Title A new algorithm for continuous-discrete filtering with randomly delayed measurements

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Abstract

The filtering of nonlinear continuous-discrete systems is widely applicable in real-life and extensive literature is available to deal with such problems. However, all of these approaches are constrained with the assumption that the current measurement is available at every time step, although delay in measurement is natural in real-life applications. To deal with this problem, we re-derive the conventional Bayesian approximation framework for solving the continuous-discrete filtering problems. In practice, the delay is often smaller than one sampling time, which is the main case considered here. During the filtering of such systems, the actual time of correspondence should be known for a measurement received at the k^{th} time instant. In this paper, a simple and intuitively justified cost function is used to decide the time to which the measurement at k^{th} time instant actually corresponds. The performance of the proposed filter is compared with a conventional filter based on numerical integration which ignores random delays for a continuous-discrete tracking problem. We show that the conventional filter fails to track the target while the modification proposed in this paper successfully deals with random delays. The proposed method may be seen as a valuable addition to the tools available for continuous-discrete filtering in nonlinear systems.

Keywords - Nonlinear filtering, Continuous-discrete Bayesian approximation framework for nonlinear filtering, Delayed measurements, Negative Gaussian log-likelihood. integration.

1 Introduction

The dynamic state space model of a system is said to be of continuous-discrete nature, if the process model is in continuous time domain and the measurement model is in discrete time domain. The filtering problems of continuous-discrete systems commonly appear in target tracking [1], navigation [2], stochastic control [3] and in many other real-life estimation problems.

In the literature, the conventional extended Kalman filter (EKF) [1, 4], unscented Kalman filter (UKF) [5–7] and cubature Kalman filter (CKF) [8, 9] are developed for the systems having discrete dynamic state space model. Recently, these are extended to deal with continuous-discrete time domain. These extensions are reported in [10], [11] and [12]. In another development, the filtering accuracy of continuous-discrete systems is enhanced in [13].

All the above mentioned continuous-discrete filters assume that the current measurement is available at every sampling time instant. But in practice, the measurements may be randomly delayed in time due to the several factors like poor transmission speed of signals, large distance between the target and the device capturing the signal, limited bandwidth *etc*. The first two factors dominate in the target tracking and navigation applications [14] while the third one dominates in the filtering problems related to stochastic control.

Although the extent of delay is mostly small for most real-life applications, ignoring it may result in the loss of data or imperfect receipt of data. Subsequently, it may sharply reduce the estimation accuracy or may even cause for divergence of the filter.

In context of linear filtering, the delayed measurement problems are often called as out of sequence measurements (OOSM) and a few methods available for filtering which are highlighted by Mahmoud *et.al.* in [15]. In a book [16] later published by the same author, a detail discussion on the sensor captured data is highlighted and few nonlinear filtering techniques are also studied under no delay condition. However in recent years, for conventional discrete time filters, enhancements have been discussed in literature to make it enable to deal with the filtering problems with one or two steps randomly delayed measurements on a discrete time scale; see [17] and [18], for example. N step randomly delayed measurement on a discrete time scale means that the measurement received at k^{th} time instant might actually belong to k^{th} or $(k-i)^{\text{th}}$ time step, where $i = 1, 2, \cdots, N$.

From the above discussion, it appears that there are two gaps in the literature. The first is that, no algorithm is available which can deal with the randomly delayed measurements case in the continuous-discrete time domain. The second and major gap is that the existing literature deals only with an integer number of random delays in discrete time, *i.e.* a random delay which is a fraction of sampling time is not addressed.

In this paper, we address these gaps by developing a new continuous-discrete filtering heuristics for randomly delayed measurements, where the delay is a fraction of sampling time. As the delay is usually small, we focus our discussion on the case when the delay is less than one sampling time, although we will briefly outline the issues in dealing with delay exceeding one sampling time later in the paper. Hence, the measurement received at k^{th} time step may actually belong to $(k-\tau)^{th}$ time step with $0 \leq \tau < 1$. The equality symbol at zero implies that delay is allowed to be zero. In this regard, the major contribution of this paper can be considered as the proposition of a continuous-discrete filtering framework to deal with the randomly delayed measurement problems. Moreover, the proposed algorithm addresses a random delay which is a fraction of sample time, which is novel in the nonlinear filtering literature.

The rest of the paper is organized as follows. The conventional Bayesian approximation framework for continuous-discrete nonlinear filtering is outlined in section-2. In section-3, the conventional Bayesian approximation framework for continuous-discrete nonlinear filtering is modified to deal with the randomly delayed measurements. The proposed method is simulated and results are provided for a nonlinear filtering problems where the delay in measurement is probable in section-4. The discussions and conclusions are provided in section-5.

2 Bayesian approximation framework for continuous-discrete filtering

In this section, we first define the dynamic state space model for continous-discrete systems and outline the conventional discrete-time Bayesian approximation framework of filtering. Then we discuss its extension for the continuous-discrete systems with non-delayed measurements for the sake of completeness; see [10–12] for more details.

2.1 State space model for continuous-discrete systems

As discussed earlier, the process model of continuous-discrete systems is defined in continuous time domain while the measurement model is of discrete nature.

The process equation can be expressed as a stochastic differential equation [12], *i.e.*

$$dx(t) = f(x(t), t)dt + \sqrt{Q}d\beta(t), \tag{1}$$

where $x(t)$ is an n-dimensional state of a system at any time t; $f : \Re^n \times \Re \rightarrow \Re^n$ is called drift function and $\sqrt{Q}$ is a symmetric, positive definite square root of positive definite matrix Q, which is regarded as the diffusion matrix. $\beta(t)$ is an n-dimensional standard Wiener process with increment $d\beta(t)$.

The noisy measurement received at any time $t_k = kT$ (T is the measurement sampling interval) is

$$y_k = \gamma_k(x_k) + v_k, \tag{2}$$

where $y_k \in \Re^d$ is the measurement at k^{th} time instant, γ is an arbitrary function and measurement noise $v_k \in \Re^d$ is assumed to be Gaussian with zero mean and known covariance R_k.

2.2 Bayesian approximation framework of filtering

Under the Bayesian paradigm, the filtering is performed in two steps:

1. *Prediction step:* In this step, the prior probability density function is evaluated by using the Chapman-Kolmogorov equation, *i.e.*

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|y_{1:k-1})dx_{k-1}. \tag{3}$$

2. *Update step:* In this step, the posterior probability density function is evaluated by using the Baye's rule, *i.e.*

$$p(x_k|y_{1:k}) = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})}, \tag{4}$$

where the normalizing constant

$$p(y_k|y_{1:k-1}) = \int p(y_k|x_k)p(x_k|y_{1:k-1})dx_k. \tag{5}$$

During filtering, the conditional pdfs are assumed to be Gaussian. Under this assumption, the integrals which appear in (3) and (4) reduce to expectations of nonlinear functions of normally distributed random variables. Such integrals are intractable for most of the nonlinear systems and hence have to be approximated. Before discussing a methodology for numerical approximation, we take a look at a dynamic state space model of continuous-discrete systems.

In continuous-discrete filtering, the process model is dicretized at a smaller scale than the sampling time T in order to capture the continuous property of the system.

