Universal Guard Problems[*]

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Abstract

We provide a spectrum of results for the *Universal Guard Problem*, in which one is to obtain a small set of points ("guards") that are "universal" in their ability to guard any of a set of possible polygonal domains in the plane. We give upper and lower bounds on the number of universal guards that are always sufficient to guard all polygons having a given set of n vertices, or to guard all polygons in a given set of k polygons on an n-point vertex set. Our upper bound proofs include algorithms to construct universal guard sets of the respective cardinalities.

1   Introduction

Problems of finding optimal covers are among the most fundamental algorithmic challenges that play an important role in many contexts. One of the best-studied prototypes in a geometric setting is the classic Art Gallery Problem (AGP), which asks for a small number of points ("guards") required for covering ("seeing") all of the points within a geometric domain. An enormous body of work on algorithmic aspects of visibility coverage and related problems (see, e.g., O'Rourke [24], Keil [19], and [25]) was spawned by Klee's question for worst-case bounds more than 40 years ago: How many guards are always sufficient to guard all of the points in a simple polygon having n vertices? The answer, as shown originally by Chvátal [4], and with a very simple and elegant proof by Fisk [11], is that $\lfloor n/3 \rfloor$ guards are always sufficient, and sometimes necessary, to guard a simple n-gon.

While Klee's question was posed about guarding an n-vertex *simple polygon*, a related question about *point sets* was posed at the 2014 NYU Goodman-Pollack Fest: Given a set S of n points in the plane, how many *universal* guards are sometimes necessary and always sufficient to guard any simple polygon with vertex set S? This problem, and several related

questions, are studied in this paper. We give the first set of results on universal guarding, including combinatorial bounds and efficient algorithms to compute universal guard sets that achieve the upper bounds we prove. We focus on the case in which guards must be placed at a subset of the input set S and thus will be vertex guards for any polygonalization of S.

A strong motivation for our study is the problem of computing guard sets in the face of uncertainty. In our model, we require that the guards are *robust* with respect to different possible polygonalizations consistent with a given set of points (e.g., obtained by scanning an environment). Our Universal Guard Problem is, in a sense, an extreme version of the problem of guarding a set of possible polygonalizations that are consistent with a given set of sample points that are the polygon vertices: In the universal setting, we require that the guards are a rich enough set to achieve visibility coverage for *all* possible polygonalizations. Another variant studied here is the *k-universal* guarding problem in which the guards must perform visibility coverage for a set of k different polygonalizations of the input points. Further, in the full version of the paper [10], we study the case in which guards are required to be placed at non-convex hull points of S, or at points of a regular rectangular grid.

Related Work

In addition to the worst-case results for the AGP, related work includes algorithmic results for computing a minimum-cardinality guard set. The problem of computing an optimal guard set is known to be NP-hard [24], even in very basic settings such as guarding a 1.5D terrain [21]. Ghosh [13, 14] observed that greedy set cover yields an $O(\log n)$-approximation for guarding with the fewest vertices. Using techniques of Clarkson [5] and Brönnimann-Goodrich [3], $O(\log OPT)$-approximation algorithms were given, if guards are restricted to vertices or points of a discrete grid [7, 8, 15]. For the special case of *rectangle visibility* in rectilinear polygons, an exact optimization algorithm is known [27]. Recently, for vertex guards (or discrete guards on the boundary) in a simple polygon P, King and Kirkpatrick [20] obtained an $O(\log \log OPT)$-approximation, by building ϵ-nets of size $O((1/\epsilon)loglog(1/\epsilon))$ for the associated hitting set instances, and applying [3]. For the special case of guarding 1.5D terrains, local search yields a PTAS [22, 12]. Experiments based on heuristics for computing upper and lower bounds on guard numbers have been shown to perform very well in practice [1]. Methods of combinatorial optimization with insights and algorithms from computational geometry have been successfully combined for the Art Gallery Problem, leading to provably optimal guard sets for instances of significant size [2, 6, 23, 26, 9].

The notion of "universality" has been studied in other contexts in combinatorial optimization [18, 16], including the traveling salesman problem (TSP), Steiner trees, and set cover. For example, in the universal TSP, one desires a single "master" tour on all input points so that, for *any* subset S of the input points, the tour obtained by visiting S in the order specified by the master tour yields a tour that approximates an optimal tour on the subset.

Our Results

We introduce a family of universal coverage problems for the classic Art Gallery Problems. We provide a spectrum of lower and upper bounds for the required numbers of guards. See Table 2 and 3 for a detailed overview, and the following Section 2 for involved notation.

2   Preliminaries

For $n \in \mathbb{N}$, let $\mathcal{S}(n)$ be the set of all discrete point sets in the plane that have cardinality n. A single *shell* of a point set S is the subset of points of S on the boundary of the convex

Table 1 The universal guard numbers considered in this paper.

universal guard numbers	$\boldsymbol{u}(n)$	$\max_{S \in \mathcal{S}(n)} w(\mathcal{P}(S))$
m-shelled universal guard numbers	$\boldsymbol{s}(n,m)$	$\max_{S \in \mathcal{S}(n,m)} w(\mathcal{P}(S))$
interior universal guard numbers	$\boldsymbol{i}(n)$	$\max_{S \in \mathcal{S}(n)} \boldsymbol{i}(\mathcal{P}(S))$
k-universal guard numbers of simple polygons	$\boldsymbol{u}_k(n)$	$\max_{S \in \mathcal{S}(n)} \max_{\substack{A \subseteq \mathcal{P}(S) \\ s.t. \ |A|=k}} w(A)$
k-universal guard numbers of polygons w. holes	$\boldsymbol{h}_k(n)$	$\max_{S \in \mathcal{S}(n)} \max_{\substack{A \subseteq \mathcal{H}(S) \\ s.t. \ |A|=k}} w(A)$
grid universal guard numbers	$\boldsymbol{g}(n)$	$\max_{S \in \mathcal{S}_g(n)} w(\mathcal{P}(S))$

Table 2 Results for simple polygons. The approaches for the upper bounds for $\boldsymbol{u}(n)$ and $\boldsymbol{s}(n,m)$ also apply to polygons with holes, yielding the same upper bounds.

