Homotheties and topology of tangent sphere bundles

R. Albuquerque*
rpa@uevora.pt

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Abstract

We prove a Theorem on homotheties between two given tangent sphere bundles S_rM of a Riemannian manifold M,g of dim ≥ 3, assuming different variable radius functions r and weighted Sasaki metrics induced by the conformal class of g. New examples are shown of manifolds with constant positive or with constant negative scalar curvature which are not Einstein. Recalling results on the associated almost complex structure I^G and symplectic structure ω^G on the manifold TM, generalizing the well-known structure of Sasaki by admitting weights and connections with torsion, we compute the Chern and the Stiefel-Whitney characteristic classes of the *manifolds* TM and S_rM.

Key Words: tangent sphere bundle, isometry, characteristic classes.

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1 Introduction

This article consists of a study of the main properties which identify the tangent sphere bundles $S_rM = \{u \in TM : \|u\| = r\}$ of a Riemannian manifold (M, g) with variable radius r and induced weighted Sasaki metric $g^{f_1,f_2} = f_1\pi^*g \oplus f_2\pi^*g$, where f_1, f_2 are $\mathbb{R}^+$-valued functions on M and $\pi : TM \to M$ is the bundle map. Recall the well-known Sasaki metric on TM is just $g^S = g^{1,1}$ induced by the Levi-Civita connection splitting of TTM. Our main results are as follows.

We consider a conformal change λg by some function λ on M, then take both Levi-Civita connections of g and λg and consider, accordingly, the lifts of these metrics to TM. We obtain very different weighted Sasaki metrics on TM and induced metrics on the sphere bundles, since the

*Departamento de Matemática da Universidade de Évora and Centro de Investigação em Matemática e Aplicações (CIMA), Rua Romão Ramalho, 59, 671-7000 Évora, Portugal.

horizontal subspaces are very different when λ is non-constant. So one wishes to compare the S_rM, with radius functions $r,s:M\to\mathbb{R}^+$ and within the same conformal class of M, through the map $u\overset{h}{\mapsto}\frac{s}{r\sqrt{\lambda}}u$. For M connected and of dimension ≥ 3 we prove:

$$(S_rM,g^{f_1,f_2})\quad\text{is homothetic via } h \text{ to}\quad (S_sM,(\lambda g)^{f'_1,f'_2}) \tag{1}$$

if and only if $\frac{f'_1}{f_1}\lambda=\frac{s^2}{r^2}\frac{f'_2}{f_2}$, the function λ is constant and one of the following conditions holds: (i) s/r is constant or (ii) rs is constant.

Equation (ii) is quite interesting, and reassuring if the reader suspects it is true. As a corollary it says that, for any positive function r on M, (S_rM,g^S) is isometric to $(S_{\frac{1}{r}}M,g^{1,r^4})$.

We give some applications in the treatment of the S_sM_R of the space-form M_R, the locus of $x_1^2+\cdots+x_m^2\pm x_{m+1}^2=R^2$, which has constant sectional curvature $\pm 1/R^2$. Using [7], we prove in Theorem 2.3 that, for $\dim M_R=m\geq 3$, no matter the sign $\pm$ or the constants $R,f_1,f_2>0$, we can always find a radius $s>0$ suficiently small such that S_sM_R has constant positive scalar curvature or suficiently large such that the same space has constant negative scalar curvature. These are examples of manifolds with constant Scal but which are not Einstein.

Proceeding with the weighted metric $G=g^{f_1,f_2}$ on TM, we define a compatible almost Hermitian structure (G,I^G,ω^G), which is a generalization of the canonical or Sasaki almost Hermitian structure on TM. In our case we also allow ∇ to have torsion. Then the integrability equations of I^G and ω^G reserve distinguished roles for the functions f_1/f_2 and f_1f_2 respectively, both implying the torsion to be of certain so-called vectorial type. In principle having no relation, notice the similarity of these equations with the two cases (i) and (ii) above! Finally, the two functions only have to be both constant, the curvature of ∇ flat and the torsion zero if and only if we require the defined structure on TM to be Kähler.

We also determine the characteristic classes of the *manifold* TM. The Chern classes of (TM,I^G) are proved to agree with the Pontryagin classes of M. Moreover, they do not depend on the metric connection ∇. The Stiefel-Whitney characteristic classes of S_rM are also found. In particular we conclude that any tangent sphere bundle of an oriented manifold is a spin manifold.

The motivation for the present article is the discovery of a natural G_2-structure on S_1M, for any M oriented of dimension 4, which is having many developments and good expectations, cf. [4, 5]. However, here we just complete an independent study of the S_rM initiated in [6, 7].

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2 Riemannian geometry of the tangent bundle

2.1 The tangent bundle

Let M be an m-dimensional smooth manifold without boundary. Let $\pi:TM\to M$ be the tangent bundle so that $\pi(u)=x,\ \forall u\in T_xM,\ x\in M$. Then $V=\ker\mathrm{d}\pi$ is known as the vertical bundle tangent to TM. There is a canonical identification $V=\pi^*TM$ and an exact sequence over the manifold TM:

$$0\longrightarrow V\longrightarrow TTM\overset{\mathrm{d}\pi}{\longrightarrow}\pi^*TM\longrightarrow 0. \tag{2}$$

The tangent bundle TM is endowed with a natural vertical vector field, denoted ξ, which is succinctly defined by $\xi_u = u$.

Let ∇ be a connection on M. Then there is a complement for V

$$H = \{X \in TTM : \ \pi^*\nabla_X\xi = 0\}. \tag{3}$$

Indeed H is m-dimensional and $\pi^*\nabla_{\cdot}\xi$ is the vertical projection onto V. For any vector field X over TM we may always find the unique decomposition (∇^* denotes the pull-back connection)

$$X = X^h + X^v = X^h + \nabla^*_X\xi. \tag{4}$$

Now, $\mathrm{d}\pi$ induces a vector bundle isomorphism between H and π^*TM, by (2), and we have $V = \pi^*TM$. Hence we may define an endomorphism

$$B : TTM \longrightarrow TTM \tag{5}$$

sending X^h to the respective $BX^h \in V$ and sending V to 0. We also define an endomorphism, *denoted* B^{ad}, which gives $B^{\mathrm{ad}}X^v \in H$ and which annihilates H. In particular $B^{\mathrm{ad}}BX^h = X^h$ and $B^2 = 0$. Sometimes we call BX^h the mirror image of X^h in V. The map B appears also in [4]. We endow TTM with the direct sum connection $\nabla^* \oplus \nabla^*$, which we sometimes denote by ∇^*. We have in particular that $\nabla^*B = \nabla^*B^{\mathrm{ad}} = 0$.

