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Universal and homogeneous embeddings of dual polar spaces of rank 3 defined over quadratic alternative division algebras

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Abstract

Suppose $\mathbb{O}$ is an alternative division algebra that is quadratic over some subfield $\mathbb{K}$ of its center $Z(\mathbb{O})$. Then with $(\mathbb{O}, \mathbb{K})$, there is associated a dual polar space. We provide an explicit representation of this dual polar space into a $(6n + 7)$-dimensional projective space over $\mathbb{K}$, where $n = \dim_{\mathbb{K}}(\mathbb{O})$. We prove that this embedding is the universal one, provided $|\mathbb{K}| > 2$. When $\mathbb{O}$ is not an inseparable field extension of $\mathbb{K}$, we show that this universal embedding is the unique polarized one. When $\mathbb{O}$ is an inseparable field extension of $\mathbb{K}$, then we determine the minimal full polarized embedding, and show that all homogeneous embeddings are either universal or minimal. We also provide explicit generators of the corresponding projective representations of the little projective group associated with the (dual) polar space.

Keywords: composition division algebra, Cayley-Dickson division algebra, (dual) polar space, (universal, polarized) projective embedding, groups of mixed type
MSC2010: 17A45, 20E42, 51E24 (primary); 17A35, 17A75, 51A45, 51A50 (secondary)

1 Introduction

The classification of spherical buildings of rank at least 3 by Jacques Tits in [27] was a milestone in the theory of groups and in the theory of incidence geometries, bridging these two worlds. One of the connections that became apparent was the one between (projective) modular representations of (covers of) the automorphism group of the building and projective embeddings of all sorts of Grassmann-point-line geometries associated to the building. Despite the connection, there is no one-to-one correspondence between these representations and these projective embeddings. Rather, representation theory gives restricted possibilities and hints of what certain (universal) projective embeddings could be, and conversely, projective embeddings provide efficient ways to describe certain representations. Hence, it is natural to try to determine the (universal) projective embeddings of point-line geometries related to a spherical building.

In this paper, we will be concerned with a class of spherical buildings of type B_3. The most important related point-line geometries here are *polar spaces*, *line Grassmannians of polar spaces* and *dual polar spaces*, all of rank 3. There has been some activity in the literature to determine the universal embeddings of the dual polar spaces related to some classes of buildings of type B_3. In the present paper, we will achieve this goal for the buildings of type B_3 associated to quadratic alternative division algebras.

These comprise every so-called non-embeddable polar space, which arises from the Tits index $\mathsf{E}_{7,3}^{28}$ in a split algebraic group of type E_7 such that the field $\mathbb{E}$ of definition is included in the Cayley-Dickson division algebra $\mathbb{O}$ over a subfield $\mathbb{K}$ of $\mathbb{E}$, and $\mathbb{E}$ is a quadratic Galois extension of $\mathbb{K}$. The corresponding dual polar space Δ is hence included as a point-line geometry in the highest weight geometry of a building of type E_7 (over $\mathbb{E}$). The latter admits an embedding in the corresponding highest weight module M, which is a 55-dimensional projective space over $\mathbb{E}$. It now follows from a construction of Bernhard Mühlherr [20] (which is already apparent in [25], see page 89) that Δ in fact lives in a $\mathbb{K}$-Baer subspace M' of M and hence is fully embedded in M'. Since it has been proved by Ronan and Smith that the embedding of the highest weight geometry of the building of type E_7 in M is universal [21], it is likely that also the embedding of Δ in M' is universal. Our results will imply that this is indeed the case. Note that, since the start in the early nineties of the determination of the universal embeddings of dual polar spaces, the case of the non-embeddable polar spaces remained open and untreated, see for instance Shult [23, p. 229].

We note that our results not only prove universality, but also provide a very explicit construction of the universal embedding. In particular it makes Mühlherr's existence proof very concrete. Also, in the real case ($\mathbb{K} = \mathbb{R}$), the embedding is known under the name of *Veronesean embedding*, and it arises from the so-called *polar representation* of the maximal compact subgroup of the simple noncompact Lie group of type $\mathsf{E}_{7(-25)}$, or, equivalently, from an *s-representation* of the latter simple Lie group, see [8, 15]. No simple model of this embedding was known to date, and so, the present paper remedies this by constructing this embedding, even over an arbitrary ground field.

Another class of buildings of type B_3 that we cover is the class of the buildings of so-called mixed type which correspond to groups of mixed type and were introduced by Jacques Tits in [27, Section 10.3]. For those dual polar spaces nothing is known besides the trivial observation that they can be included as full subgeometries in dual polar spaces of symplectic (split) type C_3, giving rise to induced projective embeddings of finite dimension. Also for these dual polar spaces we will determine the universal embeddings, which could be infinite-dimensional.

To prove our results, we rely on a recent coordinate description of the polar spaces of rank 3 defined over quadratic alternative division algebras as given in [13]. Since we will need to do some explicit computations in such algebras, we recall some crucial basic properties of those in Section 2. By providing coordinates, we present in Section 3 an explicit construction of a projective embedding of each dual polar space of rank 3 related to a quadratic alternative division algebra. We show that this embedding is polarized, and derive necessary and sufficient conditions under which the embedding is isomorphic to the minimal full polarized embedding. In Section 4 we show that the given embedding is isomorphic to the known Grassmann embedding in the case the quadratic alternative division algebra $\mathbb{O}$ is trivial, or a Galois extension of degree 2. In each of the latter two cases, it follows from [6, 7, 12, 18] that the embedding is universal if all lines of the dual polar space contain at least four points. In Section 5, we show that for a general quadratic alternative division ring $\mathbb{O}$, the given embedding is universal, except in two instances related to the field of order 2. Since all methods in the literature used so far to prove universality of a given embedding somehow use the finite-dimensionality of the embedding, and in the mixed case we deal with possibly infinite dimensions, we have to use some non-traditional ideas to finish the proof. As a corollary we know precisely when the described embedding is unique as a polarized embedding.

We end this paper with an application in the theory of projective representations of groups. We show that all homogeneous embeddings of the dual polar spaces under consideration are either universal or minimal, and provide explicit generators of the corresponding projective representations of the little projective groups of the polar spaces. In particular, we will provide projective representations for all such groups of mixed type.

2 Alternative division rings, Moufang planes and associated polar spaces

2.1 Alternative division rings and Moufang planes

An *alternative division ring* is a set $\mathbb{D}$ of size at least 2 which is endowed with two binary operations, an addition $+$ and a multiplication $\cdot$, satisfying the following properties:

- the structure $(\mathbb{D}, +)$ is a commutative group;
- the multiplication is left- and right-distributive with respect to the addition;
- there exists a (necessarily unique) neutral element 1 for the multiplication;
- if 0 denotes the neutral element for the addition, then for every $a \in \mathbb{D} \setminus \{0\}$, there exists a (necessarily unique) element $a^{-1} \in \mathbb{D}$ such that $a^{-1} \cdot a = 1 = a \cdot a^{-1}$;
- for every $a \in \mathbb{D} \setminus \{0\}$ and every $b \in \mathbb{D}$, we have $a^{-1} \cdot (a \cdot b) = b = (b \cdot a) \cdot a^{-1}$.

It is a costume to denote the product $a \cdot b$ of two elements $a, b \in \mathbb{D}$ by ab. In the literature, one can find alternative but equivalent definitions for the notion of alternative division ring, see e.g. Tits and Weiss [28].

The alternative division rings with associative multiplication are precisely the skew fields. An important class of (non-associative) alternative division rings are the so-called Cayley-Dickson division algebras. Explicit constructions of such alternative division rings can be found in Jacobson [17, p. 426] (for characteristic distinct from 2), Schafer [22, p. 5] (for characteristic distinct from 2), Tits and Weiss [28, Section 9.8] and Van Maldeghem [29, Appendix B].

The multiplication in a Cayley-Dickson division algebra is not associative. In fact, it is a result due to Bruck and Kleinfeld [3] and Kleinfeld [19] that the Cayley-Dickson division algebras are the only alternative division rings in which the multiplication is not associative. A proof of that result can also be found in Tits and Weiss [28, Chapter 20] and Van Maldeghem [29, Appendix B]. The proof given in [29] is attributed to Jacques Tits.

With every alternative division ring $\mathbb{D}$, we can associate a point-line geometry $\pi_{\mathbb{D}}$ in the following way. There are three types of points:

- a symbol (∞), where $\infty \notin \mathbb{D}$;
- symbols (s), where $s \in \mathbb{D}$;
- symbols (a, b), where $a, b \in \mathbb{D}$.

There are also three types of lines:

- the set $[\infty] := \{(\infty)\} \cup \{(\lambda) : \lambda \in \mathbb{D}\}$;
- the sets $[k] := \{(\infty)\} \cup \{(k, \lambda) : \lambda \in \mathbb{D}\}$;
- the sets $[m, k] := \{(m)\} \cup \{(\lambda, m\lambda + k) : \lambda \in \mathbb{D}\}$.

It is well-known (and straightforward to verify) that $\pi_{\mathbb{D}}$ is a projective plane. In fact, $\pi_{\mathbb{D}}$ is a Moufang plane which means that every line is a so-called translation line. It is also known that every Moufang plane can be coordinatized by an alternative division ring in the above described way. More background information on the coordinatization of (Moufang) projective planes can be found in the monograph [16] by Hughes and Piper.

Throughout this section, $\mathbb{O}$ is an alternative division ring. The *center* $Z(\mathbb{O})$ of $\mathbb{O}$ is defined to be the set of all $a \in \mathbb{O}$ such that $ab = ba$, $a(bc) = (ab)c$, $(ba)c = b(ac)$ and $(bc)a = b(ca)$ for all $b, c \in \mathbb{O}$. Clearly, $Z(\mathbb{O})$ is a field and $\mathbb{O}$ can be regarded as an algebra over $Z(\mathbb{O})$. We are especially interested in the case of *quadratic* alternative division rings, which we define next.

2.2 Quadratic alternative division rings

Suppose $\mathbb{F}$ is a subfield of $Z(\mathbb{O})$. We say that $\mathbb{O}$ *is quadratic over* $\mathbb{F}$ if there exist (necessarily unique) functions $T : \mathbb{O} \to \mathbb{F}$ and $N : \mathbb{O} \to \mathbb{F}$ such that:

- $a^2 - T(a)a + N(a) = 0$ for any $a \in \mathbb{O}$;
- $T(a) = 2a$ and $N(a) = a^2$ for any $a \in \mathbb{F}$.

It is well-known that the map $\sigma : \mathbb{O} \to \mathbb{O} : x \mapsto T(x) - x$ is an involution of $\mathbb{O}$, which is called the *standard involution* of $\mathbb{O}$ (with respect to $\mathbb{K}$). We have $T(x) = x + x^\sigma$ and $N(x) = x^{\sigma+1} := x^\sigma x = x x^\sigma$ for every $x \in \mathbb{O}$.

Examples. (1) Suppose that either $\mathbb{O}$ is a field and $\mathbb{F} = \mathbb{O}$ or $\mathbb{O}$ is a field of characteristic 2 and $\mathbb{F}$ is proper subfield of $\mathbb{O}$ containing all squares of $\mathbb{O}$. Then $\mathbb{O}$ is a quadratic alternative division ring over $\mathbb{F}$, and the associated standard involution is trivial.

(2) Suppose that $\mathbb{O}$ is a quadratic Galois extension of the field $\mathbb{F}$. Then $\mathbb{O}$ is a quadratic alternative division ring over $\mathbb{F}$, and the associated standard involution is the unique nontrivial element in $\mathsf{Gal}(\mathbb{O}/\mathbb{K})$.

(3) Suppose $\mathbb{O}'$ is an associative quadratic alternative division ring over $\mathbb{F}'$ whose standard involution σ' is nontrivial. Suppose we are given an $\ell \in \mathbb{F}'$ such that $aa^{\sigma'} - \ell bb^{\sigma'} \neq 0$ for all $a, b \in \mathbb{O}'$ with $(a, b) \neq (0, 0)$. Define $\mathbb{O} = \{(a, b) \in \mathbb{O}' \times \mathbb{O}'\}$, with termwise addition and multiplication defined by $(a, b)(c, d) = (ac + \ell \cdot db^{\sigma'}, a^{\sigma'} d + cb)$. The elements $(a, 0)$ with $a \in \mathbb{F}'$ define a subfield $\mathbb{F}$ of $\mathbb{O}$ isomorphic to $\mathbb{F}'$. Now, $\mathbb{O}$ is also a quadratic alternative division ring over $\mathbb{F}$ with associated maps $T[(a, b)] = (a + a^{\sigma'}, 0)$ and $N[(a, b)] = (aa^{\sigma'} - \ell bb^{\sigma'}, 0)$. If $\mathbb{O}'$ is a quadratic Galois extension of $\mathbb{F}'$, then $\mathbb{O}$ is a so-called *quaternion division algebra* over $\mathbb{F}$ which is a non-commutative alternative division algebra of dimension 4 over $\mathbb{F}$. If $\mathbb{O}'$ is a non-commutative quaternion division algebra over $\mathbb{F}'$, then $\mathbb{O}$ is a so-called *Cayley-Dickson division algebra* over $\mathbb{F}$ which is a non-associative alternative division algebra of dimension 8 over $\mathbb{F}$. We refer to Chapter 9 of Tits and Weiss [28] for more details.

The following proposition is precisely Theorem 20.3 of [28].

Proposition 2.1 ([28]) *Suppose $\mathbb{O}$ is an alternative division ring which is quadratic over some subfield $\mathbb{K}$ of its center $Z(\mathbb{O})$. Let $T : \mathbb{O} \to \mathbb{K}$ and $N : \mathbb{O} \to \mathbb{K}$ be the unique functions as defined above and put $a^\sigma := T(a) - a$ for all $a \in \mathbb{O}$. Then exactly one of the following holds:*

(*a*) *$\mathbb{O} = \mathbb{K}$ is a field and $\sigma = 1$;*

(*b*) *$\mathbb{O}$ and $\mathbb{K}$ are fields, $\mathbb{O}$ is a separable quadratic extension of $\mathbb{K}$ and σ is the nontrivial element of the Galois group $\mathsf{Gal}(\mathbb{O}/\mathbb{K})$;*

(*c*) *$\mathbb{O}$ is a field of characteristic 2, $\sigma = 1$ and $\mathbb{O}^2 \subseteq \mathbb{K} \neq \mathbb{O}$;*

(*d*) *$\mathbb{O}$ is a quaternion division algebra, $\mathbb{K} = Z(\mathbb{O})$ and σ is nontrivial;*

(*e*) *$\mathbb{O}$ is a Cayley-Dickson division algebra over $\mathbb{K} = Z(\mathbb{O})$ and σ is nontrivial.*

In the sequel of this section, we suppose that $\mathbb{O}$ is an alternative division ring which is quadratic over some subfield $\mathbb{K}$ of its center $Z(\mathbb{O})$. By Proposition 2.1, there are five possibilities for the pair $\mathcal{T} := (\mathbb{O}, \mathbb{K})$. With σ denoting the standard involution, we know that for each $a \in \mathbb{O}$, the elements $T(a) = a + a^\sigma$ and $N(a) = a^{\sigma+1}$ belong to $\mathbb{K}$. If $a \in \mathbb{K}$, then $a^\sigma = a$. If $a \neq 0$, then, since $a^\sigma = a^{\sigma+1} \cdot a^{-1}$ with $a^{\sigma+1} \in \mathbb{K}$, we have $(a^\sigma)^{-1} = (a^{-1})^\sigma = \frac{a}{a^{\sigma+1}}$. We denote $(a^\sigma)^{-1} = (a^{-1})^\sigma$ also by $a^{-\sigma}$.

In several proofs, we will invoke some properties of alternative division rings. In Propositions 2.2 and 2.3 below, we state some results which we will need later.

For all $a, b, c \in \mathbb{O}$, we define the *commutator* $[a, b]$ of a and b as the element $ab - ba$ and the *associator* $[a, b, c]$ of a, b and c as the element $(ab)c - a(bc)$. Since $\mathbb{O}$ is an alternative division ring, we have $[a, b] = 0$ for all $a, b \in \mathbb{O}$ for which $\{a, b\} \cap \mathbb{K} \neq \emptyset$, $[a, b, c] = 0$ for all $a, b, c \in \mathbb{O}$ for which $\{a, b, c\} \cap \mathbb{K} \neq \emptyset$ and $[a^{-1}, a, b] = [b, a, a^{-1}] = 0$ for all $a, b \in \mathbb{O}$ for which $a \neq 0$. The commutator can be regarded as a map from $\mathbb{O}^2$ to $\mathbb{O}$ and the associator can be regarded as a map from $\mathbb{O}^3$ to $\mathbb{O}$. These maps are $\mathbb{K}$-linear in each of their components.

The following properties of alternative division rings are well-known, see e.g. Bruck and Kleinfeld [3], Tits and Weiss [28, Chapter 9] and Van Maldeghem [29, Appendix B].

Proposition 2.2 (1) *If $a, b, c \in \mathbb{O}$, then $[a, b, c] = 0$ if a, b and c are not mutually distinct*[1]*.*

(2) *We have $[b, a] = -[a, b]$ for all $a, b \in \mathbb{O}$.*

(3) *If $a_1, a_2, a_3 \in \mathbb{O}$, then $[a_{\pi(1)}, a_{\pi(2)}, a_{\pi(3)}] = sgn(\pi) \cdot [a_1, a_2, a_3]$ for any permutation π of $\{1, 2, 3\}$.*

(4) *The* Moufang identities *hold in $\mathbb{O}$. This means that $a(b(ac)) = (aba)c$, $((ab)c)b = a(bcb)$ and $(ac)(ba) = a(cb)a$ for all $a, b, c \in \mathbb{O}$.*

(5) *For all $a, b, c \in \mathbb{O}$, we have $a \cdot [a, b, c] = [a, ba, c] = [a, b, ca]$ and $[a, b, c] \cdot a = [a, ab, c] = [a, b, ac]$.*

(6) *The subring generated by two distinct elements of $\mathbb{O}$ is associative.*

The following properties can be derived from Proposition 2.2, see De Bruyn and Van Maldeghem [13].

Proposition 2.3 (1) *For all $a, b, c \in \mathbb{O}$, we have $a^\sigma \cdot [a, b, c] = [a, ba^\sigma, c] = [a, b, ca^\sigma]$ and $[a, b, c] \cdot a^\sigma = [a, a^\sigma b, c] = [a, b, a^\sigma c]$.*

(2) *For all $a, b, c \in \mathbb{O}$ with $a \neq 0$, we have $a^{-1} \cdot [a, b, c] = [a, ba^{-1}, c] = [a, b, ca^{-1}]$ and $[a, b, c] \cdot a^{-1} = [a, a^{-1}b, c] = [a, b, a^{-1}c]$.*

(3) *For all $a, b, c \in \mathbb{O}$, we have $[a^\sigma, b] = [a, b^\sigma] = -[a, b]$ and $[a^\sigma, b, c] = [a, b^\sigma, c] = [a, b, c^\sigma] = -[a, b, c]$.*

(4) *For all $a, b, c \in \mathbb{O}$, we have $[a, b]^\sigma = -[a, b]$ and $[a, b, c]^\sigma = -[a, b, c]$.*

(5) *For all $a, b \in \mathbb{O}$, we have $(ab)^{\sigma+1} = a^{\sigma+1} b^{\sigma+1}$.*

(6) *Let $a, b, c \in \mathbb{O}$. Then $T(ab) = T(ba)$ and $T((ab)c) = T(a(bc))$. Hence, $T(a(bc)) = T((ab)c) = T(b(ca)) = T((bc)a) = T(c(ab)) = T((ca)b)$.*

(7) *For all $a, b, c \in \mathbb{O}$, we have $a^{\sigma+1}(b^\sigma c + c^\sigma b) = (a^\sigma b^\sigma)(ca) + (a^\sigma c^\sigma)(ba) = (b^\sigma a^\sigma)(ac) + (c^\sigma a^\sigma)(ab)$.*

[1]So, $(ab)a = a(ba)$ for all $a, b \in \mathbb{O}$. We denote this element also by aba.

