Differential forms and Clifford analysis

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Abstract. In this paper we use a calculus of differential forms which is defined using an axiomatic approach. We then define integration of differential forms over chains in a new way and we present a short proof of Stokes' formula using distributional techniques. We also consider differential forms in Clifford analysis, vector differentials and their powers. This framework enables an easy proof for a Cauchy's formula on a k-surface. Finally, we discuss how to compute winding numbers in terms of the monogenic Cauchy kernel and the vector differentials with a new approach which does not involve cohomology of differential forms.

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1. Introduction

This paper is a continuation of our former papers [9, 10, 11, 12] in which the calculus of differential forms has been combined with the Clifford algebra. Using Clifford analysis techniques, and monogenic functions in particular, we were able to establish a Cauchy-type formula for the Dirac operator on surfaces (see [10]), a theory of monogenic differential forms allowing a cohomology theory (see [9, 12]) and a formula for the winding number of a k-cycle and a $(m-k-1)$-cycle in $\mathbb{R}^m$ (see [9]). This extends the work of Hodge [7] in which the homology of a domain is measured in terms of integrals over cycles of harmonic differential forms. To understand these ideas, one has to recall that the theory of monogenic functions in Clifford analysis deals with nullsolutions of the Dirac operator $\partial_{\underline{x}}$ in $\mathbb{R}^m$, which is a higher dimensional generalization of the theory of holomorphic functions in the plane. Consider a point p in the plane (or a number of points) and a closed Jordan curve (a 1-cycle) $\Gamma \subset \mathbb{C} \setminus \{p\}$; then the winding number of Γ around p is given by the Cauchy integral

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{dz}{z-p}$$

which is a special case of the residue formula

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)\,dz}{z-p}.$$

The analog Cauchy formula for monogenic functions has the form (see [1])

$$f(\underline{x}) = \int_{\partial C} E(\underline{u}-\underline{x})\,\sigma(d\underline{u})\,f(\underline{u})$$

where C is an open bounded set in $\mathbb{R}^m$, $\underline{x}\in C$, $E(\underline{u}-\underline{x})$ is the Cauchy kernel and $\sigma(d\underline{u})$ is a suitable $(m-1)$-form with values in a Clifford algebra that represents the oriented surface measure. Using this Cauchy formula in special cases, one can establish a formula for the winding number of an $(m-1)$-cycle around one or several points.

However, in $\mathbb{R}^m$ one can also consider k-cycles C_k and $(m-k-1)$-cycles C_{m-k-1} in $\mathbb{R}^m\setminus C_k$ for which there is a winding number that can be defined in terms of the intersection number; it cannot be measured in terms of monogenic functions right away. This makes it necessary to combine a calculus of differential forms with the theory of monogenic functions, as we do in this work.

The paper consists of 5 sections, besides this introduction. In Section 2, we define the calculus of differential forms from scratch using an axiomatic approach which is inspired by the use of differential forms in analysis. In Section 3 we define integration of differential forms over chains in a novel way which also includes partial integration operators that are anti-commuting. In Section 4 we present a short proof of Stokes' formula using distributional techniques. Section 5 is devoted to differential forms in Clifford analysis, starting with a short introduction to Clifford algebras and monogenic functions. Then we introduce the vector differential $d\underline{x}=\sum_{j=1}^m e_j\,dx_j$, that generalizes the complex differential $dz=dx+idy$, and its powers $d\underline{x}^k$ represent the oriented k-dimensional surface measure. This enables an easy proof for a Cauchy's formula on a k-surface. The final Section 6 is devoted to the calculation of the winding number in terms of the monogenic Cauchy kernel and the vector differentials $d\underline{x}$, $d\underline{u}$, etc. The formulas thus obtained are easier to present and understand than the ones presented in [9], moreover the approach is new and does not involve cohomology of differential forms.

2. Differential forms

Let $\Omega\subseteq\mathbb{R}^n$ be an open set, and let $\mathcal{C}^\infty(\Omega)$ be the ring of real (or complex)-valued smooth functions on Ω. We begin by defining the algebra of differential forms:

Definition 2.1. The algebra $\Lambda(C^\infty(\Omega))$ of smooth differential forms on Ω is defined as the smallest associative algebra over $\mathcal{C}^\infty(\Omega)$ satisfying the following axioms:

(A_{-1}) $\mathcal{C}^\infty(\Omega)\subset\Lambda(C^\infty(\Omega))$;

and there is a map $d: \Lambda(C^\infty(\Omega)) \to \Lambda(C^\infty(\Omega))$ such that

(A_0) $d1 = 0$;

(A_1) for $\varphi \in \mathcal{C}^\infty(\Omega)$, $F \in \Lambda(C^\infty(\Omega))$

$$d(\varphi F) = d\varphi\, F + \varphi\, dF;$$

(A_2) for $\varphi \in \mathcal{C}^\infty(\Omega)$, $F \in \Lambda(C^\infty(\Omega))$

$$d(d\varphi\, F) = -d\varphi\, dF.$$

Let $\mathcal{P} = \text{Alg}\{x_1, \ldots, x_m\}$ be the algebra of polynomials in $x_1, \ldots, x_m$ with real (or complex) coefficients. Then the generators $x_1, \ldots, x_m$, interpreted as coordinate functions, give rise to the differential $dx_1, \ldots, dx_m$. We can then give the following:

Definition 2.2. The subalgebra $\Lambda(\mathcal{P})$ of $\Lambda(C^\infty(\Omega))$ is generated by $\mathcal{P}$ and satisfies, for any $F \in \Lambda(\mathcal{P})$, the axioms

(A'_1) for $F \in \Lambda(\mathcal{P})$

$$d(x_j F) = dx_j\, F + x_j dF;$$

(A'_2) for $F \in \Lambda(\mathcal{P})$

$$d(dx_j\, F) = -dx_j\, dF.$$

Proposition 2.3. *The following properties hold:*

(i) $d(x_k\, dx_j) = dx_k\, dx_j$;

(ii) $d(dx_j\, x_k) = -dx_j dx_k$;

(iii) $dx_j\, dx_k = -dx_j\, dx_k$.

Proof. Property (i) follows from

$$d(x_k\, dx_j) = dx_k\, dx_j + x_k d^2 x_j = dx_k\, dx_j,$$

since, by (A_0) and (A_2)

$$d^2\varphi = d(d\varphi\, 1) = -d\varphi\, d1 = 0$$

for $\varphi \in \mathcal{C}^\infty(\Omega)$.

