Models for some irreducible representations of $\mathfrak{so}(m, \mathbb{C})$ in discrete Clifford analysis

H. De Ridder[*], T. Raeymaekers[†]

Abstract

In this paper we work in the 'split' discrete Clifford analysis setting, i.e. the m-dimensional function theory concerning null-functions of the discrete Dirac operator ∂, defined on the grid $\mathbb{Z}^m$, involving both forward and backward differences. This Dirac operator factorizes the (discrete) Star-Laplacian ($\Delta^* = \partial^2$). We show how the space $\mathcal{H}_k$ of discrete k-homogeneous spherical harmonics, which is a reducible $\mathfrak{so}(m, \mathbb{C})$-representation, may explicitly be decomposed into 2^{2m} isomorphic copies of irreducible $\mathfrak{so}(m, \mathbb{C})$-representations with highest weight $(k, 0, \dots, 0)$. The key element is the introduction of 2^{2m} idempotents, dividing the discrete Clifford algebra in 2^{2m} subalgebras of dimension $\binom{k+m-1}{k} - \binom{k+m-3}{k}$.

MSC 2010: 17B15, 47A67, 20G05, 15A66, 39A12

Keywords: discrete Clifford analysis, irreducible representation, orthogonal Lie algebra, harmonic functions

1 Introduction

In classical Clifford analysis, the infinitesimal 'rotations' are given by the angular momentum operators, in our function theoretical setting denoted by the differential operators $L_{a,b} = x_a \partial_{x_b} - x_b \partial_{x_a}$. These operators satisfy the commutation relations

$$[L_{a,b}, L_{c,d}] = \delta_{b,c}\, L_{a,d} - \delta_{b,d}\, L_{a,c} - \delta_{a,c}\, L_{b,d} + \delta_{a,d}\, L_{b,c},$$

which are the defining relations of the orthogonal Lie algebra $\mathfrak{so}(m)$. Being endomorphisms of the space $\mathcal{H}_k(m, \mathbb{C})$ of scalar-valued harmonic polynomials homogeneous of degree k, this polynomial space is a model for an (irreducible) $\mathfrak{so}(m, \mathbb{C})$-representation [see e.g.[8, 1]]. To

[*]Ghent University, Department of Mathematical Analysis, Building S22, Galglaan 2, 9000 Gent, Belgium, email: hdr@cage.ugent.be

[†]Ghent University, Department of Mathematical Analysis, Building S22, Galglaan 2, 9000 Gent, Belgium, email: tr@cage.ugent.be

establish $\mathcal{M}_k$, the space of spinor-valued monogenics, homogeneous of degree k classically as $\mathfrak{so}(m,\mathbb{C})$-representation, the following operators are considered

$$dR(e_{a,b}) : \mathcal{M}_k \to \mathcal{M}_k, \qquad M_k \mapsto \left(L_{a,b} + \frac{1}{2}\, e_a\, e_b\right) M_k.$$

These operators are endomorphisms of $\mathcal{M}_k$ which also satisfy the defining relations of $\mathfrak{so}(m,\mathbb{C})$:

$$[dR(e_{a,b}), dR(e_{c,d})] = \delta_{b,c}\, dR(e_{a,d}) - \delta_{b,d}\, dR(e_{a,c}) - \delta_{a,c}\, dR(e_{b,d}) + \delta_{a,d}\, dR(e_{b,c}).$$

In [5], we developed discrete counterparts of the operators $L_{a,b}$ and $dR(e_{a,b})$ in the discrete Clifford analysis setting.

Definition 1. *The (discrete) angular momentum operators are discrete operators $L_{a,b} = \xi_a\, \partial_b + \xi_b\, \partial_a$, $1 \le a \ne b \le m$, acting on the discrete functions. For $a = b$, we define $L_{a,a} = 0$. Furthermore, let the operator $\Omega_{a,b}$ act on discrete functions f as $\Omega_{a,b}\, f = L_{a,b}\, f\, e_b\, e_a$.*

The discrete angular momentum operators also satisfy the defining relations of the orthogonal lie algebra $\mathfrak{so}(m)$ (see e.g. [10]):

$$[\Omega_{a,b}, \Omega_{c,d}] = \delta_{b,c}\, \Omega_{a,d} - \delta_{b,d}\, \Omega_{a,c} - \delta_{a,c}\, \Omega_{b,d} + \delta_{a,d}\, \Omega_{b,c}.$$

Furthermore, they are endomorphisms of the space $\mathcal{H}_k$ of Clifford-algebra valued harmonics, homogeneous of degree k, since $\Omega_{a,b}$ commutes with $\mathfrak{sl}_2 = \left\{\Delta, \xi^2, \mathbb{E} + \frac{m}{2}\right\}$, for all (a,b). We thus concluded that $\mathcal{H}_k$ is a representation of $\mathfrak{so}(m,\mathbb{C})$; however, this is not an irreducible representation, as will be shown in the following sections.

An important difference with the Euclidean Clifford setting is the addition of the basis elements $e_b\, e_a$ to the right of the considered function f. This will have consequence later on in this paper, when we describe irreducible representations by means of an idempotent; the action of $\mathfrak{so}(m,\mathbb{C})$ elements will affect this idempotent, in the sense that the representation can no longer be interpreted as a left ideal in the Clifford algebra. Another unexpected result was the possibility to rotate points of the grid $\mathbb{Z}^m$ over all real angles by rotation of the discrete delta functions (resp. distributions); we are thus not longer restricted to rotating over (integer multiples of) right angles.

In a similar manner, discrete operators $dR(e_{a,b})$ were constructed in [5], satisfying the defining relations of the orthogonal lie algebra $\mathfrak{so}(m)$ and commuting with $\mathfrak{osp}(1|2) = \left\{\partial, \xi, \mathbb{E} + \frac{m}{2}\right\}$ which makes them endomorphisms of the space $\mathcal{M}_k$ of k-homogeneous discrete monogenic polynomials. As such, $\mathcal{M}_k$ is a reducible $\mathfrak{so}(m,\mathbb{C})$-representation. The decomposition of $\mathcal{M}_k$ into irreducible representations will be the topic of an upcoming paper.

Describing the discrete harmonic spaces $\mathcal{H}_k$ as $\mathfrak{so}(m, \mathbb{C})$-representations will be most effective (from a representation-theoretic point of view) when the representations are irreducible. Only then will we be able to draw conclusions about for example Gelfand-Tsetlin bases (see e.g. [11]) An accurate description of the decomposition is thus very important, and this will be done for $\mathcal{H}_k$ in the following sections.

To keep this paper self-contained, we start with a preliminary section on the discrete Clifford analysis framework. Throughout this paper, we will use concepts regarding the Cartan subalgebra and the positive roots of the orthogonal Lie algebra $\mathfrak{so}(m, \mathbb{C})$, hence in Section 3, we briefly summarize these concepts. In Section 4, we describe the decomposition of $\mathcal{H}_k$ in irreducible components.

2 Preliminaries

Let $\mathbb{R}^m$ be the m-dimensional Euclidian space with orthonormal basis e_j, $j = 1, \ldots, m$ and consider the Clifford algebra $\mathbb{R}_{m,0}$ over $\mathbb{R}^m$, governed by the multiplication relations $e_i e_j + e_j e_i = 2\delta_{ij}$. Passing to the so-called 'split' discrete setting [6, 2], we embed this Clifford algebra into the bigger complex one $\mathbb{C}_{2m}$, the underlying vector space of which has twice the dimension, and introduce forward and backward basis elements $\mathbf{e}_j^{\pm}$ satisfying the following anti-commutator rules:

$$\left\{\mathbf{e}_j^-, \mathbf{e}_\ell^-\right\} = \left\{\mathbf{e}_j^+, \mathbf{e}_\ell^+\right\} = 0, \qquad \left\{\mathbf{e}_j^+, \mathbf{e}_\ell^-\right\} = \delta_{j\ell}, \qquad j, \ell = 1, \ldots, m.$$

The connection to the original basis e_j is given by $\mathbf{e}_j^+ + \mathbf{e}_j^- = e_j$, $j = 1, \ldots, m$. This implies $e_j^2 = 1$, in contrast to the usual Clifford setting where traditionally $e_j^2 = -1$ is chosen.