$m, n \in \mathbb{N}$	$\boldsymbol{u}(n)$	$\boldsymbol{s}(n,m)$	$\boldsymbol{g}(n)$	$\boldsymbol{i}(n)$
lower bounds	$\left(1 - \Theta\left(\frac{1}{\sqrt{n}}\right)\right)n$	$\left(1 - \frac{1}{2(m-1)} - \frac{8m}{n(m-1)}\right)n$	$\lfloor \frac{n}{2} \rfloor$	$n - \mathcal{O}(1)$
upper bounds	$\left(1 - \Theta\left(\frac{1}{n}\right)\right)n$	$\left(1 - \frac{1}{16n^{\left(1 - \frac{1}{2m}\right)}}\right)n$	$\lfloor \frac{n}{2} \rfloor$	$n - \Omega(1)$

Table 3 Overview of our results for k-universal guard numbers of simple polygons and of polygons with holes. We give a new corresponding approach for the upper bounds of $\boldsymbol{h}_1(n), \boldsymbol{h}_2(n), \ldots$. We also consider the lower bounds for $\boldsymbol{u}_1(n), \boldsymbol{u}_2(n), \ldots$ as lower bounds for $\boldsymbol{h}_1(n), \boldsymbol{h}_2(n), \ldots$.

$n \in \mathbb{N}$	$\boldsymbol{u}_2(n)$	$\boldsymbol{u}_3(n)$	$\boldsymbol{u}_4(n)$	$\boldsymbol{u}_5(n)$	$\boldsymbol{u}_k(n)$ for $k \geq 6$	$\boldsymbol{h}_k(n)$ for $k \in \mathbb{N}$
lower bounds	$\lfloor \frac{3n}{8} \rfloor$	$\frac{4n}{9}$	$\frac{n}{2} - \mathcal{O}(\sqrt{n})$	$\frac{n}{2} - \mathcal{O}(\sqrt{n})$	$\frac{5n}{9}$	$\frac{5n}{9}$
upper bounds	$\frac{5n}{9}$	$\frac{19n}{27}$	$\frac{65n}{81}$	$\frac{211n}{243}$	$(1 - (\frac{2}{3})^k)n$	$(1 - (\frac{5}{8})^k)n$

hull of S. Recursively, for $k \geq 2$, a point set lies on k shells, if removing the points on its convex hull, leaves a set that lies on $k-1$ shells. We denote by $\mathcal{S}_g(n) \subset \mathcal{S}(n)$ and $\mathcal{S}(n,m) \subset \mathcal{S}(n)$ the set of all discrete point sets that form a rectangular $a \times b$-grid of n points for $a, b, a \cdot b = n \in \mathbb{N}$, and the set of all discrete point sets that lie on m shells for $m \in \mathbb{N}$, respectively.

For $S \in \mathcal{S}(n)$, let $\mathcal{P}(S)$ (resp., $\mathcal{H}(S)$) be the set of all simple polygons (resp., polygons with holes) whose vertex set equals S.

Let P be a polygon. We say a point $p \in P$ sees (w.r.t. P) another point $q \in P$ if $pq \subset P$; we then write $p \leftrightarrow_P q$. The visible region (w.r.t. P) of a point $g \in P$ is $V_P(g) = \{a \in P : g \leftrightarrow_P a\}$. A point set $G \subseteq S$ is a guard set for P if $\bigcup_{g \in G} V_P(g) = P$. Furthermore, we say that G is an interior guard set for P if G is a guard set for P and no $g \in G$ is a vertex of the convex hull of P.

For a set A of polygons we say that $G \subseteq S$ is a(n) (interior) guard set of A if G is a(n) (interior) guard set for each $P \in A$. We denote by $w(A)$ the minimum cardinality guard set for A and by $\boldsymbol{i}(A)$ the minimum cardinality interior guard set for A. Furthermore, for any given point set S we say that $G \subseteq S$ is a guard set for S if G is a guard set for $\mathcal{P}(S)$. For $k, m, n \in \mathbb{N}$, the guard numbers are listed in Table 1.

3 Bounds for Universal Guard Numbers

In the following, we provide different lower and upper bounds for the universal guard numbers. In particular, the provided bounds can be classified by the number of shells on which the points of the considered point set are located.

3.1 Lower Bounds for Universal Guard Numbers

In this section we give lower bounds for the universal guard numbers $\boldsymbol{u}(n)$ and $\boldsymbol{s}(n,m)$ for $n \in \mathbb{N}$ and $m \geq 2$. In particular, we provide lower bound constructions that can be described by the following approach: For any given $n \in \mathbb{N}$ and $m \geq 2$, we construct a point set $S_m \in \mathcal{S}(n)$ as follows. S_m is partitioned into pairwise disjoint subsets $B_1, \ldots, B_m$, such that $\bigcup_{i=1}^m B_i = S$. For $i \in \{1, ..., m\}$, each B_i lies on a circle C_i such that C_i is enclosed by C_{i+1} for $i \in \{1, ..., m-1\}$. Furthermore, $C_1, \ldots, C_m$ are concentric and have "sufficiently large" radii; see Sections 3.1.1, 3.1.2, and 3.1.3 for details. In particular, the radii depend on the approaches that are applied for the different cases $m = 2$, $m = 3$, and $m \geq 4$. We place four equidistant points on C_m. The remaining points are placed on $C_{m-1}, \ldots, C_1$.