(8) *For all $a, b, c, d \in \mathbb{O}$, we have $a^\sigma((bc)d) + b^\sigma((ac)d) = (c(da^\sigma))b + (c(db^\sigma))a$.*

For all $a, b, c \in \mathbb{O}$, we define

$$S(a, b, c) := T((ab)c) = T(a(bc)) = T((bc)a) = T(b(ca)) = T((ca)b) = T(c(ab)).$$

Then $S(a, b, c) = S(b, c, a) = S(c, a, b) = S(c^\sigma, b^\sigma, a^\sigma) = S(a^\sigma, c^\sigma, b^\sigma) = S(b^\sigma, a^\sigma, c^\sigma)$.

Proposition 2.4 *If $a, b, c \in \mathbb{O}$, then*

(1) $S(c, a, b) \cdot c - c^{\sigma+1}(b^\sigma a^\sigma) = c(ab)c = (ca)(bc)$;

(2) $S(a, b, c)^2 = S(ab, ca, bc) + 2 \cdot a^{\sigma+1} b^{\sigma+1} c^{\sigma+1}$.

Proof. (1) The equality $c(ab)c = (ca)(bc)$ is just one of the Moufang identities. We have $S(c, a, b) \cdot c - c^{\sigma+1}(b^\sigma c^\sigma) = S(c, a, b) \cdot c - ((b^\sigma a^\sigma)c^\sigma)c = \Big(S(c, a, b) \cdot c - (b^\sigma(a^\sigma c^\sigma))c\Big) - [b^\sigma, a^\sigma, c^\sigma] \cdot c = ((ca)b)c - [c, a, b] \cdot c = (c(ab))c = c(ab)c$.

(2) We have $S(a, b, c)^2 = \Big((ab)c + c^\sigma(b^\sigma a^\sigma)\Big)\Big((ab)c + c^\sigma(b^\sigma a^\sigma)\Big) = T[((ab)c)^2] + 2 \cdot a^{\sigma+1} b^{\sigma+1} c^{\sigma+1}$. By relying on (1), we find $T[((ab)c)^2] = T[((ab)c) \cdot ((ab)c)] = T[(ab)(c(ab)c)] = T[(ab)((ca)(bc))] = S(ab, ca, bc)$. ∎

2.3 Associated polar spaces

Let $\mathbb{O}$ be a quadratic alternative division algebra over the field $\mathbb{K}$ with corresponding standard involution σ. We continue with the notation of the previous subsections.

In De Bruyn and Van Maldeghem [13], we showed that with the pair $\mathcal{T} = (\mathbb{O}, \mathbb{K})$, there is associated a polar space $\mathcal{P}_\mathcal{T}$. Let ∞ be a symbol not belonging to $\mathbb{O}$. The point set of $\mathcal{P}_\mathcal{T}$, which we will denote by Ω, is the following set:

$$\{(\infty), (x_1), (x_1, x_2), (x_1, x_2; k), (x_1, x_2, x_3; k), (x_1, x_2, x_3, x_4; k) \; : \; x_1, x_2, x_3, x_4 \in \mathbb{O}, \; k \in \mathbb{K}\}.$$

The point (∞) is called the *point of Type 0*. If $x_1, x_2, x_3, x_4 \in \mathbb{O}$ and $k \in \mathbb{K}$, then (x_1) is called a *point of Type 1*, (x_1, x_2) is called a *point of Type 2*, $(x_1, x_2; k)$ is called a *point of Type 3*, $(x_1, x_2, x_3; k)$ is called a *point of Type 4* and $(x_1, x_2, x_3, x_4; k)$ is called a *point of Type 5*. We now define a number of subsets of Ω. We first define 8 families of subsets of Ω which are the planes of $\mathcal{P}_\mathcal{T}$.

(I) We denote by $[\infty]$ the set consisting of the points

$$\begin{aligned} p_1(a, b) &:= (a, b), \\ p_2(s) &:= (s), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[\infty]$ the *plane of Type I.*

(II) For every $k \in \mathbb{K}$, we denote by $[k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a,b;k), \\ p_2(s) &:= (s), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[k]$ a *plane of Type II.*

(III) For every $x \in \mathbb{O}$ and every $k \in \mathbb{K}$, we denote by $[x;k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (x,a,b;k), \\ p_2(s) &:= (-x^\sigma, s), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x;k]$ a *plane of Type III.*

(IV) For every $x \in \mathbb{O}$ and all $k, l \in \mathbb{K}$, we denote by $[x;k,l]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, x+la, b; k + x^\sigma a + a^\sigma x + la^{\sigma+1}), \\ p_2(s) &:= (x^\sigma, s; l), \\ p_3^* &:= (\infty), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x;k,l]$ a *plane of Type IV.*

(V) For all $x_1, x_2 \in \mathbb{O}$ and every $k \in \mathbb{K}$, we denote by $[x_1, x_2; k]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (-x_2^\sigma, -x_1^\sigma, a, b; k), \\ p_2(s) &:= (s, x_1 + x_2 s), \\ p_3^* &:= (x_2), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2; k]$ a *plane of Type V.*

(VI) For all $x_1, x_2 \in \mathbb{O}$ and all $k, l \in \mathbb{K}$, we denote by $[x_1, x_2; k, l]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (-x_2^\sigma, a, x_1^\sigma + ka, b; l + x_1 a + a^\sigma x_1^\sigma + ka^{\sigma+1}), \\ p_2(s) &:= (s, x_1 + x_2 s; k), \\ p_3^* &:= (x_2), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2; k, l]$ a *plane of Type VI.*

(VII) For all $x_1, x_2, x_3 \in \mathbb{O}$ and all $k, l \in \mathbb{K}$, we denote by $[x_1, x_2, x_3; k, l]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, -x_3^\sigma + x_1 a, b, x_2^\sigma + ka - x_1^\sigma b; l + x_2 a + a^\sigma x_2^\sigma + k a^{\sigma+1}), \\ p_2(s) &:= (x_1, s, x_2 + x_3 s; k), \\ p_3^* &:= (-x_1^\sigma, x_3), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2, x_3; k, l]$ a *plane of Type VII.*

(VIII) For all $x_1, x_2, x_3 \in \mathbb{O}$ and all $k, l, m \in \mathbb{K}$, we denote by $[x_1, x_2, x_3; k, l, m]$ the set consisting of the points

$$\begin{aligned} p_1(a,b) &:= (a, b, {x_3}^\sigma + lb + x_1 a, {x_2}^\sigma + ka + x_1^\sigma b; \\ &\qquad m + x_2 a + a^\sigma x_2^\sigma + x_3 b + b^\sigma {x_3}^\sigma + k a^{\sigma+1} + l b^{\sigma+1} + (a^\sigma x_1^\sigma) b + b^\sigma (x_1 a)), \\ p_2(s) &:= (s, x_1 + ls, x_2 + x_3 s; k + x_1^\sigma s + s^\sigma x_1 + l s^{\sigma+1}), \\ p_3^* &:= (x_1^\sigma, x_3; l), \end{aligned}$$

where $a, b, s \in \mathbb{O}$. We call $[x_1, x_2, x_3; k, l, m]$ a *plane of Type VIII.*

Each plane has the natural structure of a projective plane $\pi_{\mathbb{O}}$ as described in Subsection 2.1 by putting $(\infty) = p_3^*$, $(s) = p_2(s)$ and $(a, b) = p_3(a, b)$. The lines of these planes are the lines of $\mathcal{P}_{\mathcal{T}}$. However, it is convenient to have a more direct description of the lines, independent of their embedding in the planes. That is exactly what we present now.

(A) Let L_1 be the following set of points:

$$\{(\infty)\} \cup \{(\lambda) \,:\, \lambda \in \mathbb{O}\}.$$

We call L_1 the *line of Type A*.

(B) For every $x \in \mathbb{O}$, let $L_2(x)$ denote the following set of points:

$$\{(\infty)\} \cup \{(x, \lambda) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_2(x)$ a *line of Type B*.

(C) For every $x \in \mathbb{O}$ and every $k \in \mathbb{K}$, let $L_3(x, k)$ denote the following set of points:

$$\{(\infty)\} \cup \{(x, \lambda; k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_3(x, k)$ a *line of Type C*.

(D) For all $x, y \in \mathbb{O}$ and every $k \in \mathbb{K}$, let $L_4(x, y, k)$ denote the following set of points:

$$\{(\infty)\} \cup \{(x, y, \lambda; k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_4(x, y, k)$ a *line of Type D*.

(E) For all $x, y, z \in \mathbb{O}$, let $L_5(x, y, z)$ denote the following set of points:

$$\{(x)\} \cup \{(\lambda, z + x(\lambda - y)) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_5(x, y, z)$ a *line of Type E*. The set $L_5(x, y, z)$ contains the points (x) and (y, z).

(F) For all $x, y, z \in \mathbb{O}$ and every $k \in \mathbb{K}$, let $L_6(x, y, z, k)$ be the following set of points:

$$\{(x)\} \cup \{(\lambda, z + x(\lambda - y); k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_6(x, y, z, k)$ a *line of Type F*. The set $L_6(x, y, z, k)$ contains the points (x) and $(y, z; k)$.

(G) For all $x, y, z, u \in \mathbb{O}$ and every $k \in \mathbb{K}$ satisfying $x = -y^\sigma$, let $L_7(x, y, z, u, k)$ denote the following set of points:

$$\{(x)\} \cup \{(y, z, u, \lambda; k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_7(x, y, z, u, k)$ a *line of Type G*.

(H) For all $x, y, u, v, w \in \mathbb{O}$ and every $k \in \mathbb{K}$ satisfying $u = -x^\sigma$, let $L_8(x, y, u, v, w, k)$ be the following set of points:

$$\{(x, y)\} \cup \{(u, \lambda, w + y(\lambda - v); k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_8(x, y, u, v, w, k)$ a *line of Type H*. The set $L_8(x, y, u, v, w, k)$ contains the points (x, y) and $(u, v, w; k)$.

(I) For all $x, y, z, u, v, w \in \mathbb{O}$ and every $k \in \mathbb{K}$ satisfying $y = -u^\sigma - z^\sigma x$, let $L_9(x, y, z, u, v, w, k)$ be the following set of points:

$$\{(x, y)\} \cup \{(z, u, \lambda, w + x(\lambda - v); k) \,:\, \lambda \in \mathbb{O}\}.$$

We call $L_9(x, y, z, u, v, w, k)$ *a line of Type I*. The set $L_9(x, y, z, u, v, w, k)$ contains the points (x, y) and $(z, u, v, w; k)$.

(J) For all $k_1, k_2 \in \mathbb{K}$ and all $x, y, u, v, w \in \mathbb{O}$ satisfying $v = x^\sigma + k_1 u$, let $L_{10}(x, y, u, v, w, k_1, k_2)$ be the following set of points:

$$\{(x, y; k_1)\} \cup \{(\lambda, v + k_1(\lambda - u), w + y(\lambda - u); k_2 + x(\lambda - u) + (\lambda - u)^\sigma x^\sigma + k_1(\lambda^{\sigma+1} - u^{\sigma+1})) : \lambda \in \mathbb{O}\}$$

$$= \{(x, y; k_1)\} \cup \{(\lambda, v + k_1(\lambda - u), w + y(\lambda - u); k_2 + v^\sigma(\lambda - u) + (\lambda - u)^\sigma v + k_1(\lambda - u)^{\sigma+1}) : \lambda \in \mathbb{O}\}.$$

We call $L_{10}(x, y, u, v, w, k_1, k_2)$ a *line of Type J*. The set $L_{10}(x, y, u, v, w, k_1, k_2)$ contains the points $(x, y; k_1)$ and $(u, v, w; k_2)$.

(K) For all $x, y, z, u, v, w \in \mathbb{O}$ and all $k_1, k_2 \in \mathbb{K}$ satisfying $v = x^\sigma z + y^\sigma + k_1 u$, let $L_{11}(x, y, z, u, v, w, k_1, k_2)$ be the following set of points:

$$\{(x, y; k_1)\} \cup \{(z, \lambda, v + k_1(\lambda - u), w + x(\lambda - u); k_2 + (y + z^\sigma x)(\lambda - u) + (\lambda - u)^\sigma (y^\sigma + x^\sigma z) + k_1(\lambda^{\sigma+1} - u^{\sigma+1})) : \lambda \in \mathbb{O}\}$$

$$= \{(x, y; k_1)\} \cup \{(z, \lambda, v + k_1(\lambda - u), w + x(\lambda - u); k_2 + v^\sigma(\lambda - u) + (\lambda - u)^\sigma v + k_1(\lambda - u)^{\sigma+1}) : \lambda \in \mathbb{O}\}.$$

We call $L_{11}(x, y, z, u, v, w, k_1, k_2)$ a *line of Type K*. The set $L_{11}(x, y, z, u, v, w, k_1, k_2)$ contains the points $(x, y; k_1)$ and $(z, u, v, w; k_2)$.

(L) For all $x, y, z, u, v, w, r \in \mathbb{O}$ and all $k_1, k_2 \in \mathbb{K}$ satisfying $r = z^\sigma - y^\sigma(xu - v) + k_1 u - x^\sigma w$, let $L_{12}(x, y, z, u, v, w, r, k_1, k_2)$ be the following set of points:

$$\{(x, y, z; k_1)\} \cup \{(\lambda, v + x(\lambda - u), w + y(\lambda - u), r + k_1(\lambda - u) - x^\sigma(y(\lambda - u)); k_2 + (z - (xu - v)^\sigma y)(\lambda - u) + (\lambda - u)^\sigma (z^\sigma - y^\sigma(xu - v)) + k_1(\lambda^{\sigma+1} - u^{\sigma+1})) : \lambda \in \mathbb{O}\}$$

$$= \{(x, y, z; k_1)\} \cup \{(\lambda, v + x(\lambda - u), w + y(\lambda - u), r + k_1(\lambda - u) - x^\sigma(y(\lambda - u)); k_2 + (r^\sigma + w^\sigma x)(\lambda - u) + (\lambda - u)^\sigma (r + x^\sigma w) + k_1(\lambda - u)^{\sigma+1}) : \lambda \in \mathbb{O}\}.$$

We call $L_{12}(x, y, z, u, v, w, r, k_1, k_2)$ a *line of Type L*. The set $L_{12}(x, y, z, u, v, w, r, k_1, k_2)$ contains the points $(x, y, z; k_1)$ and $(u, v, w, r; k_2)$.

Figure 1 pictures the incidence of the different types of points, lines and planes on an octahedron.

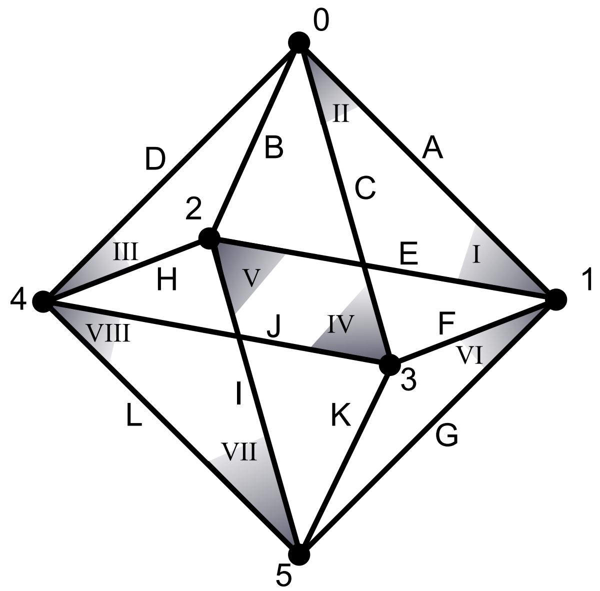

Figure 1: Incidence for types of points, lines and planes

The following proposition was proved in De Bruyn and Van Maldeghem [13]. We use the notation of the tables in Tits [26].

Proposition 2.5 (1) *If $\mathbb{O} = \mathbb{K}$ is a field, then $\mathcal{P}_\mathcal{T}$ is isomorphic to the symplectic polar space of rank 3 defined over the field $\mathbb{K}$, i.e., the polar space associated with a split algebraic group of type C_3.*

(2) *If $\mathbb{O}$ and $\mathbb{K}$ are fields such that $\mathbb{O}$ is a separable quadratic extension of $\mathbb{K}$, then $\mathcal{P}_\mathcal{T}$ is isomorphic to a so-called Hermitian rank 3 polar space associated with $(\mathbb{O}, \mathbb{K})$,i.e., the polar space associated with a quasi-split group of type ${}^2\mathsf{A}^{(1)}_{5,3}$.*

(3) *If $\mathbb{O}$ is a field of characteristic 2 and $\mathbb{O}^2 \subseteq \mathbb{K} \neq \mathbb{O}$, then $\mathcal{P}_\mathcal{T}$ is isomorphic to the polar space of rank 3 of mixed type associated with $(\mathbb{O}, \mathbb{K})$.*

(4) *If $\mathbb{O}$ is a quaternion division algebra and $\mathbb{K} = Z(\mathbb{O})$, then $\mathcal{P}_\mathcal{T}$ is isomorphic to the so-called quaternionic polar space of rank 3 associated with $\mathbb{O}$, i.e., the polar space associated with group of type ${}^1\mathsf{D}^{(2)}_{6,3}$.*

(5) *If $\mathbb{O}$ is a Cayley-Dickson division algebra and $\mathbb{K} = Z(\mathbb{O})$, then $\mathcal{P}_\mathcal{T}$ is isomorphic to the non-embeddable polar space of rank 3 associated with the Cayley-Dickson division algebra $\mathbb{O}$, i.e., the polar space associated with a group of type $\mathsf{E}^{28}_{7,3}$.*

3 A full projective embedding of the dual polar space associated with $\mathcal{P}_\mathcal{T}$

We continue with the notation introduced in Section 2. Recall that $\mathbb{O}$ is an alternative division ring which is quadratic over the field $\mathbb{K}$ which is contained in the center $Z(\mathbb{O})$ of $\mathbb{O}$. In case that $\mathbb{O}$ is not abelian, $\mathbb{K}$ coincides with $Z(\mathbb{O})$. With $\mathcal{T} := (\mathbb{O}, \mathbb{K})$, there is associated a polar space $\mathcal{P}_\mathcal{T}$ of rank 3, and we denote the corresponding dual polar space by $\Delta_\mathcal{T}$, i.e., the point-line geometry obtained from $\mathcal{P}_\mathcal{T}$ by taking as point set the set of planes of $\mathcal{P}_\mathcal{T}$ and as line set the set of lines of $\mathcal{P}_\mathcal{T}$, using symmetrized inclusion as incidence relation.

The alternative division ring $\mathbb{O}$ can be regarded in a natural way as a vector space over $\mathbb{K}$, and let n be its dimension. So, the set $V := \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{K}$ can be regarded in a natural way as an $(8 + 6n)$-dimensional vector space over $\mathbb{K}$ and hence its nonzero elements can serve as the homogeneous coordinates of the points of the projective space $\mathsf{PG}(7 + 6n, \mathbb{K}) = \mathsf{PG}(V)$. Note that n can be infinite because we also include the mixed case.

With every point α of $\Delta_\mathcal{T}$, i.e. with every plane α of $\mathcal{P}_\mathcal{T}$, we now define a certain point $e(\alpha)$ of $\mathsf{PG}(7 + 6n, \mathbb{K})$.

In what follows, x, x_1, x_2, x_3 are arbitrary elements of $\mathbb{O}$, whereas k, ℓ, m arbitrarily belong to $\mathbb{K}$.

