Clustered Planarity with Pipes*

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Abstract

We study the version of the C-Planarity problem in which edges connecting the same pair of clusters must be grouped into pipes, which generalizes the Strip Planarity problem. We give algorithms to decide several families of instances for the two variants in which the order of the pipes around each cluster is given as part of the input or can be chosen by the algorithm.

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1 Introduction

Visualizing clustered graphs is a challenging task with several applications in the analysis of networks that exhibit a hierarchical structure. The most established criterion for a readable visualization of these graphs has been formalized in the notion of *c-planarity*, introduced by Feng, Cohen, and Eades [12] in 1995. Given a *clustered graph* $\mathcal{C}(G, \mathcal{T})$ (*c-graph*), that is, a graph G equipped with a recursive clustering $\mathcal{T}$ of its vertices, problem C-Planarity asks whether there exist a planar drawing of G and a representation of each cluster as a topological disk enclosing all and only its vertices, such that no *"unnecessary"* crossings occur between disks and edges, or between disks. Ever since its introduction, this problem has been attracting a great deal of research. However, the question about its computational complexity withstood the attack of several powerful algorithmic tools, as the Hanani-Tutte theorem [13, 15], the SPQR-tree machinery [9], and the Simultaneous PQ-ordering framework [5].

The clustering of a c-graph $\mathcal{C}(G, \mathcal{T})$ is described by a rooted tree $\mathcal{T}$ whose leaves are the vertices of G and whose each internal node μ, except for the root, represents a *cluster* containing all and only the leaves of the subtree of $\mathcal{T}$ rooted at μ. A c-graph is *flat* if $\mathcal{T}$ has height 2. The *clusters-adjacency graph* G_A of a flat c-graph is the graph obtained by contracting each cluster into a single vertex and by removing multi-edges and loops.

Cortese *et al.* [10] introduced a variant of C-Planarity for flat c-graphs, which we call C-Planarity with Embedded Pipes, whose input is a flat c-graph together with a planar drawing of its clusters-adjacency graph, where vertices are represented by disks and edges by pipes. The goal is to produce a c-planar drawing in which each vertex lies inside the disk representing the cluster it belongs to and each inter-cluster edge lies inside the corresponding

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pipe. In [10] this problem is solved when the underlying graph is a cycle. Chang, Erickson, and Xu [8] observed that in this case the problem is equivalent to determining whether a closed walk of length n in a simple plane graph is weakly simple, and improved the time complexity to $O(n \log n)$. For the special case in which the clusters-adjacency graph is a path, known by the name of Strip Planarity, there exist polynomial-time algorithms when the underlying graph has a fixed planar embedding [2] and when it is a tree [13].

We remark that polynomial-time algorithms for the C-Planarity problem are known under strong limitations on the number or on the arrangement of the components in the clusters. A *component* of a cluster is a maximal connected subgraph induced by its vertices. In particular, C-Planarity can be decided in linear time when each cluster contains one connected component [9, 12] (the c-graph is *c-connected*). However, even when each cluster contains at most two connected components, polynomial-time algorithms are known only when further restrictions are imposed on the c-graph [5, 14]. The results we show in this paper are also based on imposing constraints on the number of certain types of components.

A component is *multi-edge* if it is incident to at least two inter-cluster edges, otherwise it is *single-edge*. Also, it is *passing* if it is adjacent to vertices belonging to at least two other clusters in $\mathcal{T}$, otherwise it is *originating*. For Strip Planarity the originating components can be further distinguished into *source* and *sink* components, based on whether the inter-cluster edges incident to them only belong to the lower or to the upper strip.

Our contributions. We give polynomial-time algorithms for instances of Strip Planarity with a unique source component (Section 3) and for instances of C-Planarity with Embedded Pipes with certain combinations of originating and passing multi-edge components in the clusters (Section 4). Finally, in Section 5 we introduce a generalization of C-Planarity with Embedded Pipes, which we call C-Planarity with Pipes, in which the inter-cluster edges are still required to be grouped into pipes, but the order of the pipes around each disk is not prescribed by the input. By introducing a new characterization of C-Planarity, we give an FPT algorithm for C-Planarity with Pipes that runs in $g(K, c) \cdot O(n^2)$ time, with $g(K, c) \in O(K^{c(K-2)})$, where K is the maximum number of multi-edge components in a cluster and c is the number of clusters with at least two multi-edge components. We remark that our results imply polynomial-time algorithms for all the three problems in the case in which each cluster contains at most two components.

Due to space limitations, complete proofs are deferred to the full version of the paper [1].

2 Preliminaries

For the standard definitions on planar graphs, planar drawings, planar embeddings, and connectivity we point the reader to [11]. We call *rotation scheme* the clockwise circular ordering of the edges around each vertex in a planar embedding, and refer to the containment relationships between vertices and cycles in the embedding as *relative positions*. Also, if block of a 1-connected graph consists of a single edge, we call it *trivial*, otherwise *non-trivial*.

PQ-trees. A *PQ-tree* [7] T is an unrooted tree, whose leaves are the elements of a set A and whose internal nodes are either *P-nodes* or *Q-nodes*, that can be used to represent all and only the circular orderings $\mathcal{O}(T)$ on A satisfying a given set of *consecutivity constraints* on subsets of A. The orderings in $\mathcal{O}(T)$ are all and only the circular orderings on the leaves of T obtained by arbitrarily ordering the neighbours of each P-node and by arbitrarily selecting for each Q-node a given circular ordering on its neighbours or its reverse ordering.

Connectivity. A k*-cut* of a graph is a set of at most k vertices whose removal disconnects the graph. A graph with no 1-cut is *biconnected*. The maximal biconnected components of a graph are its *blocks*. Without loss of generality, we will assume that the clusters-adjacency graph of $\mathcal{C}(G, \mathcal{T})$ is connected and that for every component c of every cluster $\mu \in \mathcal{T}$:

(i) there exists at least an inter-cluster edge incident to c,

(ii) every block of c that is a leaf in the block-cut-vertex tree of c contains at least a vertex v such that v is not a cut-vertex of c and it is incident to at least an inter-cluster edge, and

(iii) if there exists exactly one vertex in c that is incident to inter-cluster edges, then c consists of a single vertex.

