Scaling and Proximity Properties of Integrally Convex Functions*

Satoko Moriguchi[1], **Kazuo Murota**[2], **Akihisa Tamura**[3], and **Fabio Tardella**[4]

1 Department of Business Administration, Tokyo Metropolitan University, Hachioji, Japan
satoko5@tmu.ac.jp
2 Department of Business Administration, Tokyo Metropolitan University, Hachioji, Japan
murota@tmu.ac.jp
3 Department of Mathematics, Keio University, Yokohama, Japan
aki-tamura@math.keio.ac.jp
4 Department of Methods and Models for Economics, Territory and Finance, Sapienza University of Rome, Roma, Italy
fabio.tardella@uniroma1.it

Abstract

In discrete convex analysis, the scaling and proximity properties for the class of L$^\natural$-convex functions were established more than a decade ago and have been used to design efficient minimization algorithms. For the larger class of integrally convex functions of n variables, we show here that the scaling property only holds when $n \leq 2$, while a proximity theorem can be established for any n, but only with an exponential bound. This is, however, sufficient to extend the classical logarithmic complexity result for minimizing a discretely convex function in one dimension to the case of integrally convex functions in two dimensions. Furthermore, we identified a new class of discrete convex functions, called directed integrally convex functions, which is strictly between the classes of L$^\natural$-convex and integrally convex functions but enjoys the same scaling and proximity properties that hold for L$^\natural$-convex functions.

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1 Introduction

The proximity-scaling approach is a fundamental technique in designing efficient algorithms for discrete or combinatorial optimization. For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ in integer variables and a positive integer α, called scaling unit, the α-scaling of f is the function f^α defined by $f^\alpha(x) = f(\alpha x)$ $(x \in \mathbb{Z}^n)$. A proximity theorem is a result guaranteeing that a (local) minimum of the scaled function f^α is close to a minimizer of the original function f. The scaled function f^α is simpler in shape, and hence easier to minimize, whereas

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the quality of the obtained minimizer of f^α as an approximation to the minimizer of f is guaranteed by a proximity theorem. The proximity-scaling approach consists in applying this idea for a decreasing sequence of α, often by halving the scale unit α. A generic form of a proximity-scaling algorithm may be described as follows, where K_∞ denotes the ℓ_∞-size of the effective domain of f, and $B(n,\alpha)$ denotes the proximity bound in ℓ_∞-distance.

S0: Find an initial vector x with $f(x) < +\infty$, and set $\alpha := 2^{\lceil \log_2 K_\infty \rceil}$.
S1: Find a vector y with $\|\alpha y\|_\infty \leq B(n,\alpha)$ that is a (local) minimizer of $\tilde{f}(y) = f(x+\alpha y)$, and set $x := x + \alpha y$.
S2: If $\alpha = 1$, then stop (x is a minimizer of f).
S3: Set $\alpha := \alpha/2$, and go to S1.

The algorithm consists of $O(\log K_\infty)$ scaling phases. This approach has been particularly successful for resource allocation problems [6, 7, 8, 13] and for convex network flow problems (under the name of "capacity scaling") [1, 11, 12]. Different types of proximity theorems have also been investigated: proximity between integral and real optimal solutions, among others.

In discrete convex analysis [15], a variety of discrete convex functions are considered. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *integrally convex* if its local convex extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is (globally) convex in the ordinary sense, where $\tilde{f}$ is defined as the collection of convex extensions of f in each unit hypercube $[\boldsymbol{a}, \boldsymbol{a}+\mathbf{1}]_{\mathbb{R}}$ with $\boldsymbol{a} \in \mathbb{Z}^n$; see Section 2 for precise statements.

For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, $\mathrm{dom}\, f = \{x \in \mathbb{Z}^n \mid f(x) < +\infty\}$ is called the effective domain of f. Discrete midpoint convexity of f for $x, y \in \mathbb{Z}^n$ means

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x+y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right), \tag{1.1}$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the integer vectors obtained by componentwise rounding-up and rounding-down to the nearest integers, respectively. For $x, y \in \mathbb{Z}^n$, $x \vee y$ and $x \wedge y$ denote the vectors of componentwise maximum and minimum of x and y, respectively.

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *L$^\natural$-convex* if it satisfies one of the equivalent conditions (a) to (d) below:

(a) f is integrally convex and submodular: $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ $(x, y \in \mathbb{Z}^n)$.
(b) f satisfies discrete midpoint convexity (1.1) for all $x, y \in \mathbb{Z}^n$.
(c) f satisfies discrete midpoint convexity (1.1) for all $x, y \in \mathbb{Z}^n$ with $\|x - y\|_\infty \leq 2$, and the effective domain has the property: $x, y \in \mathrm{dom}\, f \Rightarrow \lceil (x+y)/2 \rceil, \lfloor (x+y)/2 \rfloor \in \mathrm{dom}\, f$.
(d) f satisfies translation-submodularity: $f(x) + f(y) \geq f((x - \mu\mathbf{1}) \vee y) + f(x \wedge (y + \mu\mathbf{1}))$ $(\mu \in \mathbb{Z}_+,\ x, y \in \mathbb{Z}^n)$, where $\mathbf{1} = (1, 1, \ldots, 1)$.

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called *M$^\natural$-convex* if it has the exchange property: For any $x, y \in \mathrm{dom}\, f$ and any $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $f(x) + f(y) \geq f(x - \mathbf{1}_i + \mathbf{1}_j) + f(y + \mathbf{1}_i - \mathbf{1}_j)$, where $\mathrm{supp}^+(z) = \{i \mid z_i > 0\}$ and $\mathrm{supp}^-(z) = \{j \mid z_j < 0\}$ for $z \in \mathbb{Z}^n$, $\mathbf{1}_i$ denotes the i-th unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ if $1 \leq i \leq n$, and $\mathbf{1}_i = \mathbf{0}$ if $i = 0$.

Integrally convex functions constitute a common framework for discrete convex functions, including separable convex, L$^\natural$-convex and M$^\natural$-convex functions as well as L$^\natural_2$-convex and M$^\natural_2$-convex functions [15], and BS-convex and UJ-convex functions [3]. The concept of integral convexity is used in formulating discrete fixed point theorems [9, 19], and designing solution algorithms for discrete systems of nonlinear equations [17, 18]. In game theory the integral concavity of payoff functions guarantees the existence of a pure strategy equilibrium in finite symmetric games [10].

The scaling operation preserves L$^\natural$-convexity, that is, if f is L$^\natural$-convex, then f^α is L$^\natural$-convex. M$^\natural$-convexity is subtle in this respect: for an M$^\natural$-convex function f, f^α remains M$^\natural$-convex if $n \leq 2$, while this is not always the case if $n \geq 3$. However, nothing is known about scaling of integrally convex functions.

As for proximity theorems, the following facts are known for L$^\natural$-convex and M$^\natural$-convex functions.

