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Event-triggered Filtering and Fault Estimation for Nonlinear Systems with Stochastic Sensor Saturations

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This paper is concerned with the filtering problem for a class of nonlinear systems with stochastic sensor saturations and event-triggered measurement transmissions. An event-triggered transmission scheme is proposed with hope to ease the traffic burden and improve the energy efficiency. The measurements are subject to randomly occurring sensor saturations governed by Bernoulli distributed sequences. Special effort is made to obtain an upper bound of the filtering error covariance in the presence of linearization errors, stochastic sensor saturations as well as event-triggered transmissions. A filter is designed to minimize the obtained upper bound at each time step by solving two sets of Riccati-like matrix equations, and thus the recursive algorithm is suitable for online computation. Sufficient conditions are established under which the filtering error is exponentially bounded in mean square. The applicability of the presented method is demonstrated by dealing with the fault estimation problem. An illustrative example is exploited to show the effectiveness of the proposed algorithm.

Keywords: Nonlinear systems; Kalman filtering; fault estimation; event-triggered transmission; sensor saturation; error boundedness.

1. Introduction

In the past few decades, the event-triggered transmission (ETT) mechanism has aroused a great deal of interest due to the rapid development of computer science and digital microprocessor Demir & Lunze (2014); Orihuela, Millan, Vivas, & Rubio (2014); Tabuada (2007); Zhang, Hao, Zhang, & Wang (2015). Compared with the conventional clock-driven strategy referring to periodic signal transmissions, in an ETT scheme, the outputs/inputs are released only when some conditions are violated. By reducing signal exchanges, the ETT could avoid some harmful transmission phenomena (e.g. data dropout, time delay and congestion), improve the energy efficiency and extend the lifetime of the services.

Recently, the event-triggered filtering (ETF) problem has started to gain some initial research attention especially for systems with wireless links and energy constraints. For example, the event-triggered H_∞ filtering problem with transmission delays has been investigated in Hu & Yue (2012) and a modified Kalman filter for linear systems with event-triggered transmissions has been designed in Suh & Nguyen (2007) where the differences between the measurements have been assumed to be uniformly distributed. In Sijs & Lazar (2009); Liang, Jia, Johansson, & Shi (2013), the event-triggered minimum-variance filter has been thoroughly studied where the probability density functions (PDFs) of the states and the innovations conditional on measurements have been

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approximated with a sum of Gaussian distributions. However, when the system model is relatively complicated, the conditional PDFs will be intrinsically non-Gaussian and the Gaussian approximations may be quite inaccurate. Therefore, there appears to be a practical need to develop an alternative approach for addressing the ETF problem without strong assumptions on the distribution of measurements.

Due to physical and technological limitations, sensors/actuators cannot provide signals with unbounded amplitudes and such saturation phenomena pose extra challenges to the systems design. The control/filtering problems with actuator/sensor saturations have drawn much research attention Ding & Zheng (2015); Turner & Tarbouriech (2009); Yang & Li (2009); Yuan & Wu (2015); Zuo, Ho, & Wang (2010) where most available literature has treated the saturations as sector-bounded nonlinearities. Nevertheless, sensors in practical systems might frequently encounter some transient phenomena especially when systems are deployed in unattended environments such as power grids Han, Xie, Chen, & Ling (2014); Kisner et al. (2010); Neuman (2009). Under the circumstances, the saturation itself may undergo random switches/changes in its occurrence/intensity because of various reasons such as random sensor failures and abrupt environmental changes Wang, Shen, & Liu (2012). As such, it would be interesting to examine the impact of both the ETT and stochastic saturations on the filter performance in the minimum variance sense. Note that the filtering problem with stochastic saturations has not received adequate research attention yet, not to mention the case when the nonlinearity and ETT are also taken into account. Note that, 1) it is novel to cope with the ETT issue without the approximated conditional PDFs of states and innovations; and 2) it would be non-trivial to include the saturation level and the statistical characteristics of the sensor saturations in the filter design.

In this paper, we aim to address the filtering problem for a class of nonlinear systems subject to event-triggered measurement transmissions and stochastic sensor saturations. Some Bernoulli-distributed sequences are introduced to govern the stochastic sensor saturations. An upper bound of the filtering error covariance is obtained and then the filter gain is determined so as to minimize the bound. The filtering performance is analyzed with respect to the error boundedness. Sufficient conditions are achieved under which the filtering error is exponentially bounded in mean square. As a consequence, the application on the fault estimation problem is investigated, since faults resulting from external disturbances and component/actuator malfunctions might still occur in the presence of ETT and stochastic sensor saturations. The main novelty of the paper lies in the following aspects: 1) a comprehensive model is established which covers nonlinearities, event-triggered measurement transmissions and stochastic sensor saturations; 2) an upper bound of the filtering error covariance is minimized by appropriately designing the recursive filter and the algorithm is applied in the fault estimation problem; and 3) the boundedness of the filtering error dynamics is analyzed.

Notations. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). I is the identity matrix with compatible dimension. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x. $\|A\|$ denotes the spectral norm of matrix A, and $\|x\|$ refers to the Euclidean norm of vector x. diag$\{\cdots\}$ stands for a block-diagonal matrix. $\circ$ is the Hadamard product defined as $[A \circ B]_{ij} = A_{ij}B_{ij}$.

2. Problem Formulation

Consider the following stochastic discrete-time nonlinear system:

$$\begin{cases} x_{k+1} = g(x_k, u_k) + D_k w_k, \\ y_k = \Lambda_{\alpha_k}\sigma(C_k x_k) + (I - \Lambda_{\alpha_k})C_k x_k + F_k v_k, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state; $u_k \in \mathbb{R}^l$ is the control; $y_k \in \mathbb{R}^m$ is the measurement; $w_k \in \mathbb{R}^p$ and $v_k \in \mathbb{R}^q$ are the mutually uncorrelated zero-mean process noise and the communication noise with $\mathbb{E}\{w_k w_k^T\} = W_k$ and $\mathbb{E}\{v_k v_k^T\} = V_k$. The initial condition x_0 is stochastic with known $\mathbb{E}\{x_0\}$ and $\mathbb{E}\{x_0 x_0^T\}$, and independent of the noises. C_k, D_k, and F_k are known matrices and the nonlinear function g is twice continuously differentiable.

For every $k \in \mathbb{N}$, $\Lambda_{\alpha_k} = \mathrm{diag}\{\alpha_{1,k}, \ldots, \alpha_{m,k}\}$ where for every $i = 1, 2, \ldots, m$, $\alpha_{i,k} \in \mathbb{R}$ is a Bernoulli distributed white sequence taking values on 0 or 1 with

$$\begin{cases} \mathrm{Prob}\{\alpha_{i,k} = 1\} = \lambda_i, \\ \mathrm{Prob}\{\alpha_{i,k} = 0\} = 1 - \lambda_i. \end{cases} \tag{2}$$

Here, $\lambda_i \in [0, 1]$ is a known scalar for every i. Denoting $\Lambda_\lambda := \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}$, it follows directly that $\mathbb{E}\{\Lambda_{\alpha_k}\} = \Lambda_\lambda$.

For a vector $r = [r_1, \ldots, r_m]^T$, the saturation function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is defined as:

$$\sigma(r) = [\sigma_1(r_1), \ldots, \sigma_m(r_m)]^T \tag{3}$$

where $\sigma_s(r_s) = \mathrm{sign}(r_s)\min(b_s, |r_s|)$ and $b_s \geq 0$ for all $s = 1, \ldots, m$. Furthermore, $\mathrm{sign}(\cdot)$ denotes the signum function and b_s represents the saturation level.