Now consider the standard equidistant lattice $\mathbb{Z}^m$; the coordinates of a Clifford vector $\underline{x}$ will thus only take integer values. We construct a discrete Dirac operator factorizing the discrete Laplacian, using both forward and backward differences $\Delta_j^{\pm}$, $j = 1, \ldots, m$, acting on Clifford-valued functions f as follows:

$$\Delta_j^+[f](\underline{x}) = f(\underline{x} + \mathbf{e}_j) - f(\underline{x}), \qquad \Delta_j^-[f](\underline{x}) = f(\underline{x}) - f(\underline{x} - \mathbf{e}_j).$$

With respect to the $\mathbb{Z}^m$-grid, the usual definition of the discrete Laplacian in $\underline{x} \in \mathbb{Z}^m$ is

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^+ \Delta_j^-[f](\underline{x}) = \sum_{j=1}^{m} \left(f(\underline{x} + \mathbf{e}_j) - f(\underline{x} - \mathbf{e}_j)\right) - 2m\, f(\underline{x}).$$

This operator is also known as "Star-Laplacian"; we will from now on simply write Δ. An appropriate definition of a discrete Dirac operator ∂ factorizing Δ, i.e. satisfying $\partial^2 = \Delta$, is obtained by combining the forward and backward basis elements with the corresponding forward and backward differences, more precisely

$$\partial = \sum_{j=1}^{m} \left(\mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^- \right).$$

In order to receive an analogue of the classical Weyl relations $\partial_{x_j} x_k - x_k \partial_{x_j} = \delta_{jk}$, the coordinate vector variable operators $\xi_j = \mathbf{e}_j^+ X_j^- + \mathbf{e}_j^- X_j^+$ are defined by their interaction with the corresponding coordinate operators $\partial_j = \mathbf{e}_j^+ \Delta_j^+ + \mathbf{e}_j^- \Delta_j^-$, according to the skew Weyl relations, cf. [2]:

$$\partial_j \, \xi_j - \xi_j \, \partial_j = 1, \; j = 1, \ldots, m,$$

which imply that $\partial_j \, \xi_j^k[1] = k \, \xi_j^{k-1}[1]$. The operators ξ_j and ∂_j furthermore satisfy the following anti-commutator relations:

$$\{\xi_j, \xi_k\} = \{\partial_j, \partial_k\} = \{\partial_j, \xi_k\} = 0, \qquad j \neq k, \; j, k = 1, \ldots, m$$

implying that $\partial_\ell \, \xi_j^k[1] = 0$, $j \neq \ell$.

The natural powers $\xi_j^k[1]$ of the operator ξ_j acting on the ground state 1 are the basic discrete k-homogeneous polynomials of degree k in the variable x_j, i.e. $\mathbb{E} \, \xi_j^k[1] = k \, \xi_j^k[1]$, where $\mathbb{E} = \sum_{j=1}^m \xi_j \, \partial_j$ is the discrete Euler operator. They constitute a basis for all discrete polynomials. Explicit formulas for $\xi_j^k[1]$ are given for example in [2, 3]; furthermore $\xi_j^k[1](x_j) = 0$ if $k \geqslant 2\,|x_j| + 1$.

A discrete function is discrete harmonic in a domain $\Omega \subset \mathbb{Z}^m$ if $\Delta f(\underline{x}) = 0$ for all $\underline{x} \in \Omega$. The space of discrete harmonic homogeneous polynomials of degree k is denoted $\mathcal{H}_k$, while the space of all discrete harmonic homogeneous polynomials is denoted $\mathcal{H}$. It is clear that

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,$$

The dimensions over the discrete Clifford algebra are

$$\dim(\mathcal{H}_k) = \binom{k+m-1}{k} - \binom{k+m-3}{k}.$$

Now that we know what discrete harmonic functions are, we can

3 Orthogonal Lie algebras

We will start by briefly introducing the orthogonal Lie algebra $\mathfrak{so}(m, \mathbb{C})$; a detailed description can be found for example in e.g. [7]. The orthogonal Lie algebra $\mathfrak{so}(m, \mathbb{C})$ is generated in even dimension $m = 2n$ by $\frac{m(m-1)}{2}$ basis elements H_a, $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$

$(1 \leqslant a, b \leqslant n)$ and in odd dimension $m = 2n+1$ these basis elements are extended to a full basis of $\mathfrak{so}(m, \mathbb{C})$ by $2n$ extra elements U_j and V_j, $1 \leqslant j \leqslant n$:

$$\mathfrak{so}(2n, \mathbb{C}) = \mathrm{span}_{\mathbb{C}} \{H_a, X_{a,b}, Y_{a,b}, Z_{a,b}, 1 \leqslant a, b \leqslant n,\ a \neq b\},$$
$$\mathfrak{so}(2n+1, \mathbb{C}) = \mathrm{span}_{\mathbb{C}} \{H_a, X_{a,b}, Y_{a,b}, Z_{a,b}, U_a, V_a, 1 \leqslant a, b \leqslant n,\ a \neq b\}.$$

These basis elements are the root vectors of the adjoint representation of $\mathfrak{so}(m, \mathbb{C})$ in accordance to [7, 9].

The Cartan subalgebra can be chosen as $\mathfrak{h} = \{H_a, 1 \leqslant a \leqslant n\}$, independent of the parity of the dimension, i.e. $\mathfrak{so}(2n, \mathbb{C})$ and $\mathfrak{so}(2n+1, \mathbb{C})$ are both Lie algebras of rank n. The roots of $\mathfrak{so}(m, \mathbb{C})$ (see also [10]) are determined by considering the adjoint representation:

$$\begin{aligned}
[H_s, Y_{a,b}] &= (\delta_{sa} + \delta_{sb})\, Y_{a,b} = ((L_a + L_b)\,(H_s))\, Y_{a,b} \\
[H_s, X_{a,b}] &= (\delta_{sa} - \delta_{sb})\, X_{a,b} = ((L_a - L_b)\,(H_s))\, X_{a,b}, \\
[H_s, Z_{a,b}] &= -\,(\delta_{sa} + \delta_{sb})\, Z_{a,b} = ((-L_a - L_b)\,(H_s))\, Z_{a,b}, \\
[H_s, U_a] &= \delta_{sa}\, U_a = (L_a(H_s))\, U_a, \\
[H_s, V_a] &= -\delta_{sa}\, U_a = (-L_a(H_s))\, U_a,
\end{aligned}$$

where $\{L_a, 1 \leqslant a \leqslant n\}$ is a basis of the dual vector space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$, i.e. $L_a\,(H_b) = \delta_{a,b}$. Note in particular that the Cartan subalgebra elements H_a can be calculated by taking the commutator of a positive root vector with its corresponding negative root vector:

$$[Y_{a,b}, Z_{a,b}] = -H_a - H_b, \qquad [X_{a,b}, X_{b,a}] = H_a - H_b.$$

We thus deduce the following roots and root vectors.

$m = 2n$

root	root vector
$L_a - L_b$	$X_{a,b}$
$L_a + L_b$	$Y_{a,b}$
$-L_a - L_b$	$Z_{a,b}$

$m = 2n+1$

root	root vector
$L_a - L_b$	$X_{a,b}$
$L_a + L_b$	$Y_{a,b}$
$-L_a - L_b$	$Z_{a,b}$
L_a	U_a
$-L_a$	V_a

To make a distinction between positive and negative roots, we consider the linear functional $l : \mathfrak{h}^* \to \mathbb{R}$ defined by fixing n different real numbers c_i such that for all $a_i \in \mathbb{R}$:

$$l\,(a_1\, L_1 + \ldots + a_n\, L_n) = a_1\, c_1 + \ldots + a_n\, c_n.$$

We choose the constants c_i such that the ordering $c_1 > c_2 > \ldots > c_n > 0$ is satisfied. With this convention, the positive roots in even dimension, i.e. roots α for which $l(\alpha) > 0$, are given by

$$\{L_a + L_b : 1 \leqslant a \neq b \leqslant n\} \cup \{L_a - L_b : 1 \leqslant a < b \leqslant n) .$$

The negative roots, i.e. roots α for which $l(\alpha) < 0$, are given by

$$\{-L_a - L_b : 1 \leqslant a \neq b \leqslant n\} \cup \{L_a - L_b : 1 \leqslant b < a \leqslant n) .$$

