_0}}{b_s^n} \to 0 \quad \text{ as } n \to \infty.$$

We finally show that x is a frequently hypercyclic vector by proving that for all $p \ge 1$ and $n \in G_p$, $\|T_n x - y^p\| < \varepsilon(p)$ with $\varepsilon(p) \to 0$ as $p \to \infty$. We have

$$\|T_n x - y^p\| = \sup_{s \notin W_p} |b_s^n x_{\phi_n(s)}|.$$

If $s \notin W_p$, then $b_s^n x_{\phi_n(s)}$ does not vanish if and only if

$$\exists q \ge 1 \ \exists m \in G_q,\ t \in W_q : \quad \phi_n(s) = \phi_m(t). \tag{2.7}$$

If (2.7) holds, then $n \ne m$, as otherwise, $p = q$ by (2.5) and $s = t$ by the injectivity of ϕ_n, which is impossible since $s \notin W_p$ and $t \in W_p = W_q$.

Hence, we can apply (d) to get

$$|b_s^n x_{\phi_n(s)}| = \left|\frac{b_s^n}{b_t^m} y^q(t)\right| \le \frac{\rho^q}{M(p)M(q)} \le \frac{\rho^q}{\rho^p \rho^q} = \frac{1}{\rho^p}.\square$$

As a corollary, we obtain a characterization of frequent hypercyclicity for weighted backward shift operators defined on $c_0(I)$ where $I \subseteq \mathbb{R}$.

Corollary 2.2. *Let I be a countably infinite subset of $\mathbb{R}$ such that $I + \mathbb{Z} \subseteq I$ (resp. $I + \mathbb{N} \subseteq I$), $I = \bigcup_{p=1}^{\infty} W_p$, where $(W_p)_p$ is an increasing sequence of finite subsets. Let $(w_i)_{i\in I}$ be a family of positive real numbers such that*

$$0 < \inf_{i \in I} w_i \le \sup_{i \in I} w_i < \infty. \tag{2.8}$$

The operator $T : c_0(I) \to c_0(I)$ defined by $T(x_i)_{i\in I} = (w_i x_{i+1})_{i \in I}$ is frequently hypercyclic on $c_0(I)$ if and only if there exist a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that

(a) *for any $p \ge 1$, $\underline{\text{dens}}(E_p) > 0$,*

(b) *for any* $p, q \geq 1$, $p \neq q$, $(E_p + W_p) \cap (E_q + W_q) = \emptyset$,

(c) *for all* $p \geq 1$ *and* $s \in W_p$, $\lim_{n\to\infty,\, n\in E_p} w_s \dots w_{s+n-1} = \infty$,

(d) *for any* $p, q \geq 1$, $n \in E_p$ *and* $m \in E_q$ *with* $m \neq n$, *and* $t \in W_q$ *(resp.* $t \in W_q$ *such that* $t + (m-n) \in I$*),*

$$\frac{w_{m-n+t} \dots w_{m+t-1}}{w_t \dots w_{t+m-1}} \leq \frac{1}{M(p)M(q)}.$$

Moreover, one can replace "there exist a sequence $(M(p))_{p\in\mathbb{N}}$ *and a sequence* $(E_p)_{p\in\mathbb{N}}$*" by "for any sequence* $(M(p))_{p\in\mathbb{N}}$ *there exists a sequence* $(E_p)_{p\in\mathbb{N}}$*".*

Proof. Observe that T is frequently hypercyclic if and only the family of its powers $(T^n)_{n\in\mathbb{N}}$ is frequently universal. If we set $T_n = T^n$, we have that

$$T_n[(x_i)_{i\in I}] = (w_i w_{i+1} \dots w_{i+n-1} x_{i+n})_{i\in I}.$$

Therefore, if we set

$$b_s^n := w_s w_{s+1} \dots w_{s+n-1}, \qquad \phi(s) := s+1, \qquad \phi_n := \underbrace{\phi \circ \dots \circ \phi}_{n}$$

and $g(s) := s$ for all $s \in I$ and $n \in \mathbb{N}$, then assumptions (i), (iii) and (iv) of Theorem 2.1 are trivially satisfied, while (ii) follows from (2.8). The characterization follows by Theorem 2.1. □

Remark 2.3. Observe that condition (d) is equivalent to saying that for any $p, q \geq 1$, $n \in E_p$ and $m \in E_q$ with $n \neq m$, and $t \in W_q$ (resp. $t \in W_q$ such that $t + (m-n) \in I$),

$$\begin{cases} w_t \dots w_{t+m-n-1} \geq M(p)M(q) & \text{if } m > n, \\ w_{t+(m-n)} \dots w_{t-2} w_{t-1} \leq \frac{1}{M(p)M(q)} & \text{if } m < n, \end{cases} \tag{2.9}$$

and we obtain the conditions of [12, Theorem 12].

3 Frequently hypercyclic translation semigroups

The purpose of this section is to obtain a characterization of frequent hypercyclicity for translation semigroups on $C_0^\rho(\mathbb{R})$ under the assumption $\sup_{k\in\mathbb{Z}} \rho(k+1)/\rho(k) < \infty$, and on $L_p^\rho(\mathbb{R})$.

In the following we set, for any $r, s \in \mathbb{Z}$, $[\![r, s]\!] = [r, s] \cap \mathbb{Z}$.

To treat the case of continuous functions, we recall some known results about the construction of a Schauder basis in $C_0(\mathbb{R})$, referring for more details to [29].