In this paper, the following standard send-on-delta Miskowicz (2006) transmission strategy is considered: the current measurement y_{k+j} would be transmitted if it satisfies

$$(y_{k+j} - y_k)^T (y_{k+j} - y_k) > \varsigma, \tag{4}$$

where y_k is the previously transmitted measurement and ς is a given positive scalar. Letting the release instants be denoted by $k_0, k_1, \cdots$, the released signal $\tilde{y}_k$ can be written as

$$\tilde{y}_k = y_{k_j}, k \in \{k_j, k_j + 1, \cdots, k_{j+1} - 1\}. \tag{5}$$

For system (1), consider a filter of the following structure:

$$\hat{x}_{k+1|k} = g(\hat{x}_{k|k}, u_k), \tag{6}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left[\tilde{y}_{k+1} - \Lambda_\lambda \sigma(C_{k+1}\hat{x}_{k+1|k}) - (I - \Lambda_\lambda)C_{k+1}\hat{x}_{k+1|k}\right], \tag{7}$$

where $\hat{x}_{k|k} \in \mathbb{R}^n$ is the estimation of x_k at time step k with $\hat{x}_{0|0} = \mathbb{E}\{x_0\}$, $\hat{x}_{k+1|k} \in \mathbb{R}^n$ is the one step prediction at time step k, and K_{k+1} is the filter gain to be determined.

Remark 1: The measurement equation in (1) is introduced to address the stochastic sensor saturations which may arise from uncertain working conditions and technological/physical limitations. The proposed transmission condition (4) means that the current measurement is released only when it changes greatly. Also note that the terms reflecting the statistics (i.e. λ_i for all $i = 1, 2, \ldots, m$) are fixed scalars, which facilitates the filter implementation. Both ETT and stochastic sensor saturations would affect the observability of the addressed system, making the filtering problem more challenging.

Denote the prediction error, the estimation error, and their covariances conditional on the received measurements as $e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}$, $e_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$, $P_{k+1|k} = \mathbb{E}\left\{e_{k+1|k}e_{k+1|k}^T | y_0, \ldots, y_k\right\}$, and $P_{k+1|k+1} = \mathbb{E}\left\{e_{k+1|k+1}e_{k+1|k+1}^T | y_0, \ldots, y_{k+1}\right\}$, respectively. The goal of the addressed problem is to design an estimator in the form of (6) and (7) for system (1) such that an upper bound of $P_{k+1|k+1}$ can be obtained and subsequently minimized.

3. Filter Design

In this section, two sets of recursive Riccati-like matrix equations are established to calculate the filter parameter in (7) in order to minimize an upper bound of the filtering error covariance for system (1). To start with, it follows from (1) and (6) that

$$e_{k+1|k} = g(x_k, u_k) - g(\hat{x}_{k|k}, u_k) + D_k w_k. \tag{8}$$

Based on the results in Calafiore (2005); Xiong, Wei, & Liu (2010), (8) can be written as:

$$e_{k+1|k} = (A_k + S_k U_k) e_{k|k} + D_k w_k, \tag{9}$$

where

$$A_k = \left. \frac{\partial g(z_k, u_k)}{\partial z_k} \right|_{z_k = \hat{x}_{k|k}},$$

S_k is a problem-dependent scaling matrix and U_k is an unknown matrix with $\|U_k\| \leq 1$. Then the following lemma can be established.

Lemma 1: *The prediction error covariance satisfies*

$$P_{k+1|k} = \mathbb{E}\left\{(A_k + S_k U_k) P_{k|k} (A_k + S_k U_k)^T\right\} + D_k W_k D_k^T, \tag{10}$$

and the estimation error covariance can be recursively calculated as follows:

$$\begin{aligned} P_{k+1|k+1} = & [I - K_{k+1}(I - \Lambda_\lambda) C_{k+1}]\, P_{k+1|k} [I - K_{k+1}(I - \Lambda_\lambda) C_{k+1}]^T + K_{k+1} \mathbb{E}\{(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda) \\ & \times [\sigma(C_{k+1} x_{k+1}) - C_{k+1} x_{k+1}][\sigma(C_{k+1} x_{k+1}) - C_{k+1} x_{k+1}]^T (\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)^T\} K_{k+1}^T \\ & + K_{k+1} \Lambda_\lambda \mathbb{E}\{[\sigma(C_{k+1} x_{k+1}) - \sigma(C_{k+1} \hat{x}_{k+1|k})][\sigma(C_{k+1} x_{k+1}) - \sigma(C_{k+1} \hat{x}_{k+1|k})]^T\} \Lambda_\lambda^T \\ & \times K_{k+1}^T + K_{k+1} \mathbb{E}\{(\tilde{y}_{k+1} - y_{k+1})(\tilde{y}_{k+1} - y_{k+1})^T\} K_{k+1}^T - \left[I - K_{k+1}(I - \Lambda_\lambda) C_{k+1}\right] \\ & \times \mathbb{E}\{e_{k+1|k} [\sigma(C_{k+1} x_{k+1}) - \sigma(C_{k+1} \hat{x}_{k+1|k})]^T\} \Lambda_\lambda^T K_{k+1}^T - K_{k+1} \Lambda_\lambda \mathbb{E}\{[\sigma(C_{k+1} x_{k+1}) \\ & - \sigma(C_{k+1} \hat{x}_{k+1|k})] e_{k+1|k}^T\} \left[I - K_{k+1}(I - \Lambda_\lambda) C_{k+1}\right]^T - [I - K_{k+1}(I - \Lambda_\lambda) C_{k+1}] \\ & \times \mathbb{E}\{e_{k+1|k} (\tilde{y}_{k+1} - y_{k+1})^T\} K_{k+1}^T - K_{k+1} \mathbb{E}\{(\tilde{y}_{k+1} - y_{k+1}) e_{k+1|k}^T\} [I - K_{k+1}(I - \Lambda_\lambda) \\ & \times C_{k+1}]^T + K_{k+1} \mathbb{E}\{(\tilde{y}_{k+1} - y_{k+1})[\sigma(C_{k+1} x_{k+1}) - \sigma(C_{k+1} \hat{x}_{k+1|k})]^T\} \Lambda_\lambda^T K_{k+1}^T + K_{k+1} \\ & \times \Lambda_\lambda \mathbb{E}\{[\sigma(C_{k+1} x_{k+1}) - \sigma(C_{k+1} \hat{x}_{k+1|k})](\tilde{y}_{k+1} - y_{k+1})^T\} K_{k+1}^T + K_{k+1} \mathbb{E}\{(\tilde{y}_{k+1} - y_{k+1}) \\ & \times v_{k+1}^T\} F_{k+1}^T K_{k+1}^T + K_{k+1} F_{k+1} \mathbb{E}\{v_{k+1} (\tilde{y}_{k+1} - y_{k+1})^T\} K_{k+1}^T + K_{k+1} \mathbb{E}\{(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda) \\ & \times [\sigma(C_{k+1} x_{k+1}) - C_{k+1} x_{k+1}](\tilde{y}_{k+1} - y_{k+1})^T\} K_{k+1}^T + K_{k+1} \mathbb{E}\{(\tilde{y}_{k+1} - y_{k+1}) \\ & \times [\sigma(C_{k+1} x_{k+1}) - C_{k+1} x_{k+1}]^T (\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)^T\} K_{k+1}^T + K_{k+1} F_{k+1} V_{k+1} F_{k+1}^T K_{k+1}^T. \end{aligned} \tag{11}$$

Proof. (10) is easily accessible from (9) and the fact that $e_{k|k}$ is independent of w_k, and now we are going to prove (11). From (1) and (7), it follows that

$$e_{k+1|k+1} = e_{k+1|k} - K_{k+1}\left[\tilde{y}_{k+1} - \Lambda_\lambda \sigma(C_{k+1} \hat{x}_{k+1|k}) - (I - \Lambda_\lambda) C_{k+1} \hat{x}_{k+1|k}\right]. \tag{12}$$

Adding the zero term

$$
\begin{aligned}
&K_{k+1}y_{k+1} - K_{k+1}y_{k+1} + K_{k+1}\Lambda_\lambda \sigma(C_{k+1}x_{k+1}) - K_{k+1}\Lambda_\lambda \sigma(C_{k+1}x_{k+1}) \\
&+ K_{k+1}(I-\Lambda_\lambda)C_{k+1}x_{k+1} - K_{k+1}(I-\Lambda_\lambda)C_{k+1}x_{k+1}
\end{aligned}
$$

to the right-hand side of (12), we have

$$
\begin{aligned}
e_{k+1|k+1} =& [I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}]e_{k+1|k} - K_{k+1}(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)[\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}] \\
&- K_{k+1}\Lambda_\lambda[\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})] - K_{k+1}(\tilde{y}_{k+1} - y_{k+1}) - K_{k+1}F_{k+1}v_{k+1}.
\end{aligned} \tag{13}
$$