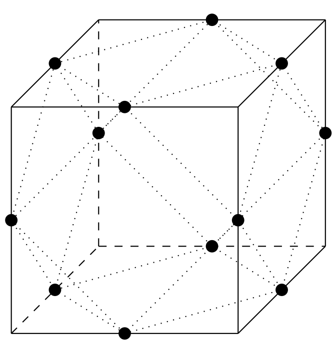
In odd dimension, one finds positive roots

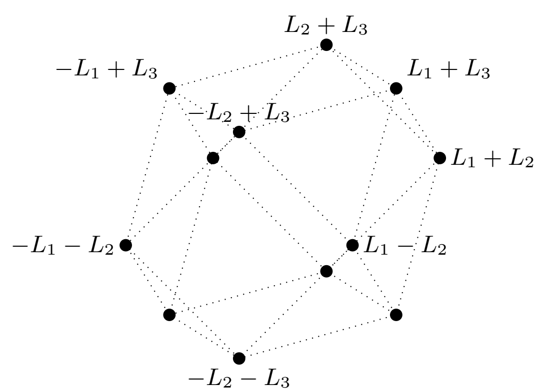
$$\{L_a + L_b : 1 \leqslant a \neq b \leqslant n\} \cup \{L_a - L_b : 1 \leqslant a < b \leqslant n) \cup \{L_a : 1 \leqslant a \leqslant n\}$$

and negative roots

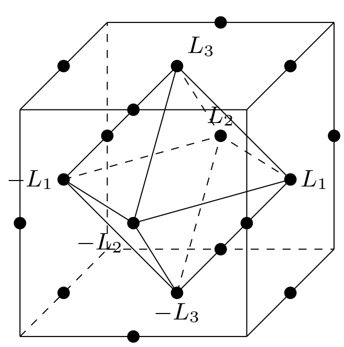
$$\{-L_a - L_b : 1 \leqslant a \neq b \leqslant n\} \cup \{L_a - L_b : 1 \leqslant b < a \leqslant n) \cup \{-L_a : 1 \leqslant a \leqslant n\} .$$

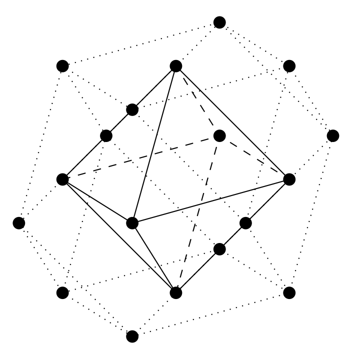
As illustrative examples, we depict the root diagrams of $\mathfrak{so}(6, \mathbb{C})$:





and of $\mathfrak{so}(7, \mathbb{C})$:





In [5], we introduced the algebra $\mathfrak{so}(m,\mathbb{C})$ (up to an isomorphism) in the discrete Clifford analysis context. The generators of $\mathfrak{so}(m,\mathbb{C})$ were not given in terms of the root vectors and Cartan subalgebra, but rather by the generators $\{\Omega_{a,b} : 1 \leqslant a \neq b \leqslant m\}$ introduced in Definition 1, satisfying the defining relations of $\mathfrak{so}(m,\mathbb{C})$:

$$[\Omega_{a,b}, \Omega_{c,d}] = \delta_{a,d}\, \Omega_{b,c} + \delta_{b,c}\, \Omega_{a,d} - \delta_{a,c}\, \Omega_{b,d} - \delta_{b,d}\, \Omega_{a,c}. \tag{1}$$

In the following sections, we will re-establish the orthogonal Lie algebra in the discrete Clifford analysis setting, but now by determining the explicit expressions of the root vectors and Cartan subalgebra.

4 Decomposition of $\mathcal{H}_k$ in irreducible representations

The space of discrete harmonic Clifford-valued homogeneous polynomials $\mathcal{H}_k$ is a representation for $\mathfrak{so}(m,\mathbb{C})$. To see this, we again consider the operators $\Omega_{a,b} : \mathcal{H}_k \to \mathcal{H}_k$:

$$\Omega_{a,b}(H_k) = L_{a,b}\, H_k\, e_b\, e_a = (\xi_b\, \partial_a + \xi_a\, \partial_b)\, H_k\, e_b\, e_a, \quad a \neq b \qquad \text{and} \qquad \Omega_{a,a} = 0.$$

By calculating the dimension, we immediately may conclude that this representation is just a model for the irreducible representation with highest weight $(k, 0, \dots, 0)$ like the classical case:

$$\dim_{\mathbb{C}}(\mathcal{H}_k) = \dim_{\mathbb{C}_{2m}}(\mathcal{H}_k)\, \dim_{\mathbb{C}}(\mathbb{C}_{2m}) = 2^{2m}\left(\binom{k+m-1}{k} - \binom{k+m-3}{k}\right),$$

while

$$\dim_{\mathbb{C}}(k, 0, \dots, 0) = \left(\binom{k+m-1}{k} - \binom{k+m-3}{k}\right).$$

Here $(k, 0, \dots, 0)$ represents the irreducible representation of $\mathfrak{so}(m,\mathbb{C})$ with highest weight $(k, 0, \dots, 0)$. This means that $\mathcal{H}_k$ is probably reducible. The remainder of this article is exactly this decomposition into irreducible representations.

Classically, one considers the scalar-valued harmonic polynomials as an irreducible representation of $\mathfrak{so}(m,\mathbb{C})$ within the space of Clifford-valued harmonic polynomials. However, note that in the discrete setting, due to the addition of the basis elements $e_b\, e_a$ in the definition of the operators $\Omega_{a,b}$, and the fact that $L_{a,b}$ itself is not scalar, the operators $\Omega_{a,b}$ are no longer scalar. Hence the subspace of $\mathcal{H}_k$ of scalar harmonics, i.e. harmonic functions that have scalar Taylor coefficients, is *not* an invariant under their action. To arrive at irreducible representations of $\mathfrak{so}(m,\mathbb{C})$ within the space of *Clifford-valued* discrete harmonics $\mathcal{H}_k$, we must thus reconsider our approach. We will do this by introducing an appropriate idempotent with which we multiply our harmonics from the right. To determine which idempotent is appropriate, we first consider the analogues of the root system of $\mathfrak{so}(m,\mathbb{C})$ in our discrete Clifford setting.

4.1 Even dimension $m = 2n$

Definition 2. *We define the operators H_a, $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b} \in \mathfrak{so}(m, \mathbb{C})$:*

$$\begin{aligned}
H_a &= i\,\Omega_{2a-1,2a}, \qquad 1 \leqslant a \leqslant n,\\
X_{a,b} &= \frac{1}{2}\left(\Omega_{2a-1,2b-1} + i\,\Omega_{2a-1,2b} - i\,\Omega_{2a,2b-1} + \Omega_{2a,2b}\right),\\
Y_{a,b} &= \frac{1}{2}\left(\Omega_{2a-1,2b-1} - i\,\Omega_{2a-1,2b} - i\,\Omega_{2a,2b-1} - \Omega_{2a,2b}\right),\\
Z_{a,b} &= \frac{1}{2}\left(\Omega_{2a-1,2b-1} + i\,\Omega_{2a-1,2b} + i\,\Omega_{2a,2b-1} - \Omega_{2a,2b}\right), \qquad 1 \leqslant a, b \leqslant n.
\end{aligned}$$

Note that, because $\Omega_{a,b} = -\Omega_{b,a}$, we find that $Y_{b,a} = -Y_{a,b}$. Furthermore, $\Omega_{a,a} = 0$ implies that $Y_{a,a} = 0$. The same holds for $Z_{a,b}$. For $X_{a,b}$, we find that $X_{b,a} \neq X_{a,b}$ and that $X_{a,a} = H_a$ hence we consider all couples (a, b) with $a \neq b$.