Let $\widetilde{D}$ be the set of dyadic numbers except 1, that is, $\widetilde{D} = \bigcup_{n=0}^{\infty} D_n$ where $D_0 = \{0\}$ and, if $n \geq 1$,

$$D_n = \left\{ \frac{2k-1}{2^n} : k = 1, \dots, 2^{n-1} \right\}.$$

For any $\tau \in D_n$, set $\tau^- = \tau - 2^{-n}$ and $\tau^+ = \tau + 2^{-n}$.

Let $\varphi(x) := \max\{0, 1 - |x|\}$ for $x \in \mathbb{R}$, and define for every $k \in \mathbb{Z}$, $\tau \in D_n$, and $x \in \mathbb{R}$,

$$\varphi_{k+\tau}(x) := \varphi(2^n(x - k - \tau)). \tag{3.1}$$

Observe that $\varphi_{k+\tau}(x) = \varphi_\tau(x - k)$ where φ_τ is the Faber–Schauder dyadic function with peak at τ.

Set $I = \mathbb{Z} + \widetilde{D}$ and consider the partition $I = \bigcup_{n \geq 0} V_n$ where $V_0 = \{0\}$, and

$$V_n = \{-n + h + D_h : h = 0, 1, \dots, n\} \cup \{h + D_{n-h} : h = 1, \dots, n\}. \tag{3.2}$$

We define an order on I assuming that the elements of V_k are earlier than those in V_n if $0 \leq k < n$, and within each V_n we keep the usual order.

The system $(\varphi_i)_{i \in I}$ is a Schauder basis in $C_0(\mathbb{R})$. More precisely, if $f \in C_0(\mathbb{R})$, then $f = \sum_{k+\tau \in \mathbb{Z}+\widetilde{D}} a_{k+\tau} \varphi_{k+\tau}$ where

$$a_{k+\tau} = \begin{cases} f(k), & k \in \mathbb{Z},\ \tau = 0, \\ f(k+\tau) - \frac{1}{2}(f(k+\tau^-) + f(k+\tau^+)), & k \in \mathbb{Z},\ \tau \neq 0. \end{cases}$$

By the construction of the functions $\varphi_{k+\tau}$, it follows that for any $n \in \mathbb{Z}_+$, for every family $(a_{k+\tau})_{k+\tau \in \mathbb{Z}+D_n}$ of real numbers, and for every $x \in \mathbb{R}$:

$$\sum_{k \in \mathbb{Z}, \tau \in D_n} |a_{k+\tau} \varphi_{k+\tau}(x)| \leq 2 \sup_{k \in \mathbb{Z}, \tau \in D_n} |a_{k+\tau}|. \tag{3.3}$$

If we set for every $n \in \mathbb{Z}_+$

$$\widetilde{D}_n := \bigcup_{h=0}^{n} D_h, \qquad W_n = [\![-n, n]\!] + \widetilde{D}_n,$$

then clearly $(W_n)_{n \geq 0}$ is an increasing sequence of finite subsets such that $\mathbb{Z} + \widetilde{D} = \bigcup_{n \geq 0} W_n$. Thus

$$\begin{aligned} c_0(\mathbb{Z} + \widetilde{D}) &= \{(a_{k+\tau})_{k,\tau} \in \mathbb{R}^{\mathbb{Z}+\widetilde{D}} \mid \forall \varepsilon > 0\ \exists n \in \mathbb{N} : k + \tau \notin W_n \Rightarrow |a_{k+\tau}| < \varepsilon\} \\ &= \{(a_{k+\tau})_{k,\tau} \in \mathbb{R}^{\mathbb{Z}+\widetilde{D}} \mid \forall \varepsilon > 0\ \exists n \in \mathbb{N} : (|k| > n \text{ or } \tau \notin \widetilde{D}_n) \Rightarrow |a_{k+\tau}| < \varepsilon\}. \end{aligned}$$

For any $x \in \mathbb{R}$, let $[x]$ denote the integer part of x.

Lemma 3.1. *Let ρ be an admissible weight function on $\mathbb{R}$ such that $\rho(x) = \rho([x])$ for any $x \in \mathbb{R}$ and let $T_1 : C_0^\rho(\mathbb{R}) \to C_0^\rho(\mathbb{R})$ be the translation operator defined as $T_1 f(x) = f(x+1)$. Then T_1 is quasiconjugate to the weighted backward shift operator $B_w : c_0(\mathbb{Z}+\widetilde{D}) \to c_0(\mathbb{Z}+\widetilde{D})$ defined by*

$$B_w[(x_{k+\tau})_{k+\tau\in\mathbb{Z}+\widetilde{D}}] = (w_{k+\tau}x_{k+\tau+1})_{k+\tau\in\mathbb{Z}+\widetilde{D}}, f$$

where

$$w_{k+\tau} := \frac{\rho(k+\tau)}{\rho(k+1+\tau)} = \frac{\rho(k)}{\rho(k+1)}, \qquad k+\tau \in \mathbb{Z}+\widetilde{D}.$$

Moreover, B_w is quasiconjugate to T_1.