(11) can be obtained directly from (13). This concludes the proof. □

Remark 2: In Lemma 1, the exact covariances of one-step prediction error and filtering error have been obtained. However, it is very difficult to determine the covariances recursively by using these two equations because of the stochastic sensor saturations and event-triggered transmissions. To handle terms related to $y_{k+1} - \tilde{y}_{k+1}$ and the saturations, we need the posteriori PDF of the states based on the PDF of states conditional on measurements. Unfortunately, since the system (1) is relatively complex that contains both the nonlinearities and the stochastic sensor saturations, the conditional PDF might be difficult to calculate or approximate. In Suh & Nguyen (2007), $y_{k+1} - \tilde{y}_{k+1}$ is assumed to be uniformly distributed, and the filtering error covariance is updated accordingly. However, such an assumption is a bit too stringent in practice. An alternative way is to find an upper bound of the filtering error covariance and then design the filter gain to minimize the upper bound at each time step. In this way, neither the conditional PDF nor the strong assumption on the distribution of $y_{k+1} - \tilde{y}_{k+1}$ will be required.

Before proceeding, the following lemma is to be introduced Hu, Wang, Gao, & Stergioulas (2012); Liu, Wang, He, & Zhou (2015).

Lemma 2: *For any two matrices $X, Y \in \mathbb{R}^{n\times n}$, the inequality $XY^T + YX^T \le \varepsilon XX^T + \varepsilon^{-1}YY^T$ holds where $\varepsilon > 0$ is a constant scalar.*

Now we are in a position to obtain an upper bound of the filtering error covariance and design the filter to minimize the bound.

Theorem 1: *Let ε_j $(j = 1,\ldots,8)$ and γ_k $(k \in \mathbb{N})$ be positive scalars. Assume that the following recursive equations*

$$
\bar{P}_{k+1|k} = (1+\varepsilon_1)A_k\bar{P}_{k|k}A_k^T + \gamma_k(1+\varepsilon_1^{-1})S_kS_k^T + D_kW_kD_k^T, \tag{14}
$$

$$
\begin{aligned}
\bar{P}_{k+1|k+1} =& (1+\varepsilon_2+\varepsilon_3)\left[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}\right]\bar{P}_{k+1|k}[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}]^T \\
&+ (1+\varepsilon_6)K_{k+1}(\tilde{\Lambda}\circ\Theta_{k+1})K_{k+1}^T + 4\bar{b}(1+\varepsilon_2^{-1}+\varepsilon_4)K_{k+1}\Lambda_\lambda\Lambda_\lambda^T K_{k+1}^T \\
&+ \varsigma(1+\varepsilon_3^{-1}+\varepsilon_4^{-1}+\varepsilon_5+\varepsilon_6^{-1})K_{k+1}K_{k+1}^T + (1+\varepsilon_5^{-1})K_{k+1}F_{k+1}V_{k+1}F_{k+1}^T K_{k+1}^T,
\end{aligned} \tag{15}
$$

have positive definite solutions with initial condition $\bar{P}_{0|0} = P_{0|0}$, where

$$\bar{b} = \sum_{s=1}^{m} b_s^2, \tag{16}$$

$$\begin{aligned}\Theta_{k+1} =& \bar{b}(1+\varepsilon_7)I + (1+\varepsilon_7^{-1})(1+\varepsilon_8)C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T \\ &+ (1+\varepsilon_7^{-1})(1+\varepsilon_8^{-1})C_{k+1}\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T C_{k+1}^T,\end{aligned} \tag{17}$$

$$\bar{P}_{k|k} \le \gamma_k I, \tag{18}$$

$$\tilde{\Lambda} = \text{diag}\{\lambda_1 - \lambda_1^2, \dots, \lambda_m - \lambda_m^2\}, \tag{19}$$

$$\begin{aligned}Y_{k+1} =& (1+\varepsilon_2+\varepsilon_3)(I-\Lambda_\lambda)C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T(I-\Lambda_\lambda)^T + 4\bar{b} \\ &\times (1+\varepsilon_2^{-1}+\varepsilon_4)\Lambda_\lambda\Lambda_\lambda^T + (1+\varepsilon_6)\tilde{\Lambda}\circ\Theta_{k+1} + \varsigma(1+\varepsilon_3^{-1} \\ &+ \varepsilon_4^{-1} + \varepsilon_5 + \varepsilon_6^{-1})I + (1+\varepsilon_5^{-1})F_{k+1}V_{k+1}F_{k+1}^T,\end{aligned} \tag{20}$$

$$Z_{k+1} = (1+\varepsilon_2+\varepsilon_3)(I-\Lambda_\lambda)C_{k+1}\bar{P}_{k+1|k}, \tag{21}$$

$$K_{k+1} = Z_{k+1}^T Y_{k+1}^{-1}. \tag{22}$$

Then $\bar{P}_{k|k}$ is an upper bound of $P_{k|k}$, and the bound $\bar{P}_{k+1|k+1}$ is minimized at each time step with the filter gain K_{k+1} given in (22).

Proof. The theorem can be proved by induction. Based on the initial condition, we have $\bar{P}_{0|0} \ge P_{0|0}$. Then, assume that $\bar{P}_{k|k} \ge P_{k|k}$, and we need to prove that $\bar{P}_{k+1|k+1} \ge P_{k+1|k+1}$. Firstly, based on $\bar{P}_{k|k} \ge P_{k|k}$, one needs to show that $\bar{P}_{k+1|k} \ge P_{k+1|k}$ and $\Theta_{k+1} \ge \mathbb{E}\{[\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}][\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}]^T\} =: \Psi_{k+1}$.

With the assumption $\bar{P}_{k|k} \ge P_{k|k}$, we have from (10) that

$$P_{k+1|k} \le \mathbb{E}\left\{(A_k + S_k U_k)\bar{P}_{k|k}(A_k + S_k U_k)^T\right\} + D_k W_k D_k^T.$$

Then, it follows from Lemma 2 that

$$P_{k+1|k} \le (1+\varepsilon_1)A_k\bar{P}_{k|k}A_k^T + (1+\varepsilon_1^{-1})\mathbb{E}\left\{S_k U_k \bar{P}_{k|k} U_k^T S_k^T\right\} + D_k W_k D_k^T. \tag{23}$$

From (18) and $\|U_k\| \le 1$, we have $S_k U_k \bar{P}_{k|k} U_k^T S_k^T \le \gamma_k S_k S_k^T$ and, subsequently, (23) can be written as

$$P_{k+1|k} \le (1+\varepsilon_1)A_k\bar{P}_{k|k}A_k^T + \gamma_k(1+\varepsilon_1^{-1})S_k S_k^T + D_k W_k D_k^T = \bar{P}_{k+1|k}.$$

Next, let us deal with Ψ_{k+1}. It follows from Lemma 2 that

$$\Psi_{k+1} \le (1+\varepsilon_7)\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\} + (1+\varepsilon_7^{-1})\mathbb{E}\{C_{k+1}x_{k+1}x_{k+1}^T C_{k+1}^T\}.$$

From the facts that $x_{k+1} = \hat{x}_{k+1|k} + e_{k+1|k}$ and $\bar{P}_{k+1|k} \ge P_{k+1|k}$, it follows that

$$\begin{aligned}\Psi_{k+1} \le& (1+\varepsilon_7)\mathbb{E}\{\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1})\} + (1+\varepsilon_7^{-1})(1+\varepsilon_8)C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T \\ &+ (1+\varepsilon_7^{-1})(1+\varepsilon_8^{-1})C_{k+1}\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T C_{k+1}^T.\end{aligned} \tag{24}$$