As a special case of (A'_2), we also have

$$d(dx_j\, x_k) = -dx_j dx_k,$$

so (ii) follows. As a consequence of (i) and (ii) we obtain $dx_j dx_k = -dx_j dx_k$. □

Remark 2.4. The previous result implies that $dx_1, \ldots, dx_m$ generate a Grassmann algebra of dimension 2^m.

From the definition of $\Lambda(\mathcal{P})$ it follows that every $F \in \Lambda(\mathcal{P})$ has the form

$$F = \sum_{A \subset M} F_A(\underline{x}) dx_A, \qquad F_A(\underline{x}) \in \mathcal{P},$$

where $M = \{1, \ldots, m\}$, $dx_A = dx_{\alpha_1} \ldots dx_{\alpha_k}$ for $A = \{\alpha_1, \ldots, \alpha_k\}$ and with $\alpha_1 < \ldots < \alpha_k$. It follows that

$$dF = \sum_{A \subset M} dF_A(\underline{x}) dx_A,$$

so it suffices to calculate $d\varphi$ for $\varphi \in \mathcal{P}$. By using iteratively the axiom (A_1') one can prove by induction on the degree of $\varphi \in \mathcal{P}$ that

$$d\varphi = \sum_{j=1}^{m} dx_j \, \partial_{x_j} \varphi.$$

Now note that $\mathcal{P}$ is dense in $\mathcal{C}^\infty(\Omega)$ and $\Lambda(\mathcal{P})$ is dense in $\Lambda(C^\infty(\Omega))$. So, it follows that every $F \in \Lambda(C^\infty(\Omega))$ is of the form

$$F = \sum_{A \subset M} F_A(\underline{x}) \, dx_A, \qquad F_A \in \mathcal{C}^\infty(\Omega),$$

and, in general,

$$dF = \sum_{j=1}^{m} dx_j \sum_{A \subset M} \partial_{x_j} F_A(\underline{x}) \, dx_A = \sum_{j=1}^{m} dx_j \, \partial_{x_j} F.$$

However, the definition of $\Lambda(\mathcal{C}^\infty(\Omega))$ and of d are independent of any coordinate system. Hence, if $(y_1, \ldots, y_m)$ is another $\mathcal{C}^\infty$-coordinate system on Ω, then we have that

$$d = \sum_{j=1}^{m} dx_j \, \partial_{x_j} = \sum_{j=1}^{m} dy_j \, \partial_{y_j},$$

so that we also have the chain rule

$$dx_j = \sum_{\ell=1}^{m} \frac{\partial x_j}{\partial y_\ell} \, dy_\ell$$

and for $A = \{\alpha_1, \ldots, \alpha_k\} \subseteq M$ with $\alpha_1 < \ldots < \alpha_k$ we have

$$\begin{aligned} dx_A &= \sum_{\ell_1 \ldots \ell_k} \frac{\partial x_{\alpha_1}}{\partial y_{\ell_1}} \cdots \frac{\partial x_{\alpha_k}}{\partial y_{\ell_k}} \, dy_{\ell_1} \ldots dy_{\ell_k} \\ &= \sum_{|B|=k} \left(\sum_{\pi \in \mathrm{Sym}(k)} \mathrm{sgn}\pi \frac{\partial x_{\alpha_1}}{\partial y_{\beta_{\pi(1)}}} \cdots \frac{\partial x_{\alpha_k}}{\partial y_{\beta_{\pi(k)}}} \right) dy_B \\ &= \sum_{|B|=k} J_{AB} \, dy_B, \qquad B = \{\beta_1, \ldots, \beta_k\}, \quad \beta_1 < \ldots < \beta_k, \end{aligned}$$

where

$$J_{AB} = \sum_{\pi \in \mathrm{Sym}(k)} \mathrm{sgn}\pi \frac{\partial x_{\alpha_1}}{\partial y_{\beta_{\pi(1)}}} \cdots \frac{\partial x_{\alpha_k}}{\partial y_{\beta_{\pi(k)}}}$$

are the generalized Jacobians. So, in the coordinates $(y_1, \ldots, y_m)$ we have

$$F = \sum_{A \subseteq M} F_A(\underline{x}) \, dx_A = \sum_{B \subseteq M} \left(\sum_{|A|=|B|} F_A(\underline{x}(y)) J_{AB} \right) dy_B.$$

Hence, the chain rule and Jacobians are an automatic consequence of the axioms.

3. Integration of differential forms

We extend the notion of differential form to the case where the components $F_A(\underline{x})$ are generalized functions or distributions in Ω. Let

$$F = F_M(x_1, \ldots, x_m)\, dx_1 \ldots dx_m$$

be a distributional form of maximum degree with $\operatorname{supp}(F_M) = K \subset \Omega$ compact. Then, the integral

$$\int_\Omega F = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F_M(\underline{x}) dx_1 \ldots dx_m$$

is well defined (note that this is a formal way of writing: we are using the density of $\mathcal{D}(\Omega)$ in $\mathcal{E}'(\Omega)$ and thus the integrals are meant in the sense of functionals, see e.g. [5]). Denote by $\Lambda_k(\mathcal{C}^\infty(\Omega))$ the subspace of k-forms, namely of elements $F = \sum_{|A|=k} F_A(\underline{x})\, dx_A$, where $F_A \in \mathcal{C}^\infty(\Omega)$ and denote by $\overline{\Lambda}_k(\mathcal{C}^\infty(\Omega))$ its closure in the distributions, namely the subspace of the k-forms $F = \sum_{|A|=k} F_A(\underline{x})\, dx_A$, with $F_A \in \mathcal{D}'(\Omega)$. Let Σ be an infinitely differentiable k-surface in $\mathbb{R}^m$ defined as the image of a $\mathcal{C}^\infty$-map:

$$\underline{x}(\cdot) : \ (u_1, \ldots, u_k) \to \underline{x}(u_1, \ldots, u_k),$$

where $\underline{u} = (u_1, \ldots, u_k) \in \Omega' \subset \mathbb{R}^k$, i.e. $\Sigma = \underline{x}(\Omega')$. Next, let $F \in \overline{\Lambda}_k(\mathcal{C}^\infty(\mathbb{R}^m))$ with $\operatorname{supp}(F) \cap \Sigma$ compact. Then we can define

$$\int_\Sigma F := \int_{\Omega'} \sum_A F_A(\underline{x}(u_1, \ldots, u_k)) J_A(\underline{u})\, du_1 \ldots du_k$$

