Capturing Extremes

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***Abstract*— The ability of the Generalised Extreme Value and Generalised Logistic distributions to describe adequately extreme financial returns is examined. The empirical results strongly reject the Generalised Extreme Value in favour of the fatter tailed Generalised Logistic. This implies that risk measurements which are based on the Generalised Extreme Value may underestimate risk since it assigns lower probabilities to the really ruinous events located deep into the tails of the returns distribution.**

***Index Terms*— Extreme Value Theory, Probability Weighted Moments, Anderson-Darling goodness of fit test, Generalised Extreme Value distribution, Generalised Logistic distribution.**

I. INTRODUCTION

Accurate modelling of the empirical distribution of financial returns is of crucial importance for risk management. A popular assumption made is that log-returns are generated by a normal distribution; however, it has long been known that the probabilities of large returns are much greater than implied by the normal. Extreme Value Theory (EVT) is a branch of statistics which studies the probabilities of extreme events. Current favoured distribution for the extremes is the Generalised Extreme Value (GEV) proposed by [1]. The GEV is made up of the Weibull, Fréchet and Gumbel distributions, and it is used to model extreme financial returns which have been collected as the minimum or maximum returns over non overlapping time intervals of equal length.

The contribution of this paper is two fold. First, the method of Probability Weighted Moments (PWM), which is consider to yield less biased parameter and quantile estimates, is proposed in order to estimate the parameters of the extremes distributions. Second, inspired by research and practise in flood frequency analysis, the ability of the Generalised Logistic (GL) distribution to fit extreme minima is examined[1]. It was found that the GL provides a considerable better fit, compared to the GEV, to the minima extremes of the FTSE100 daily returns.

II. DISTRIBUTIONS OF EXTREMES

Extremes of financial returns are defined as the minimum of the daily (or weekly, monthly or larger time periods) logarithmic returns over a given period. Denote, for example, the time series of an index daily log-returns with the variable $Y_1, Y_2, \ldots, Y_n$. If the length of the selection interval is m, we divide the series into non-overlapping time intervals of length m. The time series of the extreme minima will be $X_1 = min(Y_1, \ldots, Y_m)$, $X_2 = min(Y_{m+1}, \ldots, Y_{2m}), \ldots, X_{n/m} = min(Y_{n-m}, \ldots, Y_n)$. According to the extreme value theorem [2], the limiting distribution of the extremes, after normalised and centered, ought to be the GEV. The GEV is a three parameter distribution which has the following probability density function (pdf):

$$f(x) = \alpha^{-1} e^{-(1-\kappa)y} e^{-e^{-y}}, \text{ where}$$

$$y = \begin{cases} -\kappa^{-1} \log\{1-\kappa(x-\beta)/\alpha\}, & \kappa \neq 0 \\ (x-\beta)/\alpha, & \kappa = 0 \end{cases} \quad (1)$$

the parameters α, β and κ are called scale, location and shape, respectively. The first parameter is analogous to the standard deviation, the second is analogous to the mean, while the third governs the shape of the tail of the distribution and it is probably the most important parameter since larger values correspond to fatter tailed distributions. The Gumbel distribution is obtained for $\kappa = 0$, the Fréchet for $\kappa < 0$ and the Weibull is the special case of the reversed GEV when $\kappa > 0$.

Although its use in finance is rather limited, the GL has found to be very popular in flood frequency analysis. The pdf of the GL is given by:

$$f(x) = \alpha^{-1} e^{-(1-\kappa)y} / (1+e^{-y})^2, \quad \text{where}$$

$$y = \begin{cases} -\kappa^{-1} \log\{1-\kappa(x-\beta)/\alpha\}, & \kappa \neq 0 \\ (x-\beta)/\alpha, & \kappa = 0 \end{cases} \quad (2)$$

the logistic distribution is obtained when $\kappa = 0$.

From the parameter estimation methods2 available, the PWM method has found to provide less biased parameter and quantile estimates with lower root mean square error ([3]; [4]; [5]; [6]). [7] defined the PWM of a random variable X with a finite mean and a distribution function F to be the quantities:

$$M_{p,r,s} = E\left[X^p \{F(X)\}^r \{1-F(X)\}^s\right] \quad (3)$$

Where $E[X(\cdot)]$ is the expectation of the quantile function of X and *p*, *r*, and *s* are real numbers. It is often better to work with the $M_{1,r,s}$ because the implied relationship between

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[1] The focus is kept on describing the lower tail of the returns distribution since this is where the big losses of a long position are located. However, similar analysis can be applied to the upper tail for the case of a short position.