▶ **Theorem 1.1** ([12, 14, 15]). *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, $\alpha \in \mathbb{Z}_{++}$ (positive integer), and $x^\alpha \in \mathrm{dom}\, f$.*
(1) *Suppose that f is an L$^\natural$-convex function. If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{0,1\}^n \cup \{0,-1\}^n$, then there exists a minimizer x^* of f with $\|x^\alpha - x^*\|_\infty \leq n(\alpha - 1)$.*
(2) *Suppose that f is an M$^\natural$-convex function. If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{\mathbf{1}_i, -\mathbf{1}_i \ (1 \leq i \leq n),\ \mathbf{1}_i - \mathbf{1}_j \ (i \neq j)\}$, then there exists a minimizer x^* of f with $\|x^\alpha - x^*\|_\infty \leq n(\alpha - 1)$.*

Based on the above results, efficient algorithms for minimizing L$^\natural$-convex and M$^\natural$-convex functions have been successfully designed with the proximity-scaling approach (see [15]). Proximity theorems are also available for L$_2^\natural$-convex and M$_2^\natural$-convex functions [16] and L-convex functions on graphs [5]. However, no proximity theorem is proved for integrally convex functions.

The following are the new findings of this paper about integrally convex functions:

- A "box-barrier property" (Theorem 2.3), which allows us to restrict the search for a global minimum.
- Integral convexity is preserved under scaling if $n = 2$ (Theorem 3.1), but not when $n \geq 3$ (Example 3.3).
- A proximity theorem with an exponential bound $[(n+1)!/2^{n-1}](\alpha - 1)$ holds for all n (Theorem 4.3), but does not hold with the smaller bound $n(\alpha - 1)$ when $n \geq 3$ (Examples 4.1, 4.2).

Thus, to extend the known proximity and scaling results for L$^\natural$-convex functions to a wider class of functions, a novel concept of "directed integrally convex functions" is defined. For this new class of functions the following properties hold:

- The new class coincides with the class of integrally convex functions for $n \leq 2$, and is a proper subclass of this for $n \geq 3$ (Proposition 5.1 (1)).
- The new class is a proper superclass of L$^\natural$-convex functions for all $n \geq 2$ (Proposition 5.1 (2)).
- Directed integral convexity is preserved under scaling for all n (Theorem 5.6).
- A proximity theorem with bound $n(\alpha - 1)$ holds for all n (Theorem 5.7).

As a consequence of our proximity and scaling results, we derive that:

- When n is fixed, a (directed) integrally convex function can be minimized in $O(\log K_\infty)$ time by standard proximity-scaling algorithms, where K_∞ denotes the ℓ_∞-size of $\mathrm{dom}\, f$.

2 Integrally Convex Functions

For $x \in \mathbb{R}^n$ the integer neighborhood of x is defined as $N(x) = \{z \in \mathbb{Z}^n \mid |x_i - z_i| < 1 \ (i = 1, \ldots, n)\}$. For a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ the local convex extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of f is defined as the union of all convex envelopes of f on $N(x)$ as follows:

$$\tilde{f}(x) = \min\{ \sum_{y \in N(x)} \lambda_y f(y) \mid \sum_{y \in N(x)} \lambda_y y = x, \sum_{y \in N(x)} \lambda_y = 1, \lambda_y \geq 0 \ (\forall y \in N(x))\} \quad (x \in \mathbb{R}^n). \tag{2.1}$$

If $\tilde{f}$ is convex on $\mathbb{R}^n$, then f is said to be integrally convex. A set $S \subseteq \mathbb{Z}^n$ is said to be integrally convex if, for any $x \in \mathbb{R}^n$, $x \in \overline{S}$ implies $x \in \overline{S \cap N(x)}$, i.e., if the convex hull of S coincides with the union of the convex hulls of $S \cap N(x)$ for $x \in \mathbb{R}^n$.

Integral convexity can be characterized by a local condition. The following theorem is proved in [2] when the effective domain is an integer interval (discrete rectangle).

▶ **Theorem 2.1** ([2, Proposition 3.3]). *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a function with an integrally convex effective domain. Then the following properties,* (a) *and* (b)*, are equivalent:* (a) *f is integrally convex.* (b) *For every $x, y \in \mathrm{dom}\, f$ with $\|x - y\|_\infty = 2$ we have*

$$\tilde{f}\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}. \tag{2.2}$$

▶ **Theorem 2.2** ([2, Proposition 3.1]; see also [15, Theorem 3.21]). *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an integrally convex function and $x^* \in \mathrm{dom}\, f$. Then x^* is a minimizer of f if and only if $f(x^*) \leq f(x^* + d)$ for all $d \in \{-1, 0, +1\}^n$.*

The local characterization of global minima stated in Theorem 2.2 can be generalized to the following form, which we use in Section 5.4.

▶ **Theorem 2.3** (Box-barrier property). *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, and let $p \in (\mathbb{Z} \cup \{+\infty\})^n$ and $q \in (\mathbb{Z} \cup \{-\infty\})^n$, where $q \leq p$. Let $S = \{x \in \mathbb{Z}^n \mid q_i < x < p_i \ (\forall i)\}$, $W_i^+ = \{x \in \mathbb{Z}^n \mid x_i = p_i,\ q_j \leq x_j \leq p_j \ (j \neq i)\}$, $W_i^- = \{x \in \mathbb{Z}^n \mid x_i = q_i,\ q_j \leq x_j \leq p_j \ (j \neq i)\}$ $(i = 1, \ldots, n)$, $W = \bigcup_{i=1}^n (W_i^+ \cup W_i^-)$, and $\hat{x} \in S \cap \mathrm{dom}\, f$. If $f(\hat{x}) \leq f(y)$ for all $y \in W$, then $f(\hat{x}) \leq f(z)$ for all $z \in \mathbb{Z}^n \setminus S$.*

Proof. Let $U = \bigcup_{i=1}^n \{x \in \mathbb{R}^n \mid x_i \in \{p_i, q_i\},\ q_j \leq x_j \leq p_j \ (j \neq i)\}$, for which we have $U \cap \mathbb{Z}^n = W$. For $z \in \mathbb{Z}^n \setminus S$, the line segment connecting $\hat{x}$ and z intersects U at a point, say, $u \in \mathbb{R}^n$. Then $N(u)$ is contained in W. Since the local convex extension $\tilde{f}(u)$ is a convex combination of $f(y)$'s with $y \in N(u)$ and $f(y) \geq f(\hat{x})$ for every $y \in W$, we have $\tilde{f}(u) \geq f(\hat{x})$. On the other hand, it follows from integral convexity that $\tilde{f}(u) \leq (1-\lambda) f(\hat{x}) + \lambda f(z)$ for some λ with $0 < \lambda \leq 1$. Hence $f(\hat{x}) \leq \tilde{f}(u) \leq (1-\lambda) f(\hat{x}) + \lambda f(z)$, and therefore, $f(\hat{x}) \leq f(z)$. ◀

3 Scaling Operation for Integrally Convex Functions

For $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ and $\alpha \in \mathbb{Z}_{++}$, the α-scaling of f is the function $f^\alpha : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $f^\alpha(x) = f(\alpha x)$ $(x \in \mathbb{Z}^n)$. When $n = 2$, integral convexity is preserved under scaling.