Simultaneous Embedding with Fixed Edges. Given planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, problem SEFE asks whether there exist planar drawings Γ_1 of G_1 and Γ_2 of G_2 such that (i) any vertex $v \in V$ is mapped to the same point in Γ_1 and Γ_2 and (ii) any edge $e \in E_1 \cap E_2$ is mapped to the same curve in Γ_1 and Γ_2. Graphs $G_\cap = (V, E_1 \cap E_2)$ and $G_\cup = (V, E_1 \cup E_2)$ are the *common* and the *union* graph, respectively. See [4] for a survey.

We state here a theorem on SEFE that will be fundamental for our results. Even though this theorem has never been explicitly stated in the literature, it can be easily deduced from known results [6]. We discuss this in the full version of the paper [1].

▶ **Theorem 1.** *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two planar graphs whose common graph $G_\cap = (V, E_1 \cap E_2)$ is a forest and whose cut-vertices are incident to at most two non-trivial blocks. It can be tested in $O(|V|^2)$ time whether $\langle G_1, G_2 \rangle$ admits a SEFE.*

3 Single-source Strip Planarity

In this section we prove a result of the same flavour as that by Bertolazzi *et al.* [3] for the upward planarity testing of single-source digraphs. Namely, we show that instances of Strip Planarity with a unique source component can be tested efficiently. The Strip Planarity problem takes in input a planar graph $G = (V, E)$ and a mapping $\gamma : V \to \{1, \dots, k\}$ of each vertex to one of k unbounded horizontal strips such that, for any edge $(u, v) \in E$, it holds $|\gamma(u) - \gamma(v)| \leq 1$. The goal is to find a planar drawing of G in which vertices lie inside the corresponding strips and edges cross the boundary of any strip at most once. This problem is equivalent to C-Planarity with Embedded Pipes when G_A is a path [2].

We start with an auxiliary lemma. An instance $\langle G, \gamma \rangle$ of Strip Planarity on $k > 1$ strips is *spined* if there exists a path $(v_1, \dots, v_k)$ in G such that $\gamma(v_i) = i$, vertex v_k is the unique vertex in the k-th strip, and each vertex v_i with $i \neq 1$ induces a component in the i-th strip. Path $(v_1, \dots, v_k)$ is the *spine path* of $\langle G, \gamma \rangle$ and (v_i, v_{i+1}) is the i-th edge of this path.

▶ **Lemma 2.** *Any positive spined instance $\langle G, \gamma \rangle$ of* Strip Planarity *admits a strip-planar drawing in which the intersection point between the first edge of the spine path of $\langle G, \gamma \rangle$ and the horizontal line separating the first and the second strip is the left-most intersection point between any inter-strip edge and such a line.*

▶ **Lemma 3.** *Let $\langle G = (V, E), \gamma \rangle$ be a spined instance of* Strip Planarity *on $k > 1$ strips with a unique source component c. It is possible to construct in linear time an equivalent spined instance $\langle G' = (V', E'), \gamma' \rangle$ on $k - 1$ strips with a unique source component c'.*

Proof Sketch. First note that the source component c lies in the first strip. We construct an auxiliary planar graph G_c as follows. Initialize $G_c = c$ and add a dummy vertex v to it.

For each inter-strip edge e incident to a vertex u in c, add to G_c a dummy vertex v_e and edges (v, v_e) and (v_e, u). If G_c has cut-vertices, then let B_c be the block of G_c that contains v. Then, construct a PQ-tree $\mathcal{T}_c$ representing all possible orders of the edges around v in a planar embedding of B_c. This can be done by applying the planarity testing algorithm by Booth and Lueker [7], in such a way that v is the last vertex of the *st*-numbering of B_c. Note that the leaves of PQ-tree $\mathcal{T}_c$ are in one-to-one correspondence with the vertices v_e in B_c. We construct a *representative graph* $G_{\mathcal{T}_c}$ from $\mathcal{T}_c$, as described in [12], composed of (i) *wheel* graphs (that is, graphs consisting of a cycle, called *rim*, and of a *central vertex* connected to every vertex of the rim), of (ii) edges connecting vertices of different rims not creating any simple cycle that contains vertices belonging to more than one wheel, and of (iii) vertices of degree 1, which are in one-to-one correspondence with the leaves of $\mathcal{T}_c$ (an hence with the dummy vertices v_e in B_c), each connected to a vertex of some rim. As proved in [12], in any planar embedding of $G_{\mathcal{T}_c}$ in which all the degree-1 vertices are incident to the same face, the order in which such vertices appear in a Eulerian tour of such a face is in $O(\mathcal{T}_c)$.

Construct $\langle G', \gamma' \rangle$ as follows. For $i = 2, \dots, k$ and for each vertex v with $\gamma(v) = i$, add v to V' and set $\gamma'(v) = i - 1$, that is, assign all the vertices of the i-th strip of $\langle G, \gamma \rangle$, with $i \geq 2$, to the $(i-1)$-th strip of $\langle G', \gamma' \rangle$. Further, add to E' all edges in $E \cap (V' \times V')$. Also, add all vertices and edges of $G_{\mathcal{T}_c}$ to V' and to E', respectively, and set $\gamma'(u) = 1$ for each vertex u of $G_{\mathcal{T}_c}$. Finally, for each inter-strip edge $e = (x, y)$ in E with $\gamma(x) = 1$ and $\gamma(y) = 2$, add to E' an intra-strip edge between y and the degree-1 vertex of $G_{\mathcal{T}_c}$ corresponding to v_e.

Instance $\langle G', \gamma' \rangle$ can be constructed in linear time [7, 12] and its size is linear in the one of $\langle G, \gamma \rangle$. Further, $\langle G', \gamma' \rangle$ has a unique source component, which contains $G_{\mathcal{T}_c}$ as a subgraph, and is spined. We now show the equivalence between the two instances.

Suppose that $\langle G, \gamma \rangle$ admits a strip-planar drawing Γ. Note that all the vertices of c incident to inter-strip edges lie on the outer face of c in Γ. To construct a strip-planar drawing Γ' of $\langle G', \gamma' \rangle$, subdivide each inter-strip edge incident to c with a dummy vertex v_e lying in the interior of the first strip of Γ. By the construction of $\mathcal{T}_c$ and of $G_{\mathcal{T}_c}$, each vertex v_e corresponds to exactly one degree-1 vertex of $G_{\mathcal{T}_c}$. Let c^+ be the subgraph of G induced by the vertices in c and by all the vertices v_e. Since the order in which the vertices v_e appear in a Eulerian tour of the outer face of c^+ in Γ is in $O(\mathcal{T}_c)$, we can replace the drawing of c^+ in Γ with a drawing of $G_{\mathcal{T}_c}$ in which each degree-1 vertex is mapped to its corresponding vertex v_e. To obtain Γ', we merge the first two strips of Γ into the first strip of Γ'.