Type I. If α is the plane $[\infty]$, then $e(\alpha)$ has coordinates

$$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

Type II. If α is the plane $[k]$, then $e(\alpha)$ has coordinates

$$(0,0,0,0,0,0,0,0,1,0,0,0,0,k).$$

Type III. If α is the plane $[x;k]$, then $e(\alpha)$ has coordinates

$$(0,0,0,0,0,0,0,0,x^{\sigma+1},1,-x,0,0,k).$$

Type IV. If α is the plane $[x;k,\ell]$, then $e(\alpha)$ has coordinates

$$(0,1,0,0,0,0,0,0,k,\ell,x,0,0,\ell k-x^{\sigma+1}).$$

Type V. If α is the plane $[x_1,x_2;k]$, then $e(\alpha)$ has coordinates

$$(0,0,0,0,0,0,0,1,x_1^{\sigma+1},x_2^{\sigma+1},-x_1^{\sigma}x_2,x_2,x_1,k).$$

Type VI. If α is the plane $[x_1,x_2;k,\ell]$, then $e(\alpha)$ has coordinates

$$(0,x_2^{\sigma+1},0,1,0,x_2,0,k,\ell,kx_2^{\sigma+1},-x_1^{\sigma}x_2,kx_2,x_1,k\ell-x_1^{\sigma+1}).$$

Type VII. If α is the plane $[x_1,x_2,x_3;k,\ell]$, then $e(\alpha)$ has coordinates

$$(0,x_3^{\sigma+1},1,x_1^{\sigma+1},-x_1,-x_3x_1,x_3,k,\ell x_1^{\sigma+1}+kx_3^{\sigma+1}+x_2(x_1^{\sigma}x_3{}^{\sigma})+(x_3x_1)x_2^{\sigma},\ell,$$
$$-x_3{}^{\sigma}x_2-\ell x_1,x_2,x_2x_1^{\sigma}+kx_3,k\ell-x_2^{\sigma+1}).$$

Type VIII. If α is the plane $[x_1,x_2,x_3;k,\ell,m]$, then $e(\alpha)$ has coordinates

$$(1,m,\ell,k,x_1,x_2,x_3,\ell k-x_1^{\sigma+1},km-x_2^{\sigma+1},m\ell-x_3^{\sigma+1},mx_1-x_3{}^{\sigma}x_2,\ell x_2-x_3x_1,kx_3-x_2x_1^{\sigma},$$
$$m\ell k+x_3{}^{\sigma}(x_2x_1^{\sigma})+(x_1x_2^{\sigma})x_3-mx_1^{\sigma+1}-\ell x_2^{\sigma+1}-kx_3^{\sigma+1}).$$

Proposition 3.1 *The map e from the point set of $\Delta_{\mathcal{T}}$ to the point set of $\mathsf{PG}(7+6n,\mathbb{K})$ is injective. Moreover, the image of e generates the whole projective space $\mathsf{PG}(7+6n,\mathbb{K})$.*

Proof. Note first that the images of two planes of different types are distinct. Indeed, the image of the plane of lower type has a zero on the coordinate place where the image of the plane of higher type has a one. But also two different planes of the same type must have distinct images since all parameters (or in rare cases the negative of some) of the plane appear separately in the coordinate tuple. Hence e is injective.

For every $i\in\{1,2,\ldots,14\}$, let E_i denote the subspace of V consisting of those vectors of V all whose coordinates are zero except possibly for those in the i-th position. The subspace E_i has dimension either 1 or n. The second assertion can be easily verified starting from the image of the plane of Type I, which generates E_{14}, and then considering each type in ascending type order. Indeed, the images of planes of Type I and II clearly generate E_9 and E_{14}. Adding the images of planes of Type III, we obtain E_{10} (put $x = 0$) and E_{11} (x arbitrary). Type IV adds E_2, Type V adds E_8 (putting $x_1 = x_2 = 0$), E_{12} (putting $x_1 = 0$ and x_2 arbitrary) and E_{13} (putting $x_2 = 0$ and x_1 arbitrary). Similarly, Type VI adds E_4 and E_6, whereas Type VII adds E_3, E_5 and E_7. Finally, the image of any plane of Type VIII has a nontrivial component in E_1, which concludes the proof of the proposition. ∎

Proposition 3.2 *For every line M of $\mathcal{P}_\mathcal{T}$ the set of planes of $\mathcal{P}_\mathcal{T}$ through M has the form $\{\alpha^*\} \cup \{\alpha(k) : k \in \mathbb{K}\}$, where the types of all planes $\alpha(k)$, $k \in \mathbb{K}$, are equal to each other and distinct from the type of the plane α^*. Moreover, the set $\{e(\alpha^*)\} \cup \{e(\alpha(k)) : k \in \mathbb{K}\}$ is a line of $\mathsf{PG}(7+6n, \mathbb{K})$.*

Proof. We prove this for each type of line separately. We will perform the calculations only for the most involved case, namely the case of lines of Type L. We simply list the other cases without details, but the reader can easily reconstruct these details for himself.

(1) The line L_1 is incident with the planes $[\infty]$ and $[k]$, $k \in \mathbb{K}$. The image under e is the line generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ (0,0,0,0,0,0,0,0,1,0,0,0,0,0). \end{cases}$$

(2) The line $L_2(-x^\sigma)$, $x \in \mathbb{O}$, is incident with the planes $[\infty]$ and $[x;k]$, $k \in \mathbb{K}$. The image under e is the line generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ (0,0,0,0,0,0,0,0,x^{\sigma+1},1,-x,0,0,0). \end{cases}$$

(3) The line $L_3(x^\sigma, k)$, $x \in \mathbb{O}$ and $k \in \mathbb{K}$, is incident with the planes $[k]$ and $[x;\ell,k]$, $\ell \in \mathbb{K}$. The image under e is the line generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,1,0,0,0,0,k), \\ (0,1,0,0,0,0,0,0,0,k,x,0,0,-x^{\sigma+1}). \end{cases}$$

(4) The line $L_4(x,y,k)$, $x, y \in \mathbb{O}$ and $k \in \mathbb{K}$, is incident with the planes $[x;k]$ and $[y-\ell x; k+\ell x^{\sigma+1} - y^\sigma x - x^\sigma y, \ell]$, $\ell \in \mathbb{K}$. The image under e is generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,x^{\sigma+1},1,-x,0,0,k), \\ (0,1,0,0,0,0,0,0,k-y^\sigma x - x^\sigma y, 0, y, 0, 0, -y^{\sigma+1}). \end{cases}$$

(5) Let $x, y, z \in \mathbb{O}$. Then $L_5(x,y,z)$ is the unique line of Type E containing (x) and (y,z). Since every line of Type E contains a point of the form $(0,*)$, we may without loss of

generality suppose that $y = 0$. Now, the line $L_5(x, 0, z)$ is incident with the planes $[\infty]$ and $[z, x; k]$, $k \in \mathbb{K}$. The image under e is generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,0,0,0,0,0,1), \\ (0,0,0,0,0,0,0,1,z^{\sigma+1},x^{\sigma+1},-z^\sigma x,x,z,0). \end{cases}$$

(6) Let $x, y, z \in \mathbb{O}$ and $k \in \mathbb{K}$. Then $L_6(x, y, z, k)$ is the unique line of Type F containing (x) and $(y, z; k)$. Since every line of Type F contains a point of the form $(0, *; *)$, we may without loss of generality suppose that $y = 0$. Now, the line $L_6(x, 0, z, k)$ is incident with the planes $[k]$ and $[z, x; k, l]$, $\ell \in \mathbb{K}$. The image under e is generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,1,0,0,0,0,k), \\ (0,x^{\sigma+1},0,1,0,x,0,k,0,kx^{\sigma+1},-z^\sigma x,kx,z,-z^{\sigma+1}). \end{cases}$$

(7) The line $L_7(x, -x^\sigma, -z^\sigma, u^\sigma, k)$, $x, z, u \in \mathbb{O}$ and $k \in \mathbb{K}$, is incident with the planes $[z, x; k]$ and $[u + \ell z, x; \ell, k + \ell z^{\sigma+1} + zu^\sigma + uz^\sigma]$, $\ell \in \mathbb{K}$. The image under e is generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,1,z^{\sigma+1},x^{\sigma+1},-z^\sigma x,x,z,k), \\ (0,x^{\sigma+1},0,1,0,x,0,0,k+uz^\sigma+zu^\sigma,0,-u^\sigma x,0,u,-u^{\sigma+1}). \end{cases}$$

(8) For lines of Type H we may assume that the fourth parameter equals 0. Then the line $L_8(-x^\sigma, y, x, 0, w, k)$, $x, y, w \in \mathbb{O}$ and $k \in \mathbb{K}$, is incident with the planes $[x; k]$ and $[x, w, y; k, \ell]$, $\ell \in \mathbb{K}$. The image under e is generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,0,x^{\sigma+1},1,-x,0,0,k), \\ (0,y^{\sigma+1},1,x^{\sigma+1},-x,-yx,y,k,ky^{\sigma+1}+w(x^\sigma y^\sigma)+(yx)w^\sigma,0,-y^\sigma w,w,wx^\sigma+ky,-w^{\sigma+1}). \end{cases}$$

(9) For lines of Type I we may assume that the fifth parameter equals 0. Then the line $L_9(-x^\sigma, u - zx^\sigma, -z^\sigma, -u^\sigma, 0, w^\sigma, k)$, $x, z, u, w \in \mathbb{O}$ and $k \in \mathbb{K}$, is incident with the planes $[u, z; k]$ and $[x, w + \ell z, u - zx^\sigma; \ell, k + \ell z^{\sigma+1} + wz^\sigma + zw^\sigma]$, $\ell \in \mathbb{K}$. Making use of Proposition 2.2(6), Proposition 2.3(6) and the fact that $(wz^\sigma + zw^\sigma)x = x(wz^\sigma + zw^\sigma) = x(z^\sigma w + w^\sigma z) = (xz^\sigma)w + (xw^\sigma)z - [x, z^\sigma, w] - [x, w^\sigma, z] = (xz^\sigma)w + (xw^\sigma)z$, we see that the image under e is generated by the points

$$\begin{cases} (0,0,0,0,0,0,0,1,u^{\sigma+1},z^{\sigma+1},-u^\sigma z,z,u,k), \\ (0,(u-zx^\sigma)^{\sigma+1},1,x^{\sigma+1},-x,(zx^\sigma-u)x,u-zx^\sigma,0,kx^{\sigma+1}+w(x^\sigma u^\sigma)+(ux)w^\sigma, \\ \qquad k+wz^\sigma+zw^\sigma,-u^\sigma w-kx-(xw^\sigma)z,w,wx^\sigma,-w^{\sigma+1}). \end{cases}$$

(10) For lines of Type J we may assume that the third parameter is 0. Then the line $L_{10}(x^\sigma, y, 0, x, w, k_1, k_2)$, $x, y, w \in \mathbb{O}$ and $k_1, k_2 \in \mathbb{K}$, is incident with the planes $[x; k_2, k_1]$ and $[x, w, y; k_2, k_1, \ell]$, $\ell \in \mathbb{K}$. The image under e is generated by the points

$$\begin{cases} (0,1,0,0,0,0,0,0,k_2,k_1,x,0,0,k_1k_2-x^{\sigma+1}), \\ (1,0,k_1,k_2,x,w,y,k_1k_2-x^{\sigma+1},-w^{\sigma+1},-y^{\sigma+1},-y^\sigma w,k_1w-yx,k_2y-wx^\sigma, \\ \qquad y^\sigma(wx^\sigma)+(xw^\sigma)y-k_1w^{\sigma+1}-k_2y^{\sigma+1}). \end{cases}$$

(11) For lines of Type K, we may assume that the fourth parameter equals 0. Then the line $L_{11}(x^\sigma, y, -z^\sigma, 0, y^\sigma - xz^\sigma, w^\sigma, k_1, k_2)$, $x, y, z, w \in \mathbb{O}$ and $k_1, k_2 \in \mathbb{K}$, is incident with the planes $[y - zx^\sigma, z; k_1, k_2]$ and $[x, w + \ell z, y; \ell, k_1, k_2 + wz^\sigma + zw^\sigma + \ell z^{\sigma+1}]$, $\ell \in \mathbb{K}$. Making use of Proposition 2.2(6), we see that the image under e is generated by the points

$$\begin{cases} (0, z^{\sigma+1}, 0, 1, 0, z, 0, k_1, k_2, k_1 z^{\sigma+1}, -y^\sigma z + xz^{\sigma+1}, k_1 z, y - zx^\sigma, \\ \qquad k_1k_2 - y^{\sigma+1} - z^{\sigma+1}x^{\sigma+1} + y^\sigma(zx^\sigma) + (xz^\sigma)y), \\ (1, k_2 + wz^\sigma + zw^\sigma, k_1, 0, x, w, y, -x^{\sigma+1}, -w^{\sigma+1}, k_1k_2 + k_1wz^\sigma + k_1zw^\sigma - y^{\sigma+1}, \\ \qquad k_2x + (wz^\sigma)x + (zw^\sigma)x - y^\sigma w, k_1 w - yx, -wx^\sigma, \\ \qquad y^\sigma(wx^\sigma) + (xw^\sigma)y - (wz^\sigma + zw^\sigma)x^{\sigma+1} - k_1 w^{\sigma+1} - k_2 x^{\sigma+1}). \end{cases}$$

(12) For lines of Type L, we provide some more details. For such lines, we may assume that the fourth parameter is 0. So we may consider the line $L_{12}(x, y, z, 0, -v^\sigma, w^\sigma, z^\sigma - y^\sigma v^\sigma - x^\sigma w^\sigma, k_1, k_2)$, $x, y, z, v, w \in \mathbb{O}$ and $k_1, k_2 \in \mathbb{K}$. According to the definition this line contains the two points $p := (x, y, z; k_1)$ (Type 4) and $q := (0, -v^\sigma, w^\sigma, z^\sigma - y^\sigma v^\sigma - x^\sigma w^\sigma; k_2)$ (Type 5). Now, only planes of Type VII and VIII contain points of Types 4 and 5.

Suppose first some plane $\alpha^* := [x_1^*, x_2^*, x_3^*; k^*, \ell^*]$ of Type VII contains the two points p and q. From $p \in \alpha^*$ we deduce $x_1^* = x$ and $k^* = k_1$ (indeed, in the notation of Section 2.3 where we defined planes of Type VII, the point p coincides with the point $p_2(y)$). Also, $x_2^* + x_3^* y = z$. From $q \in \alpha^*$, we deduce $x_3^* = v$ and $\ell^* = k_2$ (since q coincides with $p_1(0, w^\sigma)$). All this implies $\alpha^* = [x, z - vy, v; k_1, k_2]$. Conversely, one easily calculates that a plane with these coordinates contains p and q.

Suppose now that some plane $\pi := [x_1, x_2, x_3; k, \ell, m]$ of Type VIII contains the two points p and q. From $p \in \pi$ we deduce, by comparing p with the point $p_2(x)$ in the definition of planes of Type VIII in Section 2.3,

$$\begin{cases} x_1 + \ell x = y, \\ x_2 + x_3 x = z, \\ k + x_1^\sigma x + x^\sigma x_1 + \ell x^{\sigma+1} = k_1. \end{cases}$$

Likewise, from $q \in \pi$, we deduce (noting $q = p_1(0, -v^\sigma)$),

$$\begin{cases} x_3^\sigma - \ell v^\sigma = w^\sigma \\ x_2^\sigma - x_1^\sigma v^\sigma = z^\sigma - y^\sigma v^\sigma - x^\sigma w^\sigma, \\ m - x_3 v^\sigma - v x_3^\sigma + \ell v^{\sigma+1} = k_2. \end{cases}$$

All this is easily seen to be equivalent with

$$\begin{cases} x_1 = y - \ell x, \\ x_2 = z - wx - \ell vx, \\ x_3 = w + \ell v, \\ k = k_1 - y^\sigma x - x^\sigma y + \ell x^{\sigma+1}, \\ m = k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1}. \end{cases}$$

Hence the plane $\alpha(\ell)$, $\ell \in \mathbb{K}$, with parameters

$$[y - \ell x, z - wx - \ell vx, w + \ell v; k_1 - y^\sigma x - x^\sigma y + \ell x^{\sigma+1}, \ell, k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1}]$$

is incident with $L_{12}(x, y, z, 0, -v^\sigma, w^\sigma, z^\sigma - y^\sigma v^\sigma - x^\sigma w^\sigma, k_1, k_2)$.
We now calculate all coordinates $a_0, a_1, \ldots, a_{13}$ of $e(\alpha(\ell))$. We find

$$\begin{aligned}
a_0 &= 1,\\
a_1 &= k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1},\\
a_2 &= \ell,\\
a_3 &= k_1 - y^\sigma x - x^\sigma y + \ell x^{\sigma+1},\\
a_4 &= y - \ell x,\\
a_5 &= z - wx - \ell vx,\\
a_6 &= w + \ell v,\\
a_7 &= \ell(k_1 - y^\sigma x - x^\sigma y + \ell x^{\sigma+1}) - (y - \ell x)^{\sigma+1} = -y^{\sigma+1} + \ell k_1,\\
a_9 &= \ell(k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1}) - (w + \ell v)^{\sigma+1} = -w^{\sigma+1} + \ell k_2,\\
a_{11} &= \ell(z - wx - \ell vx) - (w + \ell v)(y - \ell x) = -wy + \ell(z - vy).
\end{aligned}$$

For the coordinate a_8, we find

$$\begin{aligned}
a_8 &= (k_1 - y^\sigma x - x^\sigma y + \ell x^{\sigma+1})(k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1}) - (z - wx - \ell vx)^{\sigma+1}\\
&= (k_1 - y^\sigma x - x^\sigma y)(k_2 + wv^\sigma + vw^\sigma) - (z - wx)^{\sigma+1} + \ell^2\Big(x^{\sigma+1}v^{\sigma+1} - (x^\sigma v^\sigma)(vx)\Big)\\
&\quad +\ell\Big(k_2 x^{\sigma+1} + k_1 v^{\sigma+1} + x^{\sigma+1}(wv^\sigma + vw^\sigma) - v^{\sigma+1}(y^\sigma x + x^\sigma y) + (x^\sigma v^\sigma)(z - wx)\\
&\qquad\qquad +(z^\sigma - x^\sigma w^\sigma)(vx)\Big).
\end{aligned}$$

By Proposition 2.2(6) and Proposition 2.3(6)+(7), we have

$$\begin{aligned}
a_8 &= (k_1 - y^\sigma x - x^\sigma y)(k_2 + wv^\sigma + vw^\sigma) - (z - wx)^{\sigma+1}\\
&\quad +\ell\Big(k_2 x^{\sigma+1} + k_1 v^{\sigma+1} - (x^\sigma v^\sigma)(vy) - (y^\sigma v^\sigma)(vx) + (x^\sigma v^\sigma)z + z^\sigma(vx)\Big)\\
&= (k_1 - y^\sigma x - x^\sigma y)(k_2 + wv^\sigma + vw^\sigma) - (z - wx)^{\sigma+1}\\
&\quad +\ell\Big(k_2 x^{\sigma+1} + k_1 v^{\sigma+1} + (x^\sigma v^\sigma)(z - vy) + (z^\sigma - y^\sigma v^\sigma)(vx)\Big),\\
&= (k_1 - y^\sigma x - x^\sigma y)(k_2 + wv^\sigma + vw^\sigma) - (z - wx)^{\sigma+1}\\
&\quad +\ell\Big(k_2 x^{\sigma+1} + k_1 v^{\sigma+1} + (z - vy)(x^\sigma v^\sigma) + (vx)(z^\sigma - y^\sigma v^\sigma)\Big).
\end{aligned}$$

For the coordinate a_{10}, we find by relying on Propositions 2.2 and 2.3 that

$$\begin{aligned}
a_{10} &= (k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1})(y - \ell x) - (w + \ell v)^\sigma(z - wx - \ell vx)\\
&= k_2 y + (wv^\sigma + vw^\sigma)y - w^\sigma z + w^\sigma(wx) + \ell^2\Big(v^\sigma(vx) - v^{\sigma+1}x\Big)\\
&\quad +\ell\Big(v^{\sigma+1}y - k_2 x - (wv^\sigma)x - (vw^\sigma)x + w^\sigma(vx) + v^\sigma(wx) - v^\sigma z\Big).\\
&= k_2 y + (wv^\sigma + vw^\sigma)y - w^\sigma z + w^{\sigma+1}x + \ell\Big(-v^\sigma(z - vy) - k_2 x\Big).
\end{aligned}$$

In the latter equality, we have made use of the fact that $-(wv^\sigma)x - (vw^\sigma)x + w^\sigma(vx) + v^\sigma(wx) = -(v^\sigma w)x - (w^\sigma v)x + w^\sigma(vx) + v^\sigma(wx) = -[v^\sigma, w, x] - [w^\sigma, v, x] = 0$.