Notice the canonical section ξ can be mirrored by B^{ad} to give a horizontal canonical vector field $B^{\mathrm{ad}}\xi$. In the torsion free case, the latter is known as the spray of the connection, cf. [9, 13], or the geodesic field, cf. [10]. It has the further property that $\mathrm{d}\pi_u(B^{\mathrm{ad}}\xi) = u, \ \forall u \in TM$. Away from the zero section, we have a line bundle $\mathbb{R}\xi \subset V$ and therefore a line sub-bundle too of H.

2.2 Natural metrics

Suppose the previous manifold M is furnished with a Riemannian metric g and a linear connection. We also use $\langle\ ,\ \rangle$ in place of the symmetric tensor g; this same remark on notation is valid for the pull-back metric on π^*TM. We recall from [9, 14] the now called Sasaki metric in $TTM = H \oplus V$: it is given by $g^S = \pi^*g \oplus \pi^*g$ (originally, with the Levi-Civita connection). With g^S, the map $B_| : H \to V$ is an isometric morphism and B^{ad} corresponds with the adjoint of B. We stress that $\langle\ ,\ \rangle$ on TTM always refers to the Sasaki metric.

Let φ_1, φ_2 be any given functions on M and let

$$G = g^{f_1,f_2} = f_1\pi^*g \,\oplus\, f_2\pi^*g \tag{6}$$

with

$$f_1 = \mathrm{e}^{2\varphi_1}, \quad f_2 = \mathrm{e}^{2\varphi_2}. \tag{7}$$

Obviously, we convention all these functions to be composed with π on the right hand side when used on the manifold TM.

Remark. With the canonical vector field ξ we may produce other symmetric bilinear forms over TM: first the 1-forms $\eta = \xi^\flat$ and $\theta = \xi^\flat \circ B = (B^{\mathrm{ad}}\xi)^\flat$ and then the three symmetric products of these. Actually one may see that θ does not depend on a chosen connection which is metric; cf. last remark in section 3.1. The classification of all g-induced natural metrics on TM may be found e.g. in [1, 2].

2.3 Metric connections

Let us assume from now on that the connection on M is metric, which implies $\nabla^* g^S = 0$. It is well-known that $\nabla^{f_1} = \nabla + C_1$, with

$$C_1(X,Y) = X(\varphi_1)Y + Y(\varphi_1)X - \langle X,Y\rangle \mathrm{grad}\,\varphi_1, \tag{8}$$

is a metric connection for $f_1 g$ on M, with the same torsion as ∇ since C is symmetric.

For any function φ, recall the usual identities $X(\varphi) = \mathrm{d}\varphi(X) = \langle \mathrm{grad}\,\varphi, X\rangle$, adopted throughout. On TM we shall use the functions $\partial\varphi(u) = \mathrm{d}\varphi_{\pi(u)}(u),\ \forall u$. In other words,

$$\partial\varphi = \langle B\pi^*\mathrm{grad}\,\varphi, \xi\rangle \tag{9}$$

where B is the mirror map (5). And we agree on lifting gradient vector fields only to H.

We have that $\nabla^{*,f_1} = \nabla^* + \pi^* C_1$ makes $f_1\pi^* g$ parallel on H. On the vertical side, ∇^{*,f_2}, defined by

$$\nabla^{*,f_2}_X Y = \nabla^*_X Y + B\pi^* C_2(X, B^{\mathrm{ad}} Y) \tag{10}$$

$\forall X, Y$ vector fields on TM, makes $f_2\pi^* g$ parallel. Henceforth, the connection $\nabla^{*,f_1} \oplus \nabla^{*,f_2}$ is metric for $G = g^{f_1,f_2}$.

Proposition 2.1. *i) The torsion of* $\nabla^* \oplus \nabla^*$ *is* $\pi^* T^\nabla + \mathcal{R}^\xi$.
ii) The connection $\nabla^{*,f_2,'}_X Y = \nabla^*_X Y + X(\varphi_2) Y$ *is metric on* $(V, f_2\pi^* g)$.

The proof of this result is immediate. The vertical part in i) is defined via the curvature, $\mathcal{R}^\xi(X,Y) = \pi^* R^\nabla(X,Y)\xi$. We remark it is ∇^{*,f_1} and the connection in ii) which enter in the Levi-Civita connection ∇^G of G. Formulas for the curvature are well-known, cf. [2, 7, 9, 11].

2.4 Homotheties of TM

Suppose we have a conformal change of the metric g on the base M. With $\lambda = \mathrm{e}^{2\varphi}$ and $\varphi \in \mathrm{C}^\infty_M$ we pass to the metric

$$g' = \lambda g = \lambda\langle\ ,\ \rangle. \tag{11}$$

Let us distinguish by $T'M$ the tangent manifold of M with the metric g', when necessary. For the rest of the section we restrict to the Levi-Civita connection

$$\nabla = \nabla^g. \tag{12}$$

Notice $TTM = H \oplus V = H' \oplus V$ and we conform to our previous remarks on notation.

Let also $t : M \to \mathbb{R}\backslash\{0\}$ be a smooth function. Then we may consider the isomorphism (letting $\hat{h} = \mathrm{e}^{-\varphi} t$)

$$h : TM \longrightarrow T'M, \qquad h(u) = \mathrm{e}^{-\varphi} t u = \hat{h} u =: u'\ . \tag{13}$$

We treat all given scalar functions like φ or t, depending on the context, as functions composed with π. This implies, for example,

$$X(\varphi) = \mathrm{d}\varphi(X) = X^h(\varphi)\ . \tag{14}$$

Recall the 1-form θ on TM given by $\theta(X) = \langle BX, \xi\rangle$.

Proposition 2.2. *Let X be any vector field on TM and consider the differential map $h_*:TTM\to h^*TT'M$. It satisfies the identities $h_*(X^v)=\hat{h}X^v$ and, more generally,*

$$h_*X=X^{h'}+\hat{h}\big(\frac{X(t)}{t}\xi+X^v+\partial\varphi.BX-\theta(X)B\mathrm{grad}\,\varphi\big) \tag{15}$$

where B refers to the decomposition $H\oplus V$.