[2] Maximum Likelihood (ML) is probably the most popular parameter estimation method. However, in the case of small samples, which are the norm in EVT, convergence of the likelihood function is not always guaranteed to be at the global maximum; ML parameter estimates are usually obtained by finding a local maximum of the likelihood function.

parameters, quantiles and moments is linear since only the first power of X appears in the expression of $M_{1,r,s}$. In addition, when *r* and *s* are integers, $F^r(1-F)^s$ can be expressed as a linear combination of either powers of *F* or powers of *(1-F)*. [3] suggested the moments $M_{1,r,0}$ in order to summarise a distribution:

$$M_{1,r,0} = \beta_r = E\left[X\{F(X)\}^r\right], \qquad r = 0,1,\cdots \quad (4)$$

PWM involves estimating parameters by equating sample moments to those of the chosen distribution. Although, PWM may be sensitive to outliers, [8] demonstrated that there exist linear relationships between the PWM and the more robust L-moments, given by:

$$\lambda_{r+1} = \sum_{k=0}^{r} P^*_{r,k}\beta_k, \qquad r = 0,1,\cdots, \quad \text{where}$$

$$P^*_{r,k} = (-1)^{r-k}\binom{r}{k}\binom{r+k}{k} \quad (5)$$

L-moments are linear combinations of ordered data which, like the conventional moments, provide descriptions of probability distributions[3]. [8] defined the r^{th} L-moment, λ_r, for any random variable X which has a finite mean as:

$$\lambda_r \equiv r^{-1}\sum_{\kappa=0}^{r-1}(-1)^{\kappa}\binom{r-1}{\kappa}\mathrm{E}X_{(r-\kappa:r)}, \qquad r = 1,2,\cdots \quad (6)$$

where $EX_{(r-\kappa:r)}$ is the expectation of the $(r-\kappa)^{th}$ extreme order statistic[4].

For the GEV the solutions for the parameter estimates are:

$\kappa = 7.8590c + 2.9554c^2$, where

$$c = \frac{(2\beta_1 - \beta_0)}{(3\beta_2 - \beta_0)} - \frac{\ln 2}{\ln 3} \quad (7)$$

$$\alpha = \frac{\lambda_2\kappa}{(1-2^{-\kappa})\Gamma(1+\kappa)} \quad (8)$$

$$\beta = \lambda_1 - \frac{\alpha}{\kappa}\{1-\Gamma(1+\kappa)\} \quad (9)$$

and for the GL:

$$\kappa = -\tau_3 \quad (10)$$

$$\alpha = \frac{\lambda_2}{\Gamma(1-\kappa)\Gamma(1+\kappa)} \quad (11)$$

$$\beta = \lambda_1 - \frac{\alpha}{\kappa}\{1-\Gamma(1-\kappa)\Gamma(1+\kappa)\} \quad (12)$$

Once the parameters have been estimated it is important to assess the goodness of fit to the empirical data. [9] defined a goodness of fit test by:

$$A_n^2 = -n - (1/n)\sum_{i=1}^{n}\left[(2i-1)\log z_i + (2n+1-2i)\log(1-z_i)\right] \quad (13)$$

where, $z_i = F(x_i)$, $i = 1,\cdots,n$ is the empirical distribution function of a variable X *of size* n. [10] and [11] have reported that the AD test is the most powerful among a wide set of available tests for small samples.

III. Capturing the Extremes

Daily returns were collected for the FTSE100 index from 1979 to 2006 and weekly and monthly extreme minima were calculated as the minimum daily return over a week and month respectively[5]. The GL and GEV distributions were fitted to the weekly and monthly minima for the whole interval and for 10 sub-periods. The PWM parameter estimates and the *p*-value of the AD goodness of fit test are contained in Tables 1 and 2. It is only the GL that fitted adequately the weekly minima for the whole interval with an AD *p*-value of 0.152. For the case of 10 sub-periods, the GEV fitted adequately in 7 while the GL in 9 sub-periods but in comparison the GL fitted better than the GEV in 9 of the 10 sub-periods. Both distributions fitted adequately the series of monthly extremes for the whole period as well as the 10 sub-periods. However, the AD test tended to take higher *p*-values in the case of the GL. In summary, the GL fitted better than the GEV in 7 of the 10 sub-periods. Overall, it appears that the GL has the ability to fit the extreme daily returns better than the GEV.