For the coordinate a_{12}, we find again by relying on Propositions 2.2 and 2.3 that

$$\begin{aligned}
a_{12} &= (k_1 - y^\sigma x - x^\sigma y + \ell x^{\sigma+1})(w + \ell v) - (z - wx - \ell vx)(y - \ell x)^\sigma \\
&= k_1 w - zy^\sigma - (y^\sigma x + x^\sigma y)w + (wx)y^\sigma + \ell\Big(k_1 v - v(yx^\sigma + xy^\sigma) + (vx)y^\sigma + zx^\sigma\Big) \\
&= k_1 w - zy^\sigma - (y^\sigma x + x^\sigma y)w + (wx)y^\sigma + \ell\Big(k_1 v - v(yx^\sigma) + [v, x, y^\sigma] + zx^\sigma\Big) \\
&= k_1 w - zy^\sigma - (y^\sigma x + x^\sigma y)w + (wx)y^\sigma + \ell\Big(k_1 v - v(yx^\sigma) - [v, y, x^\sigma] + zx^\sigma\Big) \\
&= k_1 w - zy^\sigma - (y^\sigma x + x^\sigma y)w + (wx)y^\sigma + \ell\Big(k_1 v + (z - vy)x^\sigma\Big).
\end{aligned}$$

Finally, for the coordinate a_{13}, we find

$$\begin{aligned}
a_{13} &= \ell a_8 - (w + \ell v)^\sigma a_{12} + \Big((y - \ell x)(z - wx - \ell vx)^\sigma\Big)(w + \ell v) \\
&\quad -(k_2 + wv^\sigma + vw^\sigma + \ell v^{\sigma+1})(y - \ell x)^{\sigma+1}.
\end{aligned}$$

It is clear that the coefficient of ℓ^3 in this expression is 0. We now calculate the coefficient of ℓ^2. It is equal to

$$k_2 x^{\sigma+1} + k_1 v^{\sigma+1} + (z - vy)(x^\sigma v^\sigma) + (vx)(z^\sigma - y^\sigma v^\sigma) - k_1 v^{\sigma+1} - v^\sigma((z - vy)x^\sigma) + x^{\sigma+1}(v^\sigma w)$$

$$-(xz^\sigma)v + x^{\sigma+1}(w^\sigma v) - (y(x^\sigma v^\sigma))v + v^{\sigma+1}(x^\sigma y + y^\sigma x) - x^{\sigma+1}(k_2 + wv^\sigma + vw^\sigma).$$

In this expression, the terms containing k_1 and k_2 cancel out, just like the terms containing $x^{\sigma+1}$. This leaves the following expression, where we have separated the terms containing z, and where we have substituted $xy^\sigma + yx^\sigma$ for $x^\sigma y + y^\sigma x$:

$$z(x^\sigma v^\sigma) + (vx)z^\sigma - v^\sigma(zx^\sigma) - (xz^\sigma)v - (vy)(x^\sigma v^\sigma) - (vx)(y^\sigma v^\sigma)$$

$$+(xy^\sigma + yx^\sigma)v^{\sigma+1} + v^\sigma((vy)x^\sigma) - (y(x^\sigma v^\sigma))v.$$

By Proposition 2.3(6), the first four terms add up to 0 and by Proposition 2.3(7) the next three terms do as well. Finally, the last two terms also cancel out since, freely using Propositions 2.2 and 2.3, we have

$$\begin{aligned}
& v^\sigma((vy)x^\sigma) - (y(x^\sigma v^\sigma))v \\
=\ & v^\sigma((vy)x^\sigma) + (x(y^\sigma v^\sigma))v - (y(x^\sigma v^\sigma) + x(y^\sigma v^\sigma))v \\
=\ & (v^\sigma(vy))x^\sigma + x((y^\sigma v^\sigma)v) - ((xy^\sigma)v^\sigma - [x, y^\sigma, v^\sigma] + (yx^\sigma)v^\sigma - [y, x^\sigma, v^\sigma])v \\
=\ & (yx^\sigma + xy^\sigma)v^{\sigma+1} - ((xy^\sigma + yx^\sigma)v^\sigma)v \\
=\ & 0.
\end{aligned}$$

X	U_X	V_X	X	U_X	V_X	X	U_X	V_X
A	I	II	E	I	V	I	V	VII
B	I	III	F	II	VI	J	IV	VIII
C	II	IV	G	V	VI	K	VI	VIII
D	III	IV	H	III	VII	L	VII	VIII

Table 1: The parameters U_X and V_X

Hence a_{13} is a polynomial of degree 1 in ℓ, and one calculates now that

$$\begin{aligned} a_{13} &= w^\sigma(zy^\sigma) + (yz^\sigma)w - k_1 w^{\sigma+1} - k_2 y^{\sigma+1} - y^{\sigma+1}(wv^\sigma + vw^\sigma) \\ &\quad + \ell\Big(k_1k_2 - z^{\sigma+1} - v^{\sigma+1}y^{\sigma+1} + z^\sigma(vy) + (y^\sigma v^\sigma)z\Big). \end{aligned}$$

These calculations are similar as before, except that for the coefficient of ℓ, one encounters the expression $w^\sigma((vy)x^\sigma) + v^\sigma((wy)x^\sigma) - (y(x^\sigma w^\sigma))v - (y(x^\sigma v^\sigma))w$, which is 0 by Proposition 2.3(8).

Now one sees that the coefficient tuple of ℓ in the 14-tuple $(a_0, a_1, \ldots, a_{13})$ is exactly equal to $e(\alpha^*)$. This completes the proof of the proposition. ∎

By Propositions 3.1 and 3.2, we obtain

Corollary 3.3 *The map e defines a full projective embedding of $\Delta_{\mathcal{T}}$ into* $\mathsf{PG}(7 + 6n, \mathbb{K})$.

Let M be a line of $\mathcal{P}_{\mathcal{T}}$ and let α^* and $\alpha(k)$, $k \in \mathbb{K}$, denote the planes of $\mathcal{P}_{\mathcal{T}}$ through M as mentioned in Proposition 3.2. If $X \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{L}\}$ denotes the type of the line M, then U_X denotes the type of the plane α^* and V_X denotes the common type of the planes $\alpha(k)$, $k \in \mathbb{K}$. The values of U_X and V_X are listed in Table 1.

Let Σ be a projective space. For every point x of Σ, let x^ζ be a subspace of co-dimension at most 1 of Σ. Following Tits [27], ζ is called a *polarity* of Σ if for all points x_1 and x_2 of Σ, $e(x_1) \in e(x_2)^\zeta$ implies $e(x_2) \in e(x_1)^\zeta$. The radical of ζ consists of all points x for which $x^\zeta = \Sigma$. The polarity ζ is called *nondegenerate* if its radical is empty, otherwise it is called *degenerate*.

Suppose Δ is a dual polar space of rank $m \geq 1$. Two points of Δ are called *opposite* if they lie at maximal distance m from each other (in the collinearity graph). In the associated polar space, this means that the corresponding maximal singular subspaces do not meet. By Cameron [4], Δ is a *near polygon* which means that for every point x and every line L there exists a unique point on L nearest to x. Using the fact that Δ is a near polygon, we easily can prove the following.

Lemma 3.4 *Let e be a full embedding of a dual polar space Δ into a projective space Σ (having only thick lines). Let ζ be a possibly degenerate polarity of Σ, let x, y be points of Δ and let L be a line of Δ containing y. Suppose that for every point $z \in L \setminus \{y\}$, it* *holds that $e(z) \in e(x)^\zeta$ if and only if x and z are not opposite points. Then $e(y) \in e(x)^\zeta$ if and only if x and y are not opposite points.*

Proof. This follows from the following two facts:

- either one or all points of $e(L)$ are contained in $e(x)^\zeta$;
- either one or all points of L are not opposite to x. ∎

Following De Bruyn [11], we call a set X of points of a point-line geometry $\mathcal{S}$ a *pseudo-subspace* if no line of $\mathcal{S}$ intersects the complement of X in a singleton. The whole point set of $\mathcal{S}$ is an example of a pseudo-subspace. The intersection of any collection of pseudo-subspaces of $\mathcal{S}$ is again a pseudo-subspace, and we define $[X]$ to be the intersection of all pseudo-subspaces containing X. Clearly, $[X]$ is the smallest pseudo-subspace containing X. We call it the *pseudo-subspace generated by* X. If $[X]$ coincides with the whole point-set of $\mathcal{S}$, then X is called a *pseudo-generating set* of $\mathcal{S}$. From Table 1, we easily deduce the following.

Lemma 3.5 *The set of planes of Type* VIII *of $\mathcal{P}_\mathcal{T}$ is a pseudo-generating set of $\Delta_\mathcal{T}$.*

The following can easily be derived from Lemma 3.4.

Lemma 3.6 *Let e be a full embedding of a dual polar space Δ into a projective space Σ (having only thick lines). Let ζ be a possibly degenerate polarity of Σ and let X be a pseudo-generating set of Δ. Suppose that for all $x_1, x_2 \in X$, we have $e(x_1) \in e(x_2)^\zeta$ if and only if x_1 and x_2 are not opposite. Then for any two points y_1 and y_2 of Δ, we have $e(y_1) \in e(y_2)^\zeta$ if and only if y_1 and y_2 are not opposite.*

Proof. For every point x of Δ, let A_x denote the set of all points y of Δ such that $e(x) \in e(y)^\zeta$ if and only if x and y are not opposite. By Lemma 3.4, A_x is a pseudo-subspace of Δ. If $x \in X$, then A_x is a pseudo-subspace of Δ containing X and hence coincides with the whole point set $\mathcal{P}$ of Δ. So, if y is an arbitrary point of Δ, then A_y is a pseudo-subspace containing X and hence coincides with $\mathcal{P}$. This proves the lemma. ∎

Let e be a full embedding of a dual polar space Δ in a projective space Σ. Following Cardinali, De Bruyn and Pasini [5], we call e *polarized* if through every point $e(x)$ of the image of e there exists a (necessarily unique) hyperplane T_x of Σ such that for every point y of Δ, $e(y) \in T_x$ if and only if x and y are not opposite.

Proposition 3.7 *The embedding e of $\Delta_\mathcal{T}$ is polarized.*

Proof. We label the coordinates of $\mathsf{PG}(7+6n, \mathbb{K})$ by $Z_0, Z_1, \ldots, Z_{13}$, where $Z_0, Z_1, Z_2, Z_3, Z_7, Z_8, Z_9, Z_{13} \in \mathbb{K}$ and $Z_4, Z_5, Z_6, Z_{10}, Z_{11}, Z_{12} \in \mathbb{O}$. The symplectic $\mathbb{K}$-linear form

$$-Z_0 Z'_{13} + Z_{13} Z'_0 + \sum_{i=1}^{3} (Z_i Z'_{i+6} - Z_{i+6} Z'_i) - \sum_{i=4}^{6} (Z_i^\sigma Z'_{i+6} + Z'^\sigma_{i+6} Z_i) + \sum_{i=4}^{6} (Z'^\sigma_i Z_{i+6} + Z^\sigma_{i+6} Z'_i)$$

of V defines a (possibly degenerate) symplectic polarity ζ of $\mathsf{PG}(7+6n,\mathbb{K})$.

We first show that no point of the image of e is contained in the radical $\mathfrak{R}$ of ζ. Indeed, we note that the coordinates of every element of the image of e are nonzero in at least one of the positions 0, 1, 2, 3, 7, 8, 9 or 13. Let p be any element of the image of e and suppose the i-th coordinate of p is nonzero where i belongs to $\{0,1,2,3,7,8,9,13\}$. Put $i' := i^\tau$ where τ is the permutation of $\{0,1,\ldots,13\}$ defined by $(0,13)(1,7)(2,8)(3,9)$. Then the point of $\mathsf{PG}(7+6n,\mathbb{K})$ with all coordinates 0 except for the one in the i'-th place which is equal to 1 does not belong to p^ζ, showing that p does not belong to $\mathfrak{R}$.

In order to prove that e is polarized it suffices to prove that two points π and α of $\Delta_\mathcal{T}$ are not opposite if and only if $e(\alpha) \in e(\pi)^\zeta$. By Lemmas 3.5 and 3.6, it suffices to prove this in the special case that π and α are planes of Type VIII. By Theorem 7.3 of Blok and Brouwer [1] or Lemma 6.1 of Shult [24], the set of points of $\Delta_\mathcal{T}$ at non-maximal distance from π is a maximal proper subspace of $\Delta_\mathcal{T}$. From this and the fact that the image of e generates $\mathsf{PG}(7+6n,\mathbb{K})$, it follows that it already suffices to prove that, if π and α are two nondisjoint planes of Type VIII of $\mathcal{P}_\mathcal{T}$, then $e(\alpha) \in e(\pi)^\zeta$.

So, suppose that $\pi = [x_1,x_2,x_3;k_3,k_2,k_1]$ and $\alpha = [y_1,y_2,y_3;m_3,m_2,m_1]$ are two nondisjoint planes of Type VIII of $\mathcal{P}_\mathcal{T}$. We must show that $e(\alpha) \in e(\pi)^\zeta$, or equivalently that

$$\begin{gathered}
(k_1k_2k_3 + x_3^\sigma(x_2x_1^\sigma) + (x_1x_2^\sigma)x_3 - k_1x_1^{\sigma+1} - k_2x_2^{\sigma+1} - k_3x_3^{\sigma+1}) - (m_1m_2m_3 + y_3^\sigma(y_2y_1^\sigma) \\
+(y_1y_2^\sigma)y_3 - m_1y_1^{\sigma+1} - m_2y_2^{\sigma+1} - m_3y_3^{\sigma+1}) + k_1(m_2m_3 - y_1^{\sigma+1}) - m_1(k_2k_3 - x_1^{\sigma+1}) \\
+k_2(m_1m_3 - y_2^{\sigma+1}) - m_2(k_1k_3 - x_2^{\sigma+1}) + k_3(m_1m_2 - y_3^{\sigma+1}) - m_3(k_1k_2 - x_3^{\sigma+1}) \\
-x_1^\sigma(m_1y_1 - y_3^\sigma y_2) - (m_1y_1 - y_3^\sigma y_2)^\sigma x_1 + y_1^\sigma(k_1x_1 - x_3^\sigma x_2) + (k_1x_1 - x_3^\sigma x_2)^\sigma y_1 \\
-x_2^\sigma(m_2y_2 - y_3y_1) - (m_2y_2 - y_3y_1)^\sigma x_2 + y_2^\sigma(k_2x_2 - x_3x_1) + (k_2x_2 - x_3x_1)^\sigma y_2 \\
-x_3^\sigma(m_3y_3 - y_2y_1^\sigma) - (m_3y_3 - y_2y_1^\sigma)^\sigma x_3 + y_3^\sigma(k_3x_3 - x_2x_1^\sigma) + (k_3x_3 - x_2x_1^\sigma)^\sigma y_3 = 0.
\end{gathered}$$

Taking into account Proposition 2.3(6), the latter equation is equivalent with

$$\delta_1\delta_2\delta_3 - \delta_1\Delta_1^{\sigma+1} - \delta_2\Delta_2^{\sigma+1} - \delta_3\Delta_3^{\sigma+1} + (\Delta_2\Delta_1^\sigma)\Delta_3^\sigma + \Delta_3(\Delta_1\Delta_2^\sigma) = 0, \qquad (*)$$

where $\Delta_i := x_i - y_i$ and $\delta_i := k_i - m_i$, $i = 1,2,3$. Note that the planes π and α have a point in common which necessarily is of Type 3, 4 or 5.

Suppose π and α have a point of Type 3 in common. This point necessarily coincides with $(x_1^\sigma, x_3; k_2) = (y_1^\sigma, y_3; m_2)$. Hence, $\Delta_1 = \Delta_3 = \delta_2 = 0$ and $(*)$ is satisfied.

Suppose π and α have a point of Type 4 in common. In this case, there exists an $s \in \mathbb{O}$ such that

$$\left\{\begin{array}{rcl} 0 & = & \Delta_1 + \delta_2 s, \\ 0 & = & \Delta_2 + \Delta_3 s, \\ 0 & = & \delta_3 + \Delta_1^\sigma s + s^\sigma\Delta_1 + \delta_2 s^{\sigma+1}. \end{array}\right.$$

This implies that $\Delta_3\Delta_1 - \delta_2\Delta_2 = 0$, and so Δ_1, Δ_2 and Δ_3 associate by Proposition 2.2(6). If $\delta_2 = 0$, then $\Delta_1 = 0$ and also $\delta_3 = 0$, and so every term of $(*)$ is 0. If $\delta_2 \neq 0$, then $s = -\delta_2^{-1}\Delta_1$ and plugging this into the third equation above, this implies

$\delta_2\delta_3 - \Delta_1^{\sigma+1} = 0$. Hence the first two terms of $(*)$ cancel out. Now the third and last term of $(*)$ also cancel out, and it remains to show that $\delta_3\Delta_3^{\sigma+1} = (\Delta_2\Delta_1^\sigma)\Delta_3^\sigma$, or equivalently that $\delta_2\delta_3\Delta_3^{\sigma+1} = (\delta_2\Delta_2)(\Delta_1^\sigma\Delta_3^\sigma)$. But this follows from the equations $\delta_2\delta_3 = \Delta_1^{\sigma+1}$ and $\delta_2\Delta_2 = \Delta_3\Delta_1$.

Finally, suppose that π and α have a point of Type 5 in common. Then

$$\left\{\begin{array}{l} \Delta_3^\sigma + \delta_2 b + \Delta_1 a = 0, \\ \Delta_2^\sigma + \delta_3 a + \Delta_1^\sigma b = 0, \\ \delta_1 + \Delta_2 a + a^\sigma\Delta_2^\sigma + \Delta_3 b + b^\sigma\Delta_3^\sigma + \delta_3 a^{\sigma+1} + \delta_2 b^{\sigma+1} + (a^\sigma\Delta_1^\sigma)b + b^\sigma(\Delta_1 a) = 0. \end{array}\right.$$

for some $a, b \in \mathbb{O}$. An obvious manipulation of the first two equations yields

$$\left\{\begin{array}{lcl} (\Delta_1^{\sigma+1} - \delta_2\delta_3)a & = & \delta_2\Delta_2^\sigma - \Delta_1^\sigma\Delta_3^\sigma, \\ (\Delta_1^{\sigma+1} - \delta_2\delta_3)b & = & \delta_3\Delta_3^\sigma - \Delta_1\Delta_2^\sigma. \end{array}\right.$$

Suppose first that $\delta_2\delta_3 - \Delta_1^{\sigma+1} = 0$. Then also $\delta_2\Delta_2^\sigma - \Delta_1^\sigma\Delta_3^\sigma = 0$ and $\delta_3\Delta_3^\sigma - \Delta_1\Delta_2^\sigma = 0$. If $\delta_3 = 0$, then $\Delta_1 = \delta_2\Delta_2^\sigma = 0$ and $(*)$ is satisfied. If $\delta_3 \neq 0$, then $\Delta_3 = \delta_3^{-1}\Delta_2\Delta_1^\sigma$ and one can easily verify that $(*)$ holds in this case taking Proposition 2.2(6) into account as well as the fact that $\delta_2\delta_3 = \Delta_1^{\sigma+1}$.

