Greed is Good for Deterministic Scale-Free Networks

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Abstract

Large real-world networks typically follow a power-law degree distribution. To study such networks, numerous random graph models have been proposed. However, real-world networks are not drawn at random. In fact, the behavior of real-world networks and random graph models can be the complete opposite of one another, depending on the considered property. Brach, Cygan, Lacki, and Sankowski [SODA 2016] introduced two natural deterministic conditions: (1) a power-law upper bound on the degree distribution (PLB-U) and (2) power-law neighborhoods, that is, the degree distribution of neighbors of each vertex is also upper bounded by a power law (PLB-N). They showed that many real-world networks satisfy both deterministic properties and exploit them to design faster algorithms for a number of classical graph problems like transitive closure, maximum matching, determinant, PageRank, matrix inverse, counting triangles and maximum clique.

We complement the work of Brach et al. by showing that some well-studied random graph models exhibit both the mentioned PLB properties and additionally also a power-law lower bound on the degree distribution (PLB-L). All three properties hold with high probability for Chung-Lu Random Graphs and Geometric Inhomogeneous Random Graphs and almost surely for Hyperbolic Random Graphs. As a consequence, all results of Brach et al. also hold with high probability for Chung-Lu Random Graphs and Geometric Inhomogeneous Random Graphs and almost surely for Hyperbolic Random Graphs.

In the second part of this work we study three classical NP-hard combinatorial optimization problems on PLB networks. It is known that on general graphs, a greedy algorithm, which chooses nodes in the order of their degree, only achieves an approximation factor of asymptotically at least logarithmic in the maximum degree for Minimum Vertex Cover and Minimum Dominating Set, and an approximation factor of asymptotically at least the maximum degree for Maximum Independent Set. We prove that the PLB-U property suffices such that the greedy approach achieves a constant-factor approximation for all three problems. We also show that all three combinatorial optimization problems are APX-complete, even if all PLB-properties hold. Hence, a PTAS cannot be expected, unless P=NP.

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1 Introduction

A wide range of real-world networks, like Internet topologies [18], the Web [27, 8], social networks [1], power grids [32], and many other networks [29, 5, 30], exhibit a power-law degree distribution. Power-law degree distribution means that the number of nodes of degree k is proportional to $k^{-\beta}$, where $\beta > 1$ is the power-law exponent, a constant intrinsic to the network. Networks with a power-law degree distribution are also called scale-free networks and have been widely studied.

To capture the degree distribution and other properties of scale-free networks, a multitude of random graph models have been proposed. These models include Preferential Attachment [8], the Configuration Model [2], Chung-Lu Random Graphs [15] and Hyperbolic Random Graphs [26]. Despite the multitude of random models, none of the models truly has the same set of properties as real-world networks.

This shortcoming of random graph models motivates studying deterministic properties of scale-free models, which can be verified on real-world networks. To describe the properties of scale-free networks without the use of random graphs, Aiello et al. [4] define (α, β)-Power Law Graphs. The problem of this model is that it essentially demands a perfect power-law degree distribution, whereas the degree distributions of real networks normally exhibit slight deviations from power laws. Therefore, (α, β)-Power Law Graphs are too constrained and do not capture most real networks.

To allow for those deviations in the degree distribution Brach et al. [10] define buckets containing nodes of degrees $\left[2^i, 2^{i+1}\right)$. If the number of nodes in each bucket is at most as high as for a power-law degree sequence, a network is said to be *power-law bounded*, which we denote as a network with property **PLB-U**. They also define the property of *PLB neighborhoods*: A network has PLB neighborhoods if every node of degree k has at most as many neighbors of degree at least k, as if those neighbors were picked independently at random with probability proportional to their degree. This property we abbreviate as **PLB-N**. A formal definition of both properties can be found in Section 3. Brach et al. [10] show that various classical graph problems can be solved more efficiently in networks with properties PLB-(U,N). The graph problems addressed are transitive closure, maximum matching, determinant, PageRank, matrix inverse, counting triangles and maximum clique. Brach et al. [10] also showed experimentally that PLB-(U,N) properties hold for many real-world networks, which implies that the mentioned graph problems can be solved faster on these real-world networks than worst-case lower bounds for general graphs suggest.

1.1 Motivation and Results

PLB properties in power-law random graph models

The PLB-(U,N) properties are designed to describe power-law graphs in a way that allows analyzing algorithms deterministically. As already mentioned, there is a mutitude of random graph models [2, 15, 8, 26], which can be used to generate power-law graphs. Brach et al. [10] proved that the Erased Configuration Model [2] follows PLB-U and w. h. p. also PLB-N. Since the Erased Configuration Model has a fixed degree sequence, it is relatively easy to prove the PLB-U property, but it is quite technical to prove the PLB-N property. There are other power-law random graph models, which are based on the expected degree sequence, e.g. Chung-Lu Random Graphs [15]. Brach et al. argued that for showing the PLB-U property on these models, a typical concentration statement does not work, as it accumulates the additive error for each bucket. They leave it as a *challenging open question*, whether other

random graph models also produce graphs with PLB-(U,N) properties with high probability[1].

In section 4 we address this question and extend the list of random graph models with the PLB-U and PLB-N property. We prove that Chung-Lu Random Graphs and Geometric Inhomogeneous Random Graphs are PLB-(U,N) graphs with high probability and that Hyperbolic Random Graphs are PLB-(U,N) graphs almost surely.