Proof. We know that $\nabla'=\nabla+C$ where $C_XY=\mathrm{d}\varphi(X)Y+\mathrm{d}\varphi(Y)X-\langle X,Y\rangle\mathrm{grad}\,\varphi$ (here X,Y denote vector fields on M or on TM). Since $\pi\circ h=\pi$, then $(h_*X)^{h'}=(\mathrm{d}\pi)^{-1}(\mathrm{d}\pi(X))$ and this is the same as $X^{h'}$, the H'-part of X. Writing ξ' for the very same canonical vector field ξ on $T'M$, so that $h^*\xi'=\xi\circ h=\hat{h}\xi$, and computing,

$$\begin{aligned}\pi^*\nabla'_{h_*(X)}\xi' &= h^*\pi^*(\nabla+C)_Xh^*\xi'\\ &= \pi^*\nabla_X(\hat{h}\xi)+B\pi^*C(X,B^{\mathrm{ad}}(\hat{h}\xi))\\ &= \mathrm{d}\hat{h}(X)\xi+\hat{h}\nabla^*_X\xi+\hat{h}B\pi^*C(X,B^{\mathrm{ad}}\xi)\\ &= -X(\varphi)\hat{h}\xi+\mathrm{e}^{-\varphi}X(t)\xi+\hat{h}X^v+\hat{h}X(\varphi)\xi+\\ &\qquad\qquad +\hat{h}(B^{\mathrm{ad}}\xi)(\varphi).BX-\hat{h}\langle BX,\xi\rangle B\mathrm{grad}\,\varphi\\ &= \hat{h}\big(\frac{X(t)}{t}\xi+X^v+\partial\varphi.BX-\theta(X)B\mathrm{grad}\,\varphi\big)\end{aligned}$$

we find the vertical part. ∎

Remark. Notice any tangent vector $X=X^h+X^v=X^{h'}+X^{v'}$ has two decompositions. We have, cf. figure 1,

$$\begin{aligned}X^{v'} &= \nabla'^*_X\xi = \nabla_X\xi+B\pi^*C(X,B^{\mathrm{ad}}\xi)\\ &\qquad\qquad = X^v+\partial\varphi.BX+X(\varphi)\xi-\theta(X)B\mathrm{grad}\,\varphi,\\ X^{h'} &= X-X^{v'} = X^h-\partial\varphi.BX-X(\varphi)\xi+\theta(X)B\mathrm{grad}\,\varphi\,.\end{aligned} \tag{16}$$

Now we suppose TM is endowed with the metric $G=g^{f_1,f_2}$ introduced in previous sections and we let $T'M$ have the metric $G'=(\lambda g)^{f'_1,f'_2}$ (the four weight functions are just smooth, positive and defined on M).

Theorem 2.1. *The map h is a homothety (ie. $h^*G'=\psi G$ for some function ψ) if and only if t and λ are constants and satisfy $\frac{f'_1}{f_1}\lambda=t^2\frac{f'_2}{f_2}$. In this case, the latter is the value of ψ.*

Proof. We write $h_*X=X^{h'}+\hat{h}E(X)$ defining E from (15). Then solving the equation above with vertical vector fields X_1,X_2 we immediately find

$$h^*G'(X_1,X_2)=\psi G(X_1,X_2)\ \text{ if and only if }\ \lambda\hat{h}^2f'_2=\psi f_2\ \text{ i.e. }\ t^2f'_2=\psi f_2.$$

In particular, ψ is only defined on M. Notice we may write

$$E_{au}(X^h)=aE_u(X^h),\qquad \forall a\in\mathbb{R},$$

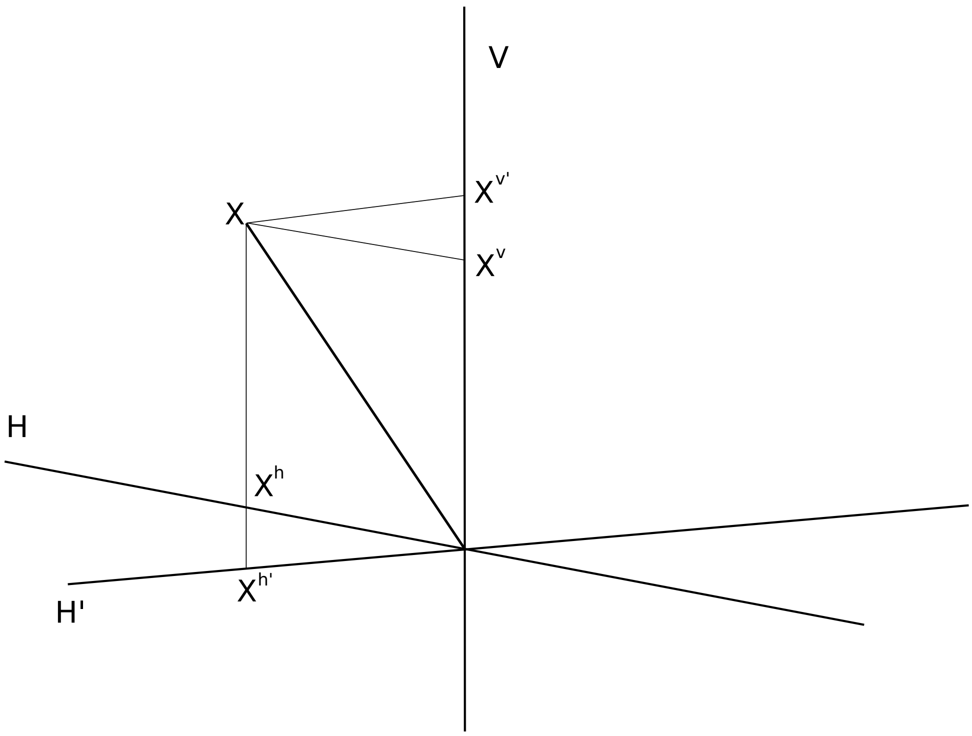

Figure 1: The connection induced projections

because ξ is also hidden linearly in $\partial\varphi$ and θ. Picking two horizontal lifts and having in mind that t and ψ are only defined on M, it is then easy to deduce that a necessary condition for h to be a homothety is that $E(X^h) = 0$ for all H-horizontal X. Now

$$t\langle E(B^{\mathrm{ad}}\xi), \xi\rangle = t\langle \frac{\partial t}{t}\xi + \partial\varphi.\xi - \|\xi\|^2 B\mathrm{grad}\,\varphi, \xi\rangle = (\partial t + t\partial\varphi - t\partial\varphi)\|\xi\|^2$$

and hence $\partial t = \mathrm{d}t(B^{\mathrm{ad}}\xi) = 0$. Choosing any X horizontal and orthogonal to $B^{\mathrm{ad}}\xi$ (recall $m > 1$), we find $0 = \langle E(X), BX\rangle = \partial\varphi\|X\|^2 = 0 \Leftrightarrow \partial\varphi = 0$, as we wished. In particular, $\nabla = \nabla'$. Finally, solving the equation above for horizontal vector fields X_1, X_2 we get $f_1'\lambda = \psi f_1$. For generic vectors the result follows. ∎

