

The Relationship Between Galerkin and Collocation Methods in Statistical Transmission Line Analysis

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Abstract—This paper discusses the relationship between two standard methods for the stochastic analysis of linear circuits, namely the stochastic Galerkin method (SGM) and the stochastic collocation method (SCM), based on a multidimensional Gaussian quadrature. It is established that the SCM corresponds to an approximate factorization of the SGM, involving matrix polynomials sharing the same coefficients as the pertinent polynomial chaos basis functions. Under certain assumptions, the two methods coincide. These findings are illustrated by means of a frequency-domain simulation of a transmission line circuit.

Index Terms—Circuit simulation, matrix factorization, polynomial chaos, statistical analysis, stochastic collocation method, stochastic Galerkin method, uncertainty quantification.

I. INTRODUCTION

The past few years have seen an evergrowing interest in uncertainty quantification techniques for electrical engineering applications. In particular, the polynomial chaos (PC) method [1] was widely adopted in the analysis of electrical circuits and interconnects [2]–[7]. The underlying idea of PC is to represent stochastic variables of interest (e.g., circuit voltages and currents) as expansions of suitable orthogonal polynomials. The determination of the expansion coefficients allows obtaining relevant statistical information and it is typically much faster than traditional approaches such as the Monte Carlo method.

Essentially there exist two classes of methods to calculate the PC expansion (PCE) coefficients, namely Galerkin and collocation-based techniques [1]. The stochastic Galerkin method (SGM) requires the single solution of a modified (augmented) problem [2], [3]. On the other hand, stochastic collocation methods (SCM) only require to sample the solution at a limited set of points in the space of the random variables [4]–[7]. Several different strategies are available for choosing the collocation points, which in most cases are (a subset of) the nodes of a multidimensional Gaussian quadrature [8].

In literature it is often stated that the SGM is more accurate in the calculation of the PCE coefficients. However, this statement often remains somewhat vague, and comparisons were only provided from a numerical standpoint (e.g., [9]). This paper aims at elaborating in a more quantitative way on this important matter, by focusing on the relationship between the full-tensor SGM and the *pseudo-spectral* SCM [1] (from now on, “SCM” will denote this particular scheme). Other stochastic collocation techniques use a reduced subset

of quadrature nodes or random collocation points, and they are therefore understood to be an approximation of the pseudo-spectral SCM. Although the formal derivations are deferred to a mathematical paper [10], it is here discussed that the SCM is an approximate factorization of the SGM problem and that the two methods coincide under certain assumptions. Moreover, the discussion, available in a rigorous mathematical sense only for the single-variable case [10], is here extended by means of numerical comparisons to the multivariate case, while refraining from deriving lengthy and tedious mathematical derivations due to space limitations.

II. STOCHASTIC SIMULATION VIA POLYNOMIAL CHAOS

For the sake of illustration, the discussion is applied to the well-known transmission-line equations [11]. Consider an interconnect of length ℓ described in the frequency domain by

$$\frac{d\mathbf{V}(z, \omega, \boldsymbol{\xi})}{dz} = -(\mathbf{R}(\omega, \boldsymbol{\xi}) + j\omega\mathbf{L}(\omega, \boldsymbol{\xi}))\mathbf{I}(z, \omega, \boldsymbol{\xi}) \quad (1a)$$

$$\frac{d\mathbf{I}(z, \omega, \boldsymbol{\xi})}{dz} = -(\mathbf{G}(\omega, \boldsymbol{\xi}) + j\omega\mathbf{C}(\omega, \boldsymbol{\xi}))\mathbf{V}(z, \omega, \boldsymbol{\xi}) \quad (1b)$$

where $z \in [0, \ell]$ denotes the longitudinal coordinate, vectors \mathbf{V} and \mathbf{I} collect the voltages and currents along the line, and \mathbf{R} , \mathbf{L} , \mathbf{G} , \mathbf{C} are the so-called per-unit-length (p.u.l.) parameters describing the electromagnetic propagation. Besides the frequency, they also depend on D stochastic variables defined by vector $\boldsymbol{\xi} = [\xi_1, \dots, \xi_D]$. In turn, the voltages and currents are also $\boldsymbol{\xi}$ -dependent.

The discussion that follows traces the one in [10], which is however limited to the univariate case ($D = 1$). The formulation is here extended to the multivariate case, but without providing any formal proof. The validity is empirically assessed via the numerical analysis of Section III.