Algorithmic Results

The above results imply that all results of Brach et al. [10] also hold w. h. p. for Chung-Lu Random Graphs and Geometric Inhomogeneous Random Graphs and almost surely for Hyperbolic Random Graphs. Therefore the problems transitive closure, maximum matching, determinant, PageRank, matrix inverse, counting triangles and maximum clique have faster algorithms on Chung-Lu and Geometric Inhomogeneous Random Graphs w. h. p. and on Hyperbolic Random Graphs almost surely.

In this work we additionally consider the three classical NP-complete problems Minimum Dominating Set(MDS), Maximum Independent Set(MIS) and Minimum Vertex Cover(MVC) on PLB-U networks. For the first two problems, positive results are already known for (α, β)-Power Law Graphs, which are a special case of graphs with the PLB-(U,L) properties. Note that this deterministic graph class is much more restrictive and does *not* cover typical real-world graphs. On the contrary, our positive results only assume the PLB-U property. Our algorithmic results can therefore be applied to real-world networks after measuring the respective constants of the PLB-model. In section 5 we prove our main lemma, Lemma 5.2 (the potential volume lemma). Using the potential volume lemma, we prove lower bounds for MDS, MIS and MVC in the order of $\Theta(n)$ on PLB-U networks with exponent $\beta > 2$. This essentially means, even taking all nodes as a solution gives a constant factor approximation. Furthermore, in Theorem 5.5 we prove that the greedy algorithm actually achieves a better constant approximation ratio. These positive results also hold for (α, β)-Power Law Graphs.

In section 6, we consider the mentioned NP-Complete problems and prove that these problems are APX-hard even for PLB-(U,L,N) networks with $\beta > 2$. As a side product we also get a lower-bound on the approximability of the respective problems under some complexity theoretical assumptions. Since the negative results for (α, β)-Power Law Graphs imply the same non-approximability on graphs with PLB-(U,L), we only consider graphs with PLB-(U,L,N) in Section 6.

Technical Ideas

The intuition behind our positive results is simple: In a power law graph with exponent $\beta > 2$, any set of $o(n)$ vertices has a volume of at most $o(n)$. The potential volume lemma gives upper bounds on $\sum_{x \in S} h(\deg(x))$ in terms of $|S|$, where S is any set with a certain minimum volume. This is done by upper-bounding the density $\sum_{x \in S} h(\deg(x))/|S|$ by the highest possible density of a set of size $|S|$ in a PLB-U graph. The lemma does not only enable us to prove the stated intuition formally, but also allows us to give upper bounds on the approximation ratios of some greedy algorithms.

Our negative results rely on the graph embedding technique introduced by Shen et al. [33] for (α, β)-Power Law Graphs.

[1] We say that an event E holds *w. h. p.*, if there exists an $\delta > 0$ such that $\Pr[E] \geqslant 1 - \mathcal{O}(n^{-\delta})$, and *almost surely* if it holds with probability $\Pr[E] \geqslant 1 - o(1)$.

Table 1 Comparison of the approximation ratios achieved by greedy algorithms on networks with an upper bound on the power-law degree distribution (PLB-U) and $\beta > 2$, and on general graphs. While on general graphs, greedy achieves only a logarithmic or polynomial approximation, greedy achieves a constant-factor-approximation on graphs with PLB-U and $\beta > 2$.

Problem	General Graph	Graphs with PLB-U
Minimum Dominating Set	$\mathcal{O}(\ln \Delta)$ [25]	$\Theta_n(1)$ [Theorem 5.5]
Minimum Vertex Cover	$\mathcal{O}(\ln \Delta)$ [34]	$\Theta_n(1)$ [Theorem 5.8]
Maximum Independent Set	$\mathcal{O}(\Delta)$ [17]	$\Theta_n(1)$ [Theorem 5.7]

2 Related Work

MDS, MVC and MIS are well studied NP-complete problems. It is know that MDS cannot be approximated within a factor of $(1-\varepsilon)\ln|V|$ for any $\varepsilon > 0$ [19] unless $\mathsf{NP} \subseteq \mathsf{DTIME}(|V|^{\log\log|V|})$ and not to within a factor of $\ln\Delta - c\ln\ln\Delta$ for some $c > 0$ [14] unless $\mathsf{P} = \mathsf{NP}$, although a simple greedy algorithm achieves an approximation ratio of $1 + \ln\Delta$ [25]. Even for sparse graphs, MDS cannot be approximated within a factor of $o(\ln(n))$, since we could have a graph with a star of $n - \sqrt{n}$ nodes to which an arbitrary graph of the $\sqrt{n}$ remaining nodes is attached [28].

MIS cannot be approximated within a factor of Δ^{ε} for some $\varepsilon > 0$ unless $\mathsf{P} = \mathsf{NP}$ [7], although a simple greedy algorithm achieves an approximation factor of $\frac{\Delta+2}{3}$ [23]. We also know from Turán's theorem that every graph with an average degree of $\bar{d}$ has a maximum independent set of size at least $\frac{n}{\bar{d}+1}$. This lower bound can already be achieved by the same greedy algorithm [23, Theorem 1].

MVC cannot be approximated within a factor of $10\sqrt{5} - 21 \approx 1.36$ unless P=NP, whereas the simple algorithm which greedily constructs a maximal matching achieves an approximation ratio of 2 [31]. The greedy algorithm based on node degrees only achieves an approximation factor of $\ln\Delta$.