Proof. To prove that T_1 is quasiconjugate to B_w, we exhibit a continuous linear operator $Q : C_0^\rho(\mathbb{R}) \to c_0(\mathbb{Z}+\widetilde{D})$ with dense range such that the following diagram commutes:

$$\begin{array}{ccc} C_0^\rho(\mathbb{R}) & \xrightarrow{T_1} & C_0^\rho(\mathbb{R}) \\ \downarrow_Q & & \downarrow_Q \\ c_0(\mathbb{Z}+\widetilde{D}) & \xrightarrow{B_w} & c_0(\mathbb{Z}+\widetilde{D}) \end{array}$$

Given $f \in C_0^\rho(\mathbb{R})$, we define $Q(f) = (a_{k+\tau})_{k+\tau\in\mathbb{Z}+\widetilde{D}}$ where

$$a_{k+\tau} = \rho(k) \cdot \begin{cases} f(k), & k \in \mathbb{Z},\ \tau = 0, \\ f(k+\tau) - \frac{1}{2}(f(k+\tau^-) + f(k+\tau^+)), & k \in \mathbb{Z},\ \tau \neq 0. \end{cases}$$

It holds that $Q(f) \in c_0(\mathbb{Z}+\widetilde{D})$. Indeed, since $f \in C_0^\rho(\mathbb{R})$, for every $\varepsilon > 0$,

(1) there exists $N_1 \in \mathbb{N}$ such that if $|x| > N_1$, then $|f(x)\rho(x)| < \frac{\varepsilon}{2}$

(2) there exists $\delta > 0$ such that, for all $x, y \in [-N_1-1, N_1+2]$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{\rho_{N_1}}$, where $\rho_{N_1} = \max_{k\in[\![-N_1-1,N_1+2]\!]} \rho(k)$.

Choose $N_2 \in \mathbb{N}$ such that $\frac{1}{2^{N_2}} < \delta$ and set $N = \max\{N_1+1, N_2\}$. Let $k+\tau \in (\mathbb{Z}+\widetilde{D}) \setminus W_N$.

If $|k| > N_1 + 1$, by (1) it is clear that for every $s \in [0,1[$,

$$|f(k+s)\rho(k)| = |f(k+s)\rho(k+s)| < \frac{\varepsilon}{2},$$

since $|k+s| > N_1$. Hence, by the continuity of f, for every $s \in [0,1]$,

$$|f(k+s)\rho(k)| \leq \frac{\varepsilon}{2}. \tag{3.4}$$

By replacing s by τ, τ^+ and τ^- in (3.4), we immediately get $|a_{k+\tau}| \leq \varepsilon$.

If $|k| \leq N_1+1$, then necessarily $\tau \in D_n$, with $n > N$, and, in particular, $\tau \neq 0$. It holds that

$$|\tau - \tau^-| < \frac{1}{2^N} < \delta, \; |\tau - \tau^+| < \frac{1}{2^N} < \delta,$$

and $k+\tau, k+\tau^-, k+\tau^+ \in [-N_1-1, N_1+2]$, thus, by (2) we get that

$$|a_{k+\tau}| = |\rho(k)(f(k+\tau) - \tfrac{1}{2}(f(k+\tau^-) + f(k+\tau^+)))| \leq \varepsilon.$$

Then $Q : C_0^\rho(\mathbb{R}) \to c_0(\mathbb{Z}+\widetilde{D})$ is well-defined and clearly linear. Moreover, for every $f \in C_0^\rho(\mathbb{R})$, $k \in \mathbb{Z}$, we have

$$|f(k+s)\rho(k)| = |f(k+s)\rho(k+s)| \leq \|f\|_\infty^\rho \quad \text{for all } s \in [0,1[,$$

hence, by the continuity of f,

$$|f(k+s)\rho(k)| \leq \|f\|_\infty^\rho \quad \text{for all } s \in [0,1]. \tag{3.5}$$

By replacing s by τ, τ^+ and τ^- in (3.5), with $\tau \in \widetilde{D}$, we immediately get $\|Q(f)\| \leq 2\|f\|_\infty^\rho$, so Q is continuous.

To prove that Q has dense range, it is enough to show that for every $h \in \mathbb{Z}$ and $\sigma \in \widetilde{D}$,

$$(x_{k+\tau}^{h+\sigma})_{k\in\mathbb{Z},\, \tau\in\widetilde{D}} \in Q(C_0^\rho(\mathbb{R})) \quad \text{where} \quad x_{k+\tau}^{h+\sigma} = \delta_{(h,\sigma)}(k,\tau),$$

and, indeed, by the definition of $\varphi_{h+\sigma}$, we have

$$Q(\rho(h)^{-1}\varphi_{h+\sigma}) = (x_{k+\tau}^{h+\sigma})_{k\in\mathbb{Z},\, \tau\in\widetilde{D}}.$$

Finally, by observing that, for every $f \in C_0^\rho(\mathbb{R})$, $Q \circ T_1(f) = (b_{k+\tau})_{k+\tau\in\mathbb{Z}+\widetilde{D}}$, where

$$b_{k+\tau} = \begin{cases} f(k+1)\rho(k) = a_{k+1}\dfrac{\rho(k)}{\rho(k+1)}, & \text{if } \tau = 0, \\ (f(k+1+\tau) - \frac{1}{2}(f(k+1+\tau^-) + f(k+1+\tau^+))\rho(k) & \\ \quad = a_{k+\tau+1}\dfrac{\rho(k)}{\rho(k+1)} & \text{if } \tau \neq 0, \end{cases}$$

for $k \in \mathbb{Z}$ and $\tau \in \widetilde{D}$, it is immediate that $Q \circ T_1(f) = B_w \circ Q(f)$.