A. Polynomial Chaos Expansion

Following PC theory [1], the RLGC parameters as well as the vectors of voltages and currents are approximated by means of PCEs, e.g.,

$$\mathbf{R}(\omega, \boldsymbol{\xi}) \approx \sum_n \mathbf{R}_n(\omega) \varphi_n(\boldsymbol{\xi}) \quad (2)$$

and

$$\mathbf{V}(z, \omega, \boldsymbol{\xi}) \approx \sum_n \mathbf{V}_n(z, \omega) \varphi_n(\boldsymbol{\xi}), \quad (3)$$

respectively, where $\mathbf{n} = [n_1, \dots, n_D]$ is a D -variate multi-index associated to the basis functions $\varphi_{\mathbf{n}}$, which are constructed as the product of the univariate polynomials $p_n(\xi)$ that are orthogonal with respect to the probability density function of the random variables in $\boldsymbol{\xi}$, i.e., $\varphi_{\mathbf{n}} = \prod_{d=1}^D p_{n_d}(\xi_d)$. The discussion in this paper assumes a full tensor-product expansion, with polynomials up to degree P in each dimension (i.e., $n_d \leq P, \forall d$). The total number of terms is thus $(P+1)^D$.

B. Stochastic Galerkin Method

The substitution of the PCEs (2) and (3) into (1) and subsequent Galerkin projection yield the following coupled and augmented, yet deterministic equations in the unknown PCE coefficients [2]:

$$\frac{d\tilde{\mathbf{V}}(z, \omega)}{dz} = -(\tilde{\mathbf{R}}(\omega) + j\omega\tilde{\mathbf{L}}(\omega))\tilde{\mathbf{I}}(z, \omega) \quad (4a)$$

$$\frac{d\tilde{\mathbf{I}}(z, \omega)}{dz} = -(\tilde{\mathbf{G}}(\omega) + j\omega\tilde{\mathbf{C}}(\omega))\tilde{\mathbf{V}}(z, \omega) \quad (4b)$$

where $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{I}}$ collect all the unknown coefficients of the voltage and current PCE, whereas $\tilde{\mathbf{R}}, \tilde{\mathbf{L}}, \tilde{\mathbf{G}}, \tilde{\mathbf{C}}$ are augmented p.u.l. matrices obtained from the PCE coefficients in (2) as

$$\tilde{\mathbf{R}}(\omega) = \sum_{\mathbf{n}} \mathbf{A}_{n_D} \otimes \dots \otimes \mathbf{A}_{n_2} \otimes \mathbf{A}_{n_1} \otimes \mathbf{R}_{\mathbf{n}}(\omega) \quad (5)$$

with \otimes denoting the Kronecker product and with the ‘‘auxiliary’’ matrix \mathbf{A}_n having entries given by

$$[\mathbf{A}_n]_{ij} = \frac{1}{\|p_i\|^2} \int_{-\infty}^{+\infty} p_n(\xi) p_j(\xi) p_i(\xi) w(\xi) d\xi \quad (6)$$

C. Stochastic Collocation Method

The SCM calculates the unknown PCE coefficients according to the classical projection theorem:

$$\begin{aligned} \mathbf{V}_{\mathbf{n}}(z, \omega) &= \frac{1}{\|\varphi_{\mathbf{n}}\|^2} \int_{-\infty}^{+\infty} \mathbf{V}(z, \omega, \boldsymbol{\xi}) \varphi_{\mathbf{n}}(\boldsymbol{\xi}) w(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &\approx \sum_{\mathbf{q}} \mathbf{V}(z, \omega, \xi_1^{(q_1)}, \dots, \xi_D^{(q_D)}) \prod_{d=1}^D p_{n_d}(\xi_d^{(q_d)}) \frac{w_{q_d}}{\|p_{n_d}\|^2} \end{aligned} \quad (7)$$

with the integral approximated by means of a multivariate P -order Gaussian quadrature [8] based on the univariate nodes $\{\xi^{(q)}\}_{q=1}^Q$ and the corresponding weights $\{w_q\}_{q=1}^Q$, with $Q = P + 1$, and indexed by $\mathbf{q} = [q_1, \dots, q_D]$.