2.3 Discretization of process model

The continuous-time process model is discretized using the Itô-Taylor expansion of order 1.5 [12, 19]. Using this approximation method over time interval $(t, t+\delta)$, the process model could be written as [12, 19]

$$x(t+\delta) = x(t) + \delta f(x(t), t) + \frac{1}{2}\delta^2(\mathbb{L}_0 f(x(t), t)) + \sqrt{Q}w + (\mathbb{L}f(x(t), t))q, \tag{6}$$

where

$$\mathbb{L}_0 = \frac{\partial}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial \mathsf{x}_i} + \frac{1}{2}\sum_{j,p,q=1}^{n} \sqrt{Q}_{p,j}\sqrt{Q}_{q,j}\frac{\partial^2}{\partial \mathsf{x}_p \partial \mathsf{x}_q}$$

$$\text{and} \qquad \mathbb{L} = \sum_{i,j=1}^{n} \sqrt{Q}_{i,j}\frac{\partial}{\partial \mathsf{x}_i}$$

with $\sqrt{Q}_{i,j}$ being the i^{th} row and j^{th} column of $\sqrt{Q}$. We can consider

$$f_d(x(t), t) = x(t) + \delta f(x(t), t) + \frac{1}{2}\delta^2(\mathbb{L}_0 f(x(t), t)) \tag{7}$$

as the noise free process function. The process noise is given by n-dimensional correlated Gaussian random variables, (w, q), which are independent of state vector $x(t)$ and distributed with zero mean and covariance matrices

$$\mathbb{E}[ww^T] = \delta I_n,$$

$$\mathbb{E}[wq^T] = \frac{1}{2}\delta^2 I_n,$$

$$\text{and} \qquad \mathbb{E}[qq^T] = \frac{1}{3}\delta^3 I_n.$$

Now, the filtering of continuous-discrete systems could be performed by recursively performing two steps as outlined in Section 2.2: time update and measurement update.

2.4 Time Update

After discretizing the process equation, to compute the predicted state and its error covariance at time t_{k+1}, an m-step iterations of length δ is performed over the time interval t_k to t_{k+1}. Hence m number of intermediate time steps are considered between the two samples.

The expressions for mean and covariances at any intermediate time instant $t_k + j\delta$ could be given as

$$\begin{aligned}\hat{x}^j_{k|k} &= E[f_d(x_{k+(j-1)\delta})] \\ &\approx \int f_d(x_{k+(j-1)\delta})\aleph(x_{k+(j-1)\delta}; \hat{x}^{j-1}_{k|k}, \mathbf{P}^{j-1}_{k|k})dx_{k+(j-1)\delta},\end{aligned} \tag{8}$$

and

$$\begin{aligned}\mathbf{P}^j_{k|k} &= E[(x_{k+j\delta} - \hat{x}_{k+j\delta|k})(x_{k+j\delta} - \hat{x}_{k+j\delta|k})^T] \\ &= \int f_d(x_{k+(j-1)\delta})f_d(x_{k+(j-1)\delta})^T\aleph(x_{k+(j-1)\delta}; \hat{x}^{j-1}_{k|k}, \mathbf{P}^{j-1}_{k|k})dx_{k+(j-1)\delta} - (\hat{x}^{j-1}_{k|k})(\hat{x}^{j-1}_{k|k})^T.\end{aligned} \tag{9}$$

where $P(\cdot)$ and $\aleph(\cdot,\cdot)$ represent the probability density function and normal density function respectively.

In the time update, $\hat{x}^j_{k|k}$ and $\mathbf{P}^j_{k|k}$ are approximated recursively for increasing j. As j reaches to m, the estimate and covariance are updated at time t_{k+1} as

$$\hat{x}_{k+1|k} = \hat{x}^m_{k|k},$$

and $$\mathbf{P}_{k+1|k} = \mathbf{P}^m_{k|k}.$$

2.5 Measurement Update

In this step, the mean and covariance of measurements are evaluated as:

$$\begin{aligned}\hat{y}_{k|k-1} &= E[\gamma_k(x_k) + v_k] \\ &\approx \int \gamma_k(x_k)\aleph(x_k; \hat{x}_{k|k-1}, \mathbf{P}_{k|k-1})dx_k,\end{aligned} \tag{10}$$

and,

$$\begin{aligned}\mathbf{P}^{yy}_{k|k-1} &= E[(y_k - \hat{y}_{k|k-1})(y_k - \hat{y}_{k|k-1})^T] \\ &\approx \int \gamma_k(x_k)\gamma_k(x_k)^T\aleph(x_k; \hat{x}_{k|k-1}, \mathbf{P}_{k|k-1})dx_k - (\hat{y}_{k|k-1})(\hat{y}_{k|k-1})^T.\end{aligned} \tag{11}$$

Based on these values, the filter gain is determined as

$$K_k = \mathbf{P}^{xy}_{k|k-1}(\mathbf{P}^{yy}_{k|k-1})^{-1},$$

where

$$\begin{aligned}\mathbf{P}^{xy}_{k|k-1} &= E[(x_k - \hat{x}_{k|k-1})(y_k - \hat{y}_{k|k-1})^T] \\ &\approx \int x_k \gamma_k(x_k)^T \aleph(x_k; \hat{x}_{k|k-1}, \mathbf{P}_{k|k-1}) dx_k - (\hat{x}_{k|k-1})(\hat{y}_{k|k-1})^T.\end{aligned} \tag{12}$$

As the measurement is received at k^{th} time instant, the mean and covariance is updated as:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}), \tag{13}$$

and

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - K_k \mathbf{P}^{yy}_{k|k-1} K_k^T. \tag{14}$$

By recursively performing these two steps, the estimates and covariances are computed at every time instant and filtering of continuous-discrete systems could be carried out. We get a closed-form solution if $f_d(\cdot)$ and $\gamma_k(\cdot)$ are linear, and this solution is also optimal in the sense of minimizing variance of the state estimates. If the system is nonlinear, the integrals in (9) to (13) are intractable in general and have to be approximated numerically with the help of the sample points χ_i and corresponding weights W_i where $i = 1, 2, \cdots, n_s$ *i.e.* n_s is number of sample points.

3 Bayesian approximation framework of filtering for continuous-discrete systems with randomly delayed measurements

As discussed earlier, the random delay is caused due to many physical factors like the varying transmission speed due to unknown on way disturbances, inappropriate bandwidth, large distance between the target and receiver. To look at a few real life examples of such problems, we can consider underwater target tracking problems, packet switching network *etc*. In under-water target tracking problems, the measurement is received as the disturbances caused in water flow due to the target motion, however the actual disturbance is interrupted by many other unknown disturbances caused due to weather factors, motion of giant fishes, motion of seller and fishing boats *etc.* which causes random delay in measurement. In packet switching network [20], if the bandwidth is insufficient then many packets are queued and the time of queuing depends on the users and completely unknown for the service provider which further causes the random delay in measurement.

A simple way to deal with the randomly delayed measurements filtering problems in continuous-discrete environment is to reduce the sampling time which may reduce the loss of data. But the sampling time is a hardware constraint and hence can not be changed easily or arbitrarily.

In this section, a theoretical aspect is discussed which enables to deal with such problems. As the delay is small in practical applications, the maximum delay is considered to be less than one time step. Then, the measurement received at $(k+1)^{th}$ time step may actually belong to $(k+1-\tau)^{th}$ time instant with $0 \leq \tau < 1$.