▶ **Theorem 3.1.** *Let $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{+\infty\}$ be an integrally convex function and $\alpha \in \mathbb{Z}_{++}$. Then the scaled function f^α is integrally convex.*

Proof. First note that a set $S \subseteq \mathbb{Z}^2$ is an integrally convex set if and only if it can be represented as $S = \{(x_1, x_2) \in \mathbb{Z}^2 \mid a_i x_1 + b_i x_2 \leq c_i \ (i = 1, \ldots, m)\}$ for some $a_i, b_i \in \{-1, 0, 1\}$ and $c_i \in \mathbb{Z}$ $(i = 1, \ldots, m)$. Hence,

$\mathrm{dom}\, f^\alpha = (\mathrm{dom}\, f \cap (\alpha\mathbb{Z})^2)/\alpha$ is an integrally convex set. By Theorem 2.1 we only have to check condition (2.2) for f^α with $x = (0, 0)$ and $y = (2, 0)$, $(2, 1)$, $(2, 2)$, i.e.,

$$f(0,0) + f(2\alpha, 0) \geq 2 f(\alpha, 0),$$
$$f(0,0) + f(2\alpha, 2\alpha) \geq 2 f(\alpha, \alpha),$$
$$f(0,0) + f(2\alpha, \alpha) \geq f(\alpha, \alpha) + f(\alpha, 0).$$

The first two inequalities follow easily from integral convexity of f, whereas the third inequality is a special case of "basic parallelogram inequality" (3.1) below with $a = b = \alpha$. ◀

▶ **Proposition 3.2.** *For an integrally convex function $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{+\infty\}$ we have*

$$f(0,0) + f(a+b,a) \geq f(a,a) + f(b,0) \qquad (a,b \in \mathbb{Z}_+). \tag{3.1}$$

Proof. We may assume $a,b \geq 1$ and $\{(0,0),(a+b,a)\} \subseteq \mathrm{dom}\, f$, which implies $k(1,1) + l(1,0) \in \mathrm{dom}\, f$ for all (k,l) with $0 \leq k \leq a$, $0 \leq l \leq b$. We use notation $f_x(z) = f(x+z)$. For each $x \in \mathrm{dom}\, f$ we have $f_x(0,0) + f_x(2,1) \geq f_x(1,1) + f_x(1,0)$ by integral convexity of f. By adding these inequalities for $x = k(1,1) + l(1,0)$ with $0 \leq k \leq a-1$ and $0 \leq l \leq b-1$, we obtain (3.1). Note that all terms involved in these inequalities are finite. ◀

If $n \geq 3$, f^α is not always integrally convex, as is demonstrated by the following example.

▶ **Example 3.3.** Consider the integrally convex function $f : \mathbb{Z}^3 \to \mathbb{R} \cup \{+\infty\}$ defined on $\mathrm{dom}\, f = [(0,0,0),(4,2,2)]_{\mathbb{Z}}$ by

$f(x_1,x_2,0)$

x_2 \ x_1	0	1	2	3	4
2	3	1	1	1	3
1	1	0	0	0	0
0	0	0	0	0	3

$f(x_1,x_2,1)$

x_2 \ x_1	0	1	2	3	4
2	2	1	0	0	0
1	1	0	0	0	0
0	0	0	0	0	0

$f(x_1,x_2,2)$

x_2 \ x_1	0	1	2	3	4
2	3	2	1	0	0
1	2	1	0	0	0
0	3	0	0	0	3

For the scaling with $\alpha = 2$, we have a failure of integral convexity: $f(0,0,0) + f(4,2,2) < \min[f(2,2,2) + f(2,0,0), f(2,2,0) + f(2,0,2)]$. The set $S = \mathrm{argmin}\, f = \{x \mid f(x) = 0\}$ is an integrally convex set, and $S^\alpha = \{x \mid \alpha x \in S\} = \{(0,0,0),(1,0,0),(1,0,1),(2,1,1)\}$ is not. ◀

Seeing that the class of L$^\natural$-convex functions is stable under scaling, while this is not true for the superclass of integrally convex functions, naturally leads to the question of finding an intermediate class of functions that is stable under scaling. An answer to this question will be given in Section 5.3.

4 Proximity Results for Integrally Convex Functions

In this section we show that a proximity theorem holds for integrally convex functions. More precisely, we show that if x^α is an α-local minimizer of an integrally convex function f, i.e., $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{-1,0,+1\}^n$, then there exists a global minimizer x^* of f for which the ℓ_∞-distance from x^α is bounded by an appropriate function $B(n,\alpha)$. However, we first show that the bounding function $B(n,\alpha)$ must be at least quadratic in n, so that $B(n,\alpha) = n(\alpha-1)$, which applies to L$^\natural$-convex functions, is not valid in this case.

4.1 Lower bounds for proximity distance

For integrally convex functions with $n \geq 3$, the bound $n(\alpha - 1)$ is not valid.

▶ **Example 4.1.** Consider the integrally convex function $f : \mathbb{Z}^3 \to \mathbb{R} \cup \{+\infty\}$ defined on $\mathrm{dom}\, f = [(0,0,0),(4,2,2)]_{\mathbb{Z}}$ by

$f(x_1,x_2,0)$

x_2 \ x_1	0	1	2	3	4
2	5	1	0	0	4
1	2	−1	−2	0	3
0	0	−1	0	1	6

$f(x_1,x_2,1)$

x_2 \ x_1	0	1	2	3	4
2	4	1	−2	−3	−1
1	2	−1	−2	−3	−1
0	2	−1	−2	0	5

$f(x_1,x_2,2)$

x_2 \ x_1	0	1	2	3	4
2	6	3	0	−3	−4
1	6	1	−2	−3	1
0	6	2	0	3	6

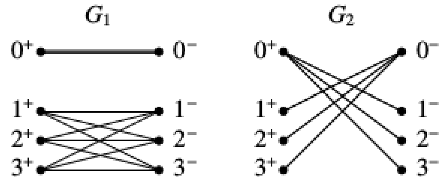

Figure 1 Example for $O(n^2)$ lower bound for proximity distance ($m = 3$).

and let $\alpha = 2$. For $x^\alpha = (0, 0, 0)$ we have $f(x^\alpha) = 0$, $f(x^\alpha) \leq f(x^\alpha + 2d)$ for $d = (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. Hence $x^\alpha = (0, 0, 0)$ is α-local minimal. A unique minimizer of f is located at $x^* = (4, 2, 2)$ with $f(x^*) = -4$ and $\|x^\alpha - x^*\|_\infty = 4$. The ℓ_∞-distance between x^α and x^* is strictly larger than $n(\alpha - 1) = 3$. ◀