Since (7) requires the solution of (1) at the quadrature nodes, the SCM problem can be rewritten by combining (1) and (7) in compact form as [10]

$$\frac{d\tilde{\mathbf{V}}(z, \omega)}{dz} = -(\mathbf{P}\hat{\mathbf{R}}(\omega)\mathbf{P}^{-1} + j\omega\mathbf{P}\hat{\mathbf{L}}(\omega)\mathbf{P}^{-1})\tilde{\mathbf{I}}(z, \omega) \quad (8a)$$

$$\frac{d\tilde{\mathbf{I}}(z, \omega)}{dz} = -(\mathbf{P}\hat{\mathbf{G}}(\omega)\mathbf{P}^{-1} + j\omega\mathbf{P}\hat{\mathbf{C}}(\omega)\mathbf{P}^{-1})\tilde{\mathbf{V}}(z, \omega) \quad (8b)$$

where the matrices denoted with a ‘‘hat’’ are block diagonal, and each block is given by the corresponding matrix evaluated at the quadrature nodes. Matrix \mathbf{P} is given by

$$\mathbf{P} = \underbrace{\mathbf{Q} \otimes \dots \otimes \mathbf{Q}}_{D \text{ times}} \otimes \mathbf{1} \quad (9)$$

with $\mathbf{1}$ the identity matrix of the same size as the RLGC matrices, and \mathbf{Q} a matrix with entries $Q_{qn} = p_n(\xi^{(q)}) \frac{w_q}{\|p_n\|^2}$ for $n = 0, \dots, P$ and $q = 1, \dots, Q$.

D. An Approximate Matrix Factorization

The problems (4) and (8) have similar form, though different matrices. In [10] it is shown, for the univariate case, that (8) is an approximate factorization of (4), obtained by replacing the auxiliary matrices (6) in (5) with $\hat{\mathbf{A}}_n = p_n(\mathbf{M})$, i.e., a matrix polynomial sharing the same coefficients as the n th-order basis function p_n , and with the argument \mathbf{M} being a tridiagonal matrix constructed from the coefficients of the three-term recursion relation that defines these polynomials [8]. Moreover, it was shown that for any polynomial basis, $\hat{\mathbf{A}}_1 \equiv \mathbf{A}_1$, which implies that the SGM and the SCM exactly coincide when the expansion (2) does not contain terms of degree higher than one.

The specific fact that the SGM and the SCM coincide for first-order parameters was already proven for the general multivariate case in [13]. In [3], the factorization (8) of the SGM was derived for the particular case of Hermite polynomials, showing that the polynomial approximation of the auxiliary matrices \mathbf{A}_n provides the best factorization in terms of error, and it was later extended to arbitrary polynomials in [12]. Here, however, it is argued for the first time that in the general multivariate case with arbitrary polynomials, the SCM can always be considered as an approximate factorization of the SGM.

III. NUMERICAL RESULTS

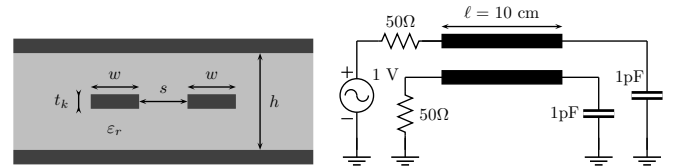


Fig. 1. Stripline cross-section (left) and terminations (right).

As a validation, the outlined considerations are applied to the frequency-domain simulation of the stripline interconnect depicted in Fig. 1. The pertinent RLGC parameters are computed by means of a field solver based on the interconnect cross-sectional properties. The dielectric is assumed to be lossless, thus $\mathbf{G} = 0$ in the following simulations.

The following two test cases are considered:

- 1) Two random parameters, i.e., conductor resistivity $\rho \in [1.7241, 32.759] \text{ n}\Omega \cdot \text{m}$ and relative dielectric permittivity $\varepsilon_r \in [2.8, 5.2]$. The remaining parameters are $w = 20 \mu\text{m}$, $t_k = 2 \mu\text{m}$, $s = 30 \mu\text{m}$, $h = 50 \mu\text{m}$.
- 2) Three random parameters, i.e., $w \in [2, 20] \mu\text{m}$, $s \in [3, 30] \mu\text{m}$, and $h \in [5, 50] \mu\text{m}$. The remaining parameters are $\rho = 17.241 \text{ n}\Omega \cdot \text{m}$, $\varepsilon_r = 4$, $t_k = 2 \mu\text{m}$.