To extract the information about the continuous delay and approximate the actual correspondence time of the measurement received at $(k+1)^{th}$ time step, we consider a sequence of m time steps between $t_k = kT$ and $t_{k+1} = (k+1)T$ as $t_k^j = t_k + j\delta$, $\delta = T/m$, $j = 1, 2, \cdots, m$. We consider the same time scale for discretization of process model as well as for generating intermediate time sequences to capture the continuous delay. Now, we can modify the previous statement and can say that the measurement received at $(k+1)^{th}$ time step may actually belong to $(k+1-(j-1)\delta)^{th}$ time instant where $j = 1, 2, ..., m$.

To model the delayed measurements, let us assume $\beta_k = [\beta_{1k} \; \beta_{2k} \; ... \; \beta_{mk}]$ be a set of Bernoulli random variables. Here, β_k is independently generated at different time steps *i.e.* β_k is independent to $\beta_i \; \forall \; i \neq k$. Moreover, each entry of β_k (*i.e.* $\beta_{jk} \; \forall \; j = 1, 2, \cdots, m$) will be either 0 or 1 with probability

$$\begin{aligned} &P(\beta_{jk} = 1) = p_j = E[\beta_{jk}], \\ &P(\beta_{jk} = 0) = 1 - p_j, \\ \text{and} \quad &E[(\beta_{jk} - p_j)^2] = p_j(1 - p_j). \end{aligned} \tag{15}$$

For any specific k, only one entry of β_k will be unity and all the remaining will be zero. The unity value at j^{th} entry represents that the measurement received at k^{th} time step actually belongs to $(k - (j-1)\delta)^{th}$ time instant. Hence, the delayed measurement model can be expressed as

$$y_k = \beta_{jk}\gamma(x_{k-(j-1)\delta}) + v_{k-(j-1)\delta}. \tag{16}$$

To this end, we define a probability vector $\Psi = [p_1, \; p_2 \;, ..., \; p_m]$ where $p_j = P(\beta_{jk} = 1)$ represents the probability that the j^{th} element of β_k is unity *i.e.* the probability that the measurement received at $(k+1)^{th}$ time step actually belongs to $(k+1-(j-1)\delta)^{th}$ instant of time. Hence, p_1 is the probability of no delay.

In later part of this section, we will see that the delayed measurement model appeared in (16) is not required for implementation of the proposed framework. However, the model is required for simulation purpose to generate the measurements in lack of real-time data.

Remark 1. *The probability of higher delay will always be less in practice. Hence,* $p_1 < p_2 < ... < p_m$*.*

In practice, the majority of measurements are received with no delay in time. In the most poor case, one out of two measurements can be considered to be delayed *i.e.* the probability of delay should not exceed 0.5. In the vector of delay probabilities, p_1 represents the probability of no delay while the probability of delay could be expressed as $(p_2 + p_3 + ... + p_m)$. Hence the numerical value of p_1 should usually be higher than 0.5.

To perform the filtering of the defined system, we modify the conventional Bayesian approximation framework of filtering. In the modified framework which is capable of dealing with the randomly delayed measurements, the filtering is carried out in four steps unlike the two steps used in conventional approach. The intractable integrals which appear during the filtering are approximated with the help of numerically chosen set of sample points and their corresponding weights

3.1 Time update

This is the first step used in the proposed method. In this step, the estimate and covariance of the states are predicted at a time step next to that for which the latest measurement is available.

To perform this step, the intractable integrals appeared in equations (8) and (9) are approximated numerically with the help of deterministically chosen points and weights. There are many filtering methods like UKF [5], CKF [8], GHF [21] *etc.* which uses different numerical techniques for generating the points set and corresponding weights. As discussed earlier, any of these conventional filters can be extended in the proposed framework. Let us consider, for a filter, the i^{th} sample point is ξ_i and the corresponding weight is W_i; $i = 1, 2, ..., n_s$ (for example, see: [5–8, 21]). Then, the estimate of states at $(kT+j\delta)^{th}$ instant of time could be approximated as

$$\hat{x}^j_{k|k} = \sum_{i=1}^{n_s} W_i \chi^j_{i,k|k}$$

$$\text{where} \qquad \chi^j_{i,k|k} = f_d(S^{j-1}_{k|k}\xi_i + \hat{x}^{j-1}_{k|k}).$$

Now, let us represent

$$\chi^j_{k|k} = [\chi^j_{1,k|k} - \hat{x}^j_{k|k} \quad \chi^j_{2,k|k} - \hat{x}^j_{k|k} \quad \dots \quad \chi^j_{n_s,k|k} - \hat{x}^j_{k|k}]W_s,$$

where W_s is a diagonal matrix of size $n_s \times n_s$ so that i^{th} diagonal element is $\sqrt{W_i}$. Then, the square-root of predicted covariance matrix, $S^j_{k|k}$ can be approximated as [12]

$$S^j_{k|k} = \text{Tria}\Big[\chi^j_{k|k} \quad \sqrt{\delta}(\sqrt{Q} + \delta/2)\mathbb{L}f(\hat{x}^{j-1}_{k|k}, kT + j\delta)\sqrt{T^3/12}\mathbb{L}f(\hat{x}^{j-1}_{k|k}, kT + j\delta)\Big].$$

where 'Tria[.]' is an operator such that Tria[A]=B if $A = B^T B$ and B is lower triangular matrix.

Remark 2. *$\hat{x}^j_{k|k}$ and $S^j_{k|k}$ are computed repeatedly for increasing j. As j reaches to m, we get the predicted values $\hat{x}_{k+1|k}$ and $S_{k+1|k}$ at $(k+1)^{th}$ time instant.*

To compute the updated estimate and the square-root factor of the covariance matrix, $\hat{x}^j_{k|k}$ and $S^j_{k|k}$ are computed repeatedly for increasing j. As j reaches to m, $\hat{x}_{k+1|k}$ and $S_{k+1|k}$ are obtained. In conventional approach, these parameters are directly used to update the measurement at t_{k+1}. But in case of the randomly delayed measurements, before proceeding to the measurement update, we need to predict the intermediate time step whose likelihood corresponding to the measurement at time t_{k+1} is maximum.

3.2 Predicting the actual correspondence time for measurement received at time t_{k+1}

This prediction is the most challenging task during filtering of continuous-discrete systems with a random delay in measurement. In this paper, we use negative Gaussian log-likelihood to choose a time instant to which the measurement corresponds, from a set of candidate time instants. To implement this heuristic, the estimate of measurements as well as its error covariance should be known at each intermediate time step.

At any intermediate time instant $t_k + j\delta$, the measurement estimate can be given as

$$\hat{y}^j_{k|k} = \sum_{i=1}^{n_s} W_i \psi^j_{i,k|k},$$

where

$$\psi^j_{i,k|k} = \gamma(S^j_{k|k}\xi_i + \hat{x}^j_{k|k}).$$

At the same time, the error covariance for the measurements can be given as

$$\mathbf{P}^j_{yy,k|k} = (\psi^j_{k|k})(\psi^j_{k|k})^T + R$$

where

$$\psi^j_{k|k} = [\psi^j_{1,k|k} - \hat{y}^j_{k|k} \ \ \psi^j_{2,k|k} - \hat{y}^j_{k|k} \ \ \ldots \ \ \psi^j_{n_s,k|k} - \hat{y}^j_{k|k}]W_s.$$

Now, at each intermediate time step, the negative Gaussian log-likelihood could be computed as

$$L^j_{k+1}(y_{k+1}) = \log(\det(\mathbf{P}^j_{yy,k|k})) + (y_{k+1} - \hat{y}^j_{k|k})^T(\mathbf{P}^j_{yy,k|k})^{-1}(y_{k+1} - \hat{y}^j_{k|k}) \tag{17}$$

Remark 3. *As the conditional density of measurements are assumed to be Gaussian, the negative Gaussian log-likelihood is a natural choice for choosing the most likely prior instant for measurement sampled at time* t_{k+1}.