where

$$J_A(\underline{u}) = \sum_\pi \operatorname{sgn}\pi \frac{\partial x_{\alpha_1}}{\partial u_{\pi(1)}} \cdots \frac{\partial x_{\alpha_k}}{\partial u_{\pi(k)}}$$

is the Jacobian that appears automatically from the chain rule. This also implies that the above definition will not depend on the coordinate system in use. Indeed, if we use another coordinate system $(y_1, \ldots, y_k)$ that locally has the same orientation as $(u_1, \ldots, u_k)$, then for any $\varphi \in \mathcal{C}^\infty(\Omega')$

$$\begin{aligned}\int_{\Omega'} \varphi(\underline{u})\, du_1 \ldots du_k &= \int_{\Omega''} \varphi(\underline{u}(\underline{y})) \left| \frac{\partial^k u_1 \ldots u_k}{\partial y_1 \ldots \partial y_k} \right| dy_1 \ldots dy_k \\ &= \int_{\Omega''} \varphi(\underline{u}(\underline{y})) \frac{\partial^k u_1 \ldots u_k}{\partial y_1 \ldots \partial y_k}\, dy_1 \ldots dy_k,\end{aligned}$$

but we also have that

$$J_A(\underline{u}(\underline{y})) = J_A(\underline{u}) \cdot \frac{\partial^k u_1 \ldots u_k}{\partial y_1 \ldots \partial y_k}.$$

In other words, the calculus with differential forms automatically keeps track of Jacobians.

In the sequel we also need partial integration of differential forms. For a form $F(\underline{y})\,dy_1\dots dy_\ell$ with compact support, this is defined as the operator

$$\int_{y_j} F(\underline{y})\,dy_1\dots dy_\ell := (-1)^{j-1}\left(\int_{-\infty}^{+\infty} F(\underline{y})\,dy_j\right)\,dy_1\dots dy_{j-1}dy_{j+1}dy_\ell$$

that transforms differential forms into differential forms. From this definition, it is clear that, as operators:

$$\int_{y_j}\int_{y_\ell}\cdot = -\int_{y_\ell}\int_{y_j}\cdot$$

and also

$$dy_j\int_{y_\ell}\cdot = -\int_{y_\ell}\cdot\, dy_j,$$

while the integral of k-forms may now be defined as

$$\begin{aligned}\int_\Sigma F &= \int_{\Omega'}\sum_A F_A(\underline{x}(\underline{u}))J_A(\underline{u})\,du_1\dots du_k\\ &= \int_{u_k}\cdots\left(\int_{u_1}\sum_A F_A(\underline{x}(\underline{u}))J_A(\underline{u})\,du_1\right)\dots du_k.\end{aligned}$$

In other words, variables of integration have to be moved to the left side of a differential form. It is important to note that the above definition of integral automatically keeps track of the orientation on Σ: it is determined by the order of the coordinates $u_1,\dots,u_k$.

4. Stokes' formula

Let $F\in\overline{\Lambda}_{k-1}(\mathcal{C}^\infty(\Omega))$ with $\mathrm{supp}F\cap\Sigma$ compact and Σ is as above. Then we have that (where $\widehat{x}$ means that x is suppressed):

$$\int_\Sigma dF = \int_{\mathbb{R}^k}\sum_{j=1}^k du_j\,\partial_{u_j}\sum_A F_A(\underline{x}(\underline{u}))\frac{\partial^{k-1}x_{\alpha_1}\dots x_{\alpha_{k-1}}}{\partial u_1\dots\widehat{\partial u_j}\dots\partial u_k}\,du_1\dots\widehat{du_j}\dots du_m$$

$$= \int_{\mathbb{R}^k}\sum_{j=1}^k\partial_{u_j}g_j(\underline{u})du_1\dots du_k = 0,$$

since $\mathrm{supp}F\cap\Sigma$ is compact, with

$$g_j(\underline{u}) = \sum_A F_A(\underline{x}(\underline{u}))\frac{\partial^{k-1}x_{\alpha_1}\dots x_{\alpha_{k-1}}}{\partial u_1\dots\widehat{\partial u_j}\dots\partial u_k}.$$

Let C be a compact set in $\mathbb{R}^m$ with nonempty interior and $\mathcal{C}^\infty$ boundary. Let $\varphi\in\mathcal{C}^\infty(\mathbb{R}^m)$ be a defining function for C, i.e. $\varphi<0$ in $\mathrm{int}(C)$, $\varphi>0$ in $\mathbb{R}^m\setminus C$ and $\varphi=0$, $\nabla\varphi\neq 0$ on ∂C. Then, if Y denotes the Heaviside function, we have that $Y(-\varphi) = \chi_C$ where χ_C is the characteristic function of C. Moreover, for $F \in \Lambda_{k-1}(\mathcal{C}^\infty(\Omega))$ with $C \subset \Omega$ we would have that

$$\int_\Sigma d(\chi_C(x)F) = 0,$$

where

$$d(\chi_C F) = dY(-\varphi)F + \chi_C dF, \qquad dY(-\varphi) = -\delta(\varphi)\, d\varphi,$$

where δ is the Dirac distribution on the real line. This leads to

Theorem 4.1 (Stokes' formula). *With the above notations, the following formula holds:*

$$\int_\Sigma \delta(\varphi)\, d\varphi F = \int_\Sigma Y(-\varphi)\, dF.$$

The formula can be also written in the more familiar form

$$\int_{\partial C \cap \Sigma} F = \int_{C \cap \Sigma} dF.$$

Here one has to choose local coordinates $(v_1, \ldots, v_{k-1})$ on $\partial C \cap \Sigma$ such that the orientation of the system of coordinates $(\varphi, v_1, \ldots, v_{k-1})$ is the same as the orientation of $(u_1, \ldots, u_k)$.
Indeed, we have that

$$\int_\Sigma \delta(\varphi)\, d\varphi\, F = \int_{v_{k-1}} \cdots \int_{v_1} \int_\varphi \delta(\varphi)\, d\varphi\, F = \int_{v_{k-1}} \cdots \int_{v_1} F_{|\varphi=0} = \int_{\partial C \cap \Sigma} F.$$