▶ **Example 4.2.** For a positive integer $m \geq 1$, we consider two bipartite graphs G_1 and G_2 on vertex bipartition $(\{0^+, 1^+, \ldots, m^+\}, \{0^-, 1^-, \ldots, m^-\})$; see Fig. 1. The edge sets of G_1 and G_2 are defined respectively as $E_1 = \{(0^+, 0^-)\} \cup \{(i^+, j^-) \mid i, j = 1, \ldots, m\}$ and $E_2 = \{(0^+, j^-) \mid j = 1, \ldots, m\} \cup \{(i^+, 0^-) \mid i = 1, \ldots, m\}$. Let $V^+ = \{1^+, \ldots, m^+\}$, $V^- = \{1^-, \ldots, m^-\}$, and $n = 2m + 2$. Consider $X_1, X_2 \subseteq \mathbb{Z}^n$ defined by

$$X_1 = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij} (\mathbf{1}_{i^+} - \mathbf{1}_{j^-}) + \lambda_0 (\mathbf{1}_{0^+} - \mathbf{1}_{0^-}) \;\middle|\; \begin{array}{l} \lambda_{ij} \in [0, \alpha - 1]_\mathbb{Z} \ (i, j = 1, \ldots, m) \\ \lambda_0 \in [0, m^2(\alpha - 1)]_\mathbb{Z} \end{array} \right\},$$

$$X_2 = \left\{ \sum_{i=1}^{m} \mu_i (\mathbf{1}_{i^+} - \mathbf{1}_{0^-}) + \sum_{j=1}^{m} \nu_j (\mathbf{1}_{0^+} - \mathbf{1}_{j^-}) \;\middle|\; \begin{array}{l} \mu_i \in [0, m(\alpha - 1)]_\mathbb{Z} \ (i = 1, \ldots, m) \\ \nu_j \in [0, m(\alpha - 1)]_\mathbb{Z} \ (j = 1, \ldots, m) \end{array} \right\},$$

where X_1 and X_2 represent the sets of boundaries of flows in G_1 and G_2, respectively. We define functions $f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f_1 = X_1$ and $\mathrm{dom}\, f_2 = X_2$ by

$$f_1(x) = \begin{cases} x(V^-) & (x \in X_1) \\ +\infty & (x \notin X_1), \end{cases} \qquad f_2(x) = \begin{cases} x(V^-) & (x \in X_2) \\ +\infty & (x \notin X_2) \end{cases} \qquad (x \in \mathbb{Z}^n),$$

where $x(U) = \sum_{u \in U} x_u$ for any set U of vertices. Both f_1 and f_2 are M-convex, and hence $f = f_1 + f_2$ is an M$_2$-convex function, which is integrally convex (see [15, Section 8.3.1]). We have $\mathrm{dom}\, f = X_1 \cap X_2$ and f is linear on $\mathrm{dom}\, f$. As is easily verified, f has a unique minimizer at x^* defined by

$$\begin{aligned} x^*_u &= m(\alpha - 1) \ (u \in V^+), \\ &= -\, m(\alpha - 1) \ (u \in V^-), \\ &= m^2(\alpha - 1) \ (u = 0^+), \\ &= -\, m^2(\alpha - 1) \ (u = 0^-), \end{aligned}$$

which corresponds to $\lambda_0 = m^2(\alpha - 1)$, $\lambda_{ij} = \alpha - 1$, $\mu_i = \nu_j = m(\alpha - 1)$ $(i, j = 1, \ldots, m)$.

Let $x^\alpha = \mathbf{0}$. This is α-local minimal, since $\mathrm{dom}\, f \cap \{-\alpha, 0, \alpha\}^n = \{\mathbf{0}\}$, which can be verified easily. With $\|x^* - x^\alpha\|_\infty = m^2(\alpha - 1) = (n - 2)^2(\alpha - 1)/4$, this example demonstrates a quadratic lower bound $(n - 2)^2(\alpha - 1)/4$ for the proximity distance for integrally convex functions. ◀

We have seen that the proximity theorem with linear bound $B(n, \alpha) = n(\alpha - 1)$, which is valid for L$^\natural$-convex functions, does not hold for all integrally convex functions. Thus, we may ask if there is a subclass of integrally convex functions, including L$^\natural$-convex functions, that admits such a linear proximity bound. An answer to this question will be given in Section 5.4. Another question is whether we can establish a proximity theorem at all by enlarging the proximity bound. This question is answered next.

4.2 Theorem

▶ **Theorem 4.3.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an integrally convex function, $\alpha \in \mathbb{Z}_{++}$, and $x^\alpha \in \mathrm{dom}\, f$.*

(1) *If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ $(\forall\ d \in \{-1, 0, +1\}^n)$, then $\mathrm{argmin}\, f \neq \emptyset$ and there exists $x^* \in \mathrm{argmin}\, f$ with $\|x^\alpha - x^*\|_\infty \leq \beta_n(\alpha - 1)$, where $\beta_1 = 1$, $\beta_2 = 2$; $\beta_n = \dfrac{n+1}{2}\beta_{n-1} + 1$ $(n = 3, 4, \ldots)$.*

(2) *$\beta_n \leq \dfrac{(n+1)!}{2^{n-1}}$ $(n = 3, 4, \ldots)$.*

To prove Theorem 4.3(1) we first note that it follows from its special case where $x^\alpha = \mathbf{0}$ and f is defined on a bounded set in the nonnegative orthant $\mathbb{Z}^n_+$. That is, the proof of Theorem 4.3(1) is reduced to proving the following proposition. We use notation $V = \{1, 2, \ldots, n\}$ and the characteristic vector of $A \subseteq V$ is denoted by $\mathbf{1}_A$.

▶ **Proposition 4.4.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an integrally convex function such that $\mathrm{dom}\, f$ is a bounded subset of $\mathbb{Z}^n_+$ containing the origin $\mathbf{0}$. If*

$$f(\mathbf{0}) \leq f(\alpha \mathbf{1}_A) \qquad (\forall A \subseteq V), \tag{4.1}$$

then there exists $x^* \in \mathrm{argmin}\, f$ with $\|x^*\|_\infty \leq \beta_n(\alpha - 1)$.

4.3 Tools for the proof: f-minimality

In this section we introduce some technical tools that we use in the proof of Proposition 4.4.

For two nonnegative integer vectors $x, y \in \mathbb{Z}^n_+$, we write $y \preceq_f x$ if $y \leq x$ and $f(y) \leq f(x)$. Note that $y \preceq_f x$ if and only if $(y, f(y)) \leq (x, f(x))$ in $\mathbb{R}^{n+1}$. We say that $x \in \mathbb{Z}^n_+$ is *f-minimal* if there exists no $y \in \mathbb{Z}^n_+$ such that $y \preceq_f x$ and $y \neq x$. That is, x is f-minimal if and only if it is the unique minimizer of the function f restricted to the integer interval $[\mathbf{0}, x]_{\mathbb{Z}}$.

▶ **Lemma 4.5.** *Assume (4.1). For any $A\ (\neq \emptyset) \subseteq V$ and $\lambda \in \mathbb{Z}_+$ we have $(\alpha - 1)\mathbf{1}_A \preceq_f (\alpha - 1)\mathbf{1}_A + \lambda \mathbf{1}_A$.*

Proof. First note that $(\alpha - 1)\mathbf{1}_A \leq (\alpha - 1)\mathbf{1}_A + \lambda \mathbf{1}_A$ for all $\lambda \in \mathbb{Z}_+$. By integral convexity of f, $g(t) = f(t\mathbf{1}_A)$ is convex in $t \in \mathbb{Z}_+$, and therefore, $g(\alpha - 1) \leq [(\alpha - 1)g(\alpha) + g(0)]/\alpha$. On the other hand, $g(0) \leq g(\alpha)$ by α-local minimality (4.1). Hence $g(\alpha - 1) \leq g(\alpha)$. By convexity of g, this implies $g(\alpha - 1) \leq g((\alpha - 1) + \lambda)$ for all $\lambda \in \mathbb{Z}_+$, i.e., $f((\alpha - 1)\mathbf{1}_A) \leq f((\alpha - 1)\mathbf{1}_A + \lambda \mathbf{1}_A)$ for all $\lambda \in \mathbb{Z}_+$. ◀