Note that $\boldsymbol{s}(n,1) = 1$ holds, because for every convex point set $S \in \mathcal{S}(n)$, $\mathcal{P}(S)$ consists of only the boundary of the convex hull of S. Thus we start with the case of $m = 2$.

3.1.1 Lower Bounds for $s(n, 2)$

We give an approach that provides a lower bound for $\boldsymbol{s}(n, 2)$. In particular, for any $n \in \mathbb{N}$, we construct a point set $S_2 \in \mathcal{S}(n)$ having $n - 4$ equally spaced points lie on circle C_1 and 4 equally spaced points on a larger concentric circle C_2, such that these 4 points form a square containing C_1; see Figure 5. In order to assure that the constructed subsets of S_2 and $S_3, S_4, \ldots$ (which are described later) are nonempty, we require $n \geq 32$ for the rest of Section 3.1.

Let v be a point from the square and let p, q be two consecutive points from the circle C_1, such that the segments vp and vq do not intersect the interior of the circle C_1; see Figure 1(a). We choose the side lengths of the square such that the cone c that is induced by p and q with apex at v contains at most $\frac{n}{8}$ points from C_1 for all choices of v, p, and q.

▶ **Lemma 1.** *Let G be a guard set of S_2. Then we have $|G| > \frac{n}{2} - 4$.*

Proof. Suppose $|G| \leq \lfloor \frac{n-4}{2} \rfloor - 1$. This implies that there are two points $p, q \in S_m \setminus G$ such that p and q lie adjacent on C_1; see Figure 1(b). Let w_1, w_2, w_3, and w_4 be the four points from the square. At most two points $v_1, v_2 \in \{w_1, w_2, w_3, w_4\}$ span a cone, such that $v_1 p, v_1 q, v_2 p, v_2 q$ do not intersect the interior of C_1. W.l.o.g., we assume that these two different cones c_1 and c_2 exist. c_1 and c_2 contain at most $\frac{n}{4}$ points from C. Thus, there is another point $w \in S_2 \setminus G$ such that $v \notin c_1 \cup c_2$. This implies that there is a polygon in which w is not seen by a guard from G; see Figure 1(b). This is a contradiction to the assumption that G is a guard set.

Thus we have $|G| > \lfloor \frac{n-4}{2} \rfloor - 1 \geq \frac{n-4}{2} - 2 = \frac{n}{2} - 4$. This concludes the proof. ◀
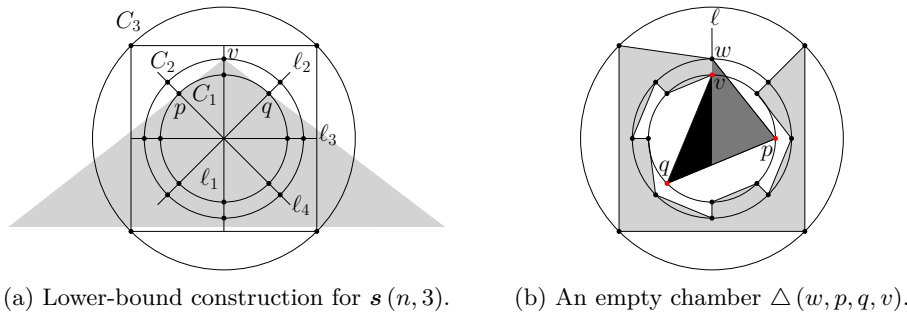
▶ **Corollary 2.** $\boldsymbol{s}(n, 2) \geq \lfloor \frac{n}{2} \rfloor - 4$

3.1.2 A First Lower Bound for $s(n, 3)$

The high-level idea is to guarantee in the construction of S_3 that at most two points on C_1 are unguarded; see Figure 2 for the idea of the proof of contradiction. By constructing

(a)   (b)
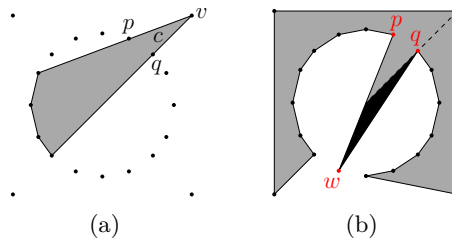
■ **Figure 1** Lower-bound construction for $s(n, 2)$.

(a) Lower-bound construction for $s(n, 3)$.   (b) An empty chamber $\triangle(w, p, q, v)$.

■ **Figure 2** The lower-bound construction for $s(n, 3)$.

$S_3 = B_1 \cup B_2 \cup B_3$ such that $|B_1| = \lfloor \frac{n-4}{2} \rfloor$, $|B_2| = \lceil \frac{n-4}{2} \rceil$, and $|B_3| = 4$, we obtain $|G| \geq \frac{n}{2} - 5$ for any guard set G of S_3.

We consider the lower-bound construction S_m for $m - 1 = 2$ and $n = (m-1)2^l + 4 = 3 \cdot 2^l + 4$ for any $l \geq 4$, i.e., for all $S_3 \in S(2 \cdot 2^l + 4)$ for any $l \geq 2$. The argument can easily be extended to $n \in \mathbb{N}$.