5. Clifford differential forms

The complex Clifford algebra $\mathbb{C}_m$ is the complex associative algebra with generators $e_1, \ldots, e_m$ together with the defining relations $e_j e_k + e_k e_j = -2\delta_{jk}$. Every element $a \in \mathbb{C}_m$ can be written in the form

$$a = \sum_{A \subset M} a_A e_A, \qquad a_A \in \mathbb{C},$$

where, as before, $M = \{1, \ldots, m\}$ and for any multi-index $A = \{\alpha_1, \ldots, \alpha_k\} \subseteq M$, with $\alpha_1 < \ldots < \alpha_k$ we put $e_A = e_{\alpha_1} \cdots e_{\alpha_k}$.
Every $a \in \mathbb{C}_m$ admits a multivector decomposition

$$a = \sum_{k=0}^m [a]_k, \quad \text{where} \quad [a]_k = \sum_{|A|=k} a_A e_A,$$

so $[\cdot]_k : \mathbb{C}_m \to \mathbb{C}_m^k$ denotes the canonical projection of $\mathbb{C}_m$ onto the space $\mathbb{C}_m^k$ of k-vectors. Note that $\mathbb{C}_m^0 = \mathbb{C}$, the set of scalars while $\mathbb{C}_m^1$ is the space of 1-vectors $\underline{v} = \sum_{j=1}^m v_j e_j$. So the map

$$(v_1, \ldots, v_m) \to \underline{v} = \sum_{j=1}^m v_j e_j$$

leads to the identification of $\mathbb{C}^m$ with $\mathbb{C}_m^1$. For any $\underline{v}, \underline{w} \in \mathbb{C}_m^1$ we have

$$\underline{v}\,\underline{w} = \underline{v} \cdot \underline{w} + \underline{v} \wedge \underline{w},$$

$$\underline{v} \cdot \underline{w} = -\langle \underline{v}, \underline{w} \rangle = -\sum_{j=1}^{m} v_j w_j,$$

$$\underline{v} \wedge \underline{w} = \sum_{j<\ell} e_{j\ell}(v_j w_\ell - v_\ell w_j) \in \mathbb{C}_m^2.$$

More in general, for $\underline{v}_1, \ldots, \underline{v}_k \in \mathbb{C}_m^1$ we define the wedge (or Grassmann) product in terms of the Clifford product by

$$\underline{v}_1 \wedge \ldots \wedge \underline{v}_k = \frac{1}{k!} \sum_{\pi \in \mathrm{Sym}(k)} \mathrm{sgn}\pi \ \underline{v}_{\pi(1)} \cdots \underline{v}_{\pi(k)} \in \mathbb{C}_m^k.$$

We also call $\underline{v}_1 \wedge \ldots \wedge \underline{v}_k$ a *k-blade*. The k-blades span $\mathbb{C}_m^k$, but not every element in $\mathbb{C}_m^k$ is a k-blade.
For $\underline{v} \in \mathbb{C}_m^1$ and $a \in \mathbb{C}_m^k$ we set

$$\underline{v}a = [\underline{v}a]_{k-1} + [\underline{v}a]_{k+1} = \underline{v} \cdot a + \underline{v} \wedge a$$

where

$$\underline{v} \cdot a = \frac{1}{2}(\underline{v}a + (-1)^{k-1} a\underline{v}),$$

$$\underline{v} \wedge a = \frac{1}{2}(\underline{v}a + (-1)^{k} a\underline{v}).$$

More in general, for $a \in \mathbb{C}_m^k$, $b \in \mathbb{C}_m^\ell$, $k \geq \ell$ we have

$$ab = [ab]_{k-\ell} + [ab]_{k-\ell+2} + \cdots + [ab]_{k+\ell}$$

and we define the wedge product as

$$[ab]_{k+\ell} = a \wedge b.$$

So we have the Grassmann product in terms of the Clifford product. The variable $(x_1, \ldots, x_m) \in \mathbb{R}^m$ is identified with the vector variable $\underline{x} = \sum_{j=1}^m x_j e_j$ and $\mathbb{C}_m$-valued functions in $\mathbb{R}^m$ are denoted by $f(\underline{x}) = \sum_A f_A(\underline{x}) e_A$, f_A are $\mathbb{C}$-valued functions.

Definition 5.1. A function $f : \Omega \subseteq \mathbb{R}^m \to \mathbb{C}_m$ real differentiable will be called left monogenic in Ω if it satisfies $\partial_{\underline{x}} f(\underline{x}) = 0$ for $\underline{x} \in \Omega$, where $\partial_{\underline{x}} = \sum_{j=1}^m e_j \partial_{x_j}$ is the Dirac operator (or vector derivative).

We have the following formulas

$$\underline{x}\,\partial_{\underline{x}} = \underline{x} \cdot \partial_{\underline{x}} + \underline{x} \wedge \partial_{\underline{x}} = -E_{\underline{x}} - \Gamma_{\underline{x}}$$

where $E_{\underline{x}} = -\underline{x} \cdot \partial_{\underline{x}} = \sum_{j=1}^m x_j \partial_{x_j}$ is the Euler operator and $\Gamma_{\underline{x}} = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{j<k}^m e_{jk} L_{jk}$, $L_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}$, are the angular momentum operators. Moreover we have the overdot notation introduced by Hestenes

$$\partial_{\underline{x}}(\underline{x}\, f) = -mf + \dot{\partial}_{\underline{x}}(\underline{x} \dot{f})$$

where

$$\dot{\partial}_{\underline{x}}(\underline{x}\dot{f}) = -\underline{x}\partial_{\underline{x}}f - 2E_{\underline{x}}f.$$

Remark 5.2. The elements $dx_1, \dots, dx_m$ generate a Grassmann algebra and also $e_1, \dots, e_m$ form a Grassmann algebra with respect to the wedge product. Yet, we do not identify dx_j with e_j as some authors do. The elements e_j are imaginary units and so symbolic constants, while the elements dx_j are the differentials of the real coordinates $x_1, \dots, x_m$. The wedge notation will be used only for Clifford numbers, not for the differential forms $dx_1, \dots, dx_m$. However, we may use it for vector differentials (see below and the last section).