▶ **Lemma 4.6.** *Let $x \in \mathbb{Z}^n_+$ and $A\ (\neq \emptyset) \subseteq V$, and assume $x \preceq_f x + \mathbf{1}_A$.*

(1) *For any $i \in V$ and $\lambda \in \mathbb{Z}_+$ we have $x + \mathbf{1}_A + \mathbf{1}_i \preceq_f (x + \mathbf{1}_A + \mathbf{1}_i) + \lambda \mathbf{1}_A$.*

(2) *For any $i \in A$ and $\lambda \in \mathbb{Z}_+$ we have $x + \mathbf{1}_A - \mathbf{1}_i \preceq_f (x + \mathbf{1}_A - \mathbf{1}_i) + \lambda \mathbf{1}_A$.*

Proof. (1) First note that $x + \mathbf{1}_A + \mathbf{1}_i \leq (x + \mathbf{1}_A + \mathbf{1}_i) + \lambda \mathbf{1}_A$ for all $\lambda \in \mathbb{Z}_+$. Define $g(\lambda) = f((x + \mathbf{1}_A + \mathbf{1}_i) + \lambda \mathbf{1}_A)$, which is a convex function in $\lambda \in \mathbb{Z}_+$ by integral convexity of f. We are to show $g(0) \leq g(\lambda)$ for all $\lambda \in \mathbb{Z}_+$, which is equivalent to $g(0) \leq g(1)$ by convexity of g. By integral convexity of f we have $f(x+2\mathbf{1}_A+\mathbf{1}_i)+f(x) \geq f(x+\mathbf{1}_A)+f(x+\mathbf{1}_A+\mathbf{1}_i)$, whereas $f(x) \leq f(x+\mathbf{1}_A)$ by the assumption. Hence $f(x+\mathbf{1}_A+\mathbf{1}_i) \leq f(x+2\mathbf{1}_A+\mathbf{1}_i)$, i.e., $g(0) \leq g(1)$. (2) Similarly. ◀

Repeated application of (1) and (2) of Lemma 4.6 yields the following general form.

▶ **Lemma 4.7.** *Let $x \in \mathbb{Z}^n_+$ and A ($\neq \emptyset$) $\subseteq V$, and assume $x \preceq_f x+\mathbf{1}_A$. For any $\lambda \in \mathbb{Z}_+$, $\mu_i^+, \mu_i^- \in \mathbb{Z}_+$ ($i \in A$) and $\mu_i^\circ \in \mathbb{Z}_+$ ($i \notin A$), we have*

$$\begin{aligned} & x+\sum_{i\in A}\mu_i^+(\mathbf{1}_A+\mathbf{1}_i)+\sum_{i\in A}\mu_i^-(\mathbf{1}_A-\mathbf{1}_i)+\sum_{i\notin A}\mu_i^\circ(\mathbf{1}_A+\mathbf{1}_i) \\ & \preceq_f \ x+\sum_{i\in A}\mu_i^+(\mathbf{1}_A+\mathbf{1}_i)+\sum_{i\in A}\mu_i^-(\mathbf{1}_A-\mathbf{1}_i)+\sum_{i\notin A}\mu_i^\circ(\mathbf{1}_A+\mathbf{1}_i)+\lambda\mathbf{1}_A. \end{aligned} \tag{4.2}$$

For A ($\neq \emptyset$) $\subseteq V$, let B_A denote the set of the generating vectors in (4.2) and C_A the set of their nonnegative integer combinations[1]:

$$B_A = \{\mathbf{1}_A\} \cup \bigcup_{i\in A}\{\mathbf{1}_A+\mathbf{1}_i, \mathbf{1}_A-\mathbf{1}_i\} \cup \bigcup_{i\notin A}\{\mathbf{1}_A+\mathbf{1}_i\}, \tag{4.3}$$

$$C_A = \{\lambda\mathbf{1}_A+\sum_{i\in A}\mu_i^+(\mathbf{1}_A+\mathbf{1}_i)+\sum_{i\in A}\mu_i^-(\mathbf{1}_A-\mathbf{1}_i)+\sum_{i\notin A}\mu_i^\circ(\mathbf{1}_A+\mathbf{1}_i) \mid \lambda, \mu_i^+, \mu_i^-, \mu_i^\circ \in \mathbb{Z}_+\}. \tag{4.4}$$

▶ **Lemma 4.8.** *Assume* (4.1). *If $y \in \mathbb{Z}^n_+$ is f-minimal, then $y \notin \alpha\mathbf{1}_A + C_A$ for any $A(\neq \emptyset) \subseteq V$.*

Proof. To prove the contraposition, suppose that $y \in \alpha\mathbf{1}_A + C_A$ for some A. Then

$$y = \alpha\mathbf{1}_A + \left(\mu\mathbf{1}_A+\sum_{i\in A}\mu_i^+(\mathbf{1}_A+\mathbf{1}_i)+\sum_{i\in A}\mu_i^-(\mathbf{1}_A-\mathbf{1}_i)+\sum_{i\notin A}\mu_i^\circ(\mathbf{1}_A+\mathbf{1}_i)\right)$$

for some $\mu, \mu_i^+, \mu_i^-, \mu_i^\circ \in \mathbb{Z}_+$. Or equivalently,

$$y = (\alpha-1)\mathbf{1}_A+\sum_{i\in A}\mu_i^+(\mathbf{1}_A+\mathbf{1}_i)+\sum_{i\in A}\mu_i^-(\mathbf{1}_A-\mathbf{1}_i)+\sum_{i\notin A}\mu_i^\circ(\mathbf{1}_A+\mathbf{1}_i)+(\mu+1)\mathbf{1}_A,$$

which is of the form of (4.2) with $x = (\alpha-1)\mathbf{1}_A$ and $\lambda = \mu+1$. Since $x = (\alpha-1)\mathbf{1}_A \preceq_f \alpha\mathbf{1}_A = x+\mathbf{1}_A$ by Lemma 4.5, Lemma 4.7 shows that y is not f-minimal. ◀

4.4 Proof of Proposition 4.4 for $n = 2$

We prove Proposition 4.4 for $n = 2$. Take $x^* = (x_1^*, x_2^*) \in \arg\min f$ that is f-minimal. We may assume $x_1^* \geq x_2^*$. Since x^* is f-minimal, Lemma 4.8 shows that x^* belongs to $X^* = \{(x_1,x_2) \in \mathbb{Z}^2_+ \mid x_1 \geq x_2\} \setminus ((\alpha\mathbf{1}_A + C_A) \cup (\alpha\mathbf{1}_V + C_V))$, where $A = \{1\}$ and $V = \{1,2\}$. On noting $C_A = \{\mu_1(1,0)+\mu_{12}(1,1) \mid \mu_1, \mu_{12} \in \mathbb{Z}_+\}$, $C_V = \{\mu_1(1,0)+\mu_2(0,1) \mid \mu_1, \mu_2 \in \mathbb{Z}_+\}$, we see that X^* consists of all integer points contained in the parallelogram with vertices $(0,0)$, $(\alpha-1,0)$, $(2\alpha-2,\alpha-1)$, $(\alpha-1,\alpha-1)$. Therefore, $\|x^*\|_\infty \leq 2(\alpha-1)$. Thus Proposition 4.4 for $n = 2$ is proved.