Now, let us prove that B_w is quasiconjugate to T_1, namely that there exists a continuous linear operator $P : c_0(\mathbb{Z}+\widetilde{D}) \to C_0^\rho(\mathbb{R})$ with dense range such that the following diagram commutes:

$$\begin{array}{ccc} c_0(\mathbb{Z}+\widetilde{D}) & \xrightarrow{B_w} & c_0(\mathbb{Z}+\widetilde{D}) \\ \downarrow P & & \downarrow P \\ C_0^\rho(\mathbb{R}) & \xrightarrow{T_1} & C_0^\rho(\mathbb{R}) \end{array}$$

Given $(a_{k+\tau})_{k\in\mathbb{Z},\tau\in\widetilde{D}} \in c_0(\mathbb{Z}+\widetilde{D})$, we define

$$P\left((a_{k+\tau})_{k\in\mathbb{Z},\tau\in\widetilde{D}}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)} \varphi_{k+\tau} \right). \tag{3.6}$$

Observe that, for every $N \in \mathbb{N}$, $x \in [-N, N]$, taking (3.3) into account,

$$\begin{aligned} \left| \sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)} \varphi_{k+\tau}(x) \right| &\le \left| \sum_{|k|\le N,\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)} \varphi_{k+\tau}(x) \right| \le \\ \le \sum_{|k|\le N,\tau\in D_n} \frac{|a_{k+\tau}|}{\rho(k)} \varphi_{k+\tau}(x) &\le 2 \frac{\|(a_{k+\tau})_{k,\tau}\|}{\min_{|k|\le N}\{\rho(k)\}}, \end{aligned}$$

thus by the Weierstrass M-test, the series on the right-hand side of (3.6) is uniformly convergent on $[-N, N]$. It follows that $P\big((a_{k+\tau})_{k\in\mathbb{Z},\tau\in\widetilde{D}}\big)$ is a continuous function on $\mathbb{R}$.

Moreover, for every $x \in \mathbb{R}$ there exists $\overline{k} \in \mathbb{Z}$ such that $\overline{k} \le x < \overline{k}+1$. Hence, by (3.3) and by (1.1) applied with $l = 1$

$$\begin{aligned} |P\left((a_{k+\tau})_{k,\tau}\right)(x)\rho(x)| &= \left| \rho(x) \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)} \varphi_{k+\tau}(x) \right) \right| = \\ = \left| \rho(\overline{k}) \left(\sum_{k\in\mathbb{Z}} \frac{a_k}{\rho(k)} \varphi_k(x) \right) \right. &+ \left. \rho(\overline{k}) \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)} \varphi_{k+\tau}(x) \right) \right| = \\ = \left| a_{\overline{k}} \varphi_{\overline{k}}(x) + \frac{\rho(\overline{k})}{\rho(\overline{k}+1)} a_{\overline{k}+1} \varphi_{\overline{k}+1}(x) \right. &+ \left. \rho(\overline{k}) \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\sum_{\tau\in D_n} \frac{a_{\overline{k}+\tau}}{\rho(\overline{k})} \varphi_{\overline{k}+\tau}(x) \right) \right| \le \\ \le \frac{\rho(\overline{k})}{\rho(\overline{k}+1)} \left| a_{\overline{k}+1} \right| &+ \left| \sum_{n=0}^{\infty} \frac{1}{2^n} \left(\sum_{\tau\in D_n} a_{\overline{k}+\tau} \varphi_{\overline{k}+\tau}(x) \right) \right| \le \\ &\le B \left| a_{\overline{k}+1} \right| + 2 \sum_{n=0}^{\infty} \frac{1}{2^n} \sup_{\tau\in D_n} |a_{\overline{k}+\tau}|. \end{aligned} \tag{3.7}$$

Thus, if $\varepsilon > 0$ and $N \in \mathbb{N}$ is such that $|a_{k+\tau}| < \varepsilon$ if $k+\tau \notin W_N$, then for any $x \in \mathbb{R}$ such that $|x| > N+1$, it holds that $|\overline{k}| > N$ and $|\overline{k}+1| > N$, thus $\overline{k}+\tau \notin W_N$ for every $\tau \in \widetilde{D}$ and $\overline{k}+1 \notin W_N$. Therefore $|a_{\overline{k}+\tau}| < \varepsilon$ and $|a_{\overline{k}+1}| < \varepsilon$. Hence

$$|P\left((a_{k+\tau})_{k,\tau}\right)(x)\rho(x)| < (B+4)\varepsilon.$$

On the other hand, (3.7) implies that

$$\|P\left((a_{k+\tau})_{k,\tau}\right)\|_\infty^\rho \le (B+4)\|(a_{k+\tau})_{k,\tau}\|,$$

hence $P : c_0(\mathbb{Z}+\widetilde{D}) \to C_0^\rho(\mathbb{R})$ is well defined, clearly linear and continuous.

Observe that clearly

$$\{\varphi_{k+\tau}, k \in \mathbb{Z}, \tau \in \widetilde{D}\} \subseteq P(c_0(\mathbb{Z}+\widetilde{D})).$$

If $f \in C_0^\rho(\mathbb{R})$, given $\varepsilon > 0$, there exists $g \in C(\mathbb{R})$ with compact support such that $\|f-g\|_\infty^\rho < \frac{\varepsilon}{2}$. Let us assume that $\operatorname{supp}(g) \subseteq [-M, M]$, $M \in \mathbb{N}$ and consider $\varphi \in \operatorname{span}\{\varphi_{k+\tau}, k \in \mathbb{Z}, \tau \in \widetilde{D}\} \subseteq P(c_0(\mathbb{Z}+\widetilde{D}))$, $\operatorname{supp}(\varphi) \subseteq [-M-1, M+1]$, such that $\|g-\varphi\|_\infty < \frac{\varepsilon}{2\max_{x\in[-M-1,M+1]}\rho(x)}$. Then

$$\|f-\varphi\|_\infty^\rho < \|f-g\|_\infty^\rho + \|g-\varphi\|_\infty^\rho < \frac{\varepsilon}{2} + \sup_{x\in[-M-1,M+1]} |g(x)-\varphi(x)|\rho(x) < \varepsilon$$

and therefore P has dense range.