Remark 1. *Note that we can reconstruct the original operators $\Omega_{a,b}$ as $\Omega_{2a-1,2a} = -i\,H_a$ and for $a \neq b$:*

$$\begin{aligned}
2\,\Omega_{2a-1,2b-1} &= X_{a,b} - X_{b,a} + Y_{a,b} + Z_{a,b},\\
-2i\,\Omega_{2a,2b-1} &= X_{a,b} + X_{b,a} + Y_{a,b} - Z_{a,b},\\
2i\,\Omega_{2a-1,2b} &= X_{a,b} + X_{b,a} - Y_{a,b} + Z_{a,b},\\
2\,\Omega_{2a,2b} &= X_{a,b} - X_{b,a} - Y_{a,b} - Z_{a,b}.
\end{aligned}$$

We will now show that these operators indeed show the expected commutator relations associated with the root system of $\mathfrak{so}(m, \mathbb{C})$:

Lemma 1. *The following commutator relations hold:*

$$\begin{aligned}
[H_j, Y_{a,b}] &= (\delta_{ja} + \delta_{jb})\,Y_{a,b} = (L_a + L_b)\,(H_j)\,Y_{a,b},\\
[H_j, X_{a,b}] &= (\delta_{ja} - \delta_{jb})\,X_{a,b} = (L_a - L_b)\,(H_j)\,X_{a,b},\\
[H_j, Z_{a,b}] &= -\,(\delta_{ja} + \delta_{jb})\,Z_{a,b} = -\,(L_a + L_b)\,(H_j)\,Z_{a,b},\\
[Y_{a,b}, Z_{c,d}] &= \delta_{ad}\,X_{b,c} + \delta_{bc}\,X_{a,d} - \delta_{ac}\,X_{b,d} - \delta_{bd}\,X_{a,c}, \qquad [X_{a,b}, Y_{c,d}] = \delta_{bc}\,Y_{a,d} - \delta_{bd}\,Y_{a,c},\\
[Y_{a,b}, Y_{c,d}] &= 0, \qquad [X_{a,b}, Z_{c,d}] = \delta_{ad}\,Z_{b,c} - \delta_{ac}\,Z_{b,d},\\
[Z_{a,b}, Z_{c,d}] &= 0, \qquad [X_{a,b}, X_{c,d}] = \delta_{bc}\,X_{a,d} - \delta_{ad}\,X_{c,b}.
\end{aligned}$$

In particular, $X_{a,b}$, $a < b$ resp. $Y_{a,b}$ are root vectors corresponding to the positive roots $L_a - L_b$, resp. $L_a + L_b$. Furthermore, $X_{a,b}$ with $a > b$ and $Z_{a,b}$ are root vectors corresponding to the negative roots $L_a - L_b$ resp. $-L_a - L_b$.

Proof. We will only write down the commutator relation $[H_j, Y_{a,b}]$ and $[X_{a,b}, Y_{c,d}]$ here, the other ones can be proven in a similar fashion.

$$[H_j, Y_{a,b}] = \frac{1}{2}\left[i\,\Omega_{2j-1,2j}, \Omega_{2a-1,2b-1} - i\,\Omega_{2a-1,2b} - i\,\Omega_{2a,2b-1} - \Omega_{2a,2b}\right].$$

Applying the commutator rule (1) results in:

$$\begin{aligned}[H_j, Y_{a,b}] &= \frac{i}{2}\,(\delta_{jb}\,\Omega_{2j,2a-1} - \delta_{ja}\,\Omega_{2j,2b-1}) + \frac{1}{2}\,(-\delta_{jb}\,\Omega_{2j-1,2a-1} - \delta_{ja}\,\Omega_{2j,2b}) \\ &+ \frac{1}{2}\,(\delta_{ja}\,\Omega_{2j-1,2b-1} + \delta_{jb}\,\Omega_{2j,2a}) - \frac{i}{2}\,(-\delta_{jb}\,\Omega_{2j-1,2a} + \delta_{ja}\,\Omega_{2j-1,2b}) \\ &= \delta_{ja}\,Y_{j,b} + \delta_{jb}\,Y_{a,j} = (\delta_{ja} + \delta_{jb})\,Y_{a,b}.\end{aligned}$$

For the second statement, we again apply (1):

$$\begin{aligned}4\,[X_{a,b}, Y_{c,d}] &= 2\,\delta_{bc}\,(\Omega_{2a-1,2d-1} - i\,\Omega_{2a-1,2d} - i\,\Omega_{2a,2d-1} - \Omega_{2a,2d}) \\ &\quad - 2\,\delta_{bd}\,(\Omega_{2a-1,2c-1} - i\,\Omega_{2a-1,2c} - \Omega_{2a,2c} - i\,\Omega_{2a,2c-1}) \\ &= 2\,\delta_{bc}\,Y_{a,d} - 2\,\delta_{bd}\,Y_{a,c}.\end{aligned}$$

□

In this discrete setting, the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{so}(m, \mathbb{C})$ is given by

$$\mathfrak{h} = \{H_a,\ 1 \leqslant a \leqslant n\}\,.$$

As the Cartan elements mutually commute, their action on any representation of $\mathfrak{so}(m, \mathbb{C})$ can be diagonalised simultaneously, since $\mathfrak{so}(m, \mathbb{C})$ is semi-simple, $m > 2$. Any finite-dimensional representation $\mathbb{V}_\mu$ of the Lie algebra $\mathfrak{so}(m, \mathbb{C})$ may thus be decomposed as eigenspaces for the subalgebra $\mathfrak{h}$. The set of n eigenvalues of such an eigenspace is also known as the weight of the considered eigenspace and the eigenspace itself is called weight space. We may decompose $\mathbb{V}_\mu$ according to a finite set of weights W:

$$\mathbb{V}_\mu = \bigoplus_{\lambda \in W} V_\lambda$$

where $V_\lambda = \{P \in \mathbb{V}_\mu : H_a\,P = \ell_i\,P,\ 1 \leqslant a \leqslant n\}$, for all $\lambda = (\ell_1, \ldots, \ell_n) \in W$.

In particular, we consider the decomposition of the representation $\mathcal{H}_k$. If we are thus to introduce an idempotent I and a harmonic polynomial P_k such that the space $\mathrm{span}_{\mathbb{C}}\,\{P_k\,I\}$ is a weight space of a representation of the orthogonal algebra with weight $(\ell_1, \ldots, \ell_n)$, it must certainly hold that $H_a\,P_k\,I = \ell_a\,P_k\,I$. Therefor, the necessary idempotents must satisfy

$$\exists\, c_a \in \mathbb{C} : \qquad I\,e_{2a-1}\,e_{2a} = c_a\,I.$$

Definition 3. *For $1 \leqslant s \leqslant n$, we denote the following Clifford elements*

$$\begin{aligned} I^{\pm}_{2s-1} &= \left(\mathbf{e}^{+}_{2s-1}\mathbf{e}^{-}_{2s-1} \pm \mathbf{e}^{+}_{2s-1}\right), & I^{\pm}_{2s} &= \left(\mathbf{e}^{+}_{2s}\mathbf{e}^{-}_{2s} \pm i\,\mathbf{e}^{+}_{2s}\right), \\ K^{\pm}_{2s-1} &= \left(\mathbf{e}^{-}_{2s-1}\mathbf{e}^{+}_{2s-1} \pm \mathbf{e}^{-}_{2s-1}\right), & K^{\pm}_{2s} &= \left(\mathbf{e}^{-}_{2s}\mathbf{e}^{+}_{2s} \pm i\,\mathbf{e}^{-}_{2s}\right). \end{aligned}$$

Lemma 2. *Let* $I_s = I^+_{2s-1}\, I^-_{2s} = \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)$ *for* $1 \leqslant s \leqslant n$. *Then the Clifford element*

$$I = \prod_{s=1}^{n} I_s = \prod_{s=1}^{n} \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)$$

is an idempotent ($I^2 = I$) *and it satisfies*

$$I\, e_{2s-1}e_{2s} = i\, I, \quad \forall\, 1 \leqslant s \leqslant n.$$

Proof. Note that

$$\begin{aligned}
I^\pm_{2s-1}\, e_{2s-1} &= \left(\mathbf{e}^+_{2s-1} \pm \mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1}\right) = \pm\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} \pm \mathbf{e}^+_{2s-1}\right) = \pm\, I^\pm_{2s-1},\\
K^\pm_{2s-1}\, e_{2s-1} &= \left(\mathbf{e}^-_{2s-1} \pm \mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1}\right) = \pm\left(\mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1} \pm \mathbf{e}^-_{2s-1}\right) = \pm\, K^\pm_{2s-1},\\
I^\pm_{2s}\, e_{2s} &= \left(\mathbf{e}^+_{2s} \pm i\,\mathbf{e}^+_{2s}\mathbf{e}^-_{2s}\right) = \pm\, i\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} \mp i\,\mathbf{e}^+_{2s}\right) = \pm\, i\, I^\mp_{2s},\\
K^\pm_{2s}\, e_{2s} &= \left(\mathbf{e}^-_{2s} \pm i\,\mathbf{e}^-_{2s}\mathbf{e}^+_{2s}\right) = \pm\, i\left(\mathbf{e}^-_{2s}\mathbf{e}^+_{2s} \mp i\,\mathbf{e}^-_{2s}\right) = \pm\, i\, K^\mp_{2s}.
\end{aligned}$$