Before receipt of the measurement y_{k+1} at time instant t_{k+1}, it's negative Gaussian log-likelihood is available at each delayed intermediate time step. The measurement y_{k+1} is most likely to belong to an intermediate time step for which the negative Gaussian log-likelihood is minimum. To this regard, let j^* be such that $y_{k+1} = y(t_k + j^*\delta)$. Then we choose j^* such that

$$j^* = \arg \min_{j \in [0,m]} L^j_{k+1}(y_{k+1}). \tag{18}$$

As the measurement received at t_{k+1} time instant actually belongs to the time instant $t_k + j^*\delta$, the measurement y_{k+1} can be used to approximate the posterior estimates at the intermediate time step $t_k + j^*\delta$, but not at t_{k+1}.

3.3 Measurement update at intermediate time step

In this step, the measurement update is performed at the intermediate time instant $(t_k + j^*)$ and based on that, the posterior estimates of states are obtained at this instant. To perform this step, we need the updates of state belonging to the same time step. It could be directly recalled from the time update step as $\hat{x}^{j^*}_{k|k}$ and $S^{j^*}_{k|k}$.

At the intermediate step $t_k + j^*$, the estimate of measurement could be approximated as

$$\hat{y}^{j^*}_{k|k} = \sum_{i=1}^{n_s} W_i \gamma(\chi^{j^*}_{i,k|k}),$$

where

$$\chi^{j^*}_{i,k|k} = S^{j^*}_{k|k} \xi_i + \hat{x}^{j^*}_{k|k}.$$

Let us say

$$\chi^{j^*}_{k|k} = [\chi^{j^*}_{1,k|k} - \hat{x}^{j^*}_{k|k} \quad \chi^{j^*}_{2,k|k} - \hat{x}^{j^*}_{k|k} \quad \dots \quad \chi_{n_s,k+1|k} - \hat{x}_{k+1|k}] W_s$$

and

$$Y^{j^*}_{k|k} = [\gamma(\chi_{1,k|k})^{j^*} - \hat{y}^{j^*}_{k|k} \quad \gamma(\chi_{2,k|k})^{j^*} - \hat{y}^{j^*}_{k|k} \quad \dots \quad \gamma(\chi_{n_s,k|k})^{j^*} - \hat{y}^{j^*}_{k|k}] W_s.$$

Then, the covariance matrices of measurement can be given as

$$\mathbf{P}^{j^*}_{yy,k|k} = (Y^{j^*}_{k|k})(Y^{j^*}_{k|k})^T + R$$

and

$$\mathbf{P}^{j^*}_{xy,k|k} = (\chi^{j^*}_{k|k})(Y^{j^*}_{k|k})^T.$$

Let us assume

$$\begin{pmatrix} G_{11} & 0_{n\times d} \\ G_{21} & G_{22} \end{pmatrix} = \text{Tria} \begin{pmatrix} Y^{j^*}_{k|k} & \sqrt{R} \\ \chi^{j^*}_{k|k} & 0_{n\times d} \end{pmatrix}.$$

Hence,

$$\begin{aligned}
\begin{pmatrix} \mathbf{P}^{j^*}_{yy,k|k} & \mathbf{P}^{j^*}_{yx,k|k} \\ \mathbf{P}^{j^*}_{xy,k|k} & \mathbf{P}^{j^*}_{k|k} \end{pmatrix} &= \begin{pmatrix} Y^{j^*}_{k|k} & \sqrt{R} \\ \chi^{j^*}_{k|k} & 0_{n\times d} \end{pmatrix} \begin{pmatrix} Y^{j^*}_{k|k} & \sqrt{R} \\ \chi^{j^*}_{k|k} & 0_{n\times d} \end{pmatrix}^T \\
&= \begin{pmatrix} G_{11}G_{11}^T & G_{11}G_{21}^T \\ G_{21}G_{11}^T & G_{21}G_{21}^T + G_{22}G_{22}^T \end{pmatrix}.
\end{aligned}$$

Subsequently, the Kalman gain can be given as

$$K_k^{j^*} = \mathbf{P}_{xy,k+1|k}^{j^*}(\mathbf{P}_{yy,k+1|k}^{j^*})^{-1} = G_{21}G_{11}^{-1}. \tag{19}$$

The updated state estimate can be given as

$$\hat{x}_{k|k+1}^{j^*} = \hat{x}_{k|k}^{j^*} + K_k^{j^*}(y_{k+1} - \hat{y}_{k|k}^{j^*}). \tag{20}$$

The updated covariance matrix can be given as

$$\mathbf{P}_{k|k+1}^{j^*} = \mathbf{P}_{k|k}^{j^*} - K_k^{j^*}\mathbf{P}_{yy,k|k}^{j^*}(K_k^{j^*})^T = (G_{22}G_{22}^T).$$

Hence, the square-root of updated covariance can be given as

$$S_{k|k+1}^{j^*} = G_{22}. \tag{21}$$

The posterior estimates of states, $\hat{x}_{k|k+1}^{j^*}$ and $\mathbf{P}_{k|k+1}^{j^*}$ belong to the time step $t_k + j^*\delta$. However, for filtering we need to compute these estimates at t_{k+1}. Hence, these parameters are further updated to get the estimates at t_{k+1}.

3.4 Updating the intermediate estimates to approximate next step estimates

To obtain $\hat{x}_{k+1|k+1}$ and $S_{k+1|k+1}$, the following steps can be performed in recursion, varying i from 1 to $(m - j^*)$:

1. Transform the sample points with the known mean and covariance at $t_k + j^*\delta$:

$$\Theta_{i,k|k+1}^{j^*+i} = S_{k|k+1}^{j^*+i}(\xi_i) + \hat{x}_{k|k+1}^{j^*+i}$$

2. Update the sample points by propagating it through the discretized process equation (or state equation):

$$\chi_{i,k|k+1}^{j^*+i+1} = f_d(\Theta_{i,k|k+1}^{j^*+i})$$

3. Compute the estimate of state at the intermediate time step:

$$\hat{x}_{k|k+1}^{j^*+i+1} = \sum_i^{n_s} W_i\chi_{i,k|k+1}^{j^*+i+1}$$

4. Compute the square-root factor of covariance matrix:

$$S_{k|k+1}^{j^*+i+1} = \text{Tria}\Big[\chi_{k|k+1}^{j^*+i+1} \quad \sqrt{\delta}(\sqrt{Q}+\delta/2)\mathbb{L}f(\hat{x}_{k|k+1}^{j^*+i}, kT+\delta(j^*+i)) \quad \sqrt{T^3/12}\mathbb{L}f(\hat{x}_{k|k+1}^{j^*+i}, kT+\delta(j^*+i))\Big]$$

where

$$\chi_{k|k+1}^{j^*+i+1} = [\chi_{1,k|k+1}^{j^*+i+1} - \hat{x}_{k|k+1}^{j^*+i+1} \quad \chi_{2,k|k+1}^{j^*+i+1} - \hat{x}_{k|k+1}^{j^*+i+1} \dots \quad \chi_{n_s,k|k+1}^{j^*+i+1} - \hat{x}_{k|k+1}^{j^*+i+1}]W_s$$

As i reaches to $(m-j^*)$, we get the next step estimate and square-root factor of covariance *i.e.* $\hat{x}_{k+1|k+1}$ and $S_{k+1|k+1}$ respectively.