Suppose therefore that $\Delta_1^{\sigma+1} - \delta_2\delta_3 \neq 0$. Then the previous equations provide explicit expressions for a and b in terms of the δ_i and Δ_i, $i = 1, 2, 3$. Plugging these expressions in the equality $\delta_1 + \Delta_2 a + a^\sigma\Delta_2^\sigma + \Delta_3 b + b^\sigma\Delta_3^\sigma + \delta_3 a^{\sigma+1} + \delta_2 b^{\sigma+1} + (a^\sigma\Delta_1^\sigma)b + b^\sigma(\Delta_1 a) = 0$, we obtain

$$\begin{gathered} \delta_1(\Delta_1^{\sigma+1} - \delta_2\delta_3)^2 - (\Delta_1^{\sigma+1} - \delta_2\delta_3)\Delta_2(\Delta_1^\sigma\Delta_3^\sigma - \delta_2\Delta_2^\sigma) - (\Delta_1^{\sigma+1} - \delta_2\delta_3)(\Delta_3\Delta_1 - \delta_2\Delta_2)\Delta_2^\sigma \\ -(\Delta_1^{\sigma+1} - \delta_2\delta_3)\Delta_3(\Delta_1\Delta_2^\sigma - \delta_3\Delta_3^\sigma) - (\Delta_1^{\sigma+1} - \delta_2\delta_3)(\Delta_2\Delta_1^\sigma - \delta_3\Delta_3)\Delta_3^\sigma \\ +\delta_3(\Delta_1^\sigma\Delta_3^\sigma - \delta_2\Delta_2^\sigma)(\Delta_3\Delta_1 - \delta_2\Delta_2) + \delta_2(\Delta_1\Delta_2^\sigma - \delta_3\Delta_3^\sigma)(\Delta_2\Delta_1^\sigma - \delta_3\Delta_3) \\ +((\Delta_3\Delta_1 - \delta_2\Delta_2)\Delta_1^\sigma)(\Delta_1\Delta_2^\sigma - \delta_3\Delta_3^\sigma) + (\Delta_2\Delta_1^\sigma - \delta_3\Delta_3)(\Delta_1(\Delta_1^\sigma\Delta_3^\sigma - \delta_2\Delta_2^\sigma)) = 0. \qquad (**) \end{gathered}$$

If we expand the left hand side as a polynomial in δ_2 and δ_3, then the coefficient of $\delta_2\delta_3$ equals

$$\begin{gathered} \Delta_2(\Delta_1^\sigma\Delta_3^\sigma) + (\Delta_3\Delta_1)\Delta_2^\sigma + \Delta_3(\Delta_1\Delta_2^\sigma) + (\Delta_2\Delta_1^\sigma)\Delta_3^\sigma - \Delta_2^\sigma(\Delta_3\Delta_1) - (\Delta_1^\sigma\Delta_3^\sigma)\Delta_2 \\ -\Delta_3^\sigma(\Delta_2\Delta_1^\sigma) - (\Delta_1\Delta_2^\sigma)\Delta_3 + (\Delta_2\Delta_1^\sigma)\Delta_3^\sigma + \Delta_3(\Delta_1\Delta_2^\sigma), \end{gathered}$$

which by Proposition 2.3(6) is equal to $(\Delta_2\Delta_1^\sigma)\Delta_3^\sigma + \Delta_3(\Delta_1\Delta_2^\sigma)$. One now calculates that $(**)$ is equivalent to

$$(\delta_2\delta_3 - \Delta_1^{\sigma+1})(\delta_1\delta_2\delta_3 - \delta_1\Delta_1^{\sigma+1} - \delta_2\Delta_2^{\sigma+1} - \delta_3\Delta_3^{\sigma+1} + (\Delta_2\Delta_1^\sigma)\Delta_3^\sigma + \Delta_3(\Delta_1\Delta_2^\sigma)) = 0.$$

Since $\delta_2\delta_3 - \Delta_1^{\sigma+1} \neq 0$ we see that $(*)$ holds. ∎

For every point p of $\Delta_{\mathcal{T}}$, let T_p denote the unique hyperplane of $\mathsf{PG}(7+6n, \mathbb{K})$ through $e(p)$ containing all points $e(q)$ where q is a point of $\Delta_{\mathcal{T}}$ not opposite to x. If ζ denotes the symplectic polarity of $\mathsf{PG}(7+6n,\mathbb{K})$ as defined in the proof of Proposition 3.7, then $T(p) = e(p)^\zeta$. Since the image of e generates $\mathsf{PG}(7+6n,\mathbb{K})$, the intersection of all hyperplanes $T(p)$ is precisely the radical $\mathfrak{R}$ of ζ.

Since $\mathfrak{R}$ contains no point of the image of e, a quotient embedding $e/\mathfrak{R}$ of $\Delta_\mathcal{T}$ can be defined in the quotient projective space of $\mathsf{PG}(7+6n,\mathbb{K})$ determined by $\mathfrak{R}$. Following the terminology of Cardinali, De Bruyn and Pasini [5], the embedding $e/\mathfrak{R}$ is precisely the minimal full polarized embedding of $\Delta_\mathcal{T}$ (up to isomorphism).

Proposition 3.8 *If σ is nontrivial or $\mathsf{char}\,\mathbb{K} \neq 2$, then $\mathfrak{R} = \emptyset$ and e is isomorphic to the minimal full polarized embedding of $\Delta_\mathcal{T}$. If σ is trivial and $\mathsf{char}\,\mathbb{K} = 2$, then the vector dimension of the minimal full polarized embedding of $\Delta_\mathcal{T}$ is equal to* 8.

Proof. If σ is trivial and $\mathsf{char}\,\mathbb{K} \neq 2$, then $\mathbb{O} = \mathbb{K}$ is commutative and ζ is a nondegenerate symplectic polarity. So, $\mathfrak{R} = \emptyset$ in this case.

Now suppose that σ is nontrivial. In order to prove that $\mathfrak{R} = \emptyset$, it clearly suffices to show that the symplectic $\mathbb{K}$-linear form $f((X,X'),(Y,Y')) = X'^\sigma Y + Y^\sigma X' - X^\sigma Y' - Y'^\sigma X$ defined over the $\mathbb{K}$-vector space $\mathbb{O} \times \mathbb{O}$ is nondegenerate. Suppose that for some nonzero vector (Y,Y'), we have $X'^\sigma Y + Y^\sigma X' - X^\sigma Y' - Y'^\sigma X = 0$ for all $X, X' \in \mathbb{O}$. Putting consecutively $X = 0$ and $X' = 0$, we see that it suffices to show that, if $X^\sigma Y + Y^\sigma X = 0$ for all $X \in \mathbb{O}$, then $Y = 0$. Suppose to the contrary that $Y \neq 0$ and $X^\sigma Y + Y^\sigma X = 0$ for all $X \in \mathbb{O}$. Putting $X = 1$, we obtain $Y = -Y^\sigma$; putting $X = Y$, we obtain $2Y^\sigma Y = 0$, implying $\mathsf{char}\,\mathbb{K} = 2$. Hence $X^\sigma = YXY^{-1}$ for all $X \in \mathbb{O}$. For all $X \in \mathbb{O}$, we then have

$$YX^\sigma = (XY)^\sigma = Y(XY)Y^{-1} = YX,$$

which implies $X = X^\sigma$ for all $X \in \mathbb{O}$, a contradiction.

Finally, suppose that σ is trivial and $\mathsf{char}\,\mathbb{K} = 2$. Then the symplectic $\mathbb{K}$-linear form defined in Proposition 3.7 reduces to

$$-Z_0 Z'_{13} + Z_{13} Z'_0 + \sum_{i=1}^{3} (Z_i Z'_{i+6} - Z_{i+6} Z'_i).$$

So, $\mathfrak{R} = \mathsf{PG}(\langle E_5, E_6, E_7, E_{11}, E_{12}, E_{13}\rangle)$, where E_i, $i \in \{1,2,\ldots,14\}$, is the subspace of V as defined in the proof of Proposition 3.1. The vector dimension of the quotient embedding $e/\mathfrak{R}$ is then equal to $\dim(\langle E_1, E_2, E_3, E_4, E_8, E_9, E_{10}, E_{14}\rangle) = 8$. ∎

We end this section by giving a list of 26 equations which completely determine the image $\mathsf{Im}(e)$ of e.

Proposition 3.9 *A point $p = (Y_1, Y_2, \ldots, Y_{14})$ of $\mathsf{PG}(V)$ belongs to $\mathsf{Im}(e)$ if and only if the following* 26 *equations are satisfied:*

$$Y_5^{\sigma+1} = Y_3 Y_4 - Y_1 Y_8, \qquad Y_{11}^{\sigma+1} = Y_9 Y_{10} - Y_2 Y_{14},$$

$$Y_6^{\sigma+1} = Y_2Y_4 - Y_1Y_9, \qquad Y_{12}^{\sigma+1} = Y_8Y_{10} - Y_3Y_{14},$$
$$Y_7^{\sigma+1} = Y_2Y_3 - Y_1Y_{10}, \qquad Y_{13}^{\sigma+1} = Y_8Y_9 - Y_4Y_{14},$$
$$Y_1^2Y_{14} = Y_2Y_3Y_4 + S(Y_7^\sigma, Y_6, Y_5^\sigma) - Y_2Y_5^{\sigma+1} - Y_3Y_6^{\sigma+1} - Y_4Y_7^{\sigma+1},$$
$$Y_{14}^2Y_1 = Y_8Y_9Y_{10} - S(Y_{13}^\sigma, Y_{12}, Y_{11}^\sigma) - Y_8Y_{11}^{\sigma+1} - Y_9Y_{12}^{\sigma+1} - Y_{10}Y_{13}^{\sigma+1},$$
$$Y_2^2Y_8 = Y_1Y_9Y_{10} - S(Y_{11}^\sigma, Y_7^\sigma, Y_6) - Y_1Y_{11}^{\sigma+1} + Y_9Y_7^{\sigma+1} + Y_{10}Y_6^{\sigma+1},$$
$$Y_8^2Y_2 = Y_3Y_4Y_{14} + S(Y_5^\sigma, Y_{13}^\sigma, Y_{12}) + Y_3Y_{13}^{\sigma+1} + Y_4Y_{12}^{\sigma+1} - Y_{14}Y_5^{\sigma+1},$$
$$Y_3^2Y_9 = Y_1Y_8Y_{10} - S(Y_{12}^\sigma, Y_7, Y_5) - Y_1Y_{12}^{\sigma+1} + Y_8Y_7^{\sigma+1} + Y_{10}Y_5^{\sigma+1},$$
$$Y_9^2Y_3 = Y_2Y_4Y_{14} + S(Y_6^\sigma, Y_{13}, Y_{11}) + Y_2Y_{13}^{\sigma+1} + Y_4Y_{11}^{\sigma+1} - Y_{14}Y_6^{\sigma+1},$$
$$Y_4^2Y_{10} = Y_1Y_8Y_9 - S(Y_{13}^\sigma, Y_6, Y_5^\sigma) - Y_1Y_{13}^{\sigma+1} + Y_8Y_6^{\sigma+1} + Y_9Y_5^{\sigma+1},$$
$$Y_{10}^2Y_4 = Y_2Y_3Y_{14} + S(Y_7^\sigma, Y_{12}, Y_{11}^\sigma) + Y_2Y_{12}^{\sigma+1} + Y_3Y_{11}^{\sigma+1} - Y_{14}Y_7^{\sigma+1},$$
$$Y_7^\sigma Y_6 = Y_2Y_5 - Y_1Y_{11}, \qquad Y_{13}^\sigma Y_{12} = Y_{14}Y_5 - Y_8Y_{11},$$
$$Y_7Y_5 = Y_3Y_6 - Y_1Y_{12}, \qquad Y_{13}Y_{11} = Y_{14}Y_6 - Y_9Y_{12},$$
$$Y_6Y_5^\sigma = Y_4Y_7 - Y_1Y_{13}, \qquad Y_{12}Y_{11}^\sigma = Y_{14}Y_7 - Y_{10}Y_{13},$$
$$Y_6Y_{11}^\sigma = Y_7Y_9 - Y_2Y_{13}, \qquad Y_{12}Y_5^\sigma = Y_8Y_7 - Y_{13}Y_3,$$
$$Y_7Y_{11} = Y_6Y_{10} - Y_2Y_{12}, \qquad Y_{13}Y_5 = Y_8Y_6 - Y_{12}Y_4,$$
$$Y_7^\sigma Y_{12} = Y_5Y_{10} - Y_3Y_{11}, \qquad Y_{13}^\sigma Y_6 = Y_9Y_5 - Y_{11}Y_4.$$

Proof. To verify that $Y_1, Y_2, \ldots, Y_{14}$ satisfy the above equations if p belongs to $\mathsf{Im}(e)$, we need to consider eight cases, corresponding to the eight possibilities for the planes. It is easy, but sometimes tedious, to verify that in each of these eight cases, the coordinates indeed satisfy the 26 above-stated equations. The only difficulties seem to occur for the type VII and VIII planes, where a (straightforward) use of the identities $S(c,a,b)\cdot c - c^{\sigma+1}(b^\sigma a^\sigma) = (ca)(bc)$ and $S(a,b,c)^2 = S(ab,ca,bc) + 2\cdot a^{\sigma+1}b^{\sigma+1}c^{\sigma+1}$ (see Proposition 2.4) seems to be necessary in some of the calculations.

Conversely, suppose that $Y_1, Y_2, \ldots, Y_{14}$ satisfy the above 26 equations. If $Y_1 \neq 0$, then we may suppose that $Y_1 = 1$. The equations $Y_5^{\sigma+1} = Y_3Y_4 - Y_1Y_8$, $Y_6^{\sigma+1} = Y_2Y_4 - Y_1Y_9$, $Y_7^{\sigma+1} = Y_2Y_3 - Y_1Y_{10}$, $Y_1^2Y_{14} = Y_2Y_3Y_4 + S(Y_7^\sigma, Y_6, Y_5^\sigma) - Y_2Y_5^{\sigma+1} - Y_3Y_6^{\sigma+1} - Y_4Y_7^{\sigma+1}$, $Y_7^\sigma Y_6 = Y_2Y_5 - Y_1Y_{11}$, $Y_7Y_5 = Y_3Y_6 - Y_1Y_{12}$ and $Y_6Y_5^\sigma = Y_4Y_7 - Y_1Y_{13}$ then allow us to express Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13} and Y_{14} in terms of Y_2, Y_3, Y_4, Y_5, Y_6 and Y_7. Putting $Y_2 = m$, $Y_3 = l$, $Y_4 = k$, $Y_5 = x_1$, $Y_6 = x_2$ and $Y_7 = x_3$, where $k, l, m \in \mathbb{K}$ and $x_1, x_2, x_3 \in \mathbb{O}$, we see that p is the image of the plane $[x_1, x_2, x_3; k, l, m]$ of Type VIII. If $Y_1 = 0$ and $Y_3 \neq 0$, then we may suppose that $Y_3 = 1$ and after putting $Y_5 = -x_1$, $Y_7 = x_3$, $Y_8 = k$, $Y_{10} = l$, $Y_{12} = x_2$ with $k, l \in \mathbb{K}$ and $x_1, x_2, x_3 \in \mathbb{O}$ in the above 26 equations, we can determine all the remaining coordinates. We then see that p is the image of the plane $[x_1, x_2, x_3; k, l]$ of Type VII. The remaining possibilities can be divided into 6 cases, and the treatment in each of these cases is similar as above. These cases are:

- $Y_1 = Y_3 = 0$ and $Y_4 \neq 0$, giving rise to a Type VI plane;
- $Y_1 = Y_3 = Y_4 = 0$ and $Y_8 \neq 0$, giving rise to a Type V plane;
- $Y_1 = Y_3 = Y_4 = Y_8 = 0$ and $Y_2 \neq 0$, giving rise to a Type IV plane;
- $Y_1 = Y_2 = Y_3 = Y_4 = Y_8 = 0$ and $Y_{10} \neq 0$, giving rise to a Type III plane;
- $Y_1 = Y_2 = Y_3 = Y_4 = Y_8 = Y_{10} = 0$ and $Y_9 \neq 0$, giving rise to a Type II plane;
- $Y_1 = Y_2 = Y_3 = Y_4 = Y_8 = Y_9 = Y_{10} = 0$, giving rise to the unique Type I plane.

Using the above 26 equations, it is indeed possible in each case to show that certain other coordinates need to be zero, and/or to express certain coordinates in terms of others. These "free coordinates" then determine the parameters of the plane (just as this was the case for the Type VII and Type VIII planes). ■

4 The case of symplectic and Hermitian dual polar spaces

Let U be a 6-dimensional vector space over a field $\mathbb{O}$. Suppose σ is an automorphism of $\mathbb{O}$ which is either trivial or an involutory automorphism. Let $\mathbb{K} \subseteq \mathbb{O}$ denote the fixed field of σ. In the former case, we have $\mathbb{K} = \mathbb{O}$ and in the latter case, $\mathbb{O}$ is a separable quadratic extension of $\mathbb{K}$. Let $(\bar{e}_1, \bar{f}_1, \bar{e}_2, \bar{f}_2, \bar{e}_3, \bar{f}_3)$ be a fixed ordered basis of U and let f be the σ-sesquilinear form on U (being linear in the second and semi-linear in the first component) such that $f(\bar{e}_i, \bar{e}_j) = f(\bar{f}_i, \bar{f}_j) = 0$ and $f(\bar{e}_i, \bar{f}_j) = -f(\bar{f}_j, \bar{e}_i) = \delta_{ij}$ for all $i, j \in \{1, 2, 3\}$. The third exterior power $\bigwedge^3 U$ can be regarded as a vector space over $\mathbb{K}$. We have $\dim_{\mathbb{K}}(\bigwedge^3 U) = 20$ if σ is trivial and $\dim_{\mathbb{K}}(\bigwedge^3 U) = 40$ if σ is nontrivial. Let W be the subspace of this $\mathbb{K}$-vector space defined by

$$W := W_1 \oplus W_2 \oplus \cdots \oplus W_{14},$$

where

$W_1 := \langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \rangle_{\mathbb{K}}$, $W_2 := \langle \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 \rangle_{\mathbb{K}}$, $W_3 := \langle \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 \rangle_{\mathbb{K}}$, $W_4 := \langle \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 \rangle_{\mathbb{K}}$,

$W_8 := \langle \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 \rangle_{\mathbb{K}}$, $W_9 := \langle \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 \rangle_{\mathbb{K}}$, $W_{10} := \langle \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 \rangle_{\mathbb{K}}$, $W_{14} := \langle \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 \rangle_{\mathbb{K}}$,

$W_5 := \{x \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 - x^\sigma \cdot \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3 \,|\, x \in \mathbb{O}\}$, $W_6 := \{x \cdot \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_1 - x^\sigma \cdot \bar{e}_3 \wedge \bar{e}_2 \wedge \bar{f}_2 \,|\, x \in \mathbb{O}\}$,

$W_7 := \{x \cdot \bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1 - x^\sigma \cdot \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3 \,|\, x \in \mathbb{O}\}$, $W_{11} := \{x \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 - x^\sigma \cdot \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3 \,|\, x \in \mathbb{O}\}$,

$W_{12} := \{x \cdot \bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_1 - x^\sigma \cdot \bar{f}_3 \wedge \bar{e}_2 \wedge \bar{f}_2 \,|\, x \in \mathbb{O}\}$, $W_{13} := \{x \cdot \bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_1 - x^\sigma \cdot \bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3 \,|\, x \in \mathbb{O}\}$.