Since the GL is a fatter tailed distribution than the GEV, the empirical findings imply that risk measurement tools based on the GEV would tend to underestimate the probabilities of extreme events. This is highlighted in Table 3 where the probabilities of obtaining a daily return of specific magnitude were estimated according to the normal, GEV and GL distributions. For that reason the following intervals were used: [μ-1σ, μ-2σ], [μ-2σ, μ-3σ], [μ-3σ, μ-4σ], [μ-4σ, μ-5σ] and [μ-5σ, μ-6σ], where μ and σ are the mean and standard deviation of the daily returns. It can be noticed that both the GEV and GL assigned more accurate probabilities to the extreme daily returns. However, the GL tends to be more accurate than the GEV, especially deep into the tails of the returns distributions, when sub-periods are examined.

IV. Conclusion

It was found that the too much celebrated GEV distribution is not the best model for the extreme minima of the FTSE100 daily returns since a fatter tailed distribution, the GL, was found to offer better tail descriptions. Considering that current applications of EVT in finance focus mainly on the GEV distribution the implication is that the probabilities of the really ruinous events maybe underestimated.

[3] The most important feature of the L-moments is that they are more robust to the presence of outliers than conventional moments. This is because the calculations of conventional moments involve powers which give greater weight to outliers that can lead to considerable bias and variance in the parameter and quantile estimates.

[4] The first two such statistics, λ1 and λ2, are measures of location and scale and the two L-moment ratios, $\tau_3 = \lambda_3/\lambda_2$ and $\tau_4 = \lambda_4/\lambda_2$ are measures of skewness and kurtosis, respectively.

[5] The daily mean return was 0.04% and the daily standard deviation 1.16%. The minimum daily return was -17.60% and the maximum was 12.67%. The skewness value of -0.561 implies that negative returns were larger than positive returns while the kurtosis value of 19.448 implies that the distribution of returns is fat tailed.

Table 1

FTSE100 weekly minima GEV and GL PWM parameter estimates and AD p-values

		GEV parameter estimates				GL parameter estimates				
Sub-periods (s)	***N***	β_s	α_s	κ_s	**AD *p*-value**	β_s	α_s	κ_s	**AD *p*-value**	**Better fit**
***s* = 1**										
1.	1423	0.007	0.006	-0.134	0.000	-0.009	0.004	0.259	0.152	**GL**
***s* = 10**										
1.	142	0.005	0.004	-0.518	0.000	-0.007	0.004	0.549	0.043	**GL**
2.	142	0.007	0.006	-0.131	0.254	-0.009	0.004	0.257	0.850	**GL**
3.	142	0.008	0.008	0.157	0.180	-0.011	0.005	0.073	0.364	**GL**
4.	142	0.007	0.007	-0.209	0.047	-0.010	0.005	0.311	0.474	**GL**
5.	142	0.008	0.007	-0.055	0.298	-0.010	0.004	0.206	0.863	**GL**
6.	142	0.006	0.005	0.071	0.488	-0.008	0.003	0.125	0.752	**GL**
7.	142	0.006	0.006	0.036	0.727	-0.008	0.004	0.147	0.866	**GL**
8.	142	0.010	0.007	0.031	0.818	-0.013	0.005	0.150	0.683	**GEV**
9.	142	0.010	0.008	-0.019	0.016	-0.013	0.006	0.182	0.475	**GL**
10.	145	0.006	0.004	-0.120	0.096	-0.007	0.003	0.249	0.215	**GL**

This table includes the PWM parameter estimates and the Anderson-Darling (AD) goodness of fit test p-values for the GEV fitted to the reverse weekly minima and for the GL fitted to the weekly minima over the period 1979 to 2006. The GEV distribution is fitted to the reverse minima because although it is not symmetric around its location, results that hold for a random variable X_n generated by the GEV can be extended for the reverse variable $-X_n$. This affects both the location and shape parameters sign. N denotes the number of extreme observations in each sub-period, and β_s, α_s and κ_s denote the location, scale and shape parameters, respectively.