The filtering of continuous-discrete systems with randomly delayed measurements could be carried out by performing these four steps in recursion.

Remark 4. *The proposed algorithm can conceptually be employed to deal with delays which exceed one sampling time, up to a fixed number of sampling times. For example, if the maximum possible random delay is two sampling times, the measurement received at $(k+1)^{th}$ time step may actually belong to $(k+1-(j-1)\delta)^{th}$ time step where $j = 1, 2, \cdots, 2m$. One can then carry out the time update and find the most likely correspondence time using the methods outlined in sections 3.1 and 3.2 respectively. Once the measurement update parameters are computed by following the section 3.3, an iterative update through the intermediate steps will be required from $j^\star$ to $2m$, similar to the section 3.4. This will, however, ignore the measurement $y(k)$ in constructing $\hat{x}_{k+1|k+1}$, which is obviously not desirable.*

Remark 5. *In practice, the exact approximation of the probability of delay i.e. $p_i \quad \forall \quad i \in [1, m]$ is very difficult. The proposed approach does not use these values during filtering. In this particular sense, the proposed method is 'model-free' when it comes to dealing with random delays, which may be seen as a major advantage.*

4 Simulation

In this paper, the proposed method is applied to solve a nonlinear filtering problem with continuous-discrete dynamic state space model and the results are compared with the existing approach which ignores the possible delay in measurements. The problem considered here is an air-traffic control problem, trajectory of which is described in [12].

The state space equation of the aircraft is

$$dx(t) = f(x(t))dt + \sqrt{Q}d\beta(t), \tag{22}$$

where $x = \begin{bmatrix} \epsilon & \dot{\epsilon} & \eta & \dot{\eta} & \zeta & \dot{\zeta} & \omega \end{bmatrix}^T$ is a seven-dimensional state vector with ϵ, η and ζ representing positions, $\dot{\epsilon}$, $\dot{\eta}$ and $\dot{\zeta}$ representing velocities in three dimensional X, Y and Z Cartesian coordinates respectively, and ω representing the turn rate. The drift function is $f(x) = \begin{bmatrix} \dot{\epsilon} & -\omega\dot{\eta} & \dot{\eta} & \omega\dot{\epsilon} & \dot{\zeta} & 0 & 0 \end{bmatrix}^T$, which shows that the motions in horizontal.

The noise vector is $\beta(t) = [\beta_1(t)\ \beta_2(t)\ \ldots\ \beta_7(t)]^T$, where $\beta_i(t)$ is standard Brownian motion independent of $\beta_k(t)$. It accounts for modeling error appeared due to the wind forces, turbulance etc.

The motion is severely nonlinear and the degree of nonlinearity depends on the turn rate parameter, ω. After applying Itô-Taylor expansion of order 1.5 to (22), the discretized process model could be obtained as

$$x_k^{j+1} = f_d(x_k^j) + \sqrt{Q}w + (\mathbb{L}f(x_k^j)q, \tag{23}$$

where

$$f_d(x) = \begin{bmatrix} \epsilon + \delta\dot{\epsilon} - \frac{\delta^2}{2}\omega\dot{\eta} \\ \dot{\epsilon} - \delta\omega\dot{\eta} - \frac{\delta^2}{2}\omega^2\dot{\epsilon} - \frac{\delta^2}{2}\sigma_1\sigma_2 \\ \eta + \delta\dot{\eta} + \frac{\delta^2}{2}\omega\dot{\epsilon} \\ \dot{\eta} + \delta\omega\dot{\epsilon} - \frac{\delta^2}{2}\omega^2\dot{\eta} + \frac{\delta^2}{2}\sigma_1\sigma_2 \\ \zeta + \delta\dot{\zeta} \\ \dot{\zeta} \\ \omega \end{bmatrix},$$

$$\mathbb{L}f(x) = \begin{bmatrix} 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_1\omega & 0 & 0 & -\sigma_2\dot{\eta} \\ 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1\omega & 0 & 0 & 0 & 0 & \sigma_2\dot{\epsilon} \\ 0 & 0 & 0 & 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and, $Q = diag\left[(0 \quad \sigma_1^2 \quad 0 \quad \sigma_1^2 \quad 0 \quad \sigma_1^2 \quad \sigma_2^2)\right]^T$ is the diffusion matrix for process noise given as with $\sigma_1 = \sqrt{2}m$, $\sigma_2 = 2.85 \times 10^{-7}\ ^o/sec$.

The trajectory of the target is plotted in Fig. (1), for two different values of ω. To plot the trajectory, we consider the number of iteration per second as 100 and $x_0 = [1000\ 0\ 2650\ 150\ 200\ 0\ \omega]^T$.

The measurements obtained at regular interval of time $T = 1sec$ is

$$\begin{bmatrix} r_k \\ \theta_k \\ \phi_k \end{bmatrix} = \begin{bmatrix} \sqrt{\epsilon_k^2 + \eta_k^2 + \zeta_k^2} \\ tan^{-1}\left(\frac{\eta_k}{\epsilon_k}\right) \\ tan^{-1}\left(\frac{\zeta_k}{\sqrt{\epsilon_k^2+\eta_k^2}}\right) \end{bmatrix} + v_k.$$

The radar is located at origin and is equipped to measure the range r, the azimuth angle θ and the elevation angle ϕ. The measurement noise is given as $v_k \sim \aleph(0, R)$ with $R = diag(\left[\sigma_r^2 \quad \sigma_\theta^2 \quad \sigma_\phi^2)\right]$, where $\sigma_r = 0.1m$, $\sigma_\theta = 0.1^o$ and $\sigma_\phi = 0.1^o$ are standard deviations for range, azimuth and elevation respectively.

The simulation is performed considering the initial estimate $\hat{x}_{0|0}$ as normally distributed with mean x_0 and covariance $\mathbf{P}_{0|0}$=diag([100 10 100 1 0 0 $(\sqrt{0.1}deg/s)^2$]). During the simulation, the number of iterations per sampling interval is considered as $m = 10$.

To compare the performance, we have considered two different scenarios with different probability vector as

Scenario 1: $\Psi = [0.60\ 0.10\ 0.06\ 0.05\ 0.04\ 0.03\ 0.03\ 0.03\ 0.03\ 0.03]$,

Scenario 2: $\Psi = [0.50\ 0.20\ 0.1\ 0.05\ 0.04\ 0.03\ 0.03\ 0.03\ 0.02\ 0.00]$.

As discussed earlier, the probability of no delay should not be less than 0.5. Hence the first entry of probability vector should not be chosen below 0.5. Further, it is reasonable to assume that probability of a longer delay is no more than that of a shorter delay in all cases, which is reflected in our choice of Ψ. As soon as Ψ is defined, $\beta_{jk}\ \forall\ j = 1, 2, \cdots, m$ could be generated and hence the delayed measurement could be obtained using equation (16).

The simulation is carried out for continuous-discrete extensions of UKF, CKF and recently introduced CQKF [22, 23]. Under no delay condition, these filters are abbreviated as UKF-CD, CKF-CD and CQKF-CD respectively. However, UKF-CD_RD, CKF-CD_RD and CQKF-CD_RD represents their extensions under the proposed framework.