Then $\dim(W) = 14$ if σ is trivial and $\dim(W) = 20$ if σ is nontrivial. Let η be the $\mathbb{K}$-linear isomorphism from W to the $\mathbb{K}$-vector space $V = \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{O} \times \mathbb{O} \times \mathbb{O} \times \mathbb{K}$, mapping the vector

$$k_1 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + k_2 \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{e}_3 + k_3 \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + k_4 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + x_5^\sigma \cdot \bar{e}_1 \wedge \bar{e}_2 \wedge \bar{f}_2$$

$-x_5 \cdot \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{f}_3 + x_6 \cdot \bar{e}_3 \wedge \bar{e}_1 \wedge \bar{f}_1 - x_6^\sigma \cdot \bar{e}_3 \wedge \bar{e}_2 \wedge \bar{f}_2 - x_7 \cdot \bar{e}_2 \wedge \bar{e}_1 \wedge \bar{f}_1 + x_7^\sigma \cdot \bar{e}_2 \wedge \bar{e}_3 \wedge \bar{f}_3$
$+k_8 \cdot \bar{e}_1 \wedge \bar{f}_2 \wedge \bar{f}_3 + k_9 \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{e}_3 + k_{10} \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_3 + x_{11}^\sigma \cdot \bar{f}_1 \wedge \bar{e}_2 \wedge \bar{f}_2 - x_{11} \cdot \bar{f}_1 \wedge \bar{e}_3 \wedge \bar{f}_3$
$+x_{12} \cdot \bar{f}_3 \wedge \bar{e}_1 \wedge \bar{f}_1 - x_{12}^\sigma \cdot \bar{f}_3 \wedge \bar{e}_2 \wedge \bar{f}_2 - x_{13} \cdot \bar{f}_2 \wedge \bar{e}_1 \wedge \bar{f}_1 + x_{13}^\sigma \cdot \bar{f}_2 \wedge \bar{e}_3 \wedge \bar{f}_3 + k_{14} \cdot \bar{f}_1 \wedge \bar{f}_2 \wedge \bar{f}_3$
of W to the vector $(k_1, k_2, k_3, k_4, x_5, x_6, x_7, k_8, k_9, k_{10}, x_{11}, x_{12}, x_{13}, k_{14})$ of V.

With the pair (U, f), there is associated a symplectic/Hermitian polar space $\mathcal{P}$ and a dual polar space Δ. The singular subspaces of $\mathcal{P}$ are the subspaces of U which are totally isotropic with respect to f. As indicated in Section 3 of De Bruyn and Van Maldeghem [13], we can coordinatize the polar space $\mathcal{P}$ in the following way.

- The point $\langle \bar{f}_1 \rangle$ of $\mathcal{P}$ is denoted by (∞).
- For every point $x \in \mathbb{O}$, the point $\langle \bar{f}_2 + x\bar{f}_1 \rangle$ of $\mathcal{P}$ is denoted by (x).
- For all $x_1, x_2 \in \mathbb{O}$, the point $\langle \bar{f}_3 + x_1\bar{f}_2 + x_2\bar{f}_1 \rangle$ of $\mathcal{P}$ is denoted by (x_1, x_2).
- For all $x_1, x_2 \in \mathbb{O}$ and every $k \in \mathbb{K}$, the point $\langle \bar{e}_3 + k\bar{f}_3 + x_1\bar{f}_2 + x_2\bar{f}_1 \rangle$ of $\mathcal{P}$ is denoted by $(x_1, x_2; k)$.
- For all $x_1, x_2, x_3 \in \mathbb{O}$ and every $k \in \mathbb{K}$, the point $\langle \bar{e}_2 + x_1\bar{e}_3 + x_2\bar{f}_3 + (k - x_1^\sigma x_2)\bar{f}_2 + x_3\bar{f}_1 \rangle$ of $\mathcal{P}$ is denoted by $(x_1, x_2, x_3; k)$.
- For all $x_1, x_2, x_3, x_4 \in \mathbb{O}$ and every $k \in \mathbb{K}$, the point $\langle \bar{e}_1 + x_1\bar{e}_2 + x_2\bar{e}_3 + x_3\bar{f}_3 + x_4\bar{f}_2 + (k - x_1^\sigma x_4 - x_2^\sigma x_3)\bar{f}_1 \rangle$ of $\mathcal{P}$ is denoted by $(x_1, x_2, x_3, x_4; k)$.

The lines and planes of $\mathcal{P}$ are then as described in Section 2. So, via this coordinatization, we can identify $\mathcal{P}$ with $\mathcal{P}_\mathcal{T}$ and Δ with $\Delta_\mathcal{T}$, where $\mathcal{T} = (\mathbb{O}, \mathbb{K})$.

We will identify each point of $\mathsf{PG}(W)$ with its corresponding point of $\mathsf{PG}(\bigwedge^3 U)$. If $\langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is a 3-dimensional subspace of U which is totally isotropic with respect to f, then $p = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is a point of Δ. By Cooperstein [6, 7] and De Bruyn [9, 10], $e''(p) := \langle \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 \rangle$ is a point of $\mathsf{PG}(W)$ and the map e'' from the point set of Δ to the point set of $\mathsf{PG}(W)$ defines a full embedding of Δ into $\mathsf{PG}(W)$, called the *Grassmann embedding* of Δ.

Let e be the full embedding of Δ into $\mathsf{PG}(7 + 6n, \mathbb{K}) = \mathsf{PG}(V)$ as described in Section 3 (via the above-mentioned identification of Δ with $\Delta_\mathcal{T}$). Note that $n = 1$ if σ is trivial and $n = 2$ if σ is nontrivial. We now construct another embedding e' of Δ into $\mathsf{PG}(7+6n, \mathbb{K}) = \mathsf{PG}(V)$.

If $p = \langle \bar{v}_1, \bar{v}_2, \bar{v}_3 \rangle$ is a point of Δ, then there exists an $x \in \mathbb{O} \setminus \{0\}$ and a $\bar{w} \in W \setminus \{\bar{o}\}$ such that $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 = x \cdot \bar{w}$. Then we define $e'(p) := \langle \eta(\bar{w}) \rangle \in \mathsf{PG}(7 + 6n, \mathbb{K})$. Since e'' is a full projective embedding and η is a linear isomorphism, e' is a full embedding of Δ into $\mathsf{PG}(7 + 6n, \mathbb{K})$.

We will now show that $e' = e$. This then implies the following.

Proposition 4.1 *The full projective embedding of Δ as described in Section 3 (via the above-mentioned identification of Δ with $\Delta_{\mathcal{T}}$) is isomorphic to the Grassmann embedding of Δ.*

The following lemma, whose proof is straightforward, says that in order to prove that $e' = e$, it suffices to show that $e'(\alpha) = e(\alpha)$ for every plane α of $\mathcal{P}$ of type VIII (recall Lemma 3.5).

Lemma 4.2 *Let e_1 and e_2 be two full embeddings of a point-line geometry $\mathcal{S}$ into a projective space Σ. If $e_1(x) = e_2(x)$ for every point x of a given pseudo-generating set of $\mathcal{S}$, then $e_1 = e_2$.*

Consider an arbitrary plane $\alpha = [x_1, x_2, x_3; k, l, m]$ of Type VIII where $x_1, x_2, x_3 \in \mathbb{O}$ and $k, l, m \in \mathbb{K}$. Then α is generated by the points $p_3^* = (x_1^\sigma, x_3; l) = \langle \bar{e}_3 + x_3\bar{f}_1 + x_1^\sigma \bar{f}_2 + l\bar{f}_3\rangle$, $p_2(0) = (0, x_1, x_2; k) = \langle \bar{e}_2 + x_2\bar{f}_1 + k\bar{f}_2 + x_1\bar{f}_3\rangle$ and $p_1(0,0) := (0, 0, x_3^\sigma, x_2^\sigma; m) = \langle \bar{e}_1 + m\bar{f}_1 + x_2^\sigma\bar{f}_2 + x_3^\sigma\bar{f}_3\rangle$. So, $e''(\alpha) = \langle \bar{w}\rangle$ where $\bar{w} = (\bar{e}_1 + m\bar{f}_1 + x_2^\sigma\bar{f}_2 + x_3^\sigma\bar{f}_3) \wedge (\bar{e}_2 + x_2\bar{f}_1 + k\bar{f}_2 + x_1\bar{f}_3) \wedge (\bar{e}_3 + x_3\bar{f}_1 + x_1^\sigma\bar{f}_2 + l\bar{f}_3)$. We have $\eta(\bar{w}) = (1, m, l, k, x_1, x_2, x_3, lk - x_1^{\sigma+1}, km - x_2^{\sigma+1}, ml - x_3^{\sigma+1}, mx_1 - x_3^\sigma x_2, lx_2 - x_3x_1, kx_3 - x_2x_1^\sigma, mlk + x_3^\sigma x_2 x_1^\sigma + x_1 x_2^\sigma x_3 - mx_1^{\sigma+1} - lx_2^{\sigma+1} - kx_3^{\sigma+1})$, which is precisely the coordinate of $e(\alpha)$ as defined in Section 3.

We have also verified for ourselves that $e(\alpha) = e'(\alpha)$ if α is a plane of one of the seven other types. But as mentioned before, this verification is in fact not necessary if one relies on Lemma 4.2.

5 Universality of the embedding e

Suppose $\mathbb{O}$ is an alternative division ring which is quadratic over some subfield $\mathbb{K}$ of its center $Z(\mathbb{O})$, and let σ denote the corresponding standard involution of $\mathbb{O}$.

As before, let n be the dimension of $\mathbb{O}$ regarded as a $\mathbb{K}$-vector space. After choosing a basis of this vector space, $\lambda^{\sigma+1}$ can be regarded as a quadratic polynomial in the coordinates of $\lambda \in \mathbb{O}$, with all coefficients belonging to $\mathbb{K}$. So, with respect to a certain reference system, the equation $X_0X_1 + X_2^{\sigma+1} = X_3X_4$, where $X_0, X_1, X_3, X_4 \in \mathbb{K}$ and $X_2 \in \mathbb{O}$, defines a nonsingular quadric of Witt index 2 of the projective space $\mathsf{PG}(n+3, \mathbb{K})$. The points and lines of $\mathsf{PG}(n+3, \mathbb{K})$ contained in this quadric define a generalized quadrangle which we will denote by $\mathcal{Q}^*$.

Put $\mathcal{T} := (\mathbb{O}, \mathbb{K})$. As explained in Section 2, with $\mathcal{T}$ there is associated a polar space $\mathcal{P}_{\mathcal{T}}$ of rank 3. In Section 3, we constructed a full embedding e of the dual polar space $\Delta_{\mathcal{T}}$ associated with $\mathcal{P}_{\mathcal{T}}$ into the projective space $\mathsf{PG}(V)$, where V is some $\mathbb{K}$-vector space of dimension $8 + 6n$.

If e' is a full embedding of $\Delta_{\mathcal{T}}$ into a projective space Σ, then the underlying division ring of Σ is uniquely determined and must therefore be isomorphic to $\mathbb{K}$. (Indeed, this is a property which is known to hold for any full projective embedding of a thick dual polar space. It follows from applying the results of Dienst [14] and Tits [27, 8.6] to a so-called quad of the dual polar space.) A result of Kasikova and Shult [18, Section 4.6] then guarantees that $\Delta_{\mathcal{T}}$ must admit the so-called *absolutely universal embedding* $\widetilde{e}$. Every other full projective embedding of $\Delta_{\mathcal{T}}$ is a quotient of $\widetilde{e}$, i.e. is obtained from $\widetilde{e}$ by "projecting". The aim of this section is to prove that the embeddings e and $\widetilde{e}$ of $\Delta_{\mathcal{T}}$ are isomorphic if $|\mathbb{K}| \neq 2$. Observe that $|\mathbb{K}| = 2$ is only possible when $\mathbb{O} = \mathbb{K}$ or when $\mathbb{O}$ is a quadratic Galois extension of $\mathbb{K}$ (case (a) or (b) of Proposition 2.1).

A *frame point* of $\mathcal{P}_{\mathcal{T}}$ (with respect to the coordinatization given in Section 2) is a point whose coordinates do not contain any element of $\mathbb{O} \setminus \{0\}$; hence the frame points are (∞), (0), $(0,0)$, $(0,0;0)$, $(0,0,0;0)$ and $(0,0,0,0;0)$.

Recall that a set X of points of a point-line geometry is called a *subspace* if every line having two of its points in X has all its points in X. If X is a subspace, then we denote by $\widetilde{X}$ the subgeometry induced on X by those lines that have all their points in X. A subspace X is called *convex* if every point on a shortest path in the collinearity graph between any two of its points also belongs to X. If p is a point of $\mathcal{P}_{\mathcal{T}}$, then the set Q_p of all planes of $\mathcal{P}_{\mathcal{T}}$ containing p is a convex subspace of diameter 2 of $\Delta_{\mathcal{T}}$, called a *quad*. The point-line geometry $\widetilde{Q}_p$ is a generalized quadrangle (GQ).

The following lemma describes the sets $e(Q_p)$ where p is one of the six frame points. As in the proof of Proposition 3.1, E_i denotes the subspace of V consisting of those vectors of V all whose coordinates are zero except possibly for those in the i-th position. We will denote a generic vector of V (or a generic point of $\mathsf{PG}(V)$) by $(Z_1, Z_2, \ldots, Z_{14})$. For the sake of brevity, we write i consecutive zeros as 0^i.

Lemma 5.1

- *If* $p = (\infty)$, *then* $e(Q_p)$ *is the quadric* $Z_2Z_{14}+Z_{11}^{\sigma+1} = Z_9Z_{10}$ *of* $\mathsf{PG}(\langle E_2, E_9, E_{10}, E_{11}, E_{14}\rangle)$.
- *If* $p = (0)$, *then* $e(Q_p)$ *is the quadric* $Z_4Z_{14}+Z_{13}^{\sigma+1} = Z_8Z_9$ *of* $\mathsf{PG}(\langle E_4, E_8, E_9, E_{13}, E_{14}\rangle)$.
- *If* $p = (0,0)$, *then* $e(Q_p)$ *is the quadric* $Z_3Z_{14}+Z_{12}^{\sigma+1} = Z_8Z_{10}$ *of* $\mathsf{PG}(\langle E_3, E_8, E_{10}, E_{12}, E_{14}\rangle)$.
- *If* $p = (0,0;0)$, *then* $e(Q_p)$ *is the quadric* $Z_1Z_9+Z_6^{\sigma+1} = Z_2Z_4$ *of* $\mathsf{PG}(\langle E_1, E_2, E_4, E_6, E_9\rangle)$.
- *If* $p = (0,0,0;0)$, *then* $e(Q_p)$ *is the quadric* $Z_1Z_{10}+Z_7^{\sigma+1} = Z_2Z_3$ *of* $\mathsf{PG}(\langle E_1, E_2, E_3, E_7, E_{10}\rangle)$.
- *If* $p = (0,0,0,0;0)$, *then* $e(Q_p)$ *is the quadric* $Z_1Z_8+Z_5^{\sigma+1} = Z_3Z_4$ *of* $\mathsf{PG}(\langle E_1, E_3, E_4, E_5, E_8\rangle)$.

Proof. From the description of the planes given in Section 2, the following explicit descriptions for the quads can easily be deduced:

$$\begin{aligned}
Q_{(\infty)} &= \{[\infty], [k], [x;k], [x;k,\ell] \ : \ x \in \mathbb{O} \text{ and } k,\ell \in \mathbb{K}\},\\
Q_{(0)} &= \{[\infty], [k], [x,0;k], [x,0;k,\ell] \ : \ x \in \mathbb{O} \text{ and } k,\ell \in \mathbb{K}\},\\
Q_{(0,0)} &= \{[\infty], [0;k], [0,x;k], [0,x,0;k,\ell] \ : \ x \in \mathbb{O} \text{ and } k,\ell \in \mathbb{K}\},\\
Q_{(0,0;0)} &= \{[0], [0;k,0], [0,x;0,k], [0,x,0;k,0,\ell] \ : \ x \in \mathbb{O} \text{ and } k,\ell \in \mathbb{K}\},\\
Q_{(0,0,0;0)} &= \{[0;0], [0;0,k], [0,0,x;0,k], [0,0,x;0,k,\ell] \ : \ x \in \mathbb{O} \text{ and } k,\ell \in \mathbb{K}\},\\
Q_{(0,0,0,0;0)} &= \{[0,0;0], [0,0;k,0], [x,0,0;k,0], [x,0,0;k,\ell,0] \ : \ x \in \mathbb{O} \text{ and } k,\ell \in \mathbb{K}\}.
\end{aligned}$$

For each of these quads, one can now verify that $e(Q)$ is the quadric described above. We will work this out for the last case. The other cases are similar.

Put $Q := Q_{(0,0,0,0;0)}$. Then the points of $e(Q)$ are $(0^7, 1, 0^6)$, $(0^3, 1, 0^3, k, 0^6)$, $(0, 0, 1, x^{\sigma+1}, -x, 0, 0, k, 0^6)$, $(1, 0, \ell, k, x, 0, 0, \ell k - x^{\sigma+1}, 0^6)$. Clearly, all these points belong to the subspace $\mathsf{PG}(\langle E_1, E_3, E_4, E_5, E_8\rangle)$ and satisfy the equation $Z_1 Z_8 + Z_5^{\sigma+1} = Z_3 Z_4$. Conversely, suppose the coordinates of some point of $\mathsf{PG}(\langle E_1, E_3, E_4, E_5, E_8\rangle)$ satisfy the equation $Z_1 Z_8 + Z_5^{\sigma+1} = Z_3 Z_4$.

If $Z_1 \neq 0$, then we can choose $Z_1 = 1$. Since $Z_8 = Z_3 Z_4 - Z_5^{\sigma+1}$, we obtain the point $(1, 0, Z_3, Z_4, Z_5, 0, 0, Z_3 Z_4 - Z_5^{\sigma+1}, 0^6)$, which is precisely $e([Z_5, 0, 0; Z_4, Z_3, 0])$.

If $Z_1 = 0$, but $Z_3 \neq 0$, then we may assume $Z_3 = 1$. Then Z_8 is arbitrary and $Z_4 = Z_5^{\sigma+1}$. We obtain $(0, 0, 1, Z_5^{\sigma+1}, Z_5, 0, 0, Z_8, 0^6)$, which is precisely $e([-Z_5, 0, 0; Z_8, 0])$.

If $Z_1 = Z_3 = 0$, but $Z_4 \neq 0$, then we may assume $Z_4 = 1$. Then Z_8 is still arbitrary and $Z_5^{\sigma+1} = 0$, hence $Z_5 = 0$. We obtain $(0^3, 1, 0^3, Z_8, 0^6)$, which is precisely $e([0, 0; Z_8, 0])$.