The vector variable $\underline{x} = \sum_{j=1}^{m} x_j e_j$ can be seen as a $\mathbb{R}^1_m$-valued function. Its differential, called *vector differential* is given by $d\underline{x} = \sum_{j=1}^{m} e_j dx_j$. Combining the Clifford product and the differential form product, we have that

$$(d\underline{x})^2 = \sum_{j,\ell}^{m} dx_j e_j dx_\ell e_\ell = 2\sum_{j<\ell}^{m} dx_j dx_\ell e_j e_\ell = d\underline{x} \wedge d\underline{x} = [d\underline{x}^2]_2,$$

and, more in general,

$$(d\underline{x})^k = k! \sum_{|A|=k} dx_A e_A = d\underline{x} \wedge \ldots \wedge d\underline{x} = [d\underline{x}^k]_k.$$

In particular

$$\frac{d\underline{x}^m}{m!} = dx_1 \ldots dx_m e_1 \ldots e_m = V(d\underline{x})e_M,$$

$$\frac{d\underline{x}^{m-1}}{(m-1)!} = \sum_{j=1}^{m} dx_{M\setminus\{j\}} e_{M\setminus\{j\}} = -\sum_{j=1}^{m} e_j(-1)^{j-1} dx_{M\setminus\{j\}} e_M = -\sigma(d\underline{x})e_M$$

where $V(d\underline{x})$ denotes the Euclidean volume form and

$$\sigma(d\underline{x}) = \sum_{j=1}^{m} (-1)^{j-1} e_j \, dx_1 \ldots \widehat{dx_j} \ldots dx_m$$

is called σ-form.
Let $f, g :\ \Omega \to \mathbb{C}_m$, then

$$\begin{aligned} d(f\,\sigma\,g) &= \sum_{j=1}^{m} \partial_{x_j}(f\,e_j\,g)\,dx_1 \ldots dx_m \\ &= (\dot{f}\dot{\partial}_{\underline{x}}g + f\dot{\partial}_{\underline{x}}\dot{g})\,V(d\underline{x}). \end{aligned}$$

Hence, for a compact subset $C \subset \Omega$ with nonempty interior and with smooth boundary, we have (see [1, 10]):

Theorem 5.3 (Cauchy-Borel-Pompeiu). *Let $\Omega \subseteq \mathbb{R}^m$ be an open set and $f, g :\ \Omega \to \mathbb{C}_m$. Let $C \subset \Omega$ with nonempty interior and with smooth boundary. Then*

$$\int_{\partial C} d(f\,\sigma\,g) = \int_C (\dot{f}\dot{\partial}_{\underline{x}}g + f\dot{\partial}_{\underline{x}}\dot{g})\,V(d\underline{x}).$$

We are now going to generalize this result to smooth k-surfaces $C \cap \Sigma$ where, as before, Σ is the infinitely differentiable image of a map $\underline{u} = (u_1, \dots, u_m) \to \underline{x}(\underline{u}) \in \Sigma$.
First of all, we have that for $\underline{x} \in \Sigma$:

$$d\underline{x} = \sum_{j=1}^{k} du_j\, \partial_{u_j}(\underline{x}) = \partial_{u_j}\underline{x}\, du_j,$$

$$\begin{aligned} d\underline{x}^2 &= \sum_{j<\ell} (\partial_{u_j}\underline{x}\,\partial_{u_\ell}\underline{x} - \partial_{u_\ell}\underline{x}\,\partial_{u_j}\underline{x}) du_j\, du_\ell \\ &= 2\sum_{j<\ell} \frac{\partial \underline{x}}{\partial u_j} \wedge \frac{\partial \underline{x}}{\partial u_\ell}\, du_j du_\ell \end{aligned}$$

$$\frac{d\underline{x}^\ell}{\ell!} = \sum_{|A|=\ell} \frac{\partial \underline{x}}{\partial u_{j_1}} \wedge \dots \wedge \frac{\partial \underline{x}}{\partial u_{j_\ell}}\, du_{j_1} \dots du_{j_\ell}$$

and eventually

$$\frac{d\underline{x}^k}{k!} = \frac{\partial \underline{x}}{\partial u_1} \wedge \dots \wedge \frac{\partial \underline{x}}{\partial u_k}\, du_1 \dots du_k \tag{5.1}$$

is the oriented k-vector valued surface element on Σ. All these surface forms are coordinate independent.
We now prove the following crucial result, see also [10]:

Lemma 5.4. *We have the formal identity*

$$d\frac{d\underline{x}^{k-1}}{(k-1)!} = -\partial_{\underline{x}} \cdot \frac{d\underline{x}^k}{k!}.$$

Proof. We have the following chain of equalities

$$\begin{aligned} -\partial_{\underline{x}} \cdot \frac{d\underline{x}^k}{k!} &= -\frac{1}{2}\left(\partial_{\underline{x}} \frac{d\underline{x}^k}{k!} + (-1)^{k-1}\frac{d\underline{x}^k}{k!}\partial_{\underline{x}}\right) \\ &= -\frac{1}{2}\Bigg(\{\partial_{\underline{x}}, d\underline{x}\}\frac{d\underline{x}^{k-1}}{k!} - d\underline{x}\{\partial_{\underline{x}}, d\underline{x}\}\frac{d\underline{x}^{k-2}}{k!} + \cdots \\ &\quad \cdots + (-1)^{k-1}\frac{d\underline{x}^{k-1}}{k!}\{\partial_{\underline{x}}, d\underline{x}\}\Bigg) \\ &= -\frac{1}{2}\{\partial_{\underline{x}}, d\underline{x}\}\frac{d\underline{x}^{k-1}}{(k-1)!} \end{aligned}$$

and clearly $\{\partial_{\underline{x}}, d\underline{x}\} = -2d$. □

Theorem 5.5 (Stokes). *Let Σ be a smooth k-surface, let C be a compact set with non empty interior whose boundary ∂C is a smooth $(n-1)$-surface, (so $C \cap \Sigma$ is a compact set). Let f, g be $\mathbb{C}_m$-valued smooth functions on Σ. Then*

$$\int_{\partial C \cap \Sigma} f \frac{d\underline{x}^{k-1}}{(k-1)!} g = -\int_{C \cap \Sigma} \left(\dot{f}\dot{\partial}_{\underline{x}} \cdot \frac{d\underline{x}^k}{k!} g + f\dot{\partial}_{\underline{x}} \cdot \frac{d\underline{x}^k}{k!}\dot{g}\right).$$