Table 2

FTSE100 monthly minima GEV and GL PWM parameter estimates and AD p-values

		GEV parameter estimates				GL parameter estimates				
Sub-periods (s)	***N***	β_s	α_s	κ_s	**AD *p*-value**	β_s	α_s	κ_s	**AD *p*-value**	**Better fit**
***s* = 1**										
1.	356	0.014	0.007	-0.254	0.167	-0.017	0.005	0.344	0.539	**GL**
***s* = 10**										
1.	36	0.013	0.012	-0.428	0.056	-0.018	0.010	0.477	0.079	**GL**
2.	36	0.015	0.007	-0.140	0.509	-0.018	0.005	0.263	0.447	**GEV**
3.	36	0.018	0.007	0.256	0.324	-0.021	0.004	0.016	0.149	**GEV**
4.	36	0.015	0.007	-0.400	0.066	-0.018	0.006	0.454	0.253	**GL**
5.	36	0.018	0.008	-0.045	0.615	-0.020	0.005	0.199	0.910	**GL**
6.	36	0.012	0.004	-0.014	0.545	-0.014	0.003	0.179	0.411	**GEV**
7.	36	0.013	0.006	-0.093	0.012	-0.016	0.004	0.231	0.019	**GL**
8.	36	0.019	0.006	-0.119	0.746	-0.021	0.004	0.249	0.822	**GL**
9.	36	0.017	0.007	-0.071	0.572	-0.020	0.005	0.217	0.602	**GL**
10.	32	0.011	0.004	-0.144	0.065	-0.012	0.003	0.266	0.394	**GL**

This table includes the PWM parameter estimates and the Anderson-Darling (AD) goodness of fit test p-values for the GEV fitted to the reverse monthly minima and for the GL fitted to the monthly minima over the period 1979 to 2006. The GEV distribution is fitted to the reverse minima because although it is not symmetric around its location, results that hold for a random variable X_n generated by the GEV can be extended for the reverse variable $-X_n$. This affects both the location and shape parameters sign. N denotes the number of extreme observations in each sub-period, and β_s, α_s and κ_s denote the location, scale and shape parameters, respectively.

Table 3

Probability (%) of obtaining a daily return within specific intervals

Interval (%)	[μ-1σ, μ-2σ]	[μ-2σ, μ-3σ]	[μ-3σ, μ-4σ]	[μ-4σ, μ-5σ]	[μ-5σ, μ-6σ]
Period 1979-2006	[-1.13, -2.29]	[-2.29, -3.45]	[-3.45, -4.61]	[-4.61, -5.77]	[-5.77, -6.93]
Empirical interval	[-1.13, -2.29]	[-2.29, -3.48]	[-3.48, -4.59]	[-4.59, -5.68]	[-4.59, -6.94]
Empirical	8.52	1.57	0.32	0.15	0.04
Normal	13.53	2.13	0.12	0.00	0.00
GL	7.22	1.15	0.26	0.09	0.04
GEV	7.37	1.90	0.43	0.10	0.03
Period 1979-1992	[-1.23, -2.50]	[-2.50, -3.78]	[-3.78, -5.05]	[-5.05, -6.32]	[-6.32, -7.60]
Empirical interval	[-1.23, -2.51]	[-2.51, -3.87]	[-3.87, -5.03]	[-5.03, -6.15]	[-6.15, -7.41]
Empirical	7.44	1.31	0.20	0.20	0.06
Normal	13.62	2.16	0.10	0.00	0.00
GL	6.52	1.28	0.32	0.13	0.07
GEV	6.32	1.74	0.42	0.13	0.04
Period 1993-2006	[-1.01, -2.05]	[-2.05, -3.09]	[-3.09, -4.13]	[-4.13, -5.17]	[-5.17, -6.21]
Empirical interval	[-1.01, -2.05]	[-2.05, -3.10]	[-3.10, -4.19]	[-4.19, -5.13]	[-5.13, -5.30]
Empirical	9.78	1.88	0.53	0.14	0.06
Normal	13.57	2.16	0.13	0.00	0.00
GL	9.64	1.71	0.39	0.10	0.01
GEV	9.01	2.23	0.44	0.06	0.00

This table includes the probabilities of obtaining a daily return contained within specific intervals under the corresponding distribution. It also includes the empirical probability (frequency). The bounds of these intervals are defined as numbers of daily standard deviations away from the daily mean. The row named *Empirical interval* contains the best approximation interval (based on the empirical returns) to the theoretical interval. For each period μ denotes the overall daily mean and σ denotes the overall daily standard deviation.

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