Since the absolute value of the ith entry of $\sigma(C_{k+1}x_{k+1})$ is less than or equal to b_i, we obtain

$$\sigma(C_{k+1}x_{k+1})\sigma^T(C_{k+1}x_{k+1}) \le \bar{b}I. \tag{25}$$

Substituting (25) into (24) yields

$$\Psi_{k+1} \leq \bar{b}(1+\varepsilon_7)I + (1+\varepsilon_7^{-1})(1+\varepsilon_8)C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T + (1+\varepsilon_7^{-1})(1+\varepsilon_8^{-1})C_{k+1}\hat{x}_{k+1|k}\hat{x}_{k+1|k}^T C_{k+1}^T = \Theta_{k+1}.$$

Now, we are going to show that $\bar{P}_{k+1|k+1} \geq P_{k+1|k+1}$. It follows from Lemma 2 that

$$\begin{aligned} P_{k+1|k+1} \leq & (1+\varepsilon_2+\varepsilon_3)\left[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}\right]P_{k+1|k}[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}]^T + (1+\varepsilon_6)K_{k+1} \\ & \times \mathbb{E}\{(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)[\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}][\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}]^T(\Lambda_{\alpha_{k+1}} \\ & - \Lambda_\lambda)^T\}K_{k+1}^T + (1+\varepsilon_2^{-1}+\varepsilon_4)K_{k+1}\Lambda_\lambda\mathbb{E}\{[\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})] \\ & \times [\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})]^T\}\Lambda_\lambda^T K_{k+1}^T + (1+\varepsilon_3^{-1}+\varepsilon_4^{-1}+\varepsilon_5+\varepsilon_6^{-1})K_{k+1} \\ & \times \mathbb{E}\{(\tilde{y}_{k+1}-y_{k+1})(\tilde{y}_{k+1}-y_{k+1})^T\}K_{k+1}^T + (1+\varepsilon_5^{-1})K_{k+1}F_{k+1}V_{k+1}F_{k+1}^T K_{k+1}^T. \end{aligned} \tag{26}$$

Considering $\bar{P}_{k+1|k} \geq P_{k+1|k}$ and $\Theta_{k+1} \geq \Psi_{k+1}$, (26) can be written as

$$\begin{aligned} P_{k+1|k+1} \leq & (1+\varepsilon_2+\varepsilon_3)[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}]\bar{P}_{k+1|k}[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}]^T + (1+\varepsilon_6)K_{k+1} \\ & \times \mathbb{E}\{(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)\Theta_{k+1}(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)^T\}K_{k+1}^T + (1+\varepsilon_2^{-1}+\varepsilon_4)K_{k+1}\Lambda_\lambda\mathbb{E}\{[\sigma(C_{k+1}x_{k+1}) \\ & - \sigma(C_{k+1}\hat{x}_{k+1|k})][\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})]^T\}\Lambda_\lambda^T K_{k+1}^T + (\varepsilon_3^{-1}+\varepsilon_4^{-1}+\varepsilon_5+\varepsilon_6^{-1} \\ & + 1)K_{k+1}\mathbb{E}\{(\tilde{y}_{k+1}-y_{k+1})(\tilde{y}_{k+1}-y_{k+1})^T\}K_{k+1}^T + (1+\varepsilon_5^{-1})K_{k+1}F_{k+1}V_{k+1}F_{k+1}^T K_{k+1}^T. \end{aligned}$$

Based on the transmission condition (4), for any $k \in \mathbb{N}$, we have

$$(\tilde{y}_k - y_k)(\tilde{y}_k - y_k)^T \leq \varsigma I. \tag{27}$$

Similar to (25), we get

$$[\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})][\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})]^T \leq 4\bar{b}I. \tag{28}$$

From (27) and (28), it follows that

$$\begin{aligned} P_{k+1|k+1} \leq & (1+\varepsilon_2+\varepsilon_3)\left[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}\right]\bar{P}_{k+1|k}[I - K_{k+1}(I-\Lambda_\lambda)C_{k+1}]^T + (1+\varepsilon_6)K_{k+1} \\ & \times (\tilde{\Lambda} \circ \Theta_{k+1})K_{k+1}^T + 4\bar{b}(1+\varepsilon_2^{-1}+\varepsilon_4)K_{k+1}\Lambda_\lambda\Lambda_\lambda^T K_{k+1}^T + \varsigma(1+\varepsilon_3^{-1}+\varepsilon_4^{-1}+\varepsilon_5+\varepsilon_6^{-1}) \\ & \times K_{k+1}K_{k+1}^T + (1+\varepsilon_5^{-1})K_{k+1}F_{k+1}V_{k+1}F_{k+1}^T K_{k+1}^T = \bar{P}_{k+1|k+1}. \end{aligned}$$

So far, $\bar{P}_{k|k}$ has been verified to be an upper bound of $P_{k|k}$, and what remains to show is that K_{k+1} in (22) minimizes the bound. With (20) and (21), $\bar{P}_{k+1|k+1}$ can be written as

$$\bar{P}_{k+1|k+1} = (1+\varepsilon_2+\varepsilon_3)\bar{P}_{k+1|k} + K_{k+1}Y_{k+1}K_{k+1}^T - Z_{k+1}^T K_{k+1}^T - K_{k+1}Z_{k+1}.$$

Noticing the fact that $Y_{k+1} = Y_{k+1}^T > 0$ and completing the square with respect to K_{k+1}, we have

$$\bar{P}_{k+1|k+1} = (K_{k+1} - Z_{k+1}^T Y_{k+1}^{-1})Y_{k+1}(K_{k+1} - Z_{k+1}^T Y_{k+1}^{-1})^T - Z_{k+1}^T Y_{k+1}^{-1} Z_{k+1} + (1+\varepsilon_2+\varepsilon_3)\bar{P}_{k+1|k}.$$

Therefore, it is straightforward to see that when $K_{k+1} = Z_{k+1}^T Y_{k+1}^{-1}$, the bound $\bar{P}_{k+1|k+1}$ is minimized and satisfies the next recursion:

$$\bar{P}_{k+1|k+1} = -\, Z_{k+1}^T Y_{k+1}^{-1} Z_{k+1} + (1 + \varepsilon_2 + \varepsilon_3)\bar{P}_{k+1|k}.$$

The proof is now complete. □

Remark 3: The filtering problem is solved in Theorem 1 in a recursive way for a class of discrete time-varying nonlinear systems with stochastic sensor saturations and event-triggered transmissions. To cope with the stochastic sensor saturations and event-triggered transmissions, special effort has been made to obtain the upper bound of the filtering error covariance and design the filter so as to minimize the bound. The matrix S_k reflects the linearization errors, the parameters Λ_λ and $\bar{b}$ represent the effects of stochastic sensor saturations, and the scalar ς quantities the influences of the event-triggered transmissions. The parameters ε_j can be determined to balance the intrinsic characteristic of the proposed filter and the impacts induced by ETT and stochastic sensor saturations. Neither the approximated PDF of states conditional on measurements nor the assumption on the distribution of $y_{k+1} - \tilde{y}_{k+1}$ is required in the presented approach. In other words, the applicability and feasibility of the algorithm have been enhanced. Furthermore, the desired filter gain is obtained via solving two sets of discrete Riccati-like equations, hence the method is suitable for online applications.

4. Boundedness Analysis

Before proceeding, the following widely used concept for the boundedness of stochastic processes is introduced.

Definition 1: Reif, Gunther, Yaz, & Unbehauen (1999) The stochastic process ζ_k is said to be exponentially bounded in mean square if there are real numbers $\eta > 0$, $\nu > 0$ and $0 < \vartheta < 1$ such that

$$\mathbb{E}\left\{\|\zeta_k\|^2\right\} \le \eta \|\zeta_0\|^2 \vartheta^k + \nu \tag{29}$$

holds for every $k > 0$.

For the boundedness analysis of the estimation error, we establish sufficient conditions under which the filtering error is exponentially bounded in mean square. For this purpose, we make the following assumption.