We start with the second statement. Choose $s = 1$, the general proof is similar:

$$\begin{aligned}
I\, e_1 e_2 &= I^+_1\, I^-_2\, e_1\, e_2 \prod_{s=2}^{n}\left(I^+_{2s-1}\, I^-_{2s}\right) = I^+_1\, e_1\, I^+_2\, e_2 \prod_{s=2}^{n}\left(I^+_{2s-1}\, I^-_{2s}\right) = i\, I^+_1\, I^-_2 \prod_{s=2}^{n}\left(I^+_{2s-1}\, I^-_{2s}\right)\\
&= i\, I.
\end{aligned}$$

The idempotency also holds:

$$I^2 = \left(\prod_{s=1}^{n} I_s\right)^2 = \left(\prod_{s=1}^{n}\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)\right)^2.$$

When we consider $s = 1$, $I^+_1 = \left(\mathbf{e}^+_1\mathbf{e}^-_1 + \mathbf{e}^+_1\right)$ has a part $\mathbf{e}^+_1\mathbf{e}^-_1$ that commutes with all $I^\pm_j$, $j \neq 1$, and a part $\mathbf{e}^+_1$ that does not. However, as $\mathbf{e}^+_1\, I^+_1 = \mathbf{e}^+_1\left(\mathbf{e}^+_1\mathbf{e}^-_1 + \mathbf{e}^+_1\right) = 0$ (because of the isotropy of $\mathbf{e}^+_1$), this part does not need to be taken into consideration. We only need to determine the result of the commutative part:

$$\begin{aligned}
I^2 &= \prod_{p=1}^{n}\left(\mathbf{e}^+_{2p-1}\mathbf{e}^-_{2p-1}\,\mathbf{e}^+_{2p}\mathbf{e}^-_{2p}\right)\prod_{s=1}^{n}\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)\\
&= \prod_{s=1}^{n}\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1}\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\mathbf{e}^+_{2s}\mathbf{e}^-_{2s}\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)\\
&= \prod_{s=1}^{n}\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)\\
&= I.
\end{aligned}$$

□

Lemma 3. *If we replace any (or multiple) I^+_{2s-1} with I^-_{2s-1}, K^+_{2s-1} or K^-_{2s-1}, the resulting Clifford element is still an idempotent. The same holds if we replace any (or multiple) terms I^-_{2s} by I^+_{2s} or $K^\pm_{2s}$. Furthermore*

$$\begin{aligned}
I^\pm_{2s-1} I^\pm_{2s}\, e_{2s-1}\, e_{2s} &= \pm(\mp i)\, I^\pm_{2s-1} I^\pm_{2s},\\
I^\pm_{2s-1}\, K^\pm_{2s}\, e_{2s-1}\, e_{2s} &= \pm(\mp i)\, I^\pm_{2s-1}\, K^\pm_{2s},\\
K^\pm_{2s-1}\, I^\pm_{2s}\, e_{2s-1}\, e_{2s} &= \pm(\mp i)\, K^\pm_{2s-1}\, I^\pm_{2s},\\
K^\pm_{2s-1}\, K^\pm_{2s}\, e_{2s-1}\, e_{2s} &= \pm(\mp i)\, K^\pm_{2s-1}\, K^\pm_{2s}.
\end{aligned}$$

Proof. We here only consider the combination of $I^\pm_{2s-1}$ with $I^\pm_{2s}$. The proofs of the other combinations are similar. So let

$$I = \prod_{s=1}^{n} I^\pm_{2s-1} I^\pm_{2s} = \prod_{s=1}^{n} \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} \pm \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} \pm i\,\mathbf{e}^+_{2s}\right).$$

Again because of the isotropy of $\mathbf{e}^+_i$ and of $\mathbf{e}^-_i$, in each case, we only need to consider the commutative part:

$$\begin{aligned}
I^2 &= \prod_{p=1}^{n} \left(\mathbf{e}^+_{2p-1}\mathbf{e}^-_{2p-1}\, \mathbf{e}^+_{2p}\mathbf{e}^-_{2p}\right) \prod_{s=1}^{n} \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} \pm \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} \pm i\,\mathbf{e}^+_{2s}\right)\\
&= \prod_{s=1}^{n} \mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1}\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} \pm \mathbf{e}^+_{2s-1}\right) \mathbf{e}^+_{2s}\mathbf{e}^-_{2s}\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} \pm i\,\mathbf{e}^+_{2s}\right)\\
&= \prod_{s=1}^{n} \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} \pm \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} \pm i\,\mathbf{e}^+_{2s}\right) = I.
\end{aligned}$$

Furthermore $I^\pm_1 I^\pm_2\, e_1\, e_2 = I^\pm_1\, e_1\, I^\mp_2\, e_2 = \pm(\mp i)\, I^\pm_1 I^\pm_2$ and similar for other $1 \leqslant s \leqslant n$. $\square$

Remark 2. *By combining the different idempotents, we can form all basis elements of the Clifford algebra. Indeed*

$$\begin{aligned}
I^+_{2s-1} + I^-_{2s-1} &= \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right) + \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} - \mathbf{e}^+_{2s-1}\right) = 2\,\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1}\\
K^+_{2s-1} + K^-_{2s-1} &= \left(\mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1} + \mathbf{e}^-_{2s-1}\right) + \left(\mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1} - \mathbf{e}^-_{2s-1}\right) = 2\,\mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1}\\
I^+_{2s-1} - I^-_{2s-1} &= \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right) - \left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} - \mathbf{e}^+_{2s-1}\right) = 2\,\mathbf{e}^+_{2s-1}\\
K^+_{2s-1} - K^-_{2s-1} &= \left(\mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1} + \mathbf{e}^-_{2s-1}\right) - \left(\mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1} - \mathbf{e}^-_{2s-1}\right) = 2\,\mathbf{e}^-_{2s-1}
\end{aligned}$$

and since $1 = \mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^-_{2s-1}\mathbf{e}^+_{2s-1}$, we can also produce scalars. Similar combinations are used to arrive at $\mathbf{e}^+_{2s}\mathbf{e}^-_{2s}$, $\mathbf{e}^-_{2s}\mathbf{e}^+_{2s}$ and $\mathbf{e}^\pm_{2s}$.

We find that $\mathcal{H}_k$ decomposes into the direct sum of $4^m = 2^{2m}$ subspaces $\mathcal{H}_k\, I$, where the I runs over all idempotents mentioned above. In the following sections, we will always denote the idempotent $\prod_{s=1}^{n}\left(\mathbf{e}^+_{2s-1}\mathbf{e}^-_{2s-1} + \mathbf{e}^+_{2s-1}\right)\left(\mathbf{e}^+_{2s}\mathbf{e}^-_{2s} - i\,\mathbf{e}^+_{2s}\right)$ by I .