All three problems have already been studied in the context of (α, β)-Power Law Graphs. Ferrante et al. [21] showed that these problems remains NP-hard for $\beta > 0$. Shen et al. [33] proved that there is no $\left(1 + \frac{1}{3120\zeta(\beta)3^{\beta}}\right)$-approximation for MDS and no $\left(1 + \frac{1}{1120\zeta(\beta)3^{\beta}} - \varepsilon\right)$-approximation for MIS when $\beta > 1$ unless $\mathsf{P} = \mathsf{NP}$, showing that in this case the problem is APX-hard. For MVC, Schen et al. [33] proved that there is no PTAS when $\beta > 1$ under the Unique Games Conjecture. They also showed that the greedy algorithm achieves a constant approximation factor for $\beta > 2$. Gast et al. [22] also proved a logarithmic lower bound on the approximation factor when $\beta \leqslant 2$ for MDS. Hauptmann et al. [24] gave the first non-constant bound on the approximation ratio for MIS when $\beta \leqslant 1$. In contrast to (α, β)-Power Law Graphs the PLB-U property captures a wide range of real networks, making it possible to transfer our results to them.

3 Preliminaries and Notation

We generally consider undirected multigraphs $G = (V, E)$ without loops, where V denotes the set of vertices and E the multiset of edges. If we consider simple graphs, we state so specifically. Throughout the paper we use $\deg(v)$ to denote the degree of node v, d_i for the set of nodes of degree i , $d_{\geqslant i}$ for the set of nodes of degree greater than or equal to i. We will also let b_i denote the set of nodes $v \in V$ with $\deg(v) \in [2^i, 2^{i+1})$ and for $v \in V$ we let $N^+(v)$ denote the *inclusive neighborhood* of v in G. We also use d_{min} and Δ to denote the minimum

Table 2 Comparison of the approximation lower bounds for polynomial-time algorithms (assuming $\mathsf{P} \neq \mathsf{NP}$) on networks with an upper (PLB-U) and lower (PLB-L) bound on the power-law degree distribution and with PLB neighborhoods (PLB-N) with the approximation lower bounds on general graphs. Even with the additional properties of PLB-L and PLB-N the problems on graphs with PLB-U remain APX-hard, i.e. these problems cannot admit a PTAS. Better lower bounds for each problem are in respective theorem, $\Omega(1)$ hides the PLB-L parameters β, t and constant c_2.

Problem	General Graph		Graph with PLB-(U,L,N)
Minimum Dominating Set (MDS)	$\Omega(\ln \Delta)$	[14]	$1 + \Omega(1)$ [Theorem 6.9]
Minimum Vertex Cover (MVC)	$\geqslant 1.3606$	[16]	$1 + \Omega(1)$ [Theorem 6.10]
Maximum Independent Set (MIS)	$\Omega(\mathsf{poly}(\Delta))$	[7]	$1 + \Omega(1)$ [Theorem 6.11]

and maximum degree of the graph respectively. For a set of nodes $S \subseteq V$, the volume of S, denoted by $\textsc{vol}(S)$ is the sum of degrees of vertices in S, $\textsc{vol}(S) = \sum_{v \in S} \deg(v)$. We denote the optimal value of an objective function f on input x by $\textsc{opt}_f(x)$. If not stated otherwise log denotes the logarithm of base 2.

Now we give a formal definition of the PLB properties for (multi-)graphs.

▶ **Definition 3.1** (PLB-U [10]). Let G be an undirected n-vertex graph and $c_1 > 0$ be a universal constant. We say that G is power law bounded (PLB-U) for some parameters $1 < \beta = \mathcal{O}(1)$ and $t \geqslant 0$ if for every integer $d \geqslant 0$, the number of vertices v, such that $\deg(v) \in \left[2^d, 2^{d+1}\right)$ is at most

$$c_1 n (t+1)^{\beta-1} \sum_{i=2^d}^{2^{d+1}-1} (i+t)^{-\beta}.$$

▶ **Definition 3.2** (PLB-L). Let G be an undirected n-vertex graph and $c_2 > 0$ be a universal constant. We say that G is power law bounded PLB-L for some parameters $1 < \beta = \mathcal{O}(1)$ and $t \geqslant 0$ if for every integer $\lfloor \log d_{min} \rfloor \leqslant d \leqslant \lfloor \log \Delta \rfloor$, the number of vertices v, such that $\deg(v) \in \left[2^d, 2^{d+1}\right)$ is at least

$$c_2 n (t+1)^{\beta-1} \sum_{i=2^d}^{2^{d+1}-1} (i+t)^{-\beta}.$$

Since the PLB-U property alone can capture a much broader class of networks, for example empty graphs and rings, this lower-bound is important to restrict networks to those with an actual (approximate) power-law degree distribution. In the definition of PLB-L d_{min} is necessary because in real-world power law-networks the minimum degree is not always 1.

▶ **Definition 3.3** (PLB-N [10]). Let G be a PLB (multi-)graph with parameters $\beta > 2$ and $t \geqslant 0$, and let $c_2 > 0$ be a universal constant. We say that G has PLB neighborhoods (PLB-N) if for every vertex v of degree k, the number of neighbors of v of degree at least k is at most $c_3 \max\left(\log n, (t+1)^{\beta-2} k \sum_{i=k}^{n-1} i(i+t)^{-\beta}\right)$.

Note that throughout the paper we assume the parameters, c_i, β, and t, of the above definitions to be constants.

▶ **Definition 3.4** (Graphical degree sequence). A graphical sequence is a sequence of numbers which can be the degree sequence of some graph.