The simulation is performed for a period of $100sec$ and 100 Monte-Carlo runs are performed to ensure the performance. To compare the results, the root mean square error (RMSE) of radial position and velocity are calculated along the steps. For $\omega = 2^o/sec$ and $\omega = 3^o/sec$, the RMSE plots of range and velocity are shown in figure-2 to figure-5, under two different scenarios mentioned earlier. In every figure, zoomed plot of RMSEs of UKF-CD_RD, CKF-CD_RD and CQKF-CD_RD are shown between 30 to 60 seconds.

The RMSEs of UKF-CD, CKF-CD and CQKF-CD are very high while the RMSEs due to UKF-CD_RD, CKF-CD_RD and CQKF-CD_RD seem to be touching the zero axis. The high RMSEs of UKF-CD, CKF-CD and CQKF-CD show that the conventional approach fails to track the path of the target, if the measurement arrives to the filter with some delay. However, small RMSEs for UKF-CD_RD, CKF-CD_RD and CQKF-CD_RD assure that the proposed extension could enable the filters to successively trace the target.

As mentioned earlier in Remark 4, the proposed algorithm can be implemented for higher delay cases as well, although it ignores $y(k)$ in constructing $\hat{x}_{k+1|k+1}$ which may lead to significantly reduced estimation accuracy. For a comparative study, the RMSEs of range and velocity for one delay and two delay cases are plotted in figure-6. The two delay extensions of UKF, CKF and CQKF has been abbreviated as UKF-CD_2RD, CKF-CD_2RD and CQKF-CD_2RD. For one delay case, the Ψ was considered as the one used in *Scenario 1* while for two delay case it was taken as $\Psi = [3/5\ 1/20\ 1/20\ 0.03\ 0.03\ 0.025\ 0.025\ 0.02\ 0.02\ 0.015\ 0.015\ 0.015\ 0.015\ 0.015\ 0.015\ 0.015\ 0.015\ 0.01\ 0.01\ 0.01]$ *i.e.* the probability of no delay was fixed at 0.6 (similar to one delay case). The RMSE for velocity was similar for one and two delay cases. However, it was significantly higher in the two delay case than in one delay case, for range. This can be expected as the intermediate measurement is ignored in the two delay case.

5 Discussions and conclusions

In this paper, a novel approach is discussed to perform the filtering under the continuous-discrete time domain if the measurement is expected to be randomly delayed on continuous time scale. In other words, the measurement received at time t_{k+1} may belong to any time t so that $t_k < t \leq t_{k+1}$. To the best of authors' knowledge, for the first time the delay is introduced in the literature of nonlinear filtering under continuous-discrete time domain. The proposed algorithm is based on an intuitively appealing idea of using maximum likelihood to choose the time instant to which an observation might actually belong.

For simulation purpose, a maneuvering target tracking problem is implemented. It is found that the conventional approach fails to trace the path of the target, while the proposed method could track it successfully. Due to the failure of conventional method in the delayed measurement environment, the authors recommend the use of the proposed approach if a small delay is probable. Combining this proposed algorithm with the existing methods for tackling delays which exceed the measurement sampling time is a topic of current research.

The proposed algorithm works if the delay is a fraction of sampling time. While it can be used for delays exceeding one sampling time, its performance is unsatisfactory due to ignored measurements. Combining the proposed algorithm for dealing with a fractional delay with the existing methods for dealing with integer number of delays, to tackle delays exceeding one measurement sampling time is a topic of current research.

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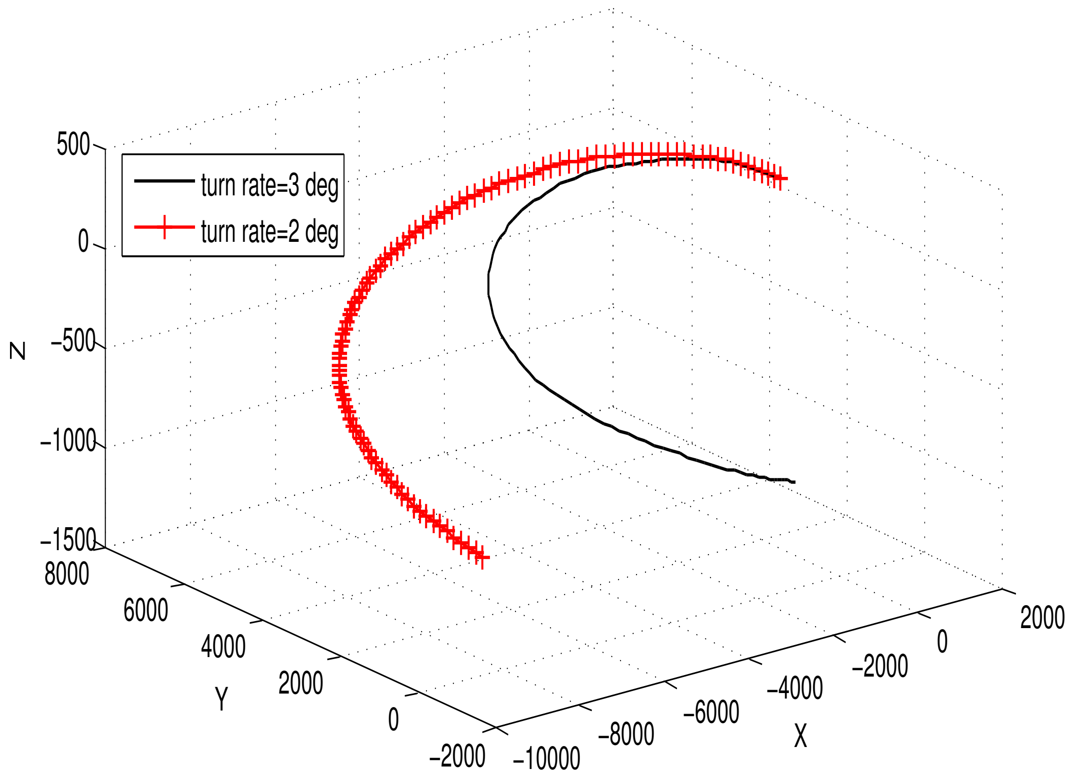
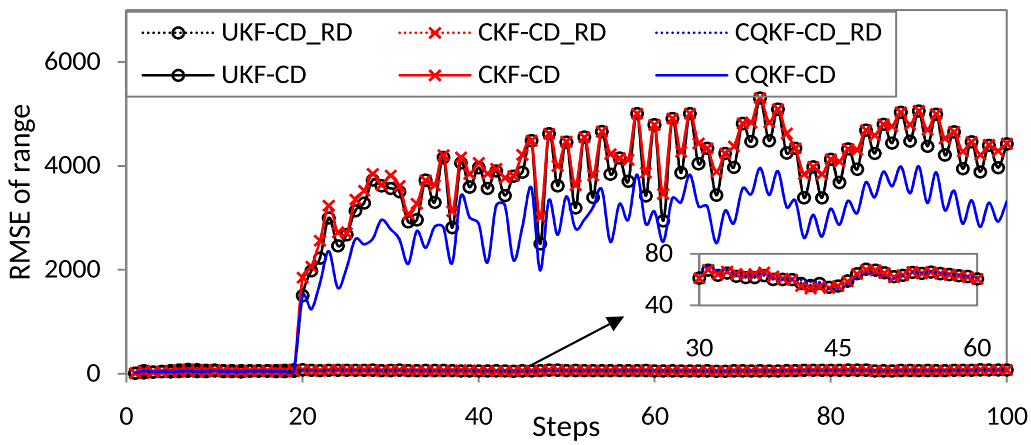
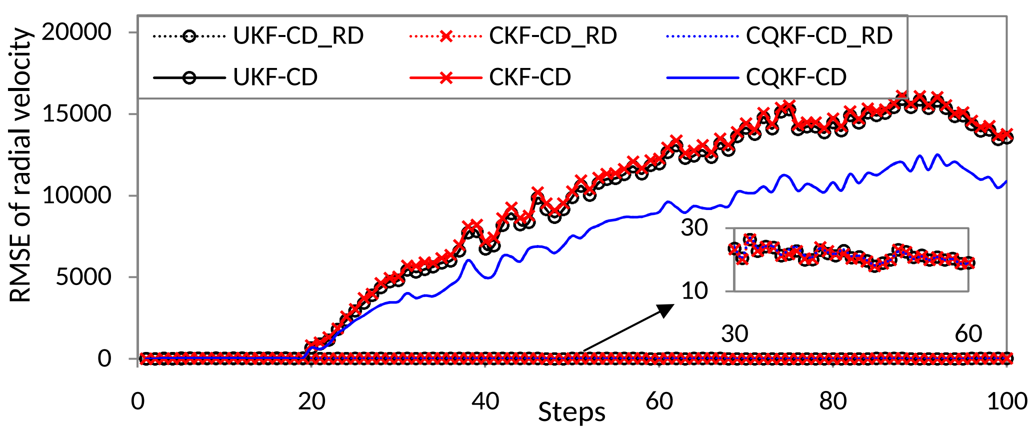