We already established that $\mathcal{H}_{2k}$ is a representation of $\mathfrak{so}(2n, \mathbb{C})$. This representation is reducible. An eigenspace of all generators of $\mathfrak{h}$ is given by $\mathrm{span}_{\mathbb{C}} \{f_{2k} I\}$ where

$$f_{2k} = ((\xi_2 + \xi_1)(\xi_2 - \xi_1))^k .$$

If we consider the representation $\mathcal{H}_{2k+1}$, then the corresponding weight space of all elements of the Cartan subalgebra is given by $\mathrm{span}_{\mathbb{C}} \{f_{2k+1} I\}$ where

$$f_{2k+1} = (\xi_2 + \xi_1)((\xi_2 - \xi_1)(\xi_2 + \xi_1))^k .$$

Lemma 4. *The subspace* $\mathrm{span}_{\mathbb{C}} \{f_{2k} I\}$ *resp.* $\mathrm{span}_{\mathbb{C}} \{f_{2k+1} I\}$ *is an eigenspace of all generators of* $\mathfrak{h}$*, and can hence be seen as (part of) a weight space of an* $\mathfrak{so}(m, \mathbb{C})$*-representation with weight* $(2k, 0, \ldots, 0)$*, resp.* $(2k+1, 0, \ldots, 0)$*.*

Proof. Since the generating elements f_{2k} and f_{2k+1} only contain ξ_1 and ξ_2, we only need to consider $H_1 = i\,\Omega_{1,2}$. The generating elements will automatically vanish under the action of the other $\mathfrak{h}$-elements. It was previously established (see [4], proof of Prop. 2) that

$$\begin{aligned} \partial_1 f_{2k} = \partial_2 f_{2k} &= 2k\,(\xi_2 - \xi_1)((\xi_2 + \xi_1)(\xi_2 - \xi_1))^{k-1} , \\ \partial_1 f_{2k+1} = \partial_2 f_{2k+1} &= (2k+1)((\xi_2 - \xi_1)(\xi_2 + \xi_1))^k . \end{aligned}$$

From this, the action of $H_1 = i\,\Omega_{12}$ follows easily:

$$\begin{aligned} H_1\, f_{2k}\, I &= i\,(\xi_1\,\partial_2 + \xi_2\,\partial_1)\, f_{2k}\, I\, e_2 e_1 \\ &= i\,(2k)(\xi_1 + \xi_2)(\xi_2 - \xi_1)((\xi_2 + \xi_1)(\xi_2 - \xi_1))^{k-1}\,(-i\,I) \\ &= (2k)((\xi_2 + \xi_1)(\xi_2 + \xi_1))^k\, I \\ &= 2k\, f_{2k}\, I, \end{aligned}$$

while

$$\begin{aligned} H_1\, f_{2k+1}\, I &= i\,(\xi_1\,\partial_2 + \xi_2\,\partial_1)\, f_{2k+1}\, I\, e_2 e_1 \\ &= i\,(2k+1)(\xi_1 + \xi_2)((\xi_2 - \xi_1)(\xi_2 + \xi_1))^k\,(-i\,I) \\ &= (2k+1)\, f_{2k+1}\, I. \end{aligned}$$

□

Lemma 5. *The 1-dimensional spaces* $\mathrm{span}_{\mathbb{C}} \{f_{2k} I\}$ *and* $\mathrm{span}_{\mathbb{C}} \{f_{2k+1} I\}$ *vanish under the action of the positive roots, i.e.*

$$\begin{aligned} Y_{a,b}\,(f_{2k}\, I) &= 0,\ \forall (a, b),\ a \neq b, \\ X_{a,b}\,(f_{2k}\, I) &= 0,\ \forall (a, b),\ a < b, \\ Y_{a,b}\,(f_{2k+1}\, I) &= 0,\ \forall (a, b),\ a \neq b, \\ X_{a,b}\,(f_{2k+1}\, I) &= 0,\ \forall (a, b),\ a < b. \end{aligned}$$

Proof. Since f_{2k} and f_{2k+1} only contain ξ_2 and ξ_1, we only need to consider (a,b) with $b=1$ and $a>1$:

$$\begin{aligned} 2\,Y_{a,1}\,(f_{2k}\,I) &= (\Omega_{2a-1,1} - i\,\Omega_{2a-1,2} - i\,\Omega_{2a,1} - \Omega_{2a,2})\,f_{2k}\,I \\ &= \xi_{2a-1}\,(\partial_1 f_{2k})\,I\,e_1 e_{2a-1} - i\,\xi_{2a-1}\,(\partial_2 f_{2k})\,I\,e_2 e_{2a-1} \\ &\quad - i\,\xi_{2a}\,(\partial_1 f_{2k})\,I\,e_1 e_{2a} - \xi_{2a}\,(\partial_2 f_{2k})\,I\,e_2 e_{2a}. \end{aligned}$$

We thus consider

$$\begin{aligned} I\,e_1 e_{2a} &= I_1^+\,I_2^-\,\ldots\,I_{2a-1}^+\,I_{2a}^-\,e_1\,e_{2a}\,I_{2a+1}^+\,I_{2a+2}^-\ldots I_{2n-1}^+ I_{2n}^- \\ &= I_1^+\,e_1\,I_2^{\color{red}+}\,I_3^{\color{red}-}\,\ldots\,I_{2a-2}^{\color{red}+}\,I_{2a-1}^{\color{red}-}\,I_{2a}^{\color{red}+}\,e_{2a}\,I_{2a+1}^+\,I_{2a+2}^-\ldots I_{2n-1}^+ I_{2n}^- \\ &= {\color{red}i}\,I_1^+\,I_2^{\color{red}+}\,I_3^{\color{red}-}\,\ldots\,I_{2a-2}^{\color{red}+}\,I_{2a-1}^{\color{red}-}\,I_{2a}^-\,I_{2a+1}^+\,I_{2a+2}^-\ldots I_{2n-1}^+ I_{2n}^-. \end{aligned}$$

Denote for now

$$I^{1,a} = I_1^+\;\overbrace{I_2^{\color{red}+}\,I_3^{\color{red}-}\,\ldots\,I_{2a-2}^{\color{red}+}\,I_{2a-1}^{\color{red}-}}^{\color{red}\text{changed sign of second term}}\;I_{2a}^-\,I_{2a+1}^+\,I_{2a+2}^-\ldots I_{2n-1}^+ I_{2n}^-.$$

Then $I\,e_1\,e_{2a} = i\,I^{1,a}$ and

$$\begin{aligned} I\,e_1\,e_{2a-1} &= I_1^+\,I_2^-\,\ldots\,I_{2a-1}^+\,e_1\,e_{2a-1}\,I_{2a}^-\ldots I_{2n-1}^+\,I_{2n}^- = I_1^+\,e_1\,I_2^{\color{red}+}\ldots I_{2a-1}^{\color{red}-}\,e_{2a-1}\,I_{2a}^-\ldots I_{2n-1}^+\,I_{2n}^- \\ &= {\color{red}-}\,I_1^+\,I_2^{\color{red}+}\ldots I_{2a-2}^{\color{red}+}\,I_{2a-1}^{\color{red}-}\,I_{2a}^-\ldots I_{2n-1}^+\,I_{2n}^- \\ &= {\color{red}-}I^{1,a}, \\ I\,e_2\,e_{2a} &= I_1^+\,I_2^-\,\ldots\,I_{2a-1}^+\,I_{2a}^-\,e_2\,e_{2a}\,I_{2a+1}^+\ldots I_{2n-1}^+\,I_{2n}^- \\ &= I_1^+\,I_2^-\,e_2\,I_3^{\color{red}-}\,\ldots I_{2a-1}^{\color{red}-}\,I_{2a}^{\color{red}+}\,e_{2a}\,I_{2a+1}^+\ldots I_{2n-1}^+\,I_{2n}^- \\ &= {\color{red}(-i)\,i}\,I_1^+\,I_2^{\color{red}+}\,I_3^{\color{red}-}\,\ldots I_{2a-1}^{\color{red}-}\,I_{2a}^-\,I_{2a+1}^+\ldots\,I_{2n-1}^+\,I_{2n}^- \\ &= I^{1,a}, \\ I\,e_2\,e_{2a-1} &= I_1^+\,I_2^-\,\ldots\,I_{2a-1}^+\,e_2\,e_{2a-1}\,I_{2a}^-\ldots I_{2n-1}^+\,I_{2n}^- = I_1^+\,I_2^-\,e_2\,I_3^{\color{red}-}\ldots I_{2a-1}^{\color{red}-}\,e_{2a-1}\,I_{2a}^-\ldots I_{2n-1}^+\,I_{2n}^- \\ &= {\color{red}(-1)\,(-i)}\,I_1^+\,I_2^{\color{red}+}\,I_3^{\color{red}-}\ldots I_{2a-2}^{\color{red}+}\,I_{2a-1}^{\color{red}-}\,I_{2a}^-\ldots I_{2n-1}^+\,I_{2n}^- \\ &= {\color{red}i}\,I^{1,a}. \end{aligned}$$

We recall that $\partial_1 f_{2k} = \partial_2 f_{2k}$ and combining all this, we find that

$$\begin{aligned} 2\,Y_{a,1}\,(f_{2k}\,I) &= -\xi_{2a-1}\,(\partial_1 f_{2k})\,I^{1,a} - i\,\xi_{2a-1}\,(\partial_1 f_{2k})\,\left(i\,I^{1,a}\right) - i\,\xi_{2a}\,(\partial_1 f_{2k})\,\left(i\,I^{1,a}\right) - \xi_{2a}\,(\partial_1 f_{2k})\,I^{1,a} \\ &= 0. \end{aligned}$$

The case of f_{2k+1} is completely similar.