Assumption 1: *There are positive real numbers $\bar{a}, \bar{c}, \underline{c}, \bar{\lambda}, \underline{\lambda}, \bar{\psi}, \bar{s}, \bar{f}, \underline{f}, \bar{d}, \underline{d}, \bar{w}, \underline{w}, \bar{v} > 0$ such that the following bounds on various matrices are fulfilled for every $1 \le i \le m$ and $k \ge 0$:*

$$\begin{aligned} &\|A_k\| \le \bar{a}, \;\; \|S_k\| \le \bar{s}, \;\; \underline{c} \le \|C_k\| \le \bar{c}, \mathrm{tr}\left\{\Psi_k\right\} \le \bar{\psi}, \;\; \underline{\lambda} \le \lambda_i \le \bar{\lambda}, \\ &\underline{d}I \le D_k D_k^T \le \bar{d}I, \;\; \underline{f} \le \|F_k\| \le \bar{f}, \underline{w}I \le W_k \le \bar{w}I, \;\; V_k \le \bar{v}I. \end{aligned} \tag{30}$$

Moreover, the following inequality holds:

$$\varrho = \left[(1+\eta)\bar{a}^2 + (1+\eta^{-1})\bar{s}^2\right]\left[1 + \frac{\bar{c}^2}{(1-\bar{\lambda})\underline{c}^2}\right]^2 < 1, \tag{31}$$

where η is a positive scalar.

Theorem 2: *Consider the time-varying system (1) with the filter given in (6) and (7) whose parameters are provided in Theorem 1. Under Assumption 1, the filtering error is exponentially bounded in mean square.*

Proof. Denote $\Xi_{k+1} = I - K_{k+1}(I - \Lambda_\lambda)C_{k+1}$, $\breve{A}_{k+1} = \Xi_{k+1}A_k$ and $\hat{A}_{k+1} = \Xi_{k+1}S_kU_k$. Substituting (9) into (13) and considering the definitions of $\breve{A}_{k+1}$ and $\hat{A}_{k+1}$, we have

$$e_{k+1|k+1} = (\breve{A}_{k+1} + \hat{A}_{k+1})e_{k|k} + p_{k+1} + q_{k+1}, \tag{32}$$

where

$$\begin{aligned} p_{k+1} =& - K_{k+1}\Lambda_\lambda[\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})] - K_{k+1}(\tilde{y}_{k+1} - y_{k+1}), \\ q_{k+1} =& \Xi_{k+1}D_kw_k - K_{k+1}(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)[\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}] - K_{k+1}F_{k+1}v_{k+1}. \end{aligned}$$

Based on (22), it follows easily from $\underline{\lambda} \le \lambda_i \le \bar{\lambda}$ and $\underline{c} \le \|C_{k+1}\| \le \bar{c}$ that

$$\begin{aligned} \|K_{k+1}\| =& \left\|Z_{k+1}^T Y_{k+1}^{-1}\right\| \\ <& \Big\| \left[(1+\varepsilon_2+\varepsilon_3)(I-\Lambda_\lambda)C_{k+1}\bar{P}_{k+1|k}\right]^T [(1+\varepsilon_2+\varepsilon_3) \\ & \times (I-\Lambda_\lambda)C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T(I-\Lambda_\lambda)^T]^{-1} \Big\| \\ \le& \frac{\bar{c}}{(1-\bar{\lambda})\underline{c}^2} =: \bar{k}, \end{aligned}$$

and

$$\begin{aligned} \|\Xi_{k+1}\| <& \Big\| I - \left[(1+\varepsilon_2+\varepsilon_3)(I-\Lambda_\lambda)C_{k+1}\bar{P}_{k+1|k}\right]^T [(1+\varepsilon_2+\varepsilon_3)(I-\Lambda_\lambda) \\ & \times C_{k+1}\bar{P}_{k+1|k}C_{k+1}^T(I-\Lambda_\lambda)^T]^{-1}(I-\Lambda_\lambda)C_{k+1} \Big\| \\ \le& 1 + \frac{\bar{c}^2}{\underline{c}^2} =: \bar{\xi}. \end{aligned}$$

Then, we have

$$\begin{aligned} \left\|\breve{A}_{k+1}\right\| &\le \|\Xi_{k+1}\| \, \|A_k\| \le \bar{\xi}\bar{a} =: \bar{a}_1, \\ \left\|\hat{A}_{k+1}\right\| &\le \|\Xi_{k+1}\| \, \|S_k\| \, \|U_k\| \le \bar{\xi}\bar{s} =: \bar{a}_2. \end{aligned}$$

Recalling Lemma 2, we can obtain

$$\begin{aligned} & \mathbb{E}\left\{p_{k+1}^T p_{k+1}\right\} \\ \le& (1+\eta_1)\mathbb{E}\big\{[\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})]^T \Lambda_\lambda^T K_{k+1}^T K_{k+1}\Lambda_\lambda[\sigma(C_{k+1}x_{k+1}) - \sigma(C_{k+1}\hat{x}_{k+1|k})]\big\} \\ & + (1+\eta_1^{-1})\mathbb{E}\left\{(\tilde{y}_{k+1} - y_{k+1})^T K_{k+1}^T K_{k+1}(\tilde{y}_{k+1} - y_{k+1})\right\}, \end{aligned} \tag{33}$$

where η_1 is a positive scalar.

Substituting (27) and (28) into (33) leads to

$$\mathbb{E}\left\{p_{k+1}^T p_{k+1}\right\} \le 4(1+\eta_1)\bar{b}\bar{\lambda}^2\bar{k}^2 + (1+\eta_1^{-1})\varsigma\bar{k}^2 =: \bar{p}^2.$$

Since w_k, v_k, and $\alpha_{i,k}$ are assumed to be mutually independent, we have

$$\begin{aligned}&\mathbb{E}\left\{q_{k+1}^T q_{k+1}\right\}\\ =&\mathbb{E}\left\{w_k^T D_k^T \Xi_{k+1}^T \Xi_{k+1} D_k w_k\right\} + \mathbb{E}\left\{v_{k+1}^T F_{k+1}^T K_{k+1}^T K_{k+1} F_{k+1} v_{k+1}\right\} + \mathbb{E}\big\{[\sigma(C_{k+1}x_{k+1})\\ &- C_{k+1}x_{k+1}]^T(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)^T K_{k+1}^T K_{k+1}(\Lambda_{\alpha_{k+1}} - \Lambda_\lambda)[\sigma(C_{k+1}x_{k+1}) - C_{k+1}x_{k+1}]\big\}\\ \leq& p\bar{\xi}^2\bar{d}^2\bar{w} + q\bar{k}^2\bar{f}^2\bar{v} + \bar{k}^2\hat{\lambda}^2\bar{\psi} =: \bar{q}^2,\end{aligned}$$

where $\hat{\lambda} = \max\{1 - \underline{\lambda}, \bar{\lambda}\}$.

Consider the following iterative matrix equation

$$\Pi_{k+1} = (1+\eta)\breve{A}_{k+1}\Pi_k\breve{A}_{k+1}^T + (1+\eta^{-1})\rho_{\max}(\Pi_k)\Xi_{k+1}S_kS_k^T\Xi_{k+1}^T + D_kW_kD_k^T,$$

with initial condition $\Pi_0 = D_0W_0D_0^T$ where $\rho_{\max}(\Pi_k)$ represents the maximum eigenvalue of Π_k. Then, it follows directly that

$$\begin{aligned}\|\Pi_{k+1}\| &\leq \|\Pi_k\| \left[(1+\eta)\left\|\breve{A}_{k+1}\right\|^2 + (1+\eta^{-1})\left\|\Xi_{k+1}S_k\right\|^2\right] + \left\|D_kW_kD_k^T\right\|\\ &\leq \varrho\|\Pi_k\| + \bar{w}\bar{d}^2.\end{aligned}$$

By iteration, we obtain

$$\|\Pi_k\| \leq \varrho^k\|\Pi_0\| + \bar{w}\bar{d}^2\sum_{i=0}^{k-1}\varrho^i.$$

With assumption (31), we have $\varrho < 1$ and then arrive at

$$\|\Pi_k\| < \|\Pi_0\| + \bar{w}\bar{d}^2\sum_{i=0}^{\infty}\varrho^i = \|\Pi_0\| + \frac{\bar{w}\bar{d}^2}{1-\varrho}. \tag{34}$$

Furthermore, since Π_k is positive definite for all k, it is straightforward to see that

$$\Pi_{k+1} \geq D_kW_kD_k^T \geq \underline{wd}^2 I. \tag{35}$$

Based on (34) and (35), it can be concluded that there are positive real numbers $\underline{\pi}, \bar{\pi} > 0$ such that the inequality $\underline{\pi}I \leq \Pi_k \leq \bar{\pi}I$ holds for every $k \geq 0$.

