The Dirichlet problem for elliptic and degenerate elliptic equations, and related results

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ABSTRACT

The purpose of this thesis is to understand three different types of problems related to weighted elliptic operators and some results related to solvability of non-degenerate elliptic equations in rough domains. First, we prove that the Dirichlet problem for degenerate elliptic equations $\operatorname{div}(A\nabla u) = 0$ in the upper-half space $(x,t) \in \mathbb{R}^{n+1}_+$ is solvable when $n \geq 2$ and the boundary data is in $L^p(\mathbb{R}^n)$ for some $0 < p < \infty$. The coefficient matrix A is only assumed to be measurable, real-valued and t-independent with a degenerate bound and ellipticity controlled by a t-independent A_2-weight μ. It is not required to be symmetric. The result is achieved by proving a Carleson measure estimate for all bounded solutions in order to deduce that harmonic measure is in the A_∞-class with respect to the μ-weighted Lebesgue measure on $\mathbb{R}^n$. The Carleson measure estimate allows us to avoid applying the method of ϵ-approximability, which simplifies the proof obtained recently in the case of uniformly elliptic coefficients. The results have natural extensions to Lipschitz graph domains. Second, We obtain Hodge-decomposition, L^p bounds semi-groups and their gradients, and then we get L^p bounds for Riesz transforms and square functions associated to a degenerate elliptic operator in divergence form, with degeneracy controlled by a weight in the Muckenhoupt class A_2. Finally, we show that for a uniformly elliptic divergence form operator L, defined in an open set Ω with Ahlfors-David regular boundary, BMO-solvability implies scale invariant quantitative absolute continuity (the weak-A_∞ property) of elliptic-harmonic measure with respect to surface measure on $\partial\Omega$. We do not impose any connectivity hypothesis, qualitative or quantitative; in particular, we do not assume the Harnack Chain condition, even within individual connected components of Ω. In this generality, our results are new even for the Laplacian. Moreover, we obtain a converse, under the additional assumption that Ω satisfies an interior Corkscrew condition, in the special case that L is the Laplacian.