Now we have to characterize the restriction to Σ of $\partial_{\underline{x}} \cdot \frac{d\underline{x}^k}{k!}$. We already know, see (5.1), that

$$\frac{d\underline{x}^k}{k!} = \mathbb{T}(\underline{x}) V(du_1, \ldots, du_k),$$

where $\mathbb{T}(\underline{x})$ is the unit k-blade tangent to Σ at the point $\underline{x}$ and $V(du_1, \ldots, du_k)$ is the Euclidean volume form. Let $p \in \Sigma$ and consider an orthonormal basis $\underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_k$ of k-planes tangent to Σ at the point p. Assume that the orthonormal basis has the same orientation as the coordinate frame $u_1, \ldots, u_k$. Then

$$\mathbb{T} = \underline{\varepsilon}_1 \ldots \underline{\varepsilon}_k = \omega \frac{\partial \underline{x}}{\partial u_1} \wedge \ldots \wedge \frac{\partial \underline{x}}{\partial u_k}$$

where ω is a positive weight, namely a function with strictly positive real values. Let $\underline{\nu}_1, \ldots, \underline{\nu}_{m-k}$ be the remaining $(m-k)$ unit vectors such that $(\underline{\varepsilon}_1, \ldots, \underline{\varepsilon}_k; \underline{\nu}_1, \ldots, \underline{\nu}_{m-k})$ is an orthonormal basis of $\mathbb{R}^m$. Then

$$\partial_{\underline{x}} = \partial_{\underline{x}\|} + \partial_{\underline{x}\perp}$$

$$\partial_{\underline{x}\|} = \sum_{j=1}^{k} \underline{\varepsilon}_j \langle \underline{\varepsilon}_j, \partial_{\underline{x}} \rangle$$

$$\partial_{\underline{x}\perp} = \sum_{j=1}^{m-k} \underline{\nu}_j \langle \underline{\nu}_j, \partial_{\underline{x}} \rangle$$

so that after restriction to Σ we have

$$\partial_{\underline{x}} \cdot \frac{d\underline{x}^k}{k!} = \partial_{\underline{x}\|} \frac{d\underline{x}^k}{k!} = (-1)^{k-1} \frac{d\underline{x}^k}{k!} \partial_{\underline{x}\|}.$$

We then have (compare with [10]):

Theorem 5.6 (Cauchy). *Let Σ be a smooth k-surface, let C be a compact set with non empty interior whose boundary ∂C is a smooth $(n-1)$-surface. Let f, g be $\mathbb{C}_m$-valued smooth functions on Σ. Then*

$$\int_{\partial C \cap \Sigma} f \frac{d\underline{x}^{k-1}}{(k-1)!} g = -\int_{C \cap \Sigma} (f \partial_{\underline{x}\|}) \frac{d\underline{x}^k}{k!} g + (-1)^k \int_{C \cap \Sigma} f \frac{d\underline{x}^k}{k!} (\partial_{\underline{x}\|} g).$$

6. Winding numbers from monogenic functions

Let us recall that the Cauchy kernel for monogenic functions is

$$E(\underline{x}) = -\frac{1}{A_m} \frac{\underline{x}}{|\underline{x}|^m}, \qquad A_m = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

The function $E(\underline{x})$ is both left and right monogenic in $\mathbb{R}^m \setminus \{0\}$ and takes values in the space of 1-vectors $\mathbb{R}^1_m$. We also have the validity of the following equalities

$$\partial_{\underline{x}} E(\underline{x}) = E(\underline{x}) \partial_{\underline{x}} = \delta(\underline{x}) = \delta(x_1) \ldots \delta(x_m)$$

which hold in the distributional sense, see also [3]. Hence, we also have that, in view of Lemma 5.4:

$$d\left(E(\underline{x})\frac{d\underline{x}^{m-1}}{(m-1)!}\right) = -\dot{E}(\underline{x})\dot{\partial}_{\underline{x}}\cdot\frac{d\underline{x}^m}{m!}$$
$$= -(E(\underline{x})\partial_{\underline{x}})\frac{d\underline{x}^m}{m!}$$
$$= -\delta(\underline{x})\frac{d\underline{x}^m}{m!},$$

whereby as before $\frac{d\underline{x}^m}{m!} = dx_1\dots dx_m e_M$. We are now going to consider two sets of coordinates $x_1,\dots,x_m$, and $u_1,\dots,u_m$ and the corresponding differentials. Then, by translation, we have that

$$\partial_{\underline{x}}E(\underline{x}-\underline{u}) = E(\underline{x}-\underline{u})\partial_{\underline{x}} = \delta(\underline{x}-\underline{u})$$
$$= \delta(x_1-u_1)\dots\delta(x_m-u_m)$$
$$= -\partial_{\underline{u}}E(\underline{x}-\underline{u}) = -E(\underline{x}-\underline{u})\partial_{\underline{u}}.$$

Now, by replacing also the vector differential

$$d\underline{x}\to d\underline{y} = d(\underline{x}-\underline{u}) = d\underline{x}-d\underline{u}, \qquad \text{where } \underline{y}=\underline{x}-\underline{u},$$

we still have that

$$d_{\underline{y}}E(\underline{y})\frac{d\underline{y}^{m-1}}{(m-1)!} = -\delta(\underline{y})\frac{d\underline{y}^m}{m!},$$

where $d_{\underline{y}} = \sum_{j=1}^m dy_j\partial_{y_j} = \sum_{j=1}^m (dx_j - du_j)\partial_{y_j}$ and also

$$\partial_{y_j}E(\underline{y}) = \partial_{x_j}E(\underline{x}-\underline{u}) = -\partial_{u_j}E(\underline{x}-\underline{u}).$$

Hence $d_{\underline{y}} = d_{\underline{x}} + d_{\underline{u}}$ and the above identity may be rewritten as

$$(d_{\underline{x}}+d_{\underline{u}})E(\underline{x}-\underline{u})\frac{(d\underline{x}-d\underline{u})^{m-1}}{(m-1)!} = -\delta(\underline{x}-\underline{u})\frac{(d\underline{x}-d\underline{u})^m}{m!} \tag{6.1}$$

where

$$\frac{(d\underline{x}-d\underline{u})^m}{m!} = (dx_1-du_1)\dots(dx_m-du_m)e_M = \sum_{k=0}^m V_k(d\underline{x},d\underline{u})e_M$$

and (see also [11])

$$V_k(d\underline{x},d\underline{u}) = (-1)^k\sum_{|A|=k}\mathrm{sgn}A\, du_A\, dx_{M\setminus A},$$

$$du_A = du_{\alpha_1}\dots du_{\alpha_k}, \qquad A=\{\alpha_1\dots\alpha_k\},\ \alpha_1<\dots<\alpha_k$$
$$dx_{M\setminus A} = dx_{\beta_1}\dots dx_{\beta_{m-k}}, \qquad M\setminus A=\{\beta_1\dots\beta_{m-k}\},\ \beta_1<\dots<\beta_{m-k}$$

and sgn$A$ denotes the signature of the permutation $(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_{m-k})$ with respect to $(1,\dots,m)$. This can also be obtained as follows:

$$d\underline{x}\cdot d\underline{u} = [d\underline{x}\,d\underline{u}]_0 = -\sum_{j=1}^m dx_j\,du_j$$
$$d\underline{x}\wedge d\underline{u} = [d\underline{x}\,d\underline{u}]_2 = \frac{1}{2}(d\underline{x}\,d\underline{u}+d\underline{u}\,d\underline{x}) = d\underline{u}\wedge d\underline{x}.$$