According to Assumption 1, we have

$$\begin{aligned}&(\breve{A}_{k+1} + \hat{A}_{k+1})^T\Pi_{k+1}^{-1}(\breve{A}_{k+1} + \hat{A}_{k+1}) - \Pi_k^{-1}\\ \leq&(\breve{A}_{k+1} + \hat{A}_{k+1})^T\left[(\breve{A}_{k+1} + \hat{A}_{k+1})\Pi_k(\breve{A}_{k+1} + \hat{A}_{k+1})^T + D_kW_kD_k^T\right]^{-1}(\breve{A}_{k+1} + \hat{A}_{k+1}) - \Pi_k^{-1}\\ =&-\left[\Pi_k + \Pi_k(\breve{A}_{k+1} + \hat{A}_{k+1})^T\left(D_kW_kD_k^T\right)^{-1}(\breve{A}_{k+1} + \hat{A}_{k+1})\Pi_k\right]^{-1}\\ =&-\left[I + (\breve{A}_{k+1} + \hat{A}_{k+1})^T\left(D_kW_kD_k^T\right)^{-1}(\breve{A}_{k+1} + \hat{A}_{k+1})\Pi_k\right]^{-1}\Pi_k^{-1}\\ \leq&-\left[\frac{(\bar{a}_1 + \bar{a}_2)^2\bar{\pi}}{\underline{wd}^2} + 1\right]^{-1}\Pi_k^{-1}.\end{aligned}$$

Define $\alpha_0 = \left[\frac{(\bar{a}_1+\bar{a}_2)^2\bar{\pi}}{\underline{w}\underline{d}^2} + 1\right]^{-1}$. Since $\alpha_0 < 1$, there always exists a positive scalar β such that $\alpha = (1+\beta)(1-\alpha_0) < 1$. Choosing a positive scalar $\gamma > 0$, and denoting $V_k\left(e_{k|k}\right) = e_{k|k}^T \Pi_k^{-1} e_{k|k}$, and $\mu = \left[(1+\beta^{-1}+\gamma)\bar{p}^2 + (1+\gamma^{-1})\bar{q}^2\right]/\underline{\pi}$, it follows from (32) that

$$
\begin{aligned}
&\mathbb{E}\left\{V_{k+1}(e_{k+1|k+1})|e_{k|k}\right\} - (1+\beta)V_k(e_{k|k}) \\
=&\mathbb{E}\Big\{ \left[(\check{A}_{k+1} + \hat{A}_{k+1})e_{k|k} + p_{k+1} + q_{k+1}\right]^T \Pi_{k+1}^{-1}\left[(\check{A}_{k+1} + \hat{A}_{k+1})e_{k|k} + p_{k+1} + q_{k+1}\right]|e_{k|k}\Big\} \\
&- (1+\beta)e_{k|k}^T \Pi_k^{-1} e_{k|k} \\
=&\mathbb{E}\Big\{e_{k|k}^T(\check{A}_{k+1} + \hat{A}_{k+1})^T \Pi_{k+1}^{-1}(\check{A}_{k+1} + \hat{A}_{k+1})e_{k|k} - (1+\beta)e_{k|k}^T \Pi_k^{-1} e_{k|k}|e_{k|k}\Big\} + 2\mathbb{E}\Big\{e_{k|k}^T(\check{A}_{k+1} \\
&+ \hat{A}_{k+1})^T \Pi_{k+1}^{-1} p_{k+1}|e_{k|k}\Big\} + \mathbb{E}\left\{p_{k+1}^T \Pi_{k+1}^{-1} p_{k+1}|e_{k|k}\right\} + 2\mathbb{E}\left\{p_{k+1}^T \Pi_{k+1}^{-1} q_{k+1}|e_{k|k}\right\} \\
&+ \mathbb{E}\left\{q_{k+1}^T \Pi_{k+1}^{-1} q_{k+1}|e_{k|k}\right\}.
\end{aligned} \tag{36}
$$

Applying Lemma 2 to (36), we have

$$
\begin{aligned}
&\mathbb{E}\left\{V_{k+1}(e_{k+1|k+1})|e_{k|k}\right\} - (1+\beta)V_k(e_{k|k}) \\
\leq&(1+\beta)\mathbb{E}\Big\{e_{k|k}^T\left[(\check{A}_{k+1} + \hat{A}_{k+1})^T \Pi_{k+1}^{-1}(\check{A}_{k+1} + \hat{A}_{k+1}) - \Pi_k^{-1}\right]e_{k|k}|e_{k|k}\Big\} + (1+\beta^{-1}+\gamma) \\
&\times \mathbb{E}\left\{p_{k+1}^T \Pi_{k+1}^{-1} p_{k+1}|e_{k|k}\right\} + (1+\gamma^{-1})\mathbb{E}\left\{q_{k+1}^T \Pi_{k+1}^{-1} q_{k+1}|e_{k|k}\right\} \\
\leq& - \alpha_0(1+\beta)V_k(e_{k|k}) + \mu.
\end{aligned}
$$

Then it follows that

$$
\mathbb{E}\left\{V_{k+1}(e_{k+1|k+1})|e_{k|k}\right\} \leq \alpha V_k(e_{k|k}) + \mu,
$$

which gives rise to

$$
\begin{aligned}
\mathbb{E}\left\{\left\|e_{k|k}\right\|^2\right\} &\leq \frac{\bar{\pi}}{\underline{\pi}}\left\|e_{0|0}\right\|^2 \alpha^k + \mu\bar{\pi}\sum_{i=0}^{\infty}\alpha^i \\
&= \frac{\bar{\pi}}{\underline{\pi}}\left\|e_{0|0}\right\|^2 \alpha^k + \frac{\mu\bar{\pi}}{1-\alpha}
\end{aligned}
$$

in which the relationships $0 < \alpha < 1$ and $\mu, \bar{\pi} > 0$ have been utilized. Therefore, the stochastic process $e_{k|k}$ is exponentially bounded in mean square and the proof is complete. □

5. Fault Estimation

In this section, we aim to show that the main results in Theorem 1 can be applied to estimate both the system state and additive faults within a unified framework.