Generalizing the Theorem for the case of two conformal changes we have: the map h from $(\lambda_1 g)^{f_1,f_2}$ to $(\lambda_2 g)^{f_1',f_2'}$ is a homothety if and only if

$$t, \frac{\lambda_2}{\lambda_1} \text{ are constants} \quad \text{and} \quad \frac{f_1'\lambda_2}{f_1\lambda_1} = t^2\frac{f_2'}{f_2}. \tag{17}$$

2.5 Homotheties of S_rM

Let $r, s \in \mathrm{C}^\infty_M(\mathbb{R}^+)$ and recall the tangent sphere bundle of radius r

$$S_rM = \{u \in TM : \ \|u\|_g^2 = r^2\} \tag{18}$$

submanifold of TM, for which we have

$$S_rM \ = \ S_1'M \tag{19}$$

using the metric λg to define S'_sM with $\lambda = r^{-2} = \mathrm{e}^{2\varphi}$. Consider the smooth function $N = r^{-2}\|\xi\|^2$ on TM, cf. formula (14). Then $S_rM = N^{-1}(1) = \{u \in TM :\ G(\xi_u, \xi_u) = 1\}$ where $G = g^{f_1, r^{-2}}$ with f_1 any positive function. Using Proposition 2.1 to differentiate $N = G(\xi, \xi)$, it is easy to deduce

$$TS_rM = \big\{X \in TTM :\ \langle X, \xi\rangle = rX(r)\big\}. \tag{20}$$

We have to assume $\varphi_2 = \varphi = -\log r$. But of course one just applies ∇^* to $\|\xi\|^2 - r^2 = 0$ to easily find the same information. Notice $X \in TS_rM \Leftrightarrow \langle X^v, \xi\rangle = rX^h(r)$.

We shall consider a more general setting: with r and φ independent.

Let λg be any conformal change of the given metric, $\lambda = \mathrm{e}^{2\varphi}$. Let s be another positive function on M and consider the map h from Proposition 2.2 with an appropriate chosen t. It restricts to a *diffeomorphism*

$$h : S_rM \longrightarrow S'_sM, \qquad h(u) = \mathrm{e}^{-\varphi}\frac{s}{r}u = \hat{h}u\ . \tag{21}$$

When is h a homothety for the induced metrics? For a start, only the metrics G, G' constructed as in section 2.4 are relevant, i.e. those induced from $\nabla = \nabla^g$ the Levi-Civita connection.

Remark. Recall the metric on the right hand side arises from $H' \oplus V$. Since $h_* : T_uS_rM \to T_{\hat{h}u}S'_sM$, it is true that we have

$$rX(r)\ =\ \langle X, \xi\rangle \quad\Leftrightarrow\quad s\,(h_*X)(s)\ =\ \langle h_*X, \hat{h}\xi\rangle'.$$

Indeed, we may write $h_*X = X^{h'} + \hat{h}E(X)$ where $E(X)$ is given in (15) as

$$EX = \frac{X(t)}{t}\xi + X^v + \partial\varphi.BX - \theta(X)B\mathrm{grad}\,\varphi. \tag{22}$$

but now with the function

$$t = \frac{s}{r}\ . \tag{23}$$

Also, on vertical vector fields the metrics agree up to the scale, so we find

$$\begin{aligned}
\langle h_*X, \hat{h}\xi\rangle' &= \hat{h}^2\langle EX, \xi\rangle' \\
&= \mathrm{e}^{2\varphi}\mathrm{e}^{-2\varphi}t\langle X(t)\xi + tX^v + t\partial\varphi.BX - t\theta(X)B\mathrm{grad}\,\varphi, \xi\rangle \\
&= t\big(\frac{X(s)r - sX(r)}{r^2}\|\xi\|^2 + t\langle X^v, \xi\rangle + t\partial\varphi.\theta(X) - t\theta(X)\partial\varphi\big) \\
&= \frac{s}{r}\big(rX(s) - sX(r) + \frac{s}{r}rX(r)\big) \\
&= sX(s)\ =\ s\,(h_*X)(s)
\end{aligned}$$

since on S_rM we have $\|\xi\|^2 = r^2$.

In the next Theorem we prove that each tangent sphere bundle S_rM with metric G induced from that of TM is quite unique, independently of any of the metric transformations above and up to the straightforward coincidences expressed in the corollaries. The reader may notice the impossibility of adapting the arguments used for Theorem 2.1. We also remark we were not able to prove the cases of $\dim M = 1, 2$.

We let $\lambda = \mathrm{e}^{2\varphi}$ and $r, s, f_1, f_2, f'_1, f'_2$ be any positive functions on M.

Until the end of this section we assume M is connected and $\dim M \geq 3$.

Theorem 2.2. *Let S_rM have the induced metric $G = g^{f_1,f_2}$ and let S'_sM have the induced metric $G' = (\lambda g)^{f'_1,f'_2}$. Then the following are equivalent:*

1. *$h : S_rM \to S'_sM$ is a homothety, ie. $h^*G' = \psi G$ for some function ψ.*
2. *λ is constant, ψ verifies simultaneously $\psi = \frac{f'_1}{f_1}\lambda = \frac{s^2}{r^2}\frac{f'_2}{f_2}$ and one of the following hold:*
   (i) s/r is constant
   (ii) rs is constant.

For the case of the identity ($\hat{h} = 1$), we have that it is a homothety if and only if $\lambda = s^2/r^2$ is a constant and $\frac{f'_1}{f_1} = \frac{f'_2}{f_2}$.