Finally, if $Z_1 = Z_3 = Z_4 = 0$, then $Z_5 = 0$ and so $Z_8 \neq 0$. Hence we obtain $(0^7, 1, 0^6)$, which is precisely $e([0, 0; 0])$. ■

Lemma 5.2 *If p, p' and p'' are three distinct frame points contained in a plane α of $\mathcal{P}_\mathcal{T}$, then $\langle e(Q_p)\rangle$ intersects $\langle e(Q_{p'})\rangle$ in the line $e(Q_p \cap Q_{p'})$ and $\langle e(Q_{p'}), e(Q_{p''})\rangle$ in the plane $\langle e(Q_p \cap Q_{p'}), e(Q_p \cap Q_{p''})\rangle$.*

Proof. There are eight possibilities for $\{p, p', p''\}$, namely $\{(\infty), (0), (0, 0)\}$, $\{(\infty), (0), (0, 0; 0)\}$, $\{(\infty), (0, 0), (0, 0, 0; 0)\}$, $\{(\infty), (0, 0; 0), (0, 0, 0; 0)\}$, $\{(0), (0, 0), (0, 0, 0, 0; 0)\}$, $\{(0), (0, 0; 0), (0, 0, 0, 0; 0)\}$, $\{(0, 0), (0, 0, 0; 0), (0, 0, 0, 0; 0)\}$ and $\{(0, 0; 0), (0, 0, 0; 0), (0, 0, 0, 0; 0)\}$. These respectively correspond to the following possibilities for α: $[\infty]$, $[0]$, $[0; 0]$, $[0; 0, 0]$, $[0, 0; 0]$, $[0, 0; 0, 0]$, $[0, 0, 0; 0, 0]$ and $[0, 0, 0; 0, 0, 0]$. The claims of the lemma are easily verified with the aid of Lemma 5.1. ■

Lemma 5.3 *Suppose Q is a subspace of $\Delta_\mathcal{T}$ such that $\widetilde{Q}$ is a GQ. Suppose $e(Q)$ is a nonsingular quadric of Witt index 2 of a certain subspace A of $\mathsf{PG}(V)$. Then e defines an isomorphism between $\widetilde{Q}$ and the GQ defined by the points and lines of A contained in $e(Q)$.*

Proof. By Proposition 3.2, e maps lines of $\widetilde{Q}$ to lines of A contained in $e(Q)$. We now show that every line of A contained in $e(Q)$ is the image of a line of $\widetilde{Q}$. It suffices to show that if x_1 and x_2 are two noncollinear points of $\widetilde{Q}$, then the points $e(x_1)$ and $e(x_2)$ are noncollinear on the quadric $e(Q)$ of A. Let L_i, $i \in \{1, 2\}$, be a line of $\widetilde{Q}$ through x_i such that L_1 and L_2 are disjoint, and let $x_1' \neq x_1$ be the unique point of L_1 collinear with x_2. Now, $e(L_1)$ and $e(L_2)$ are two disjoint lines of the quadric $e(Q)$. Since $e(Q)$ has Witt index 2 and the point $e(x_1') \in e(L_1)$ is collinear with $e(x_2)$, the points $e(x_1)$ and $e(x_2)$ cannot be collinear on the quadric $e(Q)$. ■

Corollary 5.4 *If Q is a quad of $\Delta_\mathcal{T}$, then $\widetilde{Q} \cong \mathcal{Q}^*$.*

Proof. From the theory of thick (dual) polar spaces, we know that $\widetilde{Q}_1 \cong \widetilde{Q}_2$ for any two quads Q_1 and Q_2 of $\Delta_\mathcal{T}$. Taking this fact into account, the lemma is an immediate consequence of Lemmas 5.1 and 5.3. ∎

The generalized quadrangle $\mathsf{Q}(4,\mathbb{K})$ is the quadrangle arising from a nondegenerate quadric of Witt index 2 in a projective 4-space over $\mathbb{K}$. With a *full* $\mathsf{Q}(4,\mathbb{K})$*-subquadrangle* of $\Delta_\mathcal{T}$ we mean a full subgeometry that is isomorphic to $\mathsf{Q}(4,\mathbb{K})$. Any such subgeometry is contained in a unique quad. The dual polar space $\Delta_\mathcal{T}$ has many full $\mathsf{Q}(4,\mathbb{K})$-subquadrangles. For instance, if we go back to Lemma 5.1 and intersect the quadric $e(Q_{(\infty)})$ with the subspace of $\mathsf{PG}(\langle E_2, E_9, E_{10}, E_{11}, E_{14}\rangle)$ consisting of those points whose 11th coordinate belongs to $\mathbb{K}$, then we see that the set of points $\{[\infty],[k],[x;k],[x;k,\ell] : x,k,\ell \in \mathbb{K}\}$ forms a full $\mathsf{Q}(4,\mathbb{K})$-subquadrangle $Q'_{(\infty)}$ contained inside the quad $Q_{(\infty)}$.

Lemma 5.5 *Suppose* $|\mathbb{K}| > 2$*.*

(1) *For every line L of $\Delta_\mathcal{T}$ through the point $[\infty]$ not contained in any of the quads $Q_{(\infty)}$, $Q_{(0)}$, $Q_{(0,0)}$, there exists a full $\mathsf{Q}(4,\mathbb{K})$-subquadrangle containing L and intersecting the quads $Q_{(\infty)}$, $Q_{(0)}$ and $Q_{(0,0)}$ in three distinct lines.*

(2) *For every line L of $\Delta_\mathcal{T}$ through the point $[0,0,0;0,0,0]$ not contained in any of the quads $Q_{(0,0;0)}$, $Q_{(0,0,0;0)}$, $Q_{(0,0,0,0;0)}$, there exists a full $\mathsf{Q}(4,\mathbb{K})$-subquadrangle containing L and intersecting the quads $Q_{(0,0;0)}$, $Q_{(0,0,0;0)}$ and $Q_{(0,0,0,0;0)}$ in three distinct lines.*

Proof. (1) Let $Q'_{(\infty)}$ be as above and let $\pi_\mathbb{K}$ be the subplane of $[\infty]$ obtained from the description of $[\infty]$ as a plane of Type I by restricting the parameters a,b,s to $\mathbb{K}$. When we look at the lines of $Q'_{(\infty)}$ through the point $[\infty]$, we easily see that these correspond in $\mathcal{P}_\mathcal{T}$ with the lines L_1 and $L_2(x)$, $x \in \mathbb{K}$, of respective types A and B. Now, these lines are precisely the lines of the plane $[\infty]$ (in $\mathcal{P}_\mathcal{T}$) that are generated by the lines of $\pi_\mathbb{K}$ through (∞). The automorphism group of $\mathcal{P}_\mathcal{T}$ fixing $[\infty]$ induces a group of automorphisms of the projective plane based on $[\infty]$ that contains the little projective group of $[\infty]$ as a Moufang plane (this follows directly from the Moufang property of irreducible spherical buildings of rank at least 3). The stabilizer of $\pi_\mathbb{K}$ in that little projective group induces the little projective group of $\pi_\mathbb{K}$ in $\pi_\mathbb{K}$ which is a point-transitive group. Consequently, since automorphisms preserve the family of full $\mathsf{Q}(4,\mathbb{K})$-subquadrangles, there exists for every pair (x,π), consisting of a point x inside a subplane π of $[\infty]$ that is an automorphic image of $\pi_\mathbb{K}$, a full $\mathsf{Q}(4,\mathbb{K})$-subquadrangle Q' through $[\infty]$ contained in Q_x such that the lines of Q' through $[\infty]$ are precisely those lines of $[\infty]$ generated by the lines of π through x.

Now, clearly the quads $Q_{(\infty)}$, $Q_{(0)}$, $Q_{(0,0)}$ correspond to the three points (∞), (0) and $(0,0)$ of the plane $[\infty]$ in $\mathcal{P}_\mathcal{T}$. The line L is a line of $[\infty]$ not incident with any of (∞), (0) and $(0,0)$ (because of the condition that L is not contained in any of the corresponding quads). Now, the three lines in $[\infty]$ defined by the three points (∞), (0) and $(0,0)$, together with the line L form a quadruple of lines no three of which are concurrent. Since the little projective group of a Moufang plane acts transitively on such quadruples, we may include that quadruple in a subplane π automorphic to $\pi_{\mathbb{K}}$. The three lines defined by the three points (∞), (0) and $(0,0)$ intersect L in three distinct points. Since $|\mathbb{K}| > 2$, we can select a fourth point p on L in π. This point has the property that the respective lines in π through p and the points (∞), (0) and $(0,0)$ are all distinct, and also distinct from L. Any full $Q(4,\mathbb{K})$-subquadrangle of $\Delta_{\mathcal{T}}$ associated with the pair (p,π) (see previous paragraph) satisfies the requirements stated in (1).

(2) The second claim of the lemma follows from the first claim, taking into account that the automorphism group of a polar space of rank 3 is a point-transitive group. ∎

For every point x of $\Delta_{\mathcal{T}}$, let $x^{\perp}$ denote the set of points of $\Delta_{\mathcal{T}}$ collinear with x.

Lemma 5.6 *The subspaces $\langle e([\infty]^{\perp})\rangle$ and $\langle e([0,0,0;0,0,0]^{\perp})\rangle$ are complementary subspaces of $\mathsf{PG}(V)$.*

Proof. The points of $\Delta_{\mathcal{T}}$ collinear with $[\infty]$ are the points $[\infty]$, $[k]$, $[x;k]$ and $[x_1,x_2;k]$, $x,x_1,x_2 \in \mathbb{O}$ and $k \in \mathbb{K}$. It is easily checked that $\langle e([\infty]^{\perp})\rangle$ has equations $Z_1 = Z_2 = \cdots = Z_7 = 0$.

From the explicit calculation of the lines of $\Delta_{\mathcal{T}}$ (proof of Proposition 3.2), we infer that the lines incident with $[0,0,0;0,0,0]$ are given by $L_{10}(0^7)$, $L_{11}(0,0,-z^{\sigma},0^5)$ and $L_{12}(x,0,0,0,-v^{\sigma},0^4)$, $v,x,z \in \mathbb{O}$. These are incident with the following points of $\Delta_{\mathcal{T}}$ ($\ell \in \mathbb{K}$):

$$\begin{array}{l}[0;0,0],[0,0,0;0,0,\ell],\\ [0,z;0,0],[0,\ell z,0;\ell,0,\ell z^{\sigma+1}],\\ [x,0,v;0,0],[-\ell x,-\ell vx,\ell v;\ell x^{\sigma+1},\ell,\ell v^{\sigma+1}].\end{array}$$

These correspond, respectively, to the following points of $\mathsf{PG}(V)$:

$$\begin{array}{l}(0,1,0^{12}),(1,\ell,0^{12}),\\ (0,z^{\sigma+1},0,1,0,z,0^8),(1,\ell z^{\sigma+1},0,\ell,0,\ell z,0^8),\\ (0,v^{\sigma+1},1,x^{\sigma+1},-x,-vx,v,0^7),(1,\ell v^{\sigma+1},\ell,\ell x^{\sigma+1},-\ell x,-\ell vx,\ell v,0^7).\end{array}$$

It is easy to check that these points generate the subspace of $\mathsf{PG}(V)$ given by the equations $Z_8 = Z_9 = \cdots = Z_{14} = 0$. ∎

Lemma 5.7 *Let e' be a full embedding of $\Delta_{\mathcal{T}}$ in a projective space $\mathsf{PG}(V')$. Put $x := [\infty]$, $y := [0,0,0;0,0,0]$, $Q_1 := Q_{(\infty)}$, $Q_2 := Q_{(0)}$, $Q_3 := Q_{(0,0)}$, $R_1 := Q_{(0,0;0)}$, $R_2 := Q_{(0,0,0;0)}$ and $R_3 := Q_{(0,0,0,0;0)}$. Then the following holds.*

1. *The subspaces $\langle e'(x^{\perp})\rangle$ and $\langle e'(y^{\perp})\rangle$ generate $\mathsf{PG}(V')$.*
2. *If $|\mathbb{K}| \neq 2$, then the subspaces $\langle e'(x^{\perp}\cap Q_1)\rangle$, $\langle e'(x^{\perp}\cap Q_2)\rangle$ and $\langle e'(x^{\perp}\cap Q_3)\rangle$ generate $\langle e'(x^{\perp})\rangle$.*

(3) *If* $|\mathbb{K}| \neq 2$, *then the subspaces* $\langle e'(y^\perp \cap R_1)\rangle$, $\langle e'(y^\perp \cap R_2)\rangle$ *and* $\langle e'(r^\perp \cap R_3)\rangle$ *generate* $\langle e'(y^\perp)\rangle$.

Proof. Suppose L_1, L_2 and L_3 are three distinct lines through a point z of the generalized quadrangle $\mathsf{Q}(4,\mathbb{K})$. The absolutely universal embedding of $\mathsf{Q}(4,\mathbb{K})$ is defined by a nonsingular quadric of Witt index 2 of $\mathsf{PG}(4,\mathbb{K})$. If $\mathsf{char}\,\mathbb{K} \neq 2$, then this is, up to isomorphism, the unique full projective embedding of $\mathsf{Q}(4,\mathbb{K})$. If $\mathsf{char}\,\mathbb{K} = 2$, then there is one other full projective embedding, namely the one which is obtained from the absolutely universal embedding by projecting from the nucleus of the associated quadric. If e'' is a full projective embedding of $\mathsf{Q}(4,\mathbb{K})$, then we always have $\langle e''(z^\perp)\rangle = \langle e''(L_1), e''(L_2), e''(L_3)\rangle$. Claims (2) and (3) of the lemma now follow from this fact and Lemma 5.5.

In order to show Claim (1), it suffices to show that the smallest subspace S of $\Delta_{\mathcal{T}}$ containing $x^\perp$ and $y^\perp$ is the whole set of points. Let z be an arbitrary point of $\Delta_{\mathcal{T}}$ at distance 2 from x, let Q denote the unique quad through x and z and let y' be the unique point of Q collinear with y. Since S contains the maximal proper subspace $Q \cap x^\perp$ of $\widetilde{Q}$ and the point $y' \in Q \setminus x^\perp$, the whole quad Q is contained in S, in particular the point z. So, the set S' of all points not opposite to x is contained in S. Since S' is a maximal proper subspace (recall Blok and Brouwer [1, Theorem 7.3] or Shult [24, Lemma 6.1]) and $y \in S \setminus S'$, S should contain all points of $\Delta_{\mathcal{T}}$. ∎

Theorem 5.8 *If* $|\mathbb{K}| \neq 2$, *then* e *is isomorphic to the absolutely universal embedding* $\widetilde{e}$ *of* $\Delta_{\mathcal{T}}$.

Proof. We denote by $\widetilde{\Sigma}$ the projective space over $\mathbb{K}$ which serves as co-domain for $\widetilde{e}$. There exists a subspace α of $\widetilde{\Sigma}$ disjoint from the image $\mathsf{Im}(\widetilde{e})$ of $\widetilde{e}$ such that e is isomorphic to the projective embedding of $\Delta_{\mathcal{T}}$ which arises by projecting $\mathsf{Im}(\widetilde{e})$ from α onto a subspace of $\widetilde{\Sigma}$ complementary to α. As in Lemma 5.7, we put $x := [\infty]$, $y := [0,0,0;0,0,0]$, $Q_1 := Q_{(\infty)}$, $Q_2 := Q_{(0)}$ and $Q_3 := Q_{(0,0)}$.

We show that the subspaces $\langle \widetilde{e}(x^\perp)\rangle$ and α are disjoint. Suppose α meets $\langle \widetilde{e}(Q_i)\rangle$ for some $i \in \{1,2,3\}$. Then after projecting from α, we see that the embedding of $\widetilde{Q}_i$ induced by e is not absolutely universal. This is however not possible. By Lemmas 5.1 and 5.3, the embedding of $\widetilde{Q}_i$ induced by e arises from some nonsingular quadric of Witt index 2 and such an embedding must be absolutely universal by Tits [27, Corollary 8.7]. So, α is disjoint from $\langle \widetilde{e}(Q_1)\rangle$, $\langle \widetilde{e}(Q_2)\rangle$ and $\langle \widetilde{e}(Q_3)\rangle$. By Lemma 5.7(2), we know that $\langle \widetilde{e}(Q_1 \cap x^\perp)\rangle$, $\langle \widetilde{e}(Q_2 \cap x^\perp)\rangle$ and $\langle \widetilde{e}(Q_3 \cap x^\perp)\rangle$ generate $\langle \widetilde{e}(x^\perp)\rangle$. If α would meet $\langle \widetilde{e}(x^\perp)\rangle$ nontrivially, then the property of e mentioned in Lemma 5.2 would no longer hold.

So, α is disjoint from $\langle \widetilde{e}(x^\perp)\rangle$. In a similar way one proves that α is disjoint from $\langle \widetilde{e}(y^\perp)\rangle$.

If the subspaces $\langle \widetilde{e}(x^\perp)\rangle$ and $\langle \widetilde{e}(y^\perp)\rangle$ meet nontrivially, then by Lemma 5.6, their intersection would be contained in α which is impossible as α is disjoint from $\langle \widetilde{e}(x^\perp)\rangle$ and $\langle \widetilde{e}(y^\perp)\rangle$. Lemma 5.7(1) then implies that $\langle \widetilde{e}(x^\perp)\rangle$ and $\langle \widetilde{e}(y^\perp)\rangle$ are complementary subspaces of $\widetilde{\Sigma}$. So, if $\alpha \neq \emptyset$, then every point of α would be contained in a unique line meeting $\langle \widetilde{e}(x^\perp)\rangle$

and $\langle \widetilde{e}(y^\perp)\rangle$. After projecting, we then see that the subspaces $\langle e(x^\perp)\rangle$ and $\langle e(y^\perp)\rangle$ of $\mathsf{PG}(V)$ should meet, which is in contradiction with Lemma 5.6.

We conclude that $\alpha = \emptyset$. So, e and $\widetilde{e}$ are isomorphic embeddings. ∎

The condition $|\mathbb{K}| \neq 2$ in Theorem 5.8 cannot be omitted. If $|\mathbb{K}| = 2$, then $\Delta_\mathcal{T}$ is isomorphic to one of the dual polar spaces $DW(5,2)$ or $DH(5,4)$, and e is isomorphic to the Grassmann embedding. The Grassmann embeddings of these two dual polar spaces are known not to be absolutely universal, see for instance Yoshiara [30].

Proposition 3.8 now immediately implies the following corollary.

Corollary 5.9 *Suppose $|\mathbb{K}| \neq 2$. If σ is nontrivial or $\mathsf{char}\,\mathbb{K} \neq 2$, then e is, up to isomorphism, the unique full polarized embedding of $\Delta_\mathcal{T}$.*

6 Projective representations of the little projective group

Suppose $\mathbb{O}$ is an alternative division algebra which is quadratic over some subfield $\mathbb{K}$ of its center. We denote by σ the corresponding standard involution of $\mathbb{O}$. With the pair $\mathcal{T} = (\mathbb{O}, \mathbb{K})$, there is associated a polar space $\mathcal{P}_\mathcal{T}$ and a dual polar space $\Delta_\mathcal{T}$. Regarding $\mathcal{P}_\mathcal{T}$ as a building of type C_3, the frame points determine a unique apartment (which then consists of all points, lines and planes whose coordinates do not contain any element of $\mathbb{O} \setminus \{0\}$), which can be seen as the octahedron in Figure 1, and which we call the *standard apartment*. Viewing the octahedron in real Euclidean 3-space, a *half apartment* or *root* is the set of elements of the octahedron either intersecting or lying at a fixed side of a symmetry plane. The set of elements intersecting a symmetry plane is a *wall*, and every wall hence determines two roots in the obvious way. The elements of a root not on the corresponding wall are called *interior*. With every half apartment, there is associated a *root group*, which consists of all the automorphisms of $\mathcal{P}_\mathcal{T}$ (called *root elations*) that fix every element incident with an interior element of the root. We denote by $G^\dagger$ the *little projective group* of $\mathcal{P}_\mathcal{T}$, i.e., the group generated by all root groups (in fact, it suffices to only take the root groups belonging to the roots of the standard apartment). The little projective group, which is a normal subgroup of $\mathrm{Aut}(\mathcal{P}_\mathcal{T})$, is nothing else than the standard simple group associated to the polar space $\mathcal{P}_\mathcal{T}$. For a point x of $\mathcal{P}_\mathcal{T}$, we denote by E_x the group generated by all root elations such that x is an interior point of the corresponding root. It is well-known and easy to prove that E_x acts regularly on the set of points opposite to x (that means, noncollinear with x). If θ is an automorphism of $\mathcal{P}_\mathcal{T}$, then $E_x^\theta = E_{x^\theta}$. We will refer to E_x as the *elation group with center x* (in terms of algebraic groups, it is the unipotent radical of x). If we do not want to specify x, we just call it a *central elation group*. Naturally, every such group is contained in $G^\dagger$.

Lemma 6.1 (1) *$G^\dagger$ is generated by all central elation groups.*

(2) *If ξ is an element of $G^\dagger$ mapping (∞) to $(0,0,0,0;0)$, then $G^\dagger = \langle E_{(\infty)}, \xi \rangle$.*

Proof. (1) Since $E_x^\theta = E_{x^\theta}$ for every point x and every automorphism θ of $\mathcal{P}_\mathcal{T}$, the subgroup of $G^\dagger$ generated by all central elation groups must be normal and hence equal to $G^\dagger$ as the latter is a simple group. Alternatively, every root group is contained in some central elation group.