4 Power-Law Random Graphs and the PLB properties

In this section we consider some well known power law random graph models and prove that w. h. p. or *almost surely* graphs generated by these models have PLB-U and PLB-N properties. We chose Chung-Lu Random Graphs, Geometric Inhomogeneous Random Graphs, and Hyperbolic Random Graphs, because they are common models and rather easy to analyze. Furthermore, they assume independence or some geometrically implied sparseness of edges, which is important for establishing the PLB-N property.

4.1 (α, β)-Power Law Graphs

▶ **Definition 4.1** ((α, β)-Power Law Graph [3]). An (α, β)-Power Law Graph is an undirected multigraph with the following degree distribution depending on two given values α and β. For $1 \leqslant i \leqslant \Delta = \lfloor e^{\alpha/\beta} \rfloor$ there are $y_i = \lfloor \frac{e^\alpha}{i^\beta} \rfloor$ nodes of degree i.

▶ **Theorem 4.2.** *The (α, β)-Power Law Graph with $\beta > 1$ has the PLB-U property with $c_1 = \frac{1}{\zeta(\beta)}$, $t = 0$, and exponent β and the PLB-L property with $c_2 = \frac{1}{2\zeta(\beta)}$, $t = 0$, and exponent β.*

Proof. The number of nodes of degree i is exactly $\lfloor \frac{e^\alpha}{i^\beta} \rfloor$. It holds that the number of nodes of degree between 2^d and $2^{d+1} - 1$ is at most

$$e^\alpha \sum_{i=2^d}^{2^{d+1}-1} i^{-\beta} = \frac{n}{\zeta(\beta)} \sum_{i=2^d}^{2^{d+1}-1} i^{-\beta}$$

due to the definition of the degree distribution and the fact that $n = \zeta(\beta)e^\alpha$ for $\beta > 1$. Furthermore, since $i \leqslant \lfloor e^{\alpha/\beta} \rfloor$, $\lfloor \frac{e^\alpha}{i^\beta} \rfloor$ is at least one. Therefore $\lfloor \frac{e^\alpha}{i^\beta} \rfloor \geqslant \frac{1}{2} \frac{e^\alpha}{i^\beta}$. It now holds that the number of nodes of degree between 2^d and $2^{d+1} - 1$ is at least

$$\frac{e^\alpha}{2} \sum_{i=2^d}^{2^{d+1}-1} i^{-\beta} = \frac{n}{2\zeta(\beta)} \sum_{i=2^d}^{2^{d+1}-1} i^{-\beta}.$$ ◀

▶ **Corollary 4.3.** *A random (α, β)-Power Law Graph with $\beta > 1$ created with the Erased Configuration Model has the PLB-U and PLB-N properties with high probability.*

4.2 Geometric Inhomogeneous Random Graphs

▶ **Definition 4.4** (Geometric Inhomogeneous Random Graphs (GIRGs) [11]). For $n \in \mathbb{N}$ let $w = (w_1, \cdots, w_n)$ be a sequence of positive weights. Let $W = \sum_{i=1}^n w_i$ be the total weight. For any vertex v, draw a point $x_v \in \mathbb{T}^d$ uniformly and independently at random. We connect vertices $u \neq v$ independently with probability $p_{uv} = p_{uv}(r)$, which now depends not only on the weights w_u, w_v but also on the positions x_u, x_v, more precisely, on the distance $r = \|x_u - x_v\|$. We require for some constant $\alpha > 1$ the following edge probability condition

$$p_{uv} = \Theta\Big(min\Big\{ \frac{1}{||x_u - x_v||^{\alpha d}} \Big(\frac{w_u w_v}{W}\Big)^\alpha, 1 \Big\}\Big).$$

▶ **Definition 4.5** (General Power-law [11]). A weight sequence $\vec{w}$ is said to follow a general power-law with exponent $\beta > 2$ if $w_{\min} := \min\{w_v \mid v \in V\} = \Omega(1)$ and if there is a $\bar{w} = \bar{w}(n) \geqslant n^{\omega(1/\log\log n)}$ such that for all constants $\eta > 0$ there are $c_1, c_2 > 0$ with

$$c_1 \frac{n}{w^{\beta-1+\eta}} \leqslant |\{v \in V \mid w_v \geqslant w\}| \leqslant c_2 \frac{n}{w^{\beta-1-\eta}},$$

where the first inequality holds for all $w_{\min} \leqslant w \leqslant \bar{w}$ and the second holds for all $w \geqslant w_{\min}$.

To prove that GIRGs fulfill PLB-U and PLB-N we need the following theorem and some auxiliary lemmas by Bringmann et al. [12]. For the sake of brevity these lemmas as well as the remaining proofs of this section can be found in the full version of the paper [13].

▶ **Theorem 4.6** ([12]). *Let G be a GIRG with a weight sequence that follows a general power-law with exponent β and average degree $\Theta(1)$. Then, with high probability the degree sequence of G follows a power law with exponent β and average degree $\Theta(1)$, i.e there exist constants $c_3, c_4 > 0$ such that w. h. p.*

$$c_3 \frac{n}{k^{\beta-1+\eta}} \leqslant |\{v \in V | deg(v) \geqslant k\}| \leqslant c_4 \frac{n}{k^{\beta-1-\eta}},$$

where the first inequality holds for all $1 \leqslant d \leqslant \bar{w}$ and the second holds for all $d \geqslant 1$.