[1] It can be shown that B_A is a Hilbert basis of the convex cone generated by B_A.

4.5 Proof of Proposition 4.4 for $n \geq 3$

In this section we prove Proposition 4.4 for $n \geq 3$ by induction on n. Accordingly we assume that Proposition 4.4 is true for every integrally convex function in $n-1$ variables.

Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be an integrally convex function such that $\mathrm{dom}\, f$ is a bounded subset of $\mathbb{Z}_+^n$ containing the origin $\mathbf{0}$. Take $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \arg\min f$ that is f-minimal. Then

$$[\mathbf{0}, x^*]_{\mathbb{Z}} \cap \arg\min f = \{x^*\}. \tag{4.5}$$

We may assume $x_1^* \geq x_2^* \geq \cdots \geq x_n^*$. For any $x \in \mathbb{Z}_+^n$ let $f_{[\mathbf{0},x]} : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ denote the restriction of f to the interval $[\mathbf{0}, x]_{\mathbb{Z}}$, that is, $f_{[\mathbf{0},x]}(y) = f(y)$ if $y \in [\mathbf{0}, x]_{\mathbb{Z}}$, and $= +\infty$ otherwise.

The following lemma reveals a key fact that will be used for induction on n. Note that, by (4.5), x^* satisfies the condition imposed on $x^\bullet$.

▶ **Lemma 4.9.** *Let $x^\bullet \in \mathrm{dom}\, f$ and assume $\arg\min f_{[\mathbf{0},x^\bullet]} = \{x^\bullet\}$. Then for any $i \in V$ there exists $x^\circ \in \mathrm{dom}\, f$ such that*

$$\mathbf{0} \leq x^\circ \leq x^\bullet, \quad \|x^\circ - x^\bullet\|_\infty = 1, \quad x_i^\circ = x_i^\bullet - 1, \quad \arg\min f_{[\mathbf{0},x^\circ]} = \{x^\circ\}.$$

Proof. Let x° be a minimizer of $f(x)$ among those x which satisfy the conditions: $\mathbf{0} \leq x \leq x^\bullet$, $\|x - x^\bullet\|_\infty = 1$, and $x_i = x_i^\bullet - 1$; in case of multiple minimizers, we choose a minimal minimizer. Then we can show $\arg\min f_{[\mathbf{0},x^\circ]} = \{x^\circ\}$. ◀

Lemma 4.9 can be applied repeatedly, since the resulting point x° satisfies the condition imposed on the initial point $x^\bullet$. Starting with $x^\bullet = x^*$ we apply Lemma 4.9 repeatedly with $i = n$. After x_n^* applications, we arrive at a point $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}, 0)$. We have $\arg\min f_{[\mathbf{0},\hat{x}]} = \{\hat{x}\}$ and

$$x_j^* - x_n^* \leq \hat{x}_j \qquad (j = 1, 2, \ldots, n-1). \tag{4.6}$$

We now consider a function $\hat{f} : \mathbb{Z}^{n-1} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\hat{f}(x_1, x_2, \ldots, x_{n-1}) = \begin{cases} f(x_1, x_2, \ldots, x_{n-1}, 0) & (0 \leq x_j \leq \hat{x}_j \ (j = 1, 2, \ldots, n-1)), \\ +\infty & (\text{otherwise}). \end{cases}$$

This function $\hat{f}$ is an integrally convex function in $n-1$ variables, and the origin $\mathbf{0}$ is α-local minimal for $\hat{f}$ and $\hat{x}$ is the unique minimizer of $\hat{f}$. By the induction hypothesis, we can apply Proposition 4.4 to $\hat{f}$ to obtain $\|\hat{x}\|_\infty \leq \beta_{n-1}(\alpha - 1)$. Combining this with (4.6) we obtain

$$x_1^* - x_n^* \leq \beta_{n-1}(\alpha - 1). \tag{4.7}$$

We can also show

$$x_n^* \leq \frac{n-1}{n+1} x_1^* + \frac{2(\alpha - 1)}{n+1} \tag{4.8}$$

from the f-minimality of x^*. It follows from (4.7) and (4.8) that

$$x_1^* \leq \left(\frac{n+1}{2} \beta_{n-1} + 1 \right) (\alpha - 1) = \beta_n (\alpha - 1).$$

This completes the proof of Proposition 4.4, and hence that of Theorem 4.3 (1). The bound for β_n in Theorem 4.3 (2) is a simple calculus from the recurrence.

5 Directed Integrally Convex Functions

In this section we introduce a novel class of integrally convex functions, which admits the scaling operation and the proximity theorem with the linear bound $n(\alpha - 1)$.

5.1 Definition

We call a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ *directed integrally convex* if $\mathrm{dom}\, f$ is an integer interval and discrete midpoint convexity (1.1) is satisfied by every pair $(x, y) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $\|x - y\|_\infty = 2$ (exactly equal to two).

▶ **Proposition 5.1.**

(1) *A directed integrally convex function is integrally convex.*

(2) *An* $\mathrm{L}^\natural$*-convex function (defined on an integer interval) is directed integrally convex.*

Proof. (1) Let f be a directed integrally convex function. To use Theorem 2.1, take integer points $x, y \in \mathrm{dom}\, f$ with $\|x - y\|_\infty = 2$. For $u = (x + y)/2$, both $\lceil u \rceil$ and $\lfloor u \rfloor$ belong to $N(u)$, and therefore $2\tilde{f}(u) \leq f(\lceil u \rceil) + f(\lfloor u \rfloor) \leq f(x) + f(y)$, where the second inequality is midpoint convexity.

(2) By the characterizations of $\mathrm{L}^\natural$-convex functions in Section 1. ◀

5.2 Parallelogram inequality

For directed integrally convex functions two special direction vectors play a crucial role:

$$d_1 = (\mathbf{1}^{m_1}, \mathbf{1}^{m_2}, -\mathbf{1}^{m_3}, \mathbf{0}^{m_4}, \mathbf{0}^{m_5}), \quad d_2 = (\mathbf{1}^{m_1}, \mathbf{0}^{m_2}, -\mathbf{1}^{m_3}, -\mathbf{1}^{m_4}, \mathbf{0}^{m_5}),$$

where $m_1, m_2, m_3, m_4, m_5 \geq 0$, $m_1 + m_2 + m_3 + m_4 + m_5 = n$, and $\mathbf{1}^m = (1, 1, \ldots, 1) \in \mathbb{Z}^m$ and $\mathbf{0}^m = (0, 0, \ldots, 0) \in \mathbb{Z}^m$ for $m \in \mathbb{Z}_+$ ($m = 0$ is allowed). For integers a and b we define

$$z(a, b) = a d_1 + b d_2 = \big(\ (a + b)\mathbf{1}^{m_1}, a\mathbf{1}^{m_2}, -(a + b)\mathbf{1}^{m_3}, -b\mathbf{1}^{m_4}, \mathbf{0}^{m_5}\ \big). \tag{5.1}$$

▶ **Lemma 5.2.** *Let a and b be integers.*

(1) $\|z(a + 1, b + 1) - z(a, b)\|_\infty = 2$ *if* $m_1 + m_3 \geq 1$.