Consider the following faulty system corresponding to (1):

$$
\begin{cases}
x_{k+1} = g(x_k, u_k) + D_k w_k + E_k f_k, \\
y_k = \Lambda_{\alpha_k}\sigma(C_k x_k) + (I - \Lambda_{\alpha_k})C_k x_k + F_k v_k,
\end{cases} \tag{37}
$$

where $f_k \in \mathbb{R}^l$ is the additive fault, E_k is a known matrix with appropriate dimensions, and all the other variables are the same as defined in (1). Defining an augmented state $\bar{x}_k = \left[x_k^T, f_k^T\right]^T$, (37) can be rewritten as follows:

$$\begin{cases} \bar{x}_{k+1} = \bar{g}(\bar{x}_k, u_k) + \bar{D}_k w_k, \\ y_k = \Lambda_{\alpha_k} \sigma(\bar{C}_k \bar{x}_k) + (I - \Lambda_{\alpha_k}) \bar{C}_k \bar{x}_k + F_k v_k, \end{cases} \tag{38}$$

where

$$\bar{g}(\bar{x}_k, u_k) := \begin{bmatrix} g(x_k, u_k) + E_k f_k \\ f_k \end{bmatrix}, \bar{D}_k := \begin{bmatrix} D_k \\ 0 \end{bmatrix}, \bar{C}_k := [C_k, 0].$$

Similar to (6) and (7), consider a filter of the following structure:

$$\tilde{x}_{k+1|k} = \bar{g}(\tilde{x}_{k|k}, u_k), \tag{39}$$

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{k+1|k} + \tilde{K}_{k+1} \big[\tilde{y}_{k+1} - \Lambda_\lambda \sigma(\bar{C}_{k+1} \tilde{x}_{k+1|k}) - (I - \Lambda_\lambda) \bar{C}_{k+1} \tilde{x}_{k+1|k} \big], \tag{40}$$

where $\tilde{x}_{k|k} \in \mathbb{R}^n$ is the estimation of $\bar{x}_k$ at time step k with $\tilde{x}_{0|0} = \left[\mathbb{E}\left\{ x_0^T \right\}, 0^T \right]^T$, $\tilde{x}_{k+1|k} \in \mathbb{R}^n$ is the one step prediction at time step k, and $\tilde{K}_{k+1}$ is the filter gain to be determined. Denote the prediction error, the estimation error and their covariances conditional on the received measurements as $\tilde{e}_{k+1|k} = \bar{x}_{k+1} - \tilde{x}_{k+1|k}$, $\tilde{e}_{k+1|k+1} = \bar{x}_{k+1} - \tilde{x}_{k+1|k+1}$, $Q_{k+1|k} = \mathbb{E}\left\{ \tilde{e}_{k+1|k} \tilde{e}_{k+1|k}^T | y_0, \ldots, y_k \right\}$, and $Q_{k+1|k+1} = \mathbb{E}\left\{ \tilde{e}_{k+1|k+1} \tilde{e}_{k+1|k+1}^T | y_0, \ldots, y_{k+1} \right\}$, respectively. Then, we can obtain the following theorem whose proof is similar to that of Theorem 1 and is therefore omitted here.

Theorem 3: *Let $\tilde{\varepsilon}_j (j = 1, \ldots, 8)$ and $\tilde{\gamma}_k (k \in \mathbb{N})$ be positive scalars. Assume that, with initial condition $\bar{Q}_{0|0} = Q_{0|0}$, the following equations*

$$\bar{Q}_{k+1|k} = (1 + \tilde{\varepsilon}_1) \tilde{A}_k \bar{Q}_{k|k} \tilde{A}_k^T + \tilde{\gamma}_k (1 + \tilde{\varepsilon}_1^{-1}) \tilde{S}_k \tilde{S}_k^T + \bar{D}_k W_k \bar{D}_k^T, \tag{41}$$

$$\begin{aligned} \bar{Q}_{k+1|k+1} = & (1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3) \left[I - \tilde{K}_{k+1} (I - \Lambda_\lambda) \bar{C}_{k+1} \right] \bar{Q}_{k+1|k} \left[I - \tilde{K}_{k+1} (I - \Lambda_\lambda) \bar{C}_{k+1} \right]^T + (1 + \tilde{\varepsilon}_6) \\ & \times \tilde{K}_{k+1} (\tilde{\Lambda} \circ \tilde{\Theta}_{k+1}) \tilde{K}_{k+1}^T + 4\bar{b}(1 + \tilde{\varepsilon}_2^{-1} + \tilde{\varepsilon}_4) \tilde{K}_{k+1} \Lambda_\lambda \Lambda_\lambda^T \tilde{K}_{k+1}^T + \varsigma(1 + \tilde{\varepsilon}_3^{-1} + \tilde{\varepsilon}_4^{-1} + \tilde{\varepsilon}_5 \\ & + \tilde{\varepsilon}_6^{-1}) \tilde{K}_{k+1} \tilde{K}_{k+1}^T + (1 + \tilde{\varepsilon}_5^{-1}) \tilde{K}_{k+1} F_{k+1} V_{k+1} F_{k+1}^T \tilde{K}_{k+1}^T, \end{aligned} \tag{42}$$

have positive definite solutions, where

$$\tilde{A}_k = \left. \frac{\partial \bar{g}(\bar{z}_k, u_k)}{\partial \bar{z}_k} \right|_{\bar{z}_k = \tilde{x}_{k|k}} = \begin{bmatrix} \left. \dfrac{\partial g(z_k, u_k)}{\partial z_k} \right|_{z_k = H\tilde{x}_{k|k}} & E_k \\ 0 & I \end{bmatrix}, \tag{43}$$

$$H = [I \ \ 0], \tag{44}$$

$$\begin{aligned} \tilde{\Theta}_{k+1} = & \bar{b}(1 + \tilde{\varepsilon}_7) I + (1 + \tilde{\varepsilon}_7^{-1})(1 + \tilde{\varepsilon}_8) \bar{C}_{k+1} \bar{Q}_{k+1|k} \bar{C}_{k+1}^T \\ & + (1 + \tilde{\varepsilon}_7^{-1})(1 + \tilde{\varepsilon}_8^{-1}) \bar{C}_{k+1} \tilde{x}_{k+1|k} \tilde{x}_{k+1|k}^T \bar{C}_{k+1}^T, \end{aligned} \tag{45}$$

$$\bar{Q}_{k|k} \leq \tilde{\gamma}_k I, \tag{46}$$

$$\begin{aligned} \tilde{Y}_{k+1} = & (1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3)(I - \Lambda_\lambda) \bar{C}_{k+1} \bar{Q}_{k+1|k} \bar{C}_{k+1}^T (I - \Lambda_\lambda)^T + 4\bar{b} \\ & \times (1 + \tilde{\varepsilon}_2^{-1} + \tilde{\varepsilon}_4) \Lambda_\lambda \Lambda_\lambda^T + \tilde{\Lambda} \circ \tilde{\Theta}_{k+1} + \varsigma(1 + \tilde{\varepsilon}_3^{-1} + \tilde{\varepsilon}_4^{-1} \\ & + \tilde{\varepsilon}_5) I + (1 + \tilde{\varepsilon}_5^{-1}) F_{k+1} V_{k+1} F_{k+1}^T, \end{aligned} \tag{47}$$

$$\tilde{Z}_{k+1} = (1 + \tilde{\varepsilon}_2 + \tilde{\varepsilon}_3)(I - \Lambda_\lambda) \bar{C}_{k+1} \bar{Q}_{k+1|k}, \tag{48}$$

$$\tilde{K}_{k+1} = \tilde{Z}_{k+1}^T \tilde{Y}_{k+1}^{-1}. \tag{49}$$

$\tilde{S}_k$ is a problem-dependent scaling matrix. $\bar{b}$ and $\tilde{\Lambda}$ are the same as defined in (16) and (19), respectively. Then $\bar{Q}_{k|k}$ is an upper bound of $Q_{k|k}$, and the bound $\bar{Q}_{k|k}$ is minimized at each time step with the filter gain given in (49).

The proof is similar to that of Theorem 1 and is therefore omitted here. With Theorem 3, the state estimation and fault diagnosis problem can get solved simultaneously. The possible fault and state have been regarded as an augmented state and jointly estimated in the proposed filter. In the next section, a simulation example is illustrated to show the effectiveness of the proposed filter.