So, in general, we have that

$$\begin{aligned}\frac{(d\underline{x}-d\underline{u})^k}{k!} &= \frac{[(d\underline{x}-d\underline{u})^k]_k}{k!}\\ &= \frac{(d\underline{x}-d\underline{u})\wedge\ldots\wedge(d\underline{x}-d\underline{u})}{k!}\\ &= \sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\frac{[d\underline{u}^\ell\, d\underline{x}^{k-\ell}]_k}{k!}\\ &= \sum_{\ell=0}^{k}(-1)^\ell\frac{d\underline{u}^\ell}{\ell!}\wedge\frac{d\underline{x}^{k-\ell}}{(k-\ell)!}\end{aligned}$$

is a k-vector. In particular:

$$V_k(d\underline{x},d\underline{u})e_M = (-1)^k\frac{d\underline{u}^k}{k!}\wedge\frac{d\underline{x}^{m-k}}{(m-k)!}$$

while also

$$\frac{(d\underline{x}-d\underline{u})^{m-1}}{(m-1)!} = -\sigma(d\underline{x}-d\underline{u})e_M = -\sum_{k=0}^{m-1}\sigma_k(d\underline{x},d\underline{u})$$

with

$$\sigma_k(d\underline{x},d\underline{u}) = (-1)^{k+1}\frac{d\underline{u}^k}{k!}\wedge\frac{d\underline{x}^{m-k-1}}{(m-k-1)!}.$$

Thus we arrive at the fundamental identity contained in the following result:

Theorem 6.1. *For every $k=0,\ldots,m$ the following identity holds:*

$$\begin{aligned}&d_{\underline{x}}\left[E(\underline{x}-\underline{u})\frac{d\underline{u}^k}{k!}\wedge\frac{d\underline{x}^{m-k-1}}{(m-k-1)!}\right]\\ &= d_{\underline{u}}\left[E(\underline{x}-\underline{u})\frac{d\underline{u}^{k-1}}{(k-1)!}\wedge\frac{d\underline{x}^{m-k}}{(m-k)!}\right]-\delta(\underline{x}-\underline{u})\frac{d\underline{u}^k}{k!}\wedge\frac{d\underline{x}^{m-k}}{(m-k)!}.\end{aligned}$$

Proof. The result follows by identifying differential forms with same degree in $du_1,\ldots,du_m$ within the formula (6.1). □

Let C_k be a k-chain in $\mathbb{R}^m$ that can be realized as a compact subset C_k with nonempty interior of an oriented infinitely differentiable surface of dimension k (with respect to the relative topology).

Definition 6.2. The indicatrix $I(C_k)(\underline{x})$ of the k-chain C_k in $\mathbb{R}^m$ is the $(m-k-1)$-form

$$I(C_k)(\underline{x}) = \int_{\underline{u}\in C_k}E(\underline{x}-\underline{u})\frac{d\underline{u}^k}{k!}\wedge\frac{d\underline{x}^{m-k-1}}{(m-k-1)!}.$$

The indicatrix is an $(m-k-1)$-form with left monogenic component. The above theorem clearly leads to the following result (compare with the result in [9]):

Theorem 6.3. *Let ∂C_k be the boundary of C_k with proper orientation, then*

$$(-1)^k dI(C_k)(\underline{x}) = I(\partial C_k)(\underline{x}) - \int_{\underline{u}\in C_k} \delta(\underline{x}-\underline{u})\frac{d\underline{u}^k}{k!} \wedge \frac{d\underline{x}^{m-k}}{(m-k)!}.$$

Proof. The result follows from Theorem 6.1 by integrating with respect to

$$\int_{\underline{u}\in C_k} \cdot$$

The factor $(-1)^k$ in front arises because of the anti-commutativity:

$$\int_{v_j} \cdot \, dx_j = -dx_j \int_{v_j} \cdot$$

between integral operators and differentials. □

Remark 6.4. The second term is a distributional $(m-k)$-form given by

$$-\left[\Delta(C_k)(\underline{x})\frac{d\underline{x}^{m-k}}{(m-k)!}\right]_m = (-1)^{k(m-k)+1}\left[\frac{d\underline{x}^{m-k}}{(m-k)!}\Delta(C_k)(\underline{x})\right]_m$$

where $\Delta(C_k)(\underline{x})$ denotes the distribution supported by C_k defined as

$$\Delta(C_k)(\underline{x}) = \int_{\underline{u}\in C_k} \delta(\underline{x}-\underline{u})\frac{d\underline{u}^k}{k!}.$$

Corollary 6.5. *Let $\partial C_k = 0$, i.e. let C_k be a k-cycle. Then*

$$(-1)^k dI(C_k)(\underline{x}) = -(-1)^{k(m-k)}\frac{d\underline{x}^{m-k}}{(m-k)!} \wedge \Delta(C_k)(\underline{x}),$$

which vanishes in $\mathbb{R}^m \setminus C_k$.

Now consider a k-cycle C_k and let C_{m-k} be an infinitely differentiable $(m-k)$-chain with infinitely differentiable boundary $\partial C_{m-k} \subset \mathbb{R}^m \setminus C_k$. We choose C_{m-k} such that it intersect generically C_k in finitely many points. Then, in view of Stokes' formula and using the previous corollary, we have

$$\int_{\partial C_{m-k}} I(C_k)(\underline{x}) = \int_{C_{m-k}} dI(C_k)(\underline{x})$$
$$= -(-1)^{k(m-k-1)}\int_{C_{m-k}} \frac{d\underline{x}^{m-k}}{(m-k)!} \wedge \Delta(C_k)(\underline{x}).$$

Theorem 6.6. *Under the above assumptions*

$$\int_{C_{m-k}} I(C_k)(\underline{x}) = -(-1)^k \mathrm{Int}(C_k, C_{m-k})e_M,$$

where $\mathrm{Int}(C_k, C_{m-k})$ is the intersection number of C_k with respect to C_{m-k} inside $\mathbb{R}^m$.