(2) $$\left\lceil \frac{z(a, b) + z(a + 1, b + 1)}{2} \right\rceil = z(a + 1, b), \qquad \left\lfloor \frac{z(a, b) + z(a + 1, b + 1)}{2} \right\rfloor = z(a, b + 1).$$

▶ **Proposition 5.3** (Parallelogram inequality)**.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a directed integrally convex function. For $x \in \mathrm{dom}\, f$ define $f_x(z) = f(x + z)$. If $m_1 + m_3 \geq 1$, then*

$$\begin{aligned} & f_x(\ \mathbf{0}^{m_1}, \mathbf{0}^{m_2}, \mathbf{0}^{m_3}, \mathbf{0}^{m_4}, \mathbf{0}^{m_5}\ ) + f_x(\ (a + b)\mathbf{1}^{m_1}, a\mathbf{1}^{m_2}, -(a + b)\mathbf{1}^{m_3}, -b\mathbf{1}^{m_4}, \mathbf{0}^{m_5}\ ) \\ & \geq f_x(\ a\mathbf{1}^{m_1}, a\mathbf{1}^{m_2}, -a\mathbf{1}^{m_3}, \mathbf{0}^{m_4}, \mathbf{0}^{m_5}\ ) + f_x(\ b\mathbf{1}^{m_1}, \mathbf{0}^{m_2}, -b\mathbf{1}^{m_3}, -b\mathbf{1}^{m_4}, \mathbf{0}^{m_5}\ ) \qquad (a, b \in \mathbb{Z}_+). \end{aligned} \tag{5.2}$$

Proof. Using notation (5.1) we can rewrite (5.2) as

$$f_x(z(0, 0)) + f_x(z(a, b)) \geq f_x(z(a, 0)) + f_x(z(0, b)). \tag{5.3}$$

We may assume $a, b \geq 1$ and $\{z(0, 0), z(a, b)\} \subseteq \mathrm{dom}\, f_x$, since otherwise the inequality (5.3) is trivially true. By directed integral convexity of f, we have

$$f_x(z(k, l)) + f_x(z(k + 1, l + 1)) \geq f_x(z(k + 1, l)) + f_x(z(k, l + 1)) \tag{5.4}$$

for $k, l \in \mathbb{Z}_+$. By adding these inequalities for (k, l) with $0 \leq k \leq a - 1$, $0 \leq l \leq b - 1$, we obtain (5.3). Note that all terms appearing in the above inequalities are finite. ◀

The parallelogram inequality with permutations of coordinates can be stated in an alternative form. Using notation $d_i = (d_{i1}, \ldots, d_{in})$ and $\Delta = \{(-1,-1), (0,0), (1,1), (0,-1), (1,0)\}$ we define

$$\mathcal{D} = \{(d_1, d_2) \mid (d_{1j}, d_{2j}) \in \Delta \ \ (j = 1, \ldots, n), \ \|d_1 + d_2\|_\infty = 2\}. \tag{5.5}$$

For $d_1 = (\mathbf{1}^{m_1}, \mathbf{1}^{m_2}, -\mathbf{1}^{m_3}, \mathbf{0}^{m_4}, \mathbf{0}^{m_5})$ and $d_2 = (\mathbf{1}^{m_1}, \mathbf{0}^{m_2}, -\mathbf{1}^{m_3}, -\mathbf{1}^{m_4}, \mathbf{0}^{m_5})$ we have $(d_1, d_2) \in \mathcal{D}$ as long as $m_1 + m_3 \geq 1$, and any $(d_1, d_2) \in \mathcal{D}$ can be put in this form through a suitable simultaneous permutation of coordinates of d_1 and d_2. Therefore, Proposition 5.3 can be rephrased as follows: If $(d_1, d_2) \in \mathcal{D}$, then $f(x) + f(x + ad_1 + bd_2) \geq f(x + ad_1) + f(x + bd_2)$ $(a, b \in \mathbb{Z}_+)$.

A generalized form of parallelogram inequality is given in the following proposition, where $\mathcal{D}^\top = \{(d_2, d_1) \mid (d_1, d_2) \in \mathcal{D}\}$.

▶ **Proposition 5.4.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a directed integrally convex function, $x \in \operatorname{dom} f$, and $d_1^1, \ldots, d_1^K, d_2^1, \ldots, d_2^L \in \{-1, 0, +1\}^n$, where $K \geq 1$ and $L \geq 1$. If*

$$(d_1^k, d_2^l) \in \mathcal{D} \cup \mathcal{D}^\top \qquad (k = 1, \ldots, K;\ l = 1, \ldots, L), \tag{5.6}$$

$$\operatorname{supp}^+(d_1^k) \cap \operatorname{supp}^-(d_1^{k'}) = \emptyset \quad (k \neq k'), \qquad \operatorname{supp}^+(d_2^l) \cap \operatorname{supp}^-(d_2^{l'}) = \emptyset \quad (l \neq l'), \tag{5.7}$$

then

$$f(x) + f(x + \sum_{k=1}^{K} a^k d_1^k + \sum_{l=1}^{L} b^l d_2^l) \geq f(x + \sum_{k=1}^{K} a^k d_1^k) + f(x + \sum_{l=1}^{L} b^l d_2^l) \quad (a^k, b^l \in \mathbb{Z}_+). \tag{5.8}$$

Proof. With notation $x(k,l) = x + \sum_{i=1}^{k} a^i d_1^i + \sum_{j=1}^{l} b^j d_2^j$, the inequality (5.8) is rewritten as

$$f(x(0,0)) + f(x(K,L)) \geq f(x(K,0)) + f(x(0,L)). \tag{5.9}$$

We may assume $K, L \geq 1$ and $\{x(0,0), x(K,L)\} \subseteq \operatorname{dom} f$, since otherwise (5.9) is trivially true. Since $x(k,l) = x(k-1, l-1) + a^k d_1^k + b^l d_2^l$ and $(d_1^k, d_2^l) \in \mathcal{D} \cup \mathcal{D}^\top$ by assumption, we can apply the parallelogram inequality (5.2) to obtain

$$f(x(k-1, l-1)) + f(x(k,l)) \geq f(x(k, l-1)) + f(x(k-1, l)).$$

By adding these inequalities for (k,l) with $1 \leq k \leq K$ and $1 \leq l \leq L$, we obtain (5.9). Note that all terms appearing in the above inequalities are finite by (5.6) and (5.7). ◀

The assumptions in Proposition 5.4 are met in the following case.

▶ **Lemma 5.5.** *Conditions (5.6) and (5.7) are satisfied if $\{d_1^1, \ldots, d_1^K, d_2^1, \ldots, d_2^L\} \subseteq \{\mathbf{1}_{A_1} - \mathbf{1}_{B_1}, \mathbf{1}_{A_2} - \mathbf{1}_{B_2}, \ldots, \mathbf{1}_{A_s} - \mathbf{1}_{B_s}\}$ for nested families $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_s$ and $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_s$ of subsets of $\{1, \ldots, n\}$ such that $A_s \cap B_1 = \emptyset$ and $A_1 \cup B_s \neq \emptyset$.*

5.3 Scaling operation

Directed integrally convex functions are stable under scaling for arbitrary n, just as L$^\natural$-convex functions. Recall that the scaling operation preserves (general) integral convexity only when $n \leq 2$.