Suppose that $\langle G', \gamma' \rangle$ admits a strip-planar drawing Γ', we show how to construct a strip-planar drawing Γ of $\langle G, \gamma \rangle$. First, by Lemma 2, we can assume that in Γ' the intersection point between the first edge of the spine path of $\langle G', \gamma' \rangle$ and the line separating the first and the second strip in Γ' is the left-most intersection point between any edge (x, y) with $\gamma(x) = 1$ and $\gamma(y) = 2$ and such a line. Further, we can assume the following.

▶ **Claim 4.** *For every wheel W in $G_{\mathcal{T}_c}$, the rim of W contains in its interior its central vertex and no other vertex in Γ'.*

Initialize Γ as the drawing in Γ' of the subinstance of $\langle G', \gamma' \rangle$ induced by the vertices not in $G_{\mathcal{T}_c}$, where the i-th strip in Γ' is mapped to the $(i+1)$-th strip in Γ. First, draw $G_{\mathcal{T}_c}$ in the first strip of Γ as it is drawn in Γ'. Then, draw each inter-strip edge (x, y) with y in $G_{\mathcal{T}_c}$, which corresponds to an intra-strip edge incident to $G_{\mathcal{T}_c}$ in Γ', as a curve composed of six parts. The first part coincides with the drawing of (x, y) in Γ'; the second is a curve arbitrarily close to the drawing in Γ' of a path in $G_{\mathcal{T}_c}$ from y to the first vertex v_1 of the spine path of $\langle G', \gamma' \rangle$; the third is a curve arbitrarily close to the drawing in Γ' of the first edge of the spine path of $\langle G', \gamma' \rangle$ till a point p in the interior of the first strip of Γ' and arbitrarily close to the boundary of the second strip; the fourth is a horizontal segment connecting p to a point q lying to the left of Γ'; the fifth is a vertical segment connecting q to a point r in the interior of the first strip of Γ; and the sixth is a curve connecting r to y. By Claim 4, the degree-1 vertices of $G_{\mathcal{T}_c}$ lie on its outer face in Γ' (and hence in Γ). Thus, the inter-strip edges incident to $G_{\mathcal{T}_c}$ can be drawn without crossings, as they preserve the same containment relationship between vertices and cycles in Γ as the corresponding intra-strip edges in Γ'.

Let H be the graph obtained from B_c by subdividing each edge e incident to v with a dummy vertex v_e and by removing v. Replace the drawing of $G_{\mathcal{T}_c}$ in Γ with a planar drawing of H such that the vertices v_e appear in a Eulerian tour of its outer face in the same clockwise order as the corresponding degree-1 vertices appear in a Eulerian tour of the outer face of $G_{\mathcal{T}_c}$ in Γ. Recall that these vertices are on the outer face of $G_{\mathcal{T}_c}$ in Γ, by Claim 4. Such a drawing of H exists since this order is in $O(\mathcal{T}_c)$ [12]. To complete Γ, for each cut-vertex z of G_c separating B_c from a subgraph G_z of G_c, draw graph G_z arbitrarily close to z. Note that no vertex of G_z, except possibly for z, is incident to an inter-strip edge. ◀

Let $\langle G, \gamma \rangle$ be an instance of Strip Planarity on $k > 1$ strips satisfying the properties of Lemma 3. By applying this lemma $k - 1$ times, we obtain an instance of Strip Planarity on $k = 1$ strips, that is, an instance whose strip-planarity coincides with the planarity of its underlying graph, which can be tested in linear time [7]. Hence, we get the following.

▶ **Lemma 5.** *Let $\langle G = (V, E), \gamma \rangle$ be a spined instance of* Strip Planarity *on $k > 1$ strips with a unique source component c. It is possible to decide in $O(k \times n)$ time whether $\langle G, \gamma \rangle$ admits a strip-planar drawing.*

Given an instance of Strip Planarity, one can create $O(n)$ spined instances by attaching the spine path to each of the $O(n)$ vertices in the first strip. The next theorem follows.

▶ **Theorem 6.** *Let $\langle G, \gamma \rangle$ be an instance of* Strip Planarity *on k strips such that there exists a unique source component c. It is possible to decide in $O(n^3)$ time whether $\langle G, \gamma \rangle$ admits a strip-planar drawing.*

4 C-Planarity with Embedded Pipes

In this section we show that the C-Planarity with Embedded Pipes problem is solvable in quadratic time for a notable family of instances.

Let c be an originating component belonging to a cluster $\mu \in \mathcal{T}$ and let $\nu \neq \mu \in \mathcal{T}$ be the cluster to which the vertices of c are adjacent to. We say that c is *originating from μ to ν*.

▶ **Lemma 7.** *Let $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ be an instance of* C-Planarity with Embedded Pipes *and let $\mathcal{S}$ be the maximum number of originating multi-edge components in a cluster that are incident to the same pipe. It is possible to construct in linear time an equivalent instance $\langle G_1, G_2 \rangle$ of SEFE such that (i) $G_\cap$ is a spanning forest, (ii) each cut-vertex of $G_2 = (V, E_2)$ is incident to at most one non-trivial block, and (iii) each cut-vertex of $G_1 = (V, E_1)$ is incident to at most $\mathcal{S}$ non-trivial blocks.*

Proof. We show how to construct $\langle G_1, G_2 \rangle$ starting from $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$. The *frame gadget* H is an embedded planar graph defined as follows. Refer to Fig. 1.a. For each intersection point between a disk representing a cluster $\mu \in \mathcal{T}$ and a segment delimiting a pipe representing an edge of G_A incident to μ in the drawing Γ_A of G_A, we add a vertex at this point. This results in a planar drawing of a graph; we set H to be this graph. We call *disk cycle* of μ the cycle in H obtained from the disk of μ in Γ_A. Similarly, we call *pipe cycle* of an edge (μ, ν)