Finally we observe that

$$P \circ B_w((a_{k+\tau})_{k,\tau}) = \sum_{n=0}^\infty \frac{1}{2^n}\left(\sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau+1}}{\rho(k+1)}\varphi_{k+\tau}\right).$$

On the other hand, for every $x \in \mathbb{R}$,

$$T_1 \circ P((a_{k+\tau})_{k,\tau})(x) = \sum_{n=0}^\infty \frac{1}{2^n}\left(\sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)}\varphi_{k+\tau}(x+1)\right) =$$

$$\sum_{n=0}^\infty \frac{1}{2^n}\left(\sum_{k\in\mathbb{Z},\tau\in D_n} \frac{a_{k+\tau}}{\rho(k)}\varphi_{k+\tau-1}(x)\right) = \sum_{n=0}^\infty \frac{1}{2^n}\left(\sum_{h\in\mathbb{Z},\tau\in D_n} \frac{a_{h+\tau+1}}{\rho(h+1)}\varphi_{h+\tau}(x)\right).$$

Then $P \circ B_w = T_1 \circ P$.

□

Theorem 3.2. *Let $\mathcal{T}$ be the translation semigroup on $C_0^\rho(\mathbb{R})$, where ρ is an admissible weight function and $\sup_{k\in\mathbb{Z}} \rho(k+1)/\rho(k) < \infty$. Then $\mathcal{T}$ is frequently hypercyclic on $C_0^\rho(\mathbb{R})$ if and only if there exist a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that:*

(a) *for any $p \geq 1$, $\underline{\operatorname{dens}}(E_p) > 0$,*
(b) *for any distinct $p, q \geq 1$, $(E_p + [\![-p,p]\!]) \cap (E_q + [\![-q,q]\!]) = \emptyset$,*
(c) *for any $p \geq 1$, $\lim_{n\to\infty,\, n\in E_p} \rho(n) = 0$,*
(d) *for any $p, q \geq 1$ and any $n \in E_p$ and $m \in E_q$ with $m \neq n$,*

$$\rho(m-n) \leq \frac{1}{M(p)M(q)}. \tag{3.8}$$

Moreover, one can replace "there exist a sequence $(M(p))_{p\in\mathbb{N}}$ and a sequence $(E_p)_{p\in\mathbb{N}}$" by "for any sequence $(M(p))_{p\in\mathbb{N}}$ there exists a sequence $(E_p)_{p\in\mathbb{N}}$".

Proof. Since ρ is an admissible weight function $\sup_{k\in\mathbb{Z}} \rho(k)/\rho(k+1) < \infty$, there exists $M > 1$ such that for every $k \in \mathbb{Z}$,

$$M^{-1} \leq \frac{\rho(k+1)}{\rho(k)} \leq M.$$

As a consequence we get for every $n \in \mathbb{N}$ we get, if $k \in [\![0, p]\!]$,

$$\rho(k+n) = \frac{\rho(k+n)}{\rho(k-1+n)} \cdots \frac{\rho(n+1)}{\rho(n)} \cdot \rho(n) \leq M^k \rho(n)$$

and if $k \in [\![-p, 0]\!]$,

$$\rho(k+n) = \frac{\rho(k+n)}{\rho(k+1+n)} \cdots \frac{\rho(n-1)}{\rho(n)} \cdot \rho(n) \leq M^{|k|} \rho(n),$$

thus for every $k \in [\![-p, p]\!]$,

$$\rho(k+n) \leq M^{|k|} \rho(n). \tag{3.9}$$

On the other hand, by (1.1), there exist constants $0 < A < B$ such that for every $x \in [k, k+1]$,

$$A\rho(k) \leq \rho(x) \leq B\rho(k+1) \leq BM\rho(k).$$

If we now define $\tilde{\rho}(x) = \rho([x])$ for $x \in \mathbb{R}$, then there exist constants $M_1, M_2 > 0$ such that

$$M_1 \|f\|_\infty^{\tilde{\rho}} \leq \|f\|_\infty^{\rho} \leq M_2 \|f\|_\infty^{\tilde{\rho}},$$

so $\|\cdot\|_\infty^{\tilde{\rho}}$ is an equivalent norm to $\|\cdot\|_\infty^{\rho}$. Therefore, without loss of generality we can assume in the following that $\rho(x) = \rho([x])$ for every $x \in \mathbb{R}$.