Figure 1: Trajectories of motion of aircraft for varying turn rate, ω



(a)



(b)

Figure 2: First scenario: RMSE plot for $\omega = 2^o/s$ (a) range in m (b) velocity in m/sec

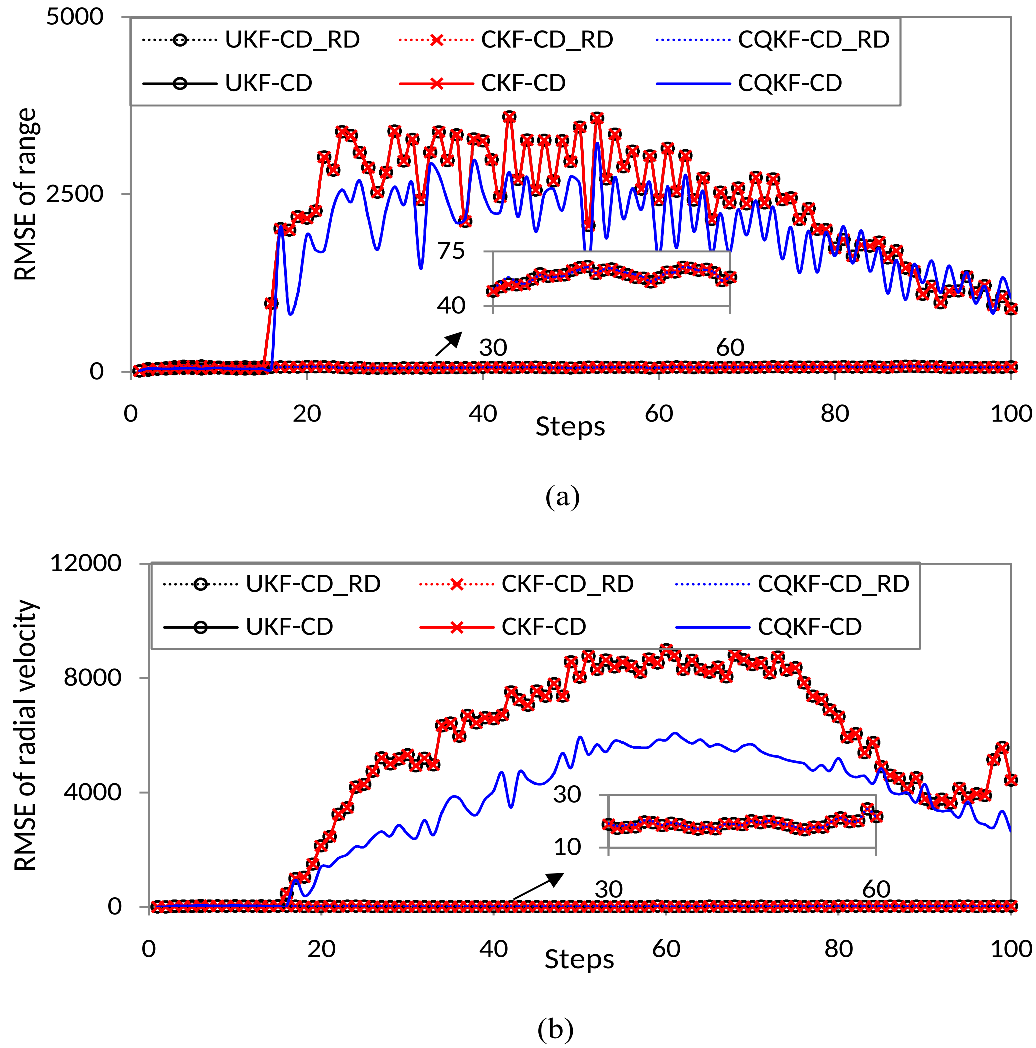
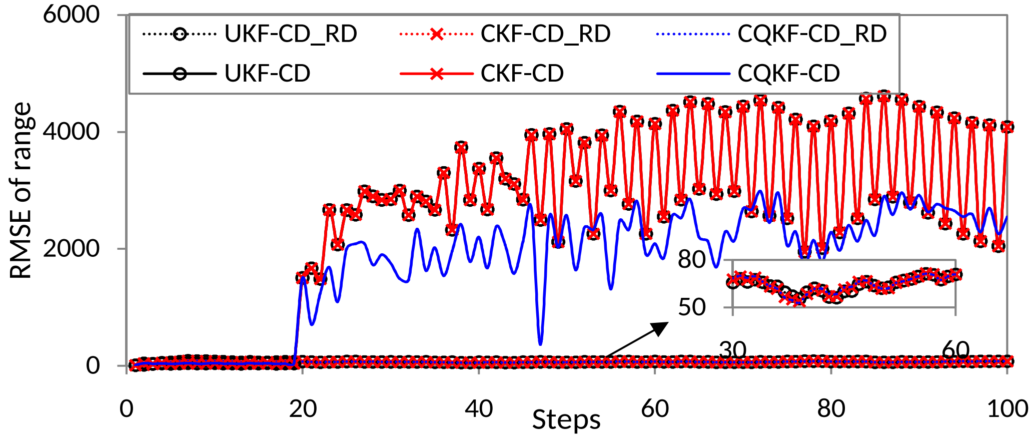
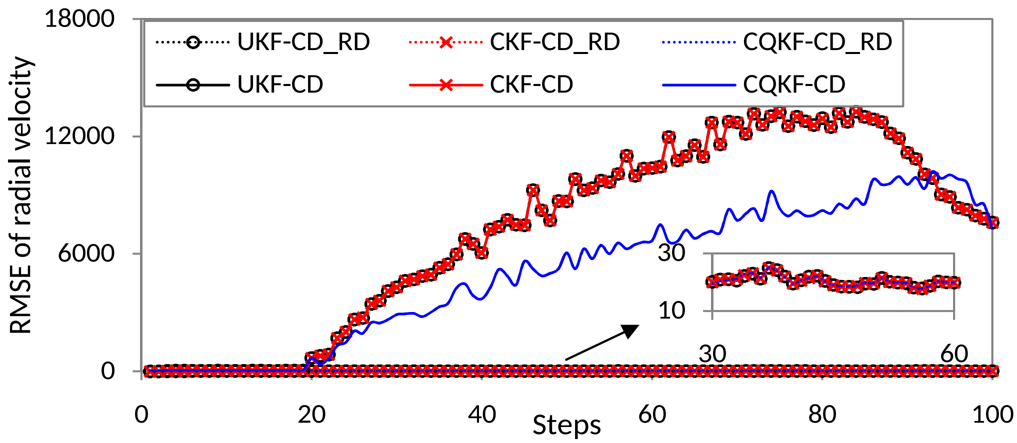