Proof. First we notice

$$\begin{aligned} G'(h_*X, h_*Y) &= f'_1\langle X^{h'}, Y^{h'}\rangle' + \hat{h}^2 f'_2\langle EX, EY\rangle' \\ &= f'_1\lambda\langle X^h, Y^h\rangle + \hat{h}^2\lambda f'_2\langle EX, EY\rangle. \end{aligned}$$

Now consider the equation $h^*G'(X,Y) = \psi G(X,Y)$. Choose one vector $X = \xi^\perp$ vertical and orthogonal to ξ, and a vector $Y = (\mathrm{grad}\, r)^\perp$ horizontal and orthogonal to $\mathrm{grad}\, r$. Then both $X, Y \in TS_rM$. Indeed, $\langle X, \xi\rangle = 0 = rX(r)$ and $\langle Y, \xi\rangle = 0 = r\langle Y, \mathrm{grad}\, r\rangle = rY(r)$. Then for two vertical vector fields, like X, we immediately get the necessary condition $\hat{h}^2\lambda f'_2 = \psi f_2 \Leftrightarrow \psi = \frac{s^2}{r^2}\frac{f'_2}{f_2}$. For X, Y we have $EX = X^v$ and $EY = \frac{Y(t)}{t}\xi + \partial\varphi.BY - \theta(Y)B\mathrm{grad}\,\varphi$, hence

$$0 = \psi G(X,Y) = G'(h_*X, h_*Y) = f'_2\lambda\hat{h}^2\big(\partial\varphi\langle X, BY\rangle - \theta(Y)\langle X, B\mathrm{grad}\,\varphi\rangle\big) \ .$$

Now we choose a point $u \in S_rM$ orthogonal to $\mathrm{grad}\, r$. Then we may take $X = B\mathrm{grad}\, r$ and $Y = u \in H$. We have $\langle BY, X\rangle = 0$ and $\theta(Y) = \langle u, u\rangle = r^2$, so our equation yields $\langle X, B\mathrm{grad}\,\varphi\rangle = 0$. Equivalently, we must have $\mathrm{grad}\, r \perp \mathrm{grad}\,\varphi$.

Now suppose $\mathrm{grad}\, r = 0$ on all points of M, ie. r is constant. Then $H \subset TS_rM$. Take any non-vanishing $Z_0 \in H$. Then we may further[1] let $Z_0 \in H \cap \{\mathrm{grad}\, s, \mathrm{grad}\,\varphi\}^\perp$. In fact, in dimension ≥ 3, we may find a point u in each fibre of S_rM such that $(\partial\varphi)_u = 0$ and a vector $Z_0 \in H_u$ such that $Z_0(s) = Z_0(\varphi) = 0$ and $\theta(Z_0) = 0$. Then on the chosen point u we get $E(Z_0) = 0$ and so $h_*Z_0 = Z_0^{h'}$. Hence our main equation yields the necessary condition $f'_1\lambda = \psi f_1$. Going back a little, we then consider any point u and any $Z_0 \in H$ perpendicular to u, ie. such that $\xi_u \perp BZ_0$. Then we deduce

$$G'(h_*Z_0, h_*Z_0) = f'_1\lambda\|Z_0\|^2 + f'_2\lambda\hat{h}^2\Big(\frac{(Z_0(s))^2}{s^2}r^2 + (\partial\varphi)^2\|Z_0\|^2\Big) \ = \ \psi f_1\|Z_0\|^2$$

This immediately implies $Z_0(s) = 0$, $\partial\varphi = 0$. Since Z_0 and u may now be put in general position, we conclude s and φ are constant on M, a connected manifold by assumption, and the Theorem follows.

So now we admit $\mathrm{grad}\, r \neq 0$ at some point $x \in M$. Recall $\mathrm{grad}\, r \perp \mathrm{grad}\,\varphi$ and let $\epsilon = \|\mathrm{grad}\, r\|$ and $\delta = \|\mathrm{grad}\,\varphi\|$.

[1] This last assumption is not plausible in dimension 2 since we want $Z_0 \neq 0$, hence the hypothesis on the dimension; although here we may assume that $\mathrm{grad}\,\varphi$ together with $\mathrm{grad}\, s$ constitute a basis of H and then try to solve the system of two quadratic equations and 4 unknowns, in the components of u and Z_0 in that basis, given by $Z_0(t)\xi + t\partial\varphi.BZ_0 - t\theta(Z_0)B\mathrm{grad}\,\varphi = 0$, for that is all we need - here, because ahead the dimension hypothesis is required again.

Hence $u_1=\frac{r}{\epsilon}\mathrm{grad}\,r\in S_rM$. Notice $\partial\varphi_{u_1}=\mathrm{d}\varphi(u_1)=0$. Consider the vector $X_0=\mathrm{grad}\,r$ and $X=X_0+\epsilon BX_0$. It is tangent to our sphere bundle at u_1 since

$$\langle X,\xi\rangle=\epsilon\frac{r}{\epsilon}\epsilon^2=r\langle X_0,X_0\rangle=rX(r).$$

And we have that

$$\begin{aligned} h_*X=X^{h'}+\hat{h}EX &= X^{h'}+\hat{h}\big(\frac{X(t)}{t}\xi_{u_1}+\epsilon BX-\theta(X)B\mathrm{grad}\,\varphi\big)\\ &= X^{h'}+\hat{h}\big(\frac{X(t)}{t}\frac{r}{\epsilon}+\epsilon\big)BX_0-\hat{h}r\epsilon\,B\mathrm{grad}\,\varphi. \end{aligned}$$

Consider also the tangent vector at u_1, $Z=B\mathrm{grad}\,\varphi$. Then $h_*Z=\hat{h}Z$. And thus $\psi G(X,Z)=\psi f_2\epsilon\langle BX_0,Z\rangle=0$; on the other hand

$$\begin{aligned} h^*G'(X,Z) &= f_2'\lambda\hat{h}^2\langle\big(\frac{X(t)}{t}\frac{r}{\epsilon}+\epsilon\big)BX_0-r\epsilon\,B\mathrm{grad}\,\varphi,B\mathrm{grad}\,\varphi\rangle\\ &= -f_2'\lambda\hat{h}^2\epsilon r\delta^2. \end{aligned}$$

This implies $\delta=0$, ie. φ and hence $\lambda=\mathrm{e}^{2\varphi}$ are constants.