▶ **Theorem 5.6.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a directed integrally convex function and $\alpha \in \mathbb{Z}_{++}$. Then the scaled function f^α is directed integrally convex.*

Proof. To show (1.1) for f^α, it suffices to consider the inequality for f with $x = \mathbf{0}$ and $y = (2\alpha\mathbf{1}^{m_1}, \alpha\mathbf{1}^{m_2}, -2\alpha\mathbf{1}^{m_3}, -\alpha\mathbf{1}^{m_4}, \mathbf{0}^{m_5}) \in \mathbb{Z}^n$, where $m_1 + m_3 \geq 1$. That is, we are to prove

$$\begin{aligned} &f(\ \mathbf{0}^{m_1}, \mathbf{0}^{m_2}, \mathbf{0}^{m_3}, \mathbf{0}^{m_4}, \mathbf{0}^{m_5}\ ) + f(\ 2\alpha\mathbf{1}^{m_1}, \alpha\mathbf{1}^{m_2}, -2\alpha\mathbf{1}^{m_3}, -\alpha\mathbf{1}^{m_4}, \mathbf{0}^{m_5}\ ) \\ &\geq f(\ \alpha\mathbf{1}^{m_1}, \alpha\mathbf{1}^{m_2}, -\alpha\mathbf{1}^{m_3}, \mathbf{0}^{m_4}, \mathbf{0}^{m_5}\ ) + f(\ \alpha\mathbf{1}^{m_1}, \mathbf{0}^{m_2}, -\alpha\mathbf{1}^{m_3}, -\alpha\mathbf{1}^{m_4}, \mathbf{0}^{m_5}\ ), \end{aligned}$$

which holds since it is a special case of the parallelogram inequality (5.2) with $a = b = \alpha$. ◀

5.4 Proximity theorem

The α-local proximity theorem with linear bound $n(\alpha-1)$ holds for directed integrally convex functions in n variables for all n. Recall that for (general) integrally convex functions the bound $n(\alpha-1)$ is valid only when $n \leq 2$, whereas it is valid for L$^\natural$-convex functions for all n.

▶ **Theorem 5.7.** *Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ be a directed integrally convex function, $\alpha \in \mathbb{Z}_{++}$, and $x^\alpha \in \mathrm{dom}\, f$. If $f(x^\alpha) \leq f(x^\alpha + \alpha d)$ for all $d \in \{-1, 0, +1\}^n$, then there exists a minimizer $x^* \in \mathbb{Z}^n$ of f with $\|x^\alpha - x^*\|_\infty \leq n(\alpha-1)$.*

To prove Theorem 5.7 we may assume $x^\alpha = \mathbf{0}$. Define $S = \{x \in \mathbb{Z}^n \mid \|x\|_\infty \leq n(\alpha-1)\}$, $W = \{x \in \mathbb{Z}^n \mid \|x\|_\infty = n(\alpha-1)+1\}$, and let μ be the minimum of $f(x)$ taken over $x \in S$ and $\hat{x}$ be a point in S with $f(\hat{x}) = \mu$. We shall show $f(y) \geq \mu$ for all $y \in W$. Then Theorem 2.3 (box-barrier property) implies that $f(z) \geq \mu$ for all $z \in \mathbb{Z}^n$.

▶ **Lemma 5.8.** *Each vector $y \in W$ can be represented as $y = (d_1^1 + d_1^2 + \cdots + d_1^{(n-1)(\alpha-1)}) + \alpha d_2$ with $(d_1^k, d_2) \in \mathcal{D} \cup \mathcal{D}^\top$ for $k = 1, 2, \ldots, (n-1)(\alpha-1)$.*

Proof. Fix $y = (y_1, \ldots y_n) \in W$ and put $m = \|y\|_\infty$, which is equal to $n(\alpha-1)+1$. With $A_k = \{i \mid y_i \geq m+1-k\}$, $B_k = \{i \mid y_i \leq -k\}$ $(k = 1, \ldots, m)$, we can represent y as $y = \sum_{k=1}^m (\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$. We have $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$, $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_m$, $A_m \cap B_1 = \emptyset$, and $A_1 \cup B_m \neq \emptyset$.

Claim 1: $\exists\, k_0 \in \{1, 2, \ldots, m-\alpha+1\}$ s.t. $(A_{k_0}, B_{k_0}) = (A_{k_0+j}, B_{k_0+j})$ for $j = 1, 2, \ldots, \alpha-1$.

Proof of Claim 1. We may assume $A_1 \neq \emptyset$. Define $(a_k, b_k) = (|A_k|, n - |B_k|)$ for $k = 1, 2, \ldots, m$ and $s = |\mathrm{supp}^+(y)|$. The sequence $(a_k, b_k)_{k=1,2,\ldots,m}$ is nondecreasing in $\mathbb{Z}^2$, satisfying $(1, s) \leq (a_1, b_1) \leq (a_2, b_2) \leq \cdots \leq (a_m, b_m) \leq (s, n)$. Since $m = n(\alpha-1)+1$ and the length of a strictly increasing chain contained in the interval $[(1, s), (s, n)]$ in $\mathbb{Z}^2$ is bounded by n, the sequence $\{(a_k, b_k)\}_{k=1,2,\ldots,m}$ must contain a constant subsequence of length $\geq \alpha$. Hence follows the claim. ◀

With reference to the index k_0 in Claim 1 we define

$$d_2 = \mathbf{1}_{A_{k_0}} - \mathbf{1}_{B_{k_0}}, \qquad d_1^k = \begin{cases} \mathbf{1}_{A_k} - \mathbf{1}_{B_k} & (1 \leq k \leq k_0 - 1), \\ \mathbf{1}_{A_{k+\alpha}} - \mathbf{1}_{B_{k+\alpha}} & (k_0 \leq k \leq m - \alpha = (n-1)(\alpha-1)). \end{cases}$$

Then we have $y = \sum_{k=1}^m (\mathbf{1}_{A_k} - \mathbf{1}_{B_k}) = (d_1^1 + d_1^2 + \cdots + d_1^{(n-1)(\alpha-1)}) + \alpha d_2$. Moreover, we have $(d_1^k, d_2) \in \mathcal{D} \cup \mathcal{D}^\top$ $(k = 1, 2, \ldots, (n-1)(\alpha-1))$ by Lemma 5.5. ◀

By Lemma 5.8 and inequality (5.8) with $K = (n-1)(\alpha-1)$ and $L = 1$ we obtain

$$f(\mathbf{0}) + f(y) \geq f(d_1^1 + d_1^2 + \cdots + d_1^{(n-1)(\alpha-1)}) + f(\alpha d_2).$$

Here we have $d_1^1 + d_1^2 + \cdots + d_1^{(n-1)(\alpha-1)} \in S$ and hence $f(d_1^1 + d_1^2 + \cdots + d_1^{(n-1)(\alpha-1)}) \geq \mu$ by the definition of μ. We also have $f(\alpha d_2) \geq f(\mathbf{0})$ by α-local minimality of $\mathbf{0}$. Therefore,

$$f(y) \geq f(d_1^1 + d_1^2 + \cdots + d_1^{(n-1)(\alpha-1)}) + [f(\alpha d_2) - f(\mathbf{0})] \geq \mu + 0 = \mu.$$

This completes the proof of Theorem 5.7.

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