▶ **Theorem 4.7.** *Let G be a GIRG whose weight sequence $\vec{w}$ follows a general power-law with exponent $\beta' > 2$ and an η with $\beta' - \eta > 2$. Then, w. h. p. G fulfills PLB-U and PLB-N with $\beta = \beta' - \eta$, $t = 0$ and some constants c_1 and c_2.*

4.3 Hyperbolic Random Graphs

▶ **Definition 4.8.** (Hyperbolic Random Graph [26]) Let $\alpha_H > 0$, $C_H \in \mathbb{R}$, $T_H > 0$, $n \in \mathbb{N}$ and $R = 2 \log n + C_H$. Then the Hyperbolic Random Graph $G_{\alpha_H, C_H, T_H}(n)$ is a graph with vertex set V=[n] and the following properties:

- Every vertex $v \in [n]$ draws coordinates (r_v, ϕ_v) independently at random, where the angle π_v is chosen uniformly at random in $[0, 2\pi)$ and the radius $r_v \in [0, R]$ is random according to density $f(r) = \frac{\alpha_H \sinh(\alpha_H r)}{\cosh(\alpha_H R) - 1}$.
- Every potential edge $e = \{u, v\} \in \binom{[n]}{2}$ is present independently with probability

$$p_H(d(u,v)) = \left(1 + e^{\frac{d(u,v)-R}{2T_H}}\right)^{-1}.$$

▶ **Lemma 4.9** ([11]). *Hyperbolic random graphs are a special case of GIRGs.*

This lemma directly leads to the following consequence.

▶ **Theorem 4.10.** *Let G be a hyperbolic random graph with $\alpha_H > \frac{1}{2}$. Then, almost surely G fulfills PLB-U and PLB-N with $\beta = 2\alpha_H + 1 - \eta$, $t = 0$, constant $\eta > 0$ and some constants c_1 and c_2.*

4.4 Chung-Lu Random Graphs

Chung-Lu Random Graphs [15] assume a sequence of expected degrees $w_1, w_2, \ldots, w_n$ and each edge (i, j) exists independently at random with probability $\min(1, \frac{w_i \cdot w_j}{W})$, where $W = \sum_{i=1}^{n} w_i$. Using exactly the same techniques as for Theorem 4.6 we can prove the theorem below.

▶ **Theorem 4.11.** *Let G be a Chung-Lu random graph whose weight sequence $\vec{w}$ follows a general power-law with exponent $\beta' > 2$ and an η with $\beta' - \eta > 2$. Then w. h. p. G fulfills PLB-U and PLB-N with $\beta = \beta' - \eta$, $t = 0$ and some constants c_1 and c_2.*

5 Greedy Algorithms

In this section we try to understand why simple greedy algorithms work efficiently in practice.

▶ **Definition 5.1.** An algorithm is an *α-approximation* for problem P if it produces a solution set S with $\alpha \geqslant \frac{|S|}{|\text{OPT}|}$ if P is a minimization problem and with $\alpha \geqslant \frac{|\text{OPT}|}{|S|}$ if P is a maximization problem.

Greedy Algorithm on PLB-U Networks

In this section we state our main lemma, Lemma 5.2, and use it to derive bounds on the solution size and approximation ratio of covering problems.

▶ **Lemma 5.2** (Potential Volume Lemma). *Let G be a (multi-)graph with the PLB-U property for some $\beta > 2$, some constant $c_1 > 0$ and some constant $t \geqslant 0$. Let S be a solution set for which we can define a function $g\colon \mathsf{R}^+ \to \mathsf{R}$ continuously differentiable and $h(x) := g(x) + C$ for some constant C such that*

1. *g non-decreasing,*
2. *$g(2x) \leqslant c \cdot g(x)$ for all $x \geqslant 2$ and some constant $c > 0$,*
3. *$g'(x) \leqslant \frac{g(x)}{x}$,*

then it holds that $\sum_{x \in S} h(\deg(x))$ is at most

$$\left(c \left(1 + \frac{\beta - 1}{\beta - 2} \frac{1}{1 - \left(\frac{t+2}{t+1}\right)^{1-\beta}} \right) g\left(\left(c_1 \tfrac{\beta-1}{\beta-2} \tfrac{n}{M} \cdot 2^{\beta-1} \cdot (t+1)^{\beta-1} \right)^{\frac{1}{\beta-2}} \right) + C \right) \cdot |S|,$$

where $M(n) \geqslant 1$ is chosen such that $\sum_{x \in S} \deg(x) \geqslant M$.

For the proof one can refer to the full version of the paper [13].

All our bounds are in terms of the following two constants, which stem from the Potential Volume Lemma and which we will define for the sake of brevity:

$$a_{\beta,t} := \left(1 + \frac{\beta - 1}{\beta - 2} \frac{1}{1 - \left(\frac{t+2}{t+1}\right)^{1-\beta}} \right) \text{ and } b_{c_1,\beta,t} := \left(c_1 \tfrac{\beta-1}{\beta-2} \cdot 2^{\beta} \cdot (t+1)^{\beta-1} \right)^{\frac{1}{\beta-2}}.$$

5.1 Minimum Dominating Set

The idea for lower-bounding the size of a dominating set is essentially the same as the one by Shen et al. [33] and by Gast et al. [22] in the context of (α, β)-Power-Law Graphs: Every set of $o(n)$ nodes in a power-law graph can dominate only $o(n)$ many nodes. For graphs with PLB-U this is implied by our Potential Volume Lemma. Finally, we will show that

▶ **Theorem 5.3.** *For a multigraph without loops and isolated vertices and with the PLB-U property with parameters $\beta > 2$, $c_1 > 0$ and $t \geqslant 0$, the minimum dominating set is of size at least*

$$(2 \cdot a_{\beta,t} \cdot b_{c_1,\beta,t} + 1)^{-1} n = \Theta(n).$$

5.1.1 The Greedy Algorithm

Theorem 5.3 implies that taking all nodes already gives a constant approximation factor, but now we want to show that using the classical greedy algorithm actually guarantees an even better approximation factor.