We will show the result just in one case. The general result follows from homological arguments. First we note that all the above results extend to the case where C_k is an unbounded chain or cycle, as long as all the needed integrals converge. Let us consider the case where $C_k = W_k$, W_k being the oriented k-space with coordinates $u_1, \dots, u_k$. In this case we have

$$\begin{aligned} I(W_k) &= \int_{u_m \in \mathbb{R}} \cdots \int_{u_1 \in \mathbb{R}} E(\underline{x} - \underline{u}) du_i \dots du_k \left(e_1 \dots e_k \wedge \frac{d\underline{x}^{m-k-1}}{(m-k-1)!} \right) \\ &= -\frac{1}{A_m} \frac{\underline{x}_\perp}{|\underline{x}_\perp|^{m-k}} \left(e_1 \dots e_k \wedge \frac{d\underline{x}_\perp^{m-k-1}}{(m-k-1)!} \right) \end{aligned}$$

where $\underline{x}_\perp = \sum_{j=k+1}^m x_j e_j$, $d\underline{x}_\perp = \sum_{j=k+1}^m dx_j e_j$, and therefore by Lemma 5.4

$$\begin{aligned} dI(W_k)(\underline{x}) &= \frac{1}{A_{m-k}} \frac{\dot{\underline{x}}_\perp}{|\underline{x}_\perp|^{m-k}} e_1 \dots e_k \left(\dot{\partial}_{\underline{x}_\perp} dx_{k+1} \dots dx_m e_{k+1} \dots e_m \right) \\ &= -(-1)^k \delta(\underline{x}_\perp) dx_{k+1} \dots dx_m e_M. \end{aligned}$$

So if $C_{m-k} = B(1) \cap W_{m-k}$, W_{m-k} being the $(m-k)$-space with coordinates $x_{k+1}, \dots, x_m$, we obtain that $\partial C_{m-k} = \mathbb{S}^{m-k-1}$, the unit sphere in W_{m-k} and

$$\begin{aligned} \int_{\mathbb{S}^{m-k-1}} I(W_k) &= \int_{|\underline{x}_\perp|<1} -(-1)^k \delta(\underline{x}_\perp) dx_{k+1} \dots dx_m e_M \\ &= -(-1)^k e_M \\ &= -(-1)^k \mathrm{Int}(W_k, W_{m-k}) e_M. \end{aligned}$$

In general, we have the following:

Definition 6.7. The intersection number $\mathrm{Int}(C_k, C_{m-k})$ is defined as

$$\sum_{p \in C_k \cap C_{m-k}} \mathrm{Int}(T_p C_k, T_p C_{m-k}),$$

where $T_p C_k$, $T_p C_{m-k}$ denote the oriented tangent spaces to C_k and C_{m-k} at the point p, respectively, and where

$$\mathrm{Int}(T_p C_k, T_p C_{m-k}) = \operatorname{sgn} \det G,$$

where $G \in \mathrm{GL}(m, \mathbb{R})$ is the matrix of a linear transformation mapping $W_k \to T_p C_k$ and $W_{m-k} \to T_p C_{m-k}$.

Definition 6.8. The intersection number $\mathrm{Int}(T_p C_k, T_p C_{m-k})$ is also called the winding number of ∂C_{m-k} around C_k.

Remark 6.9. At a given point p, the number $\mathrm{Int}(T_p C_k, T_p C_{m-k})$ gives the signature of the orientation. The sum $\sum_{p \in C_k \cap C_{m-k}} \mathrm{Int}(T_p C_k, T_p C_{m-k})$ is equal to the total intersection number between C_k and C_{m-k}; it is also equal to the number of times that ∂C_{m-k} rotates around C_k.

Proof of the theorem. As $I(C_k)$ is closed in $\mathbb{R}^m \setminus C_k$, we have that whenever C'_{m-k} is an $(m-k)$-chain for which $\partial C'_{m-k}$ is homologous to ∂C_{m-k} in $\mathbb{R}^m \setminus C_k$, then

$$\int_{\partial C'_{m-k}} I(C_k)(\underline{x}) = \int_{\partial C_{m-k}} I(C_k)(\underline{x}).$$

Moreover, C'_{m-k} may be chosen to be a sum of unit discs. Due to the symmetry, one may also change C_k to a homologous cycle C'_k inside $\mathbb{R}^m \setminus \partial C'_{m-k}$ and choose C'_k to be a sum of spheres (or even oriented k-spaces). This reduces to the general case to a k-space W_k and a disc $B(1) \cap W_{m-k}$ for which we have the result. $\square$

Remark 6.10. For the indicatrix of C_k we have the expressions

$$I(C_k)(\underline{x}) = \int_{\underline{u} \in C_k} E(\underline{x} - \underline{u}) \frac{d\underline{u}^k}{k!} \wedge \frac{d\underline{x}^{m-k-1}}{(m-k-1)!}$$

from which we get

$$\begin{aligned}\int_{\underline{x} \in \partial C_{m-k}} I(C_k)(\underline{x}) &= \left[\int_{\underline{x} \in \partial C_{m-k}} \int_{\underline{u} \in C_k} E(\underline{x} - \underline{u}) \frac{d\underline{u}^k}{k!} \frac{d\underline{x}^{m-k-1}}{(m-k-1)!} \right]_m \\ &= (-1)^{(k+1)(m-k-1)} \left[\int_{\underline{x} \in \partial C_{m-k}} \frac{d\underline{x}^{m-k-1}}{(m-k-1)!} \int_{\underline{u} \in C_k} E(\underline{x} - \underline{u}) \frac{d\underline{u}^k}{k!} \right]_m \\ &= \pm \left[\int_{\partial C_{m-k}} \frac{d\underline{x}^{m-k-1}}{(m-k-1)!} M(C_k)(\underline{x}) \right]_m\end{aligned}$$

whereby $M(C_k)$ is a left monogenic function in $\mathbb{R}^m \setminus C_k$ given by

$$M(C_k)(\underline{x}) = \int_{\underline{u} \in C_k} E(\underline{x} - \underline{u}) \frac{d\underline{u}^k}{k!} = [M(C_k)]_{k+1} + [M(C_k)]_{k-1}$$

and in fact

$$\int_{\underline{x} \in \partial C_{m-k}} I(C_k)(\underline{x}) = (-1)^{(k+1)(m-k-1)} \int_{\underline{x} \in C_{m-k}} \frac{d\underline{x}^{m-k-1}}{(m-k-1)!} \wedge [M(C_k)]_{k+1}.$$

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