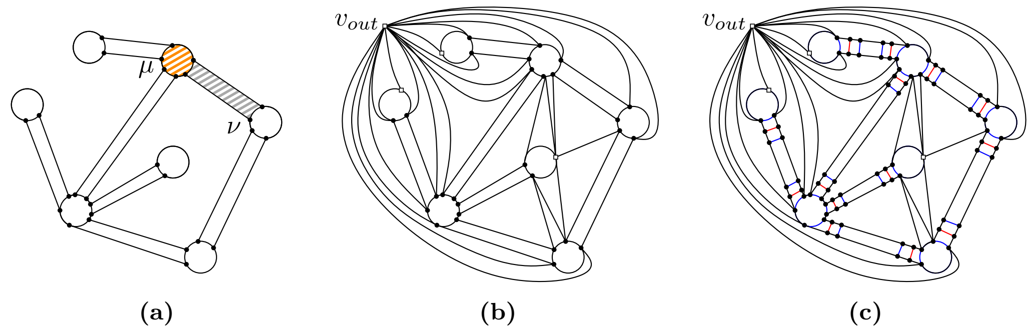


Figure 1 (a) Drawing Γ_A of the clusters-adjacency graph G_A, with vertices at the intersections of disks and pipes. The disk cycle for cluster μ and the pipe cycle for edge (μ,ν) of G_A are orange and gray tiled regions, respectively. (b) Frame gadget H. (c) Partial instance $\langle G_1, G_2\rangle$ of SEFE constructed from Γ_A; graphs G_1, G_2, and $G_\cap$ are subdivisions of triconnected planar graphs.

of G_A the cycle in H obtained from the pipe representing edge (μ,ν) in Γ_A. Note that, for clusters incident to exactly one pipe, this operation introduced two copies of the same edge; subdivide with a dummy vertex the copy that is not incident to the interior of this pipe. Then, add a vertex v_{out} in the outer face of H, connected to all the vertices incident to this face, and triangulate all the faces of H not corresponding to the interior of any cluster cycle or of any pipe cycle, hence obtaining a triconnected embedded planar graph. See Fig. 1.b.

Initialize $G_\cap = H$. For each edge $e \in E(H)$ separating a pipe from a disk, remove e from G_1 (not from G_2); this implies that disk cycles and pipe cycles only belong to G_2. Further, for each two edges e' and e'' corresponding to the two segments $(u_{\mu,\nu}, u_{\nu,\mu})$ and $(v_{\mu,\nu}, v_{\nu,\mu})$ delimiting a pipe representing an edge (μ,ν) of G_A, subdivide e' with four dummy vertices $a'_{\mu,\nu}, b'_{\mu,\nu}, b'_{\nu,\mu}, a'_{\nu,\mu}$, and e'' with four dummy vertices $a''_{\mu,\nu}, b''_{\mu,\nu}, b''_{\nu,\mu}, a''_{\nu,\mu}$, and add edges $(a'_{\mu,\nu}, a''_{\mu,\nu})$ and $(a'_{\nu,\mu}, a''_{\nu,\mu})$ to G_1 and edges $(b'_{\mu,\nu}, b''_{\mu,\nu})$ and $(b'_{\nu,\mu}, b''_{\nu,\mu})$ to G_2.

For each cluster $\mu \in \mathcal{T}$, augment $\langle G_1, G_2\rangle$ as follows; see Fig. 2.a. Subdivide an edge of $G_\cap$ that corresponds to a portion of the boundary of the disk representing μ in Γ_A with a dummy vertex γ_μ, and add to $G_\cap$ a star C_μ, whose central vertex is adjacent to γ_μ, with a leaf $z(c_i)$ for each multi-edge component c_i of μ. Also, add to $G_\cap$ each component c_i of μ. Finally, for each edge (μ,ν) of G_A, subdivide $(v_{\mu,\nu}, a'_{\mu,\nu})$ with a dummy vertex $\alpha_{\mu,\nu}$ and $(a''_{\mu,\nu}, b''_{\mu,\nu})$ with a dummy vertex $\beta_{\mu,\nu}$. Add to $G_\cap$ a star $A_{\mu,\nu}$ ($B_{\mu,\nu}$), whose central vertex is adjacent to $\alpha_{\mu,\nu}$ (is identified with $\beta_{\mu,\nu}$), with a leaf $a_\mu(e)$ (a leaf $b_\mu(e)$) for each inter-cluster edge e incident to a component of μ and to a component in ν. To complete $\langle G_1, G_2\rangle$, add the following edges only belonging to G_1 and to G_2. For each inter-cluster edge $e = (x, y)$ with $x \in \mu$ and $y \in \nu$, add to G_1 edges $(x, a_\mu(e))$, $(y, a_\nu(e))$, and $(b_\mu(e), b_\nu(e))$, and add to G_2 edges $(a_\mu(e), b_\mu(e))$ and $(a_\nu(e), b_\nu(e))$. Also, for each vertex x of a component c_i of a cluster μ such that x is incident to at least an inter-cluster edge, add to G_2 an edge $(x, z(c_i))$.

Clearly, $\langle G_1, G_2\rangle$ can be constructed in linear time. We now prove that G_1 and G_2 satisfy the properties of the lemma. We note that G_1 and G_2 are connected, since each vertex of a component c_i is connected to the frame gadget by means of paths in G_1 and in G_2 passing through stars $A_{\mu,\nu}$ and C_μ, respectively. Also, for each cluster $\mu \in \mathcal{T}$, graph G_2 contains cut-vertices γ_μ, the center of star C_μ, and vertices $z(c_i)$, for each component c_i of μ. However, vertex γ_μ is incident to exactly one non-trivial block, that is, the one containing all the vertices and edges of the frame gadget; the center of C_μ is incident only to non-trivial blocks; and vertices $z(c_i)$, for each component c_i of μ, are incident to at most one non-trivial block, that is, the one containing all the vertices and edges in c_i. Also, for each cluster $\mu \in \mathcal{T}$,