Therefore the map h verifies $h_*X=X^h+\hat{h}(\frac{X(t)}{t}\xi+X^v)$, for any vector field $X\in TS_rM$. Now we consider any horizontal vector $X\in\ker\mathrm{d}r\cap\ker\mathrm{d}s$, in particular also tangent to S_rM and orthogonal to $\mathrm{grad}\,t$ (recall $n\geq2$). Then $X(t)=0$ and, just as we had the result $\hat{h}^2\lambda f_2'=\psi f_2$ using vertical vectors, we have the similar result with horizontal: $\lambda f_1'=\psi f_1$.

Next, we use two generic tangent vectors $X,Y\in TS_rM$. It is easy to see the conformality equation $h^*G'=\psi G$ is finally equivalent to

$$\langle\frac{X(t)}{t}\xi+X^v,\frac{Y(t)}{t}\xi+Y^v\rangle=\langle X^v,Y^v\rangle,$$

$$\frac{X(t)Y(t)}{t^2}r^2+\frac{X(t)rY(r)}{t}+\frac{Y(t)rX(r)}{t}=0$$

or

$$X(t)Y(t)r+X(t)Y(r)t+X(r)Y(t)t\;=\;0.$$

Notice this last equation only involves the horizontal part of the vectors, so we assume X,Y as such. Now if we take X orthogonal to $\mathrm{grad}\,t$, ie. satisfying $X(t)=0$, and take $Y=\mathrm{grad}\,t$, then we find that $X(r)=0$ or that X is also orthogonal to $\mathrm{grad}\,r$. Henceforth, $\mathrm{grad}\,t$ and $\mathrm{grad}\,r$ are proportional, ie. lie on the same line. In other terms,

$$\mathrm{d}t=a\mathrm{d}r$$

for some function a on M. Clearly the equation above may be written as

$$r\mathrm{d}t\otimes\mathrm{d}t+t\mathrm{d}t\otimes\mathrm{d}r+t\mathrm{d}t\otimes\mathrm{d}r=0.$$

Hence we have $(ra^2+2ta)\mathrm{d}r\otimes\mathrm{d}r=0$. Recalling r is not constant, we either have t constant or $ra+2t=0$. We have both

$$\mathrm{d}t=-\frac{2t}{r}\mathrm{d}r=-\frac{2s}{r^2}\mathrm{d}r\qquad\text{and}\qquad\mathrm{d}t=\frac{r\mathrm{d}s-s\mathrm{d}r}{r^2}.$$

Hence $-2s\mathrm{d}r=r\mathrm{d}s-s\mathrm{d}r\;\Leftrightarrow\;r\mathrm{d}s+s\mathrm{d}r=0$, from which we find $sr=$constant.

Finally all conditions are fulfilled for h to be the expected homothety of ratio ψ. The identity map case is trivial. ∎

Let $g^S = g^{1,1}$ denote the induced Sasaki metric on the tangent sphere bundle and recall we are only considering $\dim M \geq 3$.

Corollary 2.1. *The Riemannian manifold (S_rM, g^S) is homothetic to $(S'_sM, (\lambda g)^S)$ via h if and only if $\psi = \lambda = \frac{s^2}{r^2}$ and this is a constant. In this case, h is the identity and $s = \sqrt{\lambda}r$; in other words $S_rM = S'_sM$. In particular, two tangent sphere bundles both with the induced Sasaki metric are homothetic if and only if they have exactly the same radius function, ie., they coincide.*

Corollary 2.2. *Other particular cases are as follows: the Riemannian manifold (S_rM, g^{f_1,f_2}) is isometric via h to $(S'_rM, (\lambda g)^{1,f_2})$ if $f_1 = \lambda$ is constant. And (S_rM, g^{f_1,f_2}) is isometric to $(S'_1M, (\lambda g)^{1,r^2f_2})$ if $f_1 = \lambda$ and both r, f_1 are constant. Moreover, (S_rM, g^S) is isometric to (S_1M, g^{1,r^2}) if r is constant.*

We have used the metric $G = g^{f_1,r^{-2}}$ on S_rM. So we study this case separately.

Corollary 2.3. *Let S_rM be given the metric $G = g^{1,\frac{1}{r^2}}$ and let S'_sM be with the metric $G' = (\lambda g)^{f'_1,\frac{1}{s^2}}$. Then the following three conditions are equivalent:*

1. *the map $h : S_rM \to S'_sM$ is a homothety.*
2. *the functions verify: $\psi = f'_1\lambda = 1$, λ is a constant and s/r or sr is a constant.*
3. *the map h is an isometry.*

In particular, for any s, r positive constants, $(S_rM, g^{1,r^{-2}}) \simeq (S_sM, g^{1,s^{-2}}) \simeq (S_1M, g^S)$.

Proof. Indeed we have $\psi = f'_1\lambda = \frac{s^2}{r^2}\frac{r^2}{s^2} = 1$. ■

Corollary 2.4. *Let r be any function on M. Then (S_rM, g^S) is isometric to $(S_{\frac{1}{r}}M, g^{1,r^4})$.*

Proof. This is due to the second particular case found in the Theorem. We are taking $\lambda = 1$ and $s = \frac{1}{r}$ and indeed $\psi = \frac{f'_1}{f_1}\lambda = 1 = \frac{r^4}{r^4} = \frac{s^2}{r^2}\frac{f'_2}{f_2}$. Also notice we have sr constant. ■

2.6 Applications to space-forms

Formulas for the curvature of tangent sphere bundles of space-forms are finally studied here. Let $R > 0$ and let

$$M_R = \left\{x \in \mathbb{R}^{m+1} :\ x_1^2 + \cdots + x_m^2 \pm x_{m+1}^2 = R^2\right\} \tag{24}$$

be an m-dimensional space-form with the induced metric g from Euclidean space. M_R has constant sectional curvature $\pm 1/R^2$. If we conformally change the metric $g \rightsquigarrow \lambda g$ by a constant, then clearly $\pm\frac{1}{R^2} \rightsquigarrow \pm\frac{1}{\lambda R^2}$.

Having another $R_1 > 0$, the map $F : M_{R_1} \longrightarrow M_R$ defined by $x \in \mathbb{R}^{m+1} \longmapsto F(x) = \frac{R}{R_1}x$ induces the following isometry through differentiation. Writing $f_1 = f_2 = \frac{R^2}{R_1^2}$ and $s = \frac{R_1 r}{R}$, we have

$$F_* : (S_sM_{R_1}, g^{f_1,f_2}) \longrightarrow (S_rM_R, g^S), \qquad F_*(x,u) = \frac{R}{R_1}(x,u). \tag{25}$$

Indeed, $(F_*)^*g^S = g^{f_1,f_2}$. This isometry and corollaries 2.2,2.3 give us the next quite interesting result.