6. Illustrations

Inspired by the model proposed in Li & Shi (2012), the following inverted pendulum example is considered in this section:

$$\begin{bmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{T}{ml^2} \\ -\kappa T + Tk_1 & 1 - \dfrac{T\chi}{ml^2} + Tk_2 \end{bmatrix} \begin{bmatrix} x_k^{(1)} \\ x_k^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ Tmgl\sin(x_k^{(1)}) \end{bmatrix} + \begin{bmatrix} 0 \\ 2T \end{bmatrix} w_k.$$

where $x_1 = \theta$, $x_2 = ml^2\dot{\theta}$, m is the mass, l is the length of the inverted pendulum, T is the sampling period, g is the gravitation coefficient, θ is the inclination angle, χ is the spring coefficient, κ is the damping parameter. The output measurement with stochastic sensor saturation can be written as

$$y_k = \lambda_k \sigma\left(0.1x_k^{(1)} + 0.1x_k^{(2)}\right) + (1-\lambda_k)\left(0.1x_k^{(1)} + 0.1x_k^{(2)}\right) + v_k.$$

The system parameters are $m = 0.5$kg, $l = 0.5$m, $\chi = 0.25$, $k_1 = -49.5$, $k_2 = -167.5$, sampling period $T = 0.01$s, and $\kappa = 0.5$N/m. The variances of w_k and v_k are 0.25 and 9×10^{-4}, respectively. $\text{Prob}\{\lambda_k = 1\} = 0.8$. The saturation level is 0.2. The transmission threshold is set to be $\varsigma = 0.002$. The initial states are uniformly distributed over [0.5,1.5]. $\varepsilon_i(i = 1, 2, 3, 5, 7)$ are selected as 0.1, and $\varepsilon_i(i = 4, 6, 8)$ are determined as 1. For all $i = 1, 2, \ldots, 8$, $\tilde{\varepsilon}_i = \varepsilon_i$.

In the fault-free case, Fig. 1 and Fig. 2 show the systems states and their estimates. Fig. 3 illustrates the real filtering errors and the bound calculated from Theorem 2. It can be seen that acceptable estimation performance is achieved.

When the inverted pendulum is subject to unexpected torques, additive faults may occur. Consider a fault f_k in the following form in $x_k^{(1)}$:

$$f_k = \begin{cases} -0.9, & \text{if} \quad k \geq 26, \\ 0, & \text{otherwise.} \end{cases}$$

Fig. 4 depicts the actual fault and its estimate obtained from Theorem 3. It can be observed that the proposed filter could estimate the additive fault well.

7. Conclusion

In this paper, the filtering problem has been investigated for a class of time-varying nonlinear systems with stochastic sensor saturations and event-triggered measurement transmissions. Special effort has been made to obtain an upper bound of the filtering error covariance and then minimize such an upper bound by solving two sets of discrete matrix equations. The presented method has been utilized to estimate the additive faults. Future research topics would include the

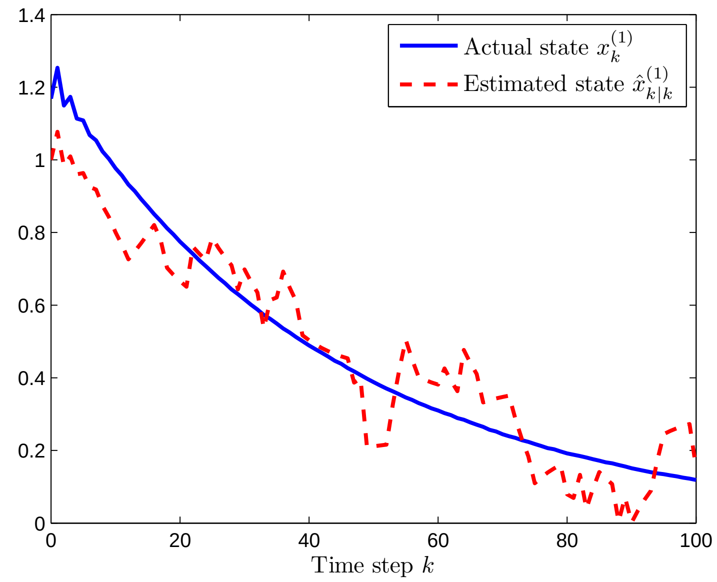

Figure 1. State 1 and its estimate

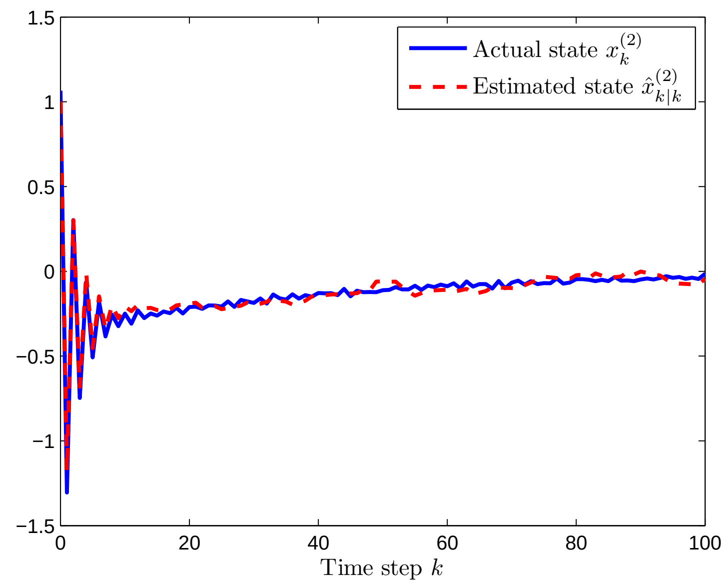

Figure 2. State 2 and its estimate

extension of our results to more complex systems such as nonlinear polynomial systems Basin & Rodriguez-Ramirez (2012), delayed sensing systems Caballero-Águila, Hermoso-Carazo, Jiménez-López, Linares-Pérez, & Nakamori (2010), networked control systems Karimi (2009) and two-dimensional systems Li & Gao (2012).

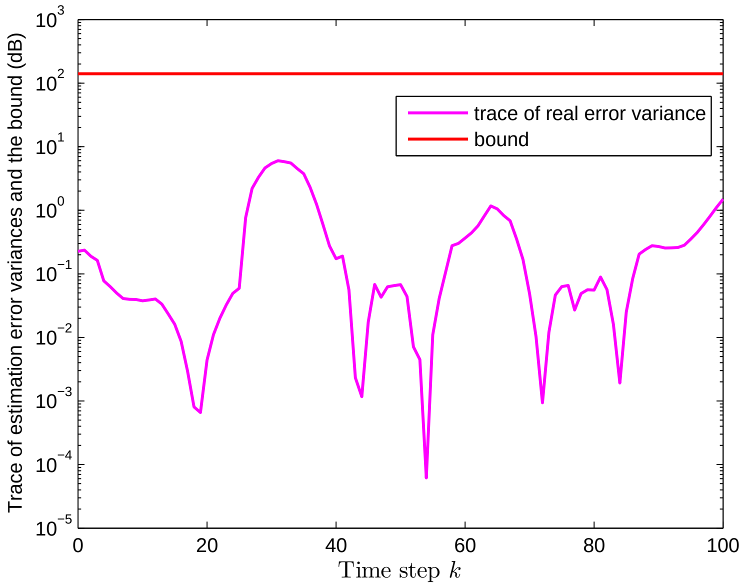

Figure 3. The estimation error and bound

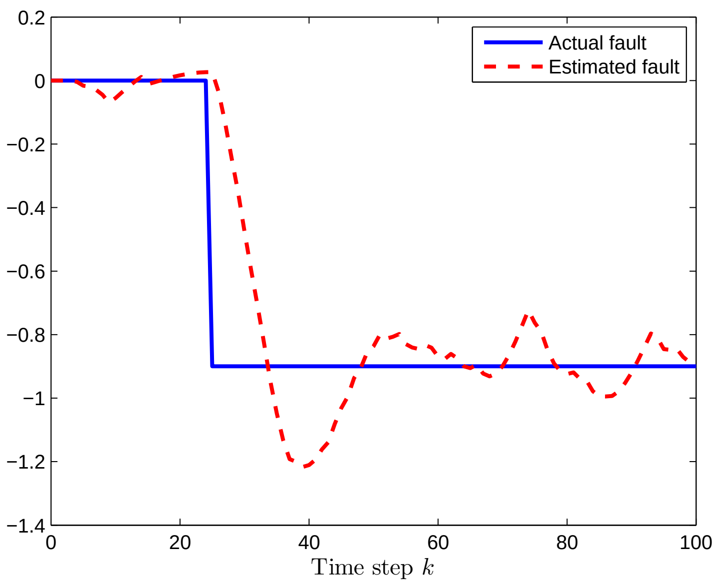

Figure 4. The fault and its estimate

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