The points of B_2 and B_3 are placed on C_2 and C_3, such that they lie on 2^{l-1} lines; see Figure 2(a). Let $v \in B_2$ be chosen arbitrarily and $p, q \in B_1$ such that p and q are the neighbors of the point from B_1 that corresponds to $v \in B_2$. We choose the radius of C_2 such that the cone that is induced by p and q and with apex at v contains all points from B_1; see the gray cone in Figure 2(a). Furthermore, we choose the radius of C_1 such that the square that is induced by the four points from B_1 contains all points from $B_1 \cup B_2$.

The key construction that we apply in the proofs of our lower bounds are *chambers*.

▶ **Definition 3.** Let S be an arbitrary discrete point set in the plane. Four points $p_1, p_2, p_3, p_4 \in S$ form a *chamber, denoted* $\triangle(p_1, p_2, p_3, p_4)$, if (1) p_1 and p_2 lie on different sides of the line $p_3 p_4$ and (2) p_3 and p_4 lie on the same side of the line $p_1 p_2$, and (3) there is no point from S that lies inside the polygon that is bounded by the polygonal chain $\langle p_1, p_2, p_3, p_4 \rangle$.

Let $G \subseteq S$. We say that $\triangle(p_1, p_2, p_3, p_4)$ is *empty* (w.r.t. G) if $p_2, p_3, p_4 \notin G$. Let $P \in \mathcal{P}(S)$. We say that $\triangle(p_1, p_2, p_3, p_4)$ *is part of* P if $p_1 p_2, p_2 p_3, p_3 p_4 \subset \partial P$.

Our proofs are based on the following simple observation.

▶ **Observation 4.** *Let G be a guard set for a polygon P. There is no empty chamber that is part of P.*

Based on Observation 4 we prove the following lemma, which we then apply to the construction above to obtain our lower bound for $s(n, m)$.

▶ **Lemma 5.** *Let G be a guard set for $\mathcal{P}(S_3)$. Then we have $|B_1 \setminus G| \leq 2$.*

Proof. Suppose there are three points $v, q, p \in B_1 \setminus G$. W.l.o.g., we assume that q and p lie on different sides w.r.t. the line ℓ that corresponds to the placement of v; see Figure 2(b). Furthermore, we denote the point from B_2 that lies above v by w. By construction it follows that w, p, q, and v form an empty chamber $\triangle(w, p, q, v)$. Furthermore, we construct a polygon $P \in \mathcal{P}(S_3)$ such that $\triangle(w, p, q, v)$ is part of P; see Figure 2(b). By Observation 4 it follows that G is not a guard set for P, a contradiction. This concludes the proof.   ◄

There is a corresponding construction for all other values $n \in \mathbb{N}$. In particular, we place four points equidistant on C_3, $\lceil \frac{n-4}{2} \rceil$ equidistant points on C_2, and $\lfloor \frac{n-4}{2} \rfloor$ points on C_1, such that each point from C_1 lies below a point from C_2. The same argument as above applies to the resulting construction of a point set. The constructions of S_m can be modified so that no three points lie on the same line, by a slight perturbation. Thus, S_3 can be assumed to be in general position. We obtain the following corollary.

▶ **Corollary 6.** $s(n, 3) \geq \frac{n}{2} - 5$.

Proof. Lemma 5 implies that in the construction S_3 at least $\lfloor \frac{n-4}{2} \rfloor - 2$ points from B_1 are guarded. Let G be an arbitrarily chosen guard set for $\mathcal{P}(S_3)$. Thus we obtain $|G| \geq \lfloor \frac{n-4}{2} \rfloor - 2 \geq \frac{n-4}{2} - 3 = \frac{n}{2} - 5$.   ◄

In the following section we generalize the above approach from the case of three shells to the case of m shells and combine that argument with the approach that we applied for the case of $m = 2$. This also leads to the improved lower bound $\boldsymbol{u}_3(n) \geq (\frac{3}{4} - \mathcal{O}(\frac{1}{n}))n$.

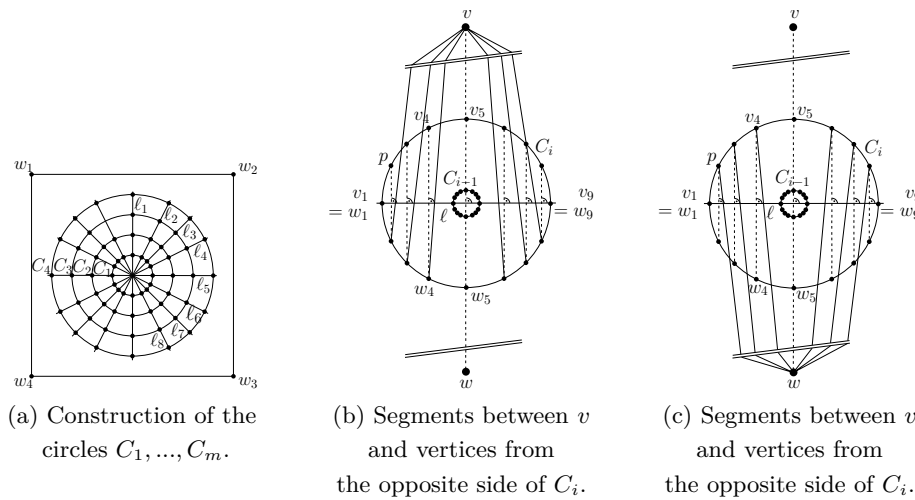### 3.1.3   (Improved) Lower Bounds for $u(n)$ and $s(n, m)$ for $m \geq 3$