For $X_{a,b}$ with $a < b$ we only need to consider the case where $a = 1$ and $1 < b$:

$$\begin{aligned}
2\, X_{1,b}\, (f_{2k}\, I) &= (\Omega_{1,2b-1} + i\,\Omega_{1,2b} - i\,\Omega_{2,2b-1} + \Omega_{2,2b})\, f_{2k}\, I \\
&= \xi_{2b-1}\, \partial_1 f_{2k}\, I\, e_{2b-1} e_1 + i\, \xi_{2b}\, \partial_1 f_{2k}\, I\, e_{2b} e_1 \\
&\quad - i\, \xi_{2b-1}\, \partial_2 f_{2k}\, I\, e_{2b-1} e_2 + \xi_{2b}\, \partial_2 f_{2k}\, I\, e_{2b} e_2 \\
&= \xi_{2b-1}\, \partial_1 f_{2k}\, I^{1,b} - i^2\, \xi_{2b}\, \partial_1 f_{2k}\, I^{1,b} + (-i)^2\, \xi_{2b-1}\, \partial_1 f_{2k}\, I^{1,b} - \xi_{2b}\, \partial_1 f_{2k}\, I^{1,b} \\
&= 0.
\end{aligned}$$

The case of f_{2k+1} is again completely similar. □

Corollary 1. *The space* $\mathrm{span}_{\mathbb{C}}\, \{f_k\, I\}$ *is a 1-dimensional highest weight space with weight* $(k, 0, \ldots, 0)$. *As such, it generates an irreducible representation of* $\mathfrak{so}(m, \mathbb{C})$, *see for example [7].*

We will from now on denote $(k) = (k, 0, \ldots, 0)$.

Remark 3. *The space* $\mathcal{H}_k\, I$ *is not a left* $\mathfrak{so}(m, \mathbb{C})$*-module, i.e. the image of the space* $\mathcal{H}_k\, I$ *under the action of a rotations* $\Omega_{a,b}$ *does not belong to* $\mathcal{H}_k\, I$, *but to some* $\mathcal{H}_k\, J$ *with* J *a different idempotent. As such, direct calculations with the given irreducible representations may become somewhat more trickier (although not impossible) than in the classical Clifford setting.*

Each space $\mathrm{span}_{\mathbb{C}}\, \{f_k\, J\}$, where J runs over all possible idempotents such that the highest weight is (k), generates an independent isomorphic irreducible representation since the heighest weight space of such an irreducible representation is one-dimensional. The element $f_k\, J$ of weight (k) can thus not be found in the irreducible representation spanned by another element $f_k\, J'$ (with J' a different idempotent).

Note that not every combination of f_k with an idempotent J delivers the weight vector (k). Half of all idempotents J delivers a weight space $\mathrm{span}_{\mathbb{C}}\, \{f_k\, J\}$ with weight (k); the other half delivers a weight space $\mathrm{span}_{\mathbb{C}}\, \{f_k\, J\}$ with weight $(-k)$ (which is in fact a lowest weight space, see remark below). However, in those cases, the vector $g_k\, J$ where we denoted

$$\begin{aligned}
g_{2k} &= ((\xi_2 - \xi_1)\, (\xi_2 + \xi_1))^k\,, \\
g_{2k+1} &= (\xi_2 - \xi_1)\, ((\xi_2 + \xi_1)\, (\xi_2 - \xi_1))^k
\end{aligned}$$

is a highest weight vector with weight (k) for our choice of positive root system. Again, these weight vectors will all generate independent isomorphic $\mathfrak{so}(m, \mathbb{C})$-representations. We thus get 2^{2m} different highest weight vectors which generate 2^{2m} different isomorphic irreducible representations.

Remark 4. *When the space* $\mathrm{span}_{\mathbb{C}}\, \{f_k\, J\}$ *spans a weight space with weight* $(-k)$, *the negative roots act trivially on this space, as one would expect.*

Each of these representations has dimension $\binom{k+m-1}{k} - \binom{k+m-3}{k}$ (see e.g. [7]). By considering all 2^{2m} idempotents and as

$$\begin{aligned}\dim_{\mathbb{C}}\left(\mathcal{H}_k(m,\mathbb{C}_{2m})\right) &= \left(\binom{k+m-1}{k} - \binom{k+m-3}{k}\right)\dim_{\mathbb{C}}(\mathbb{C}_{2m})\\ &= 2^{2m}\left(\binom{k+m-1}{k} - \binom{k+m-3}{k}\right),\end{aligned}$$

we find that we can decompose the space $\mathcal{H}_k(m,\mathbb{C}_{2m})$ into 2^{2m} irreducible isomorphic representations of $\mathfrak{so}(m,\mathbb{C})$.

4.2 Odd dimension $m = 2n+1$

In odd dimension, we extend the set of generators H_a, $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$ of the root system with the $2n$ mappings:

$$\begin{aligned}U_s &= \frac{1}{\sqrt{2}}\left(\Omega_{2s-1,m} - i\,\Omega_{2s,m}\right),\\ V_s &= \frac{1}{\sqrt{2}}\left(\Omega_{2s-1,m} + i\,\Omega_{2s,m}\right),\end{aligned}$$

where $1 \leqslant s \leqslant n$. With the addition of these $2n$ mappings, we are again able to reconstruct all original Ω_{ij}'s since $\sqrt{2}\,\Omega_{2s-1,m} = U_s + V_s$ and $-\sqrt{2}i\,\Omega_{2s,m} = U_s - V_s$.

Lemma 6. *For $1 \leqslant s,j \leqslant n$, it holds that*

$$\begin{aligned}[H_s, U_j] &= \delta_{sj}\,U_j = L_j(H_s)\,U_j,\\ [H_s, V_j] &= -\delta_{sj}\,V_j = -L_j(H_s)\,V_j.\end{aligned}$$

In particular, U_j is a root vector corresponding to the positive root L_j and V_j is a root vector corresponding with the negative root $-L_j$, $\forall 1 \leqslant j \leqslant n$.

Proof. Take $1 \leqslant s,j \leqslant n$:

$$\begin{aligned}\sqrt{2}\,[H_s, U_j] &= i\,[\Omega_{2s-1,2s}, \Omega_{2j-1,m}] + [\Omega_{2s-1,2s}, \Omega_{2j,m}] = i\,(-\delta_{sj}\,\Omega_{2s,m}) + (\delta_{sj}\,\Omega_{2s-1,m})\\ &= \sqrt{2}\,\delta_{sj}\,U_s.\end{aligned}$$

and

$$\begin{aligned}\sqrt{2}\,[H_s, V_j] &= i\,[\Omega_{2s-1,2s}, \Omega_{2j-1,m}] - [\Omega_{2s-1,2s}, \Omega_{2j,m}] = i\,(-\delta_{sj}\,\Omega_{2s,m}) - (\delta_{sj}\,\Omega_{2s-1,m})\\ &= -\sqrt{2}\,\delta_{sj}\,V_s.\end{aligned}$$

□

Lemma 7. *The operators U_j and V_j satisfy the following additional commutator relations with $X_{a,b}$, $Y_{a,b}$ and $Z_{a,b}$:*

$$\begin{aligned}
[U_j, X_{a,b}] &= -\delta_{jb}\, U_a, & [V_j, X_{a,b}] &= \delta_{ja}\, V_b,\\
[U_j, Y_{a,b}] &= 0, & [V_j, Y_{a,b}] &= \delta_{ja}\, U_b - \delta_{jb}\, U_a,\\
[U_j, Z_{a,b}] &= -\delta_{jb}\, V_a + \delta_{ja}\, V_b, & [V_j, Z_{a,b}] &= 0,\\
[U_j, U_\ell] &= -Y_{j,\ell},\ j \neq \ell, & [V_j, V_\ell] &= -Z_{j,\ell},\ j \neq \ell\\
[U_j, V_\ell] &= \begin{cases} -X_{j,\ell}, & j \neq \ell,\\ -H_j, & j = \ell. \end{cases}
\end{aligned}$$