Proposition 2.3. *We have the following isometries*

$$(S_1M_R, g^S) \simeq (S_{\frac{1}{R}}M_1, g^{R^2,R^2}) \simeq (S'_{\frac{1}{R}}M_1, (R^2g)^{1,R^2}) \simeq (S'_1M_1, (R^2g)^{1,1}) = (S_1M_1, (R^2g)^S). \quad (26)$$

Now we apply a general formula from [7, Proposition 1.6] on the scalar curvature Scal of (S_sM, g^{f_1,f_2}) for any given constants f_1, f_2:

$$\mathrm{Scal}_{(S_sM,g^{f_1,f_2})} = \frac{1}{f_1}\mathrm{Scal}_{(M,g)} - \frac{f_2}{4f_1^2}\sum_{i,j,k=1}^{m}(\mathcal{R}^\xi{}_{ijk})^2 + \frac{(n-1)n}{f_2s^2}. \quad (27)$$

Since $\sum_{i,j,k=1}^m(\mathcal{R}^\xi{}_{ijk})^2 = \frac{s^2}{R^4}2n$, where $n = m-1$, we have

$$\mathrm{Scal}_{(S_sM_R,g^{f_1,f_2})} = \pm\frac{n(n+1)}{f_1R^2} - \frac{f_2}{4f_1^2}\frac{s^2}{R^4}2n + \frac{(n-1)n}{f_2s^2}. \quad (28)$$

In particular the scalar curvature of (26) is $\pm\frac{n(n+1)}{R^2} - \frac{n}{2R^4} + (n-1)n$. We may say it is rewarding to see the same value in (28) for any of the forms in (26). Notice we can also write the scalar curvature of $(S_rM_R, g^S) \simeq (S_{\frac{1}{r}}M_R, g^{1,r^4})$.

Theorem 2.3. *For any $m \geq 3$ and both cases $\pm$, for any scalars $R, f_1, f_2 > 0$, we can always find a sufficiently small or large radius s in order to have (S_sM_R, g^{f_1,f_2}) with, respectively, positive or negative scalar curvature.*

The proof is clear just by looking at s in (28); the result is partially corroborated by two Theorems in [11].

We further remark that the formulas in [7] show the Riemannian metrics we are considering are never Einstein, though $\mathrm{Scal}_{(S_sM_R,g^{f_1,f_2})}$ is constant.

3 Characteristic classes

We know not of any reference for the fundamental questions solved in this section. We extend our study to problems of topology of the tangent and tangent sphere bundles. The first stems from the Riemannian structure.

3.1 Almost Hermitian structure on TM

The pair TM, g^S admits a compatible almost complex structure, also attributed to Sasaki. It was first studied in [9, 14] and gave origin in [15] to an almost contact structure on the unit tangent sphere bundle S_1M. For M oriented and dimension 4 we discovered a natural G_2-structure always existing on S_1M with the very same metric, cf. [4, 5].

We continue the study of TM with the metric $G = g^{f_1,f_2}$ where $f_1 = \mathrm{e}^{2\varphi_1}$ and $f_2 = \mathrm{e}^{2\varphi_2}$. We let ∇ denote a metric connection on M with torsion T^∇. The almost complex structure of Sasaki may be now written as the bundle endomorphism $I^S = B^{\mathrm{ad}} - B$.

Let

$$\psi = \varphi_2 - \varphi_1, \qquad \overline{\psi} = \varphi_2 + \varphi_1. \quad (29)$$

We then define

$$I^G = \mathrm{e}^{\psi}B^{\mathrm{ad}} - \mathrm{e}^{-\psi}B. \quad (30)$$

It is easy to see the endomorphism I^G is an almost complex structure compatible with the metric G. We consider also the associated non-degenerate 2-form ω^G defined by $\omega^G(X,Y)=G(I^GX,Y)$, $\forall X,Y\in TTM$. Since $f_1\mathrm{e}^{\psi}=f_2\mathrm{e}^{-\psi}=\mathrm{e}^{\overline{\psi}}$, it follows that $\omega^G=\mathrm{e}^{\overline{\psi}}\omega^S$ where ω^S is the 2-form associated to the Sasaki structure g^S and $I^S=B^{\mathrm{ad}}-B$.

The next Theorem is shown for completeness of exposition. For the Cartan classification of torsions of metric connections see [3]. Notice the presence *again* of the quotient and product of the weights f_1, f_2!

Theorem 3.1 ([6])**.** *i) The almost complex structure I^G is integrable if and only if ∇ is flat and has the vectorial type torsion $T^{\nabla}=\mathrm{d}\psi\wedge 1$. In particular, if ∇ is torsion free, then I^G is integrable if and only if M is Riemannian flat and f_2/f_1 =constant.*
ii) (TM,ω^G) is a symplectic manifold if and only if $T^{\nabla}=\mathrm{d}\overline{\psi}\wedge 1$. In particular, with ∇ the Levi-Civita, $\mathrm{d}\omega^G=0$ if and only if f_2f_1 =constant.

We observe that in the strict case of the Sasaki metric we have $T^{\nabla}=0$ as *necessary* condition for both integrability of I^S and $\mathrm{d}\omega^S=0$. In the general case, the two equations are distinguished, as they should, by ψ and $\overline{\psi}$. Clearly we may draw the following conclusion.

Corollary 3.1 ([6])**.** *The almost Hermitian structure (TM,G,I^G,ω^G) is Kähler if and only if M is a Riemannian flat manifold ($T^{\nabla}=0,\ R^{\nabla}=0$) and f_1,f_2 are constants. In this case, TM is flat.*

The last assertion follows indirectly from Proposition 2.1.