In this section we give general constructions $S_3, S_4, \ldots$ of the point sets that yield our lower bounds for $\boldsymbol{s}(n, m)$ for $m \geq 3$. The main difference in the construction of S_m for $m \geq 3$, compared to the previous section, is the choice of the radii of $C_1, ..., C_m$. Similar as in the previous section, we guarantee that on each circle $C_3, C_4, \ldots$ at most constant many points are unguarded. Roughly speaking, the general idea is to choose five arbitrary points q_1, q_2, q_3, q_4, q_5 on C_i for $i \in \{3, 4, \ldots\}$. There are three points $u_1, u_2, u_3 \in \{q_1, q_2, q_3, q_4, q_5\}$, such that the triangle induced by u_1, u_2, u_3 does not contain the common mid point of $C_1, C_2, \ldots$. By choosing the radius of C_{i+1} sufficiently large, we obtain that there is a chamber $\triangle(u_1, u_2, u_3, p)$, where p is a point on C_{i+1}. This implies that $\triangle(u_1, u_2, u_3, p)$ is empty if q_1, q_2, q_3, q_4, q_5 are unguarded. Thus, at most four points on C_i are allowed to be unguarded; see Corollary 9.

Finally, we show how the arguments for S_m yield lower bounds for $\boldsymbol{s}(n, m)$ and $\boldsymbol{u}(n)$.

Similar to the approach of the previous section, the constructed point sets $S_3, S_4, \ldots$ can be modified to be in general position.

The Construction of S_m for $m \geq 3$:   We construct S_m such that $|B_1| = \cdots = |B_{m-1}| = 2^l$, $|B_m| = 4$, and hence $n = (m-1)2^l + 4$ for $l \geq 4$. In particular, similar as for the construction of S_3 from the previous section, we place the points of $B_1, \ldots, B_{m-1}$ equidistant on the circles $C_1, \ldots, C_{m-1}$, such that the points lie on 2^{l-1} lines $\ell_1, \ldots, \ell_{2^{l-1}}$; see Figure 3(a).

In order to apply an argument that makes use of chambers, we need the following notation of points on a circle C_i. Let $n' := 2^l$. Let $v_1, ..., v_{1+n'/2}$ be the points on C_i to one side or on $\ell \in \{\ell_1, ..., \ell_{n'/2}\}$. Let $w_1, ..., w_{1+n'/2}$ be their reflection across ℓ; see Figure 3(b)+(c). Let $v_1, ..., v_{1+n'/2}$ and $w_1, ..., w_{1+n'/2}$ be the points that lie not below and not above ℓ; see Figure 3(b)+(c). Let $v, w \in C_{i+1}$ be the points that correspond to $v_{1+n'/4}$ and $w_{1+n'/4}$.

(a) Construction of the circles $C_1, ..., C_m$.

(b) Segments between v and vertices from the opposite side of C_i.

(c) Segments between v and vertices from the opposite side of C_i.

Figure 3 Construction of S_m for $n = 68$. For a simplified illustration we changed the ratios of the circles' radii and we shortened the lines adjacent to v.

For $i \in \{1, \ldots, m-1\}$, we choose the radius of C_{i+1} compared to the radius of C_i sufficiently large, such that the following conditions are fulfilled; see Figure 3(b)+(c):

- vw_j intersects $v_j v_{j+1}$ in its interior for all $j \in \{1, ..., n/4 + 1\}$,
- vw_j intersects $v_{j-1} v_j$ in its interior for all $j \in \{n/4 + 2, ..., n/2 + 1\}$,
- wv_j intersects the segment $w_j w_{j+1}$ in its interior for all $j \in \{1, ..., n/4 + 1\}$, and
- wv_j intersects the segment $w_{j-1} w_j$ in its interior for all $j \in \{n/4 + 2, ..., n/2 + 1\}$.

Finally, we place the four points $w_1, w_2, w_3, w_4 \in B_m$ such that all circles lie in the convex hull of w_1, w_2, w_3, and w_4; see Figure 3(a).

The Analysis of S_m for $m \geq 3$: First we show that we can choose three points u_1, u_2, u_3 from five arbitrarily chosen points from C_i, such that there is another point $u \in C_{i+1}$ with $\triangle(u, u_1, u_2, u_3)$ being a chamber; see Lemma 7. Next, we construct a polygon $P \in \mathcal{P}(S_m)$, such that $\triangle(u, u_1, u_2, u_3)$ is a part of P; see Lemma 8. Finally, by combining Lemma 7 and Lemma 8 we establish that on each C_i, at most four points are allowed to be unguarded; see Corollary 9. This leads to several lower bounds for $s(n, m)$ and $u(n)$.

▶ **Lemma 7.** *Let $q_1, q_2, q_3, q_4, q_5 \in A_i$ be chosen arbitrarily. There are three points $u_1, u_2, u_3 \in \{q_1, q_2, q_3, q_4, q_5\}$ and a point $u \in A_{i+1}$, such that $\triangle(u, u_1, u_2, u_3)$ is a chamber.*

Proof. We choose u_1, u_2, u_3 from $\{q_1, q_2, q_3, q_4, q_5\}$, such that u_1, u_2, u_3 lie in the same half of C_i, i.e., such that the midpoint of C_i does not lie inside the triangle t that is induced by u_1, u_2, u_3; see Figure 4. W.l.o.g., we assume that u_2 lies between u_1 and u_3. Otherwise, we rename the points appropriately.