Proof. We only show the first proof, as the other relations use similar arguments.

$$\begin{aligned}
\sqrt{2}\,[U_j, X_{a,b}] &= \delta_{jb}\,(\Omega_{m,2a-1} - i\,\Omega_{m,2a}) = \delta_{jb}\,(-\Omega_{2a-1,m} + i\,\Omega_{2a,m})\\
&= -\sqrt{2}\;\delta_{jb}\,U_a.
\end{aligned}$$

□

To establish highest weight vectors in the odd dimensional case, we further introduce

$$I_m^{\pm} = \left(\mathbf{e}_m^{+}\mathbf{e}_m^{-} \pm \mathbf{e}_m^{+}\right), \qquad K_m^{\pm} = \left(\mathbf{e}_m^{-}\mathbf{e}_m^{+} \pm \mathbf{e}_m^{-}\right)$$

and denote $I = \prod_{s=1}^{n}\left(I_{2s-1}^{+}\, I_{2s}^{-}\right) I_m^{+}$. Then the elements $f_{2k}\, I$ and $f_{2k+1}\, I$ are still weight vectors with weights $(2k, 0, \ldots, 0)$ resp. $(2k+1, 0, \ldots, 0)$ which vanish under the positive roots $Y_{a,b}$ and $X_{a,b}$ $(a < b)$. We only need to consider the positive roots involving the extra factor m.

Lemma 8. *The generators of the representation $f_{2k}\, I$ and $f_{2k+1}\, I$ vanish under the operator U_j, $1 \leqslant j \leqslant n$, i.e. $f_{2k}\, I$ and $f_{2k+1}\, I$ are highest weight vectors:*

$$U_j\,(f_{2k}\, I) = 0, \qquad U_j\,(f_{2k+1}\, I) = 0, \forall\, 1 \leqslant j \leqslant n.$$

Proof. Since

$$\sqrt{2}\, U_j\,(f_{2k}\, I) = (\Omega_{2j-1,m} - i\,\Omega_{2j,m})\, f_{2k}\, I$$

and f_{2k} contains only ξ_1 and ξ_2, we find that $U_j\,(f_{2k}\, I)$ will immediately be zero unless $j = 1$. Then

$$\sqrt{2}\, U_1\,(f_{2k}\, I) = \xi_m\, \partial_1\, f_{2k}\, I\, e_m\, e_1 - i\,\xi_m\, \partial_2 f_{2k}\, I\, e_m\, e_2 = 2k\,\xi_m\, g_{2k-1}\,(I\, e_m\, e_1 - i\, I\, e_m\, e_2)\,.$$

We complete the proof by noting that

$$\begin{aligned}
I\, e_m\, e_1 &= -I_1^{+}\; e_1\; I_2^{+}\, I_3^{-} \ldots I_{2n-1}^{-}\, I_{2n}^{+}\, I_m^{-}\, e_m = I_1^{+}\, I_2^{+}\, I_3^{-} \ldots I_{2n-1}^{-}\, I_{2n}^{+}\, I_m^{-},\\
I\, e_m\, e_2 &= -I_1^{+}\, I_2^{-}\;\; e_2\; I_3^{-} \ldots I_{2n-1}^{-}\, I_{2n}^{+}\, I_m^{-}\, e_m = (-i)\, I_1^{+}\, I_2^{+}\, I_3^{-} \ldots I_{2n-1}^{-}\, I_{2n}^{+}\, I_m^{-}.
\end{aligned}$$

The proof for $f_{2k+1}\, I$ is completely similar. □

Remark 5. *Let $m = 2n+1$. We compare the dimensions:*

$$\dim_{\mathbb{C}} \mathcal{H}_k = 2^{2m} \left(\binom{k+m-1}{k} - \binom{k+m-3}{k} \right)$$

On the other hand, we have found $4^{2n+1} = 2^{2m}$ different highest weight vectors and thus 2^{2m} isomorphic irreducible representations with combined dimension

$$2^{2m} \left(\binom{k+m-1}{k} - \binom{k+m-3}{k} \right).$$

We may thus conclude that $\mathcal{H}_k$ decomposes into 2^{2m} isomorphic irreducible representations of $\mathfrak{so}(m, \mathbb{C})$.

5 Conclusion and future research

The space $\mathcal{H}_k$ of discrete k-homogeneous harmonic polynomials is a reducible representation of $\mathfrak{so}(m, \mathbb{C})$, which can be decomposed into 2^{2m} isomorphic copies of irreducible $\mathfrak{so}(m, \mathbb{C})$-representations with highest weight $(k, 0, \ldots, 0)$. The discrete Clifford element

$$I = \prod_{s=1}^{n} \left(\mathbf{e}_{2s-1}^{+} \mathbf{e}_{2s-1}^{-} + \mathbf{e}_{2s-1}^{+} \right) \left(\mathbf{e}_{2s}^{+} \mathbf{e}_{2s}^{-} - i\, \mathbf{e}_{2s}^{+} \right)$$

is an idempotent. Let

$$f_{2k} = \left((\xi_2 + \xi_1)(\xi_2 - \xi_1) \right)^k,$$
$$f_{2k+1} = (\xi_2 + \xi_1) \left((\xi_2 - \xi_1)(\xi_2 + \xi_1) \right)^k$$

be discrete $2k$ (resp. $2k+1$)-homogeneous harmonic functions, then each subspace $\mathrm{span}_{\mathbb{C}} \{ f_j\, I \}$ generates an irreducible $\mathfrak{so}(m, \mathbb{C})$-representation under the action of the negative roots with highest weight $(j, 0, \ldots, 0)$ for j either $2k$ or $2k+1$.

In an upcoming paper, we decompose the space of discrete k-homogeneous monogenic polynomials in irreducible $\mathfrak{so}(m, \mathbb{C})$-representation, hereby creating a notion of discrete spinors.

6 Acknowledgments

The first author acknowledges the support of the Research Foundation - Flanders (FWO), grant no. FWO13\PDO\039.

References

[1] D. Constales, F. Sommen, P. Van Lancker, *Models for irreducible representations of Spin*(m), Adv. Appl. Clifford Algebras **11** No. S1, pp. 271-289 (2001).

[2] H. De Ridder, H. De Schepper, U. Kähler, F. Sommen. *Discrete function theory based on skew Weyl relations,* Proc. Amer. Math. Soc. 138, 2010, pp. 3241 – 3256.

[3] H. De Ridder, H. De Schepper, F. Sommen. *The Cauchy-Kovalevskaya Extension Theorem in Discrete Clifford Analysis,* Comm. Pure Appl. Math. **10 (4)**, 2011, pp. 1097 – 1109.

[4] H. De Ridder, H. De Schepper, F. Sommen, *Fueter Polynomials in Discrete Clifford Analysis*, Mathematische Zeitschrift **272 (1-2)**, 2012, pp. 253268.

[5] H. De Ridder, T. Raeymaekers, F. Sommen, $\mathfrak{so}(m, \mathbb{C})$*-representations in discrete Clifford analysis*, submitted for publication.

[6] H. De Schepper, F. Sommen, L. Van de Voorde. *A basic framework for discrete Clifford analysis*, Experimental Mathematics **18 (4)**, 2009, pp. 385 – 395.

[7] W. Fulton, J. Harris, *Representation theory*, Graduate Texts in Mathematics **129**, 1991, Springer-Verlag, New York. A first course, Readings in Mathematics.

[8] J. Gilbert, M.A.M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press, Cambridge (1991).

[9] J. Humphreys, *Introduction to Lie algebra and representation theory*, Springer-Verlag, New York (1972).

[10] A. W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics **140**, 1996, Birkhäuser Boston, Inc., Boston, MA.

[11] A.I. Molev, *Yangians and classical Lie algebras*, Mathematical surveys and monographs **143**, AMS Bookstore (2007).