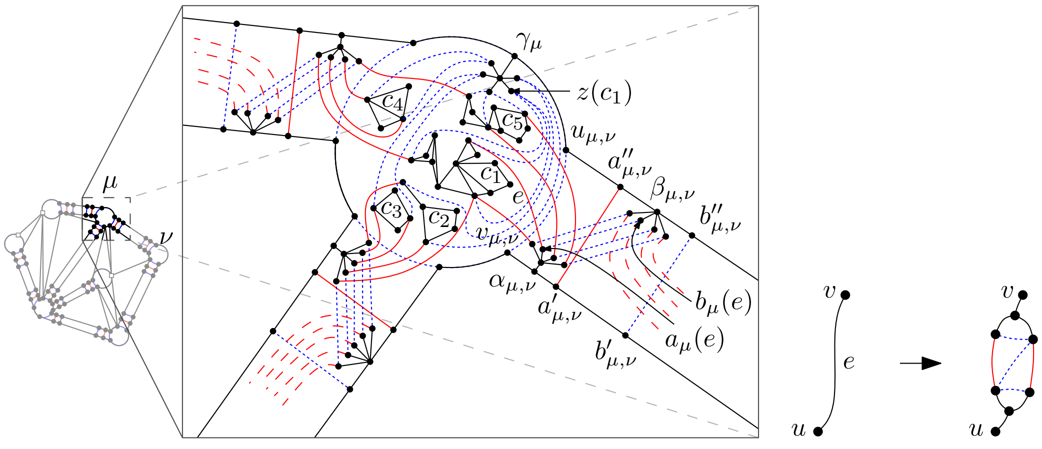


Figure 2 (a) Augmentation of instance $\langle G_1, G_2 \rangle$ focused on cluster $\mu \in \mathcal{T}$. (b) Replacing an edge $e = (u, v)$ to make $G_\cap$ acyclic.

all the passing components in μ belong to the biconnected component of G_1 containing all the vertices and edges of the frame gadget, while each multi-edge component originating from μ to a cluster ν determines a non-trivial block incident to $\alpha_{\mu,\nu}$, and each single-edge originating component from μ to a cluster ν determines a trivial block incident to $\alpha_{\mu,\nu}$. Since the number of multi-edge components originating from any cluster to any other cluster is at most $\mathcal{S}$, graph G_1 satisfies the required properties. The following claim implies that $G_\cap$ can be transformed into a spanning forest without altering the properties of $\langle G_1, G_2 \rangle$.

▶ **Claim 8.** *Each cycle of $G_\cap$ can be removed without altering the properties of $\langle G_1, G_2 \rangle$ by replacing one of its edges with the gadget in Fig. 2.b.*

We now prove the equivalence. Suppose that $\langle G_1, G_2 \rangle$ admits a SEFE $\langle \Gamma_1, \Gamma_2 \rangle$. We show how to construct a c-planar drawing with embedded pipes Γ of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$. Without loss of generality, assume that vertex v_{out} is embedded on the outer face of $\langle \Gamma_1, \Gamma_2 \rangle$. Note that the paths in $G_\cap$ corresponding to the segments delimiting the pipes representing an edge of G_A incident to a cluster $\mu \in \mathcal{T}$ appear in $\langle \Gamma_1, \Gamma_2 \rangle$ in the same clockwise circular order as the corresponding pipes appear around the disk representing μ in Γ_A. This is due to the fact that the frame gadget is a triconnected planar graph whose unique planar embedding is the one obtained from Γ_A. Note that in $\langle \Gamma_1, \Gamma_2 \rangle$ all the vertices in V appear either in the interior or on the boundary of disk cycles or of pipe cycles. This is due to the fact that removing all the vertices on the boundary of such cycles leaves a connected subgraph of $G_\cup$ and that there exists a unique face of H to which all the vertices belonging to such cycles are incident.

The proof is based on the fact that any SEFE of $\langle G_1, G_2 \rangle$ has the following properties.

1. For each cluster $\mu \in \mathcal{T}$, the central vertex of star C_μ lies in the interior of the disk cycle of μ, and hence all the vertices and edges of the components c_i of μ lie in the interior of such a cycle, since G_2 is connected.
2. For each two clusters $\mu, \nu \in \mathcal{T}$, the vertices of the components of μ and of the those of ν lie in the interior of different cycles of G_1, since all the components of each cluster μ are connected by paths in G_1 to the leaves of a star $A_{\mu,\xi}$, where ξ is a cluster adjacent to μ. Also, all the leaves of these stars lie in the interior of a cycle of G_1 delimited by edges of $G_\cap$ and by edges $(a'_{\mu,\xi_i}, a''_{\mu,\xi_i})$, for all the clusters ξ_i adjacent to μ.

3. For each inter-cluster edge e connecting a vertex v of a component c_i of μ to a cluster ν, edge $(v, a_\mu(e))$ in G_1 crosses edge $(u_{\mu,\nu}, v_{\mu,\nu})$. This is due to the previous two points and the fact that the leaves of $A_{\mu,\nu}$ lie outside the disk cycle of μ. We can assume that each of these edges crosses edge $(u_{\mu,\nu}, v_{\mu,\nu})$ exactly once, as otherwise we could redraw them to fulfill this requirement.
4. For two adjacent clusters $\mu, \nu \in \mathcal{T}$, the order in which the edges in G_1 incident to the leaves of $A_{\mu,\nu}$ cross $(u_{\mu,\nu}, v_{\mu,\nu})$ from $u_{\mu,\nu}$ to $v_{\mu,\nu}$ is the reverse of the order in which the edges in G_1 incident to the leaves of $A_{\nu,\mu}$ cross $(u_{\nu,\mu}, v_{\nu,\mu})$ from $u_{\nu,\mu}$ to $v_{\nu,\mu}$, where the identification between an edge incident to a leaf $a_\mu(e)$ of $A_{\mu,\nu}$ and an edge incident to a leaf $a_\nu(e)$ of $A_{\nu,\mu}$ is based on the inter-cluster edge e they correspond to. In fact, the order in which the edges in G_1 incident to the leaves of $A_{\mu,\nu}$ cross $(u_{\mu,\nu}, v_{\mu,\nu})$ is transmitted to the leaves of $B_{\mu,\nu}$ via edges in G_2, then it is transmitted to the leaves of $B_{\nu,\mu}$ via edges in G_1, and finally to the leaves of $A_{\nu,\mu}$ via edges in G_2. Note that all the leaves of these stars lie in the interior of the pipe cycle corresponding to edge (μ, ν) of G_A.