We distinguish two cases. (C1) The number of points between u_1 and u_3 is odd and (C2) the number of points between u_1 and u_3 is even. For (C1) and (C2) we use different chambers for achieving the required contradiction; see Figure 4. A detailed analysis can be found in the full paper [10]. ◀

▶ **Lemma 8.** *There is a polygon $P \in \mathcal{P}(S_m)$ such that $\triangle(u, u_1, u_2, u_3)$ is part of P.*
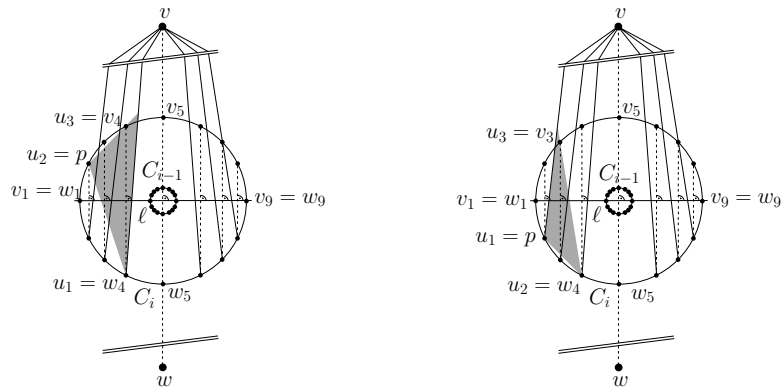
(a) Chambers for (C1).
of points between u_1 and u_2 is odd.

(b) Chambers for (C2).
of points between u_1 and u_2 is even.

Figure 4 Configuration of Lemma 7: three points from C_i in the same half of C_i imply a chamber.
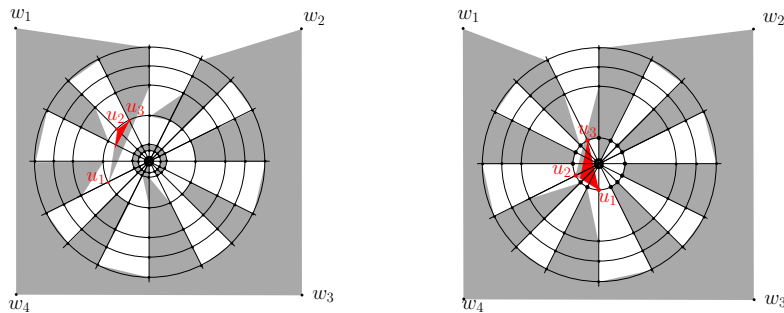
(a) The case in which the number
of points between u_1 and u_2 is odd.

(b) The case in which the number
of points between u_1 and u_2 is even.

Figure 5 Construction of $\mathcal{P}$ for $k = 6$ and $n = 16$. For a simplified illustration we changed the ratios of the circles' radii (otherwise the figure would become too large).

Proof. We construct P for the cases (C1) and (C2) separately; see Figure 5. In both cases we walk upwards on the line $\ell \in \{\ell_1, \ldots, \ell_{n'/2}\}$ until we reach C_1. Next we orbit C_i in a zig-zag approach and finally connect all points from $C_{i-1}, \ldots, C_1$ in a similar manner; see Figure 5. ◀

The combination of Lemma 7 and Lemma 8 implies the following corollary.

▶ **Corollary 9.** *Let $G \subset S_m$ be a guard set of $\mathcal{P}(S_m)$. Then $|B_i \setminus G| \leq 4$, for $i \in \{1, \ldots, m-2\}$.*

Lower bounds for $s(n, m)$ and $u(n)$ which are implied by Corollary 9: We combine the approach for $s(n, 2)$ with Corollary 9, which yields the following lower bound for $s(n, m)$ for $m \geq 3$.

▶ **Corollary 10.** *Let $m \geq 3$ and $n' = 2^l$ with $l \geq 4$. Furthermore, let $G \subset S_m$ be a guard set of S_m. Then we have $|G| \geq \left(1 - \frac{1}{2(m-1)} + \frac{8m}{n(m-1)}\right)|S_m|$.*

Proof. By Corollary 9 it follows that $(m-2)(n'-4)$ points from $B_1 \cup \cdots \cup B_{m-2}$ are guarded. Furthermore, by applying the approach of Lemma 1 to B_{m-1} and B_m yields that at least

$\frac{n'}{2} - 4$ points from $B_{m-1} \cup B_m$ are guarded. Thus we obtain $|G| \geq (m-2)(n'-4) + \frac{n'}{2} - 4$ which is upper-bounded by $|S_m| \left(1 - \frac{1}{2(m-1)} - \frac{8m}{|S_m|(m-1)}\right)$ because $n' = \frac{|S_m|-4}{m-1}$.   ◄

▶ **Theorem 11.** $s(n, m) \geq n \left(1 - \frac{1}{2(m-1)} + \frac{8m}{n(m-1)}\right)$ *for* $m \geq 3$.

By choosing m appropriately, we obtain the following lower bound:

▶ **Lemma 12.** *For any $c < 1$ and any guard set G for S_m there is an $m \in \mathbb{N}$ with $|G| > c|S_m|$.*

Proof. The approach is to choose $m := \lceil \frac{2n'}{n'-4-cn'} \rceil$, which will imply $|G| > c|S_m|$.