The proof of the following theorem is an adaptation of the proof for the greedy Set Cover algorithm to the case of unweighted Dominating Set.

▶ **Theorem 5.4** ([25]). *Let S the solution of the greedy algorithm and* OPT *an optimal solution for* Dominating Set. *Then it holds that*

$$|C| \leqslant \sum_{x \in \text{OPT}} H_{\deg(x)+1},$$

where H_k is the k-th harmonic number.

The interested reader can find the proof of the above theorem in the full version [13] of the paper. By using the inequality from Theorem 5.4 together with the Potential Volume Lemma 5.2, we can derive the following approximation factor for the greedy algorithm.

▶ **Theorem 5.5.** *For a multigraph without loops and isolated vertices and with the PLB-U property with parameters $\beta > 2$, $c_1 > 0$ and $t \geqslant 0$, the classical greedy algorithm for* Minimum Dominating Set *(cf. [17]) has an approximation factor of at most*

$$\log_3(5) \cdot a_{\beta,t} \cdot \ln\left(b_{c_1,\beta,t} + 1\right) + 1 = \Theta(1).$$

Proof. From the analysis of the greedy algorithm we know that for its solution C and an optimal solution OPT it holds that

$$|C| \leqslant \sum_{x \in \text{OPT}} H_{\deg(x)+1} \leqslant \sum_{x \in \text{OPT}} \ln(\deg(x) + 1) + 1,$$

where H_k denotes the k-th harmonic number. We can now choose $h(x) = g(x) + 1$ with $g(x) = \ln(x + 1)$. $g(x)$ satisfies (i), (ii) with $c = \log_3(5)$ and (iii). As we assume there to be no nodes of degree 0, it holds that

$$\sum_{x \in \text{OPT}} \deg(x) \geqslant \frac{n}{2} =: M,$$

since all nodes have to be covered. We can now use Lemma 5.2 with $S = \text{OPT}$ to derive that

$$|C| \leqslant \left(\log_3(5) \cdot a_{\beta,t} \cdot \ln\left(b_{c_1,\beta,t} + 1\right) + 1\right) |\text{OPT}|.$$ ◀

Note that in PLB networks the maximum degree can be $\Delta = \Theta(n^{\frac{1}{\beta-1}})$. That means the simple bound for the greedy algorithm gives us only an approximation ratio of $\ln(\Delta + 1) = \Theta(\log n)$.

5.2 Maximum Independent Set

For networks with the PLB-L property only, we can already derive the following lower bound on the size of an optimal solution.

▶ **Lemma 5.6.** *A graph with the PLB-L property with parameters $\beta > 2$, $c_2 > 0$ and $t \geqslant 0$, has an independent set of size at least $\frac{c_2(t+1)^{\beta-1}}{(t+d_{min})^{\beta}(d_{min}+1)} \cdot n$ or of size at least $\frac{c_2}{(t+1)} \cdot n$ if we assume G to be connected and $d_{min} = 1$.*

We can even go a step further and show that *all* maximal independent sets have to be quite big, even if we only have the PLB-U property. Since the PLB-U property with $\beta > 2$ induces a constant average degree, this already gives us a constant approximation factor for Maximum Independent Set on networks with this property due to Turán's theorem. Although we can not give better bounds for the maximum independent set, Theorem 5.3 immediately implies a lower bound for the size of *all maximal independent sets*:

▶ **Theorem 5.7.** *In a multigraph without loops and isolated vertices and with the PLB-U property with parameters $\beta > 2$, $c_1 > 0$ and $t \geqslant 0$, every maximal independent set is of size at least*

$$(2 \cdot a_{\beta,t} \cdot b_{c_1,\beta,t} + 1)^{-1} n = \Theta(n).$$

Especially, it holds that computing any maximal independent set gives an approximation factor of at most $2 \cdot a_{\beta,t} \cdot b_{c_1,\beta,t} + 1$. The above theorem holds since every maximal independent set is also a dominating set. It is easy to see that these lower bounds do not hold in sparse graphs in general, since in a star the center node also constitutes a maximal independent set.

5.3 Vertex Cover

From the results we know about Dominating Set, we can also derive some results about Vertex Cover in graphs without isolated vertices.

▶ **Theorem 5.8.** *In a multigraph without loops and isolated vertices and with the PLB-U property with parameters $\beta > 2$, $c_1 > 0$ and $t \geqslant 0$, the minimum vertex cover is of size at least*

$$(2 \cdot a_{\beta,t} \cdot b_{c_1,\beta,t} + 1)^{-1} n.$$

The above theorem follows because every vertex cover in a graph without isolated vertices is also a dominating set. Again, the theorem immediately implies an approximation factor of at most $2 \cdot a_{\beta,t} \cdot b_{c_1,\beta,t} + 1$.

6 Approximation Hardness for Simple Graphs

To show the actual non-approximability and APX hardness, we use the embedding framework by Shen et al. [33].

▶ **Definition 6.1** (Embedded-Approximation-Preserving Reduction [33])**.** Given an optimal substructure problem O, a reduction from an instance on graph $G = (V, E)$ to another instance on a (power law) graph $G' = (V', E')$ is called *embedded approximation-preserving* if it satisfies the following properties:

1. G is a subset of maximal connected components of G';
2. The optimal solution of O on G', $\text{OPT}(G')$, is upper bounded by $C \cdot \text{OPT}(G)$ where C is a constant correspondent to the growth of the optimal solution.