We describe how to obtain a c-planar drawing with embedded pipes Γ of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ from $\langle \Gamma_1, \Gamma_2 \rangle$. For each $\mu \in \mathcal{T}$, draw region $R(\mu)$ as the simple closed region whose boundary coincides with the drawing in Γ_2 of the disk cycle of μ. Each component c_i of a cluster μ has the same drawing in Γ as c_i in $\langle \Gamma_1, \Gamma_2 \rangle$. For each inter-cluster edge $e = (x, y)$ with $x \in \mu$ and $y \in \nu$, the portion of e in the interior of $R(\mu)$ (of $R(\nu)$) coincides with the drawing of edge $(x, a_\mu(e))$ (of edge $(y, a_\nu(e))$) between x (between y) and the intersection point of this edge with edge $(u_{\mu,\nu}, v_{\mu,\nu})$ (with edge $(u_{\nu,\mu}, v_{\nu,\mu})$). To complete the drawing of all the inter-cluster edges between μ and ν in the interior of the pipe representing edge (μ, ν) of G_A, connect the intersection points between the corresponding edges in G_1 and edges $(u_{\mu,\nu}, v_{\mu,\nu})$ and $(u_{\nu,\mu}, v_{\nu,\mu})$ by means of a set of non-intersecting curves. This is possible since the order in which the edges in G_1 incident to the leaves of $A_{\mu,\nu}$ cross $(u_{\mu,\nu}, v_{\mu,\nu})$ from $u_{\mu,\nu}$ to $v_{\mu,\nu}$ is the reverse of the order in which the edges in G_1 incident to the leaves of $A_{\nu,\mu}$ cross $(u_{\nu,\mu}, v_{\nu,\mu})$ from $u_{\nu,\mu}$ to $v_{\nu,\mu}$. This implies that Γ is a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$. The fact that Γ can be continuously deformed into a c-planar drawing with embedded pipes of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$ is due to the fact that the paths in $G_\cap$ corresponding to the segments delimiting the pipes incident to each cluster $\mu \in \mathcal{T}$ appear in $\langle \Gamma_1, \Gamma_2 \rangle$ in the same clockwise order as the corresponding pipes appear around the disk representing μ in Γ_A.

For the other direction, the goal is to construct a SEFE $\langle \Gamma_1, \Gamma_2 \rangle$ of $\langle G_1, G_2 \rangle$ that satisfies all the properties describe above starting from a c-planar drawing with pipes Γ of $\langle \mathcal{C}(G, \mathcal{T}), \Gamma_A \rangle$. For each cluster $\mu \in \mathcal{T}$, draw the disk cycle of μ as the boundary of the disk of μ in Γ_A. Also, for each edge (μ, ν) of G_A, draw the corresponding pipe cycle as the boundary of the pipe of edge (μ, ν) in Γ_A. For each cluster $\mu \in \mathcal{T}$, each component c_i of μ has the same drawing in $\langle \Gamma_1, \Gamma_2 \rangle$ as c_i in Γ. For each edge (μ, ν) of G_A, the stars $A_{\mu,\nu}$, $B_{\mu,\nu}$, $A_{\nu,\mu}$, and $B_{\nu,\mu}$ are drawn in $\langle \Gamma_1, \Gamma_2 \rangle$ so that the order of their leaves is the same or the reverse of the order in which the inter-cluster edges between μ and ν traverse the boundary of the disk of μ in Γ. Note that this order is the reverse of the order in which these edges traverse the boundary of the disk of ν in Γ. This allows to draw all the edges in G_1 and in G_2 that are incident to such leaves without introducing crossings between edges of the same graph. The drawing of star C_μ, for each cluster $\mu \in \mathcal{T}$, and of the edges in G_2 incident to its leaves can be easily obtained to respect the circular order of the inter-cluster edges incident to each of the components of μ. This concludes the proof of the lemma. ◀

By Lemma 7 and Theorem 1 we have the following main result.

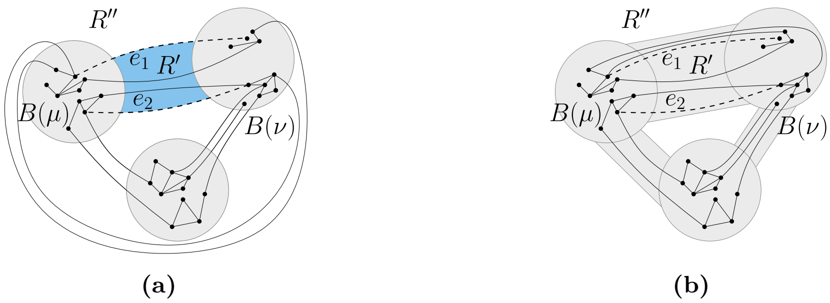


Figure 3 (a) A c-planar drawing with pipes Γ', with regions R' (blue) and R'' delimited by $B(\mu)$ and $B(\nu)$, and by e_1 and e_2 (dashed), where region R' does not contain any vertex of $G \setminus (\mu \cup \nu)$. (b) A c-planar drawing with pipes Γ^* corresponding to Γ' in which inter-cluster edges are inside pipes.

▶ **Theorem 9.** C-PLANARITY WITH EMBEDDED PIPES *can be solved in $O(n^2)$ time for instances $\langle \mathcal{C}(G,\mathcal{T}), \Gamma_A \rangle$ such that for each cluster $\mu \in \mathcal{T}$ and for each edge (μ,ν) in G_A either (CASE 1) cluster μ contains at most one originating multi-edge component from μ to ν or (CASE 2) cluster μ contains at most two multi-edge originating components from μ to ν and does not contain any passing component that is incident to ν.*

5 C-Planarity with Pipes

In this section we introduce and study problem C-PLANARITY WITH PIPES. A c-planar drawing Γ of a flat c-graph $\mathcal{C}(G,\mathcal{T})$ is a *c-planar drawing with pipes* if, for any two clusters $\mu, \nu \in \mathcal{T}$ that are adjacent in G_A and for any two inter-cluster edges e_1 and e_2 that are incident to both μ and ν, one of the two regions delimited by $B(\mu)$, by $B(\nu)$, by e_1, and by e_2 does not contain any vertex of $G \setminus (\mu \cup \nu)$; see Fig. 3.