By the results in [16, 26], it is known that $\mathcal{T}$ is a frequently hypercyclic semigroup if and only if T_1 is a frequently hypercyclic operator. Since T_1 is quasiconjugate to the operator B_w defined as in Lemma 3.1 and B_w is quasiconjugate to T_1, by [24, Proposition 9.4] we find that $\mathcal{T}$ is frequently hypercyclic if and only if B_w is frequently hypercyclic, hence, by Corollary 2.2, if and only if there exist a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{N}$ (or equivalently, for any $(M(p))_{p\in\mathbb{N}}$ there exists $(E_p)_{p\in\mathbb{N}}$) such that

(a1) for any $p \geq 1$, $\underline{\text{dens}}(E_p) > 0$,
(b1) for any distinct $p, q \geq 1$, $(E_p + W_p) \cap (E_q + W_q) = \emptyset$,
(c1) for every $p \geq 1$ and every $s \in W_p$,

$$\lim_{n\to\infty,\, n\in E_p} w_s \dots w_{s+n-1} = \lim_{n\to\infty,\, n\in E_p} \frac{\rho(s)}{\rho(s+n)} = \infty,$$

(d1) for any $p, q \geq 1$, $n \in E_p$ and $m \in E_q$ with $m \neq n$, and $t \in W_q$,

$$\frac{w_{m-n+t} \dots w_{m+t-1}}{w_t \dots w_{t+m-1}} = \frac{\rho(m-n+t)}{\rho(t)} \leq \frac{1}{M(p)M(q)},$$

where $W_p = [\![-p, p]\!] + \widetilde{D}_p$.

Clearly (a1)–(d1) imply (a)–(d) simply by observing that $[\![-p, p]\!] \subseteq W_p$, and in particular $0 \in W_p$, for every $p \in \mathbb{N}$.

Conversely, assume that (a)–(d) hold. Passing to a subsequence of $(E_p)_{p\in\mathbb{N}}$ if necessary, we can choose $(M(p))_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \frac{M(p) \cdot \min\{\rho(k) : k \in [\![-p, p]\!]\}}{M^p} = \infty.$$

If

$$(E_p + W_p) \cap (E_q + W_q) \neq \emptyset,$$

then there exist $n \in E_p$, $s \in [\![-p, p]\!]$, $\sigma \in \widetilde{D}$, $m \in E_q$, $t \in [\![-q, q]\!]$ and $\tau \in \widetilde{D}$, such that

$$n + s + \sigma = m + t + \tau;$$

taking integer parts yields $n+s = m+t$, hence $(E_p+[\![-p, p]\!]) \cap (E_q+[\![-q, q]\!]) \neq \emptyset$. Thus $p = q$. So (b1) is satisfied.

By (3.9) and (c), we get $\lim_{n\to\infty, n\in E_p} \rho(n+k) = 0$ for every $k \in [\![-p, p]\!]$, and (c1) follows by observing that for every $s \in W_p$ there exists $k \in [\![-p, p]\!]$ such that $\rho(n+s) = \rho(n+k)$.

Let $p, q \geq 1$ and $n \in E_p$, $m \in E_q$, $m \neq n$. By (3.9) and (d), for every $k \in [\![-q, q]\!]$,

$$\begin{aligned}\frac{\rho(m-n+k)}{\rho(k)} &\leq \frac{\rho(m-n)}{\rho(k)} M^{|k|} \leq \frac{M^q}{\min\{\rho(k) : k \in [\![-q, q]\!]\} M(p)M(q)} \\ &\leq \frac{1}{K(p)K(q)}\end{aligned}$$

where

$$K(q) = \min\left\{ \frac{M(q) \cdot \min\{\rho(k) : k \in [\![-q, q]\!]\}}{M^q}, M(q) \right\}.$$

We get (d1) by observing that for every $t \in W_q$ there exists $k \in [\![-q, q]\!]$ such that

$$\frac{\rho(m-n+t)}{\rho(t)} = \frac{\rho(m-n+k)}{\rho(k)}. \square$$

Remark 3.3. As an immediate consequence, if ρ is an admissible weight function on $\mathbb{R}$ such that $\sup_{k\in\mathbb{Z}} \rho(k+1)/\rho(k) < \infty$ and we set $w_k = \rho(k)/\rho(k+1)$ for $k \in \mathbb{Z}$, then, by the characterization given in [12, Theorem 9] and by Theorem 3.2, B_w is frequently hypercyclic on $c_0(\mathbb{Z})$ if and only if the translation semigroup is frequently hypercyclic on $C_0^\rho(\mathbb{R})$.

Proposition 3.4. *If the translation semigroup $\mathcal{T}$ is mixing* (*equivalently chaotic*) *on $C_0^\rho(\mathbb{R})$, then it is frequently hypercyclic.*

Proof. As is proved in [13, 27], chaos and mixing are equivalent properties for the translation C_0-semigroup on $C_0^\rho(\mathbb{R})$, and this happens if and only if $\lim_{x\to\pm\infty}\rho(x)=0$.

As already observed, it is enough to prove that T_1 satisfies the Frequent Hypercyclicity Criterion for operators (see [14]). Let $X_0=\operatorname{span}\{\varphi_{k+\tau}:k\in\mathbb{Z},\ \tau\in\widetilde{D}\}$, where $\varphi_{k+\tau}$ is defined in (3.1). Every continuous function on $\mathbb{R}$ with dense support can be approximated in the uniform norm with elements of X_0, and therefore X_0 is dense in $C_0^\rho(\mathbb{R})$. Moreover, if we define $S:X_0\to X_0$ by $Sf(x)=f(x-1)$, it is clear that $T_1Sf=f$.