Remark. Recall T^*M has a natural symplectic structure. It arises as $\mathrm{d}\lambda$ where λ is the Liouville 1-form ([10]): the unique 1-form λ on T^*M such that on a section α

$$\lambda_\alpha=\alpha\circ\pi_* \tag{31}$$

When we introduce the metric, the tangent and cotangent (sphere) bundles become isometric bundles. With a little computation we find that the 1-form $\theta=\xi^\flat\circ B=(B^{\mathrm{ad}}\xi)^\flat$ corresponds with the Liouville form, so it does not depend on the connection. Knowing the torsion of $\nabla^*\oplus\nabla^*$ for any metric connection on M, it is easy to deduce, cf. [4], that for any radius function we have:

$$\mathrm{d}\theta=\omega^S+\theta\circ T^{\nabla}. \tag{32}$$

The same is to say ω^S corresponds with the pull-back of the Liouville symplectic 2-form if and only if $T^{\nabla}=0$. Then a Hamiltonian theory of the geodesic flow is manageable. We also remark that the geodesic vector field in the sense e.g. of [10], i.e. the vector field $B^{\mathrm{ad}}\xi$ in our setting, is just the same as the geodesic spray in the sense e.g. of [9, 13].

3.2 Chern and Stiefel-Whitney classes of TM

Let us continue with the structures G, I^G on the tangent bundle, induced from any metric connection ∇, and the same notation from above.

By a deformation retract on the fibres of $\pi:TM\to M$, there is an identification of cohomology spaces $H^*(M)=H^*(TM)$. This is valid for any coefficient ring. In particular $H^i(TM)=0$, $\forall i>m$. Let w_j denote the j-th Stiefel-Whitney class of M — which is a Stiefel-Whitney class of TM as a vector bundle. Let $w=\sum w_j$ denote the total Stiefel-Whitney class.

Theorem 3.2. *For any manifold M of dimension m, the Euler class of the manifold TM vanishes and the total Stiefel-Whitney class is*

$$w(TTM) = w^2 = \sum_{j=0}^{[m/2]} w_j^2. \tag{33}$$

Proof. Being a top degree class, the Euler class must vanish. Since $TTM = \pi^*TM \oplus \pi^*TM$, the Whitney product Theorem and the naturality of the characteristic classes ([12]) immediately give $w(TTM) = w(TM)w(TM) = w^2$. Recall the coefficients are in $\mathbb{Z}_2$, hence the second identity in (33). ■

Theorem 3.3. *The Chern classes of the* manifold TM *with almost complex structure* I^G *are the Chern classes of the complexified* tangent bundle, $TM \otimes_{\mathbb{R}} \mathbb{C} \to M$.

Proof. The complex structure I^G in TTM is equivalent to I^S. One complex isomorphism is given by $f : X \mapsto X^h + e^{\psi}X^v$. Indeed, $\forall X \in TTM$,

$$\begin{aligned} I^S \circ f(X) = (B^{\mathrm{ad}} - B)(X^h + e^{\psi}X^v) \ &= \ -BX^h + e^{\psi}B^{\mathrm{ad}}X^v \\ &= e^{\psi}B^{\mathrm{ad}}X^v - e^{\psi}e^{-\psi}BX^h \ = \ f \circ I^G(X). \end{aligned}$$

By the functorial properties, we just have to compute the Chern classes of I^S. (Another argument: the homotopy induced by $t\psi$, $t \in [0, 1]$, preserves the Chern classes.) Now, the Chern classes of an almost complex manifold (N, J) are the Chern classes of the $\mathbb{C}$-vector bundle T^+N, the $+i$-eigenbundle of J where $i = \sqrt{-1}$. In our case,

$$T^+TM = H^c = \pi^*TM^c$$

where c denotes complexification, because of the $\mathbb{C}$-isomorphism induced from $X \in H \mapsto X + iBX \in T^+TM$. Indeed $I^S(X + iBX) = -BX + iB^{\mathrm{ad}}BX = i(X + iBX)$. Finally, by trivial reasons, we have $c_j(T^+TM) = c_j(TM^c)$. ■

We recall the Chern classes c_{2j} define the Pontryagin classes of M, cf. [12],

$$p_j(M) = (-1)^j c_{2j}(TM \otimes \mathbb{C}). \tag{34}$$

Moreover, the Chern classes of (TM, I^G) do not depend on the connection ∇.

3.3 Stiefel-Whitney classes of S_rM

Now let $m = n+1$ and let $r > 0$ be a scalar function on M. We continue to denote by $w = \sum_{j=1}^{m} w_j$ the total Stiefel-Whitney class of M.

Theorem 3.4. *The total Stiefel-Whitney class of the manifold S_rM is*

$$w(S_rM) = \sum_{j=0}^{n} \pi^* w_j^2 \tag{35}$$

and in particular its mod 2 Euler class vanishes.

Proof. First suppose r is constant. Then the n-vector bundle $\kappa := \xi^\perp \subset V$ sits in $TS_rM = H \oplus \kappa$ where we assume e.g. the Sasaki metric. We have $w(\pi^*TM) = w(H) = \pi^*w$. Clearly $w(\pi^*TM) = w(\kappa \oplus \mathbb{R}\xi) = w(\kappa)$. Hence

$$w(TS_rM) = w(H \oplus \kappa) = w(\pi^*TM)^2 = \pi^*w^2.$$

Notice $w_m(\kappa) = 0$ due to rank of κ being just n. Independently, $0 = w_{2n+1}(S_rM) = e(S_rM) \mod 2$. Using the homeomorphism $h : S_1M \longrightarrow S_rM,\ h(u) = ru$, we have the result for any function r. ■

Remark. 1. The results show that the odd degree Stiefel-Whitney classes of the manifolds TM and S_rM vanish.
2. We observe the independence of (35) from r. Moreover, always $w_1(S_rM) = 0$, as expected because TM is always oriented and ξ induces an orientation on the submanifold.
3. If M has a finite good cover, is oriented, and admits a non-vanishing vector field, then we deduce $H^*(S_rM) = H^*(M) \otimes H^*(S^n)$ by the Theorem of Leray-Hirsh (cf. [8]). In particular π^* is an isomorphism $H^i(S_rM) = H^i(M)$ of cohomology spaces up to degree $i \leq n-1 = m-2$. By contrast, we have proved $\pi^*(w_m) = 0$.

Since $w_2(S_rM) = w_1^2$, we have the following conclusion.

Corollary 3.2. *For any oriented Riemannian manifold M, the manifold S_rM is spin.*

Recall w_2 is also the obstruction for a closed 7-manifold to admit a G_2-structure. We have explicitly constructed a natural G_2-structure on S_1M, for any oriented Riemannian 4-manifold M, cf. [4, 5] and the references therein.

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