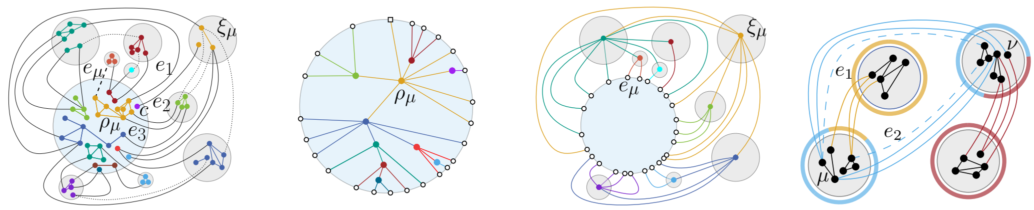
Problem C-PLANARITY WITH PIPES asks for the existence of a c-planar drawing with pipes of a given flat c-graph. We first prove that this problem is a generalization of C-PLANARITY WITH EMBEDDED PIPES.

▶ **Lemma 10.** C-PLANARITY WITH EMBEDDED PIPES *reduces in linear time to* C-PLANARITY WITH PIPES. *The reduction does not increase the number of multi-edge components in any cluster.*

We now present an FPT algorithm for C-PLANARITY WITH PIPES with two parameters, namely the maximum number K of multi-edge components in a cluster and the number c of clusters with at least two multi-edge components. Our result builds on a characterization of C-PLANARITY of flat c-graphs in terms of a new constrained embedding problem.

5.1 A Characterization of Flat C-Planarity

We start with some definitions. Let $\mathcal{C}(G,\mathcal{T})$ be a flat c-graph. A *components tree* X_μ of a cluster $\mu \in \mathcal{T}$ is a rooted tree in which every internal vertex is a multi-edge component c of μ and in which every leaf $x_\mu(e)$ corresponds to an inter-cluster edge e incident to one of such components. A *neighbor-clusters tree* Y_μ of μ is a rooted tree in which there exists an internal vertex ν for each cluster ν adjacent to μ, plus a set of additional internal vertices, and every leaf $y_\mu(e)$ corresponds to an inter-cluster edge e incident to μ. Let Γ be a c-planar drawing of $\mathcal{C}(G,\mathcal{T})$, let X_μ be a components tree of μ rooted at a multi-edge component ρ_μ, and let Y_μ be a neighbor-clusters tree of μ rooted at a cluster ξ_μ, such that there exists an



(a) **(b)** **(c)** **(d)**

Figure 4 (a) A c-planar drawing Γ focused on cluster μ. Edges incident to μ are solid. Component c is nested into component ρ_μ. Trees (b) X_μ and (c) Y_μ such that Γ is consistent with X_μ and Y_μ. (d) A c-planar drawing that is not a c-planar drawing with pipes, even if the inter-cluster edges incident to the same cluster are consecutive (see the annuli around clusters), due to the presence of trivial block (μ, ν).

inter-cluster edge e_μ incident to both ρ_μ and ξ_μ. Let $\mathcal{O}_\mu$ be the clockwise linear order in which the edges incident to μ traverse $B(\mu)$ in Γ, starting from and ending at e_μ. Drawing Γ is *consistent with* X_μ if, for each vertex $c \in X_\mu$, the leaves of the subtree of X_μ rooted at c are consecutive in the restriction of $\mathcal{O}_\mu$ to the inter-cluster edges incident to multi-edge components of μ. Also, Γ is *consistent with* Y_μ if, for each vertex $\nu \in Y_\mu$, the leaves of the subtree of Y_μ rooted at ν are consecutive in $\mathcal{O}_\mu$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets containing a components tree X_μ and a neighbor-clusters tree Y_μ, respectively, for each $\mu \in \mathcal{T}$. Drawing Γ is *consistent with* $\langle \mathcal{X}, \mathcal{Y} \rangle$ if, for each $\mu \in \mathcal{T}$, it is consistent with both X_μ and Y_μ.

Given a flat c-graph $\mathcal{C}(G, \mathcal{T})$, together with two sets $\mathcal{X}$ and $\mathcal{Y}$ of components trees and of neighbor-clusters trees, respectively, for all the clusters in $\mathcal{T}$, problem Inclusion-Constrained C-Planarity asks whether a c-planar drawing of $\mathcal{C}(G, \mathcal{T})$ exists that is consistent with $\langle \mathcal{X}, \mathcal{Y} \rangle$.

▶ **Theorem 11.** *A flat c-graph $\mathcal{C}(G, \mathcal{T})$ is c-planar if and only if there exist two sets $\mathcal{X}$ and $\mathcal{Y}$ of components trees and of neighbor-clusters trees, respectively, for all the clusters in $\mathcal{T}$, such that $\langle \mathcal{C}(G, \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ is a positive instance of* Inclusion-Constrained C-Planarity.

Proof Sketch. The "only if part" trivially follows from the definition of Inclusion-Constrained C-Planarity. For the "if part", let Γ be any c-planar drawing of $\mathcal{C}(G, \mathcal{T})$ and let μ be a cluster in $\mathcal{T}$. Suppose that μ contains at least a multi-edge component ρ_μ, as otherwise X_μ is the empty tree and Γ is consistent with it. Let e_μ be any inter-cluster edge incident to ρ_μ and to a cluster ξ_μ. Let $\mathcal{O}_\mu$ be the clockwise linear order of the edges incident to μ starting from e_μ and ending at e_μ. Since Γ is c-planar, no two pairs of edges incident to two different components of μ (two different clusters adjacent to μ) alternate in $\mathcal{O}_\mu$. Hence, order $\mathcal{O}_\mu$ defines a hierarchical inclusion of the components of μ and of the clusters adjacent to μ with respect to ρ_μ and to ξ_μ, respectively, which can be described by means of two trees X_μ and Y_μ; see Fig. 4.a–4.c. Clearly, Γ is consistent with such trees. ◀

In the following theorem, whose proof is deferred to the full version of the paper [1], we show that the Inclusion-Constrained C-Planarity problem can be solved efficiently.

▶ **Theorem 12.** Inclusion-Constrained C-Planarity *can be solved in quadratic time.*

In the following we prove that, for each cluster μ of a c-graph $\mathcal{C}(G, \mathcal{T})$, there exists a neighbor-clusters tree Y_μ such that every c-planar drawing with pipes of $\mathcal{C}(G, \mathcal{T})$ is consistent with Y_μ. Hence, an FPT algorithm for C-Planarity with Pipes can be based on generating, for each cluster, all the possible components trees and its unique neighbor-clusters tree, and on testing these instances of Inclusion-Constrained C-Planarity by Theorem 12.