Suppose $|G| \leq c|S_m|$. Corollary 9 implies that at most four points on each circle $C_i \in \{C_1, ..., C_k\}$ are unguarded. This leads to a contradiction as follows. We have $|S_m| = 4 + (m-1)n'$. On $C_1, ..., C_{m-2}$ there are at most four vertices that are unguarded. W.l.o.g., we assume that w_1, w_2, w_3, w_4, and all points on C_m are unguarded. Thus, $|G| \geq (m-2)(n'-4)$. By assumption we know $|G| \leq c(4 + (m-1)n')$. By applying $m = \lceil \frac{2n'}{n'-4-cn'} \rceil$, we obtain a contradiction as follows: $(m-2)(n'-4) \leq c(4 + (m-1)n')$ implies that $8 \leq 4$, since $m = \lceil \frac{2n'}{n'-4-cn'} \rceil$.   ◄

By choosing c appropriately, Lemma 12 leads to our general upper bound for $u(n)$.

▶ **Theorem 13.** *There is an $m \in \mathbb{N}$ such that $|G| > (1 - \frac{10}{\sqrt{|S_m|}})|S_m|$ holds for any guard set G for $\mathcal{P}(S_m)$.*

Proof. Choose $c := (1 - \frac{5}{n'})$ in the approach of Lemma 12. This implies that at least $(1 - \frac{5}{n'})|S_m|$ points have to be guarded. Furthermore, we have $|S_m| = 4 + (m-1)n'$ and $m = \lceil \frac{2n'}{n'-4-cn'} \rceil$. This implies $m \leq \frac{2n'}{n'-4-\left(1-\frac{5}{n'}\right)n'} + 1 = 2n' + 1$. Furthermore, $|S_m| \leq 4 + 2(n')^2$ implies $\sqrt{|S_m|/2}/ - 1 \leq n$. Finally, applying Lemma 12 yields $|G| > \left(1 - \frac{5\sqrt{2}}{\sqrt{|S_m|-1}}\right)|S_m| > \left(1 - \frac{10}{\sqrt{|S_m|}}\right)|S_m|$.   ◄

▶ **Theorem 14.** $u(n) \geq \left(1 - \frac{10}{\sqrt{n}}\right)n$.

3.2 Upper Bounds for Universal Guard Numbers

In the following we give an approach to computing a non-trivial guard set of a given point set. The number of the computed guards depends on the number m of shells of the considered point set S. This approach yields upper bounds for $s(n, m)$ for $m \geq 2$.
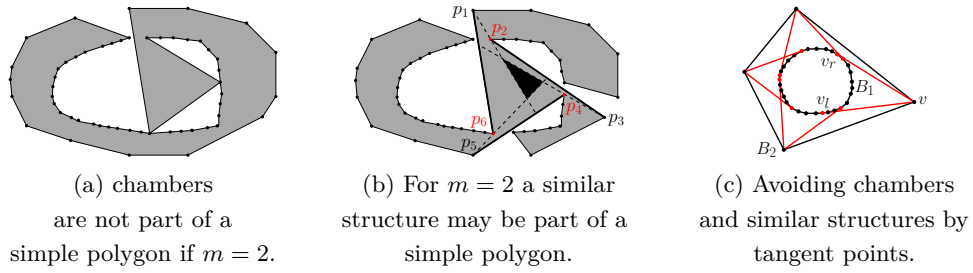
For the case of $m = 1$, a naïve approach is simply to select one arbitrarily chosen guard from S. In that case, $\mathcal{P}(S)$ just consists of the polygon that corresponds to the boundary of the convex hull of S and an arbitrarily chosen point from S sees all points from all polygons of $\mathcal{P}(S)$.

In the following, we first give an approach for the case of $m = 2$. Then, we generalize that approach to the case of $m \geq 3$.

3.2.1 Upper Bounds for $s(n, 2)$

First we describe the approach, followed by showing that the computed point set G is a guard set for the considered point set. This leads to upper bounds for $|G|$ which imply the required upper bounds for $s(n, m)$.

The high-level idea is to avoid areas that are unguarded by structures similar to chambers. In particular, in the case of $m = 2$, a chamber cannot be part of a simple polygon; otherwise,

(a) chambers
are not part of a
simple polygon if $m = 2$.

(b) For $m = 2$ a similar
structure may be part of a
simple polygon.

(c) Avoiding chambers
and similar structures by
tangent points.

Figure 6 Possible chambers in case of two shells and how we avoid them.

the boundary of P meets points at least twice (Figure 6(a)). However, there is another structure that has an effect similar to that of chambers and that also may cause unguarded areas; see Figure 6(b). In the example of Figure 6(b), our approach guarantees that p_2 or p_6, p_2 or p_4, and p_4 or p_6 is guarded. More generally, for p_1, p_3, and p_5 we guarantee that the unguarded points lie on one side w.r.t. the tangent points; see Figure 6(c).

In particular, let B_1 be the points on the inner shell and B_2 be the points on the outer shell of the input point set S. If $|B_2| \geq \sqrt{|B_1|}/2$ we set $G = B_1$. Otherwise, we choose all points from B_2 and every second point from B_1. Furthermore, we compute for each $v \in B_2$ the two tangent points v_l and v_r to B_1 (see Figure 6(c)) and insert v_l and v_r into G. Let $\langle v_1, \ldots, v_k \rangle \subset B_1$ be a sequence of maximal length that does not contain any tangent point as previously computed. We insert all remaining points from $B_1 \setminus \{v_1, \ldots, v_k\}$ that were not already inserted in G.