(2) Obviously, $G := \langle E_{(\infty)}, \xi \rangle \subseteq G^\dagger$. In order to show equality, it suffices by (1) to show that $E_x \subseteq G$ for every point x of $\mathcal{P}_\mathcal{T}$. This holds for the point $x_1 := (0,0,0,0;0) = (\infty)^\xi$ since $E_{x_1} = E_{(\infty)}^\xi$. If y is a point opposite to x, then there exists a unique $\theta \in E_{(\infty)}$ such that $y = x_1^\theta$ and hence $E_y = E_{x_1}^\theta \subseteq G$. If z is a point not opposite to (∞), then there exists a point y' that is opposite to both (∞) and z, and a unique $\theta' \in E_{y'}$ mapping (∞) to z. Since $E_{y'} \subseteq G$, we must have $E_z = E_{(\infty)}^{\theta'} \subseteq G$. ■

A full embedding e' of $\Delta_\mathcal{T}$ into a projective space $\mathsf{PG}(V')$ is called *homogeneous* if for every automorphism θ of $\Delta_\mathcal{T}$, there exists a (necessarily unique) automorphism $\widetilde{\theta}$ of $\mathsf{PG}(V')$ such that $e' \circ \theta = \widetilde{\theta} \circ e'$. If this is the case, then the map $\mathrm{Aut}(\Delta_\mathcal{T}) \to P\Gamma L(V') : \theta \mapsto \widetilde{\theta}$ is a group monomorphism. The absolutely universal embedding and the minimal full polarized embedding of $\Delta_\mathcal{T}$ are examples of homogeneous embeddings. If $\Delta_\mathcal{T}$ is a symplectic or Hermitian dual polar space, then also the Grassmann embedding of $\Delta_\mathcal{T}$ is homogeneous. This implies by Proposition 4.1 and Theorem 5.8 that the full projective embedding e is always homogeneous. If $|\mathbb{K}| \geq 3$, then we will determine in this section all homogeneous full projective embeddings of $\Delta_\mathcal{T}$ and give descriptions of the corresponding projective representations of $G^\dagger$. If $G^\dagger \to P\Gamma L(V) : \theta \to \widetilde{\theta}$ is such a representation, then it suffices by Lemma 6.1 to give a description of $\widetilde{E_{(\infty)}}$ and $\widetilde{\xi}$, where ξ is a some specific element of $G^\dagger$ mapping (∞) to $(0,0,0,0;0)$.

In Propositions 4.9 till 4.13 of De Bruyn and Van Maldeghem [13] a number of automorphisms of $\mathcal{P}_\mathcal{T}$ are described. The group generated by these automorphisms is precisely the central elation group $E_{(\infty)}$. The description in [13] only mentions the actions of the automorphisms on the points of $\mathcal{P}_\mathcal{T}$, but the actions on the planes and the corresponding automorphisms of $\mathsf{PG}(V)$ can easily be derived from that. The corresponding automorphisms of $\mathsf{PG}(V)$ are mentioned in the first five columns of Table 2, where we present their action on the coordinates with respect to the standard basis of $V = \mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{O} \times \mathbb{K}$. The lifting $\widetilde{E_{(\infty)}}$ of $E_{(\infty)}$ to a group of automorphisms of $\mathsf{PG}(V)$ is thus precisely the subgroup $\langle \theta_1(\eta), \theta_2(\eta), \theta_3(\eta), \theta_4(\eta), \theta_5(k^*) \,|\, \eta \in \mathbb{O},\ k^* \in \mathbb{K} \rangle$ of $PGL(V)$.

We will now describe a number of other automorphisms of $\Delta_\mathcal{T}$ (or equivalently, of $\mathcal{P}_\mathcal{T}$). These automorphisms will be most easily described by their induced action on $\mathsf{PG}(V)$. Before describing the automorphisms, we need to mention a lemma and a corollary.

Lemma 6.2 *Let x_1 and x_2 be two distinct points of $\Delta_\mathcal{T}$. Then the unique line L of $\mathsf{PG}(V)$ through $e(x_1)$ and $e(x_2)$ is contained in $\mathsf{Im}(e)$ if and only if x_1 and x_2 are collinear.*

	$\theta_1(\eta)$	$\theta_2(\eta)$	$\theta_3(\eta)$
X_1	X_1	X_1	X_1
X_2	$X_2 - X_6\eta - \eta^\sigma X_6^\sigma + \eta^{\sigma+1}X_4$	$X_2 - X_7\eta - \eta^\sigma X_7^\sigma + \eta^{\sigma+1}X_3$	X_2
X_3	X_3	X_3	X_3
X_4	X_4	X_4	X_4
X_5	X_5	X_5	X_5
X_6	$X_6 - \eta^\sigma X_4$	$X_6 - \eta^\sigma X_5$	X_6
X_7	$X_7 - \eta^\sigma X_5^\sigma$	$X_7 - \eta^\sigma X_3$	$X_7 + \eta^\sigma X_1$
X_8	X_8	X_8	X_8
X_9	X_9	$X_9 - X_{13}\eta - \eta^\sigma X_{13}^\sigma + \eta^{\sigma+1}X_8$	X_9
X_{10}	$X_{10} - X_{12}\eta - \eta^\sigma X_{12}^\sigma + \eta^{\sigma+1}X_8$	X_{10}	$X_{10} - \eta X_7 - X_7^\sigma\eta^\sigma - \eta^{\sigma+1}X_1$
X_{11}	$X_{11} + X_{13}^\sigma\eta^\sigma$	$X_{11} + \eta X_{12}$	$X_{11} - \eta X_6$
X_{12}	$X_{12} - \eta^\sigma X_8$	X_{12}	$X_{12} - \eta^\sigma X_5$
X_{13}	X_{13}	$X_{13} - \eta^\sigma X_8$	$X_{13} + \eta^\sigma X_4$
X_{14}	X_{14}	X_{14}	$X_{14} - \eta X_{13} - X_{13}^\sigma\eta^\sigma - \eta^{\sigma+1}X_4$
	$\theta_4(\eta)$	$\theta_5(k^*)$	θ^*
X_1	X_1	X_1	X_{14}
X_2	X_2	$X_2 + k^*X_1$	$-X_8$
X_3	X_3	X_3	$-X_9$
X_4	X_4	X_4	$-X_{10}$
X_5	X_5	X_5	X_{11}
X_6	$X_6 + \eta^\sigma X_1$	X_6	$-X_{12}$
X_7	X_7	X_7	$-X_{13}$
X_8	X_8	X_8	X_2
X_9	$X_9 - \eta X_6 - X_6^\sigma\eta^\sigma - \eta^{\sigma+1}X_1$	$X_9 + k^*X_4$	X_3
X_{10}	X_{10}	$X_{10} + k^*X_3$	X_4
X_{11}	$X_{11} - X_7^\sigma\eta^\sigma$	$X_{11} + k^*X_5$	$-X_5$
X_{12}	$X_{12} + \eta^\sigma X_3$	X_{12}	X_6
X_{13}	$X_{13} - \eta^\sigma X_5^\sigma$	X_{13}	X_7
X_{14}	$X_{14} - \eta X_{12} - X_{12}^\sigma\eta^\sigma - \eta^{\sigma+1}X_3$	$X_{14} + k^*X_8$	$-X_1$

Table 2: Generators for the projective representation Θ of $G^\dagger$.

Proof. If x_1 and x_2 are incident with the same line K, then $L \cap \mathsf{Im}(e) = L = e(K)$. Suppose now that x_1 and x_2 are not collinear. Then we show that $L \cap \mathsf{Im}(e) = \{e(x_1), e(x_2)\}$. Since e is a homogeneous embedding, we may without loss of generality suppose that one of the following cases occurs:

(1) $x_1 = [\infty]$ and $x_2 = [0; 0, 0]$ (when $\mathrm{d}(x_1, x_2) = 2$);

(2) $x_1 = [\infty]$ and $x_2 = [0, 0, 0; 0, 0, 0]$ (when $\mathrm{d}(x_1, x_3) = 3$).

In each of the two cases, it is readily verified that $L \cap \mathsf{Im}(e) = \{e(x_1), e(x_2)\}$. ■

The following is an immediate corollary of Lemma 6.2.

Corollary 6.3 (1) *The points and lines of* $\mathsf{PG}(V)$ *contained in* $\mathsf{Im}(e)$ *define a point-line geometry* $\Delta'_{\mathcal{T}}$*, and the map* $x \mapsto e(x)$ *defines an isomorphism between* $\Delta_{\mathcal{T}}$ *and* $\Delta'_{\mathcal{T}}$*.*

(2) *The map* $\theta \to \widetilde{\theta}$ *defines an isomorphism between* $\mathrm{Aut}(\Delta_{\mathcal{T}})$ *and the stabilizer of* $\mathsf{Im}(e)$ *inside* $P\Gamma L(V)$*.*

Lemma 6.4 *The involutions* ϕ_1*,* ϕ_2 *and* ϕ_3 *of* $PGL(V)$ *defined by*

$$\phi_1 : (X_1, \ldots, X_{14}) \mapsto (X_{14}, -X_8, -X_9, -X_{10}, X_{11}, X_{12}, X_{13}, X_2, X_3, X_4, -X_5, -X_6, -X_7, -X_1),$$

$$\phi_2 : (X_1, \ldots, X_{14}) \mapsto (X_{14}, X_9, X_8, X_{10}, X_{12}, X_{11}, -X_{13}^\sigma, X_3, X_2, X_4, X_6, X_5, -X_7^\sigma, X_1),$$
$$\phi_3 : (X_1, \ldots, X_{14}) \mapsto (X_{14}, X_{10}, X_9, X_8, X_{13}^\sigma, -X_{12}^\sigma, X_{11}^\sigma, X_4, X_3, X_2, X_7^\sigma, -X_6^\sigma, X_5^\sigma, X_1),$$

stabilize $\mathsf{Im}(e)$ *and hence correspond to automorphisms of* $\mathcal{P}_{\mathcal{T}}$ *and* $\Delta_{\mathcal{T}}$*.*

Proof. That the above involutions stabilize $\mathsf{Im}(e)$ can be verified with the aid of Proposition 3.9 (each of the involutions will give rise to a permutation of the 26 equations). Since $\mathsf{Im}(e)$ is stabilized, the involutions correspond to automorphisms of $\mathcal{P}_{\mathcal{T}}$ and $\Delta_{\mathcal{T}}$ by Corollary 6.3. ■

Lemma 6.5 *The involution* θ^* *of* $PGL(V)$ *defined by*

$$(X_1, \ldots, X_{14}) \mapsto (X_{14}, -X_8, -X_9, -X_{10}, X_{11}, -X_{12}, -X_{13}, X_2, X_3, X_4, -X_5, X_6, X_7, -X_1)$$

stabilizes $\mathsf{Im}(e)$ *and induces an automorphism* ξ *of* $\mathcal{P}_{\mathcal{T}}$ *that belongs to* $G^\dagger$ *and interchanges* (∞) *with* $(0, 0, 0, 0; 0)$*,* (0) *with* $(0, 0, 0; 0)$*,* $(0, 0)$ *with* $(0, 0; 0)$*,* $[0, 0, 0; 0, 0, 0]$ *with* $[\infty]$*,* $[0, 0, 0; 0, 0]$ *with* $[0]$*,* $[0, 0; 0, 0]$ *with* $[0; 0]$ *and* $[0, 0; 0]$ *with* $[0; 0, 0]$*.*

Proof. We define the following elements of $PGL(V)$:

$$\eta_1 := \theta_5(1)\phi_1\theta_5(1)\phi_1\theta_5(1), \qquad \eta_2 := \phi_2\eta_1\phi_2, \qquad \eta_3 := \phi_3\eta_1\phi_3.$$

Then

$$\eta_1 : (X_1, \ldots, X_{14}) \mapsto (X_2, -X_1, X_{10}, X_9, X_{11}, -X_6, -X_7, X_{14}, -X_4, -X_3, -X_5, -X_{12}, -X_{13}, -X_8),$$

$$\eta_2 : (X_1, \ldots, X_{14}) \mapsto (-X_3, -X_{10}, X_1, -X_8, -X_5, -X_{12}, -X_7, X_4, -X_{14}, X_2, -X_{11}, X_6, -X_{13}, X_9),$$

$$\eta_3 : (X_1, \ldots, X_{14}) \mapsto (-X_4, -X_9, -X_8, X_1, -X_5, -X_6, -X_{13}, X_3, X_2, -X_{14}, -X_{11}, -X_{12}, X_7, X_{10})$$

and one verifies that $\theta^* = \eta_1\eta_2\eta_3$. Since the automorphism of $\mathcal{P}_{\mathcal{T}}$ corresponding to $\theta_5(1)$ belongs to $G^\dagger$ and ϕ_1 induces an involution of $\mathrm{Aut}(\mathcal{P}_{\mathcal{T}})$, the element $\eta_1 = \theta_5(1)(\phi_1\theta_5(1)\phi_1)\theta_5(1)$ also induces an element of $G^\dagger$. Since ϕ_2 and ϕ_3 are involutions, also η_2 and η_3 induce automorphisms of $\mathcal{P}_{\mathcal{T}}$ that belong to $G^\dagger$. Hence, the automorphism ξ of $\mathcal{P}_{\mathcal{T}}$ induced by $\theta^* = \eta_1\eta_2\eta_3$ also belongs to $G^\dagger$.

By looking at the action on the embedded points, we see that the automorphism ξ of $\mathcal{P}_{\mathcal{T}}$ induced by θ^* interchanges $[0,0,0;0,0,0]$ with $[\infty]$, $[0,0,0;0,0]$ with $[0]$, $[0,0;0,0]$ with $[0;0]$ and $[0,0;0]$ with $[0;0,0]$. This implies that ξ interchanges (∞) with $(0,0,0,0;0)$, (0) with $(0,0,0;0)$ and $(0,0)$ with $(0,0;0)$. Indeed, each of these points can be obtained as the intersection of three of the mentioned planes and so its image can be computed. For instance, the point (∞) is the unique point in the intersection of the planes $[\infty]$, $[0]$, $[0;0]$, and hence must be mapped to the unique point in the intersection of the planes $[0,0,0;0,0,0]$, $[0,0,0;0,0]$ and $[0,0;0,0]$, namely the point $(0,0,0;0,0,0)$. ■

We have mentioned the involution θ^* in the last column of Table 2. By Lemma 6.1, $G^\dagger = \langle E_{(\infty)}, \xi\rangle$ and hence Table 2 provides generators for the projective representation Θ of $G^\dagger$ defined by the embedding e. In case σ is trivial and $\mathsf{char}\,\mathbb{K} = 2$, Table 2 also provides generators for the projective representation Θ' of $G^\dagger$ defined by the minimal full polarized embedding of $\Delta_{\mathcal{T}}$. To obtain the latter generators, one needs to consider only those rows in Table 2 that correspond to coordinates X_i where $i \in \{1,2,3,4,8,9,10,14\}$.

Suppose now that $|\mathbb{K}| \geq 3$. Then e is universal and the problem of finding all homogeneous embeddings of $\Delta_{\mathcal{T}}$ is equivalent with the determination of all proper subspaces of $\mathsf{PG}(V)$ that are invariant under the induced action of $\mathrm{Aut}(\Delta_{\mathcal{T}})$ on $\mathsf{PG}(V)$. If U is such an invariant subspace, then the quotient embedding e/U is a homogeneous embedding. By the very definition of $\mathfrak{R}$, we know that it is an invariant subspace. The corresponding homogeneous embedding $e/\mathfrak{R}$ is precisely the minimal full polarized embedding of $\Delta_{\mathcal{T}}$. By Theorem 1.2[2] of Blok et al. [2], every proper subspace of $\mathsf{PG}(V)$ that is invariant under the induced action of $G^\dagger$ on $\mathsf{PG}(V)$ is contained in $\mathfrak{R}$. So, if σ is nontrivial or $\mathsf{char}\,\mathbb{K} \neq 2$, then e is, up to isomorphism, the unique full homogeneous embedding of $\Delta_{\mathcal{T}}$. The following proposition discusses the case where σ is trivial and $\mathsf{char}\,\mathbb{K} = 2$.

Proposition 6.6 *Let $G \cong G^\dagger$ be the subgroup $\langle \theta_1(\eta), \theta_2(\eta), \theta_3(\eta), \theta_4(\eta), \theta_5(k), \theta^* \,|\, \eta \in \mathbb{O}, k \in \mathbb{K}\rangle$ of $PGL(V)$. Suppose $\mathsf{char}\,\mathbb{K} = 2$, σ is trivial and $\mathsf{PG}(W)$ is a subspace of $\mathfrak{R} = \mathsf{PG}(\langle E_5, E_6, E_7, E_{11}, E_{12}, E_{13}\rangle)$ stabilized by G. Then either $\mathsf{PG}(W) = \emptyset$ or $\mathsf{PG}(W) = \mathfrak{R}$.*

[2] The proof of Theorem 1.2 in [2] only uses the Moufang property of the polar spaces involved.

Proof. We will make use of the maps mentioned in Table 2 and regard them as elements of $GL(V)$. Suppose there exists a vector $\bar{w}_1 \in W$ not contained in $U := \langle E_6, E_7, E_{11}, E_{12}, E_{13} \rangle$. Then $\bar{w}_2 := \bar{w}_1^{\theta_5(1)} - \bar{w}_1 \in E_{11} \setminus \{\bar{o}\}$, implying that $\bar{w} := \bar{w}_2^{\theta^*}$ is a nonzero vector of W contained in E_5. Since $\bar{w}^{\theta_3(\eta)} - \bar{w}$ and $\bar{w}^{\theta_4(\eta)} - \bar{w}$ belong to W for all $\eta \in \mathbb{O}$, we see that the subspaces E_{12} and E_{13} are contained in W. Since $\bar{v}^{\theta_1(1)} - \bar{v} \in W$ for all $\bar{v} \in E_{13}$, we see that also E_{11} is contained in W. Hence, also $E_5 = E_{11}^{\theta^*}$, $E_6 = E_{12}^{\theta^*}$ and $E_7 = E_{13}^{\theta^*}$ are contained in W. We conclude that $\mathsf{PG}(W) = \mathfrak{R}$.

Suppose now that $W \subseteq U$. Then $W \subseteq U \cap U^{\theta^*} = U' := \langle E_6, E_7, E_{12}, E_{13} \rangle$. The only vectors $\bar{v}$ of U' for which none of the vectors $\bar{v}^{\theta_i(1)} - \bar{v}$, $i \in \{1, 2, 3\}$, belongs to $E_{11} \setminus \{\bar{o}\}$ are those of the subspace E_7. So, $W \subseteq E_7$. Hence, $W \subseteq E_7 \cap E_7^{\theta^*} = \{\bar{o}\}$, i.e. $W = \{\bar{o}\}$. ■

By Proposition 6.6 and the above discussion, we now know that the following should hold.

Corollary 6.7 *Suppose* $|\mathbb{K}| \neq 2$.

(1) *If σ is nontrivial or $\mathsf{char}\,\mathbb{K} \neq 2$, then e is, up to isomorphism, the unique full homogeneous embedding of $\Delta_{\mathcal{T}}$. If σ is trivial and $\mathsf{char}\,\mathbb{K} = 2$, then, up to isomorphism, e and the minimal full polarized embedding are the only two homogeneous full embeddings of $\Delta_{\mathcal{T}}$.*

(2) *The projective representation Θ is irreducible and has no nontrivial invariant subspaces if $\sigma \neq 1$ or $\mathsf{char}\,\mathbb{K} \neq 2$. If $\sigma = 1$ and $\mathsf{char}\,\mathbb{K} = 2$, then $\mathfrak{R}$ is the unique nontrivial invariant subspace.*

The conclusion of Corollary 6.7(1) is not valid if $|\mathbb{K}| = 2$. Indeed, in this case the dual polar space is isomorphic to either $DW(5, 2)$ or $DH(5, 4)$ and there is precisely one other homogeneous full embedding (up to isomorphism), namely the universal embedding (which is nonisomorphic to e).

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