5.2 Neighbor-clusters Trees in C-Planar Drawings with Pipes

In the following theorem we give a characterization of the c-graphs that are positive instances of C-Planarity with Pipes based on the possible orders of inter-cluster edges around each cluster in any c-planar drawing. We first consider only c-graphs whose clusters-adjacency graph G_A has no trivial blocks; however, we prove later that this is not a restriction.

▶ **Theorem 13.** *Let $\mathcal{C}(G,\mathcal{T})$ be a flat c-graph such that G_A has no trivial block. Then, $\mathcal{C}(G,\mathcal{T})$ is a positive instance of* C-Planarity with Pipes *if and only if $\mathcal{C}(G,\mathcal{T})$ admits a c-planar drawing Γ in which, for each cluster $\mu \in \mathcal{T}$, the inter-cluster edges between μ and any cluster ν adjacent to μ in G_A are consecutive in the order in which the inter-cluster edges incident to μ cross $B(\mu)$ in Γ.*

Proof Sketch. The "only if part" descends from the definition of a c-planar drawing with pipes. We prove the "if part"; see Fig. 4.d. Let Γ be a c-planar drawing of $\mathcal{C}(G,\mathcal{T})$ satisfying the conditions of the theorem and consider two clusters $\mu, \nu \in \mathcal{T}$ with two inter-cluster edges e_1 and e_2 incident to μ and ν. If both the regions delimited by e_1, e_2, $B(\mu)$, and $B(\nu)$ contain vertices in $G \setminus (\mu \cup \nu)$, then all the clusters in one of the regions are only connected to μ and all the clusters in the other region are only connected to ν, due to the conditions of the lemma. Hence, (μ, ν) is a trivial block of G_A, a contradiction. ◀

We exploit Theorem 13 to construct a neighbor-clusters tree Y°_μ of each cluster $\mu \in \mathcal{T}$ such that any c-planar drawing with pipes of $\mathcal{C}(G,\mathcal{T})$ is consistent with Y°_μ. Tree Y°_μ is rooted at a vertex ω_μ. There exists a child ν of ω_μ for each cluster ν adjacent to μ, having a leaf $y_\mu(e)$ for each inter-cluster edge e incident to μ and to ν. We call Y°_μ the *pipe-neighbor-clusters tree* of μ. Theorem 13 and the construction of Y°_μ, for each cluster $\mu \in \mathcal{T}$, imply the following.

▶ **Corollary 14.** *Let $\mathcal{C}(G,\mathcal{T})$ be a c-graph whose clusters-adjacency graph has no trivial blocks. Then, $\mathcal{C}(G,\mathcal{T})$ admits a c-planar drawing with pipes if and only if $\mathcal{C}(G,\mathcal{T})$ admits a c-planar drawing Γ in which, for each $\mu \in \mathcal{T}$, drawing Γ is consistent with Y°_μ.*

By Corollary 14 we can reduce C-Planarity with Pipes for a c-graph whose clusters-adjacency graph G_A has no trivial blocks to Inclusion-Constrained C-Planarity, where the role played by the neighbor-clusters trees is taken by the pipe-neighbor-clusters trees. In the full paper [1] we explain how to overcome the requirement that G_A has no trivial block.

▶ **Lemma 15.** *Let $\mathcal{C}(G,\mathcal{T})$ be an instance of* C-Planarity with Pipes *in which G_A contains trivial blocks. It is possible to construct in linear time an equivalent instance $\mathcal{C}^*(G^*,\mathcal{T}^*)$ of* C-Planarity with Pipes *in which G^*_A has no trivial block. Further, $K_* = K$ and $c_* = c$, where K (K_*) is the maximum number of multi-edge components in a cluster of $\mathcal{C}(G,\mathcal{T})$ (of $\mathcal{C}^*(G^*,\mathcal{T}^*)$) and c (c_*) is the number of clusters of $\mathcal{C}(G,\mathcal{T})$ (of $\mathcal{C}^*(G^*,\mathcal{T}^*)$) with at least two multi-edge components.*

5.3 An FPT Algorithm for C-Planarity with Pipes

In the following we prove the main result of the section.

▶ **Theorem 16.** C-Planarity with Pipes *can be tested in $O(K^{c(K-2)}) \cdot O(n^2)$ time, where K is the maximum number of multi-edge components in a cluster and c is the number of clusters with at least two multi-edge components.*

Proof Sketch. Let $\mathcal{C}(G,\mathcal{T})$ be a c-graph, which can be assumed to have no trivial block by Lemma 15. Construct the set $\mathcal{Y}$ containing the unique pipe-neighbor-clusters tree Y°_μ of each cluster $\mu \in \mathcal{T}$. Then, construct all the possible sets $\mathcal{X}$ of components trees, over all clusters in $\mathcal{T}$. For each pair $\langle \mathcal{X}, \mathcal{Y} \rangle$, apply Theorem 12 to test whether $\langle \mathcal{C}(G, \mathcal{T}), \mathcal{X}, \mathcal{Y} \rangle$ is a positive instance of Inclusion-Constrained C-Planarity. By Theorem 13 and Corollary 14, c-graph $\mathcal{C}(G, \mathcal{T})$ is a positive instance if and only if at least one of such tests succeeds. ◀

We observe two notable corollaries of Theorem 16 (for the second, see Lemma 10).

▶ **Corollary 17.** Strip Planarity *can be tested in* $O(K^{s(K-2)}) \cdot O(n^2)$ *time, where* K *is the maximum number of multi-edge components in a strip and* s *is the number of strips containing at least two multi-edge components.*

▶ **Corollary 18.** C-Planarity with Embedded Pipes *can be tested in* $K^{c(K-2)} \cdot O(n^2)$ *time, where* K *is the maximum number of multi-edge components in a cluster and* c *is the number of clusters with at least two multi-edge components.*

6 Conclusions and Open Problems

In this paper we studied the problem of constructing c-planar drawings with pipes of flat c-graphs. We presented algorithms to test the existence of such drawings when the number of certain components is small, in different scenarios, namely when the clusters-adjacency graph is a path (Strip Planarity), when it has a fixed embedding (C-Planarity with Embedded Pipes), and when it has no restrictions (C-Planarity with Pipes).

Several questions are left open. We find particularly interesting to determine whether there exist combinatorial properties of the nesting of the components allowing us to reduce the number of possible components trees, analogous to the ones we could prove for the pipe-neighbor-clusters trees. We remark that the introduction of the components trees makes the running time of our FPT algorithms independent of the size of each component.

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