Figure 3: First scenario: RMSE plot for $\omega = 3^o/s$ (a) range in m (b) velocity in m/sec

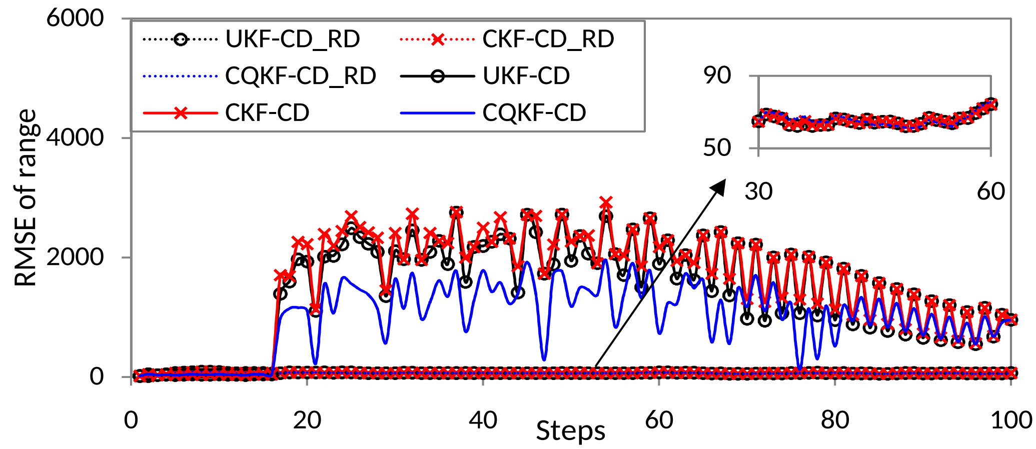


(a)

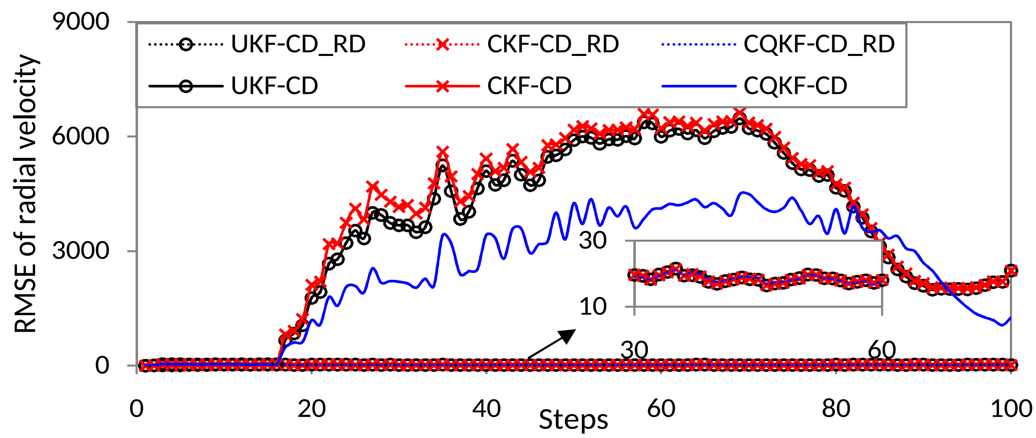


(b)

Figure 4: Second scenario: RMSE plot for $\omega = 2^o/s$ (a) range in m (b) velocity in m/sec

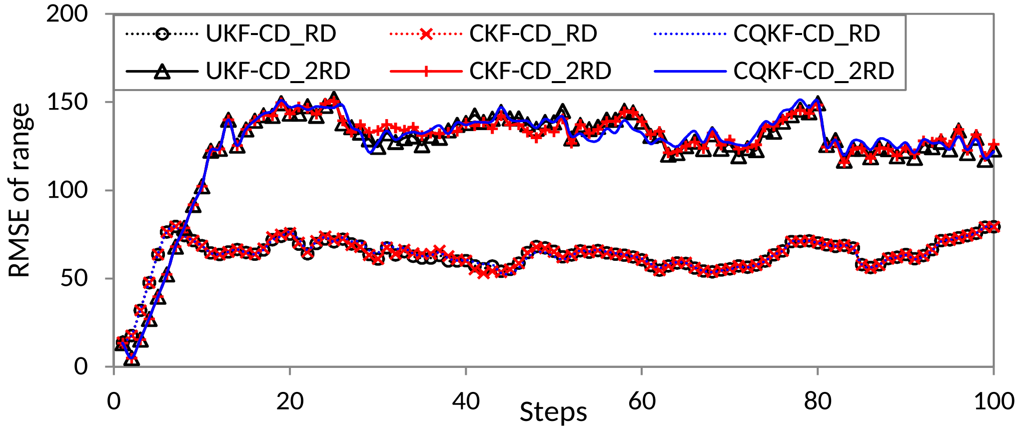


(a)

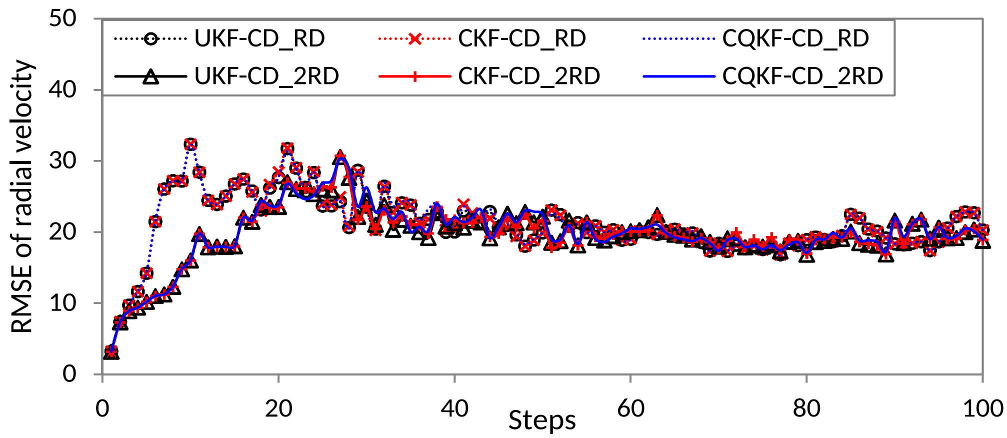


(b)

Figure 5: Second scenario: RMSE plot for $\omega = 3^o/s$ (a) range in m (b) velocity in m/sec



(a)



(b)

Figure 6: RMSE plot for $\omega = 2^o/s$ (a) range in m (b) velocity in m/sec