The first test case is designed to exhibit a linear (first-order) variation of both \mathbf{R} (the resistance is proportional to the

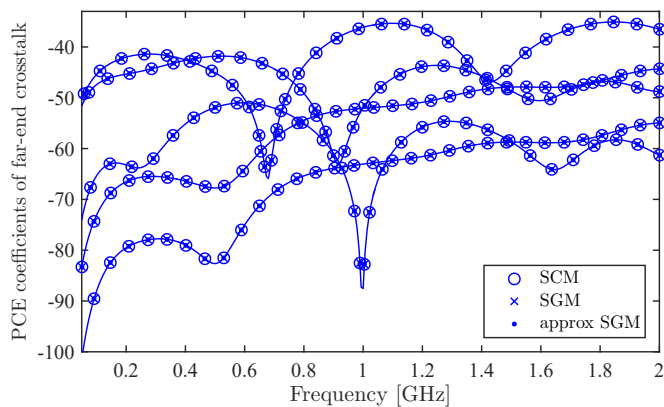


Fig. 2. A subset of the PCE coefficients of the far-end crosstalk for case 1), computed with the SCM (circles), the SGM (crosses) and the approximate factorization of the SGM (dots).

resistivity) and C (the capacitance matrix is proportional to the relative permittivity due to the homogeneity of the structure), while L is not affected by the parameter variations. No difference is therefore expected between the SGM and the SCM. In the second test case, a nonlinear variation of all the RLC parameters occurs, which will produce differences in the SGM and SCM. The parameter variations are taken deliberately very large to make this difference visually appreciable.

Uniform variations are considered and a second-order PCE with *orthonormal* Legendre polynomials is used, meaning that $p_0 = 1$, $p_1 = \sqrt{3} \cdot \xi$ and $p_2 = \sqrt{5} \cdot (\frac{3}{2}\xi^2 - \frac{1}{2})$. With this choice, the approximate auxiliary matrices that factorize the SGM into the SCM are $\hat{A}_0 = \mathbf{1}$ (identity matrix), $\hat{A}_1 = \sqrt{3} \cdot M$ and $\hat{A}_2 = \sqrt{5} \cdot (\frac{3}{2}M^2 - \frac{1}{2} \cdot \mathbf{1})$. Since $\hat{A}_1 = A_1$, it can be computed via (6) and the argument of the matrix polynomials is readily obtained as $M = A_1/\sqrt{3}$, without the need for deriving the recursion coefficients of the normalized Legendre polynomials.

Figs. 2 and 3 show the magnitude of a subset of the PCE coefficients of the far-end crosstalk voltage for cases 1) and 2). The figures compare the PCE coefficients computed with (7) or, equivalently, with (8) (SCM, circles), with (4)–(6) (SGM, crosses), and with the factorized SGM that uses the polynomial approximation of the auxiliary matrices (dots). The results confirm that, also for the multivariate case, the SCM is an approximate factorization of the SGM (Fig. 3), and the two methods coincide only for linear (first-order) variations of the input parameters (Fig. 2). The maximum difference in the estimated variance for case 2) is 0.154 dB. It is important to note that with smaller variations of the parameters (as occurs in practice) the difference between the two methods becomes much smaller.

IV. CONCLUSIONS

This paper discusses the equivalence of the SCM to an approximate factorization of the SGM and extends the mathematical formulation to multiple random variables. The validity is empirically assessed by means of an ad-hoc numerical example.

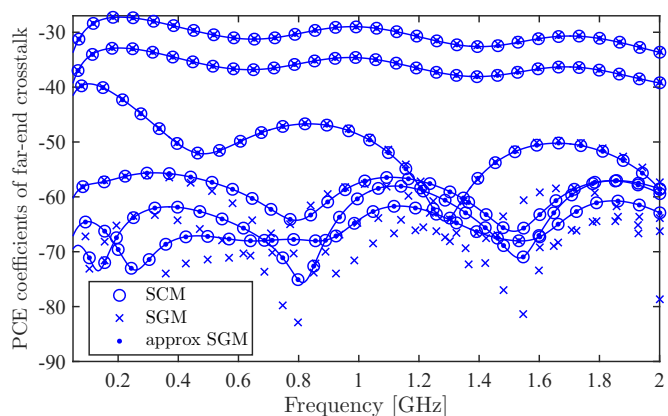


Fig. 3. A subset of the PCE coefficients of the far-end crosstalk for case 2).

ACKNOWLEDGMENT

This work was supported by the Research Foundation Flanders (FWO-Vlaanderen) and by the Interuniversity Attraction Poles Program initiated by the Belgian Science Policy Office.

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