Let us prove that $\sum_{n=1}^\infty T_1^n f$ and $\sum_{n=1}^\infty S^n f$ are unconditionally convergent for all $f\in X_0$. It is enough to consider $f=\varphi_{k+\tau}\in X_0$; then $\operatorname{supp}(f)\subseteq[a,b]$ with $b-a\le 2$, and for every $n\in\mathbb{N}$,

$$\operatorname{supp}(T_1^n(f))\subseteq[a-n,b-n];$$

thus for every $n,m\in\mathbb{N}$, if $\operatorname{supp}(T_1^n f)\cap\operatorname{supp}(T_1^m f)\neq\emptyset$, then it is immediate that $|n-m|\le|b-a|\le 2$, hence either $m=n$, or $m=n\pm1$, or $m=n\pm2$. This implies that if $J\subseteq\mathbb{N}$ is finite, then

$$\Big\|\sum_{n\in J}T_1^n f\Big\|_\infty^\rho\le 4\sup_{n\in J}\|T_1^n f\|_\infty^\rho.$$

Let $\varepsilon>0$ and let $M>0$ be such that $\rho(x)<\varepsilon$ for every $|x|>M$. For every finite set $F\subseteq\mathbb{N}\cap\,]M+b,\infty[$ and every $x\in[a,b]$ and $n\in F$, we have $|x-n|=n-x>M$, so

$$\Big\|\sum_{n\in F}T_1^n f\Big\|_\infty^\rho\le 4\sup_{x\in\mathbb{R},\,n\in F}|f(x+n)\rho(x)|=2\sup_{x\in[a,b],\,n\in F}|f(x)\rho(x-n)|\le 2\varepsilon.$$

The argument for $\sum_{n=1}^\infty S^n f$ is similar. □

Remark 3.5. The converse of the previous proposition does not hold. Indeed, let $(w_k)_{k\in\mathbb{Z}}$ be one of the sequences of weights constructed in [12] such that B_w is frequently hypercyclic on $c_0(\mathbb{Z})$ and $w_1\cdots w_k=1$ for infinitely many k. Define $\rho(k)=(w_1\cdots w_k)^{-1}$ if $k\ge1$, $\rho(k)=w_kw_{k+1}\cdots w_0$ if $k\le0$, and $\rho(x)=\rho([x])$ for any $x\in\mathbb{R}$. Then $\sup_{k\in\mathbb{Z}}\rho(k+1)/\rho(k)<\infty$, due to the fact that $1/2\le w_k\le 2$ for all $k\in\mathbb{Z}$, as shown in [12]. By Remark 3.3 the translation semigroup is frequently hypercyclic on $C_0^\rho(\mathbb{R})$, while clearly it is not mixing, since $\rho(k)=1$ for infinitely many k.

Finally, set $J = \mathbb{Z}_+ + \widetilde{D}$ and consider the partition $J = \bigcup_{n\geq 0} J_n$ where $J_0 = \{0\}$ and

$$J_n = \{h + D_{n-h} : h = 0, 1, \ldots, n\} \quad \text{for } n \geq 1. \tag{3.10}$$

Define an order on J assuming that the elements J_k are earlier than those in J_n if $0 \leq k < n$, and within each J_n we keep the usual order.

The system $(\psi_i)_{i\in J}$, where $\psi_0(x) := \max\{0, 1 - x\}$ and $\psi_{k+\tau} = \varphi_{k+\tau}$ if $\tau \neq 0$ or $k \neq 0$, is a Schauder basis on $C_0([0, \infty[)$.

Reasoning analogously to the case of translation semigroups in $C_0^\rho(\mathbb{R})$, we also get the following characterization of frequently hypercyclic translation semigroups on $C_0^\rho([0, \infty[)$:

Theorem 3.6. *Let $\mathcal{T}$ be the translation semigroup on $C_0^\rho([0, \infty[)$, where ρ is an admissible weight function and $\sup_{k\in\mathbb{N}} \rho(k + 1)/\rho(k) < \infty$. Then $\mathcal{T}$ is frequently hypercyclic on $C_0^\rho([0, \infty[)$ if and only if there exist a sequence $(M(p))_{p\in\mathbb{N}}$ of positive real numbers tending to ∞ and a sequence $(E_p)_{p\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that:*

(a) *for any $p \geq 1$, $\underline{\mathrm{dens}}(E_p) > 0$,*
(b) *for any distinct $p, q \geq 1$, $(E_p + [\![0, p]\!]) \cap (E_q + [\![0, q]\!]) = \emptyset$,*
(c) *for every $p \geq 1$, $\lim_{n\to\infty,\, n\in E_p} \rho(n) = 0$,*
(d) *for any $p, q \geq 1$ and any $n \in E_p$ and $m \in E_q$ with $m > n$,*

$$\rho(m - n) \leq \frac{1}{M(p)M(q)}, \tag{3.11}$$

Moreover, one can replace "there exist a sequence $(M(p))_{p\in\mathbb{N}}$ and a sequence $(E_p)_{p\in\mathbb{N}}$" by "for any sequence $(M(p))_{p\in\mathbb{N}}$ there exists a sequence $(E_p)_{p\in\mathbb{N}}$".

The final part of the paper will be devoted to characterizing frequently hypercyclic semigroups on $L_p^\rho(\mathbb{R})$. Also in this case we will first establish a relation between the discrete and the continuous cases. We recall that the relation between the discrete and the continuous cases for Devaney chaos was studied in [8] and for distributional chaos in [7]. The following lemma follows immediately from the conjugacy of the backward shift B on $\ell_p^v = \{(x_k)_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |x_k|^p v_k < \infty\}$ and the weighted backward shift B_w on ℓ_p where $w_k = (v_k/v_{k+1})^{1/p}$ for $k \in \mathbb{Z}$, and from the characterization of frequently hypercyclic weighted backward shifts on ℓ_p proved in [12, Theorem 3].