▶ **Theorem 15.** *For each point set S that lies on two convex hulls, we can compute in* $\mathcal{O}(|S| \log |S|)$ *time a guard set G with* $|G| \leq (1 - \frac{1}{\sqrt{8|S|}})|S|$.

▶ **Corollary 16.** $s(n, 2) \leq (1 - \frac{1}{\sqrt{8n}})n$

3.2.2   Upper Bounds for $s(n, m)$ for $m \geq 3$

In this section we generalize the above approach to the case of $m \geq 3$.

Let $B_1, \ldots, B_m$ be the pairwise disjoint subsets of S that lie on the m shells of S. The high-level idea of the approach is the following. If B_m is "large enough" (larger than a value λ), we set $G = \bigcup_{\ell=1}^{m-1} B_\ell$. Otherwise, we carefully choose one subset B_j for $j \in \{1, \ldots, m\}$ and select partially its points as unguarded. All the remaining points are selected for G. In particular, we set $\bigcup_{\ell \in \{1, \ldots, m\} \setminus \{j\}} B_\ell \subset G$. Then, we compute the tangent points on B_j for all points from $\bigcup_{\ell=j+1}^{m} B_\ell$. Finally, we apply the same subroutine as in the case $m = 2$.

We choose $j := \arg\max_{\ell \in \{1, \ldots, m-1\}} \left(\frac{n_\ell}{2 \sum_{i=\ell+1}^{m} n_i} - 1 \right)$ and $\lambda := \frac{n_j}{2 \sum_{i=j+1}^{m} n_i} - 1$. We refer to the full paper for the detailed steps of the approach.

By applying a similar argument as for the case of $m = 2$, we can show that the computed point set $G \subseteq S$ is a guard set for $\mathcal{P}(S)$. For details, see the full paper.

▶ **Theorem 17.** *For any point set S that lies on m convex hulls we can compute in* $\mathcal{O}(n \log n)$ *time a guard set G with* $|G| \leq \left(1 - \frac{1}{16|S|^{\left(1 - \frac{1}{2m}\right)}} \right) |S|$

This leads to our generalized upper bound for $s(n, m)$ for $m \geq 3$.

▶ **Corollary 18.** $s(n, m) \leq \left(1 - \frac{1}{16n^{\left(1 - \frac{1}{2m}\right)}} \right) n$.

4   Bounds for the k-Universal Guard Numbers

In the following we state several lower and upper bounds for various k-universal guard numbers; proof details are in the full paper.

4.1   Lower bounds for $u_k(n)$

▶ **Theorem 19.** $u_2(n) \geq \lfloor \frac{3n}{8} \rfloor$

▶ **Theorem 20.** $u_3(n) \geq \lfloor \frac{4n}{9} \rfloor$.

▶ **Theorem 21.** $u_5(n) \geq u_4(n) \geq \frac{n}{2} - 8\sqrt{n} - 23$.

▶ **Theorem 22.** $u_k(n) \geq \lfloor \frac{5n}{9} \rfloor$ for $k \geq 6$.

4.2   Upper Bounds for k-Universal Guard Numbers

We give non-trivial upper bounds for $u_k(n)$ and $h_k(n)$, for all values $n, k \in \mathbb{N}$. In particular, we provide algorithms that efficiently compute guard sets for $\mathcal{P}(S)$ and $\mathcal{H}(S)$ for any given $S \in \mathcal{S}(n)$ and analyze the computed guard sets.

▶ **Theorem 23.** $u_k(n) \leq \left(1 - \left(\frac{2}{3}\right)^k\right)$.

Hoffmann et al. [17] showed $h_1(n) \leq \lfloor \frac{3n}{8} \rfloor$. Our approach implies for the traditional guard number $h_1(n) \leq \lfloor \frac{n}{2} \rfloor$.

The following theorem shows that we can combine our approach with the method from [17].

▶ **Theorem 24.** $h_k(n) \leq \left(1 - \left(\frac{5}{8}\right)^k\right)n$.

5   Other Variants

Due to limited space, we state two variants of the Universal Art Gallery Problem but defer the technical details to the full paper.

5.1   Interior Guards

In the Interior Universal Guards Problem (UGPI) we allow guards to be placed only at points of S that are not convex hull vertices of S. For this case, we obtain an asymptotically tight bound on the number of universal guards:

▶ **Theorem 25.** $i(n) = n - \Theta(1)$

5.2   Full Grid Sets

A natural special case arises when considering universal guards for a full set of $n = a \times b$ grid points on an integer lattice. We are also able in this case to achieve a tight worst-case bound:

▶ **Theorem 26.** $g(n) = \lfloor \frac{n}{2} \rfloor$.

6   Conclusion

There are many open problems that are interesting challenges for future work. In particular, can the upper bound approaches for $\boldsymbol{u}_k(n)$ and $\boldsymbol{h}_k(n)$ be improved by making use of the number of shells? Can the general approach of Theorem 23 be improved? What about lower bounds for k-UGP for $k \geq 7$?

The quest for better bounds is also closely related to other combinatorial challenges. Is an instance of the 2-UGP 5-colorable? If so, our results give a first trivial upper bound of $\frac{3}{5}n$ for the 2-UGP, which would be of independent interest. Is the bound of $\frac{1}{2}n$ for the intersection-free k-UGP tight? Further questions consider the setting in which each vertex v has a bounded candidate set of vertices that may be adjacent to v. Other variants arise when the ratio of the lengths of the edges of the considered polygons is upper- and lower-bounded by given constants. It may also be interesting to explore possible relations between universal guard problems and universal graphs.

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