▶ **Theorem 6.2** ([6])**.** Minimum vertex cover, Maximum independent set *and* Minimum dominating set *are* APX*-complete for cubic simple graphs.*

Having shown an embedded-approximation-preserving reduction, we can use the following lemma to show hardness of approximation.

▶ **Lemma 6.3** ([33]). *Given an optimal substructure problem O, if there exists an embedded-approximation-preserving reduction from a graph G to another graph G' and if O is ε-inapproximable on G, then O is δ-inapproximable on G', where δ is lower bounded by $\frac{\varepsilon C}{(C-1)\varepsilon+1}$ if O is a maximization problem and by $\frac{\varepsilon+C-1}{C}$ if O is a minimization problem.*

▶ **Theorem 6.4** ([6, 14]). *In 3-bounded simple graphs it is* NP*-hard to approximate MDS within a factor of $\frac{391}{390}$.*

▶ **Theorem 6.5** ([6, 9]). *In 3-bounded simple graphs it is* NP*-hard to approximate MIS within a factor of $\frac{140}{139} - \gamma$ for any $\gamma > 0$.*

▶ **Theorem 6.6** ([16, 20]). *In regular simple graphs MVC is hard to approximate within a factor of $10\sqrt{5} - 21 \approx 1.3606$ unless* P = NP.

We will use this framework as follows: First, we show how to embed cubic graphs into simple graphs with PLB-U, PLB-L and PLB-N. Then, we derive the value of C as in Definition 6.1 for each problem we consider. Last, we use Lemma 6.3 together with the known inapproximability results for the considered problems on cubic graphs to derive the approximation hardness on graphs with PLB-U, PLB-L and PLB-N.

We start by showing the embedding of cubic simple graphs into simple graphs with PLB-U, PLB-L and PLB-N. To this end we use stars as the gadgets for our embeddings. The following is a simple observation and is therefore stated without a formal proof.

▶ **Lemma 6.7.** *A star of size n has a minimum dominating set and a minimum vertex cover of size 1 and maximum independent set of size $n-1$. Also, these can be computed in polynomial time.*

▶ **Lemma 6.8.** *Any cubic simple graph G can be embedded into a simple graph G_{PLB} having the PLB-U, PLB-L and PLB-N properties for any $\beta > 2$ and any $t \geqslant 0$.*

Proof. Suppose we are given β and t. Again, we want to determine c_1 and c_2 of PLB-U and PLB-L respectively. Let n be the number of nodes in graph G and let $N = cn$ be the number of nodes in G_{PLB} for some constant c to be determined. Like in Lemma 6.3 we have to ensure a number of conditions to get a graphical degree sequence. To hide a cubic graph in the respective bucket of G_{PLB}, we need

$$c_1 N(t+1)^{\beta-1} \sum_{i=2}^{3} (i+t)^{-\beta} = c_1 N(t+1)^{\beta-1} \left(\frac{1}{(2+t)^\beta} + \frac{1}{(3+t)^\beta} \right) \geqslant n.$$

As we will see, we can choose the constant c_1 arbitrarily large, so the former condition is no real restriction. Then we choose the maximum degree Δ such that

$$d_{max(G_{PLB})} = (cn)^{\frac{1}{\beta-1}}.$$

In our embedding we just fill each bucket $i \geqslant 2$ with the number of stars of size $2^i + 1$ it needs to reach its lower bound. Bucket 1 can get up to n nodes, since we hide the graph G in it and bucket 0 gets all the degree-one nodes of our star gadgets. By filling a bucket (other than buckets 0 and 1) we might deviate by at most one from the lower bound of that bucket. Then, we add additional stars within the bounds of our buckets until we have exactly N nodes. If we only need one more node, we just add it and connect it to an arbitrary star. This does not change the properties of the star or the degree of its center enough to make it change its bucket.

In order for this to be possible we need to ensure that after filling all buckets to their lower bound, there is still some slack until we reach N. This is the case if the following inequality holds true

$$\begin{aligned}
&n + \sum_{i=0}^{\lfloor \log \Delta \rfloor} \left((2^i + 1) \left(1 + c_2 N (t+1)^{\beta-1} \sum_{j=2^i}^{2^{i+1}-1} (j+t)^{-\beta} \right) \right) \\
&\leqslant \frac{N}{c} + \log N^{\frac{1}{\beta-1}} + \frac{c_2}{(t+1)} N + \frac{c_2}{\beta-1} N + 2\Delta + c_2 N + \frac{c_2}{\beta-2} N(t+1) \\
&\leqslant N \left(\frac{1}{c} + \eta + \frac{c_2}{t+1} + \frac{c_2}{\beta-1} + \eta + c_2 + \frac{c_2}{\beta-2}(t+1) \right) \\
&\leqslant N,
\end{aligned}$$

where in the second line we used the inequalities

$$\begin{aligned}
\sum_{i=0}^{\lfloor \log \Delta \rfloor} \left(1 + c_2 N (t+1)^{\beta-1} \sum_{j=2^i}^{2^{i+1}-1} (j+t)^{-\beta} \right) &\leqslant \log N^{\frac{1}{\beta-1}} + \frac{c_2}{(t+1)} N + \frac{c_2}{\beta-1} N, \\
\sum_{i=0}^{\lfloor \log \Delta \rfloor} \left(2^i \left(1 + c_2 N (t+1)^{\beta-1} \sum_{j=2^i}^{2^{i+1}-1} (j+t)^{-\beta} \right) \right) &\leqslant 2\Delta + c_2 N + \frac{c_2}{\beta-2} N(t+1)
\end{aligned}$$

and choose a constant $\eta > 0$ arbitrarily small.