Lemma 3.7. *Let $v = (v_k)_{k\in\mathbb{Z}}$ be a sequence of strictly positive weights such that $(v_k/v_{k+1})_k$ is bounded. Then the backward shift operator B is frequently hypercyclic on ℓ_p^v if and only if $\sum_{k\in\mathbb{Z}} v_k < \infty$.*

Theorem 3.8. *Let ρ be an admissible weight function on $\mathbb{R}$. If the translation semigroup $\mathcal{T}$ is frequently hypercyclic on $L^\rho_p(\mathbb{R})$, then the backward shift operator B is frequently hypercyclic on ℓ^v_p, where $v_k = \rho(k)$ for all $k \in \mathbb{Z}$.*

Proof. Since ρ is an admissible weight function, by (1.1) there exist $A, B \geq 0$ such that $A\rho(k) \leq \rho(t) \leq B\rho(k+1)$ for all $t \in [k, k+1]$. If $(T_t)_{t\geq 0}$ is frequently hypercyclic, then T_1 is frequently hypercyclic [16]. Hence there exists $f \in L^\rho_p(\mathbb{R})$ such that for all $g \in L^\rho_p(\mathbb{R})$ and for all $\varepsilon > 0$,

$$\underline{\text{dens}}(\{n \in \mathbb{N} : \|T_1^n f - g\| < \varepsilon\}) > 0.$$

Since $f \in L^\rho_p(\mathbb{R})$ we have $|f|\rho^{1/p} \in L_p([k, k+1]) \subseteq L_1([k, k+1])$ for every $k \in \mathbb{Z}$. As ρ is strictly positive and locally bounded below by (1.1), we infer that $f \in L_1([k, k+1])$ for all $k \in \mathbb{Z}$. Therefore we can define $x_k = \int_k^{k+1} f(t)\, dt$ for all $k \in \mathbb{Z}$. We have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |x_k|^p \rho(k) &= \sum_{k\in\mathbb{Z}} \Big| \int_k^{k+1} f(t)\, dt\Big|^p \rho(k) \leq \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(t)|^p \rho(k)\, dt \\ &\leq \frac{1}{A} \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(t)|^p \rho(t)\, dt = \frac{1}{A}\|f\|_p^p < \infty.\end{aligned}$$

So $x = (x_k)_{k\in\mathbb{Z}} \in \ell^p_v$ with $v_k = \rho(k)$.

Let

$$y = (0, \ldots, y_{-N}, \ldots, y_0, \ldots, y_M, 0, \ldots, 0)$$

and let $\varepsilon > 0$. Set $g = \sum_{k=-N}^{M} y_k \chi_{[k,k+1]} \in L^\rho_p(\mathbb{R})$. We will show that

$$\{n \in \mathbb{N} : \|T_1^n f - g\| < A^{1/p}\varepsilon\} \subseteq \{n \in \mathbb{N} : \|B^n x - y\| < \varepsilon\},$$

and therefore

$$\underline{\text{dens}}(\{n \in \mathbb{N} : \|B^n x - y\| < \varepsilon\}) > 0$$

because f is a frequently hypercyclic vector. We have

$$\begin{aligned}\|B^n x - y\|^p &= \sum_{k\in\mathbb{Z}} |x_{n+k} - y_k|^p \rho(k) \\ &\leq \frac{1}{A} \sum_{k\in\mathbb{Z}} \int_k^{k+1} |f(t+n) - g(t)|^p \rho(t)\, dt \leq \varepsilon^p.\end{aligned}$$

By the density of finite sequences in ℓ^v_p we conclude that B is frequently hypercyclic. □

Finally, we are able to characterize frequently hypercyclic translation semigroups on $L^\rho_p(\mathbb{R})$.

Theorem 3.9. *Let ρ be an admissible weight function on $\mathbb{R}$. The following assertions are equivalent:*

(1) *The translation semigroup $\mathcal{T}$ is frequently hypercyclic on $L_p^\rho(\mathbb{R})$.*

(2) $\sum_{k\in\mathbb{Z}} \rho(k) < \infty$.

(3) $\int_{-\infty}^{\infty} \rho(t)\,dt < \infty$.

(4) *$\mathcal{T}$ is chaotic on $L_p^\rho(\mathbb{R})$.*

(5) *$\mathcal{T}$ satisfies the Frequent Hypercyclicity Criterion.*

Proof. Observe that $(\rho(k)/\rho(k+1))_{k\in\mathbb{Z}}$ is bounded by the admissibility of ρ. By Theorem 3.8 and Lemma 3.7, we have (1)⇒(2). The equivalence of (2) and (3) follows from the properties of ρ, by comparing integrals and series. The equivalence of (4) and (5) can be proved with the same argument as in [26, Proposition 3.3]. (5)⇒(1) is proved in [26, Theorem 2.2], while (3)⇒(5) can be proved as in [26, Proposition 3.4]. □

With minor changes we also get a characterization of frequently hypercyclic translation semigroups on $L_p^\rho([0,\infty[)$:

Theorem 3.10. *Let ρ be an admissible weight function on $[0,\infty[$. The following assertions are equivalent:*

(1) *The translation semigroup $\mathcal{T}$ is frequently hypercyclic on $L_p^\rho([0,\infty[)$.*

(2) $\sum_{k\in\mathbb{N}} \rho(k) < \infty$.

(3) $\int_0^{\infty} \rho(t)\,dt < \infty$.

(4) *$\mathcal{T}$ is chaotic on $L_p^\rho([0,\infty[)$.*

(5) *$\mathcal{T}$ satisfies the Frequent Hypercyclicity Criterion.*

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