From this last condition we can derive

$$c \geqslant 1 + \frac{\eta' + c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)}{1 - \eta' - c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)},$$

since η and therefore η' can be arbitrarily small. We choose $\eta' = c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)$ to get

$$c = 1 + \frac{2c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)}{1 - 2c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)} = \frac{1}{1 - 2c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)}.$$

The use of star gadgets means we also have to guarantee that c_1 is big enough for all degree-one nodes to fit into bucket 0. Since c_1 can be arbitrarily large, this is no problem.

Now we can essentially choose c_1 arbitrarily large and c_2 arbitrarily small, guaranteeing $c > 1$ and a large enough gap to have a graphical degree sequence. At the same time our choice of c guarantees that we can fill the graph with exactly N nodes. Furthermore, since every node has a constant number of neighbors of equal or higher degree, G_{PLB} also fulfills PLB-N, which always allows us at least $c_3 \log N$ many neighbors. ◀

6.1 Dominating Set

▶ **Theorem 6.9.** *For every $\beta > 2$ and every $t \geqslant 0$* Minimum Dominating Set *cannot be approximated to within a factor of* $1 + \left(130 \cdot \left(4 \frac{1 - \frac{c_2}{t+1}}{1 - 2c_2 \left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1 \right)} + 1 \right) \right)^{-1}$ *on simple graphs with PLB-U, PLB-L and PLB-N unless* $\mathsf{P} = \mathsf{NP}$.

Proof. Lemma 6.8 gives us an L-reduction from a cubic graph G to a simple graph G_{PLB} with the PLB-U, PLB-L and PLB-N properties. The L-reduction from a cubic graph G to a simple graph G_{PLB} together with Theorem 6.2 implies that MDS is APX hard for simple graphs with PLB-(U,L,N). Let $\text{OPT}(G)$ and $\text{OPT}(G_{PLB})$ denote the size of a minimum dominating set for G and G_{PLB} respectively. Let b_i be the set of nodes in PLB bucket i, i.e. the set of nodes $v \in V$ with $\deg(v) \in \left[2^i, 2^{i+1}-1\right]$. We know that $\text{OPT}(G) \geqslant \frac{n}{4}$ and from Lemma 6.7 we can derive $\text{OPT}\left(G_{PLB} \backslash G\right) = N - n - |b_0|$. It now holds that

$$\begin{aligned}
\text{OPT}(G_{PLB}) &= \text{OPT}(G) + \text{OPT}(G_{PLB} \backslash G) \\
&= \text{OPT}(G) + N - n - |b_0| \\
&\leqslant \text{OPT}(G) + N - n - \frac{c_2}{t+1} N \\
&= \text{OPT}(G) + \left(c - 1 - c\frac{c_2}{t+1}\right) n \\
&\leqslant \text{OPT}(G) + \left(c - 1 - c\frac{c_2}{t+1}\right) 4\text{OPT}(G) \\
&= \left(4c\left(1 - \frac{c_2}{t+1}\right) - 3\right) \text{OPT}(G).
\end{aligned}$$

In the context of Definition 6.1 and Lemma 6.3 this means $C = 4c\left(1 - \frac{c_2}{t+1}\right) - 3$. Due to Theorem 6.4 it also holds that $\varepsilon = \frac{391}{390}$ in the context of Lemma 6.3. This gives us an approximation hardness of

$$\begin{aligned}
1 + \frac{\varepsilon - 1}{C} &= 1 + \frac{3}{390 \cdot (4c\left(1 - \frac{c_2}{t+1}\right) - 3)} \\
&= 1 + \frac{3}{390 \cdot \left(4\frac{1 - \frac{c_2}{t+1}}{1 - 2c_2\left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1\right)} + 1\right)} \\
&= 1 + \left(130 \cdot \left(4\frac{1 - \frac{c_2}{t+1}}{1 - 2c_2\left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1\right)} + 1\right)\right)^{-1}
\end{aligned}$$

due to our choice of c in Lemma 6.8. ◀

By using similar arguments as for Theorem 6.9 we can prove Theorem 6.10 and Theorem 6.11.

▶ **Theorem 6.10.** *For every $\beta > 2$ and every $t \geqslant 0$* Minimum Vertex Cover *cannot be approximated to within a factor of* $1 + \frac{\left(1 - 2c_2\left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1\right)\right)\left(10\sqrt{5} - 22\right)}{2c_2\left(\frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1\right) + 1}$ *on simple graphs with PLB-U, PLB-L and PLB-N unless* $\mathsf{P} = \mathsf{NP}$.

▶ **Theorem 6.11.** *For every $\beta > 2$ and every $t \geqslant 0$* Maximum Independent Set *cannot be approximated to within a factor of* $1 + \frac{\left(\frac{1}{139} - \gamma\right)\left((t+1)\left(1 - 2c_2\left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1\right)\right)\right)}{4c_1\left(\frac{140}{139} - \gamma\right) + (t+1)\left(1 - 2c_2\left(\frac{1}{t+1} + \frac{1}{\beta-1} + \frac{t+1}{\beta-2} + 1\right)\right)}$ *for any $\gamma > 0$ on simple graphs with PLB-U, PLB-L and PLB-N unless* $\mathsf{P} = \mathsf{NP}$.

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